TA section 10

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Homework 7

§3.4 #2,#3,#4,#5

2. 設$Y|N$有$\mathcal{B}(N,p)$分佈, $N|\Lambda$有$P(\Lambda)$分佈, 且Λ有$\Gamma(\alpha,\beta)$分佈, 試求Y之非條件分佈。

3. 設$Y|P$有$\mathcal{B}(n,P)$分佈, P有$\mathcal{B}e(\alpha,\beta)$分佈。

   (i) 試證Y之非條件分佈爲

$$P(Y=y)=\binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(\alpha+\beta+n)}, y=0,1,\cdots,n;$$

   (ii) 試求$E(Y)$及$\mathrm{Var}(Y)$。

4. 設$\mathrm{Var}(X)=E(\mathrm{Var}(X|Y))$。試證存在常數$c$, 使得$P(E(X|Y)=c)=1$。

5. 試證或否證若$E(Y|X)=c$, 其中c若爲常數, 導致X與Y獨立。

§3.4 #2

- $$\mathbb{P}(Y=y)=\int_0^\infty f_{Y,\Lambda}(y,\lambda)d\lambda$$
- $$\mathbb{P}(Y=y)=\int_0^\infty f_{Y|\Lambda}(y|\lambda)f_\Lambda(\lambda)d\lambda$$
- $$\mathbb{P}(Y=y)=\int_0^\infty \sum_{n=\lambda}^{\infty}\mathbb{P}(Y=y|N=n,\lambda)\,\mathbb{P}(N=n|\Lambda=\lambda)f_\Lambda(\lambda)d\lambda$$

- $$\begin{aligned} f_{Y|\Lambda}(y|\lambda) &= \sum_{n=\lambda}^{\infty} \mathbb{P}(Y=y|N=n,\lambda)\,\mathbb{P}(N=n|\Lambda=\lambda) \\ &= \sum_{n=\lambda}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \lambda^n e^{-\lambda}/n! \\ &= \frac{e^{-\lambda}}{y!}(\frac{p}{1-p})^y((1-p)\lambda)^y \Big[\sum_{k=0}^{\infty} \frac{((1-p)\lambda)^k}{k!}\Big] \text{ (let } k=n-y) \\ &= \frac{(p\lambda)^y e^{-p\lambda}}{y!}. \end{aligned}$$

- $$\mathbb{P}(Y=y) = \int_0^{\infty} f_{Y|\Lambda}(y|\lambda) f_\Lambda(\lambda) d\lambda = \int_0^{\infty} \frac{(p\lambda)^y e^{-p\lambda}}{y!} \frac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha} d\lambda$$

- For $y = 0, 1, \cdots,$

$$
\begin{aligned}
\mathbb{P}(Y = y) &= \frac{p^y}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{y+\alpha-1} e^{-\lambda(1+p\beta)/\beta} d\lambda \\
&= \frac{p^y}{y!\Gamma(\alpha)\beta^\alpha} \Gamma(y+\alpha) \Big(\frac{\beta}{1+p\beta} \Big)^{y+\alpha} \\
&= \binom{y+\alpha-1}{y} \Big[\frac{p\beta}{1+p\beta} \Big]^y \Big[\frac{1}{1+p\beta} \Big]^\alpha, \quad \text{for } \alpha \in \mathbb{N}
\end{aligned}
$$

- $Y \sim NB(\alpha, 1/(1+p\beta))$.

§3.4 #3

- $f_{Y,P}(y,p) = f_{Y|P}(y|p)f_P(p) = \binom{n}{y}p^y(1-p)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$
- $$\mathbb{P}(Y=y) = \int_0^1 f_{Y,P}(y,p)dp = \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 p^{y+\alpha-1}(1-p)^{n+\beta-y-1}dp$$
- Thus,
$$\mathbb{P}(Y=y) = \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(y+\alpha)\Gamma(n+\beta-y)}{\Gamma(n+\alpha+\beta)},\ y=0,1,\cdots,n.$$
- Note: $\int_0^1 p^{y+\alpha-1}(1-p)^{n+\beta-y-1}dp = \frac{\Gamma(y+\alpha)\Gamma(n+\beta-y)}{\Gamma(n+\alpha+\beta)}$.

- $$\mathbb{E}\, Y = \mathbb{E}\, \mathbb{E}[Y|P] = \mathbb{E}[nP] = n\, \mathbb{E}[P] = n\alpha/(\alpha+\beta)$$

- $$\begin{aligned} \operatorname{Var}[Y] &= \mathbb{E}[\operatorname{Var}[Y|P]] + \operatorname{Var}[\mathbb{E}[Y|P]] \\ &= \mathbb{E}[nP(1-P)] + \operatorname{Var}[nP] \\ &= n(\mathbb{E}[P] - \mathbb{E}[P^2]) + n^2 \operatorname{Var}[P] = \cdots \text{ (自己算)} \end{aligned}$$

§3.4 #4

- $\mathrm{Var}[X] = \mathbb{E}[\mathrm{Var}[X|Y]] + \mathrm{Var}[\mathbb{E}[X|Y]] = \mathbb{E}[\mathrm{Var}[X|Y]]$, whenever $\mathrm{Var}[\mathbb{E}[X|Y]] = 0$ which implies that $\mathbb{P}(\mathbb{E}[X|Y] = c) = 1$.

