# Improved Out-of-sample Equity Premium Forecasts with Many Mixed-root Predictors

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# Contents

# Prologue

# Elusive predictability of equity premium return ?

- **Predictability** Early Evidence emerged that stock returns are in-sample predictable even the weak predictability is revealed, e.g., Campbell and Shiller (1988), Fama and French (1988, 1989), Cochrane (2008), and Timmermann (2008).
- **Unpredictability** Goyal and Welch (2008) and Goyal et al. (2024) demonstrate that the historical mean benchmark is hard to beat in out-of-sample forecasting.
- Many works consider various predictors and methods to evidence the equity premium return forecasting. For examples, **(economic constraints)** Campbell and Thompson (2008), **(forecast combination)** Rapach, et al. (2010), **(option data)** Alexandridis et al. (2023), **(text data)** Lima and Godeiro (2022),

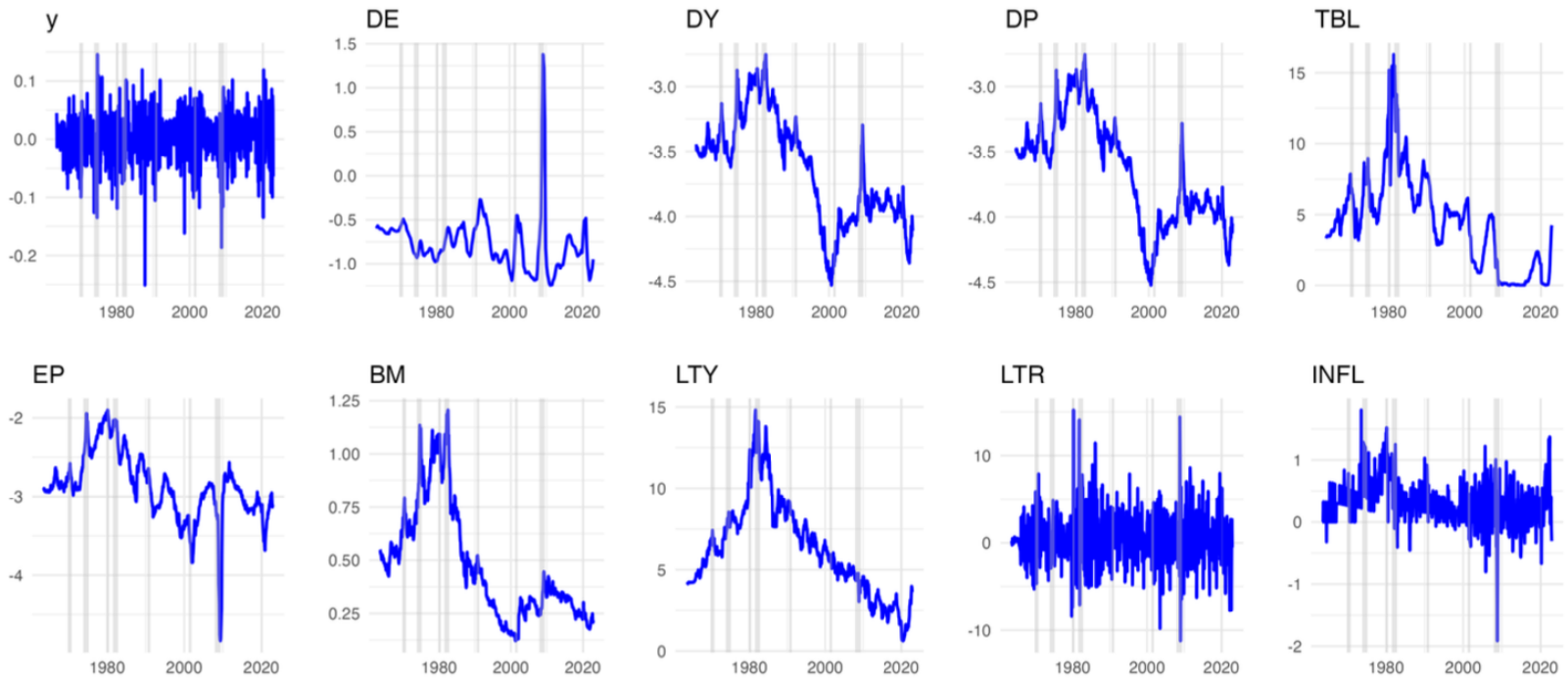
# A fundamental fact about financial predictors:

