

## TA section 6

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## Homework 3: part (I)

## §7.2 #2, #9, #11, #13, #18; §7.3 #1

2. 設  $T_1, T_2$  皆為  $\theta$  之不偏估計量, 且  $T_1$  與  $T_2$  獨立, 變異數分別為  $\sigma_1^2$  及  $\sigma_2^2$ 。  
 試求  $a, b$ , 使得  $aT_1 + bT_2$  為  $\theta$  之不偏估計量中 MSE 最小者。(解.  $a$   
 $= \sigma_2^2/(\sigma_1^2 + \sigma_2^2), b = 1 - a$ )

- WLOG, assume  $\theta \neq 0$ .
- $T_1 \perp\!\!\!\perp T_2$ . Let  $Z := aT_1 + bT_2$ .
- $\mathbb{E}[T_1] = \mathbb{E}[T_2] = \theta$ ,  $\mathbb{E}[Z] = a\mathbb{E}[T_1] + b\mathbb{E}[T_2] = \theta \Rightarrow (a + b)\theta = \theta$ . Thus,  $b = 1 - a$ .
- MSE:  $R(\theta, Z) = \mathbb{E}[(Z - \theta)^2] = \text{Var}[Z] = a^2\sigma_1^2 + (1 - a)^2\sigma_2^2$  ( $\text{Bias}(Z) = 0$ )
- $\frac{dR}{da}|_{a=a^*} = 0$ ,  $\frac{d^2R}{da^2}|_{a=a^*} > 0$ .
- Solve for  $a^* = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$  and  $b^* = 1 - a^* = \sigma_1^2/(\sigma_1^2 + \sigma_2^2)$ .

9. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{U}[1, 1 + \theta]$  分佈所產生之隨機樣本,  $\theta > 0$ 。

- (i) 試求  $\theta$  之 MLE  $T_1$ , 並問  $T_1$  是否為不偏的;
- (ii) 試求  $\theta$  之動差估計量  $T_2$ , 並問  $T_2$  是否為不偏的;
- (iii) 試比較  $T_1$  與  $T_2$  之 MSE。

- $L(\theta) = \theta^{-n} \prod_{i=1}^n \mathbf{I}(1 \leq x_i \leq 1 + \theta), i = 1, 2, \dots, n.$
- $T_1 := \hat{\theta}_{MLE} = X_{(n)} - 1.$
- $f_{X_{(n)}}(t) = n(t-1)^{n-1}/\theta^n, \theta \in [1, 1 + \theta].$
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$$\mathbb{E}[T_1] = \mathbb{E}[X_{(n)}] - 1 = n\theta^{-n} \int_1^{1+\theta} t(t-1)^{n-1} dt - 1 = 1 + \frac{n\theta}{n+1} - 1 = \frac{n\theta}{n+1}.$$

So,  $T_1$  is biased for  $\theta$ ,  $Bias(T_1) = -\theta/(n+1).$

- $\mathbb{E}[X] = (1 + 1 + \theta)/2 = 1 + \theta/2 \Rightarrow \hat{\theta}_{MME} = 2(\bar{X} - 1) =: T_2.$
- $\mathbb{E}[T_2] = 2(\mathbb{E}[\bar{X}] - 1) = 2(1 + \theta/2 - 1) = \theta$ , i.e.,  $T_2$  is unbiased for  $\theta$ .
- $\mathbb{E}[X_{(n)}^2] = 1 + 2n\theta/(n+1) + n\theta^2/(n+2),$   
 $\text{Var}[X_{(n)}] = \mathbb{E}[X_{(n)}^2] - \mathbb{E}[X_{(n)}]^2 = n\theta^2/(n+1)^2(n+2).$
- MSE:

$$R(\theta, T_1) = \text{Bias}(T_1)^2 + \text{Var}[X_{(n)}] = \frac{2\theta^2}{(n+1)^2(n+2)}.$$

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$$R(\theta, T_2) = \text{Var}[T_2] = 4 \text{Var}[\bar{X}] = \frac{\theta^2}{3n} > R(\theta, T_1).$$

11. 設  $X$  有  $\mathcal{P}(\lambda)$  分佈,  $\lambda > 0$ 。令  $\theta = P(X = 0) = e^{-\lambda}$ 。

(i) 試問  $T_1 = e^{-X}$  是否為  $\theta$  之不偏估計量;

