

Due: 10/18, 0:00am

Problem 1

Consider the following linear regression:

$$Y_i = X_i' \beta + e_i,$$

where Y_i is the equity premium of the i th month, and

$$X_i := (1, x_dfy, x_infl, x_svar, x_tms, x_tbl, x_dfr, x_dp, x_ltr, x_ep, x_bmr, x_ntis)',$$

for $i = 1, 2, \dots, 504$, based on the data `Equity_Premium.csv`. Let $\hat{\beta}_j$ be the j th element of the LS estimator $\hat{\beta}$ for β . Please check the null hypothesis:

$$H_o : \beta_j = 0$$

using the t test at the size $\alpha = 5\%$ for each j , based on the asymptotic normality of the LS estimator.

Problem 2

Let $\{(Y_i, X_i')'\}_{i=1}^n$ be an i.i.d. sequence of Gaussian random vectors. Assume that

$$Y_i = X_i' \beta + e_i,$$

where $e_i | \mathbf{X} \sim \mathcal{N}(0, \sigma^2)$ and $\mathbf{X} := (X_1, \dots, X_n)'$, and $\sum_{i=1}^n X_i X_i'$ is positive definite. Denote $\hat{\beta} := (\sum_{i=1}^n X_i X_i')^{-1} (\sum_{i=1}^n X_i Y_i)$, $\hat{e}_i = Y_i - X_i' \hat{\beta}$, and $\hat{\sigma}^2 = \sum_{i=1}^n \hat{e}_i^2 / n$.

1. Please show the asymptotic distribution of $n^{1/2}(\hat{\beta} - \beta)$, as $n \rightarrow \infty$.
2. Please show the asymptotic distribution of $n^{1/2}(\hat{\sigma}^2 - \sigma^2)$, as $n \rightarrow \infty$.
3. Please show the asymptotic distribution of $n^{1/2}(\hat{\sigma} - \sigma)$, as $n \rightarrow \infty$.

Solution:

1.

$$\hat{\beta} - \beta = \left[\sum_{i=1}^n X_i X_i' \right]^{-1} \left[\sum_{i=1}^n X_i e_i \right]$$

$$\sqrt{n}(\hat{\beta} - \beta) = \left[\frac{1}{n} \sum_{i=1}^n X_i X_i' \right]^{-1} \left[\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i e_i \right] \xrightarrow{d} \mathcal{N}(0, \sigma^2 \mathbb{E}[X_i X_i']^{-1}),$$

as $n \rightarrow \infty$.

2.

$$n^{1/2}(\hat{\sigma}^2 - \sigma^2) = n^{-1/2} \sum_{i=1}^n (\hat{e}_i^2 - \sigma^2) = n^{-1/2} \sum_{i=1}^n (e_i^2 - \sigma^2) + n^{-1/2} \sum_{i=1}^n (\hat{e}_i^2 - e_i^2)$$

and

$$\frac{1}{n} \sum_{i=1}^n (\hat{e}_i^2 - e_i^2) = -\frac{2}{n} \sum_{i=1}^n X_i e_i (\hat{\beta} - \beta) + (\hat{\beta} - \beta)' \left[\frac{1}{n} \sum_{i=1}^n X_i X_i' \right] (\hat{\beta} - \beta)$$

and hence

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{e}_i^2 - e_i^2) &= -2 \left[\frac{1}{n} \sum_{i=1}^n X_i e_i \right] \sqrt{n}(\hat{\beta} - \beta) + (\hat{\beta} - \beta)' \left[\frac{1}{n} \sum_{i=1}^n X_i X_i' \right] \sqrt{n}(\hat{\beta} - \beta) \\ &= o_p(1)O_p(1) + o_p(1)O_p(1)O_p(1) = o_p(1). \end{aligned}$$

Therefore,

$$\begin{aligned} n^{-1/2} \sum_{i=1}^n (\hat{e}_i^2 - \sigma^2) &= n^{-1/2} \sum_{i=1}^n (e_i^2 - \sigma^2) + o_p(1) \\ &\xrightarrow{d} \mathcal{N}(0, \mathbb{E}[e_1^4] - \sigma^4) \stackrel{d}{=} \mathcal{N}(0, 2\sigma^4), \text{ as } n \rightarrow \infty, \end{aligned}$$

provided by $\mathbb{E}[e_1^4] = 3\sigma^4$.

