

TA section 3

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Review: Completeness

完備性

• Definition

設 $T := T(\mathbf{X})$ 為一統計量, T 之 pdf 為 $f(t; \theta)$, $\theta \in \Omega$. 當 T 為一完備統計量, 對任一函數 g , 若

$$\mathbb{E}[g(T)|\theta] = 0, \quad \forall \theta \in \Omega,$$

則

$$\mathbb{P}(g(T) = 0) = 1, \quad \forall \theta \in \Omega,$$

i.e., $g(T) = 0$ almost surely.

定理 3.2

• Theorem

令 X_1, \dots, X_n 為一組由 k 個參數之指數族分佈所產生之隨機樣本, 其 pdf 可表示成:

$$f(x; \theta) = h(x)c(\theta) \exp \left(\sum_{j=1}^k w_j(\theta) t_j(x) \right),$$

其中 $C := \{w_1(\theta), \dots, w_k(\theta)\} \subset \mathbb{R}^k$ 其值域包含一非空開矩形 (nonempty open set in \mathbb{R}^k), 則統計量 $T(\mathbf{X}) = (\sum_{i=1}^n t_1(X_i), \dots, \sum_{i=1}^n t_k(X_i))$ 為一完備充份統計量。

定理 3.1

- Theorem

若 $T(\mathbf{X})$ 為一完備、充份統計量 (C.S.S.), 則 $T(\mathbf{X})$ 與每一個輔助統計量 (A.S.) 獨立。

Example

18. 設 X_1, \dots, X_n 為一組由 $\mathcal{U}(\theta, 2\theta)$ 分佈所產生之隨機樣本, $\theta > 0$ 。試求 θ 之一最小充分統計量, 並問此統計量是否具有完備性。

- 證 $T = (X_{(1)}, X_{(n)})$ is not complete.
- 利用定義找反例。Consider $g(T) = R - \mathbb{E}_\theta[R]$, with $R = X_{(n)} - X_{(1)}$ which is an A.S.
- $\mathbb{E}_\theta[g(T)] = \mathbb{E}_\theta[R] - \mathbb{E}_\theta[R] = 0$, but $\mathbb{P}(g(T) \neq 0) > 0$. So, T is not complete.
- 或可直接利用定理 3.1, R is not independent of T , 得到 T is not complete.

- 另解: 因為

$$f(x|\theta) = \frac{1}{\theta} \mathbf{I}(\theta < x < 2\theta) = h(x/\theta)/\theta,$$

where $h(x/\theta) = \mathbf{I}(1 < x/\theta < 2)$, which is free of θ . So, $\mathbf{X} \in$ scale family with the scale parameter θ . 可得知 $U := X_{(n)}/X_{(1)}$ is scale invariant, and is thus an A.S.

[proof] Let $X_i = \theta Z_i \Rightarrow X_{(n)} = \theta Z_{(n)}$ and $X_{(1)} = \theta Z_{(1)}$, where $Z \sim f(z)$ which is free of θ . So, $\mathbb{P}(U(\mathbf{X}) \leq u) = \mathbb{P}(X_{(n)}/X_{(1)} \leq u) = \mathbb{P}(Z_{(n)}/Z_{(1)} \leq u) = \mathbb{P}(U(\mathbf{Z}) \leq u)$, which is free of θ . \square

- 令 $g(T) := U - \mathbb{E}_\theta[U]$, 其分佈也與 θ 無關。則 $\mathbb{E}_\theta[g(T)] = 0$ but $\mathbb{P}(g(T) \neq 0) > 0$. Thus, T is not complete.
- 或利用定理 3.1, U is not independent of T , 得到 T is not complete.

注意

- 定理 3.2 只能來找完備充份統計量, 不能用來證不完備性。(以往很多同學常犯的錯誤)
- $P \rightarrow Q \iff \neg Q \rightarrow \neg P$. (若 P 則 $Q \iff$ 若非 Q 則非 P)
- but NOT: $\neg P \rightarrow \neg Q$.
- 換言之, 即使不滿足定理 3.2 的條件 (不包含 k 維開矩形), 則可能有、也可能沒有完備性。

Example: (課本例 3.12)

- $X_1, \dots, X_n \sim i.i.d. \mathcal{N}(\theta, \theta^2)$. 可知 $T := (T_1, T_2) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ 是 θ 之一最小充份統計量。
- (curved exponential family) $C := (w_1(\theta), w_2(\theta)) = (1/\theta, -1/(2\theta^2))$, 其圖形為二維平面上的曲線, 未包含二維開矩形, 不能用定理 3.2.
- 依定義: 取 $g(T) = 2T_1^2 - (n+1)T_2$, 故知 $\mathbb{E}[g(T)|\theta] = 0 \ \forall \theta \in \mathbb{R}$. 但, $\mathbb{P}(g(T) \neq 0) > 0$. 即 T is not complete.