(ii) 試證  $T_2 = I_{\{X=0\}}$  為  $\theta$  之不偏估計量;

(iii) 試分別求  $T_1$  及  $T_2$  之 MSE。

- By MGF  $M_X(t) = \mathbb{E}[e^{tX}] = e^{\lambda(e^t-1)}$ .
- 令  $t = -1$ ,  $\mathbb{E}[T_1] = \mathbb{E}[e^{-X}] = e^{-\lambda(1-1/e)}$ ,  $T_1$  is biased for  $\theta$ .
- $\mathbb{E}[T_2] = \mathbb{P}(X = 0) = e^{-\lambda} = \theta$ ,  $T_2$  is unbiased for  $\theta$ .

MSE:



$$\begin{aligned}
 R(\theta, T_1) &= \text{Bias}(T_1)^2 + \text{Var}[T_1] \\
 &= e^{-2\lambda}[e^{\lambda/e} - 1] + [e^{\lambda/e^2} - e^{2\lambda/e}]e^{-2\lambda} \\
 &= (1 - 2e^{\lambda/e} + e^{\lambda/e^2})e^{-2\lambda}.
 \end{aligned}$$

- $R(\theta, T_2) = \text{Var}[T_2] = \mathbb{E}[T_2^2] - \mathbb{E}[T_2]^2 = \mathbb{P}(X = 0) - \mathbb{P}(X = 0)^2 = e^{-\lambda}(1 - e^{-\lambda}).$

13. 設  $X_1, \dots, X_n$  為一組由某一期望值為  $\mu$ , 變異數為  $\sigma^2$  之分佈所產生之隨機樣本,  $\mu, \sigma^2$  皆設為未知。令  $T(\mathbf{X}) = \sum_{i=1}^n c_i X_i$ , 其中  $c_1, \dots, c_n$  為常數。

(i) 試證  $T$  為  $\mu$  之不偏估計量, 若且唯若  $\sum_{i=1}^n c_i = 1$ ;

(ii) 試證在型如  $\sum_{i=1}^n c_i X_i$  之  $\mu$  的不偏估計量中,  $\bar{X}_n$  為一致最小變異不偏估計量。

•  $\mathbb{E}[T] = \sum_{i=1}^n c_i \mathbb{E}[X_i] = \mu \sum_{i=1}^n c_i = \mu$  iff  $\sum_{i=1}^n c_i = 1$ .



- $\text{Var}[T] = \sum_{i=1}^n c_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^n c_i^2 =: Q(c).$

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$$c_i^* = \arg \min_{c_i} \{Q(c) : \sum_{i=1}^n c_i = 1, i = 1, \dots, n\}.$$

- By Lagrangian multiplier:

$$L(c, \lambda) = Q(c) + \lambda \left( \sum_{i=1}^n c_i - 1 \right),$$

- $\partial L / \partial c_i = 0 \Rightarrow c_i = -\lambda / 2\sigma^2;$

- $\partial L / \partial \lambda = 0 \Rightarrow \sum_{i=1}^n c_i = 1 \Rightarrow \lambda = -2\sigma^2/n;$

- So,  $c_i^* = 1/n,$

i.e.,  $T(\mathbf{X}) = n^{-1} \sum_{i=1}^n X_i = \bar{X}$  is an UMVUE of  $\mu$ .

18. 設  $X_1, \dots, X_n$  為一組由 p.d.f.  $f(x|\theta) = \theta(1+x)^{-(1+\theta)}$ ,  $x > 0$ ,  $\theta > 0$ , 所產生之隨機樣本。

(i) 試求  $\theta$  之一不偏估計量;

(ii) 是否存在  $-g(\theta) = \theta^{-1}$  之不偏估計量? 若有則給出一個。

- $f(x|\theta) = \theta \exp[-(1+\theta) \log(1+x)] =: h(x)c(\theta) \exp(w(\theta)t(x)) \mathbf{I}_A(x)$ , where  $h(x) = \mathbf{I}(x_i > 0)$ ,  $c(\theta) = \theta$ ,  $w(\theta) = -(1+\theta)$ , and  $t(x) = \log(1+x)$ . So,  $C = \{w(\theta) : \theta \in \Omega\}$  contains a nonempty open set,  $T(\mathbf{X}) = \sum_{i=1}^n \log(1+X_i)$  is a C.S.S.

