

TA section 6

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Review Exercises:
§3.1-#2, #5, #8, #10, §3.2-#13

§3.1-#2

2. 設 (X, Y) 之聯合 p.d.f. 為 $f(x, y) = 6xy^2, 0 < x, y < 1$ 。

(i) 試驗證 f 為一 p.d.f.;

(ii) 試求 $P(X + Y \geq 1)$;

(iii) 試求 $P(1/2 < X < 3/4)$ 。

• (i). check pdf: (a): $6xy^2 \geq 0, \forall x, y \in (0, 1)$; (b) $\int_0^1 \int_0^1 6xy^2 dx dy = 1$.

• (ii). $\mathbb{P}(X + Y \geq 1) = \int \int_{\{x+y \geq 1; 0 < x, y < 1\}} f(x, y) dx dy$

$$= \int_0^1 \int_{1-y}^1 6xy^2 dx dy = \int_0^1 (3x^2 y^2) \Big|_{x=1-y}^{x=1} dy = \int_0^1 (6y^3 - 3y^4) dy = 9/10.$$

$$(\text{Or, } = \int_0^1 \int_{1-x}^1 6xy^2 dy dx)$$

• (iii).

$$\begin{aligned}\mathbb{P}(1/2 < X < 3/4) &= \mathbb{P}(1/2 < X < 3/4, 0 < Y < 1) \\ &= \int_0^1 \int_{1/2}^{3/4} 6xy^2 dx dy \\ &= \int_0^1 3y^2 (x^2) \Big|_{x=1/2}^{x=3/4} dy \\ &= (5/16) \int_0^1 3y^2 dy = 5/16.\end{aligned}$$

§3.1 #5

5. 設 (X, Y) 之聯合 p.d.f. 為

$$f(x, y) = c(x + 2y), 0 < x < 2, 0 < y < 1.$$

- (i) 試決定常數 c 之值;
- (ii) 試求 X 之邊際 p.d.f.;
- (iii) 試求 $Z = 9/(X + 1)^2$ 之 p.d.f.;
- (iv) 試求 (X, Y) 之聯合分佈函數。

$$\bullet \text{ (i). } \int_0^1 \int_0^2 c(x + 2y) dx dy = 1 \Rightarrow c \int_0^1 (x^2/2 + 2xy) \Big|_{x=0}^{x=2} dy = \\ c \int_0^1 (2 + 4y) dy = c(2y + 2y^2) \Big|_{y=0}^{y=1} = 4c = 1 \Rightarrow c = 1/4.$$

- (ii). For $0 < x < 2$,

$$f_X(x) = \frac{1}{4} \int_0^1 (x + 2y) dy = (x + 1)/4, \quad 0 < x < 2.$$

- (iii). For $0 < x < 2 \Rightarrow 1 < z < 9$.

$$\mathbb{P}(Z \leq z) = \mathbb{P}\left(\frac{9}{(X+1)^2} \leq z\right) = \mathbb{P}(3z^{-1/2} - 1 \leq X) = 1 - \mathbb{P}(X \leq 3z^{-1/2} - 1), \text{ then}$$

$$\begin{aligned} f_Z(z) &= f_X(3z^{-1/2} - 1)(3/2)z^{-3/2} \\ &= (1/4)(3z^{-1/2})(3/2)z^{-3/2} \\ &= (9/8)z^{-2}, \quad 1 < z < 9. \end{aligned}$$

- check: $\int_1^9 (9/8)z^{-2} dz = 1$.

- (iv). $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$.

$$F_{X,Y}(x,y) = \frac{1}{4} \int_0^y \int_0^x (t+2s) dt ds = \cdots = \frac{1}{8}x^2y + \frac{1}{4}xy^2,$$

for $0 < x < 2$, $0 < y < 1$.

