TA section 9

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Homework 5:
§3.1-#2, #10, §3.2-#4, #11, #20, #23; §3.3-#7, #12; §3.4-#3; §3.5-#12, #13

§3.1-#2

2. 設(X,Y)之聯合p.d.f.爲$f(x,y)=6xy^2, 0<x,y<1$ 。

   (i) 試驗證f爲一p.d.f.;

   (ii) 試求$P(X+Y\geq 1)$;

   (iii) 試求$P(1/2<X<3/4)$ 。

- (i). check pdf: (a): $6xy^2 \geq 0, \forall x, y \in (0,1)$; (b) $\int_0^1 \int_0^1 6xy^2 dxdy = 1$.
- (ii). $\mathbb{P}(X+Y\geq 1) = \int\int_{\{x+y\geq 1; 0<x,y<1\}} f(x,y)dxdy$

$$= \int_0^1 \int_{1-y}^1 6xy^2 dxdy = \int_0^1 (3x^2y^2)\Big]_{x=1-y}^{x=1} dy = \int_0^1 (6y^3 - 3y^4)dy = 9/10.$$

(Or, $= \int_0^1 \int_{1-x}^1 6xy^2 dydx$)

- (iii).

$$\begin{aligned}\mathbb{P}(1/2 < X < 3/4) &= \mathbb{P}(1/2 < X < 3/4, 0 < Y < 1) \\ &= \int_0^1 \int_{1/2}^{3/4} 6xy^2 dxdy \\ &= \int_0^1 3y^2 (x^2\Big]_{x=1/2}^{x=3/4}) dy \\ &= (5/16) \int_0^1 3y^2 dy = 5/16.\end{aligned}$$

10. 設(X,Y)之聯合p.d.f.爲$f(x,y)=\lambda^2 e^{-\lambda(x+y)}, x,y \ge 0$。試求$P(X \ge 2Y)$。

- $\mathbb{P}(X \ge 2Y) = \mathbb{P}(X > 0, Y \le X/2) =$

$$\begin{aligned}
\int_0^\infty \int_0^{x/2} \lambda^2 e^{-\lambda(x+y)} dydx &= \int_0^\infty \lambda e^{-\lambda x}\Big(\int_0^{x/2} \lambda e^{-\lambda y} dy\Big) dx \\
&= \int_0^\infty \lambda e^{-\lambda x} \mathbb{P}(Y < x/2) dx, \;\; Y \sim \epsilon(\lambda), \\
& \int_0^\infty \lambda e^{-\lambda x}\Big(1 - e^{-\lambda x/2}\Big) dx \\
& \int_0^\infty \lambda e^{-\lambda x} dx - \lambda \int_0^\infty e^{-3\lambda x/2} dx \\
&= 1 - 2/3 = 1/3.
\end{aligned}$$

- or,

$$\int_0^\infty \int_{2y}^\infty \lambda^2 e^{-\lambda(x+y)} dxdy = \lambda^2 \int_0^\infty (-\lambda^{-1} e^{-\lambda(x+y)}) \Big]_{2y}^\infty dy$$
$$= \lambda \int_0^\infty e^{-3\lambda y} dy = 1/3.$$

- NOTE: useful tool by the definition of pdf:

$$\int f(z)dz = 1,$$

e.g., $\int_0^\infty (3\lambda/2) e^{-3\lambda x/2} dx = 1 \Rightarrow \int_0^\infty e^{-3\lambda x/2} dx = 2/(3\lambda)$.

§3.2-#4

4. 設(X, Y)之聯合p.d.f.爲

$$f(x, y) = \frac{2+x+y}{8}, -1 < x, y < 1 \text{。}$$

試分別求出X, Y之邊際p.d.f., 並因此驗證X與Y不獨立 。

- $f_X(x) = (1/8)\left[\int_{-1}^{1}(2+x)dy + \int_{-1}^{1} ydy\right] = (1/8)[4+2x] = (2+x)/4, \ -1 < x < 1.$
- $f_Y(y) = (2+y)/4, \ -1 < y < 1.$
- $f(x, y) \neq f_X(x) f_Y(y)$. So, $X \not\perp\!\!\!\perp Y$.

