TA section 9

JERRY C.

Email: 108354501@nccu.edu.tw

Website: jerryc520.github.io/teach/MS.html

December 10, 2024

Homework 5:
§3.1-#2, #10, §3.2-#4, #11, #20, #23; §3.3-#7, #12; §3.4-#3;
§3.5-#12, #13

§3.1-#2

2. 設(X,Y)之聯合p.d.f.為$f(x,y)=6xy^2, 0<x,y<1$。

   (i) 試驗證f為一p.d.f.;

   (ii) 試求$P(X+Y\geq 1)$;

   (iii) 試求$P(1/2<X<3/4)$。

- (i). check pdf: (a): $6xy^2 \geq 0, \forall x,y \in (0,1)$; (b) $\int_0^1 \int_0^1 6xy^2 dxdy = 1$.
- (ii). $\mathbb{P}(X+Y \geq 1) = \int\int_{\{x+y\geq 1; 0<x,y<1\}} f(x,y)dxdy$

$$= \int_0^1 \int_{1-y}^1 6xy^2 dxdy = \int_0^1 (3x^2y^2)\Big]_{x=1-y}^{x=1} dy = \int_0^1 (6y^3 - 3y^4)dy = 9/10.$$

(Or, $= \int_0^1 \int_{1-x}^1 6xy^2 dydx$)

- (iii).

$$
\begin{aligned}
\mathbb{P}(1/2 < X < 3/4) &= \mathbb{P}(1/2 < X < 3/4, 0 < Y < 1) \\
&= \int_0^1 \int_{1/2}^{3/4} 6xy^2 dx dy \\
&= \int_0^1 3y^2 (x^2\Big]_{x=1/2}^{x=3/4}) dy \\
&= (5/16) \int_0^1 3y^2 dy = 5/16.
\end{aligned}
$$

§3.1 #10

10. 設(X,Y)之聯合p.d.f.爲$f(x,y)=\lambda^2 e^{-\lambda(x+y)}, x,y\geq 0$。試求$P(X\geq 2Y)$。

- $\mathbb{P}(X \geq 2Y) = \mathbb{P}(X > 0, Y \leq X/2) =$

$$
\begin{aligned}
\int_0^\infty \int_0^{x/2} \lambda^2 e^{-\lambda(x+y)}\,dydx &= \int_0^\infty \lambda e^{-\lambda x}\Big(\int_0^{x/2} \lambda e^{-\lambda y}dy\Big)dx \\
&= \int_0^\infty \lambda e^{-\lambda x}\mathbb{P}(Y < x/2)dx,\ Y \sim \epsilon(\lambda), \\
&\int_0^\infty \lambda e^{-\lambda x}\Big(1 - e^{-\lambda x/2}\Big)dx \\
&\int_0^\infty \lambda e^{-\lambda x}dx - \lambda \int_0^\infty e^{-3\lambda x/2}dx \\
&= 1 - 2/3 = 1/3.
\end{aligned}
$$

- or,

$$\int_0^\infty \int_{2y}^\infty \lambda^2 e^{-\lambda(x+y)} dxdy = \lambda^2 \int_0^\infty (-\lambda^{-1} e^{-\lambda(x+y)})\Big]_{2y}^\infty dy$$
$$= \lambda \int_0^\infty e^{-3\lambda y} dy = 1/3.$$

- NOTE: useful tool by the definition of pdf:

$$\int f(z)dz = 1,$$

e.g., $\int_0^\infty (3\lambda/2)e^{-3\lambda x/2}dx = 1 \Rightarrow \int_0^\infty e^{-3\lambda x/2}dx = 2/(3\lambda)$.

§3.2-#4

4. 設(X, Y)之聯合p.d.f.為

$$f(x, y) = \frac{2+x+y}{8}, -1 < x, y < 1 。$$

試分別求出X, Y之邊際p.d.f., 並因此驗證X與Y不獨立。

- $f_X(x) = (1/8)\left[\int_{-1}^{1}(2+x)dy + \int_{-1}^{1} y dy\right] = (1/8)[4+2x] = (2+x)/4, \; -1 < x < 1.$
- $f_Y(y) = (2+y)/4, \; -1 < y < 1.$
- $f(x, y) \neq f_X(x) f_Y(y)$. So, $X \not\perp Y$.

