

TA section 9

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Homework 4: (part I)

§8.2 #1, #4, #19, #20, #21

1. 設 X 有 $\mathcal{N}(\mu, 16)$ 分佈。欲檢定 $H_0: \mu = 10$, vs. $H_a: \mu = 11$, 取樣本數 $n = 25$ 之一組隨機樣本。試決定型 I 錯誤機率 $\alpha = 0.05$ 下的一拒絕域, 並求此時之型 II 錯誤之機率。(解. $\{\bar{X} > 11.316\}$, 0.654)

- Consider $T(\mathbf{X}) = \bar{X}_n$, $\bar{X}_n \sim \mathcal{N}(\mu, \sigma^2/n) = \mathcal{N}(\mu, 0.8^2)$.
- the rejection region $C := \{\bar{X}_n > c | \mu = 10\} = \{Z > z_{0.05} | \mu = 10\}$.
- $\alpha = 0.05 = \mathbb{P}(\text{type I error}) = \mathbb{P}(\text{reject } H_0 | H_0) = \mathbb{P}(\bar{X}_n \in C | \mu = 10) = \mathbb{P}(Z > (c - 10)/0.8) = \mathbb{P}(Z > z_{0.05}) \Rightarrow c^* = 10 + (1.645) \times (0.8) = 11.316$.
So, $C^* = \{\bar{X}_n > 11.316\}$.
- $\mathbb{P}(\text{type II error}) = \mathbb{P}(\text{not reject } H_0 | H_1) = \mathbb{P}(\bar{X}_n \leq c^* | \mu = 11) = \mathbb{P}(Z \leq (11.316 - 11)/0.8) = 0.654$.

4. 在第1題中, 設拒絕域為 $\{\bar{X} > c\}$ 。試求 $\bar{X} = 11.40$ 時之 p -值。(解.
0.0401)

- $T(\boldsymbol{x}) := \bar{X}_0 = 11.40$. Under H_o , $z_0 = (\bar{X}_0 - 10)/0.8 = 1.75$.
- $p(\boldsymbol{x}) = \mathbb{P}(T(\mathbf{X}) > T(\boldsymbol{x})|H_o) = \mathbb{P}(Z > z_0) = \mathbb{P}(Z > 1.75) = 0.0401$.

19. 設 X_1, \dots, X_n 為一組由 $\mathcal{N}(\mu, \sigma^2)$ 分佈所產生之隨機樣本, σ^2 為已知。欲檢定 $H_0: \mu = \mu_1$, vs. $H_a: \mu = \mu_2$ 。試證只要 n 夠大, $K(\mu_1)$ 可任意小, 而 $K(\mu_2)$ 可任意大。

- Claim: $\lim_{n \rightarrow \infty} K(\mu_1) = 0$, and $\lim_{n \rightarrow \infty} K(\mu_2) = 1$.
- By CLT, $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} \mathcal{N}(0, 1)$, for some μ , as $n \rightarrow \infty$.
- For $H_1: \mu_2 > \mu_1$: $C^* = \{\bar{X}_n > c_1\}$, $c_1 > \mu_1$.
For $H_1: \mu_2 < \mu_1$: $C^* = \{\bar{X}_n < c_2\}$, $c_2 < \mu_1$.

$$K(\mu) = \begin{cases} \mathbb{P}(\bar{X}_n > c_1 | \mu), & [\mu_2 > \mu_1] \\ \mathbb{P}(\bar{X}_n < c_2 | \mu), & [\mu_2 < \mu_1] \end{cases}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} K(\mu_1) &= \begin{cases} \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma), & [\mu_2 > \mu_1] \\ \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma < \sqrt{n}(c_2 - \mu_1)/\sigma), & [\mu_2 < \mu_1] \end{cases} \\ &= \begin{cases} 1 - \Phi(\infty) = 0, & [\mu_2 > \mu_1] \\ \Phi(-\infty) = 0, & [\mu_2 < \mu_1] \end{cases} \end{aligned}$$