§3.2-#11

11. 設X, Y之聯合p.d.f.為$f(x,y) = 2e^{-x-y}, 0 \leq x < y < \infty$。

(i) 試求$P(Y > 2X)$;

(ii) 試證X之邊際分佈為$\mathcal{E}(1/2)$;

(iii) 試求$f_{Y|X}(y|x)$及$E(Y|X = x)$;

(iv) 試檢驗X與Y是否獨立。

- $\mathbb{P}(Y > 2X) = \int_0^\infty \int_0^{y/2} 2e^{-x}e^{-y}dxdy = \int_0^\infty 2e^{-y} \int_0^{y/2} e^{-x}dxdy = 2/3.$
- $f_X(x) = \int_x^\infty 2e^{-x-y}dy = 2e^{-x} \int_x^\infty e^{-y}dy = 2e^{-2x}.$
- $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = e^{x-y}.$$

$$
\begin{aligned}
\mathbb{E}[Y|X=x] &= \int_x^\infty y f_{Y|X}(y|x)dy \\
&= \int_x^\infty y e^{x-y} dy \\
&= e^x \int_x^\infty y e^{-y} dy \\
&= e^x(xe^{-x} + e^{-x}) = x + 1.
\end{aligned}
$$

- $f_Y(y) = \int_0^y 2e^{-x-y}dx = 2e^{-y}\int_0^y e^{-x}dx = 2e^{-y}(1-e^{-y})$, $y \geq 0$.
- $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, i.e., $X \not\perp\!\!\!\perp Y$.

§3.2-#20

20. 設(X,Y)之聯合p.d.f.爲$f(x,y) = 2e^{-x-y}$, $0 < x < y < \infty$。試求(X,Y)之聯合動差母函數。(**解.** $M(s,t) = 2((1-t)(2-s-t))^{-1}$, $t < 1, s+t < 2$)

- $M(s,t) = \mathbb{E}[e^{sX+tY}] = 2\int_0^\infty \int_0^y e^{(s-1)x+(t-1)y} dxdy =$ $2\int_0^\infty e^{(t-1)y} \int_0^y e^{(s-1)x} dxdy = 2\int_0^\infty e^{(t-1)y} \left[\frac{e^{(s-1)y}-1}{s-1}\right] dy =$ $\frac{2}{s-1}\left[\int_0^\infty e^{(s+t-2)y} dy - \int_0^\infty e^{(t-1)y} dy\right] = \frac{2}{s-1}\left[\frac{1}{2-s-t} - \frac{1}{1-t}\right] =$ $\frac{2}{s-1}\left[\frac{s-1}{(1-t)(2-s-t)}\right]$.
- $$M(s,t) = \frac{2}{(1-t)(2-s-t)},\ t < 1, s+t < 2.$$

23. 設X與Y獨立, 令$S = X + Y$。又知$X \sim \chi^2_m$, $S \sim \chi^2_{m+n}$。試問Y是否有卡方分佈, 若是則給出其自由度。(**解.**是, $Y \sim \chi^2_n$)

- By uniqueness of MGF, 給定 $M_X(t) = (1-2t)^{-m/2}$ and

  $M_S(t) = (1-2t)^{-(m+n)/2} = (1-2t)^{-m/2}(1-2t)^{-n/2} = M_X(t)(1-2t)^{-n/2}$.

- $\because \ X \perp\!\!\!\perp Y$,

$$M_S(t) = M_{X+Y}(t) = M_X(t)M_Y(t),$$

  so that $M_Y(t) = (1-2t)^{-n/2}$ holds. That is, $Y \sim \chi^2(n)$.

Note: $\#22$: $U := X - Y$ does not have χ^2 distribution.