11. 設X, Y之聯合p.d.f.為$f(x,y) = 2e^{-x-y}, 0 \le x < y < \infty$。

(i) 試求$P(Y > 2X)$;

(ii) 試證X之邊際分佈為$\mathcal{E}(1/2)$;

(iii) 試求$f_{Y|X}(y|x)$及$E(Y|X = x)$;

(iv) 試檢驗X與Y是否獨立。

- $\mathbb{P}(Y > 2X) = \int_0^\infty \int_0^{y/2} 2e^{-x}e^{-y}dxdy = \int_0^\infty 2e^{-y} \int_0^{y/2} e^{-x}dxdy = 2/3.$
- $f_X(x) = \int_x^\infty 2e^{-x-y}dy = 2e^{-x} \int_x^\infty e^{-y}dy = 2e^{-2x}.$
- $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = e^{x-y}.$$

$$
\begin{aligned}
\mathbb{E}[Y|X=x] &= \int_x^\infty y f_{Y|X}(y|x)dy \\
&= \int_x^\infty y e^{x-y} dy \\
&= e^x \int_x^\infty y e^{-y} dy \\
&= e^x(xe^{-x} + e^{-x}) = x + 1.
\end{aligned}
$$

- $f_Y(y) = \int_0^y 2e^{-x-y}dx = 2e^{-y}\int_0^y e^{-x}dx = 2e^{-y}(1-e^{-y})$, $y \geq 0$.
- $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, i.e., $X \not\perp\!\!\!\perp Y$.

§3.2-#20

20. 設(X,Y)之聯合p.d.f.爲$f(x,y) = 2e^{-x-y}$, $0 < x < y < \infty$。試求(X,Y)之聯合動差母函數。(**解.** $M(s,t) = 2((1-t)(2-s-t))^{-1}$, $t < 1, s+t < 2$)

- $M(s,t) = \mathbb{E}[e^{sX+tY}] = 2\int_0^\infty \int_0^y e^{(s-1)x+(t-1)y} dxdy = 2\int_0^\infty e^{(t-1)y} \int_0^y e^{(s-1)x} dxdy = 2\int_0^\infty e^{(t-1)y}\left[\frac{e^{(s-1)y}-1}{s-1}\right]dy = \frac{2}{s-1}\left[\int_0^\infty e^{(s+t-2)y}dy - \int_0^\infty e^{(t-1)y}dy\right] = \frac{2}{s-1}\left[\frac{1}{2-s-t} - \frac{1}{1-t}\right] = \frac{2}{s-1}\left[\frac{s-1}{(1-t)(2-s-t)}\right]$.
- $$M(s,t) = \frac{2}{(1-t)(2-s-t)},\ t < 1, s+t < 2.$$

23. 設X與Y獨立，令$S = X + Y$。又知$X \sim \chi^2_m$，$S \sim \chi^2_{m+n}$。試問Y是否有卡方分佈，若是則給出其自由度。(**解.**是，$Y \sim \chi^2_n$)

- By uniqueness of MGF, 給定 $M_X(t) = (1-2t)^{-m/2}$ and

  $M_S(t) = (1-2t)^{-(m+n)/2} = (1-2t)^{-m/2}(1-2t)^{-n/2} = M_X(t)(1-2t)^{-n/2}$.
- $\because$ $X \perp\!\!\!\perp Y$,

  $$M_S(t) = M_{X+Y}(t) = M_X(t)M_Y(t),$$

  so that $M_Y(t) = (1-2t)^{-n/2}$ holds. That is, $Y \sim \chi^2(n)$.

Note: $\#22$: $U := X - Y$ does not have χ^2 distribution.

