TA section 10

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Homework 5

§8.2 #5

5. 設$X_1, \cdots, X_{36}$爲一組由$\mathcal{N}(\theta, 36)$分佈所產生之隨機樣本。欲檢定$H_0: \theta = 10$, vs. $H_a: \theta = 12$。拒絕域取爲$\{\overline{X} > 11\}$。試決定型 I 及型 II 錯誤之機率α, β。(**解.** $\alpha = \beta \doteq 0.1587$)

- $\bar{X} \sim \mathcal{N}(\theta, 1)$.
- $\alpha = \mathbb{P}(\bar{X} > 11 | \theta = 10) = \mathbb{P}(Z > 1) = 1 - \Phi(1) \approx 0.1587$.
- $\beta = \mathbb{P}(\bar{X} \leq 11 | \theta = 12) = \mathbb{P}(Z \leq -1) = \Phi(-1) = 1 - \Phi(1) \approx 0.1587$. $\square$

§8.2 #16

16. 設X有$\mathcal{B}(10,\theta)$分佈，$\theta \in \Omega = \{1/4, 1/2\}$。基於$X$，欲檢定$H_0 : \theta = 1/2$, vs. $H_a : \theta = 1/4$。拒絕域取爲$\{X \leq 3\}$。試求此檢定之檢力函數。(**解.** $K(1/4) = 11/64$, $K(1/2) = 31 \cdot 3^8/4^9$)

- $K(\theta) = \mathbb{P}(\text{reject } H_o \mid \theta) = \mathbb{P}(X \leq 3 \mid \theta \in \{1/4, 1/2\})$.
- $K(1/4) = \sum_{k=0}^{3} \binom{10}{k} (1/4)^k (3/4)^{10-k} \approx 31.3^8/4^9$.
- $K(1/2) = \sum_{k=0}^{3} \binom{10}{k} (1/2)^{10} = 11/64$. $\square$

§8.2 #20

20. 設$X_1, \cdots, X_n$為一組由$\mathcal{U}(0,\theta)$分佈所產生之隨機樣本。欲檢定H_0：$\theta \le \theta_0$, vs. $H_a: \theta > \theta_0$。取拒絕域為$\{X_{(n)} \ge c\}$。

(i) 試求檢力函數$K(\theta)$, 並證明此為θ之一增函數;

(ii) 設$\theta_0 = 1/2$, 試求c之值, 使顯著水準為0.05;

- $f_{X_{(n)}}(t) = nt^{n-1}/\theta^n$
- $K(\theta) = \mathbb{P}(X_{(n)} > c|\theta) = 1 - \theta^{-n}\int_0^c nx^{n-1}dx = 1 - (c/\theta)^n$. □
- $dK(\theta)/d\theta = nc^n/\theta^{n+1} > 0$ for all $0 < \theta_0 < c < \theta$.
- $\lim_{n\to\infty} K(\theta) = 1$ as $c < \theta$.
- $K(\theta_0 = 1/2) = 1 - (2c)^n = 0.05 \Rightarrow c^* = \frac{(0.95)^{1/n}}{2}$. □

(iii) 試決定n要多大, 使得對(ii)中之檢定, $K(0.75) = 0.98$;

(iv) 設$n = 20$, 且$X_{(n)} = 0.48$, 試求此時之p-值。

- $K(\theta = 0.75) = 1 - \frac{(0.95)/2^n}{(0.75)^n} = 0.98 \Rightarrow n \approx 10.$ □
- p-value: $p(\boldsymbol{x}) = \mathbb{P}(T(\boldsymbol{X}) > T(\boldsymbol{x}) \mid H_o) = \mathbb{P}(X_{(n)} > 0.48 \mid H_o : \theta = \theta_0) = 1 - (0.48/0.5)^{20} \approx 0.558.$ □

§8.3 #7

7. 設$X_1, \cdots, X_{30}$爲一組由$\mathcal{N}(\mu, \sigma^2)$分佈所產生之隨機樣本, μ設爲未知。欲檢定$H_0 : \sigma^2 = 8$, vs. $H_a : \sigma^2 < 8$。假設樣本變異數$S^2 = 6.4$。試在$\alpha = 0.05$之下, 作一檢定。(**解.** $K = 23$, 不能接受H_0)

- Let a test statistic $K(\boldsymbol{X}) := (n-1)S^2/\sigma^2 \sim \chi^2_{n-1} = \chi^2_{29}$.
- Under H_o, a testing rule:

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(K(\boldsymbol{X}) < \chi^2_{0.05,29}),$$

  such that $0.05 = \sup_{\sigma^2=8} \mathbb{P}(K(\boldsymbol{X}) < \chi^2_{0.05,29} \mid H_o)$.
- $K = (29) \times (6.4)/(8) \approx 23.2 > \chi^2_{0.05,29} = 17.71 \Rightarrow$ do not reject H_o under the level $\alpha = 0.05$. □
- level: 事前設定的允許型一誤差之最大機率上限; size: 在所有原假設範圍內實際達到的可能出現最大型一誤差機率, 通常等於或小於 level。若為簡單假設, size = level.

