

## TA section 7

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## Homework 3: part (II)

## §7.3 #6, #10; §7.4 #2, #11

6. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{N}(\mu, \sigma^2)$  分佈所產生之隨機樣本, 欲估計  $\mu^2$ 。

試證

(i) 若  $\sigma$  已知, 則  $T_1 = \bar{X}_n^2 - \sigma^2/n$  為 UMVUE;

(ii) 若  $\sigma$  未知, 則  $T_2 = \bar{X}_n^2 - S_n^2/n$  為 UMVUE。

- Key: (1) C.S.S.; (2) (conditional) unbiasedness
- 若  $\sigma^2$  已知: let  $\theta = \mu$ ,

$$\begin{aligned} f(x|\theta) &= (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} e^{-\mu^2/2\sigma^2} \exp\{(\mu/\sigma^2)x\} \\ &=: h(x)c(\theta) \exp\{w(\theta)t(x)\}, \end{aligned}$$

belongs to the one-dimensional exponential family, where

$h(x) = (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} \mathbf{I}(-\infty < x < \infty)$ ,  $c(\theta) = e^{-\mu^2/2\sigma^2}$ ,  $t(x) = x$ ,  
and  $w(\theta) = \mu/\sigma^2$ .

- $C = \{w(\theta) = \mu/\sigma^2 : \mu \in \mathbb{R}\}$  contains an open set in  $\mathbb{R}$ . By theorem,  $T(\mathbf{X}) = \sum_{i=1}^n X_i$  is a C.S.S. of  $\mu$ , as well as  $\bar{X}$  (equivalent statistics).

- 若  $\sigma^2$  未知: let  $\theta = (\mu, \sigma^2)$ ,

$$\begin{aligned} f(x|\theta) &= (2\pi\sigma^2)^{-1/2} e^{-\mu^2/2\sigma^2} \exp\{(\mu/\sigma^2)x - x^2/(2\sigma^2)\} \\ &=: h(x)c(\theta) \exp\{w_1(\theta)t_1(x) + w_2(\theta)t_2(x)\}, \end{aligned}$$

belongs to the two-dimensional exponential family, where

$$h(x) = (2\pi\sigma^2)^{-1/2} \mathbf{I}(-\infty < x < \infty), \quad c(\theta) = e^{-\mu^2/2\sigma^2}, \quad w_1(\theta) = \mu/\sigma^2, \\ w_2(\theta) = -1/\sigma^2, \quad t_1(x) = x, \quad \text{and} \quad t_2(x) = x^2.$$

- $C = \{(w_1(\theta), w_2(\theta)) = (\mu/\sigma^2, -1/\sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+\}$ , contains an open set on  $\mathbb{R}^2$ .
- By theorem,  $T(\mathbf{X}) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$  is a C.S.S. of  $(\mu, \sigma^2)$ , and so is  $(\bar{X}, S_n^2)$  (equivalent statistics).

- $\sigma^2$  已知: let  $T(\mathbf{X}) = \bar{X}$ , and  $S(\mathbf{X}) = \bar{X}^2 - \sigma^2/n$ . Consider  $T_1 = h(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X})$  ( $\because S(\mathbf{X})$  is also a function of  $T(\mathbf{X})$ )  
 $\Rightarrow \mathbb{E}[T_1] = \mathbb{E}[\bar{X}^2 - \sigma^2/n] = \mathbb{E}[\bar{X}^2] - \sigma^2/n = \sigma^2/n + \mu^2 - \sigma^2/n = \mu^2$ .
- So,  $T_1$  is the UMVUE of  $\mu^2$  by R.-B. Theorem & L.-S. Theorem.
- $\sigma^2$  未知: let  $T(\mathbf{X}) = (\bar{X}, S_n^2)$ . Consider  $T_2 = h(T(\mathbf{X})) = \bar{X}^2 - S_n^2/n \Rightarrow \mathbb{E}[T_2] = \mathbb{E}[\bar{X}^2] - \mathbb{E}[S_n^2/n] = \sigma^2/n + \mu^2 - \sigma^2/n = \mu^2$ .
- So,  $T_2$  is the UMVUE of  $\mu^2$  by R.-B. Theorem & L.-S. Theorem.

10. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{P}(\lambda)$  分佈所產生之隨機樣本,  $\lambda > 0$ 。欲

求  $\theta = P(X_1 = 0) = e^{-\lambda}$  之 UMVUE。

(i) 先找出一完備充分統計量  $T$ ;

(ii) 利用  $I_{\{X_1=0\}}$  為  $\theta$  之一不偏估計量, 以求出 UMVUE;

(iii) 利用找一  $T$  的函數  $h(T)$ , 滿足  $E(h(T)) = \theta$ , 以求出 UMVUE。

• (i)

$$\begin{aligned} f(x|\lambda) &= (x!)^{-1} e^{-\lambda} \exp\{(\log \lambda)x\} \\ &=: h(x)c(\lambda) \exp\{w(\lambda)t(x)\}, \end{aligned}$$

belongs to the one-dimensional exponential family, where

$h(x) = (x!)^{-1} \mathbf{I}(x = 0, 1, \dots)$ ,  $c(\lambda) = \exp(\lambda)$ ,  $w(\lambda) = \log \lambda$ , and  $t(x) = x$ .