反例:

- 令 $Z_i, W_i \sim i.i.d.\ N(0,1)$, 且 $A_i := Z_i^2 \sim \chi^2(1), B_i := W_i^2 \sim \chi^2(1)$.
  則, $X = \sum_{i=1}^m A_i \sim \chi^2(m)$ 與 $Y = \sum_{i=1}^n B_i \sim \chi^2(n)$.
- For $m > n$, let $C_i := A_i - B_i$,
  $U := X - Y = \sum_{i=1}^m A_i - \sum_{i=1}^n B_i = \sum_{i=1}^n A_i + \sum_{n+1}^m A_i - \sum_{i=1}^n B_i = \sum_{i=1}^n C_i + \sum_{i=n+1}^m A_i \sim \square + \chi^2(m-n) \neq \chi^2(m-n)$.
- 雖然 A_i, B_i 獨立且相同分佈, 但是不一定 $A_i - B_i = 0$, 即, $\mathbb{P}(C_i = 0) = 0$ (連續隨機變數的單點機率是零).

7. 設X與Y獨立, 且分別有$\mathcal{E}(\lambda)$及$\mathcal{E}(\mu)$分佈。令

$$Z = \min\{X, Y\}, \quad W = \begin{cases} 1 & , Z = X, \\ 0 & , Z = Y \text{。} \end{cases}$$

(i) 試求Z, W之聯合p.d.f.(注意Z爲連續型, W爲離散型);

(ii) 試證$P(Z \le z | W = i) = P(Z \le z), i = 0, 1$, 即$Z$與$W$獨立。

- (ii). $f_{X,Y}(x, y) = \lambda\mu \exp(-\lambda x - \mu y)$.
- $\mathbb{P}(Z > z) = \mathbb{P}(Z > z, W = 0) + \mathbb{P}(Z > z, W = 1) = \mathbb{P}(X > z, Y > z, Y < X) + \mathbb{P}(X > z, Y > z, Y \ge X) = \mathbb{P}(X > z, Y > z) = \exp(-(\lambda + \mu)z)$, $z > 0$.
- $\mathbb{P}(Z > z | W = 1) = \mathbb{P}(Z > z, W = 1) / \mathbb{P}(W = 1) = \mathbb{P}(X > z, Y > z, Y > X) / \mathbb{P}(Y > X)$.

- $\mathbb{P}(X > z, Y > z, Y > X) = \int_z^\infty \int_z^y \lambda\mu e^{-\lambda x} e^{-\mu y} dx dy = \int_z^\infty \mu e^{-\mu y}(-e^{-\lambda x})|_z^y dy = \int_z^\infty \mu e^{-\mu y}(e^{-\lambda z} - e^{-\lambda y}) dy = \frac{\lambda}{\lambda+\mu} \exp(-(\lambda+\mu)z)$.
- Also, $\mathbb{P}(Y > X) = \int_0^\infty \mathbb{P}(X > y) f_Y(y) dy = \int_0^\infty (1 - e^{-\lambda y}) \mu e^{-\mu y} dy = 1 - \frac{\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$.
  Thus, $\mathbb{P}(Z > z | W = 1) = \exp(-(\lambda+\mu)z) = \mathbb{P}(Z > z)$. Similarly, $\mathbb{P}(Z > z | W = 0) = \mathbb{P}(Z > z)$. As such,

  $$\mathbb{P}(Z \leq z | W = 1) = \mathbb{P}(Z \leq z),$$

  $$\mathbb{P}(Z \leq z | W = 0) = \mathbb{P}(Z \leq z);$$

  i.e., $W \perp\!\!\!\perp Z$.