3. Let $g(t) := t^{1/2}$, such that $dg/dt = (1/2)t^{-1/2}$ with $t \neq 0$, then

$$\sigma = (\sigma^2)^{1/2}$$

and

$$\frac{d\sigma}{d(\sigma^2)} = \frac{1}{2}(\sigma^2)^{-1/2} = \frac{1}{2\sigma} \neq 0,$$

for $\sigma > 0$. By the 1st-order delta method,

$$\sqrt{n}(\hat{\sigma} - \sigma) = \frac{1}{2\sigma} \sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4\sigma^2}(\mathbb{E}[e_1^4] - \sigma^4)\right) \stackrel{d}{=} \mathcal{N}\left(0, \frac{\sigma^2}{2}\right),$$

as $n \rightarrow \infty$.

Remark. Recall the delta-method theorem:

(a) (1st-order δ -method) If $g \in C^1(\mathbb{R})$ and $a_n(X_n - \mu) \xrightarrow{d} Z$, and let $G(\mu) := \nabla g(\mu) \neq 0$, then

$$a_n(g(X_n) - g(\mu)) \xrightarrow{d} G(\mu)^\top Z.$$

In particular, if $Z \sim \mathcal{N}(0, V)$ and $\sqrt{n}(X_n - \mu) \xrightarrow{d} \mathcal{N}(0, V)$, then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \mathcal{N}(0, G(\mu)^\top V G(\mu)).$$

(b) (2nd-order δ -method) If $G(\mu) = 0$, but $\nabla G(\mu) \neq 0$, then

$$a_n(g(X_n) - g(\mu)) = a_n \frac{\nabla G(\mu)}{2} (X_n - \mu)^2 + o_p(1).$$

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Problem 3

Following #2, let $\hat{\beta}_{ML}$ and $\hat{\sigma}_{ML}^2$ be the Gaussian MLEs of β and σ^2 , respectively.

1. Please derive $\hat{\beta}_{ML}$ and $\hat{\sigma}_{ML}^2$.
2. Please show the asymptotic distribution of

$$\begin{bmatrix} n^{1/2}(\hat{\beta}_{ML} - \beta) \\ n^{1/2}(\hat{\sigma}_{ML}^2 - \sigma^2) \end{bmatrix},$$

as $n \rightarrow \infty$.

3. Please verify the information matrix equality: $B + H = 0$.

Solution:

1.

$$(\hat{\beta}'_{ML}, \hat{\sigma}_{ML}^2)' = \arg \max_{\beta \in \mathbb{R}^k, \sigma^2 > 0} \log L(\beta, \sigma^2).$$

The Gaussian log-likelihood function is

$$\log L(\theta) \propto -\frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} (\mathbf{Y} - \mathbf{X}\beta)'(\mathbf{Y} - \mathbf{X}\beta),$$

where $\theta := (\beta', \sigma^2)'$. Denote $\hat{\theta} := (\hat{\beta}'_{ML}, \hat{\sigma}_{ML}^2)'$. Via the first order conditions (F.O.C.)