- If we pre-specified a c satisfying a fixed $\alpha = 1 - \Phi(c) = \Phi(-c)$, then $\lim_{n \rightarrow \infty} K(\mu_1) = \mathbb{P}(Z > c) = \mathbb{P}(Z < -c) = \alpha$.
- For $\mu_2 > \mu_1$:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} K(\mu_2) &= \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma | H_1) \\
 &= \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_2)/\sigma + \sqrt{n}(\mu_2 - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma) \\
 &= \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_2)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma - \sqrt{n}(\mu_2 - \mu_1)/\sigma) \\
 &= 1 - \Phi(\sqrt{n}(c_1 - \mu_2)/\sigma) \\
 &= 1 - \Phi(-\infty), \text{ if } \mu_1 < c_1 < \mu_2 \text{ (critical value cannot be too high)} \\
 &= 1.
 \end{aligned}$$

Similarly, for $\mu_2 < \mu_1$, $\lim_{n \rightarrow \infty} K(\mu_2) = \Phi(\infty) = 1$ if $\mu_1 > c_2 > \mu_2$ (critical value cannot be too small).

20. 設 X_1, \dots, X_n 為一組由 $\mathcal{U}(0, \theta)$ 分佈所產生之隨機樣本。欲檢定 $H_0 :$

$\theta \leq \theta_0$, vs. $H_a : \theta > \theta_0$ 。取拒絕域為 $\{X_{(n)} \geq c\}$ 。

(i) 試求檢力函數 $K(\theta)$, 並證明此為 θ 之一增函數;

(ii) 設 $\theta_0 = 1/2$, 試求 c 之值, 使顯著水準為 0.05;

- $f_{X_{(n)}}(t) = nt^{n-1}/\theta^n$
- $K(\theta) = \mathbb{P}(X_{(n)} > c | \theta) = 1 - \theta^{-n} \int_0^c nx^{n-1} dx = 1 - (c/\theta)^n.$
- $dK(\theta)/d\theta = nc^n/\theta^{n+1} > 0$ for all $0 < \theta_0 < c \leq \theta$.
- $\lim_{n \rightarrow \infty} K(\theta) = 1$ as $c < \theta$.
- $K(\theta = 1/2) = 1 - (2c)^n = 0.05 \Rightarrow c^* = \frac{(0.95)^{1/n}}{2}.$

21. 設 X_1, \dots, X_n 為一組由 $\mathcal{E}(\lambda)$ 分佈所產生之隨機樣本。令 $\mu = 1/\lambda$ 。欲檢定 $H_0 : \mu \leq \mu_0$, vs. $H_a : \mu > \mu_0$ 。

(i) 試證對 $\forall 0 < \alpha < 1$, 拒絕域 $\{\bar{X} \geq \mu_0 \chi_{1-\alpha, 2n}^2 / (2n)\}$, 為一顯著水準 α 之檢定;

(ii) 試以 χ_{2n}^2 之分佈函數 F , 表示此檢定之檢力函數。

- Let $S_n = \sum_{i=1}^n X_i \sim \Gamma(n, \lambda) \Rightarrow 2n\lambda\bar{X} = 2n\bar{X}/\mu \sim \Gamma(n, 1/2) = \chi^2(2n)$.
- (i) $\alpha = \sup_{\mu \leq \mu_0} \mathbb{P}(2n\bar{X}/\mu \geq \chi_{1-\alpha, 2n}^2 | H_0 : \mu \leq \mu_0) \Rightarrow \alpha = \mathbb{P}(\bar{X} \geq \mu_0 \chi_{1-\alpha, 2n}^2 / (2n)), \forall \alpha \in (0, 1)$.
- (ii) Given the CDF of $\chi^2(2n)$ is $F(\cdot)$, then $K(\mu) = \mathbb{P}(\bar{X} \geq \mu_0 \chi_{1-\alpha, 2n}^2 / (2n) | H_1 : \mu > \mu_0) = \mathbb{P}(2n\bar{X}/\mu \geq (\mu_0/\mu) \chi_{1-\alpha, 2n}^2) = 1 - F((\mu_0/\mu) \chi_{1-\alpha, 2n}^2) (\because 2n\bar{X}/\mu \sim \chi^2(2n))$.

