# TA section 7

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Homework 4 (I):
§2.2-#31, #32, #33, §2.3-#3, #5, #7

31. 設$X$有$\mathcal{P}(\lambda)$分佈，且知$P(X=1)=P(X=2)$。試求$P(X<10)$。

- For $\lambda > 0$,
  $\mathbb{P}(X=1) = e^{-\lambda}\lambda^1/1! = \mathbb{P}(X=2) = e^{-\lambda}\lambda^2/2! \Rightarrow \lambda = 2.$
- $\mathbb{P}(X<10) = \sum_{k=0}^{9} \mathbb{P}(X=k) = \sum_{k=0}^{9} e^{-2}2^k/k! = \cdots$ (查表 or 按計算機)
- Note: $e \approx 2.7182$

# §2.2-#32

32. 設$X$有$\mathcal{P}(\lambda)$分佈。試求$P(X$爲偶數$)$之機率。

- $\mathbb{P}(X \in \{\text{even}\}) = \sum_{k=0}^{\infty} \mathbb{P}(X = 2k) = e^{-\lambda} \sum_{k=0}^{\infty} \lambda^{2k}/(2k)!$.
- $\because\ e^{\lambda} + e^{-\lambda} = \sum_{n=0}^{\infty} (\lambda^n/n! + (-\lambda)^n/n!) = \sum_{k=0}^{\infty} 2\lambda^{2k}/(2k)!$ since
  $\lambda^n + (-\lambda)^n = 0$ whenever $n = 2k+1$ (odd);
  $\lambda^n + (-\lambda)^n = 2\lambda^n$ whenever $n = 2k$ (even), for $k = 0, 1, 2, \cdots$;
  $$\therefore\ \sum_{k=0}^{\infty} \lambda^{2k}/(2k)! = \frac{e^{\lambda} + e^{-\lambda}}{2},$$
- $$\therefore\ \mathbb{P}(X \in \{\text{even}\}) = e^{-\lambda}(e^{\lambda} + e^{-\lambda})/2 = \frac{1 + e^{-2\lambda}}{2}.$$

33. 設$X$有$\mathcal{P}(\lambda)$分佈。試求$E((1+X)^{-1})$。

- Let $Y := 1 + X$, $Y \geq 1$.

$$\begin{aligned}
\mathbb{E}[(1+X)^{-1}] &= \sum_{x=0}^{\infty} (1+x)^{-1} e^{-\lambda} \lambda^x / x! \\
&= \lambda^{-1} \sum_{x=0}^{\infty} e^{-\lambda} \lambda^{1+x} / (1+x)! \\
&= \lambda^{-1} \sum_{y=1}^{\infty} e^{-\lambda} \lambda^y / y! \\
&= \lambda^{-1} (\sum_{y=0}^{\infty} e^{-\lambda} \lambda^y / y! - e^{-\lambda}) \\
&= (1 - e^{-\lambda}) / \lambda.
\end{aligned}$$

# §2.3 #3

3. 設$X$有$\mathcal{U}(-a,a)$分佈, $a>0$。試分別對下述二情況決定$a$之值, 使等式成立。

(i) $P(-1<X<2)=0.75$;

(ii) $P(|X|<1)=P(|X|>2)$。

- (i) $\mathbb{P}(-1<X<2)=3/2a=0.75 \Rightarrow a=2.$
- (ii) $\mathbb{P}(|X|<1)=\mathbb{P}(-1<X<1)=1/a$, $\mathbb{P}(|X|>2)=1-\mathbb{P}(|X|\leq 2)=1-\mathbb{P}(-2<X<2)=1-2/a \Rightarrow 1/a=1-2/a \quad \therefore \; a=3.$

5. 設$X$有$\Gamma(\alpha,\beta)$分佈。試求$Y=\sqrt{X}$之分佈。又問$\alpha,\beta$為何值時, $Y$有韋伯分佈, 並給出參數。

- $Y = X^{1/2}$,

$$f_Y(y) = f_X(x = y^2)|dx/dy| = \frac{(y^2)^{\alpha-1}e^{-y^2/\beta}}{\beta^{\alpha}\Gamma(\alpha)} \cdot 2y = \frac{2\beta^{-\alpha}y^{2\alpha-1}e^{-y^2/\beta}}{(\alpha-1)!}.$$

- Let $\alpha = 1, \beta = \kappa$, for all $\kappa > 0$.
  $\Rightarrow f_Y(y) = 2(\kappa^{-1})ye^{-y^2(\kappa^{-1})} \sim \mathcal{W}(a = \kappa^{-1}, b = 2)$.

Note: $Z \sim \mathcal{W}(a,b)$, if $f_Z(z) = a\beta z^{b-1}e^{-az^b}$, $a,b > 0$.

# §2.3 #7

7. 設$X$有$\mathcal{E}(1)$分佈, 試證$Y = (X/\alpha)^{1/\beta}$有$\mathcal{W}(\alpha, \beta)$分佈 。

- $F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}((X/\alpha)^{1/\beta} \le y) = \mathbb{P}(X \le \alpha y^\beta) = 1 - e^{-\alpha y^\beta}$.
- $$f_Y(y) = dF_Y(y)/dy = \alpha\beta y^{\beta-1} e^{-\alpha y^\beta}.$$
- i.e., $Y \sim \mathcal{W}(\alpha, \beta)$.