TA section 7

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Homework 3: part (II)

§7.3 #6, #10; §7.4 #2, #11

6. 設 $X_1, \cdots, X_n$ 爲一組由 $\mathcal{N}(\mu, \sigma^2)$ 分佈所產生之隨機樣本, 欲估計 μ^2 。

試證

(i) 若 σ 已知, 則 $T_1 = \overline{X}_n^2 - \sigma^2/n$ 爲 UMVUE;

(ii) 若 σ 未知, 則 $T_2 = \overline{X}_n^2 - S_n^2/n$ 爲 UMVUE 。

- Key: (1) C.S.S.; (2) (conditional) unbiasedness
- 若 σ^2 已知: let $\theta = \mu$,

$$f(x|\theta) = (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} e^{-\mu^2/2\sigma^2} \exp\{(\mu/\sigma^2)x\}$$
$$=: h(x)c(\theta)\exp\{w(\theta)t(x)\},$$

  belongs to the one-dimensional exponential family, where
  $h(x) = (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} \boldsymbol{I}(-\infty < x < \infty)$, $c(\theta) = e^{-\mu^2/2\sigma^2}$, $t(x) = x$,
  and $w(\theta) = \mu/\sigma^2$.
- $C = \{w(\theta) = \mu/\sigma^2 : \mu \in \mathbb{R}\}$ contains an open set in $\mathbb{R}$. By theorem,
  $T(\boldsymbol{X}) = \sum_{i=1}^{n} X_i$ is a C.S.S. of μ, as well as $\bar{X}$ (equivalent statistics).

- 若 σ^2 未知: let $\theta = (\mu, \sigma^2)$,

$$f(x|\theta) = (2\pi\sigma^2)^{-1/2}e^{-\mu^2/2\sigma^2}\exp\{(\mu/\sigma^2)x - x^2/(2\sigma^2)\}$$
$$=: h(x)c(\theta)\exp\{w_1(\theta)t_1(x) + w_2(\theta)t_2(x)\},$$

  belongs to the two-dimensional exponential family, where
  $h(x) = (2\pi\sigma^2)^{-1/2}\boldsymbol{I}(-\infty < x < \infty)$, $c(\theta) = e^{-\mu^2/2\sigma^2}$, $w_1(\theta) = \mu/\sigma^2$,
  $w_2(\theta) = -1/\sigma^2$, $t_1(x) = x$, and $t_2(x) = x^2$.

- $C = \{(w_1(\theta), w_2(\theta)) = (\mu/\sigma^2, -1/\sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+\}$, contains an open set on $\mathbb{R}^2$.

- By theorem, $T(\boldsymbol{X}) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a C.S.S. of (μ, σ^2), and so is $(\bar{X}, S_n^2)$ (equivalent statistics).

- 注意: 若 μ 已知, 且 σ^2 未知, 則
  $C = \{(w_1(\theta), w_2(\theta)) = (\mu/\sigma^2, -1/\sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+\}$, does NOT contain an open set on $\mathbb{R}^2$.

- σ^2 已知: let $T(\boldsymbol{X}) = \bar{X}$, and $S(\boldsymbol{X}) = \bar{X}^2 - \sigma^2/n$. Consider $T_1 = h(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X})$ ($\because S(\boldsymbol{X})$ is also a function of $T(\boldsymbol{X})$)
$\Rightarrow \mathbb{E}[T_1] = \mathbb{E}[\bar{X}^2 - \sigma^2/n] = \mathbb{E}[\bar{X}^2] - \sigma^2/n = \sigma^2/n + \mu^2 - \sigma^2/n = \mu^2$.
- So, T_1 is the UMVUE of μ^2 by R.-B. Theorem & L.-S. Thorem.
- σ^2 未知: let $T(\boldsymbol{X}) = (\bar{X}, S_n^2)$. Consider $T_2 = h(T(\boldsymbol{X})) = \bar{X}^2 - S_n^2/n \Rightarrow \mathbb{E}[T_2] = \mathbb{E}[\bar{X}^2] - \mathbb{E}[S_n^2/n] = \sigma^2/n + \mu^2 - \sigma^2/n = \mu^2$.
- So, T_2 is the UMVUE of μ^2 by R.-B. Theorem & L.-S. Theorem.

10. 設 $X_1, \cdots, X_n$ 為一組由 $\mathcal{P}(\lambda)$ 分佈所產生之隨機樣本, $\lambda > 0$ 。欲
求 $\theta = P(X_1 = 0) = e^{-\lambda}$ 之 UMVUE 。

   (i) 先找出一完備充分統計量 T;

   (ii) 利用 $I_{\{X_1=0\}}$ 為 θ 之一不偏估計量, 以求出 UMVUE;

   (iii) 利用找一 T 的函數 $h(T)$, 滿足 $E(h(T)) = \theta$, 以求出 UMVUE 。

- (i)

$$f(x|\lambda) = (x!)^{-1} e^{\lambda} \exp\{(\log \lambda)x\}$$
$$=: h(x)c(\lambda)\exp\{w(\lambda)t(x)\},$$

belongs to the one-dimensional exponential family, where
$h(x) = (x!)^{-1} \boldsymbol{I}(x = 0, 1, \cdots), c(\lambda) = \exp(\lambda), w(\lambda) = \log \lambda$, and $t(x) = x$.