- Many financial predictors are mixed-root, with nearly non-stationary $I(1)$, but not purely $I(1)$.
- But, the equity premium return process is weakly stationary $I(0)$.
- This results in an unbalance of integration order in predictive regression

$$r_{t+1} = \alpha + \beta X_t + u_{t+1},$$

  forcing the true value of $\beta$ to be zero, or very small if not zero (Phillips, 2015). The true predictability may be hidden.
- In large-dimensional scenarios with many predictors, this may lead to **inconsistent variable selection** with the conventional model selection techniques, e.g., BIC, LOOCV, LASSO since incorrect zero coefficients may be excluded.

| Var. | Mean | S.D. | P25 | P50 | P75 | Sk. | Kur. | ADF.pv | PP.pv | $AC_{(1)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $r(\%)$ | 0.20 | 4.37 | -2.14 | 0.64 | 3.03 | -0.67 | 5.29 | 0.01 | 0.01 | 0.02 |
| DE | -0.77 | 0.31 | -0.94 | -0.81 | -0.62 | 2.93 | 9.48 | 0.01 | 0.01 | 0.99 |
| DY | -3.64 | 0.41 | -3.96 | -3.56 | -3.35 | 0.01 | 2.10 | 0.38 | 0.41 | 0.99 |
| DP | -3.64 | 0.44 | -3.97 | -3.57 | -3.36 | 0.02 | 2.08 | 0.35 | 0.42 | 0.99 |
| TBL(%) | 4.46 | 3.26 | 1.69 | 4.64 | 6.13 | 0.66 | 3.63 | 0.05 | 0.20 | 0.99 |
| EP | -2.87 | 0.43 | -3.11 | -2.90 | -2.67 | -0.51 | 5.43 | 0.01 | 0.02 | 0.99 |
| BM | 0.47 | 0.26 | 0.27 | 0.39 | 0.64 | 0.90 | 2.83 | 0.48 | 0.52 | 0.99 |
| LTY(%) | 6.22 | 2.89 | 4.23 | 5.99 | 8.02 | 0.51 | 2.98 | 0.44 | 0.43 | 0.99 |
| LTR(%) | 0.56 | 3.02 | -1.22 | 0.38 | 2.31 | 0.37 | 5.24 | 0.01 | 0.01 | 0.06 |
| INFL(%) | 0.32 | 0.36 | 0.10 | 0.30 | 0.52 | 0.01 | 5.85 | 0.01 | 0.01 | 0.58 |
| DFY(%) | 1.02 | 0.44 | 0.73 | 0.91 | 1.20 | 1.74 | 7.45 | 0.01 | 0.01 | 0.96 |
| DFR(%) | -5.62 | 3.90 | -7.77 | -5.54 | -3.36 | -0.28 | 4.86 | 0.01 | 0.01 | 0.53 |
| NTIS | 0.01 | 0.02 | -0.01 | 0.01 | 0.02 | -0.48 | 2.91 | 0.01 | 0.02 | 0.98 |
| TMS | 0.02 | 0.01 | 0.01 | 0.02 | 0.03 | -0.24 | 2.70 | 0.04 | 0.01 | 0.96 |
| RVOL | 0.14 | 0.06 | 0.10 | 0.13 | 0.18 | 0.46 | 3.47 | 0.01 | 0.01 | 0.98 |


y
DE
DY
DP
TBL
EP
BM
LTY
LTR
INFL
1980
2000
2020

- Why do we not take the first differences of the predictors?
  "*In empirical finance, many financial predictors or factors are proxied and constructed through somewhat linear or nonlinear transformations based on specific financial theories and models, e.g., dividend-price ratio (Gordon growth theory), Tobin Q ratio (Tobin's Q theory), value premium (HML) (3-factor model), momentum (4-factor model),..., etc.*
  *As such, we* **avoid using differencing filters**, *as they can result in the loss of the original information contained in these variables*" (suggested by prof. Ethan Chiang (UNC)).

- Chen (2024) shows that such unbalance of integration orders has a **negative** effect on out-of-sample forecasts in terms of MSFEs.
  - $r_t$ is weakly stationary $I(0)$, and $X_t$ is nearly unit-root or purely $I(1)$,

$$r_{t+1} = \alpha + \beta X_t + u_{t+1}, \tag{1}$$

$$X_{t+1} = (1 - c/T)X_t + v_{t+1},$$

    with $\beta = bT^{-\eta}$ and $c \geq 0$,
  - $(u_t, v_t)^\top \sim$ white noise $\mathcal{N}(0, \Sigma)$,

$$\Sigma := \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{vu} & \sigma_v^2 \end{bmatrix},$$

    and $u_{t+1} = (\sigma_{uv}/\sigma_v^2)v_{t+1} + \epsilon_{t+1}$, with $\mathbb{E}[\epsilon_{t+1}|v_{t+1}] = 0$.

