TA section 9

JERRY C.

jerryc520.github.io/teach/MS.html

June 1, 2024

Homework 4: (part I)

§8.2 #1, #4, #19, #20, #21

1. 設X有$\mathcal{N}(\mu, 16)$分佈。欲檢定$H_0: \mu = 10$, vs. $H_a: \mu = 11$, 取樣本數$n = 25$之一組隨機樣本。試決定型 I 錯誤機率$\alpha = 0.05$下的一拒絕域, 並求此時之型II錯誤之機率。(**解.** $\{\overline{X} > 11.316\}$, 0.654)

- Consider $T(\boldsymbol{X}) = \bar{X}_n$, $\bar{X}_n \sim \mathcal{N}(\mu, \sigma^2/n) = \mathcal{N}(\mu, 0.8^2)$.
- the rejection region $C := \{\bar{X}_n > c | \mu = 10\} = \{Z > z_{0.05} | \mu = 10\}$.
- $\alpha = 0.05 = \mathbb{P}(\text{type I error}) = \mathbb{P}(\text{reject } H_o | H_o) = \mathbb{P}(\bar{X}_n \in C | \mu = 10) = \mathbb{P}(Z > (c - 10)/0.8) = \mathbb{P}(Z > z_{0.05}) \Rightarrow c^* = 10 + (1.645) \times (0.8) = 11.316$. So, $C^* = \{\bar{X}_n > 11.316\}$.
- $\mathbb{P}(\text{type II error}) = \mathbb{P}(\text{not reject } H_o | H_1) = \mathbb{P}(\bar{X}_n \leq c^* | \mu = 11) = \mathbb{P}(Z \leq (11.316 - 11)/0.8) = 0.654$.

4. 在第1題中，設拒絕域爲$\{\overline{X} > c\}$。試求$\overline{X} = 11.40$時之p-值。(**解.** 0.0401)

- $T(\boldsymbol{x}) := \bar{X}_0 = 11.40$. Under H_o, $z_0 = (\bar{X}_0 - 10)/0.8 = 1.75$.
- $p(\boldsymbol{x}) = \mathbb{P}(T(\boldsymbol{X}) > T(\boldsymbol{x})|H_o) = \mathbb{P}(Z > z_0) = \mathbb{P}(Z > 1.75) = 0.0401$.

19. 設$X_1, \cdots, X_n$為一組由$\mathcal{N}(\mu, \sigma^2)$分佈所產生之隨機樣本，$\sigma^2$為已知。欲檢定$H_0 : \mu = \mu_1$, vs. $H_a : \mu = \mu_2$。試證只要n夠大，$K(\mu_1)$可任意小，而$K(\mu_2)$可任意大。

- Claim: $\lim_{n\to\infty} K(\mu_1) = 0$, and $\lim_{n\to\infty} K(\mu_2) = 1$.
- By CLT, $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{\text{d}} \mathcal{N}(0, 1)$, for some μ, as $n \to \infty$.
- For $H_1 : \mu_2 > \mu_1$: $C^* = \{\bar{X}_n > c_1\}$, $c_1 > \mu_1$.
  For $H_1 : \mu_2 < \mu_1$: $C^* = \{\bar{X}_n < c_2\}$, $c_2 < \mu_1$.
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$$K(\mu) = \begin{cases} \mathbb{P}(\bar{X}_n > c_1|\mu), & [\mu_2 > \mu_1] \\ \mathbb{P}(\bar{X}_n < c_2|\mu), & [\mu_2 < \mu_1] \end{cases}$$
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$$\begin{aligned} \lim_{n\to\infty} K(\mu_1) &= \begin{cases} \lim_{n\to\infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma), & [\mu_2 > \mu_1] \\ \lim_{n\to\infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma < \sqrt{n}(c_2 - \mu_1)/\sigma), & [\mu_2 < \mu_1] \end{cases} \\ &= \begin{cases} 1 - \Phi(\infty) = 0, & [\mu_2 > \mu_1] \\ \Phi(-\infty) = 0, & [\mu_2 < \mu_1] \end{cases} \end{aligned}$$