- $C = \{w(\lambda) = \log \lambda : \lambda \in \mathbb{R}_+\}$  contains an open set in  $\mathbb{R}$ . So,  $T(\mathbf{X}) = \sum_{i=1}^n X_i$  is a C.S.S. of  $\lambda$ .
- (ii) Let  $S(\mathbf{X}) = \mathbf{I}(X_1 = 0)$ , such that  $\mathbb{E}[S(\mathbf{X})] = e^{-\lambda} = \theta$ . Then,  $U(T(\mathbf{X})) := \mathbb{E}[S(\mathbf{X})|T = t]$  is an UMVUE of  $\theta$ , where

$$\begin{aligned}
 U(t) &= \mathbb{E}[S(\mathbf{X})|T = t] = \frac{\mathbb{P}(X_1 = 0, T = t)}{\mathbb{P}(T = t)} \\
 &= \frac{\mathbb{P}(X_1 = 0, \sum_{i=2}^n X_i = t - 0)}{\mathbb{P}(T = t)} \\
 &= \frac{(e^{-\lambda} \lambda^0 / 0!)(e^{-(n-1)\lambda} ((n-1)\lambda)^{(t-0)} / (t-0)!)}{e^{-n\lambda} (n\lambda)^t / t!} \\
 &= \left( \frac{n-1}{n} \right)^t.
 \end{aligned}$$

So,  $U(\mathbf{X}) = [(n-1)/n]^{n\bar{X}}$  is the UMVUE of  $\theta$  by R.-B. Theorem & L.-S. Theorem.

- Note: the conditional distribution  $X_i|T = t \sim \text{Bin}(t, 1/n)$  (it's easy to check by yourself).
- verify:

$$\begin{aligned}
 \mathbb{E}[U] &= \mathbb{E}\left[\left(\frac{n-1}{n}\right)^T\right] = \sum_{k=0}^{\infty} \left(\frac{n-1}{n}\right)^k \frac{e^{-n\lambda} (n\lambda)^k}{k!} \\
 &= e^{-n\lambda} \sum_{k=0}^{\infty} \frac{(n\lambda(n-1)/n)^k}{k!} = e^{-n\lambda} e^{n\lambda((n-1)/n)} = e^{-\lambda} = \theta.
 \end{aligned}$$

- Consider

$$U = h(T) = \left( \frac{n-1}{n} \right)^T,$$

such that  $E[U] = \sum_{k=0}^{\infty} \left( \frac{n-1}{n} \right)^k \frac{e^{-n\lambda} (n\lambda)^k}{k!} = e^{-n\lambda} \sum_{k=0}^{\infty} \frac{(n\lambda(n-1)/n)^k}{k!} = \theta$ .  
So,  $U$  is the UMVUE of  $\theta$ .



2. 設  $X$  與  $Y$  獨立, 皆有  $\mathcal{N}(\mu, \sigma^2)$  分佈。分別對參數為  $\mu$  及  $\sigma^2$  時, 求 (i)  $X + Y$ , (ii)  $X - Y$  之資訊數。(解. (i)  $I_{X+Y}(\mu) = 2/\sigma^2$ ,  $I_{X+Y}(\sigma^2) = 1/(2\sigma^4)$ , (ii)  $I_{X-Y}(\mu) = 0$ ,  $I_{X-Y}(\sigma^2) = 1/(2\sigma^4)$ )

- $T_1 := X + Y \sim \mathcal{N}(2\mu, 2\sigma^2)$ ,  $T_2 := X - Y \sim \mathcal{N}(0, 2\sigma^2)$ .
- $\log L(\mu, \sigma^2; T_1) = -2^{-1} \log(4\pi\sigma^2) - (t_1 - 2\mu)^2/(4\sigma^2)$ .
- $\partial \log L(\mu)/\partial \mu = (t_1 - 2\mu)/(\sigma^2)$ ;  
 $\partial \log L(\sigma^2)/\partial \sigma^2 = -1/(2\sigma^2) + (t_1 - 2\mu)^2/(4\sigma^4)$ .
- $I_{T_1}(\mu) = \mathbb{E}[(\partial \log L(\mu)/\partial \mu)^2] = \mathbb{E}[(T_1 - 2\mu)^2]/\sigma^4 = 2/(\sigma^2)$ ;
- $I_{T_1}(\sigma^2) = \mathbb{E}[(\partial \log L(\sigma^2)/\partial \sigma^2)^2] = \mathbb{E}[(-1/(2\sigma^2) + (T_1 - 2\mu)^2/4\sigma^4)^2] = \mathbb{E}\left[1/(4\sigma^4) + (T_1 - 2\mu)^4/(16\sigma^8) - (T_1 - 2\mu)^2/(4\sigma^6)\right] = 1/(2\sigma^4)$ .
- Note:  $Z \sim \mathcal{N}(\mu, \sigma_Z^2) \Rightarrow \mathbb{E}[Z - \mu] = 0$ ,  $\mathbb{E}[(Z - \mu)^2] = \sigma_Z^2$ ,  $\mathbb{E}[(Z - \mu)^3] = 0$ ,  $\mathbb{E}[(Z - \mu)^4] = 3\sigma_Z^4$ .

