TA section 4

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Homework 2

§5.2 #12(i)(iv), #16, #22(i); §5.3 #11, #20(i)(ii)

12. 設$X_1, \cdots, X_n$爲一組由$\mathcal{B}e(\alpha, \beta)$分佈所產生之隨機樣本。試證

(i) 若α, β皆未知, 則$(\prod_{i=1}^n X_i, \prod_{i=1}^n (1 - X_i))$爲$(\alpha, \beta)$之一充分統計量;

(ii) 若β已知, 則$\prod_{i=1}^n X_i$爲α之一充分統計量;

(iii) 若α已知, 則$\prod_{i=1}^n (1 - X_i)$爲β之一充分統計量;

(iv) 若$\beta = \alpha$, 則$\prod_{i=1}^n (X_i(1 - X_i))$爲$\alpha$之一充分統計量。

(i)

- Let $\theta := (\alpha, \beta)$,

$$f(\boldsymbol{x}|\theta) = B(\alpha, \beta)^{-n} \left(\prod_{i=1}^{n} x_i \right)^{\alpha - 1} \left(\prod_{i=1}^{n} (1 - x_i) \right)^{\beta - 1} =: g(T(\boldsymbol{x}|\theta))h(\boldsymbol{x}),$$

where $h(\boldsymbol{x}) = 1$.
Thus,

$$T(\boldsymbol{X}) := \left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} (1 - X_i) \right)$$

is a 2-dimensional S.S. for θ, by 分解定理。 $\square$

(iv)

- Let $\theta := \alpha = \beta$,

$$f(\boldsymbol{x}|\theta) = (\Gamma(\theta)/\Gamma(2\theta))^n \left(\prod_{i=1}^n x_i(1-x_i)\right)^{\theta-1} =: g(T(\boldsymbol{x}|\theta))h(\boldsymbol{x}),$$

where $h(\boldsymbol{x}) = 1$.
Thus,

$$T(\boldsymbol{X}) := \prod_{i=1}^n X_i(1-X_i)$$

is a S.S. for θ, by 分解定理。 $\square$

§5.2 #12(i)(iv), #16, #22(i); §5.3 #11, #20(i)(ii)

16. 設$X_1, \cdots, X_n$爲一組由$\mathcal{P}(\theta)$分佈所產生之隨機樣本, $\theta > 0$。試求θ之一最小充分統計量。

• Theorem (最小充份統計量)

令 $\boldsymbol{X} := (X_1, \cdots, X_n)$ 之 joint pdf 為 $f(\boldsymbol{x}|\theta)$。假設存在一函數 $T(\boldsymbol{X})$, 使得對任二樣本點 $\boldsymbol{x}$ 及 $\boldsymbol{y}$,

$$\frac{f(\boldsymbol{x}|\theta)}{f(\boldsymbol{y}|\theta)}$$

與 θ 無關, 若且唯若

$$T(\boldsymbol{x}) = T(\boldsymbol{y}).$$

則 $T(\boldsymbol{X})$ 為 θ 之一最小充份統計量。

- Want to show:

$$f(\boldsymbol{x}|\theta)/f(\boldsymbol{y}|\theta),$$

which is free of θ if and only if

$$T(\boldsymbol{x}) = T(\boldsymbol{y}).$$

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$$\frac{f(\boldsymbol{x}|\theta)}{f(\boldsymbol{y}|\theta)} = \frac{\prod_{i=1}^{n} \frac{1}{x_i!} e^{-n\theta} \theta^{-\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} \frac{1}{y_i!} e^{-n\theta} \theta^{-\sum_{i=1}^{n} y_i}} = \prod_{i=1}^{n} \frac{y_i!}{x_i!} \theta^{\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i}$$

is free of θ iff

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Let $T(\boldsymbol{x}) := \sum_{i=1}^{n} x_i$ and $T(\boldsymbol{y}) := \sum_{i=1}^{n} y_i$.

