

## TA section 8

JERRY C.

[jerryc520.github.io/teach/MS.html](https://jerryc520.github.io/teach/MS.html)

May 28, 2024

## Review: Hypothesis Testing

- Given  $\theta \in \Theta \subseteq \Omega = \Omega_0 \cup \Omega_1$  (parametric space). Define  $H_o : \theta \in \Omega_0$  (null space) vs.  $H_1 : \theta \in \Omega_1 := \Omega \setminus \Omega_0$  (alternative space),  $\Omega_0 \cap \Omega_1 = \emptyset$ .
- $C := \{\mathbf{X} : T(\mathbf{X}) \geq d | H_o\}$ : rejection region. We say, we reject  $H_o$  iff  $T(\mathbf{X}) \geq d$  for some  $d > 0$  (critical value) (落在拒絕域).
- testing rule (decision rule):

$$\phi(\mathbf{X}) = \mathbf{I}(T(\mathbf{X}) \in C),$$

is a testing function.

- We want to control the probabilities of two errors (risks): for  $\alpha, \beta \in [0, 1]$ ,

$$\alpha := \mathbb{P}(\text{type I error}) = \mathbb{P}(\text{reject } H_o | H_o \text{ is true}) = \mathbb{E}[\phi(\mathbf{X}) | H_o];$$

$$\beta := \mathbb{P}(\text{type II error}) = \mathbb{P}(\text{NOT reject } H_o | H_o \text{ is false (or } H_1 \text{ is true)}).$$

- $\alpha \uparrow (\downarrow) \Rightarrow \beta \downarrow (\uparrow)$ ;  $\alpha + \beta \neq 1$ ;
- 犯 type I error 較嚴重 (本身無罪卻被判有罪), 對犯 type II error 較容忍 (本身有罪卻沒被判有罪); 即:  $\mathbb{P}(\text{type I error})$  應優先控制!
- Do not reject  $H_o$*  is preferred over *Accept  $H_1$*  (why ?);
- Accept  $H_1$*  is at a risk of a type II error (沒控制犯 type II error 的機率).

# Decision rule

- There are two possible decisions:  
Conclude that there is enough evidence to reject  $H_o$  (support  $H_1$  is true);  
Conclude that there is not enough evidence to reject  $H_o$ .

# Types of Hypotheses

- “=” 放在 null;
- composite hypothesis:  $H_o : \theta \in \Omega_o$  vs.  $H_1 : \theta \in \Omega_1$ ;
- simple hypothesis:  $H_o : \theta = \theta_0$  vs.  $H_1 : \theta = \theta_1$ , 其中  $\theta_0, \theta_1 \in \{ \text{singleton} \}$ ;
- two-sided (two-tailed):  $H_o : \theta = \theta_0$  vs.  $H_1 : \theta \neq \theta_0$ ;
- left-sided (left-tailed):  $H_o : \theta \geq \theta_0$  vs.  $H_1 : \theta < \theta_0$ ;
- right-sided (right-tailed):  $H_o : \theta \leq \theta_0$  vs.  $H_1 : \theta > \theta_0$ .

# Size vs. Power

- power function  $K(\theta) := \mathbb{P}(\text{reject } H_o \mid \theta) = \mathbb{P}_\theta(T(\mathbf{X}) \in C)$ ;
- $K(\theta)$  is an increasing function of  $\theta$ ,  $\lim_{\theta \rightarrow -\infty} K(\theta) = 0$  and  $\lim_{\theta \rightarrow \infty} K(\theta) = 1$ ;
- size  $\alpha$  test:  $\alpha = \sup_{\theta \in \Omega_0} K(\theta)$ ;
- level  $\alpha$  test:  $\alpha \geq \sup_{\theta \in \Omega_0} K(\theta)$ ;  $\alpha$  is called the significant level;
- power of a test:  $1 - \beta := K(\theta \in \Omega_1) = \mathbb{P}(\text{reject } H_o \mid H_1 \text{ is true})$ ;
- **Consistent Test**: If a test with the sequence of power functions  $\{K_n(\theta)\}$ , such that, for any fixed  $\theta \in \Omega_1$ ,  $\lim_{n \rightarrow \infty} K_n(\theta) = 1$ .
- **Unbiased Test**: If a test with power function  $K(\theta)$ , such that for every  $\theta' \in \Omega_1$  and  $\theta'' \in \Omega_0$ ,  $K(\theta') \geq K(\theta'')$ .

# P-value

- p-value is a test statistic  $p(\mathbf{X})$ , such that  $p(\mathbf{x}) \in [0, 1]$  for any  $\mathbf{X} = \mathbf{x}$ , if for every  $\theta \in \Omega_0$  (i.e., under  $H_o$ ),  $\alpha \in [0, 1]$ ,

$$\mathbb{P}_\theta(p(\mathbf{X}) \leq \alpha) \leq \alpha,$$

(對應之 type I error 發生機率要小於  $\alpha$ ), then we say  $p(\mathbf{X})$  is valid.

- A test rejecting  $H_o$  is a level  $\alpha$  test if and only if  $p(\mathbf{X}) \leq \alpha$  (決定出要 reject  $H_o$  之最小的顯著水準).