§3.4 #5

- No. 反例: 假設有 Z 且 $Z \perp\!\!\!\perp Y$,$Z, Y \neq 0$, 而 $\mathbb{E}[Z] = \mathbb{E}[Y] = 0$。令 $X = YZ$, 則
$$\mathbb{E}[X|Y] = \mathbb{E}[YZ|Y] = Y\,\mathbb{E}[Z|Y] = Y\,\mathbb{E}[Z] = 0.$$
但,$X \not\perp\!\!\!\perp Y$.
- NOTE:
$$X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[X|Y] = \mathbb{E}[X],$$
反之不一定成立; 除非 (X, Y) ~jointly normal.

§3.5 #7

7. 設X之p.d.f.爲一偶函數，且$E(X^4) < \infty$。試證X與X^2無相關。

- $\operatorname{cov}(X, X^2) = \mathbb{E}[X^3] - \mathbb{E}[X]\,\mathbb{E}[X^2] = \mathbb{E}[X^3] (\because \mathbb{E}[X] = 0)$
-
$$\operatorname{cov}(X, X^2) = \int_{\mathbb{R}} x^3 f(x) dx =: \int_{\mathbb{R}} h(x) dx = 0$$
  for $h(x) := x^3 f(x)$ is an odd function. (why)
- $h(-x) = -x^3 f(-x) = -x^3 f(x) = -h(x)$, so $h(.)$ is odd.

§3.5 #12

12. 設X與Y獨立，且令$E(X)=\mu_X, E(Y)=\mu_Y, \mathrm{Var}(X)=\sigma_X^2, \mathrm{Var}(Y)$
$=\sigma_Y^2$。試以μ_X, μ_Y, σ_X,及σ_Y表示$\rho(XY,Y)$。

- $$\rho(XY,Y)=\frac{\mathrm{cov}[XY,Y]}{\sigma_{XY}\sigma_Y}$$

- $X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mu_X \mu_Y$ and
  $\mathbb{E}[X^2Y^2] = (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2 + \mu_X^2\mu_Y^2$.
  So, $\sigma_{XY}^2 = \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2$.
- $\operatorname{cov}[XY, Y] = \operatorname{cov}[\mathbb{E}[XY|Y], Y] = \operatorname{cov}[Y\,\mathbb{E}[X|Y], Y] = \operatorname{cov}[Y\,\mathbb{E}[X], Y] = \mathbb{E}[X]\operatorname{Var}[Y] = \mu_X\sigma_Y^2$.
- $$\operatorname{cov}[XY, Y] = \frac{\mu_X \sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}}$$

§3.5 #16

16. 設X, Y爲二隨機變數, 且$E(X) = E(Y) = 0, \mathrm{Var}(X) = \mathrm{Var}(Y) = 1$。試證

(i) $|E(XY)| \leq 1$;

(ii) $E(XY) = 1$, 若且唯若$P(Y = X) = 1$;

(iii) $E(XY) = -1$, 若且唯若$P(Y = -X) = 1$。

- By Cauchy-Schwarz's inequality,

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]} = 1$$

- ($\Leftarrow$): Let $\mathbb{P}(Y = \pm X) = 1$, then LHS=RHS= $\mathbb{E}[X^2] = 1$;
  ($\Rightarrow$): Let $\mathbb{E}[XY] = \pm 1$,
  $\mathbb{E}[(Y \mp X)^2] = \mathbb{E}[Y^2] + \mathbb{E}[X^2] \mp 2\,\mathbb{E}[XY] = 2(1 \mp \mathbb{E}[XY]) = 0$. That is,
  $\mathbb{P}(Y \mp X = 0) = 1$.

§3.6 #15

15. 設(X, Y)之聯合p.d.f.爲

$$f(x,y)=\frac{1}{2\pi}e^{-(x^2+y^2)/2}(1+xye^{-(x^2+y^2-2)/2}),\ x,y\in R\text{。}$$

試證

(i) $f(x,y)$確爲一二變數之p.d.f.;

(ii) X, Y皆以$\mathcal{N}(0,1)$爲邊際p.d.f.。

- check:
  (1) $f(x,y)\geq 0\forall x,y\in\mathbb{R}$;
  (2) $\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy+\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2\pi}xye^{-(x^2+y^2-1)}dxdy=1.$ (自行驗證)
  So, it is a bivariate pdf.

