TA section 7

JERRY C.

108354501@nccu.edu.tw

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Review: Point Estimation

估計量的優劣評估

- 不偏性: $\mathbb{E}[T(\boldsymbol{X})] = q(\theta)$.
  – 不偏估計量非唯一。
- 效率性: (相對有效性)
  - 均方差較佳估計量: $re(T_2, T_1) = R(\theta, T_1)/R(\theta, T_2) \leq 1$ (T_1 較佳) – $q(\theta)$ 之均方差 (MSE):

$$MSE_\theta(T) = R(\theta, T) = \mathbb{E}[(T(\boldsymbol{X}) - q(\theta))^2] = \mathrm{Var}[T(\boldsymbol{X})] + Bias_\theta^2(T(\boldsymbol{X})).$$

    – $\mathrm{Var}[T(\boldsymbol{X})] = \mathbb{E}[(T(\boldsymbol{X}) - \mathbb{E}\, T(\boldsymbol{X}))^2]$, 且 $Bias_\theta(T(\boldsymbol{X})) = \mathbb{E}[T(\boldsymbol{X})] - q(\theta)$.
  - 最佳不偏估計量 (best unbiased estimator, BUE): CRLB (Cramer-Rao lower bound): $(q'(\theta))^2/I(\theta)$; "效率可達性 (Efficiency Attainment)"。
  - 均勻最小變異不偏估計量 (uniformly minimum variance unbiased estimator, UMVUE):
    – [Rao-Blackwell Theorem]: (充份 + 不偏 $\Rightarrow$ 效率)
    – [Lehmann-Scheffé (-Rao-Blackwell) Theorem]: (完備 + 充份 + 不偏 $\Rightarrow$ 效率)
  - BUE $\Rightarrow$ UMVUE. ,i.e., $\mathrm{Var}[\text{BUE}] \leq \mathrm{Var}[\text{UMVUE}]$. (即: UMVUE 可能達不到 CRLB)

大樣本性質

- 一致性:
  - (weak consistency) $T_n \xrightarrow{\text{p}} q(\theta)$ as $n \to \infty$ if
    $$\lim_{n\to\infty} \mathbb{P}(|T_n - q(\theta)| < \epsilon) = 1, \forall \epsilon > 0.$$
  - (mean-squares consistency/L_2-norm consistency) $T_n \xrightarrow{\text{m.s.}} q(\theta)$ as $n \to \infty$ if
    $$\mathbb{E}[(T_n - q(\theta))^2] \to 0, \text{ as } n \to \infty.$$
    或 $\lim_{n\to\infty} \mathrm{Var}[T_n] = 0$, 且 $\lim_{n\to\infty} Bias_\theta(T_n) = 0$.
- 漸近不偏性: $\lim_{n\to\infty} \mathbb{E}[T_n] = q(\theta)$.
- 漸近效率性: $are(T_{2n}, T_{1n}) = \lim_{n\to\infty}[\mathrm{Var}[T_{1n}]/\mathrm{Var}[T_{2n}]] \le 1$ for T_{1n}, T_{2n} two (asymptotic) unbiased estimators, 則 T_{1n} 具漸近有效。
- 漸近常態性: 給定一估計量, 其漸近變異數 $\mathrm{Avar}[T_n] = \sigma_n^2(\theta)$ 與漸近期望值 $\mathrm{Asy.}\,\mathbb{E}[T_n] = \mu_n(\theta)$, 則 $(T_n - \mu_n(\theta))/\sigma_n(\theta) \xrightarrow{\text{d}} \mathcal{N}(0,1)$.
- 最佳漸近常態性: 若 $n\sigma_n^2(\theta) \to v^2(\theta)$ 且 $n^{1/2}(\mu_n(\theta) - q(\theta)) \to 0$, 其中 $v^2(\theta) > 0$ 為某一 θ 之函數, 則稱 T_n 具有最佳漸近常態性。即, 此時漸近變異數為 $v^2(\theta)/n$。

Example

- $X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, (μ, σ^2) 未知。Let $q(\sigma^2) := \sigma^2$, consider two estimators of $q(\sigma^2)$,

$$T_1 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / n,$$

$$T_2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1) = nT_1/(n-1).$$

- $\mathbb{E}[T_1] = (n-1)\sigma^2/n \neq \sigma^2$ (biased for σ^2), $\mathrm{Var}[T_1] = 2(n-1)\sigma^4/n^2$. Also,

$$R(\theta, T_1) = \mathrm{Var}[T_1] + Bias^2(\theta, T_1) = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} = \frac{(2n-1)\sigma^4}{n^2}.$$

