

## TA section 9

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June 3, 2024

## Homework 4: (part I)

## §8.2 #1, #4, #19, #20, #21

1. 設  $X$  有  $\mathcal{N}(\mu, 16)$  分佈。欲檢定  $H_0: \mu = 10$ , vs.  $H_a: \mu = 11$ , 取樣本數  $n = 25$  之一組隨機樣本。試決定型 I 錯誤機率  $\alpha = 0.05$  下的一拒絕域, 並求此時之型 II 錯誤之機率。(解.  $\{\bar{X} > 11.316\}$ , 0.654)

- Consider  $T(\mathbf{X}) = \bar{X}_n$ ,  $\bar{X}_n \sim \mathcal{N}(\mu, \sigma^2/n) = \mathcal{N}(\mu, 0.8^2)$ .
- the rejection region  $C := \{\bar{X}_n > c | \mu = 10\} = \{Z > z_{0.05} | \mu = 10\}$ .
- $\alpha = 0.05 = \mathbb{P}(\text{type I error}) = \mathbb{P}(\text{reject } H_0 | H_0) = \mathbb{P}(\bar{X}_n \in C | \mu = 10) = \mathbb{P}(Z > (c - 10)/0.8) = \mathbb{P}(Z > z_{0.05}) \Rightarrow c^* = 10 + (1.645) \times (0.8) = 11.316$ .  
So,  $C^* = \{\bar{X}_n > 11.316\}$ .
- $\mathbb{P}(\text{type II error}) = \mathbb{P}(\text{not reject } H_0 | H_1) = \mathbb{P}(\bar{X}_n \leq c^* | \mu = 11) = \mathbb{P}(Z \leq (11.316 - 11)/0.8) = 0.654$ .

4. 在第1題中, 設拒絕域為  $\{\bar{X} > c\}$ 。試求  $\bar{X} = 11.40$  時之  $p$ -值。(解.  
0.0401)

- $T(\mathbf{x}) := \bar{X}_0 = 11.40$ . Under  $H_o$ ,  $z_0 = (\bar{X}_0 - 10)/0.8 = 1.75$ .
- $p(\mathbf{x}) = \mathbb{P}(T(\mathbf{X}) > T(\mathbf{x})|H_o) = \mathbb{P}(Z > z_0) = \mathbb{P}(Z > 1.75) = 0.0401$ .

19. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{N}(\mu, \sigma^2)$  分佈所產生之隨機樣本,  $\sigma^2$  為已知。欲檢定  $H_0: \mu = \mu_1$ , vs.  $H_a: \mu = \mu_2$ 。試證只要  $n$  夠大,  $K(\mu_1)$  可任意小, 而  $K(\mu_2)$  可任意大。

- Claim:  $\lim_{n \rightarrow \infty} K(\mu_1) = 0$ , and  $\lim_{n \rightarrow \infty} K(\mu_2) = 1$ .
- By CLT,  $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} \mathcal{N}(0, 1)$ , for some  $\mu$ , as  $n \rightarrow \infty$ .
- For  $H_1: \mu_2 > \mu_1$ :  $C^* = \{\bar{X}_n > c_1\}$ ,  $c_1 > \mu_1$ .  
For  $H_1: \mu_2 < \mu_1$ :  $C^* = \{\bar{X}_n < c_2\}$ ,  $c_2 < \mu_1$ .

$$K(\mu) = \begin{cases} \mathbb{P}(\bar{X}_n > c_1 | \mu), & [\mu_2 > \mu_1] \\ \mathbb{P}(\bar{X}_n < c_2 | \mu), & [\mu_2 < \mu_1] \end{cases}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} K(\mu_1) &= \begin{cases} \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma), & [\mu_2 > \mu_1] \\ \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma < \sqrt{n}(c_2 - \mu_1)/\sigma), & [\mu_2 < \mu_1] \end{cases} \\ &= \begin{cases} 1 - \Phi(\infty) = 0, & [\mu_2 > \mu_1] \\ \Phi(-\infty) = 0, & [\mu_2 < \mu_1] \end{cases} \end{aligned}$$

