

TA section 5

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Homework 3

§1.5-#7(iii) (iv), §1.6#10, #23; §2.2#8; #12

7. 分別對有下述所給p.d.f.之各隨機變數 X 及變換, 試求 Y 之p.d.f. f_Y 。

(i) $f_X(x) = 42x^5(1-x)$, $0 < x < 1$, $Y = X^3$;

(ii) $f_X(x) = 1$, $0 < x < 1$, $Y = X^5$;

(iii) $f_X(x) = \frac{1}{\sigma} x e^{-x^2/(2\sigma)}$, $x > 0$, $\sigma > 0$ 爲一常數, $Y = e^X$;

(iv) $f_X(x) = \frac{1}{3}(\frac{2}{3})^x$, $x = 0, 1, 2, \dots$, $Y = X/(X+1)$;

• (iii). $\mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \log y)$,

$$f_Y(y) = f_X(\log y) y^{-1} = \frac{\log y}{y\sigma} e^{-(\log y)^2/2\sigma}, \quad y > 1.$$

- (iv).

$$\begin{aligned}f_Y(y) &= \mathbb{P}(Y = y) = \mathbb{P}(X/(X+1) = y) \\&= \mathbb{P}(X = y/(1-y)) \\&= f_X(y/(1-y)) = (1/3)(2/3)^{y/(1-y)}, \quad y = 0, 1/2, 2/3, \dots\end{aligned}$$

10. 設隨機變數 X 有 $\mathcal{U}(0, 1)$ 分佈。試求 $\log X$ 之期望值、二次動差, 及變異數。

- Let $Y = \log X$, by the integration by parts,

$$\mathbb{E} Y = \int_0^1 \log x f_X(x) dx = (x \log x - x) \Big|_{x=0}^{x=1} = -1.$$

- By the integration by parts, again,

$$\mathbb{E} Y^2 = \int_0^1 (\log x)^2 f_X(x) dx = (x(\log x)^2 - 2x \log x + 2x) \Big|_{x=0}^{x=1} = 2.$$

- $\text{Var}[Y] = \mathbb{E} Y^2 - (\mathbb{E} Y)^2 = 1.$

- 另解:

$$\mathbb{P}(Y \leq y) = \mathbb{P}(X \leq \exp(y)) = e^y, \text{ for } y \leq 0 \text{ (不是指數分佈)}$$

$$\mathbb{E} Y = \int_{-\infty}^0 y e^y dy = (y e^y - e^y) \Big|_{y=0} - \lim_{b \rightarrow \infty} (y e^y - e^y) \Big|_{y=-b} = -1.$$

23. 設隨機變數 X 之 p.d.f. f 為一偶函數。試證

(i) $E(X)$ 若存在必為 0;

(ii) $M_X(t)$ 若存在, 則滿足 $M_X(t) = M_X(-t)$ 。

- (i). $\mathbb{E}[X] = \int_{-\infty}^{\infty} sf_X(s)ds = \int_0^{\infty} sf_X(s)ds + \int_{-\infty}^0 sf_X(s)ds = \int_0^{\infty} sf_X(s)ds + \int_{-\infty}^0 sf_X(-s)ds = \int_0^{\infty} sf_X(s)ds - \int_0^{\infty} tf_X(t)dt = 0.$
(let $t = -s$, $\int_{-\infty}^0 sf_X(-s)ds = \int_{\infty}^0 tf_X(t)dt = -\int_0^{\infty} tf_X(t)dt$)
- $M_X(t) = \mathbb{E}[e^{tX}] = (\int_0^{\infty} + \int_{-\infty}^0) e^{tx} f_X(x)dx = (\int_0^{\infty} + \int_{-\infty}^0) e^{tx} f_X(-x)dx = (\int_0^{-\infty} + \int_{\infty}^0) - e^{-ty} f_X(y)dy = (\int_0^{\infty} + \int_{-\infty}^0) e^{-ty} f_X(y)dy = M_X(-t).$
(let $y = -x$)

8. 設 X 有 $B(n, p)$ 分佈。試證 $\text{Var}(X) \leq n/4$, 且等號成立若且唯若 $p = 1/2$ 。

- $X \sim B(n, p), \text{Var}[X] = np(1 - p), n \geq 0.$



$$\begin{aligned}\text{Var}[X] &= np(1 - p) = -n(p - 1/2)^2 + n/4 \leq n/4, \\ &= n/4, \text{ iff } p = 1/2.\end{aligned}$$

12. 投擲一公正的骰子, 直到出現一大於4的點數才停止, 令 X 表總共之投擲數。試求 $P(X \geq 3)$, $E(X)$ 及 $\text{Var}(X)$ 。

- 即 X 是第一次試驗成功所需的次數, 成功事件: 出現 5 或 6 的點數;
- 成功機率: $p = 2/6 = 1/3$, $X \sim \text{Geo}(p)$,

$$f_X(x) = p(1-p)^{x-1}, \quad x = 1, 2, 3, \dots$$

- $\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 1 - 1/3 - 2/9 = 4/9$.
- $\mathbb{E} X = 1/p = 3$.
- $\text{Var}[X] = (1-p)/p^2 = 6$.