

## TA section 2

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# HW 1: Part (I)

## §4.2#1, #9, #14, §4.3#6, #10

1. 設  $X_n$  有  $P(n\lambda)$  分佈,  $n \geq 1$ ,  $\lambda > 0$ 。試證  $X_n/n \xrightarrow[n \rightarrow \infty]{P} \lambda$ , 且  $(X_n - n\lambda)/(\sqrt{n\lambda}) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1)$ 。

- $\mathbb{E}[X_n] = \text{Var}[X_n] = n\lambda$ ;  $\mathbb{E}[X_n/n] = \lambda$ ,  $\text{Var}[X_n/n] = \lambda/n$ , and by Chebyshev's inequality,

$$\mathbb{P}(|X_n/n - \mathbb{E}[X_n/n]| > \epsilon) \leq \text{Var}[X_n/n]/\epsilon^2, \quad \forall \epsilon > 0.$$

Then,

$$\mathbb{P}(|X_n/n - \lambda| > \epsilon) \leq \lambda/(n\epsilon^2) \rightarrow 0$$

as  $n \rightarrow \infty$ . So,  $X_n/n \xrightarrow{P} \lambda$ .

- 另解:  $X_n := \xi_1 + \cdots + \xi_n$   $\xi_i \sim i.i.d. P(\lambda)$  with  $\mathbb{E}[\xi_i] = \lambda$ . By LLN,  $X_n/n \xrightarrow{P} \mathbb{E}[\xi_i] = \lambda$ . So,  $X_n/n \xrightarrow{P} \lambda$ .

- Let  $X_n := \xi_1 + \cdots + \xi_n$ ,  $\xi_i \sim i.i.d.P(\lambda)$  with  $\mathbb{E}[\xi_i] = \text{Var}[\xi_i] = \lambda$ .
- By CLT,

$$(X_n - n\lambda)/\sqrt{n\lambda} = (X_n/n - \lambda)/\sqrt{\lambda/n} \xrightarrow{d} N(0, 1)$$

as  $n \rightarrow \infty$ .

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9. 設  $X_n$  有  $\Gamma(n, \lambda)$  分佈,  $n \geq 1$ 。試證  $X_n/n \xrightarrow[n \rightarrow \infty]{P} \lambda$ , 並利用中央極限定理, 估計  $n$  很大時,  $P(X_n \leq x)$  之值,  $x \in R$ 。

- $\mathbb{E}[X_n] = n\lambda$ ,  $\text{Var}[X_n] = n\lambda^2$ ;  $\mathbb{E}[X_n/n] = \lambda$ ,  $\text{Var}[X_n/n] = \lambda^2/n$ , and by Chebyshev's inequality,

$$\mathbb{P}(|X_n/n - \mathbb{E}[X_n/n]| > \epsilon) \leq \text{Var}[X_n/n]/\epsilon^2, \forall \epsilon > 0.$$

Then,

$$\mathbb{P}(|X_n/n - \lambda| > \epsilon) \leq \lambda^2/(n\epsilon^2) \rightarrow 0$$

as  $n \rightarrow \infty$ . So,  $X_n/n \xrightarrow{P} \lambda$ .

- 另解:  $X_n := \xi_1 + \cdots + \xi_n$   $\xi_i \sim i.i.d. \Gamma(1, \lambda)$  with  $\mathbb{E}[\xi_i] = \lambda$ . By LLN,  $X_n/n \xrightarrow{P} \mathbb{E}[\xi_i] = \lambda$ . So,  $X_n/n \xrightarrow{P} \lambda$ .

- $X_n := \xi_1 + \cdots + \xi_n$   $\xi_i \sim i.i.d.\Gamma(1, \lambda)$  with  $\mathbb{E}[\xi_i] = \lambda$  and  $\text{Var}[\xi_i] = \lambda^2$ .

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$$\frac{X_n/n - \lambda}{\lambda/\sqrt{n}} = \frac{(X_n - n\lambda)}{\lambda\sqrt{n}}.$$

- By CLT,

$$\frac{X_n/n - \lambda}{\lambda/\sqrt{n}} \xrightarrow{d} N(0, 1)$$

as  $n \rightarrow \infty$ .

- So,  $\mathbb{P}(X_n \leq x) \approx \Phi\left(\frac{x - n\lambda}{\lambda\sqrt{n}}\right)$ , as  $n \rightarrow \infty$ .

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14. 設  $X_1 \sim \mathcal{U}[0, 1]$ , 令  $X_n = X_1^n$ ,  $n \geq 1$ 。試證  $X_n \xrightarrow[n \rightarrow \infty]{p} 0$ 。

- Given any  $\epsilon > 0$ ,  
 $\mathbb{P}(|X_n| > \epsilon) = \mathbb{P}(X_1 > \epsilon^{1/n}) = 1 - \epsilon^{1/n}.$

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$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n| > \epsilon) = 1 - \lim_{n \rightarrow \infty} \epsilon^{1/n} = 0.$$

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6. 設  $X_1, X_2, X_3$  為獨立的隨機變數, 且  $X_i$  有  $\mathcal{N}(i, i^2)$  分佈,  $i = 1, 2, 3$ 。試利用  $X_1, X_2, X_3$  的函數, 分別造出有如下的分佈。

(i)  $\chi_3^2$ , (ii)  $\mathcal{T}_2$ , (iii)  $\mathcal{F}_{1,2}$ 。

(iii)

- Let  $Z_i := (X_i - i)/i \sim N(0, 1)$ , for  $i = 1, 2, 3$ .
- $\because Z_1^2 \sim \chi^2(1), Z_2^2 + Z_3^2 \sim \chi^2(2), \therefore F = \frac{Z_1^2}{Z_2^2 + Z_3^2} \sim F(1, 2)$ .
- So, let  $U := Z_1^2 = (X_1 - 1)^2/1, V := Z_2^2 = (X_2 - 2)^2/2, W := (X_3 - 3)^2/3$ , we have  $\frac{U}{V+W} \sim F(1, 2)$ .



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10. 設  $X_1$  與  $X_2$  獨立, 且皆有  $\mathcal{N}(0, 25)$  分佈。令  $D = \sqrt{X_1^2 + X_2^2}$ 。試求  $P(D \leq 12.25)$ 。(解. 約0.95)

- $X_i = 5Z_i, Z_i \sim N(0, 1)$  for  $i = 1, 2$ . Then,  
 $D = (X_1^2 + X_2^2)^{1/2} = 5(Z_1^2 + Z_2^2)^{1/2} = 5(\chi^2(2))^{1/2}$
- $\mathbb{P}(D \leq 12.25) = \mathbb{P}(\chi^2(2) \leq (12.25/5)^2) = \mathbb{P}(\chi^2(2) \leq (2.45)^2) = 1 - \exp(-(2.45)^2/2) \approx 0.95$ .  
 (or 查卡方機率值表)
- Note 1:  $W := \chi^2(k), f_W(w) = (1/2) \exp(-w/2)$ . Then,  
 $\mathbb{P}(W \leq x) = \int_{w=0}^x f_W(w) dw = 1 - \exp(-x/2)$ .
- Note 2: by CLT,

$$\mathbb{P}(\chi^2(2) \leq (2.45)^2) \approx \Phi(((2.45)^2 - 2)/\sqrt{4}) \approx 0.977.$$

不建議用 CLT 近似, 因為自由度只有 2.