- If we pre-specified a  $c$  satisfying a fixed  $\alpha = 1 - \Phi(c) = \Phi(-c)$ , then  $\lim_{n \rightarrow \infty} K(\mu_1) = \mathbb{P}(Z > c) = \mathbb{P}(Z < -c) = \alpha$ .
- For  $\mu_2 > \mu_1$ :

$$\begin{aligned}
 \lim_{n \rightarrow \infty} K(\mu_2) &= \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma | H_1) \\
 &= \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_2)/\sigma + \sqrt{n}(\mu_2 - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma) \\
 &= \lim_{n \rightarrow \infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_2)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma - \sqrt{n}(\mu_2 - \mu_1)/\sigma) \\
 &= 1 - \Phi(\sqrt{n}(c_1 - \mu_2)/\sigma) \\
 &= 1 - \Phi(-\infty), \text{ if } \mu_1 < c_1 < \mu_2 \text{ (critical value cannot be too high)} \\
 &= 1.
 \end{aligned}$$

Similarly, for  $\mu_2 < \mu_1$ ,  $\lim_{n \rightarrow \infty} K(\mu_2) = \Phi(\infty) = 1$  if  $\mu_1 > c_2 > \mu_2$  (critical value cannot be too small).

20. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{U}(0, \theta)$  分佈所產生之隨機樣本。欲檢定  $H_0$  :

$\theta \leq \theta_0$ , vs.  $H_a : \theta > \theta_0$ 。取拒絕域為  $\{X_{(n)} \geq c\}$ 。

(i) 試求檢力函數  $K(\theta)$ , 並證明此為  $\theta$  之一增函數;

(ii) 設  $\theta_0 = 1/2$ , 試求  $c$  之值, 使顯著水準為 0.05;

- $f_{X_{(n)}}(t) = nt^{n-1}/\theta^n$
- $K(\theta) = \mathbb{P}(X_{(n)} > c | \theta) = 1 - \theta^{-n} \int_0^c nx^{n-1} dx = 1 - (c/\theta)^n.$
- $dK(\theta)/d\theta = nc^n/\theta^{n+1} > 0$  for all  $0 < \theta_0 < c \leq \theta$ .
- $\lim_{n \rightarrow \infty} K(\theta) = 1$  as  $c < \theta$ .
- $K(\theta = 1/2) = 1 - (2c)^n = 0.05 \Rightarrow c^* = \frac{(0.95)^{1/n}}{2}.$

21. 設  $X_1, \dots, X_n$  為一組由  $\mathcal{E}(\lambda)$  分佈所產生之隨機樣本。令  $\mu = 1/\lambda$ 。欲

檢定  $H_0 : \mu \leq \mu_0$ , vs.  $H_a : \mu > \mu_0$ 。

(i) 試證對  $\forall 0 < \alpha < 1$ , 拒絕域  $\{\bar{X} \geq \mu_0 \chi_{1-\alpha, 2n}^2 / (2n)\}$ , 為一顯著水準  $\alpha$  之檢定;

(ii) 試以  $\chi_{2n}^2$  之分佈函數  $F$ , 表示此檢定之檢力函數。

- Let  $S_n = \sum_{i=1}^n X_i \sim \Gamma(n, 1/\lambda) \Rightarrow 2n\lambda\bar{X} = 2n\bar{X}/\mu \sim \Gamma(n, 2) = \chi^2(2n)$ .
- (i)  $\alpha = \sup_{\mu \leq \mu_0} \mathbb{P}(2n\bar{X}/\mu \geq \chi_{1-\alpha, 2n}^2 | H_0 : \mu \leq \mu_0) \Rightarrow \alpha = \mathbb{P}(\bar{X} \geq \mu_0 \chi_{1-\alpha, 2n}^2 / (2n)), \forall \alpha \in (0, 1)$ .
- (ii) Given the CDF of  $\chi^2(2n)$ ,  $F(\cdot)$ , then  

$$K(\mu) = \mathbb{P}(\bar{X} \geq \mu_0 \chi_{1-\alpha, 2n}^2 / (2n) | H_1 : \mu > \mu_0) = \mathbb{P}(2n\bar{X}/\mu \geq (\mu_0/\mu) \chi_{1-\alpha, 2n}^2) = 1 - F((\mu_0/\mu) \chi_{1-\alpha, 2n}^2) \quad (\because 2n\bar{X}/\mu \sim \chi^2(2n)).$$