Review: Estimation

找估計量的方法

- 動差法 (method of moments estimator, MME)
- 最大概似法 (maximum likelihood estimator, MLE)
- 貝氏估計法 (bayesian estimator, BE)

估計量的優劣評估

- 不偏性
- 效率性
 - 相對效率性
 - 均方差 (MSE)
 - 最佳不偏估計量 (best unbiased estimator, BUE)
 - 一致最小變異不偏估計量 (uniformly minimum variance unbiased estimator, UMVUE)
 - [Rao-Blackwell Theorem]: (充份 + 不偏 \Rightarrow 效率)
 - [Lehmann-Scheffé (-Rao-Blackwell) Theorem]: (完備 + 充份 + 不偏 \Rightarrow 效率)
- 一致性
 - 漸近不偏性
 - 漸近效率性
 - 漸近常態性
 - 最佳漸近常態性

Homework 2 (part I)

§6.2 #2, #5, #6

2. 設 X_1, \dots, X_n 為一組由 $P(\theta)$ 分佈所產生之隨機樣本, $\theta > 0$ 。

(i) 試以動差法求兩種 θ 之估計量;

(ii) 試利用(i)給出 $P(X \neq 0)$ 之兩種動差估計量。

- Method of Moment Estimator **Idea: “Matching the moments”**.
- $\mathbb{E}[X] = \text{Var}[X] = \lambda$. Let $m_k = n^{-1} \sum_{i=1}^n X_i^k$, for $k \geq 1$.
- (1) $m_1 = \mathbb{E}[X] \Rightarrow \widehat{E}[X] = m_1$, i.e., $\widehat{\lambda}_1 = \overline{X}_n$.
- (2) matching: $m_1 = \mathbb{E}[X]$ and $m_2 = \mathbb{E}[X^2] \Rightarrow \widehat{E}[X] = m_1$ and $m_2 = \widehat{\mathbb{E}}[X^2] = \widehat{\text{Var}}[X] + \widehat{\mathbb{E}}[X]^2 = \widehat{\lambda} + m_1^2$. Thus, $\widehat{\lambda}_2 = m_2 - m_1^2$.
- $\mathbb{P}(\widehat{X} \neq 0) = 1 - \exp(-\widehat{\lambda})$, with plugging in (1) and (2).

5. 設 X_1, \dots, X_n 為一組由 $U(-\theta, \theta)$ 分佈所產生之隨機樣本, $\theta > 0$ 。試求 θ 之最低次的動差估計量。

- $m_1 = \mathbb{E} X = 0$ and $m_2 = \mathbb{E} X^2 = \theta^2/3$.

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$$\hat{\theta}^2 = \frac{3}{n} \sum_{i=1}^n X_i^2,$$

or

$$\hat{\theta} = + \sqrt{\frac{3}{n} \sum_{i=1}^n X_i^2} \quad (\because \theta > 0).$$

6. 設 X_1, \dots, X_n 為一組由 $Be(\alpha, \beta)$ 分佈所產生之隨機樣本, $\alpha, \beta > 0$ 。
試求 α, β 之動差估計量。

- matching: $m_1 = \mathbb{E}[X] = \alpha/(\alpha + \beta)$ and $m_2 = \mathbb{E}[X^2] = \alpha(\alpha + 1)/(\alpha + \beta)(\alpha + \beta + 1)$.

