

# TA section 10

JERRY C.

May 27, 2025

# Homework 5

## §8.2 #5

5. 設  $X_1, \dots, X_{36}$  為一組由  $\mathcal{N}(\theta, 36)$  分佈所產生之隨機樣本。欲檢定  $H_0: \theta = 10$ , vs.  $H_a: \theta = 12$ 。拒絕域取為  $\{\bar{X} > 11\}$ 。試決定型 I 及型 II 錯誤之機率  $\alpha, \beta$ 。(解.  $\alpha = \beta \doteq 0.1587$ )

- $\bar{X} \sim \mathcal{N}(\theta, 1)$ .
- $\alpha = \mathbb{P}(\bar{X} > 11 | \theta = 10) = \mathbb{P}(Z > 1) = 1 - \Phi(1) \approx 0.1587$ .
- $\beta = \mathbb{P}(\bar{X} \leq 11 | \theta = 12) = \mathbb{P}(Z \leq -1) = \Phi(-1) = 1 - \Phi(1) \approx 0.1587$ . □

## §8.2 #16

16. 設  $X$  有  $B(10, \theta)$  分佈,  $\theta \in \Omega = \{1/4, 1/2\}$ 。基於  $X$ , 欲檢定  $H_0 : \theta = 1/2$ , vs.  $H_a : \theta = 1/4$ 。拒絕域取為  $\{X \leq 3\}$ 。試求此檢定之檢力函數。(解.  $K(1/4) = 11/64$ ,  $K(1/2) = 31 \cdot 3^8/4^9$ )

- $K(\theta) = \mathbb{P}(\text{reject } H_0 \mid \theta) = \mathbb{P}(X \leq 3 \mid \theta \in \{1/4, 1/2\})$ .
- $K(1/4) = \sum_{k=0}^3 \binom{10}{k} (1/4)^k (3/4)^{10-k} \approx 31.3^8/4^9$ .
- $K(1/2) = \sum_{k=0}^3 \binom{10}{k} (1/2)^{10} = 11/64$ .



## §8.2 #20

20. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{U}(0, \theta)$  分佈所產生之隨機樣本。欲檢定  $H_0 :$

$\theta \leq \theta_0$ , vs.  $H_a : \theta > \theta_0$ 。取拒絕域為  $\{X_{(n)} \geq c\}$ 。

(i) 試求檢力函數  $K(\theta)$ , 並證明此為  $\theta$  之一增函數;

(ii) 設  $\theta_0 = 1/2$ , 試求  $c$  之值, 使顯著水準為 0.05;

- $f_{X_{(n)}}(t) = nt^{n-1}/\theta^n$
- $K(\theta) = \mathbb{P}(X_{(n)} > c | \theta) = 1 - \theta^{-n} \int_0^c nx^{n-1} dx = 1 - (c/\theta)^n.$  □
- $dK(\theta)/d\theta = nc^n/\theta^{n+1} > 0$  for all  $0 < \theta_0 < c < \theta$ .
- $\lim_{n \rightarrow \infty} K(\theta) = 1$  as  $c < \theta$ .
- $K(\theta_0 = 1/2) = 1 - (2c)^n = 0.05 \Rightarrow c^* = \frac{(0.95)^{1/n}}{2}.$  □

(iii) 試決定 $n$ 要多大, 使得對(ii)中之檢定,  $K(0.75) = 0.98$ ;

(iv) 設 $n = 20$ , 且 $X_{(n)} = 0.48$ , 試求此時之 $p$ -值。

- $K(\theta = 0.75) = 1 - \frac{(0.95)/2^n}{(0.75)^n} = 0.98 \Rightarrow n \approx 10.$  □
- p-value:  $p(\mathbf{x}) = \mathbb{P}(T(\mathbf{X}) > T(\mathbf{x}) \mid H_o) = \mathbb{P}(X_{(n)} > 0.48 \mid H_o : \theta = \theta_0) = 1 - (0.48/0.5)^{20} \approx 0.558.$  □

## §8.3 #7

7. 設  $X_1, \dots, X_{30}$  為一組由  $\mathcal{N}(\mu, \sigma^2)$  分佈所產生之隨機樣本,  $\mu$  設為未知。欲檢定  $H_0: \sigma^2 = 8$ , vs.  $H_a: \sigma^2 < 8$ 。假設樣本變異數  $S^2 = 6.4$ 。試在  $\alpha = 0.05$  之下, 作一檢定。(解.  $K = 23$ , 不能接受  $H_0$ )

- Let a test statistic  $K(\mathbf{X}) := (n - 1)S^2/\sigma^2 \sim \chi_{n-1}^2 = \chi_{29}^2$ .
- Under  $H_0$ , a testing rule:

$$\phi(\mathbf{X}) = I(K(\mathbf{X}) < \chi_{0.05, 29}^2),$$

such that  $0.05 = \sup_{\sigma^2=8} \mathbb{P}(K(\mathbf{X}) < \chi_{0.05, 29}^2 \mid H_0)$ .