evaluated at the $\widehat{\boldsymbol{\theta}}$, we have the estimating functions for $\widehat{\boldsymbol{\theta}}$ are

$$\mathbf{s}(\widehat{\boldsymbol{\theta}}) := \begin{pmatrix} \nabla_{\boldsymbol{\beta}} \log L(\widehat{\boldsymbol{\theta}}) \\ \nabla_{\sigma^2} \log L(\widehat{\boldsymbol{\theta}}) \end{pmatrix} = \sum_{i=1}^n \begin{pmatrix} \frac{X_i \hat{e}_i}{\widehat{\sigma}_{ML}^2} \\ -\frac{1}{2\widehat{\sigma}_{ML}^2} + \frac{\hat{e}_i^2}{2\widehat{\sigma}_{ML}^4} \end{pmatrix} \triangleq 0,$$

where $\hat{e}_i = (Y_i - X_i' \widehat{\boldsymbol{\beta}}_{ML})$. Also, the second-order conditions (S.O.C.) evaluated at the $\widehat{\boldsymbol{\theta}}$ are held, $\nabla^2 \log L(\widehat{\boldsymbol{\theta}}) < 0$. Then, we can solve for

$$\widehat{\boldsymbol{\beta}}_{ML} = [\sum_{i=1}^n X_i X_i']^{-1} \sum_{i=1}^n X_i Y_i = \widehat{\boldsymbol{\beta}}_{LS}$$

and

$$\widehat{\sigma}_{ML}^2 = \frac{1}{n} \sum_{i=1}^n \hat{e}_i^2 = \widehat{\sigma}_{LS}^2.$$

2. We first denote the correct specification of the log-likelihood function at $\boldsymbol{\theta}_0$ by,

$$\log L(\boldsymbol{\theta}_0) = \sum_{i=1}^n \log f_i(\boldsymbol{\theta}_0),$$

where $\boldsymbol{\theta}_0 := (\boldsymbol{\beta}', \sigma^2)'$. Let $s_i(\boldsymbol{\theta}) := \nabla \log f_i(\boldsymbol{\theta})$, such that $\mathbf{s}(\boldsymbol{\theta}) := \nabla \log L(\boldsymbol{\theta}) = \sum_{i=1}^n s_i(\boldsymbol{\theta})$. When the true value $\boldsymbol{\theta}_0$ is in the interior of a compact parameter space Θ , the mean-value expansion of $\mathbf{s}(\widehat{\boldsymbol{\theta}})$ about $\boldsymbol{\theta}_0$ gives that

$$\mathbf{s}(\widehat{\boldsymbol{\theta}}) = \mathbf{s}(\boldsymbol{\theta}_0) + \nabla \mathbf{s}(\boldsymbol{\theta}^+)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0),$$

for some $\boldsymbol{\theta}^+ \in (\boldsymbol{\theta}_0, \widehat{\boldsymbol{\theta}})$. Note that requiring $\boldsymbol{\theta}_0$ in the interior of Θ is to ensure that the mean-value expansion is valid in the parameter space, and the left hand side of the equality is zero since $\widehat{\boldsymbol{\theta}}$ satisfies the first-order conditions under maximization problem. Given $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = o_p(1)$ (consistency), we have that $0 < |\boldsymbol{\theta}^+ - \boldsymbol{\theta}_0| < |\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0|$, such that $\boldsymbol{\theta}^+ - \boldsymbol{\theta}_0 = o_p(1)$, and by uniformly weak law of large numbers,

$$\sup_{\boldsymbol{\theta} \in \Theta} |n^{-1} \sum_{i=1}^n \nabla s_i(\boldsymbol{\theta}) - \mathbb{E}[\nabla s_i(\boldsymbol{\theta})]| = o_p(1),$$

then $\nabla \mathbf{s}(\boldsymbol{\theta}^+)/n \xrightarrow{p} \mathbb{E}[\nabla s_i(\boldsymbol{\theta}_0)] =: H_1$. In addition, assume that $\nabla \mathbf{s}(\boldsymbol{\theta}^+)$ is invertible a.s., and

$$\mathbf{s}(\boldsymbol{\theta}_0)/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, B_1), \text{ as } n \rightarrow \infty,$$