§3.1 #8

8. 設 (X, Y) 之聯合p.d.f.為 $f(x, y) = x + y, 0 \leq x, y \leq 1$ 。試求 $P(X > \sqrt{Y})$ 。

$$\begin{aligned} \bullet \mathbb{P}(X > \sqrt{Y}) &= \int_0^1 \int_{\sqrt{y}}^1 (x + y) dx dy = \int_0^1 \left[\frac{x^2}{2} + xy \right]_{\sqrt{y}}^1 dy = \\ &= \int_0^1 (1/2 + y/2 - y^{3/2}) dy = 7/20. \\ (\text{or, } \mathbb{P}(X^2 > Y) &= \int_0^1 \int_0^{x^2} (x + y) dy dx = \int_0^1 (xy + y^2/2) \Big|_0^{x^2} dx = \\ &= \int_0^1 (x^3 + x^4/2) dx = 7/20.) \end{aligned}$$

§3.1 #10

10. 設 (X, Y) 之聯合p.d.f.為 $f(x, y) = \lambda^2 e^{-\lambda(x+y)}$, $x, y \geq 0$ 。試求 $P(X \geq 2Y)$ 。

• $\mathbb{P}(X \geq 2Y) = \mathbb{P}(X > 0, Y \leq X/2) =$

$$\begin{aligned} \int_0^\infty \int_0^{x/2} \lambda^2 e^{-\lambda(x+y)} dy dx &= \int_0^\infty \lambda e^{-\lambda x} \left(\int_0^{x/2} \lambda e^{-\lambda y} dy \right) dx \\ &= \int_0^\infty \lambda e^{-\lambda x} \mathbb{P}(Y < x/2) dx, \quad Y \sim \epsilon(\lambda), \\ &= \int_0^\infty \lambda e^{-\lambda x} (1 - e^{-\lambda x/2}) dx \\ &= \int_0^\infty \lambda e^{-\lambda x} dx - \lambda \int_0^\infty e^{-3\lambda x/2} dx \\ &= 1 - 2/3 = 1/3. \end{aligned}$$

- or,

$$\begin{aligned}\int_0^\infty \int_{2y}^\infty \lambda^2 e^{-\lambda(x+y)} dx dy &= \lambda^2 \int_0^\infty (-\lambda^{-1} e^{-\lambda(x+y)}) \Big|_{2y}^\infty dy \\ &= \lambda \int_0^\infty e^{-3\lambda y} dy = 1/3.\end{aligned}$$

- NOTE: useful tool by the definition of pdf:

$$\int f(z) dz = 1,$$

$$\text{e.g., } \int_0^\infty (3\lambda/2) e^{-3\lambda x/2} dx = 1 \Rightarrow \int_0^\infty e^{-3\lambda x/2} dx = 2/(3\lambda).$$

§3.2 #13

13. 設 X 有 $\mathcal{N}(0, 1)$ 分佈, 令 $Y = X^2$ 。試證 Y 有 χ_1^2 分佈。又問其逆是否成立? 證明或否證之。

- $Y = X^2 \Rightarrow X = \pm\sqrt{Y}$.
- $J = \det(\partial x / \partial y) = dx/dy = \pm 1/(2\sqrt{y})$.
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$$f_Y(y) = f_X(x = \sqrt{y})|J| + f_X(x = -\sqrt{y})|J| = \frac{1}{\sqrt{y}\sqrt{2\pi}}e^{-y/2} = \frac{y^{1/2-1}e^{-y/2}}{\Gamma(1/2)2^{1/2}}.$$

- Thus, $Y \sim \chi_1^2$, $y > 0$.
- Or, using CDF, $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \Rightarrow f_Y(y) = (f_X(\sqrt{y}) + f_X(-\sqrt{y})) / (2\sqrt{y}) = \frac{1}{\sqrt{y}\sqrt{2\pi}}e^{-y/2}$.

- The converse statement:
 “ $Y = X^2$ with $Y \sim \chi_1^2$, then $X \sim \mathcal{N}(0, 1)$ ” may **NOT** be true.
- $\because x = \sqrt{y} > 0$ or $x = -\sqrt{y} < 0$, the support of X is not in \mathbb{R} .

• Proposition

Let X have pdf $f_X(x)$ and $Y := g(X)$, where $g : \mathcal{X} \rightarrow \mathcal{Y}$, is a monotone function. Suppose that $f_X(x)$ is continuous on the support $\mathcal{X} := \{x : f_X(x) > 0\}$ and that $g^{-1}(y)$ has a continuous derivative on a subset \mathcal{Y} , where the inverse mapping $g^{-1}(y) := \{x \in \mathcal{X} : g(x) = y\}$. Then the pdf of Y is given by:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \quad y \in \mathcal{Y}.$$