TA section 4

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October 22, 2024

Homework 2 (II)

§1.3-#3, §1.4#9, #11, #13 (iv); #16 (iv)

3. 投擲一公正的骰子一次，並獲出現點數之二倍的賞金。令X表所得賞金。試求$P(5 \le X \le 10)$。

- 令 $Y \in \{1,2,3,4,5,6\}$ 為出現的可能點數，$X := 2Y \in \{2,4,6,8,10,12\}$ 為可能得到的金額。
- $$\begin{aligned} \mathbb{P}(5 \le X \le 10) &= \mathbb{P}(5 \le 2Y \le 10) \\ &= \mathbb{P}(2.5 \le Y \le 5) = \mathbb{P}(Y \in \{3,4,5\}) \\ &= \mathbb{P}(Y=3) + \mathbb{P}(Y=4) + \mathbb{P}(Y=5) \\ &= 3/6 = 1/2. \end{aligned}$$

9. 設隨機變數X之機率密度函數爲

$$f(x) = \begin{cases} 3x^2 & , 0 < x < 1, \\ 0 & ,\text{其他}。 \end{cases}$$

(i) 試求X之分佈函數$F(x)$;

(ii) 試求$P(-1 < X \leq 1/2)$, 及$P(1 \leq X \leq 3/4)$。

- $$F_X(x) = \int_0^x 3t^2 dt = x^3,\ 0 < x < 1.$$
- $\mathbb{P}(-1 \leq X \leq 1/2) = \mathbb{P}(0 \leq X \leq 1/2) = F_X(1/2) - F_X(0) = 1/8.$
- $\mathbb{P}(3/4 \leq X \leq 1) = F_X(1) - F_X(3/4) = 1 - 27/64 = 37/64.$

11. 設某種燈泡的壽命X(單位爲月), 以$f(x) = (1+x)^{-2}$, $x > 0$, 爲其機率密度函數。試求壽命至少是3年之機率。

- $$F_X(x) = \int_0^x f(t)dt = \int_0^x (1+t)^{-2}dt = 1 - \frac{1}{(x+1)}.$$
- $\mathbb{P}(X \geq 36) = 1 - F_X(36) = 1/37 \approx 0.027.$

13. 試證下述各函數皆爲分佈函數, 並繪其圖形 。

(i) $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$, $x \in R$;

(ii) $F(x) = e^{-e^{-x}}$, $x \in R$;

(iii) $F(x) = 1 - e^{-x}$, $x > 0$;

(iv) $F(x) = 1 - e^{-\lambda x^2}$, $x > 0$, 其中$\lambda > 0$爲一常數;

(iv):

- $\because$ $\exp(z) \in C^1$ (continuously differentiable), for all $z \in \mathbb{R}$, so is $F_X(x) = 1 - \exp(-\lambda x^2) \in C^1$ (continuously differentiable), for all $x > 0$, $\lambda > 0$.
- Also, for all $x > 0$, $dF_X(x)/dx = 2\lambda x \exp(-\lambda x^2) > 0$, then $F_X(x)$ is nondecreasing function for all $x > 0$.
- $\lim_{x\to\infty} F(x) = 1 - \lim_{x\to\infty} \exp(-\lambda x^2) = 1$,
  $\lim_{x\to 0} F_X(x) = 1 - \lim_{x\to 0} \exp(-\lambda x^2) = 0$.
  So, $F_X(x)$ is a distribution function, for all $x > 0$.

16. 試對下述各函數, 分別決定常數c, 使其成爲一機率密度函數。

(i) $f(x) = c\sin x$, $0 < x < \pi/2$;

(ii) $f(x) = ce^{-|x|}$, $x \in R$;

(iii) $f(x) = cx^2e^{-x^3}$, $x > 0$;

(iv) $f(x) = \dfrac{c}{x(x+1)}$, $x = 2, 3, 4, \cdots$;

(iv):

- check: (i) $f(x) \geq 0$, (ii) $\sum_{x=2}^{\infty} f(x) = 1$.
- $$c\sum_{x=2}^{\infty}\frac{1}{x(x+1)} = c\sum_{x=2}^{\infty}\left(\frac{1}{x} - \frac{1}{x+1}\right) = \frac{c}{2} = 1,$$

So, $c^* = 2$.