- first note that

$$\alpha = \frac{m_1 \beta}{(1 - m_1)} \quad (1)$$

and

$$\frac{m_1}{m_2} = \frac{\alpha + \beta + 1}{\alpha + 1}. \quad (2)$$

- Thus, by (1) & (2), solve for

$$\hat{\alpha} = \frac{m_1(m_1 - m_2)}{m_2 - m_1^2}$$

and

$$\hat{\beta} = \frac{(1 - m_1)(m_1 - m_2)}{m_2 - m_1^2}.$$

§6.3 #6, #15, #21, #31, #35

6. 設 X_1, \dots, X_n 為一組由對數常態分佈所產生之隨機樣本, 即設 $\log X$ 有 $\mathcal{N}(\mu, \sigma^2)$ 分佈, $\mu \in R, \sigma > 0$ 。試求 μ 及 σ^2 之 MLE。

- Maximum Likelihood Estimator **Idea: "Maximizing the likelihoods (probs.)"**
- (Lognormal distribution) let $\theta := (\mu, \sigma^2)$.

$$f_X(x|\theta) = (2\pi\sigma^2)^{-1/2} x^{-1} \exp\left(-(\log(x) - \mu)^2/2\sigma^2\right).$$

- log-likelihood function:
 $\log L(\theta) = -\frac{n}{2} \log(2\pi\sigma^2) - \sum_{i=1}^n \log x_i - \sum_{i=1}^n (\log x_i - \mu)^2/2\sigma^2.$
- F.O.C.: $\partial \log L / \partial \mu \triangleq 0$, and $\partial \log L / \partial \sigma^2 \triangleq 0$. Solve for

$$\hat{\mu} = \sum_{i=1}^n \log x_i / n$$

and

$$\hat{\sigma}^2 = \sum_{i=1}^n (\log x_i - \hat{\mu})^2 / n.$$

- S.O.C.: Hessian matrix

$$\begin{aligned}
 H(\hat{\theta}) &:= \begin{pmatrix} \partial^2 \log L / \partial \mu^2 & \partial^2 \log L / \partial \mu \partial \sigma^2 \\ \partial^2 \log L / \partial \sigma^2 \partial \mu & \partial^2 \log L / \partial (\sigma^2)^2 \end{pmatrix} \big|_{\mu=\hat{\mu}, \sigma^2=\hat{\sigma}^2} \\
 &= \begin{pmatrix} -n/\hat{\sigma}^2 & 0 \\ 0 & -\sum_{i=1}^n (\log x_i - \hat{\mu})^2 / 2(\hat{\sigma}^2)^3 \end{pmatrix}
 \end{aligned}$$

is negative-definite ($H(1, 1) < 0, |H| > 0$), providing a strictly local maximum.

15. 設 X_1, X_2 為由 $\mathcal{N}(\mu, \sigma^2)$ 分佈所產生之隨機樣本, 其中 μ 為已知, $\sigma^2 > 0$ 。若只記錄 $Y = X_1 - X_2$, 試求 σ^2 之 MLE。

- (two-points sample) $Y \sim \mathcal{N}(0, 2\sigma^2)$.

$$f_Y(y|\sigma^2) = (4\pi\sigma^2)^{-1/2} \exp(-y^2/4\sigma^2).$$

- $\log L(\sigma^2) = -\frac{1}{2} \log(4\pi\sigma^2) - y^2/4\sigma^2$.
- F.O.C.:

$$d \log L / d\sigma^2 = -1/2\sigma^2 + y^2/4\sigma^4 \stackrel{\Delta}{=} 0 \Rightarrow \hat{\sigma}^2 = y^2/2 = (x_1 - x_2)^2/2.$$

- S.O.C.:

$$d^2 \log L / d(\sigma^2)^2|_{\sigma^2=\hat{\sigma}^2} = 1/2\sigma^4 - y^2/2(\sigma^2)^3|_{\sigma^2=\hat{\sigma}^2} = -2/y^4 < 0.$$

21. 設 X_1, \dots, X_n 為一組由 p.d.f. $f(x|\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, $0 \leq \theta \leq 1/2$, 所產生之隨機樣本。試分別求 θ 之動差估計量及 MLE。

