

TA section 8

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Review: Point Estimation

估計量的優劣評估

■ **不偏性:** $\mathbb{E}[T(\mathbf{X})] = q(\theta)$.

– 不偏估計量非唯一。

■ **效率性:** (相對有效性)

- **均方差較佳估計量:** $re(T_2, T_1) = R(\theta, T_1)/R(\theta, T_2) \leq 1$ (T_1 較佳) – $q(\theta)$ 之均方差 (MSE):

$$MSE_{\theta}(T) = R(\theta, T) = \mathbb{E}[(T(\mathbf{X}) - q(\theta))^2] = \text{Var}[T(\mathbf{X})] + \text{Bias}_{\theta}^2(T(\mathbf{X})).$$

– $\text{Var}[T(\mathbf{X})] = \mathbb{E}[(T(\mathbf{X}) - \mathbb{E}T(\mathbf{X}))^2]$, 且 $\text{Bias}_{\theta}(T(\mathbf{X})) = \mathbb{E}[T(\mathbf{X})] - q(\theta)$.

- **最佳不偏估計量 (best unbiased estimator, BUE):** CRLB (Cramer-Rao lower bound): $(q'(\theta))^2/I(\theta)$; “效率可達性 (Efficiency Attainment)”。
- **均勻最小變異不偏估計量 (uniformly minimum variance unbiased estimator, UMVUE):**
 - [Rao-Blackwell Theorem]: (充份 + 不偏 \Rightarrow 效率)
 - [Lehmann-Scheffé (-Rao-Blackwell) Theorem]: (完備 + 充份 + 不偏 \Rightarrow 效率)
- **BUE \Rightarrow UMVUE.** ,i.e., $\text{Var}[\text{BUE}] \leq \text{Var}[\text{UMVUE}]$. (即: UMVUE 可能達不到 CRLB)

大樣本性質

■ 一致性:

- (weak consistency) $T_n \xrightarrow{P} q(\theta)$ as $n \rightarrow \infty$ if

$$\lim_{n \rightarrow \infty} \mathbb{P}(|T_n - q(\theta)| < \epsilon) = 1, \forall \epsilon > 0.$$

- (mean-squares consistency/ L_2 -norm consistency) $T_n \xrightarrow{\text{m.s.}} q(\theta)$ as $n \rightarrow \infty$ if

$$\mathbb{E}[(T_n - q(\theta))^2] \rightarrow 0, \text{ as } n \rightarrow \infty.$$

或 $\lim_{n \rightarrow \infty} \text{Var}[T_n] = 0$, 且 $\lim_{n \rightarrow \infty} \text{Bias}_\theta(T_n) = 0$.

- 漸近不偏性: $\lim_{n \rightarrow \infty} \mathbb{E}[T_n] = q(\theta)$.
- 漸近效率性: $\text{are}(T_{2n}, T_{1n}) = \lim_{n \rightarrow \infty} [\text{Var}[T_{1n}] / \text{Var}[T_{2n}]] \leq 1$ for T_{1n}, T_{2n} two (asymptotic) unbiased estimators, 則 T_{1n} 具漸近有效。
- 漸近常態性: 給定一估計量, 其漸近變異數 $\text{Avar}[T_n] = \sigma_n^2(\theta)$ 與漸近期望值 $\text{Asy. } \mathbb{E}[T_n] = \mu_n(\theta)$, 則 $(T_n - \mu_n(\theta)) / \sigma_n(\theta) \xrightarrow{d} \mathcal{N}(0, 1)$.
- 最佳漸近常態性: 若 $n\sigma_n^2(\theta) \rightarrow v^2(\theta)$ 且 $n^{1/2}(\mu_n(\theta) - q(\theta)) \rightarrow 0$, 其中 $v^2(\theta) > 0$ 為某一 θ 之函數, 則稱 T_n 具有最佳漸近常態性。即, 此時漸近變異數為 $v^2(\theta)/n$ 。