# Illustration (Chen, 2024)

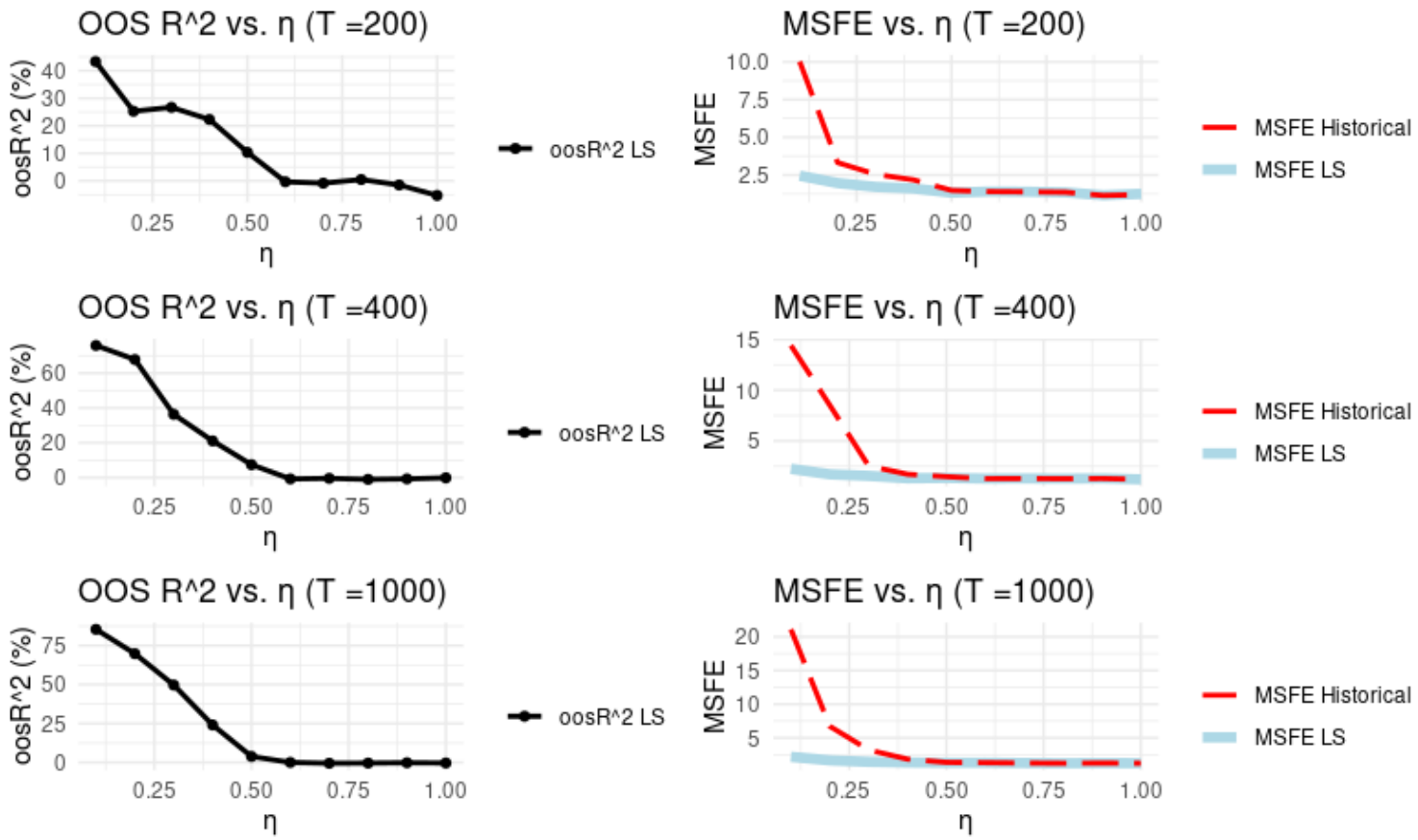



Figure: The out-of-sample $R^2$ of recursive LS estimates, and the corresponding MSFEs of recursive LS estimates and historical mean benchmark, respectively. The DGP is set up by: $r_{t+1} = 0.3 + \beta_T X_t + u_{t+1}$, $X_{t+1} = (1 - 1.1/T)X_t + v_{t+1}$, where $(u_t, v_t)^\top \sim$ white noise $\mathcal{N}(0, \Sigma)$, with $\Sigma := (1, -0.95; -0.95, 1)$. The true value $\beta_T = T^{-\eta}$. $T = \{200, 400, 1000\}$ and the number of replication is 1000. The recursive-window LS estimation with the first rolling window size is 180. The closer the value of $\beta_T$ is to zero (fixed $T$), the more the behavior of $r_t$ resembles the white noise $u_t$, which follows a weakly stationary $I(0)$.