## • Theorem

Let  $T(\mathbf{X})$  be a testing statistic, with the rejection region  $C = \{\mathbf{X} : T(\mathbf{X}) \geq d | H_o : \theta \in \Omega_0\}$ . Then, for any  $\mathbf{X} = \mathbf{x}$ , define

$$p(\mathbf{x}) := \sup_{\theta \in \Omega_0} \mathbb{P}_\theta(T(\mathbf{X}) \geq T(\mathbf{x})),$$

then  $p(\mathbf{X})$  is a valid p-value.

# Uniformly Most Powerful (UMP) test

- simple hypothesis: Neyman-Pearson lemma  $\Rightarrow$  MP test
- composite hypothesis: monotone likelihood ratio (MLR) family  $\Rightarrow$  UMP test
- UMP test  $\Rightarrow$  MP test



# Example 1

- $X_1, X_2 \sim \text{i.i.d. } U[\theta, \theta + 1]$ . Under  $H_o : \theta = 0$  vs.  $H_1 : \theta = 0.5$ , consider two testing rules  $\phi_1(X_1) = \mathbf{I}(X_1 > 0.95)$  and  $\phi_2(X_1, X_2) = \mathbf{I}(X_1 + X_2 > k)$ , for  $k \in [1, 2]$ .
- size:  $\alpha_1 = \mathbb{E}[\phi_1(X_1)|H_o] = \mathbb{P}(X_1 > 0.95|\theta = 0) = 0.05$ , and

$$\alpha_2 = \mathbb{P}(X_1 + X_2 > k|\theta = 0) = \int_{1-k}^1 \int_{k-x_1}^1 1 dx_2 dx_1 = (2-k)^2/2,$$

so,  $k^* = 2 - \sqrt{2\alpha_2}$ .

- If  $\alpha_1 = \alpha_2$ ,  $k^* = 2 - \sqrt{(0.1)} \approx 1.68$ .
- power of  $\phi_1$ :

$$K_1(\theta) = \mathbb{P}_\theta(X_1 > 0.95) = \begin{cases} 0, & [\theta \leq -0.05] \\ \theta + 0.05, & [-0.05 < \theta \leq 0.95] \\ 1, & [0.95 < \theta]. \end{cases}$$

So, power of  $\phi_1$ :  $K_1(\theta = 0.5) = 0.55$ .

- let  $Y = X_1 + X_2$ , where  $X_i \sim U[0.5, 1.5]$  under  $H_1$ ,  
let  $Z = X_1 \Rightarrow X_2 = Y - Z$ , Jacobian  $|J| = |\partial(X_1, X_2)/\partial(Z, Y)| = 1$ ,

$$f_{X_1, X_2}(x_1, x_2) = 1, \quad 0.5 \leq x_1, x_2 \leq 1.5$$

$$\Rightarrow f_{Y, Z}(y, z) = 1, \quad 0.5 \leq z \leq 1.5, 0.5 \leq y - z \leq 1.5$$

$$\Rightarrow f_Y(y) = \int_{\max(0.5, y-1.5)}^{\min(1.5, y-0.5)} 1 \, dz = \begin{cases} y - 1, & y \in [1, 2] \\ 3 - y, & y \in (2, 3] \\ 0, & o.w. \end{cases}$$

So, power of  $\phi_2$ : for  $k \in [1, 2]$ ,

$$K_2(\theta = 0.5) = \mathbb{P}(Y > k | \theta = 0.5) = \int_k^2 (y - 1) dy + \int_2^3 (3 - y) dy = 0.5 - k^2/2 + k.$$

- Take  $k^* = 1.68$ ,  $K_2(\theta = 0.5) = 0.7688$ .
- 即, 當考慮兩個檢定函數  $\phi_1, \phi_2$ , 控制相同的 size 之下,  $\phi_2$  的 power 較大。

## Example 2

- $X \sim P(\lambda)$ , for testing  $H_o : \lambda \leq 1$  vs.  $H_1 : \lambda > 1$ . Consider one sample observed  $X = 3$ ,
- the p-value of  $X$  is:  
$$p(\mathbf{x}) = \mathbb{P}(X \geq 3 | \lambda = 1) = 1 - \mathbb{P}(X < 3 | \lambda = 1) = 1 - e^{-1}1^2/2! - e^{-1}1^1/1! - e^{-1}1^0/0! \approx 0.0803.$$

# Example 3

- 假設  $X_1, \dots, X_n$ , 為服從  $\mathcal{N}(\mu, \sigma^2)$  分佈之隨機樣本。考慮檢定虛無假設  $H_0: \mu = \mu_0$ , 且檢定統計量為  $\bar{X}_n$ 。令  $z_0 = \sqrt{n}(\bar{x}_0 - \mu_0)/\sigma$ , 此時假設  $Z \sim \mathcal{N}(0, 1)$ ,
- 則可以求得 p-value 為:

$$p(\mathbf{x}) = \mathbb{P}(\bar{X}_n \geq \bar{x}_0 | \mu_0) = \begin{cases} 2(1 - \Phi(|z_0|)), & H_1: \mu \neq \mu_0 \\ 1 - \Phi(z_0), & H_1: \mu > \mu_0 \\ \Phi(z_0), & H_1: \mu < \mu_0. \end{cases}$$