Example

- $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, (μ, σ^2) 未知。 Let $q(\sigma^2) := \sigma^2$, consider two estimators of $q(\sigma^2)$,

$$T_1 = \sum_{i=1}^n (X_i - \bar{X})^2 / n,$$

$$T_2 = \sum_{i=1}^n (X_i - \bar{X})^2 / (n-1) = nT_1 / (n-1).$$

- $\mathbb{E}[T_1] = (n-1)\sigma^2/n \neq \sigma^2$ (biased for σ^2), $\text{Var}[T_1] = 2(n-1)\sigma^4/n^2$. Also,

$$R(\theta, T_1) = \text{Var}[T_1] + \text{Bias}^2(\theta, T_1) = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} = \frac{(2n-1)\sigma^4}{n^2}.$$

- $\mathbb{E}[T_2] = \sigma^2$ (unbiased for σ^2), $\text{Var}[T_2] = 2\sigma^4/(n-1) > \text{Var}[T_1]$. So,

$$R(\theta, T_2) = 2\sigma^4/(n-1) > R(\theta, T_1).$$

(T_1 的 MSE 比較小, 可是它是偏的。)

- (若不考慮不偏性)

$$\text{Var}[T_1] < \text{CRLB}(\sigma^2) = 2\sigma^4/n < \text{Var}[T_2].$$

- 換言之, 即使 T_2 可知是 UMVUE, 但卻達不到 CRLB. (效率不可達)。
- 若不追求不偏性, 則永遠可找到一個更有效率的估計量 (如: T_1)。

• Theorem (Efficiency Attainment)

假設 $\mathbf{X} = (X_1, \dots, X_n)$ 有一 joint pdf $f(\mathbf{x}|\theta)$, 與其對應之 likelihood function $L(\theta|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta)$, 則對 $q(\theta)$ 的任一不偏估計量 $U(\mathbf{X})$ 滿足 $\text{Var}[U(\mathbf{X})] < \infty$ 且

$$\frac{d}{d\theta} \mathbb{E} U(\mathbf{X}) = \int \frac{\partial}{\partial \theta} U(\mathbf{X}) f(\mathbf{x}|\theta) d\theta,$$

其 $U(\mathbf{X})$ 之變異數可達到 CRLB, 若且唯若存在一個 θ 的函數 $a(\theta)$ 使得以下等式成立:

$$a(\theta) [U(\mathbf{X}) - q(\theta)] = \frac{\partial}{\partial \theta} \log L(\theta|\mathbf{x}).$$



$$\frac{\partial}{\partial \theta} \log L(\theta | \mathbf{x}) = \frac{n}{2\sigma^4} \left(\frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 - \sigma^2 \right) =: a(\theta) [T_2 + (\bar{X}^2 - \mu^2) - \sigma^2]$$

無法寫成 T_2 的不偏函數, 且 μ 未知 (除非 $\mathbb{E}[\bar{X}^2] = \mu^2$)

大樣本之下呢? ($n \rightarrow \infty$)

- (小樣本) T_2 不偏, 但 MSE 卻較大 (因變異數大的幅度超過 T_1 偏誤的幅度), T_2 真的比較差嗎? 我們來看大樣本之下:
- $\lim_{n \rightarrow \infty} \mathbb{E}[T_1] = \lim_{n \rightarrow \infty} \mathbb{E}[T_2] = \sigma^2$ (T_1, T_2 都是漸近不偏),
 $\lim_{n \rightarrow \infty} \text{Var}[T_1] = \lim_{n \rightarrow \infty} \text{Var}[T_2] = 0$ (均方差一致性);

$$\text{are}(T_2, T_1) = \lim_{n \rightarrow \infty} \text{Var}[T_1] / \text{Var}[T_2] = \lim_{n \rightarrow \infty} \frac{2(n-1)\sigma^4/n^2}{2\sigma^4/(n-1)} = 1,$$

且 T_2 是 UMVUE, 故 T_2 具漸近有效性。

§7.3 #1

1. 設 X_1, \dots, X_n 為一組由 $Ber(\theta)$ 分佈所產生之隨機樣本, $0 \leq \theta \leq 1$ 。試分別求 $\theta, \theta^2, \theta(1 - \theta)$ 之 UMVUE。

求 UMVUE 三招:

- 「充份 + 不偏」: 定理 3.1: Rao-Blackwell Theorem (非唯一解)
- 「完備充份 + 不偏」: 定理 3.2: Lehmann-Scheffé Theorem (唯一解)
- 「不偏 + CRLB」: (i) “Efficiency Attainment” (最效率可達性); (ii) 定理 4.3: 滿足 CRLB 的不偏之一個參數 (one-dimensional) 指數族。

使用 R-B Thm. & L-S Thm 原則:

- 給定 $T(\mathbf{X})$ is a C.S.S.,
- 設法找一個 $S(T(\mathbf{X}))$ (完備充份統計量的函數) 為 $q(\theta)$ 之不偏估計量, 則利用條件期望值性質, 取條件不偏式

$$h(T(\mathbf{X})) = \mathbb{E}[S(T(\mathbf{X}))|T(\mathbf{X})]$$

為 $q(\theta)$ 之唯一 UMVUE。 (定理 3.2)

- 若 $h(T(\mathbf{X}))$ 不易找出, 則設法造出:
 - (1): 任找一個 $q(\theta)$ 之不偏估計量: $S(\mathbf{X})$ (不一定是 C.S.S. 的函數, 若是 C.S.S. 的函數, 則同上);
 - (2): 取條件不偏式, 造出:

$$\mathbb{E}[S(\mathbf{X})|T(\mathbf{X})], T(\mathbf{X}) : S.S.,$$

此即 $q(\theta)$ 之一 UMVUE。 (定理 3.1)

定理 3.1.: 充份 + 不偏 \Rightarrow 有效

• Theorem

設 $T(\mathbf{X})$ 為 θ 之一充份統計量, 設 $S(\mathbf{X})$ 為 $q(\theta)$ 為任一不偏估計量, 且 $\mathbb{E}|S(\mathbf{X})| < \infty, \forall \theta \in \Omega$. 令

$$T^*(\mathbf{X}) := \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})],$$

則 $\forall \theta \in \Omega$,

$$R(\theta, T^*) \leq R(\theta, S).$$

定理 3.2.: 完備充份 + 不偏 \Rightarrow 有效

• Theorem

設 $T(\mathbf{X})$ 為一完備充份統計量, 且 $S = S(\mathbf{X})$ 為 $q(\theta)$ 之一不偏估計量。則 $T^*(\mathbf{X}) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})]$ 為 $q(\theta)$ 之一 UMVUE; 若 $\text{Var}[T^*] < \infty, \forall \theta \in \Omega$, 則 T^* 為 $q(\theta)$ 唯一之 UMVUE。

定理 4.3.

• Theorem

設 $T(\mathbf{X})$ 為 $q(\theta)$ 一不偏估計量, $\mathbb{E}[T(\mathbf{X})] = q(\theta)$ 。設一分佈族 $\{P_\theta; \theta \in \Omega\}$ 滿足正規條件, 且為一個參數之指數族, 有 pdf 如下式:

$$f(\mathbf{x}|\theta) = h(\mathbf{x}) \exp(w(\theta)T(\mathbf{x}))I_A(\mathbf{x}), \quad \theta \in \Omega,$$

其中 $w(\theta)$ 有一連續且不為零之導數, $\forall \theta \in \Omega$, 若且唯若 $\text{Var}[T(\mathbf{X})]$ 達到 CRLB, 且 $T(\mathbf{X})$ 為 $q(\theta)$ 之一 UMVUE。

- $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a C.S.S.
- Let $S(\mathbf{X}) = \bar{X}$, such that $\mathbb{E}[S(\mathbf{X})] = \theta$.
- Then, let

$$h_1(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X}),$$

$\mathbb{E}[h_1(T(\mathbf{X}))] = \mathbb{E}[\bar{X}] = \theta$. 故, $S(\mathbf{X}) = \bar{X}$ 為 θ 之一不偏估計量, 且為 $T(\mathbf{X})$ 的函數。So,

$$h_1(T(\mathbf{X})) = \bar{X}$$

is an UMVUE of θ by R-B Thm & L-S Thm.