- Matching: $m_1 = \mathbb{E}[X] = \theta$. So, $\hat{\theta}_{MME} = m_1 = \bar{X}_n$.
- log-likelihood function:
 $\log L(\theta) = \sum_{i=1}^n x_i \log \theta + (n - \sum_{i=1}^n x_i) \log(1 - \theta)$, $0 \leq \theta \leq 1/2$. Then, F.O.C.:

$$d \log L / d\theta = \sum_{i=1}^n x_i / \theta - (n - \sum_{i=1}^n x_i) / (1 - \theta) \stackrel{\Delta}{=} 0, \quad 0 \leq \theta \leq 1/2,$$

so

$$\hat{\theta}_{MLE} = \bar{X}_n \mathbf{I}(0 \leq \bar{X}_n \leq 1/2) = (\bar{X}_n \wedge 1/2),$$

or $\min\{\bar{X}_n, 1/2\}$, i.e., when $\bar{X}_n \leq 1/2$, $\hat{\theta}_{MLE} = \bar{X}_n$; when $\bar{X}_n > 1/2$, $\hat{\theta}_{MLE} = 1/2$.

- S.O.C.: $\partial^2 \log L / \partial \theta^2|_{\theta=\hat{\theta}} < 0$.

31. 設 X_1, \dots, X_n 為一組由 $\mathcal{NB}(r, \theta)$ 分佈所產生之隨機樣本, 其中 $r \geq 1$ 為已知, $0 < \theta < 1$ 。試證 θ 之 MLE 為 $\hat{\theta} = nr / (nr + \sum_{i=1}^n X_i)$ 。

- log-likelihood function:

$$\log L(\theta) = \sum_{i=1}^n \log \binom{x_i + r - 1}{x_i} + nr \log \theta + \sum_{i=1}^n x_i \log(1 - \theta).$$

- F.O.C.:

$$d \log L / d\theta = \frac{nr}{\theta} - \frac{\sum_{i=1}^n x_i}{1 - \theta} \triangleq 0,$$

solve for

$$\hat{\theta} = \frac{nr}{nr + \sum_{i=1}^n x_i}.$$

- S.O.C.: $\partial^2 \log L / \partial \theta^2|_{\theta=\hat{\theta}} < 0$.

35. 設 X_1, \dots, X_n 為一組由 p.d.f. $f(x|\theta) = 2x/\theta^2, 0 \leq x \leq \theta$, 所產生之隨機樣本, $\theta > 0$ 。試求 X 之中位數的 MLE, 並證明此估計量為一最小充分統計量。

- $L(\theta) = 2^n \theta^{-2n} \prod_{i=1}^n x_i \mathbf{I}(0 < x_i < \theta), i = 1, 2, \dots, n.$
- log-likelihood function:
 $\log L(\theta) = n \log 2 - 2n \log \theta + \sum_{i=1}^n \log x_i, 0 < x_i < \theta.$
- F.O.C.: $d \log L / d\theta = -2n/\theta < 0$ for all $\theta > x_{(n)}.$
- So, $\hat{\theta}_{MLE} = x_{(n)}.$
- the median $M(\theta)$:

$$1/2 = \mathbb{P}(X \leq M(\theta)) = \int_0^{M(\theta)} \frac{2x}{\theta^2} dx = M(\theta)^2 / \theta^2.$$

So,

$$\widehat{M}(\theta)_{MLE} = M(\hat{\theta}_{MLE}) = \hat{\theta}_{MLE} / \sqrt{2} = x_{(n)} / \sqrt{2},$$

by the **invariance principle**.



$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{2^n \theta^{-2n} \prod_{i=1}^n x_i \mathbf{I}(x_{(n)} < \theta)}{2^n \theta^{-2n} \prod_{i=1}^n y_i \mathbf{I}(y_{(n)} < \theta)},$$

which is free of θ iff $x_{(n)} = y_{(n)}$.

- 令 $T(\mathbf{x}) := x_{(n)}$ 且 $U(\mathbf{x}) := x_{(n)}/\sqrt{2}$ 是 $T(\mathbf{x})$ 之一等價統計量。Thus, $U(\mathbf{X}) = X_{(n)}/\sqrt{2}$ is also a M.S.S. for θ .