## • Proposition (Chen, 2024)

- $\widehat{\beta}_{LS} = O_p(T^{-\eta}) = o_p(1)$.
- $T\widehat{\beta}_{LS} \Rightarrow \frac{\sigma_{uv}^2}{\sigma_v^2} \frac{\int_0^1 J_v^c(s) dB_v(s)}{\int_0^1 J_v^c(s)^2} + \frac{\sigma_\epsilon \int_0^1 J_v^c(s) dB_\epsilon(s)}{\sigma_v \int_0^1 J_v^c(s)^2}$, where $J_v^c(t)$ is a Ornstein–Uhlenbeck process with respect to $B_v(t)$, and $B_\xi(.)$ is a standard Brownian motion with respect to the stochastic process $\xi$.
- $t(\beta) \Rightarrow \sqrt{(1-\gamma_{uv}^2)} Z + \gamma_{uv} \frac{\int_0^1 J_v^c(s) dB_v(s)}{\sqrt{\int_0^1 J_v^c(s)^2 ds}}$, where $\gamma_{uv} := \sigma_{uv}/\sigma_u\sigma_v$ and $Z$ is a standard Gaussian.
- $\text{MSFE}(\bar{r}_t) = \sigma_u^2 + \frac{1}{T}\sigma_u^2 + O(T^{1-2\eta})$,
  $\text{MSFE}(\widehat{r}_{t+1|t}^{LS}) = \sigma_u^2 + \frac{K(c)}{T}\sigma_u^2 + \frac{\sigma_{uv}^2}{\sigma_v^2 T} \frac{4(1-e^{-2c})}{2c} + O(T^{-2\eta})$, where $K(c) \geq 1$, $\forall c > 0$, $\bar{Y}_t = t^{-1} \sum_{s=1}^t Y_s$ be the historical sample average, and $\widehat{Y}_{t+1|t}^{LS} = \widehat{\beta}_{LS} X_t$.
-

$$\text{MSFE}(\bar{r}_T) \leq \text{MSFE}(\widehat{r}_{T+1|T}^{LS}), \quad \eta \geq 1/2$$
$$\text{MSFE}(\bar{r}_T) > \text{MSFE}(\widehat{r}_{T+1|T}^{LS}), \quad \eta < 1/2.$$

# What this talk tries to do

- Chen, Chen, and Vincent (2024) argue that this may be induced from a problem of omitted variables, and propose a new robust model framework that incorporates highly persistent predictors with approximately co-integrations regardless of various levels of high persistence.
- A disadvantage of Chen, Chen, and Vincent (2024) is that the dimension of predictors $k$ cannot be too large, particularly $k < T/2$.
- In this project, I extend the work of Chen et al. (2024) that a **c**ontinuously-updating **a**daptive **L**ASSO **m**echanism (CALM) is proposed to select the effective predictors for the model of Chen et al. (2024) when the number of mixed-root predictors is sufficiently large. From empirical evidence, the out-of-sample equity premium forecasts can be significantly improved.
- The idea is based on the FWL (Frisch–Waugh–Lovell) theorem and the adaptive LASSO (Zou, 2006).

# Model Framework

# DGP

- Let $r_{t+1}$ be the equity premium return at $t+1$, $t. = 1, 2, \cdots, T$, and denote the potential predictors $\boldsymbol{X}_t := (\boldsymbol{X}_{1t}, \boldsymbol{X}_{2t}, \cdots, \boldsymbol{X}_{kt})^\top \in \mathcal{X} \subseteq \mathbb{R}^k$, $k < T/2$, $\mathcal{X} := \mathcal{X}^U \uplus \mathcal{X}^N$, $\mathcal{X}^U \cap \mathcal{X}^N = \phi$, where $\mathcal{X}^U$ is the set of $m$ $I(1)$ predictors, and $\mathcal{X}^N$ is the set of non-$I(1)$ predictors, including those nearly $I(1)$ predictors and weakly stationary $I(0)$ predictors.
- DGP:

$$\begin{pmatrix} r_{t+1} \\ \boldsymbol{X}_{t+1} \end{pmatrix} = \begin{pmatrix} \mu_o \\ \boldsymbol{\rho}_o \end{pmatrix} + \begin{pmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \\ \boldsymbol{\rho} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_t \\ \boldsymbol{X}_{t-1} \end{pmatrix} + \begin{pmatrix} u_{t+1} \\ \boldsymbol{\nu}_{t+1} \end{pmatrix},$$

where $(u_{t+1}, \boldsymbol{\nu}_{t+1}^\top)^\top \sim$ martingale difference sequences with possibly nonzero contemporaneous correlation.