where $B_1 := \text{Var}[\mathbf{s}(\boldsymbol{\theta}_0)/\sqrt{n}] = \mathbb{E}[s_i(\boldsymbol{\theta}_0)s_i(\boldsymbol{\theta}_0)']$. Then,

$$\begin{aligned} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= -[\nabla \mathbf{s}(\boldsymbol{\theta}^+)/n]^{-1}[\mathbf{s}(\boldsymbol{\theta}_0)/\sqrt{n}] \\ &= -H_1^{-1}[\mathbf{s}(\boldsymbol{\theta}_0)/\sqrt{n}] + o_p(1) \\ &= -H_1^{-1}B_1^{1/2}[B_1^{-1/2}\mathbf{s}(\boldsymbol{\theta}_0)/\sqrt{n}] + o_p(1). \end{aligned}$$

Thus,

$$H_1^{1/2} B_1^{-1/2} H_1^{1/2} \sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(0, \mathbf{I}_k), \text{ as } n \rightarrow \infty.$$

Now,

$$s_i(\boldsymbol{\theta}_0) = \nabla \log f_i(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \begin{pmatrix} X_i'(Y_i - X_i'\beta)/\sigma^2 \\ -1/2\sigma^2 + (Y_i - X_i'\beta)^2/2\sigma^4 \end{pmatrix}.$$

Since that $\{Y_i, X_i'\}$ are i.i.d., and the model is correctly specified,

$$\begin{aligned} H_1 &:= \mathbb{E}[\nabla s_i(\boldsymbol{\theta}_0)] = \begin{pmatrix} -\mathbb{E}[X_i X_i']/\sigma^2 & -\mathbb{E}[X_i e_i]/\sigma^4 \\ -\mathbb{E}[e_i X_i']/\sigma^4 & -1/2\sigma^4 \end{pmatrix} = \begin{pmatrix} -\mathbb{E}[X_i X_i']/\sigma^2 & 0 \\ 0 & -1/2\sigma^4 \end{pmatrix} \\ &= -\mathbb{E}[s_i(\boldsymbol{\theta}_0) s_i(\boldsymbol{\theta}_0)'] =: -B_1. \end{aligned}$$

We thus have that

$$\begin{aligned} \sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= \begin{pmatrix} n^{1/2}(\hat{\beta}_{ML} - \beta) \\ n^{1/2}(\hat{\sigma}_{ML}^2 - \sigma^2) \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, B_1^{-1}) \\ &\stackrel{d}{=} \mathcal{N}\left(0, \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right), \end{aligned}$$

as $n \rightarrow \infty$.

Remark. Alternatively, one can utilize the results in Problem 2.1 and 2.2, since $\hat{\beta}_{ML} = \hat{\beta}_{LS}$ and $\hat{\sigma}_{ML}^2 = \hat{\sigma}_{LS}^2$. Also, the off-diagonal elements of variance-covariance matrix of $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$ are zeros from the fact that $\mathbb{E}[(\hat{\beta} - \beta)\hat{e}'|\mathbf{X}] = \mathbb{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'ee'\mathbf{M}|\mathbf{X}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{M} = 0$, in which $\hat{e} = \mathbf{M}e$ with $\mathbf{M} = \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

Or, in a direct way,

$$\begin{aligned} \sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= \begin{pmatrix} (\mathbf{X}'\mathbf{X}/n)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X}'e/\sqrt{n} \\ \hat{e}'\hat{e}/\sqrt{n} - \sqrt{n}\sigma^2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X}'e/\sqrt{n} \\ e'e/\sqrt{n} - \sqrt{n}\sigma^2 \end{pmatrix} + o_p(1) \\ &\xrightarrow{d} \mathcal{N}\left(0, \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_i X_i']\sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} & 0 \\ 0 & 1 \end{pmatrix}'\right) \\ &\stackrel{d}{=} \mathcal{N}\left(0, \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right), \text{ as } n \rightarrow \infty, \end{aligned}$$

since (do it by yourself)

$$\begin{pmatrix} \mathbf{X}'e/\sqrt{n} \\ e'e/\sqrt{n} - \sqrt{n}\sigma^2 \end{pmatrix} = \sqrt{n} \begin{pmatrix} \mathbf{X}'e/n \\ e'e/n - \sigma^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(0, \begin{pmatrix} \mathbb{E}[X_i X_i']\sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right), \text{ as } n \rightarrow \infty.$$