- $\mathbb{E}[T_2] = \sigma^2$ (unbiased for σ^2), $\mathrm{Var}[T_2] = 2\sigma^4/(n-1) > \mathrm{Var}[T_1]$. So,

$$R(\theta, T_2) = 2\sigma^4/(n-1) > R(\theta, T_1).$$

(T_1 的 MSE 比較小, 可是它是偏的。)

- (若不考慮不偏性)

$$\operatorname{Var}[T_1] < CRLB(\sigma^2) = 2\sigma^4/n < \operatorname{Var}[T_2].$$

- 換言之, 即使 T_2 可知是 UMVUE, 但卻達不到 CRLB. (效率不可達)。
- 若不追求不偏性, 則永遠可找到一個更有效率的估計量 (如: T_1)。

• Theorem (Efficiency Attainment)

假設 $\boldsymbol{X} = (X_1, \cdots, X_n)$ 有一 joint pdf $f(\boldsymbol{x}|\theta)$, 與其對應之 likelihood function $L(\theta|\boldsymbol{x}) = \prod_{i=1}^n f(x_i|\theta)$, 則對 $q(\theta)$ 的任一不偏估計量 $U(\boldsymbol{X})$ 滿足 $\operatorname{Var}[U(\boldsymbol{X})] < \infty$ 且

$$\frac{d}{d\theta}\mathbb{E}\,U(\boldsymbol{X}) = \int \frac{\partial}{\partial\theta}U(\boldsymbol{X})f(\boldsymbol{x}|\theta)d\theta,$$

其 $U(\boldsymbol{X})$ 之變異數可達到 CRLB, 若且唯若存在一個 θ 的函數 $a(\theta)$ 使得以下等式成立:

$$a(\theta)\Big[U(\boldsymbol{X}) - q(\theta)\Big] = \frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}).$$

- $$\frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}) = \frac{n}{2\sigma^4}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\sigma^2\right) =: a(\theta)\left[T_2+(\bar{X}^2-\mu^2)-\sigma^2\right]$$

  無法寫成 T_2 的不偏函數, 且 μ 未知 (除非 $\mathbb{E}[\bar{X}^2]=\mu^2$)

大樣本之下呢? $(n \to \infty)$

- (小樣本) T_2 不偏, 但 MSE 卻較大 (因變異數大的幅度超過 T_1 偏誤的幅度), T_2 真的比較差嗎? 我們來看大樣本之下:
- $\lim_{n\to\infty} \mathbb{E}[T_1] = \lim_{n\to\infty} \mathbb{E}[T_2] = \sigma^2$ (T_1, T_2 都是漸近不偏), $\lim_{n\to\infty} \mathrm{Var}[T_1] = \lim_{n\to\infty} \mathrm{Var}[T_2] = 0$ (均方差一致性);

$$are(T_2, T_1) = \lim_{n\to\infty} \mathrm{Var}[T_1] / \mathrm{Var}[T_2] = \lim_{n\to\infty} \frac{2(n-1)\sigma^4/n^2}{2\sigma^4/(n-1)} = 1,$$

且 T_2 是 UMVUE, 故 T_2 具漸近有效性。

§7.3 #1

1. 設$X_1, \cdots, X_n$爲一組由$\mathcal{B}er(\theta)$分佈所產生之隨機樣本, $0 \le \theta \le 1$。試分別求θ, θ^2, $\theta(1-\theta)$之UMVUE。

求 UMVUE 三招:

- 「充份 + 不偏」: 定理 3.1: Rao-Blackwell Theorem (非唯一解)
- 「完備充份 + 不偏」: 定理 3.2: Lehmann-Scheffé Theorem (唯一解)
- 「不偏 + CRLB」: (i) "Efficiency Attainment" (最效率可達性); (ii) 定理 4.3: 滿足 CRLB 的不偏之一個參數 (one-dimensional) 指數族。

使用 R-B Thm. & L-S Thm 原則:

- 給定 $T(\boldsymbol{X})$ is a C.S.S.,
- 設法找一個 $h(T(\boldsymbol{X}))$ (完備充份統計量的函數) 為 $q(\theta)$ 之不偏估計量, 則利用條件期望值性質, 取條件不偏式

$$h(T(\boldsymbol{X})) = \mathbb{E}[h(T(\boldsymbol{X}))|T(\boldsymbol{X})]$$

  為 $q(\theta)$ 之一 UMVUE。(定理 3.2)
- 若 $h(T(\boldsymbol{X}))$ 不易找出, 則設法造出:
  - (1): 任找一個 $q(\theta)$ 之不偏估計量: $S(\boldsymbol{X})$ (不一定是 $T(\boldsymbol{X})$ 的函數, 若是 $T(\boldsymbol{X})$ 的函數, 則同上);
  - (2): 取條件不偏式, 造出:

$$\mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})],$$

    此即 $q(\theta)$ 之一 UMVUE。(定理 3.1)

定理 3.1.: 充份 + 不偏 ⇒ 有效

• Theorem

設 $T(\boldsymbol{X})$ 為 θ 之一充份統計量, 設 $S(\boldsymbol{X})$ 為 $q(\theta)$ 為任一不偏估計量, 且 $\mathbb{E}\,|S(\boldsymbol{X})| < \infty$, $\forall\theta \in \Omega$。令

$$T^*(\boldsymbol{X}) := \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})],$$

則 $\forall\theta \in \Omega$,

$$R(\theta, T^*) \leq R(\theta, S).$$

定理 3.2.: 完備充份 $+$ 不偏 $\Rightarrow$ 有效

- **Theorem**

設 $T(\boldsymbol{X})$ 為一完備充份統計量, 且 $S = S(\boldsymbol{X})$ 為 $q(\theta)$ 之一不偏估計量。則 $T^*(\boldsymbol{X}) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})]$ 為 $q(\theta)$ 之一 UMVUE; 若 $\mathrm{Var}[T^*] < \infty, \forall\theta \in \Omega$, 則 T^* 為 $q(\theta)$ 唯一之 UMVUE。

定理 4.3.

Theorem

設 $T(\boldsymbol{X})$ 為 $q(\theta)$ 一不偏估計量,$\mathbb{E}[T(\boldsymbol{X})] = q(\theta)$。設一分佈族 $\{P_\theta; \theta \in \Omega\}$ 滿足正規條件, 且為一個參數之指數族, 有 pdf 如下式:

$$f(\boldsymbol{x}|\theta) = h(\boldsymbol{x}) \exp(w(\theta)T(\boldsymbol{x}))\boldsymbol{I}_A(\boldsymbol{x}),\ \ \theta \in \Omega,$$

其中 $w(\theta)$ 有一連續且不為零之導數, $\forall\theta \in \Omega$, 若且唯若 $\mathrm{Var}[T(\boldsymbol{X})]$ 達到 CRLB, 且 $T(\boldsymbol{X})$ 為 $q(\theta)$ 之一 UMVUE。

- $T(\boldsymbol{X}) = \sum_{i=1}^{n} X_i$ is a C.S.S.
- Let $S(\boldsymbol{X}) = \bar{X}$, such that $\mathbb{E}[S(\boldsymbol{X})] = \theta$.
- Then, let

$$h_1(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X}),$$

$\mathbb{E}[h_1(T(\boldsymbol{X}))] = \mathbb{E}[\bar{X}] = \theta$. 故, $S(\boldsymbol{X}) = \bar{X}$ 為 θ 之一不偏估計量, 且為 $T(\boldsymbol{X})$ 的函數。So,

$$h_1(T(\boldsymbol{X})) = \bar{X}$$

is an UMVUE of θ by R-B Thm & L-S Thm.

- Let $S(\boldsymbol{X}) = (n/(n-1))\bar{X}(1-\bar{X})$, such that $\mathbb{E}[S(\boldsymbol{X})] = \theta(1-\theta)$.
- Let $h_2(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X})$,

$$\mathbb{E}[h_2(T(\boldsymbol{X}))] = \mathbb{E}[S(\boldsymbol{X})] = \mathbb{E}\left[(\frac{n}{n-1})\bar{X}(1-\bar{X})\right] = \theta(1-\theta).$$

Since that $S(\boldsymbol{X})$ is unbiased for $\theta(1-\theta)$ and is a function of $T(\boldsymbol{X})$. So,

$$h_2(T(\boldsymbol{X})) = (n/(n-1))\bar{X}(1-\bar{X})$$

is an UMVUE of $\theta(1-\theta)$ by R-B Thm & L-S Them.