§8.3 #5, #9

5. 設 X_1, \dots, X_{18} 為一組由 $\mathcal{N}(\mu, \sigma^2)$ 分佈所產生之隨機樣本。欲檢
定 $H_0 : \sigma^2 = 0.36$, vs. $H_a : \sigma^2 > 0.36$ 。假設樣本變異數 $S^2 =$
 0.68 。試在 $\alpha = 0.05$ 之下，作一檢定。(解. $K \doteq 32.1$, $\chi_{0.05,17}^2 \doteq$
 27.59 , 拒絕 H_0)

- Let a test statistic $K(\mathbf{X}) := (n-1)S^2/\sigma^2 \sim \chi_{n-1}^2 = \chi_{17}^2$.
- Under H_o , a test function

$$\phi(\mathbf{X}) = I(K(\mathbf{X}) > \chi_{0.05,17}^2),$$

such that $0.05 = \mathbb{P}(K(\mathbf{X}) > \chi_{0.05,17}^2 | H_o : \sigma^2 = 0.36)$.

- $K = (17) \times (0.68)/(0.36) \approx 32.1 > \chi_{0.05,17}^2 = 27.59 \Rightarrow$ reject H_o .

9. 設 X_1, \dots, X_9 為一組由 $\mathcal{N}(\mu_1, \sigma_1^2)$ 分佈所產生之隨機樣本, Y_1, \dots, Y_9 為一組由 $\mathcal{N}(\mu_2, \sigma_2^2)$ 分佈所產生之隨機樣本。又設觀測到 $\bar{X} = 16$, $\bar{Y} = 10$, $S_1^2 = 36$, $S_2^2 = 45$ 。

(i) 設 $\sigma_1^2 = \sigma_2^2$, 試給 $-\alpha = 0.10$ 下之 $H_0 : \mu_1 = \mu_2$, vs. $H_a : \mu_1 \neq \mu_2$ 之檢定;

(ii) 試給 $-\alpha = 0.05$ 下之 $H_0 : \sigma_2^2/\sigma_1^2 \leq 1$, vs. $H_a : \sigma_2^2/\sigma_1^2 > 1$ 之檢定。

- (i) $\bar{Y} - \bar{X} \sim \mathcal{N}(\mu_2 - \mu_1, \sigma_1^2/n_1 + \sigma_2^2/n_2)$.
- $Z = ((\bar{Y} - \bar{X}) - (\mu_2 - \mu_1)) / \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} \sim \mathcal{N}(0, 1)$, and $S_p^2 := (n_1 - 1)S_1^2/\sigma_1^2 + (n_2 - 1)S_2^2/\sigma_2^2 \sim \chi_{n_1+n_2-2}^2 \Rightarrow$ a test statistic

$$T := \frac{(\bar{Y} - \bar{X}) - (\mu_2 - \mu_1)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} = t_{16}.$$

- Under H_o , a test function

$$\phi = \mathbf{I}(|T(0)| > t_{0.95,16}),$$

such that $0.1 = \mathbb{P}(|T(0)| > t_{16} | H_o : \mu_2 - \mu_1 = 0)$, where

$$T(0) := \frac{(\bar{Y} - \bar{X})}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

- $|T(0)| = |-6/3| = 2 > t_{16} = 1.746 \Rightarrow$ reject H_o .

- (ii) a test statistic

$$F := \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \sim F_{(n_2-1, n_1-1)} = F_{(8,8)}.$$

- Under H_o , a test function

$$\phi = \mathbf{I}(F(0) > F_{0.95,(8,8)}),$$

such that $0.05 = \mathbb{P}(F(0) > F_{0.95,(8,8)} | H_o : \sigma_2^2/\sigma_1^2 \leq 1)$, where $F(0) := S_2^2/S_1^2$.

- $F(0) = 45/36 = 1.25 < F_{0.95,(8,8)} = 3.44 \Rightarrow$ do not reject H_o .