- $C = \{w(\lambda) = \log \lambda : \lambda \in \mathbb{R}_+\}$ contains an open set in $\mathbb{R}$. So,
  $T(\boldsymbol{X}) = \sum_{i=1}^{n} X_i$ is a C.S.S. of λ.

- (ii) Let $S(\boldsymbol{X}) = \boldsymbol{I}(X_1 = 0)$, such that $\mathbb{E}[S(\boldsymbol{X})] = e^{-\lambda} = \theta$. Then,
  $U(T(\boldsymbol{X})) := \mathbb{E}[S(\boldsymbol{X})|T = t]$ is an UMVUE of θ, where

-

$$
\begin{aligned}
U(t) = \mathbb{E}[S(\boldsymbol{X})|T = t] &= \frac{\mathbb{P}(X_1 = 0, T = t)}{\mathbb{P}(T = t)} \\
&= \frac{\mathbb{P}(X_1 = 0, \sum_{i=2}^{n} X_i = t - 0)}{\mathbb{P}(T = t)} \\
&= \frac{(e^{-\lambda}\lambda^0/0!)(e^{-(n-1)\lambda}((n-1)\lambda)^{(t-0)}/(t-0)!)}{e^{-n\lambda}(n\lambda)^t/t!} \\
&= \left(\frac{n-1}{n}\right)^t.
\end{aligned}
$$

So, $U(\boldsymbol{X}) = [(n-1)/n]^{n\bar{X}}$ is the UMVUE of θ by R.-B. Theorem & L.-S. Theorem.

- Note: the conditional distribution $X_i|T = t \sim Bin(t, 1/n)$ (it's easy to check by yourself).
- verify:

$$
\begin{aligned}
\mathbb{E}[U] = \mathbb{E}\left[\left(\frac{n-1}{n}\right)^T\right] &= \sum_{k=0}^{\infty} (\frac{n-1}{n})^k \frac{e^{-n\lambda}(n\lambda)^k}{k!} \\
&= e^{-n\lambda} \sum_{k=0}^{\infty} \frac{(n\lambda(n-1)/n)^k}{k!} = e^{-n\lambda}e^{n\lambda((n-1)/n)} = e^{-\lambda} = \theta.
\end{aligned}
$$

- Consider

$$U = h(T) = \Big(\frac{n-1}{n} \Big)^T,$$

such that $E[U] = \sum_{k=0}^{\infty} (\frac{n-1}{n})^k \frac{e^{-n\lambda}(n\lambda)^k}{k!} = e^{-n\lambda} \sum_{k=0}^{\infty} \frac{(n\lambda(n-1)/n)^k}{k!} = \theta$.
So, U is the UMVUE of θ.

2. 設 X 與 Y 獨立, 皆有 $\mathcal{N}(\mu, \sigma^2)$ 分佈。分別對參數爲 μ 及 σ^2 時, 求 (i) $X + Y$, (ii) $X - Y$ 之資訊數。(**解.** (i) $I_{X+Y}(\mu) = 2/\sigma^2$, $I_{X+Y}(\sigma^2) = 1/(2\sigma^4)$, (ii) $I_{X-Y}(\mu) = 0$, $I_{X-Y}(\sigma^2) = 1/(2\sigma^4)$)

- $T_1 := X + Y \sim \mathcal{N}(2\mu, 2\sigma^2)$, $T_2 := X - Y \sim \mathcal{N}(0, 2\sigma^2)$.
- $\log L(\mu, \sigma^2; T_1) = -2^{-1} \log(4\pi\sigma^2) - (t_1 - 2\mu)^2/(4\sigma^2)$.
- $\partial \log L(\mu)/\partial \mu = (t_1 - 2\mu)/(\sigma^2)$;
  $\partial \log L(\sigma^2)/\partial \sigma^2 = -1/(2\sigma^2) + (t_1 - 2\mu)^2/(4\sigma^4)$.
- $I_{T_1}(\mu) = \mathbb{E}[(\partial \log L(\mu)/\partial \mu)^2] = \mathbb{E}[(T_1 - 2\mu)^2]/\sigma^4 = 2/(\sigma^2)$;
- $I_{T_1}(\sigma^2) = \mathbb{E}[(\partial \log L(\sigma^2)/\partial \sigma^2)^2] = \mathbb{E}[(-1/(2\sigma^2) + (T_1 - 2\mu)^2/4\sigma^4)^2] = \mathbb{E}\left[1/(4\sigma^4) + (T_1 - 2\mu)^4/(16\sigma^8) - (T_1 - 2\mu)^2/(4\sigma^6)\right] = 1/(2\sigma^4)$.
- Note: $Z \sim \mathcal{N}(\mu, \sigma_Z^2) \Rightarrow \mathbb{E}[Z - \mu] = 0, \mathbb{E}[(Z - \mu)^2] = \sigma_Z^2, \mathbb{E}[(Z - \mu)^3] = 0, \mathbb{E}[(Z - \mu)^4] = 3\sigma_Z^4$.