- $\log L(\mu, \sigma^2; T_2) = -2^{-1} \log(2\sigma^2) - y^2/(4\sigma^2).$
- $\partial \log L(\mu)/\partial \mu = t_2/(\sigma^2);$   
 $\partial \log L(\sigma^2)/\partial \sigma^2 = -1/(2\sigma^2) + t_2^2/(4\sigma^4).$
- $I_{T_2}(\mu) = \mathbb{E}[(\partial \log L(\mu)/\partial \mu)^2] = 0;$   
 $I_{T_2}(\sigma^2) = \mathbb{E}[(\partial \log L(\sigma^2)/\partial \sigma^2)^2] = \mathbb{E} \left[ (-1/(2\sigma^2) + T_2^2/(4\sigma^4))^2 \right] =$   
 $\mathbb{E} \left[ 1/(4\sigma^4) + T_2^4/(16\sigma^8) - T_2^2/(4\sigma^6) \right] = 1/(2\sigma^4).$

11. 設  $X_1, \dots, X_n$  為一組由 p.d.f.  $f_i(x|\theta)$  所產生之隨機樣本,  $i = 1, 2$ , 其中

$$(i) f_1(x|\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0;$$

$$(ii) f_2(x|\theta) = \theta^x \log \theta / (\theta - 1), 0 < x < 1, \theta > 1.$$

試分別對此二 p.d.f., 討論是否有一  $\theta$  之函數  $q(\theta)$ , 使得存在一  $q(\theta)$  之不偏估計量, 且其變異數達到 CRLB。若有則找出來, 若沒有亦證明之。

### • Theorem (Efficiency Attainment)

假設  $\mathbf{X} = (X_1, \dots, X_n)$  有一 joint pdf  $f(\mathbf{x}|\theta)$ , 與其對應之 likelihood function  $L(\theta|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta)$ , 則對  $q(\theta)$  的任一不偏估計量  $U(\mathbf{X})$  之變異數可達到 CRLB, 若且唯若存在一個  $(\theta, n)$  的函數  $a(\theta, n)$  使得以下等式成立:

$$a(\theta, n) \left[ U(\mathbf{X}) - q(\theta) \right] = \frac{\partial}{\partial \theta} \log L(\theta|\mathbf{x}).$$

- (i)

$$\begin{aligned}
 \partial \log L / \partial \theta &= \frac{\partial}{\partial \theta} \sum_{i=1}^n [\log \theta + (\theta - 1) \log x_i] \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \theta} [\log \theta + (\theta - 1) \log x_i] \quad (\text{why}) \\
 &= \sum_{i=1}^n [\theta^{-1} + \log x_i] \\
 &= -n \left[ - \sum_{i=1}^n \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) [U(\mathbf{X}) - q(\theta)].
 \end{aligned}$$

- Then,  $a(\theta, n) = -n$ ,  $q(\theta) = 1/\theta$ , and  $U(\mathbf{X}) = -\sum_{i=1}^n \log X_i / n$  is the UMVUE of  $1/\theta$ , which attains the CRLB.

- (i)

$$\begin{aligned}
 \partial \log L / \partial \theta &= \frac{\partial}{\partial \theta} \sum_{i=1}^n [\log \theta + (\theta - 1) \log x_i] \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \theta} [\log \theta + (\theta - 1) \log x_i] \quad (\text{why}) \\
 &= \sum_{i=1}^n [\theta^{-1} + \log x_i] \\
 &= -n \left[ - \sum_{i=1}^n \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) [U(\mathbf{X}) - q(\theta)].
 \end{aligned}$$

- Then,  $a(\theta, n) = -n$ ,  $q(\theta) = 1/\theta$ , and  $U(\mathbf{X}) = -\sum_{i=1}^n \log X_i/n$  is the UMVUE of  $1/\theta$ , which attains the CRLB.
- Note:  $\nexists a(\cdot)$  such that  $\partial \log L / \partial \theta = a(\cdot)[U(\mathbf{X}) - \theta]$ , i.e., no UMVUE of  $\theta$  that can attain the CRLB.