- Let  $Y = \log(1+X) \Rightarrow X = e^Y - 1$ ,

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$$f_Y(y) = f_X(e^y - 1) \cdot e^y = \theta e^{-\theta y},$$

i.e.,  $Y \sim \mathcal{E}(\theta) = \Gamma(1, 1/\theta)$ .

- $T \sim \sum_{i=1}^n Y_i \sim \Gamma(n, 1/\theta)$ .

- $\mathbb{E}[T^{-1}] = \theta/(n-1) \Rightarrow \mathbb{E}[(n-1)/T] = \theta$ .  
(recall:  $Z \sim \Gamma(\alpha, \beta) \Rightarrow \mathbb{E}[Z^k] = \beta^k \Gamma(\alpha+k)/\Gamma(\alpha)$ ).

- Let

$$T_1 := \frac{(n-1)}{\sum_{i=1}^n \log(1+X_i)},$$

which is unbiased for  $\theta$ , such that  $\mathbb{E}[T_1] = \theta$ .

- $\mathbb{E}[T] = n/\theta \Rightarrow \mathbb{E}[T/n] = 1/\theta$ , i.e., let

$$T_2 := \sum_{i=1}^n \log(1+X_i)/n,$$

which is unbiased for  $1/\theta$ , such that  $\mathbb{E}[T_2] = 1/\theta$ .

1. 設  $X_1, \dots, X_n$  為一組由  $Ber(\theta)$  分佈所產生之隨機樣本,  $0 \leq \theta \leq 1$ 。試分別求  $\theta, \theta^2, \theta(1 - \theta)$  之 UMVUE。

求 UMVUE 三招:

- 「充份 + 不偏」: 定理 3.1: Rao-Blackwell Theorem (非唯一解)
- 「完備充份 + 不偏」: 定理 3.2: Lehmann-Scheffé Theorem (唯一解)
- 「不偏 + CRLB」: (i) “Efficiency Attainment” (最效率可達性); (ii) 定理 4.3: 滿足 CRLB 的不偏之一個參數 (one-dimensional) 指數族。

# 使用 R-B Thm. & L-S Thm 原則:

- 給定  $T(\mathbf{X})$  is a C.S.S.,
- 設法找一個  $h(T(\mathbf{X}))$  為  $q(\theta)$  之不偏估計量, 則  $h(T(\mathbf{X})) = \mathbb{E}[h(T(\mathbf{X}))|T(\mathbf{X})]$  為  $q(\theta)$  之一 UMVUE。
- 若  $h(T(\mathbf{X}))$  不易找出, 則設法造出:
  - (1): 找一個  $q(\theta)$  之不偏估計量,  $S(\mathbf{X})$  (不一定是  $T(\mathbf{X})$  的函數, 若是  $T(\mathbf{X})$  的函數, 則同上);
  - (2): 造出  $\mathbb{E}[S(\mathbf{X})|T(\mathbf{X})]$ , 此即  $q(\theta)$  之一 UMVUE。

# 定理 3.1.: 充份 + 不偏 $\Rightarrow$ 有效

## • Theorem

設  $T(\mathbf{X})$  為  $\theta$  之一充份統計量, 設  $S(\mathbf{X})$  為  $q(\theta)$  之一不偏估計量, 且  $\mathbb{E}|S(\mathbf{X})| < \infty, \forall \theta \in \Omega$ . 令  $T^*(\mathbf{X}) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})]$ , 則  $\forall \theta \in \Omega$ ,

$$R(\theta, T^*) \leq R(\theta, S).$$

## 定理 3.2.: 完備充份 + 不偏 $\Rightarrow$ 有效

### • Theorem

設  $T(\mathbf{X})$  為一完備充份統計量, 且  $S = S(\mathbf{X})$  為  $q(\theta)$  之一不偏估計量。則  $T^*(\mathbf{X}) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})]$  為  $q(\theta)$  之一 UMVUE; 若  $\text{Var}[T^*] < \infty, \forall \theta \in \Omega$ , 則  $T^*$  為  $q(\theta)$  唯一之 UMVUE。