- (i). $f_{Z,W}(z,w) = f_Z(z)\,\mathbb{P}(W = w)$, $w \in \{0, 1\}$.
- $F_Z(z) = \mathbb{P}(Z \le z) = 1 - \mathbb{P}(Z > z) = 1 - \exp(-(\lambda + \mu)z) \Rightarrow f_Z(z) = (\lambda + \mu)\exp(-(\lambda + \mu)z),\ z > 0.$
  $\mathbb{P}(W = 1) = \mathbb{P}(Y > X) = \frac{\lambda}{\mu+\lambda}$ and $\mathbb{P}(W = 0) = \mathbb{P}(Y \le X) = \frac{\mu}{\mu+\lambda}$.
-
$$f_{Z,W}(z,w) = \begin{cases} \lambda\exp(-(\lambda + \mu)z), & w = 1 \\ \mu\exp(-(\lambda + \mu)z), & w = 0, \end{cases}$$

  for $z > 0$.

12. 設X與Y獨立, 且皆有$\mathcal{U}(\alpha, \alpha+1)$分佈, 其中$\alpha \in R$爲一常數。令$U = X+Y$, $V = X-Y$。

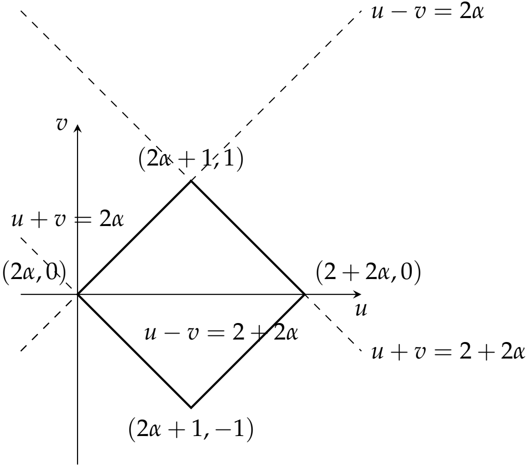
(i) 試求U與V之聯合p.d.f.;

(ii) 試檢驗U與V是否獨立。

- (i). $f_{XY}(x,y) = f_X(x)f_Y(y) = 1,\ \alpha < x < \alpha+1, \alpha < y < \alpha+1.$
- Let $U = X+Y, V = X-Y \Rightarrow X = (U+V)/2, Y = (U-V)/2.$
- $J = |\frac{\partial(X,Y)}{\partial(U,V)}| = 1/2.$
- $$f_{U,V}(u,v) = f_{X,Y}(x = \frac{u+v}{2}, y = \frac{u-v}{2})|J| = 1/2,$$
  $2\alpha < u+v < 2\alpha+2, 2\alpha < u-v < 2\alpha+2.$

- (ii). 範圍請參考下頁圖示。
  - fixed u, $\max(2\alpha - u, u - 2\alpha - 2) < v < \min(2\alpha + 2 - u, u - 2\alpha)$;
  (for $u \in [2\alpha, 2\alpha + 1]$, $2\alpha - u < v < u - 2\alpha$;
  for $u \in (2\alpha + 1, 2\alpha + 2]$, $u - 2\alpha - 2 < v < 2\alpha + 2 - u$).
  - fixed v, $\max(2\alpha - v, 2\alpha + v) < u < \min(2\alpha + 2 - v, 2\alpha + v + 2)$
  (for $v \in [-1, 0]$, $2\alpha - v < u < 2\alpha + 2 + v$;
  for $v \in (0, 1]$, $2\alpha + v < u < 2\alpha + 2 - v$).
- $f_U(u) = \int f_{UV}(u, v)dv = 1 - |u - 2\alpha - 1|,\ 2\alpha < u < 2\alpha + 2.$
- $f_V(v) = \int f_{UV}(u, v)du = 1 - |v|,\ -1 < v < 1.$
- So, $f_{UV}(u, v) \neq f_U(u)f_V(v),\ U \not\perp V.$

圖示



$u - v = 2\alpha$
v
$(2\alpha + 1, 1)$
$u + v = 2\alpha$
$(2\alpha, 0)$
$(2 + 2\alpha, 0)$
u
$u - v = 2 + 2\alpha$
$u + v = 2 + 2\alpha$
$(2\alpha + 1, -1)$