- $$f_X(x) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dy + \int_{\mathbb{R}} \frac{1}{2\pi} xye^{-(x^2+y^2-1)} dy =: A + B,$$

  where $A = \frac{1}{2\pi} e^{-x^2/2} \int_{\mathbb{R}} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and
  $B = \frac{x}{2\pi} e^{-x^2+1} \int_{\mathbb{R}} ye^{-y^2} dy = 0$.
- So,

  $$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \ x \in \mathbb{R}.$$

- $$f_Y(y) = \int_{\mathbb{R}} f(x,y) dx = \cdots = \frac{e^{-y^2/2}}{\sqrt{2\pi}}, \ y \in \mathbb{R}.$$

  (類似做法)

§3.7 #9

9. 設$X_1, \cdots, X_n$爲相互獨立的隨機變數, 且皆有$\mathcal{U}(0,1)$分佈。令$Z = \prod_{i=1}^n X_i$。試求Z之p.d.f.。

- $Z_i = \prod_{i=1}^n X_i =: e^{-W_i}$, $W_i = \sum_{i=1}^n -\log X_i$.
- Let $Y_i = -\log X_i \Rightarrow Y_i \sim \mathcal{E}(1)$, so $W_i = \sum_{j=1}^n Y_j \sim \Gamma(n, 1)$,

$$f_W(w) = w^{n-1}e^{-w}/(n-1)!, w > 0$$

- $Z = e^{-W}$,

$$\begin{aligned} f_Z(z) &= f_W(w = -\log z)|d(-\log z)/dz| \\ &= \frac{(-\log z)^{n-1} z}{(n-1)!}\frac{1}{z} = \frac{(-\log z)^{n-1}}{(n-1)!}, z \in [0,1]. \end{aligned}$$

§3.7 #12

12. 對X, Y, Z三隨機變數, 假設底下提到的各量(期望值、變異數及共變異數等)皆存在。試證

(i) X與$Y - E(Y|X)$爲無相關;

(ii) $\mathrm{Var}(Y - E(Y|X)) = E(\mathrm{Var}(Y|X))$;

(iii) $\mathrm{Cov}(X, Y) = 0 \Rightarrow E(\mathrm{Cov}(X, Y|Z)) = -\mathrm{Cov}(E(X|Z), E(Y|Z))$;

(iv) $\mathrm{Cov}(Z, E(Y|Z)) = \mathrm{Cov}(Z, Y)$。

- (i) $\mathrm{cov}[X, Y - \mathbb{E}[Y|X]] = \mathrm{cov}[X, Y] - \mathrm{cov}[X, \mathbb{E}[Y|X]] = \mathrm{cov}[X, Y] - \mathrm{cov}[X, Y] = 0$ by (iv)

- (ii) $\mathrm{Var}[Y - \mathbb{E}[Y|X]] = \mathbb{E}[(Y - \mathbb{E}[Y|X])^2] - (\mathbb{E}[Y - \mathbb{E}[Y|X]])^2 = \mathbb{E}[(Y - \mathbb{E}[Y|X])^2] - (\mathbb{E}[Y] - \mathbb{E}[Y])^2 = \mathbb{E}[(Y - \mathbb{E}[Y|X])^2] = \mathbb{E}[\mathbb{E}[(Y - \mathbb{E}[Y|X])^2|X]] = \mathbb{E}[\mathrm{Var}[Y|X]]$.
- (iii) Claim: $\mathrm{cov}[X, Y] = \mathbb{E}[\mathrm{cov}[X, Y|Z]] + \mathrm{cov}[\mathbb{E}[X|Z], \mathbb{E}[Y|Z]]$.
  Note that, $\mathrm{cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] =$
  $\mathbb{E}[\mathbb{E}[XY|Z]] - \mathbb{E}[\mathbb{E}[X|Z]]\,\mathbb{E}[\mathbb{E}[Y|Z]] =$
  $\mathbb{E}(\mathrm{cov}[X, Y|Z] + \mathbb{E}[X|Z]\,\mathbb{E}[Y|Z]) - \mathbb{E}[\mathbb{E}[X|Z]]\,\mathbb{E}[\mathbb{E}[Y|Z]] =$
  $\mathbb{E}[\mathrm{cov}[X, Y|Z]] + (\mathbb{E}[\mathbb{E}[X|Z]\,\mathbb{E}[Y|Z]]] - \mathbb{E}[\mathbb{E}[X|Z]]\,\mathbb{E}[\mathbb{E}[Y|Z]]) =$
  $\mathbb{E}[\mathrm{cov}[X, Y|Z]] + \mathrm{cov}[\mathbb{E}[X|Z], \mathbb{E}[Y|Z]]$.
  Thus,
  $$\mathrm{cov}[X, Y] = 0 \Rightarrow \mathbb{E}[\mathrm{cov}[X, Y|Z]] = -\,\mathrm{cov}[\mathbb{E}[X|Z], \mathbb{E}[Y|Z]].$$
- (iv) $\mathrm{cov}[Z, \mathbb{E}[Y|Z]] = \mathbb{E}[Z\,\mathbb{E}[Y|Z]] - \mathbb{E}[Z]\,\mathbb{E}[\mathbb{E}[Y|Z]] = \mathbb{E}[\mathbb{E}[ZY|Z]] - \mathbb{E}[Z]\,\mathbb{E}[Y] = \mathbb{E}[ZY] - \mathbb{E}[Z]\,\mathbb{E}[Y] = \mathrm{cov}[Z, Y]$.