# 定理 4.3.

## • Theorem

設  $T(\mathbf{X})$  為  $q(\theta)$  一不偏估計量,  $\mathbb{E}[T(\mathbf{X})] = q(\theta)$ 。設一分佈族  $\{P_\theta; \theta \in \Omega\}$  滿足正規條件, 且為一個參數之指數族, 有 pdf 如下式:

$$f(\mathbf{x}|\theta) = h(\mathbf{x}) \exp(w(\theta)T(\mathbf{x}))I_A(\mathbf{x}), \quad \theta \in \Omega,$$

其中  $w(\theta)$  有一連續且不為零之導數,  $\forall \theta \in \Omega$ , 若且唯若  $\text{Var}[T(\mathbf{X})]$  達到 CRLB, 且  $T(\mathbf{X})$  為  $q(\theta)$  之一 UMVUE。



### • Theorem (Efficiency Attainment)

假設  $\mathbf{X} = (X_1, \dots, X_n)$  有一 joint pdf  $f(\mathbf{x}|\theta)$ , 與其對應之 likelihood function  $L(\theta|\mathbf{x}) = \prod_{i=1}^n f(x_i|\theta)$ , 則對  $q(\theta)$  的任一不偏估計量  $U(\mathbf{X})$  之變異數可達到 CRLB, 若且唯若存在一個  $(\theta, n)$  的函數  $a(\theta, n)$  使得以下等式成立:

$$a(\theta, n) \left[ U(\mathbf{X}) - q(\theta) \right] = \frac{\partial}{\partial \theta} \log L(\theta|\mathbf{x}).$$

- $T(\mathbf{X}) = \sum_{i=1}^n X_i$  is a C.S.S.
- Let  $S(\mathbf{X}) = \bar{X}$ , such that  $\mathbb{E}[S(\mathbf{X})] = \theta$ .
- Then, let

$$h_1(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X}),$$

$\mathbb{E}[h_1(T(\mathbf{X}))] = \mathbb{E}[\bar{X}] = \theta$ . 故,  $S(\mathbf{X}) = \bar{X}$  為  $\theta$  之一不偏估計量, 且為  $T(\mathbf{X})$  的函數。So,  $h_1(T(\mathbf{X})) = \bar{X}$  is an UMVUE of  $\theta$  by R-B Thm & L-S Thm.

- Let  $S(\mathbf{X}) = (n/(n-1))\bar{X}(1 - \bar{X})$ , such that  $\mathbb{E}[S(\mathbf{X})] = \theta(1 - \theta)$ .
- Let  $h_2(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X})$  ( $\because S(\mathbf{X})$  is a function of  $T(\mathbf{X})$ ),

$$\mathbb{E}[h_2(T(\mathbf{X}))] = \mathbb{E}[S(\mathbf{X})] = \mathbb{E}[(\frac{n}{n-1})\bar{X}(1 - \bar{X})] = \theta(1 - \theta).$$

Since that  $S(\mathbf{X})$  is unbiased for  $\theta(1 - \theta)$  and is a function of  $T(\mathbf{X})$ . So,  $h_2(T(\mathbf{X})) = (n/(n-1))\bar{X}(1 - \bar{X})$  is an UMVUE of  $\theta(1 - \theta)$  by R-B Thm & L-S Them.

- Let  $S(\mathbf{X}) = T(\mathbf{X})(T(\mathbf{X}) - 1)/n(n - 1)$ , such that  $\mathbb{E}[S(\mathbf{X})] = \theta^2$ .
- Let  $h_3(T(\mathbf{X})) = \mathbb{E}[S(\mathbf{X})|T(\mathbf{X})] = S(\mathbf{X})$ .
- Since  $\mathbb{E}[T^2(\mathbf{X})] - \mathbb{E}[T(\mathbf{X})] = \text{Var}[T(\mathbf{X})] + \mathbb{E}[T(\mathbf{X})]^2 - \mathbb{E}[T(\mathbf{X})] = n\theta(1 - \theta) + n^2\theta^2 - n\theta = n(n - 1)\theta^2$ , then

$$\mathbb{E}[h_3(T(\mathbf{X}))] = \mathbb{E}[S(\mathbf{X})] = \mathbb{E}\left[\frac{T(\mathbf{X})(T(\mathbf{X}) - 1)}{n(n - 1)}\right] = \theta^2.$$

So,  $h_3(T(\mathbf{X})) = \sum_{i=1}^n X_i(\sum_{i=1}^n X_i - 1)/n(n - 1)$  is an UMVUE of  $\theta^2$  by R-B Thm & L-S Thm.