- VAR (1) coefficient matrix (e.g., Phillips, 2023):

$$\boldsymbol{\rho} = \boldsymbol{I}_k - \boldsymbol{C}/T^s,\ \boldsymbol{C} := \text{diag}(c_1, c_2, \cdots, c_k),$$

for all $c_i > 0$, and $s \geq 0$. This allows for various levels of persistence.

- VEC representation for $CI(1,1)$ among $I(1)$ predictors: $\boldsymbol{X}_t^U \in \mathcal{X}^U$,

$$\Delta \boldsymbol{X}_{t+1}^U = \boldsymbol{\Gamma} \boldsymbol{X}_t^U + \boldsymbol{\nu}_{t+1}^U,\ \boldsymbol{\Gamma} = \boldsymbol{B}\boldsymbol{A}^\top,$$

with $rank(\boldsymbol{\Gamma}) = r < m$, $\boldsymbol{B}$ is the $m \times r$ adjusting factor, representing that how predictors adjusts deviations back toward the long-run equilibrium, and $\boldsymbol{A} = (\boldsymbol{\Theta}_1, \cdots, \boldsymbol{\Theta}_r)$ is $m \times r$ co-integrating matrix with $r$ co-integrating vectors which captures the long-run equilibrium.

# Model specification

Similar to Chen et al. (2024), a constrained framework can be constructed by:

$$r_{t+1} = \boldsymbol{\mu}_o + \boldsymbol{X}_t^\top \boldsymbol{\beta}_1 + \boldsymbol{X}_{t-1}^\top \boldsymbol{\beta}_2 + e_{t+1}, \tag{2}$$

subject to two constraints:

1. (contemporaneous co-integrations)

$$\boldsymbol{R}\boldsymbol{\beta}_{1m} = \mathbf{0}_{(m-1)\times 1}, \ \boldsymbol{R} = (\widehat{\boldsymbol{\theta}}_{(m-1)\times 1}, \boldsymbol{I}_{m-1})_{(m-1)\times m}, \tag{3}$$

   where $\widehat{\boldsymbol{\Theta}} =: (1, -\widehat{\boldsymbol{\theta}}^\top)^\top$ is the $m \times 1$ normalizing estimated co-integrating vector with largest eigenvalue, among $m$ $I(1)$ predictors, which are identified by the ADF and PP tests and then selected by Johansen's standard procedure;

2. (approximately intertemporal co-integrations)

$$\boldsymbol{\beta}_2 + \widehat{\boldsymbol{\rho}}^\top \boldsymbol{\beta}_1 = \mathbf{0}_{k\times 1}, \tag{4}$$

   where $\widehat{\boldsymbol{\rho}}$ is the VAR (1) estimator for $\boldsymbol{\rho}$, componentwisely.

# CALM algorithm

Given $\widehat{\alpha}_j^{(0)}$ is the initial LS or ridge estimator. At the $i$th iteration: for $i \geq 2$,

- $$(\widehat{\boldsymbol{\mu}}^{(i)^\top}, \widehat{\boldsymbol{\alpha}}^{(i)^\top})^\top = \underset{\mu^{(i)} \in \mathbb{R}, \boldsymbol{\alpha}^{(i)} \in \mathbb{R}^{k_{(i)}}}{\operatorname{argmin}} \left\{ ||r_{t+1} - \ddot{\boldsymbol{v}}_t^\top \boldsymbol{\alpha}^{(i)}||_2^2 + \lambda^{(i)} \sum_{j=1}^{k_{(i)}} \widehat{\tau}_j^{(i-1)} |\alpha_j^{(i)}| \right\},$$

  where $\ddot{\boldsymbol{v}}_t$ is the standardized residual from regressing $\boldsymbol{X}_t$ on $\boldsymbol{X}_{t-1}$, and $\widehat{\tau}_j^{(i-1)} = 1/|\widehat{\alpha}_j^{(i-1)}|$ is the adaptive penalty weight.