反例:

- 令 $Z_i, W_i \sim i.i.d.\ N(0,1)$, 且 $A_i := Z_i^2 \sim \chi^2(1), B_i := W_i^2 \sim \chi^2(1)$.
  則, $X = \sum_{i=1}^m A_i \sim \chi^2(m)$ 與 $Y = \sum_{i=1}^n B_i \sim \chi^2(n)$.
- For $m > n$, let $C_i := A_i - B_i$,
  $U := X - Y = \sum_{i=1}^m A_i - \sum_{i=1}^n B_i = \sum_{i=1}^n A_i + \sum_{n+1}^m A_i - \sum_{i=1}^n B_i = \sum_{i=1}^n C_i + \sum_{i=n+1}^m A_i \sim \square + \chi^2(m-n) \neq \chi^2(m-n)$.
- 雖然 A_i, B_i 獨立且相同分佈, 但是不一定 $A_i - B_i = 0$, 即, $\mathbb{P}(C_i = 0) = 0$ (連續隨機變數的單點機率是零).

7. 設X與Y獨立, 且分別有$\mathcal{E}(\lambda)$及$\mathcal{E}(\mu)$分佈。令

$$Z=\min\{X,Y\},\quad W=\begin{cases}1 & ,Z=X,\\ 0 & ,Z=Y。\end{cases}$$

(i) 試求Z,W之聯合p.d.f.(注意Z爲連續型, W爲離散型);

(ii) 試證$P(Z\le z|W=i)=P(Z\le z), i=0,1$, 即$Z$與$W$獨立。

- (ii). $f_{X,Y}(x,y)=\lambda\mu\exp(-\lambda x-\mu y)$.
- $\mathbb{P}(Z>z)=\mathbb{P}(Z>z,W=0)+\mathbb{P}(Z>z,W=1)=\mathbb{P}(X>z,Y>z,Y<X)+\mathbb{P}(X>z,Y>z,Y\ge X)=\mathbb{P}(X>z,Y>z)=\exp(-(\lambda+\mu)z),\ z>0$.
- $\mathbb{P}(Z>z|W=1)=\mathbb{P}(Z>z,W=1)/\,\mathbb{P}(W=1)=\mathbb{P}(X>z,Y>z,Y>X)/\,\mathbb{P}(Y>X)$.

- $\mathbb{P}(X > z, Y > z, Y > X) = \int_z^\infty \int_z^y \lambda\mu e^{-\lambda x} e^{-\mu y} dx dy =$
$\int_z^\infty \mu e^{-\mu y}(-e^{-\lambda x})|_z^y dy = \int_z^\infty \mu e^{-\mu y}(e^{-\lambda z} - e^{-\lambda y}) dy =$
$\frac{\lambda}{\lambda+\mu} \exp(-(\lambda+\mu)z)$.
- Also, $\mathbb{P}(Y > X) = \int_0^\infty \mathbb{P}(X > y) f_Y(y) dy = \int_0^\infty (1 - e^{-\lambda y}) \mu e^{-\mu y} dy =$
$1 - \frac{\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$.
Thus, $\mathbb{P}(Z > z | W = 1) = \exp(-(\lambda+\mu)z) = \mathbb{P}(Z > z)$. Similarly, $\mathbb{P}(Z > z | W = 0) = \mathbb{P}(Z > z)$. As such,

$$\mathbb{P}(Z \leq z | W = 1) = \mathbb{P}(Z \leq z),$$

$$\mathbb{P}(Z \leq z | W = 0) = \mathbb{P}(Z \leq z);$$

i.e., $W \perp\!\!\!\perp Z$.