## §8.3 #5, #9

5. 設  $X_1, \dots, X_{18}$  為一組由  $\mathcal{N}(\mu, \sigma^2)$  分佈所產生之隨機樣本。欲檢  
定  $H_0 : \sigma^2 = 0.36$ , vs.  $H_a : \sigma^2 > 0.36$ 。假設樣本變異數  $S^2 =$   
 $0.68$ 。試在  $\alpha = 0.05$  之下，作一檢定。(解.  $K \doteq 32.1$ ,  $\chi_{0.05,17}^2 \doteq$   
 $27.59$ , 拒絕  $H_0$ )

- Let a test statistic  $K(\mathbf{X}) := (n-1)S^2/\sigma^2 \sim \chi_{n-1}^2 = \chi_{17}^2$ .
- Under  $H_o$ , a test function

$$\phi(\mathbf{X}) = I(K(\mathbf{X}) > \chi_{0.05,17}^2),$$

such that  $0.05 = \sup_{\sigma^2=0.36} \mathbb{P}(K(\mathbf{X}) > \chi_{0.05,17}^2 | H_o : \sigma^2 = 0.36)$ .

- $K = (17) \times (0.68)/(0.36) \approx 32.1 > \chi_{0.05,17}^2 = 27.59 \Rightarrow \text{reject } H_o$ .

9. 設  $X_1, \dots, X_9$  為一組由  $\mathcal{N}(\mu_1, \sigma_1^2)$  分佈所產生之隨機樣本,  $Y_1, \dots, Y_9$  為一組由  $\mathcal{N}(\mu_2, \sigma_2^2)$  分佈所產生之隨機樣本。又設觀測到  $\bar{X} = 16$ ,  $\bar{Y} = 10$ ,  $S_1^2 = 36$ ,  $S_2^2 = 45$ 。

(i) 設  $\sigma_1^2 = \sigma_2^2$ , 試給  $-\alpha = 0.10$  下之  $H_0 : \mu_1 = \mu_2$ , vs.  $H_a : \mu_1 \neq \mu_2$  之檢定;

(ii) 試給  $-\alpha = 0.05$  下之  $H_0 : \sigma_2^2/\sigma_1^2 \leq 1$ , vs.  $H_a : \sigma_2^2/\sigma_1^2 > 1$  之檢定。

- (i)  $\bar{Y} - \bar{X} \sim \mathcal{N}(\mu_2 - \mu_1, \sigma_1^2/n_1 + \sigma_2^2/n_2)$ .
- $Z = ((\bar{Y} - \bar{X}) - (\mu_2 - \mu_1))/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} \sim \mathcal{N}(0, 1)$ , and  $S_p^2 := (n_1 - 1)S_1^2/\sigma_1^2 + (n_2 - 1)S_2^2/\sigma_2^2 \sim \chi_{n_1+n_2-2}^2 \Rightarrow$  a test statistic

$$T := \frac{(\bar{Y} - \bar{X}) - (\mu_2 - \mu_1)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} = t_{16}.$$

- Under  $H_o$ , a test function

$$\phi = \mathbf{I}(|T(0)| > t_{0.95,16}),$$

such that  $0.1 = \sup_{\mu_2=\mu_1} \mathbb{P}(|T(0)| > t_{16} | H_o : \mu_2 - \mu_1 = 0)$ , where

$$T(0) := \frac{(\bar{Y} - \bar{X})}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

- $|T(0)| = |-6/3| = 2 > t_{16} = 1.746 \Rightarrow$  reject  $H_o$ .

- (ii) a test statistic

$$F := \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \sim F_{(n_2-1, n_1-1)} = F_{(8,8)}.$$

- Under  $H_o$ , a test function

$$\phi = \mathbf{I}(F(0) > F_{0.95,(8,8)}),$$

such that  $0.05 = \sup_{\sigma_2^2/\sigma_1^2 \leq 1} \mathbb{P}(F(0) > F_{0.95,(8,8)} | H_o : \sigma_2^2/\sigma_1^2 \leq 1)$ , where  $F(0) := S_2^2/S_1^2$ .

- $F(0) = 45/36 = 1.25 < F_{0.95,(8,8)} = 3.44 \Rightarrow$  do not reject  $H_o$ .