- Let $S(\mathbf{X}) = (n/(n-1))\bar{X}(1 - \bar{X})$, such that $\mathbb{E}[S(\mathbf{X})] = \theta(1 - \theta)$.
- Let $h_2(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X})$,

$$\mathbb{E}[h_2(T(\mathbf{X}))] = \mathbb{E}[S(\mathbf{X})] = \mathbb{E}\left[\left(\frac{n}{n-1}\right)\bar{X}(1 - \bar{X})\right] = \theta(1 - \theta).$$

Since that $S(\mathbf{X})$ is unbiased for $\theta(1 - \theta)$ and is a function of $T(\mathbf{X})$. So,

$$h_2(T(\mathbf{X})) = (n/(n-1))\bar{X}(1 - \bar{X})$$

is an UMVUE of $\theta(1 - \theta)$ by R-B Thm & L-S Them.

- Let $S(\mathbf{X}) = T(\mathbf{X})(T(\mathbf{X}) - 1)/n(n - 1)$, such that $\mathbb{E}[S(\mathbf{X})] = \theta^2$.
- Let $h_3(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X})$, such that $\mathbb{E}[h_3(\mathbf{X})] = \mathbb{E}[S(\mathbf{X})]$.
- Since $\mathbb{E}[T^2(\mathbf{X})] - \mathbb{E}[T(\mathbf{X})] = \text{Var}[T(\mathbf{X})] + \mathbb{E}[T(\mathbf{X})]^2 - \mathbb{E}[T(\mathbf{X})] = n\theta(1 - \theta) + n^2\theta^2 - n\theta = n(n - 1)\theta^2$, then

$$\mathbb{E}[h_3(T(\mathbf{X}))] = \mathbb{E}[S(\mathbf{X})] = \mathbb{E}\left[\frac{T(\mathbf{X})(T(\mathbf{X}) - 1)}{n(n - 1)}\right] = \theta^2.$$

So,

$$h_3(T(\mathbf{X})) = \sum_{i=1}^n X_i (\sum_{i=1}^n X_i - 1) / n(n - 1)$$

is an UMVUE of θ^2 by R-B Thm & L-S Thm.

§7.2 #5

5. 設 X_1, \dots, X_n 為一組由 $\mathcal{U}[\theta-1, \theta+1]$ 分佈所產生之隨機樣本, $\theta \in R$ 。

(i) 試證 $T_1 = \bar{X}_n$, $T_2 = (X_{(1)} + X_{(n)})/2$ 皆為 θ 之不偏估計量;

(ii) 試分別求 T_1 及 T_2 之 MSE。

(解. (ii) $R(\theta, T_1) = 1/(3n)$, $R(\theta, T_2) = 2/((n+1)(n+2))$)

(i) $\mathbb{E}[T_1] = \mathbb{E}[X_i] = ((\theta+1) + (\theta-1))/2 = \theta$.

Let $W = X - (\theta-1) \sim U[0, 2]$. $\mathbb{E}[W_{(1)}] = 2/(n+1)$ and $\mathbb{E}[W_{(n)}] = 2n/(n+1) \Rightarrow \mathbb{E}[X_{(1)}] = \mathbb{E}[W_{(1)}] + (\theta-1)$, and $\mathbb{E}[X_{(n)}] = \mathbb{E}[W_{(n)}] + (\theta-1)$.

$\mathbb{E}[T_2] = \mathbb{E}[X_{(1)} + X_{(n)}]/2 = \mathbb{E}[W_{(1)} + W_{(n)}]/2 + (\theta-1) = 1 + \theta - 1 = \theta$.

□

§7.2 #5

- (ii) $\text{Var}[T_1] = \text{Var}[X_i]/n = (4/12)/n = 1/3n \Rightarrow R(\theta, T_1) = B[T_1]^2 + \text{Var}[T_1] = \text{Var}[T_1] = 1/3n$.
 $\text{Var}[T_2] = (1/4) [\text{Var}[X_{(1)}] + \text{Var}[X_{(n)}] + 2 \text{cov}(X_{(1)}, X_{(n)})]$. So, by pp.462–463, and $\text{Var}[X_{(1)}] = \text{Var}[W_{(1)}]$, $\text{Var}[X_{(n)}] = \text{Var}[W_{(n)}]$,

$$\text{Var}[X_{(1)}] = \text{Var}[X_{(n)}] = \frac{4n}{(n+1)^2(n+2)}$$

and

$$\text{cov}[X_{(1)}, X_{(n)}] = \frac{4}{(n+1)^2(n+2)}.$$

Then,

$$R(\theta, T_2) = \text{Var}[T_2] = \frac{2}{(n+1)(n+2)}.$$



§7.2 #6

6. 設 X_1, \dots, X_n 為一組由 $\mathcal{U}[-\theta, \theta]$ 分佈所產生之隨機樣本, $\theta > 0$ 。設 $n \geq 2$, 試求常數 c , 使得 $c(X_{(n)} - X_{(1)})$ 為 θ 之一不偏估計量。(解. $(n+1)/(2(n-1)))$)