- (i). $f_{Z,W}(z,w) = f_Z(z)\,\mathbb{P}(W=w)$, $w \in \{0,1\}$.
- $F_Z(z) = \mathbb{P}(Z \le z) = 1 - \mathbb{P}(Z > z) = 1 - \exp(-(\lambda+\mu)z) \Rightarrow f_Z(z) = (\lambda+\mu)\exp(-(\lambda+\mu)z),\ z > 0.$
  $\mathbb{P}(W=1) = \mathbb{P}(Y > X) = \frac{\lambda}{\mu+\lambda}$ and $\mathbb{P}(W=0) = \mathbb{P}(Y \le X) = \frac{\mu}{\mu+\lambda}$.
- $$f_{Z,W}(z,w) = \begin{cases} \lambda \exp(-(\lambda+\mu)z), & w = 1 \\ \mu \exp(-(\lambda+\mu)z), & w = 0, \end{cases}$$

  for $z > 0$.

12. 設X與Y獨立, 且皆有$\mathcal{U}(\alpha, \alpha+1)$分佈, 其中$\alpha \in R$爲一常數。令$U = X+Y$, $V = X-Y$。

(i) 試求U與V之聯合p.d.f.;

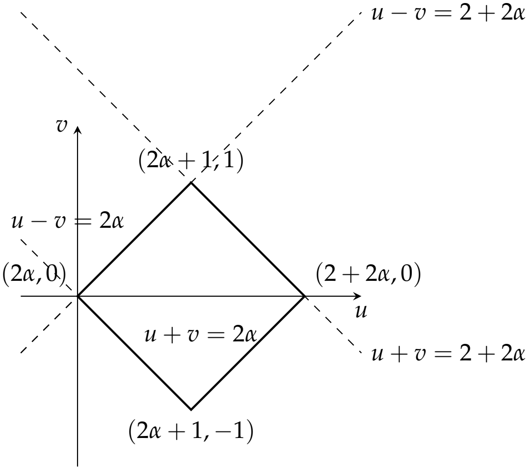
(ii) 試檢驗U與V是否獨立。

- (i). $f_{XY}(x,y) = f_X(x)f_Y(y) = 1,\ \alpha < x < \alpha+1, \alpha < y < \alpha+1.$
- Let $U = X+Y, V = X-Y \Rightarrow X = (U+V)/2, Y = (U-V)/2.$
- $J = |\frac{\partial(X,Y)}{\partial(U,V)}| = 1/2.$
- $$f_{U,V}(u,v) = f_{X,Y}(x = \frac{u+v}{2}, y = \frac{u-v}{2})|J| = 1/2,$$

  $2\alpha < u+v < 2\alpha+2, 2\alpha < u-v < 2\alpha+2.$

- (ii). 範圍請參考下頁圖示。
  - fixed u, $\max(2\alpha - u, u - 2\alpha - 2) < v < \min(2\alpha + 2 - u, u - 2\alpha)$;
  (for $u \in [2\alpha, 2\alpha + 1]$, $2\alpha - u < v < u - 2\alpha$;
  for $u \in (2\alpha + 1, 2\alpha + 2]$, $u - 2\alpha - 2 < v < 2\alpha + 2 - u$).
  - fixed v, $\max(2\alpha - v, 2\alpha + v) < u < \min(2\alpha + 2 - v, 2\alpha + v + 2)$
  (for $v \in [-1, 0]$, $2\alpha - v < u < 2\alpha + 2 + v$;
  for $v \in (0, 1]$, $2\alpha + v < u < 2\alpha + 2 - v$).
- $f_U(u) = \int f_{UV}(u,v)dv = 1 - |u - 2\alpha - 1|,\ 2\alpha < u < 2\alpha + 2.$
- $f_V(v) = \int f_{UV}(u,v)du = 1 - |v|,\ -1 < v < 1.$
- So, $f_{UV}(u,v) \neq f_U(u) f_V(v)$, $U \not\perp\!\!\!\perp V$.