- Let $S(\boldsymbol{X}) = T(\boldsymbol{X})(T(\boldsymbol{X}) - 1)/n(n-1)$, such that $\mathbb{E}[S(\boldsymbol{X})] = \theta^2$.
- Let $h_3(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X})$, such that $\mathbb{E}[h_3(\boldsymbol{X})] = \mathbb{E}[S(\boldsymbol{X})]$.
- Since $\mathbb{E}[T^2(\boldsymbol{X})] - \mathbb{E}[T(\boldsymbol{X})] = \mathrm{Var}[T(\boldsymbol{X})] + \mathbb{E}[T(\boldsymbol{X})]^2 - \mathbb{E}[T(\boldsymbol{X})] = n\theta(1-\theta) + n^2\theta^2 - n\theta = n(n-1)\theta^2$, then

$$\mathbb{E}[h_3(T(\boldsymbol{X}))] = \mathbb{E}[S(\boldsymbol{X})] = \mathbb{E}\left[\frac{T(\boldsymbol{X})(T(\boldsymbol{X}) - 1)}{n(n-1)}\right] = \theta^2.$$

So,

$$h_3(T(\boldsymbol{X})) = \sum_{i=1}^{n} X_i (\sum_{i=1}^{n} X_i - 1)/n(n-1)$$

is an UMVUE of θ^2 by R-B Thm & L-S Thm.

§7.2 #5

5. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}[\theta-1, \theta+1]$分佈所產生之隨機樣本, $\theta \in R$。

(i) 試證$T_1 = \overline{X}_n$, $T_2 = (X_{(1)} + X_{(n)})/2$皆爲$\theta$之不偏估計量;

(ii) 試分別求T_1及T_2之MSE。

(**解.**(ii) $R(\theta, T_1) = 1/(3n)$, $R(\theta, T_2) = 2/((n+1)(n+2))$)

(i) $\mathbb{E}[T_1] = \mathbb{E}[X_i] = ((\theta+1) + (\theta-1))/2 = \theta$.
Let $W = X - (\theta - 1) \sim U[0,2]$. $\mathbb{E}[W_{(1)}] = 2/(n+1)$ and $\mathbb{E}[W_{(n)}] = 2n/(n+1) \Rightarrow \mathbb{E}[X_{(1)}] = \mathbb{E}[W_{(1)}] + (\theta - 1)$, and $\mathbb{E}[X_{(n)}] = \mathbb{E}[W_{(n)}] + (\theta - 1)$.

$$\mathbb{E}[T_2] = \mathbb{E}[X_{(1)} + X_{(n)}]/2 = \mathbb{E}[W_{(1)} + W_{(n)}]/2 + (\theta - 1) = 1 + \theta - 1 = \theta.$$

§7.2 #5

(ii) $\text{Var}[T_1] = \text{Var}[X_i]/n = (4/12)/n = 1/3n \Rightarrow R(\theta, T_1) = B[T_1]^2 + \text{Var}[T_1] = \text{Var}[T_1] = 1/3n$.
$\text{Var}[T_2] = (1/4)\left[\text{Var}[X_{(1)}] + \text{Var}[X_{(n)}] + 2\,\text{cov}(X_{(1)}, X_{(n)})\right]$. So, by pp.462–463, and $\text{Var}[X_{(1)}] = \text{Var}[W_{(1)}]$, $\text{Var}[X_{(n)}] = \text{Var}[W_{(n)}]$,

$$\text{Var}[X_{(1)}] = \text{Var}[X_{(n)}] = \frac{4n}{(n+1)^2(n+2)}$$

and

$$\text{cov}[X_{(1)}, X_{(n)}] = \frac{4}{(n+1)^2(n+2)}.$$

Then,

$$R(\theta, T_2) = \text{Var}[T_2] = \frac{2}{(n+1)(n+2)}.$$

§7.2 #6

6. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}[-\theta,\theta]$分佈所產生之隨機樣本, $\theta > 0$。設$n \geq 2$, 試求常數c, 使得$c(X_{(n)} - X_{(1)})$爲θ之一不偏估計量。(**解.** $(n+1)/(2(n-1))$)