- $K = (29) \times (6.4)/(8) \approx 23.2 > \chi_{0.05, 29}^2 = 17.71 \Rightarrow$  do not reject  $H_0$  under the level  $\alpha = 0.05$ . □
- level: 事前設定的允許型一誤差之最大機率上限; size: 在所有原假設範圍內實際達到的可能出現最大型一誤差機率, 通常等於或小於 level。若為簡單假設, size = level.

## §8.4 #6, #8

6. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{U}[0, \theta]$  分佈所產生之隨機樣本。欲檢定  $H_0 : \theta = \theta_0$ , vs.  $H_a : \theta \neq \theta_0$ 。試給  $-\alpha$  下之 LRT。(解. 拒絕域為  $\{X_{(n)} < \theta_0 \alpha^{1/n}\}$ )

- likelihood  $L(\theta) = \theta^{-n} \mathbf{I}(0 < x_i < \theta)$ ,  $\forall i = 1, 2, \dots, n$ .  $\Omega := \{\theta : \theta > 0\}$ ,  $\hat{\theta}_{ML} = X_{(n)} \Rightarrow L(\hat{\theta}) = (1/X_{(n)})^n$ .
- Under  $\Omega_o = \{\theta : \theta = \theta_0\}$ ,  $\ddot{\theta}_{ML} = \theta_0 \Rightarrow L(\ddot{\theta}) = (1/\theta_0)^n$ .
- LRT statistic:  $\lambda(\mathbf{X}) = L(\ddot{\theta})/L(\hat{\theta}) = (X_{(n)}/\theta_0)^n \in (0, 1]$ , for  $X_{(n)} \leq \theta_0$ ,
- a testing rule:  $\phi(\mathbf{X}) = \mathbf{I}(\lambda(\mathbf{X}) \leq c) \iff$

$$\phi(\mathbf{X}) = \mathbf{I}(X_{(n)} \leq c' = c^{1/n} \theta_0),$$

such that  $c \in (0, 1)$  satisfying  $\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\lambda(\mathbf{X}) \leq c | H_o : \theta = \theta_0)$ .

- $\alpha = \theta_0^{-n} \int_0^{c^{1/n} \theta_0} n x^{n-1} dx \Rightarrow c = \alpha$  and  $c' = \alpha^{1/n} \theta_0$ .
- So, the rejection region  $C = \{X_{(n)} \leq \alpha^{1/n} \theta_0\}$ .





8. 設  $X_1, \dots, X_n$  為一組由  $Be(\theta, 1)$  分佈所產生之隨機樣本。欲檢定  $H_0 : \theta = \theta_0$ , vs.  $H_a : \theta \neq \theta_0$ 。試給  $-\alpha$  下之 LRT。又當  $n$  很大時, 利用定理 4.2, 得到近似的拒絕域。

- Under  $H_a : \theta \in \Omega$ ,  $\hat{\theta} = -n / \sum_{i=1}^n \log X_i$ ; under  $H_o : \theta \in \Omega_o$ ,  $\ddot{\theta} = \theta_0$ .
- (i) Let  $T(\mathbf{X}) = -\sum_{i=1}^n \log X_i$ , a S.S. of  $\theta$ ,
- LRT statistic:

$$\lambda(T(\mathbf{x})) := L(\ddot{\theta}) / L(\hat{\theta}) = (\theta_0 T / n)^n \exp(n - \theta_0 T)$$

- $\log \lambda(T(\mathbf{x})) = n \log \theta + n \log T - n \log n + n - \theta_0 T$ ,

$$d \log \lambda(T(\mathbf{x})) / dT = n/T - \theta_0 \begin{cases} > 0, & [T < n/\theta_0] \\ = 0, & [T = n/\theta_0] \\ < 0, & [T > n/\theta_0], \end{cases}$$

so, a testing rule:  $\phi(\mathbf{X}) = \mathbf{I}(\lambda(\mathbf{X}) \leq c) \iff$

$$\phi(\mathbf{X}) = \mathbf{I}(\{T < c_1\} \cup \{T > c_2\}),$$

such that  $c_1, c_2 \in (0, 1)$  with

$$\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\{T < c_1\} \cup \{T > c_2\} | H_o : \theta = \theta_0).$$

- $\because 2\theta_0 T \sim \Gamma(n, 2) = \chi^2_{(2n)}$  under  $H_o$ ,  
choose  $\alpha/2 = \mathbb{P}(2\theta_0 T < c'_1 = \chi^2_{\alpha/2, 2n})$  and  
 $\alpha/2 = \mathbb{P}(2\theta_0 T > c'_2 = \chi^2_{1-\alpha/2, 2n})$ . Thus, a testing rule:

$$\phi(\mathbf{X}) = \mathbf{I}(\{T < \chi^2_{\alpha/2, 2n}/(2\theta_0)\} \cup \{T > \chi^2_{1-\alpha/2, 2n}/(2\theta_0)\})$$

is a size  $\alpha$  LRT for  $H_o : \theta = \theta_0$ . □

- (ii) Since that  $-2 \log \lambda \xrightarrow{d} \chi^2_1$ , as  $n \rightarrow \infty$ , the rejection of region of an approximate size  $\alpha$  LRT can be:

$$C^* = \{-2 \log \lambda > \chi^2_{1-\alpha/2, 1} \vee -2 \log \lambda < \chi^2_{\alpha/2, 1}\}.$$
□