- If we pre-specified a c satisfying a fixed $\alpha = 1 - \Phi(c) = \Phi(-c)$, then $\lim_{n\to\infty} K(\mu_1) = \mathbb{P}(Z > c) = \mathbb{P}(Z < -c) = \alpha$.
- For $\mu_2 > \mu_1$:

$$\begin{aligned}
\lim_{n\to\infty} K(\mu_2) &= \lim_{n\to\infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma | H_1) \\
&= \lim_{n\to\infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_2)/\sigma + \sqrt{n}(\mu_2 - \mu_1)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma) \\
&= \lim_{n\to\infty} \mathbb{P}(\sqrt{n}(\bar{X}_n - \mu_2)/\sigma > \sqrt{n}(c_1 - \mu_1)/\sigma - \sqrt{n}(\mu_2 - \mu_1)/\sigma) \\
&= 1 - \Phi(\sqrt{n}(c_1 - \mu_2)/\sigma) \\
&= 1 - \Phi(-\infty), \text{if } \mu_1 < c_1 < \mu_2 \text{ (critical value cannot be too high)} \\
&= 1.
\end{aligned}$$

Similarly, for $\mu_2 < \mu_1$, $\lim_{n\to\infty} K(\mu_2) = \Phi(\infty) = 1$ if $\mu_1 > c_2 > \mu_2$ (critical value cannot be too small).

20. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}(0,\theta)$分佈所產生之隨機樣本。欲檢定H_0：$\theta \le \theta_0$, vs. $H_a : \theta > \theta_0$。取拒絕域爲$\{X_{(n)} \ge c\}$。

(i) 試求檢力函數$K(\theta)$, 並證明此爲θ之一增函數;

(ii) 設$\theta_0 = 1/2$, 試求c之值, 使顯著水準爲0.05;

- $f_{X_{(n)}}(t) = nt^{n-1}/\theta^n$
- $K(\theta) = \mathbb{P}(X_{(n)} > c|\theta) = 1 - \theta^{-n} \int_0^c nx^{n-1} dx = 1 - (c/\theta)^n$.
- $dK(\theta)/d\theta = nc^n/\theta^{n+1} > 0$ for all $0 < \theta_0 < c \le \theta$.
- $\lim_{n\to\infty} K(\theta) = 1$ as $c < \theta$.
- $K(\theta = 1/2) = 1 - (2c)^n = 0.05 \Rightarrow c^* = \frac{(0.95)^{1/n}}{2}$.

21. 設$X_1, \cdots, X_n$爲一組由$\mathcal{E}(\lambda)$分佈所產生之隨機樣本。令$\mu = 1/\lambda$。欲檢定$H_0 : \mu \leq \mu_0$, vs. $H_a : \mu > \mu_0$。

(i) 試證對$\forall 0 < \alpha < 1$, 拒絕域$\{\overline{X} \geq \mu_0 \chi^2_{1-\alpha,2n}/(2n)\}$, 爲一顯著水準$\alpha$之檢定;

(ii) 試以χ^2_{2n}之分佈函數F, 表示此檢定之檢力函數。

- Let $S_n = \sum_{i=1}^n X_i \sim \Gamma(n, \lambda) \Rightarrow 2n\lambda\bar{X} = 2n\bar{X}/\mu \sim \Gamma(n, 1/2) = \chi^2(2n)$.
- (i) $\alpha = \sup_{\mu \leq \mu_0} \mathbb{P}(2n\bar{X}/\mu \geq \chi^2_{1-\alpha,2n} | H_o : \mu \leq \mu_0) \Rightarrow \alpha = \mathbb{P}(\bar{X} \geq \mu_0 \chi^2_{1-\alpha,2n}/(2n))$, $\forall \alpha \in (0, 1)$.
- (ii) Given the CDF of $\chi^2(2n)$ is $F(.)$, then $K(\mu) = \mathbb{P}(\bar{X} \geq \mu_0 \chi^2_{1-\alpha,2n}/(2n) | H_1 : \mu > \mu_0) = \mathbb{P}(2n\bar{X}/\mu \geq (\mu_0/\mu)\chi^2_{1-\alpha,2n}) = 1 - F((\mu_0/\mu)\chi^2_{1-\alpha,2n})$ $(\because 2n\bar{X}/\mu \sim \chi^2(2n))$.