- (i)

$$\begin{aligned}
 \partial \log L / \partial \theta &= \frac{\partial}{\partial \theta} \sum_{i=1}^n [\log \theta + (\theta - 1) \log x_i] \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \theta} [\log \theta + (\theta - 1) \log x_i] \quad (\text{why}) \\
 &= \sum_{i=1}^n [\theta^{-1} + \log x_i] \\
 &= -n \left[ - \sum_{i=1}^n \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) [U(\mathbf{X}) - q(\theta)].
 \end{aligned}$$

- Then,  $a(\theta, n) = -n$ ,  $q(\theta) = 1/\theta$ , and  $U(\mathbf{X}) = -\sum_{i=1}^n \log X_i / n$  is the UMVUE of  $1/\theta$ , which attains the CRLB.
- Note:  $\nexists a(\cdot)$  such that  $\partial \log L / \partial \theta = a(\cdot)[U(\mathbf{X}) - \theta]$ , i.e., no UMVUE of  $\theta$  that can attain the CRLB.
- Note: 「有限項部份和」與微分 (或極限) 可以直接交換符號。

• (i)

$$\begin{aligned}
 \partial \log L / \partial \theta &= \frac{\partial}{\partial \theta} \sum_{i=1}^n [\log \theta + (\theta - 1) \log x_i] \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \theta} [\log \theta + (\theta - 1) \log x_i] \quad (\text{why}) \\
 &= \sum_{i=1}^n [\theta^{-1} + \log x_i] \\
 &= -n \left[ - \sum_{i=1}^n \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) [U(\mathbf{X}) - q(\theta)].
 \end{aligned}$$

- Then,  $a(\theta, n) = -n$ ,  $q(\theta) = 1/\theta$ , and  $U(\mathbf{X}) = -\sum_{i=1}^n \log X_i / n$  is the UMVUE of  $1/\theta$ , which attains the CRLB.
- Note:  $\nexists a(\cdot)$  such that  $\partial \log L / \partial \theta = a(\cdot)[U(\mathbf{X}) - \theta]$ , i.e., no UMVUE of  $\theta$  that can attain the CRLB.
- Note: 「有限項部份和」與微分 (或極限) 可以直接交換符號。
- Note: 「無窮項和」與微分 (或極限) 若可交換符號, 當此級數是均勻收斂 (uniform convergence) 或 控制收斂 (dominated (bounded) convergence)。

• (i)

$$\begin{aligned}
 \partial \log L / \partial \theta &= \frac{\partial}{\partial \theta} \sum_{i=1}^n [\log \theta + (\theta - 1) \log x_i] \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \theta} [\log \theta + (\theta - 1) \log x_i] \quad (\text{why}) \\
 &= \sum_{i=1}^n [\theta^{-1} + \log x_i] \\
 &= -n \left[ - \sum_{i=1}^n \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) [U(\mathbf{X}) - q(\theta)].
 \end{aligned}$$

- Then,  $a(\theta, n) = -n$ ,  $q(\theta) = 1/\theta$ , and  $U(\mathbf{X}) = -\sum_{i=1}^n \log X_i / n$  is the UMVUE of  $1/\theta$ , which attains the CRLB.
- Note:  $\nexists a(\cdot)$  such that  $\partial \log L / \partial \theta = a(\cdot)[U(\mathbf{X}) - \theta]$ , i.e., no UMVUE of  $\theta$  that can attain the CRLB.
- Note: 「有限項部份和」與微分 (或極限) 可以直接交換符號。
- Note: 「無窮項和」與微分 (或極限) 若可交換符號, 當此級數是均勻收斂 (uniform convergence) 或 控制收斂 (dominated (bounded) convergence)。



- (ii)

$$\begin{aligned}
 \partial \log L / \partial \theta &= \frac{\partial}{\partial \theta} \sum_{i=1}^n [\log \log \theta - \log(\theta - 1) + x_i \log \theta] \\
 &= \left( \frac{n}{\theta \log \theta} - \frac{n}{\theta - 1} \right) + \theta^{-1} \sum_{i=1}^n x_i \\
 &= \frac{n}{\theta} \left[ \bar{x} - \left( \frac{\theta}{\theta - 1} - \frac{1}{\log \theta} \right) \right] = a(\theta, n) \left[ U(\mathbf{X}) - q(\theta) \right].
 \end{aligned}$$

- Then,  $a(\theta, n) = n/\theta$ , and  $q(\theta) = \theta/(\theta - 1) - 1/(\log \theta)$ ;
- $U(\mathbf{X}) = \bar{X}$  is the UMVUE of  $q(\theta)$ , which attains the CRLB.
- Note: No UMVUE of  $\theta$  that attains the CRLB.