- $\log L(\mu, \sigma^2; T_2) = -2^{-1} \log(2\sigma^2) - y^2/(4\sigma^2)$.
- $\partial \log L(\mu)/\partial\mu = t_2/(\sigma^2)$;
  $\partial \log L(\sigma^2)/\partial\sigma^2 = -1/(2\sigma^2) + t_2{}^2/(4\sigma^4)$.
- $I_{T_2}(\mu) = \mathbb{E}[(\partial \log L(\mu)/\partial\mu)^2] = 0$;
  $I_{T_2}(\sigma^2) = \mathbb{E}[(\partial \log L(\sigma^2)/\partial\sigma^2)^2] = \mathbb{E}\left[(-1/(2\sigma^2) + T_2^2/(4\sigma^4))^2\right] =$
  $\mathbb{E}\left[1/(4\sigma^4) + T_2^4/(16\sigma^8) - T_2^2/(4\sigma^6)\right] = 1/(2\sigma^4)$.

11. 設 $X_1, \cdots, X_n$ 爲一組由 p.d.f. $f_i(x|\theta)$ 所產生之隨機樣本, $i = 1, 2$, 其中

(i) $f_1(x|\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0$;

(ii) $f_2(x|\theta) = \theta^x \log \theta / (\theta - 1), 0 < x < 1, \theta > 1$。

試分別對此二 p.d.f., 討論是否有一 θ 之函數 $q(\theta)$, 使得存在一 $q(\theta)$ 之不偏估計量, 且其變異數達到 CRLB。若有則找出來, 若沒有亦證明之。

- Theorem (Efficiency Attainment)

假設 $\boldsymbol{X} = (X_1, \cdots, X_n)$ 有一 joint pdf $f(\boldsymbol{x}|\theta)$, 與其對應之 likelihood function $L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i|\theta)$, 則對 $q(\theta)$ 的任一不偏估計量 $U(\boldsymbol{X})$ 之變異數可達到 CRLB, 若且唯若存在一個 (θ, n) 的函數 $a(\theta, n)$ 使得以下等式成立:

$$a(\theta, n)\Big[U(\boldsymbol{X}) - q(\theta)\Big] = \frac{\partial}{\partial \theta} \log L(\theta|\boldsymbol{x}).$$

- (i)

$$\partial \log L / \partial \theta = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} [\log \theta + (\theta - 1) \log x_i]$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \theta} [\log \theta + (\theta - 1) \log x_i] \quad \text{(why)}$$

$$= \sum_{i=1}^{n} \left[\theta^{-1} + \log x_i \right]$$

$$= -n \left[-\sum_{i=1}^{n} \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) \left[U(\boldsymbol{X}) - q(\theta) \right].$$

- Then, $a(\theta, n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n} \log X_i / n$ is the UMVUE of $1/\theta$, which attains the CRLB.

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$$\partial \log L / \partial \theta = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} [\log \theta + (\theta - 1) \log x_i]$$

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$$= -n \left[- \sum_{i=1}^{n} \log x_i / n - \frac{1}{\theta} \right] = a(\theta, n) \left[U(\boldsymbol{X}) - q(\theta) \right].$$

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- Note: $\nexists \, a(.)$ such that $\partial \log L / \partial \theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.

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- Note:「有限項部份和」與微分 (或極限) 可以直接交換符號。

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- Then, $a(\theta, n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n} \log X_i / n$ is the UMVUE of $1/\theta$, which attains the CRLB.
- Note: $\nexists \, a(.)$ such that $\partial \log L / \partial \theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.
- Note:「有限項部份和」與微分 (或極限) 可以直接交換符號。
- Note:「無窮項和」與微分 (或極限) 若可交換符號, 當此級數是均勻收斂 (uniform convergence) 或 控制收斂 (dominated (bounded) convergence)。

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- Note: $\nexists \, a(.)$ such that $\partial \log L / \partial \theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.
- Note:「有限項部份和」與微分 (或極限) 可以直接交換符號。
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- (ii)

$$\partial \log L / \partial \theta = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} [\log \log \theta - \log(\theta - 1) + x_i \log \theta]$$

$$= (\frac{n}{\theta \log \theta} - \frac{n}{\theta - 1}) + \theta^{-1} \sum_{i=1}^{n} x_i$$

$$= \frac{n}{\theta} \Big[\bar{x} - (\frac{\theta}{\theta - 1} - \frac{1}{\log \theta}) \Big] = a(\theta, n) \Big[U(\boldsymbol{X}) - q(\theta) \Big].$$

- Then, $a(\theta, n) = n/\theta$, and $q(\theta) = \theta/(\theta - 1) - 1/(\log \theta)$;
- $U(\boldsymbol{X}) = \bar{X}$ is the UMVUE of $q(\theta)$, which attains the CRLB.
- Note: No UMVUE of θ that attains the CRLB.