3. Under the correct model specification, the log-likelihood function at θ_0 is

$$\log L(\theta_0) \propto -\frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} (\mathbf{Y} - \mathbf{X}\beta)'(\mathbf{Y} - \mathbf{X}\beta),$$

where $\theta_0 := (\beta', \sigma^2)'$. Via the first order derivatives evaluated at the true value θ_0 , we have the score functions evaluated at θ_0 as

$$\mathbf{s}(\theta_0) := \begin{pmatrix} \nabla_{\beta} \log L(\theta) \\ \nabla_{\sigma^2} \log L(\theta) \end{pmatrix} \Big|_{\theta=\theta_0} = \sum_{i=1}^n \begin{pmatrix} \frac{X_i e_i}{\sigma^2} \\ -\frac{1}{2\sigma^2} + \frac{e_i^2}{2\sigma^4} \end{pmatrix}, \quad e_i = (Y_i - X_i' \beta), \quad i = 1, \dots, n,$$

so that the conditional Information Matrix under i.i.d. assumption can be given by

$$B := \mathbb{E} \left[\mathbf{s}(\theta_0) \mathbf{s}(\theta_0)' | \mathbf{X} \right] = \sum_{i=1}^n \begin{pmatrix} \frac{\mathbb{E}(e_i^2 | \mathbf{X}) X_i X_i'}{\sigma^4} & -\frac{\mathbb{E}(X_i e_i | \mathbf{X})}{2\sigma^4} + \frac{\mathbb{E}(X_i e_i^3 | \mathbf{X})}{2\sigma^6} \\ -\frac{\mathbb{E}(X_i e_i | \mathbf{X})}{2\sigma^4} + \frac{\mathbb{E}(X_i e_i^3 | \mathbf{X})}{2\sigma^6} & \frac{1}{4\sigma^4} - \frac{\mathbb{E}(e_i^2 | \mathbf{X})}{2\sigma^6} + \frac{\mathbb{E}(e_i^4 | \mathbf{X})}{4\sigma^8} \end{pmatrix}.$$

Since that $\mathbb{E}(e_i | \mathbf{X}) = 0$, $\mathbb{E}(e_i^2 | \mathbf{X}) = \sigma^2$, $\mathbb{E}(e_i^3 | \mathbf{X}) = 0$ and $\mathbb{E}(e_i^4 | \mathbf{X}) = 3\sigma^4$, then the off-diagonal entries are zero and we have that

$$B = \begin{pmatrix} \frac{\mathbf{X}' \mathbf{X}}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}.$$

Also, the corresponding negative of conditional expected Hessian matrix can be obtained by

$$\begin{aligned} -H &:= -\mathbb{E} \left[\nabla \mathbf{s}(\theta_0) | \mathbf{X} \right] = -\sum_{i=1}^n \begin{pmatrix} -\frac{X_i X_i'}{\sigma^2} & -\frac{\mathbb{E}(X_i e_i | \mathbf{X})}{\sigma^4} \\ -\frac{\mathbb{E}(e_i X_i' | \mathbf{X})}{\sigma^4} & \frac{1}{2\sigma^4} - \frac{\mathbb{E}(e_i^2 | \mathbf{X})}{\sigma^6} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\mathbf{X}' \mathbf{X}}{\sigma^2} & 0 \\ 0 & -\frac{n}{2\sigma^4} + \frac{n}{\sigma^4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\mathbf{X}' \mathbf{X}}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix} \\ &= B, \end{aligned}$$

then the desired result follows. ■