- Let $W = X + \theta \sim U[0, 2\theta] \Rightarrow \mathbb{E}[X_{(1)}] = \mathbb{E}[W_{(1)}] - \theta = 2\theta/(n+1) - \theta$ and $\mathbb{E}[X_{(n)}] = \mathbb{E}[W_{(n)}] - \theta = 2n\theta/(n+1) - \theta$.

$$c \mathbb{E}[X_{(n)} - X_{(1)}] = c \frac{2n-2}{n+1} \theta \stackrel{!}{=} \theta \Rightarrow c = \frac{n+1}{2n-2}.$$



§7.2 #11

11. 設 X 有 $\mathcal{P}(\lambda)$ 分佈, $\lambda > 0$ 。令 $\theta = P(X = 0) = e^{-\lambda}$ 。

(i) 試問 $T_1 = e^{-X}$ 是否為 θ 之不偏估計量;

(ii) 試證 $T_2 = I_{\{X=0\}}$ 為 θ 之不偏估計量;

(iii) 試分別求 T_1 及 T_2 之 MSE。

(i) By MGF $M_X(t) = \mathbb{E}[e^{tX}] = e^{\lambda(e^t-1)}$. □

Let $t = -1$, $\mathbb{E}[T_1] = \mathbb{E}[e^{-X}] = e^{-\lambda(1-1/e)} \neq e^{-\lambda}$, T_1 is biased for θ .

(ii) $\mathbb{E}[T_2] = \mathbb{P}(X = 0) = e^{-\lambda} = \theta$, T_2 is unbiased for θ . □

MSE:

(iii)

$$\begin{aligned}
 R(\theta, T_1) &= \text{Bias}(T_1)^2 + \text{Var}[T_1] \\
 &= e^{-2\lambda}[e^{\lambda/e} - 1] + [e^{\lambda/e^2} - e^{2\lambda/e}]e^{-2\lambda} \\
 &= (1 - 2e^{\lambda/e} + e^{\lambda/e^2})e^{-2\lambda}.
 \end{aligned}$$

- $R(\theta, T_2) = \text{Var}[T_2] = \mathbb{E}[T_2^2] - \mathbb{E}[T_2]^2 = \mathbb{P}(X = 0) - \mathbb{P}(X = 0)^2 = e^{-\lambda}(1 - e^{-\lambda}).$



§7.3 #9

9. 設 X_1, \dots, X_n 為一組由 $P(\lambda)$ 分佈所產生之隨機樣本。令 $\theta = P(X = 1)$ 。試求 θ 之一 UMVUE。

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$$\begin{aligned} f(x|\lambda) &= (x!)^{-1} e^\lambda \exp\{(\log \lambda)x\} \\ &=: h(x)c(\lambda) \exp\{w(\lambda)t(x)\}, \end{aligned}$$

belongs to the one-dimensional exponential family, where

$h(x) = (x!)^{-1} \mathbf{I}(x = 0, 1, \dots)$, $c(\lambda) = \exp(\lambda)$, $w(\lambda) = \log \lambda$, and $t(x) = x$.