- Thus,

$$T(\boldsymbol{X}) = \sum_{i=1}^{n} X_i$$

is a M.S.S. for θ. $\square$

§5.2 #12(i)(iv), #16, #22(i); §5.3 #11, #20(i)(ii)

22. 設$X_1, \cdots, X_n$爲一組由p.d.f.$f(x|\boldsymbol{\theta})$所產生之隨機樣本, 其中

$$f(x|\boldsymbol{\theta}) = \begin{cases} \frac{1}{\sigma} e^{-(x-\mu)/\sigma} & , x \geq \mu, \\ 0 & ,\text{其他}, \end{cases}$$

$\boldsymbol{\theta} = (\mu, \sigma)$, $\mu \in R$, $\sigma > 0$ 。

(i) 若μ, σ皆未知, 試求$\boldsymbol{\theta}$之一最小充分統計量;

(ii) 若σ已知, 試求μ之一最小充分統計量;

(iii) 若μ已知, 試求σ之一最小充分統計量 。

(i)

- $$f(x_1,\ldots,x_n \mid \mu,\sigma) \;=\; \mathbf{I}\{\mu \le x_{(1)}\}\,\frac{1}{\sigma^n}\,\exp\Big[-\tfrac{1}{\sigma}\big(\textstyle\sum_{i=1}^n x_i - n\,\mu\big)\Big].$$

- $$\frac{f(\boldsymbol{x}|\theta)}{f(\boldsymbol{y}|\theta)} = \frac{\boldsymbol{I}(\mu \le x_{(1)})}{\boldsymbol{I}(\mu \le y_{(1)})}\exp\left(-\sigma^{-1}(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i)\right),$$

  which is free of θ iff

  $$x_{(1)} = y_{(1)},\ \text{and}\ \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

  Thus, $T(\boldsymbol{X}) := (X_{(1)}, \sum_{i=1}^n X_i)$ is a M.S.S. for θ. $\square$

§5.2 #12(i)(iv), #16, #22(i); §5.3 #11, #20(i)(ii)

11. 設$X_1, \cdots, X_n$爲一組由$\mathcal{G}e(\theta)$分佈所產生之隨機樣本, $0 < \theta < 1$, 令$\boldsymbol{X} = (X_1, \cdots, X_n)$。試證$T(\boldsymbol{X}) = \sum_{i=1}^n X_i$爲$\theta$之一充分統計量。又試判定$T$是否有完備性。

- $$f(x;\theta) = \theta \exp(x \log(1-\theta)) =: h(x)c(\theta)\exp(t(x)w(\theta)),$$
  belongs to the 1-dimensional exponential family, where $h(x) = 1$, $x = 0, 1, 2, \cdots$; $c(\theta) = \theta/(1-\theta)$; $t(x) = x$; $w(\theta) = \log(1-\theta)$.
  $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is a S.S.
- $$C := \{\log(1-\theta) : \theta \in (0,1)\} = (-\infty, 0) \subset \mathbb{R},$$
  contains an open interval in $\mathbb{R}$.
  So, $T(\boldsymbol{X})$ is a C.S.S. for θ by 課本定理 **3.2**. □

定理 3.2

• Theorem

令 $X_1, \cdots, X_n$ 為一組由 k 個參數之指數族分佈所產生之隨機樣本, 其 pdf 可表示成:

$$f(x;\theta) = h(x)c(\theta)\exp\left(\sum_{j=1}^{k} w_j(\theta)t_j(x)\right),$$

其中 $C := \{w_1(\theta), \cdots, w_k(\theta)\} \subset \mathbb{R}^k$ 其值域包含一非空開矩形 (nonempty open set in $\mathbb{R}^k$), 則統計量 $T(\boldsymbol{X}) = (\sum_{i=1}^n t_1(X_i), \cdots, \sum_{i=1}^n t_k(X_i))$ 為一完備充份統計量。

By definition...

Definition

設 $T := T(X)$ 為一統計量, T 之 pdf 為 $f(t;\theta)$, $\theta \in \Omega$. 對任一函數 g, 若

$$\mathbb{E}_\theta[g(T)] = 0, \ \forall\theta \in \Omega,$$

則 $\mathbb{P}(g(T) = 0) = 1, \ \forall\theta \in \Omega$, i.e., $g(T) = 0$ almost surely。故稱 T 為一完備統計量。