圖示



$u - v = 2 + 2\alpha$
v
$(2\alpha + 1, 1)$
$u - v = 2\alpha$
$(2\alpha, 0)$
$(2 + 2\alpha, 0)$
u
$u + v = 2\alpha$
$u + v = 2 + 2\alpha$
$(2\alpha + 1, -1)$

3. 設$Y|P$有$\mathcal{B}(n,P)$分佈, P有$\mathcal{B}e(\alpha,\beta)$分佈 。

(i) 試證Y之非條件分佈爲

$$P(Y=y)=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n)}, y=0,1,\cdots,n;$$

(ii) 試求$E(Y)$及$\mathrm{Var}(Y)$ 。

- (i). $f_{Y,P}(y,p)=f_{Y|P}(y|p)f_P(p)=\binom{n}{y}p^y(1-p)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$

$$\mathbb{P}(Y=y)=\int_0^1 f_{Y,P}(y,p)dp=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 p^{y+\alpha-1}(1-p)^{n+\beta-y-1}dp,$$

So,

$$\mathbb{P}(Y=y)=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n+\beta-y)}{\Gamma(n+\alpha+\beta)},\ y=0,1,\cdots,n.$$

- (ii).

$$\mathbb{E}\,Y = \mathbb{E}\,\mathbb{E}[Y|P] = \mathbb{E}[nP] = n\,\mathbb{E}[P] = n\alpha/(\alpha+\beta)$$

- $\operatorname{Var}[Y] = \mathbb{E}[\operatorname{Var}[Y|P]] + \operatorname{Var}[\mathbb{E}[Y|P]] = (n\,\mathbb{E}[P] - n\,\mathbb{E}[P^2]) + n^2\operatorname{Var}[P]$

$$= \frac{n^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{n\alpha}{\alpha+\beta} + \frac{n\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$= \frac{n\alpha\beta(n+\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

12. 設X與Y獨立，且令$E(X)=\mu_X, E(Y)=\mu_Y, \mathrm{Var}(X)=\sigma_X^2, \mathrm{Var}(Y)$
$=\sigma_Y^2$。試以μ_X, μ_Y, σ_X,及σ_Y表示$\rho(XY, Y)$。

- $$\rho(XY, Y) = \frac{\mathrm{cov}[XY, Y]}{\sigma_{XY}\sigma_Y}$$

- $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mu_X \mu_Y$ and

$$\mathbb{E}[X^2Y^2] = (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2 + \mu_X^2\mu_Y^2.$$

  So, $\sigma_{XY}^2 = \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2$.
- $\mathrm{cov}[XY, Y] = \mathrm{cov}[\mathbb{E}[XY|Y], Y] = \mathrm{cov}[Y\,\mathbb{E}[X|Y], Y] = \mathrm{cov}[Y\,\mathbb{E}[X], Y] = \mathbb{E}[X]\,\mathrm{Var}[Y] = \mu_X\sigma_Y^2$.
- $$\mathrm{cov}[XY, Y] = \frac{\mu_X\sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}}.$$
- Note: $\mathrm{cov}[XY, Y] = \mathbb{E}[XY^2] - \mathbb{E}[XY]\,\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[XY^2|Y]] - \mathbb{E}[\mathbb{E}[XY|Y]]\,\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[XY|Y]Y] - \mathbb{E}[\mathbb{E}[XY|Y]]\,\mathbb{E}[Y] = \mathrm{cov}[\mathbb{E}[XY|Y], Y]$.

13. 設X, Y, Z為兩兩無相關之隨機變數，期望值皆為μ，變異數皆為σ^2。試以μ及σ^2表示$\mathrm{Cov}(X+Y, Y+Z)$及$\mathrm{Cov}(X+Y, X-Y)$。

- $$\mathrm{cov}[X+Y, Y+Z] = \mathrm{cov}[X,Y] + \mathrm{cov}[X,Z] + \mathrm{cov}[Y,Y] + \mathrm{cov}[Y,Z] = \sigma^2.$$
  ($\because$ X, Y, Z are pairwise independent.)
- $\mathrm{cov}[X+Y, X-Y] = \mathrm{cov}[X,X] - \mathrm{cov}[X,Y] + \mathrm{cov}[Y,X] - \mathrm{cov}[Y,Y] = \sigma^2 + 0 + 0 - \sigma^2 = 0$.