- Let $W = X + \theta \sim U[0, 2\theta] \Rightarrow \mathbb{E}[X_{(1)}] = \mathbb{E}[W_{(1)}] - \theta = 2\theta/(n+1) - \theta$ and $\mathbb{E}[X_{(n)}] = \mathbb{E}[W_{(n)}] - \theta = 2n\theta/(n+1) - \theta.$
- $$c\,\mathbb{E}\left[X_{(n)} - X_{(1)}\right] = c\frac{2n-2}{n+1}\theta \overset{let}{=} \theta \Rightarrow c = \frac{n+1}{2n-2}.$$

§7.2 #11

11. 設X有$\mathcal{P}(\lambda)$分佈，$\lambda > 0$。令$\theta = P(X=0) = e^{-\lambda}$。

(i) 試問$T_1 = e^{-X}$是否爲θ之不偏估計量;

(ii) 試證$T_2 = I_{\{X=0\}}$爲θ之不偏估計量;

(iii) 試分別求T_1及T_2之MSE。

(i) By MGF $M_X(t) = \mathbb{E}[e^{tX}] = e^{\lambda(e^t-1)}$.
Let $t=-1$, $\mathbb{E}[T_1] = \mathbb{E}[e^{-X}] = e^{-\lambda(1-1/e)} \neq e^{-\lambda}$, T_1 is biased for θ.

(ii) $\mathbb{E}[T_2] = \mathbb{P}(X=0) = e^{-\lambda} = \theta$, T_2 is unbiased for θ.

MSE:

(iii)

$$
\begin{aligned}
R(\theta, T_1) &= Bias(T_1)^2 + \mathrm{Var}[T_1] \\
&= e^{-2\lambda}[e^{\lambda/e} - 1] + [e^{\lambda/e^2} - e^{2\lambda/e}]e^{-2\lambda} \\
&= (1 - 2e^{\lambda/e} + e^{\lambda/e^2})e^{-2\lambda}.
\end{aligned}
$$

- $R(\theta, T_2) = \mathrm{Var}[T_2] = \mathbb{E}[T_2^2] - \mathbb{E}[T_2]^2 = \mathbb{P}(X = 0) - \mathbb{P}(X = 0)^2 = e^{-\lambda}(1 - e^{-\lambda})$.

§7.3 #9

9. 設$X_1, \cdots, X_n$爲一組由$\mathcal{P}(\lambda)$分佈所產生之隨機樣本。令$\theta = P(X = 1)$。試求θ之一UMVUE。

- $$\begin{aligned} f(x|\lambda) &= (x!)^{-1} e^{\lambda} \exp\{(\log \lambda) x\} \\ &=: h(x) c(\lambda) \exp\{w(\lambda) t(x)\}, \end{aligned}$$

  belongs to the one-dimensional exponential family, where $h(x) = (x!)^{-1} \boldsymbol{I}(x = 0, 1, \cdots), c(\lambda) = \exp(\lambda), w(\lambda) = \log \lambda$, and $t(x) = x$.
- $C = \{w(\lambda) = \log \lambda : \lambda \in \mathbb{R}_+\}$ contains an open set in $\mathbb{R}$. So, $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is a C.S.S. of λ.
- Let $S(\boldsymbol{X}) = \boldsymbol{I}(X_1 = 1)$, such that $\mathbb{E}[S(\boldsymbol{X})] = \lambda e^{-\lambda} =: \theta$. Then, $U(T(\boldsymbol{X})) := \mathbb{E}[S(\boldsymbol{X}) | T = t]$ is an UMVUE of θ, where

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$$
\begin{aligned}
U(t) = \mathbb{E}[S(\boldsymbol{X})|T=t] &= \frac{\mathbb{P}(X_1=1, T=t)}{\mathbb{P}(T=t)} \\
&= \frac{\mathbb{P}(X_1=1, \sum_{i=2}^{n} X_i = t-0)}{\mathbb{P}(T=t)} \\
&= \frac{(e^{-\lambda}\lambda^1/1!)(e^{-(n-1)\lambda}((n-1)\lambda)^{(t-1)}/(t-1)!)}{e^{-n\lambda}(n\lambda)^t/t!} \\
&= (\frac{t}{n})\left(\frac{n-1}{n}\right)^{t-1}.
\end{aligned}
$$

So, $U(\boldsymbol{X}) = \bar{X}[(n-1)/n]^{n\bar{X}-1}$ is the UMVUE of θ by R.-B. Theorem & L.-S.