- Denote $\widehat{\mathcal{A}}_{(i)}^{\boldsymbol{X}} := \{X_{j,t} : \widehat{\alpha}_j^{(i)} \neq 0\}$ as the selected set of predictors at the $i$th iteration. Let $k_{(i)} := |\widehat{\mathcal{A}}_{(i)}^{\boldsymbol{X}}|$ be the cardinality of $\widehat{\mathcal{A}}_{(i)}^{\boldsymbol{X}}$ at the $i$th iteration. The iteration stops at $i = 2$ if $k_{(0)} < [T/2]$, and succeeds if $k_{(0)} \geq T/2$. When $k^* = k_{(i)} = k_{(i+1)} < [T/2]$, the iteration stops. Then, redo the estimation procedure of (2) to (4) with predictors of dimension $k^*$ .
- $\lambda^{(i)}$ is the non-negative tuning parameter selected by the stationary time-series cross validation (TSCV) procedure proposed by Racine (2000).

# Data and Predictors

- The equity premium return $r_{t+1}$: log excess return.
- The set of financial predictors: $14$ predictors, conducted from Goyal and Welch (2008).
- The set of technical indices: $14$ trading indices based on three types of trading strategies, following Neely et al. (2014).
- The set of anomaly portfolios: $516$ predictors, conducted from Chen and Zimmermann (2022).
- Overall, total number of predictors $\boldsymbol{X}_t$ is $544$.
- The sample period covers from 1963/7-2022/12, monthly frequency. The full sample length is: $T = 713$.
- Recursive-window estimation with the initial window size is $180$ (15 years).

Table: Definitions of Financial Predictors

| Predictor | Definition |
|---|---|
| DP | log dividends to price ratio of the S&P 500 index |
| DY | log dividend yield of the CRSP index |
| EP | log earnings to price ratio of the S&P 500 index |
| DE | log dividends to earnings ratio of the S&P 500 index |
| BM | book-to-market ratio of the Dow Jones Industrial Average Index |
| RVOL | square root of the sum of squared daily returns on the S&P 500 indexs |
| NTIS | aggregate net equity issues by the NYSE listed stocks |
| TBL | three-month Treasury-bill rate |
| LTY | long-term government bond yield |
| LTR | long-term government bond returns |
| TMS | long-term government bond yield minus the Treasury bill rate |
| DFY | difference between BAA and AAA-rated corporate bond yields |
| DFR | difference between corporate bond and long-term government bond returns |
| INFL | monthly rate of change of CPI |

# Empirical Evidence

# Evaluation

1. Out-of-sample $R^2$ (Campbell and Thomason, 2008):

$$oosR^2(\%) := 1 - \frac{\sum_{t=R}^{T-1}(r_{t+1} - \widehat{r}_{t+1|t})^2}{\sum_{t=R}^{T-1}(r_{t+1} - \bar{r}_{t+1|t})^2} \in (-\infty, \infty),$$

where $\widehat{r}_{t+1|t}$ and $\bar{r}_{t+1|t} = \sum_{s=1}^{t} r_s/t$ are the conditional forecasts based on monthly point forecasts of $r_{t+1}$ at time $t$ under the the alternative and the historical mean benchmark models, respectively.

2. Cumulative Differences of Sum of Forecast Errors (CDSFE):

$$CDSFE_t := \sum_{h=1}^{t}(r_{h+1} - \bar{r}_{h+1|h})^2 - \sum_{h=1}^{t}(r_{h+1} - \widehat{r}_{h+1|h})^2.$$

- The positive value of $CDSFE_t$ implies the competing model beats the historical mean benchmark in terms of lower MSFEs.
- A positive (negative) slope of $CDSFE_t$ indicates a dynamic trend of improving (worsening) forecasts.