- $T = \sum_{i=1}^{n} X_i \sim NB(n, \theta)$, i.e., $f_T(t|\theta) = \binom{t+n-1}{n-1}\theta^n(1-\theta)^t$, for $t = 0, 1, 2, \cdots$ (you can use MGF to prove it).
- $0 = \mathbb{E}_\theta[g(T)] = \sum_{t=0}^{\infty} g(t)\binom{t+n-1}{n-1}\theta^n(1-\theta)^t = \theta^n \sum_{t=0}^{\infty} a_t u^t < \infty \ \ \forall\theta$, where $a_t := g(t)\binom{t+n-1}{n-1}$ and $u := 1 - \theta \in (0, 1)$.
- $g(t)$ must be 0 for all $t \geq 0$ for the power series to sum to zero. That is, $\mathbb{P}(g(T) = 0) = 1 \ \forall\theta \in (0, 1)$.
- So, T is complete. $\square$

§5.2 #12(i)(iv), #16, #22(i); §5.3 #11, #20(i)(ii)

20. 設$X_1, \cdots, X_n$爲一組由p.d.f.$f(x|\theta) = e^{-(x-\theta)}$所產生之隨機樣本, $-\infty < \theta < x < \infty$。

(i) 試證$X_{(1)}$爲一最小充分統計量;

(ii) 利用巴蘇定理證明$X_{(1)}$與S_n^2獨立。

• Definition

A statistic $A(\boldsymbol{X})$ is ancillary if the distribution of $A(\boldsymbol{X})$ does not depend on the unknown parameter θ.

• Theorem (定理 3.1)

設 $T(\boldsymbol{X})$ 為一完備、充份統計量 (C.S.S.), 則 $T(\boldsymbol{X})$ 與每一個輔助統計量 (A.S.) 獨立。

(i)

- $$\frac{f(\boldsymbol{x}|\theta)}{f(\boldsymbol{y}|\theta)} = \exp\left(-\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right) \frac{\boldsymbol{I}(\theta < x_{(1)})}{\boldsymbol{I}(\theta < y_{(1)})},$$

  is free of θ iff

  $$x_{(1)} = y_{(1)}.$$

  Thus, $T(\boldsymbol{X}) = X_{(1)}$ is a M.S.S. for θ. $\square$

(ii)

- $X_{(1)}$ is C.S.S.:
  $X \in$ Exponential family since

$$f(x \mid \theta) = \boldsymbol{I}(\theta < x)e^{\theta}\exp(-x) =: h(x)c(\theta)\exp\left(w(\theta)t(x)\right),$$

where $h(x) = \boldsymbol{I}(\theta < x_{(1)})$, $c(\theta) = e^{\theta}$, $w(\theta) = 1$, and $t(x) = -x_{(1)}$. Let $C := \{w(\theta) = 1\}$, which is degenerate, does not contain an open interval in $\mathbb{R}$. So, 定理 3.2 不適用。

- Consider a continuous function g, and a sufficient statistic $T := X_{(1)}$:
  $X_{(1)}$ 的 pdf: $f_{X_{(1)}}(t) = n\exp(n(\theta - t))\boldsymbol{I}(t > \theta)$ (check). Then,

$$0 = \mathbb{E}_{\theta}[g(T)] = e^{n\theta}\int_{\theta}^{\infty} g(t)e^{-nt}dt, \ \forall \theta \in \mathbb{R}$$

.

$$\Rightarrow g(t)\exp(-nt) = 0, \ \forall \theta$$

by Fundamental Theorem of Calculus $\Rightarrow \mathbb{P}(g(T) = 0) = 1$ for all θ. Thus, $T = X_{(1)}$ is complete. $\square$

- S_n^2 is an A.S.:
  $X \in \{f(x-\theta) : \theta < x\}$, location family with the location parameter θ.
  Let $Z := X - \theta$, then $Z \sim \mathcal{E}(1)$ which is free of θ.
-

$$
\begin{aligned}
F_{S_n^2}(s) = \mathbb{P}(S_n^2 \leq s) &= \mathbb{P}\Big(\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1) \leq s\Big) \\
&= \mathbb{P}\Big(\sum_{i=1}^{n}(Z_i - \bar{Z})^2/(n-1) \leq s\Big),
\end{aligned}
$$

  which does NOT depend on θ. Thus, S_n^2 is an A.S. of θ.
- By Basu Theorem, $X_{(1)} \perp\!\!\!\perp S_n^2$. $\square$