3. 設$Y|P$有$\mathcal{B}(n,P)$分佈, P有$\mathcal{B}e(\alpha,\beta)$分佈 。

(i) 試證Y之非條件分佈爲

$$P(Y=y)=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n)}, y=0,1,\cdots,n;$$

(ii) 試求$E(Y)$及$\mathrm{Var}(Y)$ 。

- (i). $f_{Y,P}(y,p)=f_{Y|P}(y|p)f_P(p)=\binom{n}{y}p^y(1-p)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$

$$\mathbb{P}(Y=y)=\int_0^1 f_{Y,P}(y,p)dp=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 p^{y+\alpha-1}(1-p)^{n+\beta-y-1}dp,$$

So,

$$\mathbb{P}(Y=y)=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n+\beta-y)}{\Gamma(n+\alpha+\beta)},\ y=0,1,\cdots,n.$$

- (ii).

$$\mathbb{E}\,Y = \mathbb{E}\,\mathbb{E}[Y|P] = \mathbb{E}[nP] = n\,\mathbb{E}[P] = n\alpha/(\alpha+\beta)$$

- $\operatorname{Var}[Y] = \mathbb{E}[\operatorname{Var}[Y|P]] + \operatorname{Var}[\mathbb{E}[Y|P]] = (n\,\mathbb{E}[P] - n\,\mathbb{E}[P^2]) + n^2\operatorname{Var}[P]$

$$= \frac{n^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{n\alpha}{\alpha+\beta} + \frac{n\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$= \frac{n\alpha\beta(n+\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

§3.5-#12

12. 設X與Y獨立, 且令$E(X) = \mu_X, E(Y) = \mu_Y, \mathrm{Var}(X) = \sigma_X^2, \mathrm{Var}(Y)$
$= \sigma_Y^2$。試以$\mu_X, \mu_Y, \sigma_X,$及σ_Y表示$\rho(XY, Y)$。

- $$\rho(XY, Y) = \frac{\mathrm{cov}[XY, Y]}{\sigma_{XY}\sigma_Y}$$

- $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mu_X \mu_Y$ and

$$\mathbb{E}[X^2Y^2] = (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2 + \mu_X^2\mu_Y^2.$$

  So, $\sigma_{XY}^2 = \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2$.
- $\mathrm{cov}[XY, Y] = \mathrm{cov}[\mathbb{E}[XY|Y], Y] = \mathrm{cov}[Y\,\mathbb{E}[X|Y], Y] = \mathrm{cov}[Y\,\mathbb{E}[X], Y] = \mathbb{E}[X]\,\mathrm{Var}[Y] = \mu_X\sigma_Y^2$.
- $$\mathrm{cov}[XY, Y] = \frac{\mu_X\sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}}.$$
- Note: $\mathrm{cov}[XY, Y] = \mathbb{E}[XY^2] - \mathbb{E}[XY]\,\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[XY^2|Y]] - \mathbb{E}[\mathbb{E}[XY|Y]]\,\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[XY|Y]Y] - \mathbb{E}[\mathbb{E}[XY|Y]]\,\mathbb{E}[Y] = \mathrm{cov}[\mathbb{E}[XY|Y], Y]$.

13. 設X, Y, Z爲兩兩無相關之隨機變數, 期望值皆爲μ, 變異數皆爲σ^2。試以μ及σ^2表示$\text{Cov}(X+Y, Y+Z)$及$\text{Cov}(X+Y, X-Y)$。

- $$\text{cov}[X+Y, Y+Z] = \text{cov}[X,Y] + \text{cov}[X,Z] + \text{cov}[Y,Y] + \text{cov}[Y,Z] = \sigma^2.$$
  ($\because$ X, Y, Z are pairwise independent.)
- $\text{cov}[X+Y, X-Y] = \text{cov}[X,X] - \text{cov}[X,Y] + \text{cov}[Y,X] - \text{cov}[Y,Y] = \sigma^2 + 0 + 0 - \sigma^2 = 0$.