TA section 5

JERRY C.

Email: 108354501@nccu.edu.tw

Website: jerryc520.github.io/teach/MS.html

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Homework 3

7. 分別對有下述所給p.d.f.之各隨機變數X及變換，試求Y之p.d.f. f_Y。

(i) $f_X(x) = 42x^5(1-x)$, $0 < x < 1$, $Y = X^3$;

(ii) $f_X(x) = 1$, $0 < x < 1$, $Y = X^5$;

(iii) $f_X(x) = \frac{1}{\sigma}xe^{-x^2/(2\sigma)}$, $x > 0, \sigma > 0$爲一常數, $Y = e^X$;

(iv) $f_X(x) = \frac{1}{3}(\frac{2}{3})^x$, $x = 0, 1, 2, \cdots$, $Y = X/(X+1)$;

- (iii). $\mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \log y)$,

$$f_Y(y) = f_X(\log y)y^{-1} = \frac{\log y}{y\sigma}e^{-(\log y)^2/2\sigma}, \ y > 1.$$

- (iv).

$$
\begin{aligned}
f_Y(y) &= \mathbb{P}(Y = y) = \mathbb{P}(X/(X+1) = y) \\
&= \mathbb{P}(X = y/(1-y)) \\
&= f_X(y/(1-y)) = (1/3)(2/3)^{y/(1-y)},\ y = 0, 1/2, 2/3, \cdots .
\end{aligned}
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10. 設隨機變數X有$\mathcal{U}(0,1)$分佈。試求$\log X$之期望值、二次動差, 及變異數。

- Let $Y = \log X$, by the integration by parts,
$\mathbb{E}\, Y = \int_0^1 \log x f_X(x)dx = (x \log x - x)|_{x=0}^{x=1} = -1.$
- By the integration by parts, again,
$\mathbb{E}\, Y^2 = \int_0^1 (\log x)^2 f_X(x)dx = (x(\log x)^2 - 2x \log x + 2x)|_{x=0}^{x=1} = 2.$
- $\mathrm{Var}[Y] = \mathbb{E}\, Y^2 - (\mathbb{E}\, Y)^2 = 1.$
- 另解:
$\mathbb{P}(Y \le y) = \mathbb{P}(X \le \exp(y)) = e^y$, for $y \le 0$ (不是指數分佈)
$\mathbb{E}\, Y = \int_{-\infty}^0 y e^y dy = (ye^y - e^y)|_{y=0} - \lim_{b\to\infty}(ye^y - e^y)|_{y=-b} = -1.$
- 但是, 可令 $Z := -Y > 0$, $f_Z(z) = e^{-z}$ for $z > 0$, Z 是指數分佈,
$\mathbb{E}[Z] = 1 = -\mathbb{E}[Y]$, 故 $\mathbb{E}[Y] = -1$ 且 $\mathrm{Var}[Z] = \mathrm{Var}[Y] = 1$.

23. 設隨機變數X之p.d.f. f爲一偶函數。試證

(i) $E(X)$若存在必爲0;

(ii) $M_X(t)$若存在，則滿足$M_X(t) = M_X(-t)$。

- (i). $\mathbb{E}[X] = \int_{-\infty}^{\infty} s f_X(s) ds = \int_0^{\infty} s f_X(s) ds + \int_{-\infty}^{0} s f_X(s) ds = \int_0^{\infty} s f_X(s) ds + \int_{-\infty}^{0} s f_X(-s) ds = \int_0^{\infty} s f_X(s) ds - \int_0^{\infty} t f_X(t) dt = 0.$
(let $t = -s$, $\int_{-\infty}^{0} s f_X(-s) ds = \int_{\infty}^{0} t f_X(t) dt = -\int_0^{\infty} t f_X(t) dt$)
- $M_X(t) = \mathbb{E}[e^{tX}] = (\int_0^{\infty} + \int_{-\infty}^{0}) e^{tx} f_X(x) dx = (\int_0^{\infty} + \int_{-\infty}^{0}) e^{tx} f_X(-x) dx = (\int_0^{-\infty} + \int_{\infty}^{0}) - e^{-ty} f_X(y) dy = (\int_0^{\infty} + \int_{-\infty}^{0}) e^{-ty} f_X(y) dy = M_X(-t).$
(let $y = -x$)

8. 設X有$\mathcal{B}(n,p)$分佈。試證$\mathrm{Var}(X) \le n/4$, 且等號成立若且唯若$p = 1/2$。

- $X \sim B(n,p)$, $\mathrm{Var}[X] = np(1-p)$, $n \ge 0$.
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$$\begin{aligned} \mathrm{Var}[X] = np(1-p) &= -n(p-1/2)^2 + n/4 \le n/4, \\ &= n/4, \text{iff } p = 1/2. \end{aligned}$$

12. 投擲一公正的骰子, 直到出現一大於4的點數才停止, 令X表總共之投擲數。試求$P(X \geq 3)$, $E(X)$及$\mathrm{Var}(X)$。

- 即 X 是第一次試驗成功所需的次數, 成功事件: 出現 5 或 6 的點數;
- 成功機率: $p = 2/6 = 1/3$, $X \sim \mathrm{Geo}(p)$,

$$f_X(x) = p(1-p)^{x-1},\ x = 1, 2, 3, \cdots$$

- $\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 1 - 1/3 - 2/9 = 4/9.$
- $\mathbb{E}\, X = 1/p = 3.$
- $\mathrm{Var}[X] = (1-p)/p^2 = 6.$