§6.4 #1, #3

1. 設 X_1, \dots, X_n 為一組由 $\mathcal{N}(\theta, \sigma^2)$ 分佈所產生之隨機樣本, $\sigma > 0$ 為已知。又設 θ 有 $\mathcal{D}\text{-}\mathcal{E}(a)$ 分佈, 其中 $a > 0$ 為一常數。試求

(i) 在給定 $\mathbf{X} = \mathbf{x}$ 之下, θ 之事後分佈;

(ii) θ 之貝氏估計量。

- Bayes Estimator in mean squares risk: “Finding the posterior mean” $\mathbb{E}[\theta|\mathbf{x}]$, where the posterior pdf:

$$\pi(\theta|\mathbf{x}) = \frac{f(\mathbf{x}|\theta)\pi(\theta)}{m(\mathbf{x})},$$

$\pi(\theta)$ is a prior distribution, and $m(\mathbf{x})$ is the marginal pdf of \mathbf{X} .

$$\theta \sim \pi(\theta) = (a/2)e^{-a|\theta|}$$

$$\begin{aligned} \Rightarrow f(\mathbf{x}, \theta) &= f(\mathbf{x}|\theta)\pi(\theta) = (2\pi\sigma^2)^{-n/2} \exp\left(-\sum_{i=1}^n (x_i - \theta)^2/2\sigma^2\right) \times (a/2)e^{-a|\theta|} \\ &\propto \exp\left(-\sum_{i=1}^n (x_i - \theta)^2/2\sigma^2 - a|\theta|\right) \\ &\propto \exp\left(-\frac{n\theta^2}{2\sigma^2} + \frac{n\bar{x}_n\theta}{\sigma^2} - a|\theta|\right) \\ &\propto \exp\left(-\frac{n}{2\sigma^2}(\theta - \xi_{\pm}(\mathbf{x}))^2\right), \text{ where } \xi_{\pm}(\mathbf{x}) := \bar{x}_n \pm \sigma^2 a/n. \end{aligned}$$

$$\bullet \quad m(\mathbf{x}) = \int_{\mathbb{R}} \exp\left(-\frac{n}{2\sigma^2}(\theta - \xi_{\pm}(\mathbf{x}))^2\right) d\theta \text{ and } \pi(\theta|\mathbf{x}) = f(\mathbf{x}, \theta)/m(\mathbf{x}).$$

$$\begin{aligned} m(\mathbf{x}) &= \int_{\mathbb{R}} \exp\left(-\frac{n}{2\sigma^2}(\theta - \xi_{\pm}(\mathbf{x}))^2\right) d\theta = \int_0^{\infty} \exp\left(-\frac{n}{2\sigma^2}(\theta - \xi_{\pm}(\mathbf{x}))^2\right) d\theta \\ &\quad + \int_{-\infty}^0 \exp\left(-\frac{n}{2\sigma^2}(\theta - \xi_{\pm}(\mathbf{x}))^2\right) d\theta = \sqrt{\frac{2\pi\sigma^2}{n}}. \end{aligned}$$

- Thus,

$$\pi(\theta|\mathbf{x}) = \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n}{2\sigma^2}(\theta - \xi_{\pm}(\mathbf{x}))^2\right).$$

i.e., $\theta|\mathbf{x} \sim \mathcal{N}(\xi_{\pm}(\mathbf{x}), \sigma^2/n)$.

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$$\begin{aligned}\hat{\theta}_{BE} &= \mathbb{E}[\theta|\mathbf{x}] = \mathbb{P}(\theta > 0) \mathbb{E}[\theta|\mathbf{x}, \theta \geq 0] + \mathbb{P}(\theta < 0) \mathbb{E}[\theta|\mathbf{x}, \theta < 0] \\ &= 0.5\xi_{-}(\mathbf{x}) + 0.5\xi_{+}(\mathbf{x}). \quad \square\end{aligned}$$

3. 設 X 有 $\mathcal{U}(0, \theta)$ 分佈, θ 之事前分佈為 $\mathcal{E}(1)$ 。試求

(i) 在給定 $X = x$ 之下, θ 之事後分佈;

(ii) θ 之貝氏估計量。

- $X|\theta \sim f(x|\theta) = \theta^{-1} \mathbf{I}(0 < x < \theta).$

- 題目有誤: $\theta \sim \pi(\theta) = \theta e^{-\theta}, \theta > 0.$

則

$$m(x) = \int_x^\infty e^{-\theta} d\theta = e^{-x} \Rightarrow \pi(\theta|x) = e^{x-\theta} \mathbf{I}(0 < x < \theta).$$

故, $\hat{\theta}_{BE} = \mathbb{E}[\theta|x] = \int_x^\infty \theta e^{x-\theta} d\theta = x + 1.$