§8.4 #6, #8

6. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}[0,\theta]$分佈所產生之隨機樣本。欲檢定H_0 : $\theta = \theta_0$, vs. $H_a : \theta \neq \theta_0$。試給一$\alpha$下之LRT。(**解.** 拒絕域爲$\{X_{(n)} < \theta_0 \alpha^{1/n}\}$)

- likelihood $L(\theta) = \theta^{-n} \boldsymbol{I}(0 < x_i < \theta),\ \forall i = 1, 2, \cdots, n.\ \Omega := \{\theta : \theta > 0\}$, $\widehat{\theta}_{ML} = X_{(n)} \Rightarrow L(\widehat{\theta}) = (1/X_{(n)})^n$.
- Under $\Omega_o = \{\theta : \theta = \theta_0\}$, $\ddot{\theta}_{ML} = \theta_0 \Rightarrow L(\ddot{\theta}) = (1/\theta_0)^n$.
- LRT statistic: $\lambda(\boldsymbol{X}) = L(\ddot{\theta})/L(\widehat{\theta}) = (X_{(n)}/\theta_0)^n \in (0, 1]$, for $X_{(n)} \leq \theta_0$,
- a testing rule: $\phi(\boldsymbol{X}) = \boldsymbol{I}(\lambda(\boldsymbol{X}) \leq c) \iff$

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(X_{(n)} \leq c' = c^{1/n}\theta_0),$$

  such that $c \in (0, 1)$ satisfying $\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\lambda(\boldsymbol{X}) \leq c | H_o : \theta = \theta_0)$.
- $\alpha = \theta_0^{-n} \int_0^{c^{1/n}\theta_0} n x^{n-1} dx \Rightarrow c = \alpha$ and $c' = \alpha^{1/n}\theta_0$.
- So, the rejection region $C = \{X_{(n)} \leq \alpha^{1/n}\theta_0\}$. □

8. 設$X_1, \cdots, X_n$爲一組由$\mathcal{B}e(\theta, 1)$分佈所產生之隨機樣本。欲檢定H_0 : $\theta = \theta_0$, vs. $H_a : \theta \neq \theta_0$。試給一$\alpha$下之LRT。又當$n$很大時, 利用定理4.2, 得到近似的拒絕域。

- Under $H_a : \theta \in \Omega$, $\widehat{\theta} = -n / \sum_{i=1}^n \log X_i$; under $H_o : \theta \in \Omega_o$, $\ddot{\theta} = \theta_0$.

(i) Let $T(\boldsymbol{X}) = -\sum_{i=1}^n \log X_i$, a S.S. of θ,

- LRT statistic:

$$\lambda(T(\boldsymbol{x})) := L(\ddot{\theta})/L(\widehat{\theta}) = (\theta_0 T/n)^n \exp(n - \theta_0 T)$$

- $\log \lambda(T(\boldsymbol{x})) = n \log \theta_0 + n \log T - n \log n + n - \theta_0 T$,

$$d \log \lambda(T(\boldsymbol{x}))/dT = n/T - \theta_0 \begin{cases} > 0, & [T < n/\theta_0] \\ = 0, & [T = n/\theta_0] \\ < 0, & [T > n/\theta_0], \end{cases}$$

so, a testing rule: $\phi(\boldsymbol{X}) = \boldsymbol{I}(\lambda(\boldsymbol{X}) \leq c) \iff$

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\{T < c_1\} \cup \{T > c_2\}),$$

such that $c_1, c_2 \in (0, 1)$ with
$\alpha = \sup_{\theta = \theta_0} \mathbb{P}(\{T < c_1\} \cup \{T > c_2\} | H_o : \theta = \theta_0)$.

- $\because 2\theta_0 T \sim \Gamma(n,2) = \chi^2_{(2n)}$ under H_o,
  choose $\alpha/2 = \mathbb{P}(2\theta_0 T < c_1' = \chi^2_{\alpha/2,2n})$ and
  $\alpha/2 = \mathbb{P}(2\theta_0 T > c_2' = \chi^2_{1-\alpha/2,2n})$. Thus, a testing rule:

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\{T < \chi^2_{\alpha/2,2n}/(2\theta_0)\} \cup \{T > \chi^2_{1-\alpha/2,2n}/(2\theta_0)\})$$

  is a size α LRT for $H_o : \theta = \theta_0$. $\square$

(ii) Since that $-2\log\lambda \xrightarrow{\text{d}} \chi^2_1$, as $n \to \infty$, the rejection of region of an approximate size α LRT can be:

$$C^* = \{-2\log\lambda > \chi^2_{1-\alpha,1}\}$$

or

$$C^* = \{-2n(\log(\theta_0/\hat{\theta}) - (\theta_0/\hat{\theta}) + 1) > \chi^2_{1-\alpha,1}\}.$$

$\square$