TA section 3

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HW 1: Part (II)

§4.5#8

8. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}(\alpha, \beta)$分佈所產生之隨機樣本, $\alpha < \beta$。令$R = X_{(n)} - X_{(1)}$。試求

(i) $X_{(1)}, X_{(n)}$之聯合p.d.f.;

(ii) R之p.d.f.;

(iii) $E(R)$。

(i) Given $x \le y$,

- $F_{X_{(1)},X_{(n)}}(x,y) = \mathbb{P}(X_{(1)} \le x, X_{(n)} \le y) = \mathbb{P}(X_{(n)} \le y) - \mathbb{P}(X_{(1)} > x, X_{(n)} \le y) = F(y)^n - \mathbb{P}(x < X_1 \le y)^n = F(y)^n - [F(y) - F(x)]^n$.
- $f_{X_{(1)},X_{(n)}}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X_{(1)},X_{(n)}}(x,y) = \frac{\partial}{\partial x}\Big[nF(y)^{n-1}f(y) - n[F(y) - F(x)]^{n-1}f(y)\Big] = n(n-1)[F(y) - F(x)]^{n-2}f(x)f(y) = \frac{n(n-1)}{(\beta-\alpha)^n}\,(y-x)^{n-2}, \quad \alpha \le x \le y \le \beta.$

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(ii)

- Let $V = (X_{(1)} + X_{(n)})/2 \Rightarrow X_{(1)} = V - R/2, X_{(n)} = V + R/2$. Then, $J = |\frac{\partial(x,y)}{\partial(r,v)}| = \begin{pmatrix} -1/2 & 1 \\ 1/2 & 1 \end{pmatrix} = 1.$
- $$f_{R,V}(r,v) = f_{X_{(1)},X_{(n)}}(x = v - r/2, y = v + r/2)|J| = \frac{n(n-1)}{(\beta-\alpha)^n} r^{n-2},$$

  for $0 \le r \le \beta - \alpha,\ \alpha + r/2 \le v \le \beta - r/2.$
  So,

  $$f_R(r) = \int_{\alpha+r/2}^{\beta-r/2} f_{R,V}(r,v)dv \ = \ \frac{n(n-1)}{(\beta-\alpha)^n}\ r^{n-2}\left[(\beta-\alpha)-r\right], \quad 0 \le r \le \beta-\alpha.$$

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(iii)

- $$\mathbb{E}[R] = \int_0^{\beta-\alpha} r f_R(r) dr = (\beta - \alpha)\frac{n-1}{n+1}.$$

- 另解: 令 $U_1, \cdots, U_n \sim U[0,1]$. Then, $X_i = \alpha + (\beta - \alpha)U_i$.
  令 $W = U_{(n)} - U_{(1)} \Rightarrow f_W(w) = n(n-1)(1-w)w^{n-2},\ 0 \le r \le 1$
  $\Rightarrow \mathbb{E}[W] = \frac{n-1}{n+1}$ (例 5.4).
  令 $R = X_{(n)} - X_{(1)} = (\beta - \alpha)W$
  $\Rightarrow f_R(r) = \frac{n(n-1)}{(\beta-\alpha)^n}\, r^{n-2}\left[(\beta - \alpha) - r\right],\ 0 \le r \le \beta - \alpha$

$$\Rightarrow \mathbb{E}[R] = (\beta - \alpha)\,\mathbb{E}[W] = (\beta - \alpha)\frac{n-1}{n+1}.$$

§4.5#9

9. 設$X_1, \cdots, X_n$爲一組由$\mathcal{E}(\lambda)$分佈所產生之隨機樣本。試證$nX_{(1)}$與X
有相同的分佈。

- Claim: $nX(1) \sim \mathcal{E}(\lambda)$.
- $\mathbb{P}(nX(1) \leq t) = \mathbb{P}(X(1) \leq t/n) = 1 - \mathbb{P}(X_1 > t/n, \cdots, X_n > t/n) = 1 - \mathbb{P}(X_1 > t/n)^n = 1 - \exp(-n\lambda(t/n)) = 1 - \exp(-\lambda t) = \mathbb{P}(X \leq t)$. We are done.

§4.5#10

10. 設X, Y爲二獨立的隨機變數, 且皆有$\mathcal{N}(0,1)$分佈 。又令$Z = \min\{X, Y\}$ 。試證Z^2有χ_1^2分佈, 即$Z^2 \stackrel{d}{=} X^2$ 。

- $\mathbb{P}(Z^2 \le s) = \mathbb{P}(-\sqrt{s} \le Z \le \sqrt{s}) = F_Z(\sqrt{s}) - F_Z(-\sqrt{s})$.
- $F_Z(\sqrt{s}) = \mathbb{P}(Z \le \sqrt{s}) = 1 - \mathbb{P}(Z > \sqrt{s}) = 1 - \mathbb{P}(X > \sqrt{s}, Y > \sqrt{s}) = 1 - \mathbb{P}(X > \sqrt{s})\,\mathbb{P}(Y > \sqrt{s}) = 1 - [1 - \Phi(\sqrt{s})]^2 = 1 - \Phi(-\sqrt{s})^2$.
- $F_Z(-\sqrt{s}) = 1 - [1 - \Phi(-\sqrt{s})]^2$. Then,

$$\mathbb{P}(Z^2 \le s) = F_Z(\sqrt{s}) - F_Z(-\sqrt{s}) = 2\Phi(\sqrt{s}) - 1.$$

- $$\mathbb{P}(X^2 \le s) = \Phi(\sqrt{s}) - \Phi(-\sqrt{s}) = 2\Phi(\sqrt{s}) - 1.$$

So, $Z^2 \stackrel{\mathrm{d}}{=} X^2$.