- $C = \{w(\lambda) = \log \lambda : \lambda \in \mathbb{R}_+\}$ contains an open set in \mathbb{R} . So, $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a C.S.S. of λ .
- Let $S(\mathbf{X}) = \mathbf{I}(X_1 = 1)$, such that $\mathbb{E}[S(\mathbf{X})] = \lambda e^{-\lambda} =: \theta$. Then, $U(T(\mathbf{X})) := \mathbb{E}[S(\mathbf{X})|T = t]$ is an UMVUE of θ , where

$$\begin{aligned}
 U(t) &= \mathbb{E}[S(\mathbf{X})|T = t] = \frac{\mathbb{P}(X_1 = 1, T = t)}{\mathbb{P}(T = t)} \\
 &= \frac{\mathbb{P}(X_1 = 1, \sum_{i=2}^n X_i = t - 0)}{\mathbb{P}(T = t)} \\
 &= \frac{(e^{-\lambda} \lambda^1 / 1!)(e^{-(n-1)\lambda} ((n-1)\lambda)^{(t-1)} / (t-1)!)}{e^{-n\lambda} (n\lambda)^t / t!} \\
 &= \left(\frac{t}{n}\right) \left(\frac{n-1}{n}\right)^{t-1}.
 \end{aligned}$$

So, $U(\mathbf{X}) = \bar{X}[(n-1)/n]^{n\bar{X}-1}$ is the UMVUE of θ by R.-B. Theorem & L.-S.



§7.3 #12

12. 設 X_1, \dots, X_n 為一組由 p.d.f. $f(x|\theta)$ 所產生之隨機樣本。試分別對下述三個 $f(x|\theta)$, 求 θ 之 UMVUE。

(i) $f(x|\theta) = 1/\theta, 0 < x < \theta;$

(ii) $f(x|\theta) = e^{-(x-\theta)}, x > \theta;$

(ii) $T = X_{(1)}$ is a C.S.S. (請自證) Then,

let $W_i = X_i - \theta \sim \mathcal{E}(1) \Rightarrow W_{(1)} = T - \theta \sim \mathcal{E}(n)$ with $\mathbb{E}[W_{(1)}] = 1/n$.

• Let $h(\mathbf{X}) = T - 1/n$, $\mathbb{E}[h(\mathbf{X})] = \mathbb{E}[T] - 1/n = \mathbb{E}[W_{(1)} + \theta] - 1/n = \theta$. So,

$$h(\mathbf{X}) = X_{(1)} - 1/n$$

is an UMVUE of θ by R-B Thm & L-S Thm.

§7.4 #10

10. 設 X 有 $Ge(\theta)$ 分佈, $\theta \in (0, 1)$ 。取 $q(\theta) = \theta$ 。試判斷 θ 之 UMVUE 之變異數是否達到 CRLB。

- $f_X(x) = \theta(1 - \theta)^{x-1}, x = 1, 2, 3, \dots$
- $\because T := \sum_{i=1}^n X_i$: a C.S.S. of θ , then the UMVUE of θ :
 $U(\mathbf{X}) := (n - 1)/(T - 1)$ (利用條件不偏式, 請自證), such that
 $\mathbb{E}[U(\mathbf{X})] = \theta$.

$$\begin{aligned} \frac{\partial}{\partial \theta} \log L(\theta | \mathbf{x}) &= \frac{n}{\theta} - \frac{T - n}{1 - \theta} \\ &= -\frac{n}{\theta^2} \left[\left(\frac{(T - n)\theta^2}{1 - \theta} \right) - \theta \right] \\ &=: a(\theta) \left[\left(\frac{(T - n)\theta^2}{1 - \theta} \right) - q(\theta) \right] \end{aligned}$$

- 無法寫成 $U(\mathbf{X})$ 的不偏函數。故, UMVUE of θ 變異數無法達到 CRLB.

§7.5 #15

15. 設 X_1, \dots, X_n 為一組由 $\mathcal{N}(\mu, \sigma^2)$ 分佈所產生之隨機樣本, $\theta = \sigma^2 > 0$, 而 μ 設為已知。試證 S_n^2 對 σ^2 之 UMVUE $T_n = \sum_{i=1}^n (X_i - \mu)^2$ 之漸近有效性為 1。

- μ 已知:

$$S_n^2 = \sum_{i=1}^n (X_i - \mu)^2 / (n - 1) = nT_n / (n - 1),$$

$$T_n = \sum_{i=1}^n (X_i - \mu)^2 / n.$$

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$$are(\theta; S_n^2, T_n) = \lim_{n \rightarrow \infty} \frac{n \operatorname{Var}[T_n]}{n \operatorname{Var}[S_n^2]} = \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{n}\right)^2 \right] \operatorname{Var}[T_n] / \operatorname{Var}[T_n] = 1.$$