# Main result

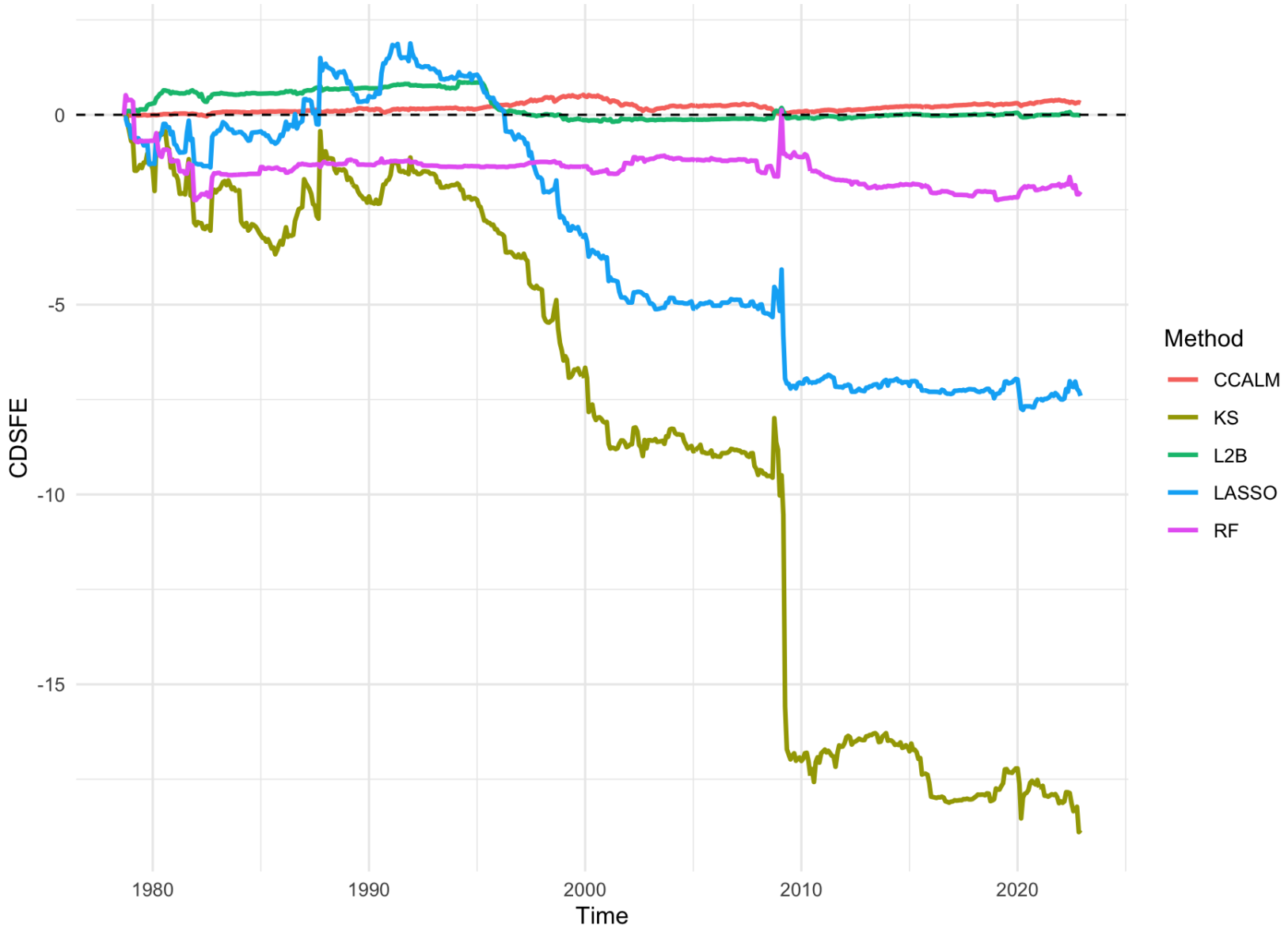
- Denote the proposed estimator as: CCALM (constrained CALM)
- Baseline model: conventional kitchen-sink (KS) model:
  $r_{t+1} = \mu_o + \boldsymbol{X}_t^\top \boldsymbol{\beta} + u_{t+1}, \ \forall \boldsymbol{X}_t \in \mathcal{X}$.
- Other competing models:
  - simple forecast combination (SFC, Rapach, et al., 2010),
  - diffusion index factor model (DI, Bai and Ng, 2008) with pre-screening by the highest in-sample adjusted $R^2$ of PCs,
  - random forests (RF, Breiman, 2001) with moving block bootstraps for 500 random trees,
  - $L_2$-boosting (L2B, Bühlmann and Yu, 2003),
  - LASSO (Tibshirani, 1996), and
  - elastic-net (EN, Zou et al., 2005).
- LASSO, EN, and CCALM utilize the time-series cross validation to select tuning parameters.