§8.3 #5, #9

5. 設$X_1, \cdots, X_{18}$爲一組由$\mathcal{N}(\mu, \sigma^2)$分佈所產生之隨機樣本。欲檢定$H_0 : \sigma^2 = 0.36$, vs. $H_a : \sigma^2 > 0.36$。假設樣本變異數$S^2 = 0.68$。試在$\alpha = 0.05$之下，作一檢定。(**解.** $K \doteq 32.1$, $\chi^2_{0.05,17} \doteq 27.59$, 拒絕$H_0$)

- Let a test statistic $K(\boldsymbol{X}) := (n-1)S^2/\sigma^2 \sim \chi^2_{n-1} = \chi^2_{17}$.
- Under H_o, a test function

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(K(\boldsymbol{X}) > \chi^2_{0.05,17}),$$

  such that $0.05 = \mathbb{P}(K(\boldsymbol{X}) > \chi^2_{0.05,17} | H_o : \sigma^2 = 0.36)$.
- $K = (17) \times (0.68)/(0.36) \approx 32.1 > \chi^2_{0.05,17} = 27.59 \Rightarrow$ reject H_o.

9. 設$X_1, \cdots, X_9$爲一組由$\mathcal{N}(\mu_1, \sigma_1^2)$分佈所產生之隨機樣本, $Y_1, \cdots, Y_9$爲一組由$\mathcal{N}(\mu_2, \sigma_2^2)$分佈所產生之隨機樣本。又設觀測到$\overline{X} = 16$, $\overline{Y} = 10$, $S_1^2 = 36$, $S_2^2 = 45$。

   (i) 設$\sigma_1^2 = \sigma_2^2$, 試給一$\alpha = 0.10$下之$H_0 : \mu_1 = \mu_2$, vs. $H_a : \mu_1 \neq \mu_2$之檢定;

   (ii) 試給一$\alpha = 0.05$下之$H_0 : \sigma_2^2/\sigma_1^2 \leq 1$, vs. $H_a : \sigma_2^2/\sigma_1^2 > 1$之檢定。

- (i) $\bar{Y} - \bar{X} \sim \mathcal{N}(\mu_2 - \mu_1, \sigma_1^2/n_1 + \sigma_2^2/n_2)$.
- $Z = ((\bar{Y} - \bar{X}) - (\mu_2 - \mu_1))/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} \sim \mathcal{N}(0, 1)$, and $S_p^2 := (n_1 - 1)S_1^2/\sigma_1^2 + (n_2 - 1)S_2^2/\sigma_2^2 \sim \chi^2_{n_1+n_2-2} \Rightarrow$ a test statistic

$$T := \frac{(\bar{Y} - \bar{X}) - (\mu_2 - \mu_1)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1+n_2-2)} = t_{16}.$$

- Under H_o, a test function

$$\phi = \boldsymbol{I}(|T(0)| > t_{0.95,16}),$$

  such that $0.1 = \mathbb{P}(|T(0)| > t_{16}|H_o : \mu_2 - \mu_1 = 0)$, where $T(0) := \frac{(\bar{Y} - \bar{X})}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$.
- $|T(0)| = |-6/3| = 2 > t_{16} = 1.746 \Rightarrow$ reject H_o.

- (ii) a test statistic

$$F := \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} \sim F_{(n_2-1,n_1-1)=F_{(8,8)}}.$$

- Under H_o, a test function

$$\phi = \boldsymbol{I}(F(0) > F_{0.95,(8,8)}),$$

such that $0.05 = \mathbb{P}(F(0) > F_{0.95,(8,8)}|H_o : \sigma_2^2/\sigma_1^2 \leq 1)$, where $F(0) := S_2^2/S_1^2$.

- $F(0) = 45/36 = 1.25 < F_{0.95,(8,8)} = 3.44 \Rightarrow$ do not reject H_o.