Table: Out-of-sample Performance

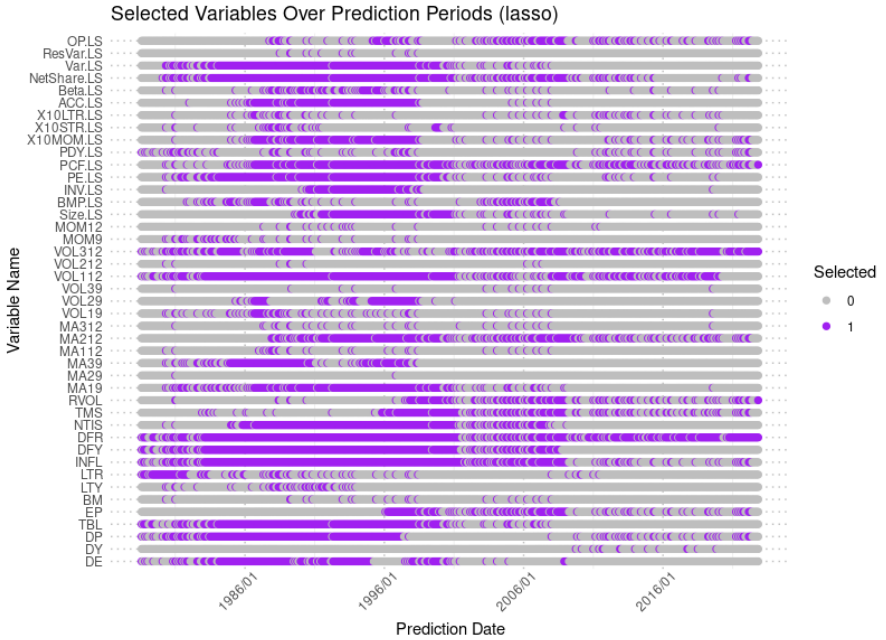
| Criteria | KS | SFC | DI | RF | L2B | LASSO | EN | CCALM |
|---|---|---|---|---|---|---|---|---|
| oos$R^2(\%)$ | -18.94 | -0.13 | -1.61 | -1.91 | 0.26 | -5.95 | -6.12 | 0.67 |
| in$\bar{R}^2(\%)$ | 0.09 | 0.11 | 0.10 | -0.09 | 0.13 | 0.09 | 0.12 | 2.65 |

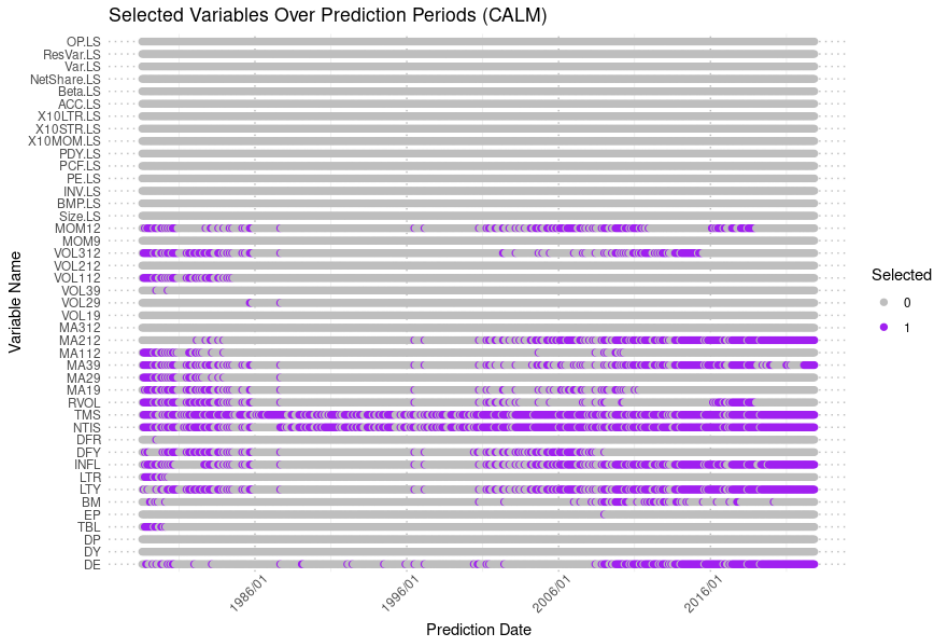
Note: oos$R^2(\%)$ denotes the out-of-sample $R^2$, and in$\bar{R}^2(\%)$ measures the in-sample adjusted $R^2$.

# CDSFEs


0
-5
-10
-15
CDSFE
1980
1990
2000
2010
2020
Time
Method
CCALM
KS
L2B
LASSO
RF

# Illustration: variable selection for the first 40 predictors



# Concluding Remarks

# Summary

- **To accurately forecast a weakly stationary $I(0)$ target variable, it is recommended to use a set of weakly stationary $I(0)$ predictors. However, if the use of (nearly) $I(1)$ non-stationary predictors becomes unavoidable, the users must proceed carefully.**
- This project proposes a new variable selector, **CALM**, to select effective predictors to examine the weakly stationary $I(0)$ equity premium forecasts when dealing with many mixed-root predictors.
- From the preliminary empirical findings, the new approach outperforms conventional methods (e.g., Kitchen-sink, LASSO) when the predictor set consists of many possibly stationary and non-stationary predictors for out-of-sample equity premium forecasts.
- The theoretical properties of the proposed estimator are yet to be established in my future work.

# THANK YOU FOR YOUR LISTENING