TA section 7

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May 14, 2024

Homework 3: part (II)

§7.3 #6, #10; §7.4 #2, #11

6. 設$X_1, \cdots, X_n$爲一組由$\mathcal{N}(\mu, \sigma^2)$分佈所產生之隨機樣本, 欲估計$\mu^2$。
試證
(i) 若σ已知, 則$T_1 = \overline{X}_n^2 - \sigma^2/n$爲UMVUE;
(ii) 若σ未知, 則$T_2 = \overline{X}_n^2 - S_n^2/n$爲UMVUE。

- Key: (1) C.S.S.; (2) (conditional) unbiasedness
- 若 σ^2 已知: let $\theta = \mu$,

$$\begin{aligned} f(x|\theta) &= (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} e^{-\mu^2/2\sigma^2} \exp\{(\mu/\sigma^2)x\} \\ &=: h(x)c(\theta)\exp\{w(\theta)t(x)\}, \end{aligned}$$

belongs to the one-dimensional exponential family, where $h(x) = (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2} \boldsymbol{I}(-\infty < x < \infty)$, $c(\theta) = e^{-\mu^2/2\sigma^2}$, $t(x) = x$, and $w(\theta) = \mu/\sigma^2$.

- $C = \{w(\theta) = \mu/\sigma^2 : \mu \in \mathbb{R}\}$ contains an open set in $\mathbb{R}$. By theorem, $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is a C.S.S. of μ, as well as $\bar{X}$ (equivalent statistics).

- 若 σ^2 未知: let $\theta = (\mu, \sigma^2)$,

$$\begin{aligned} f(x|\theta) &= (2\pi\sigma^2)^{-1/2} e^{-\mu^2/2\sigma^2} \exp\{(\mu/\sigma^2)x - x^2/(2\sigma^2)\} \\ &=: h(x)c(\theta)\exp\{w_1(\theta)t_1(x) + w_2(\theta)t_2(x)\}, \end{aligned}$$

  belongs to the two-dimensional exponential family, where $h(x) = (2\pi\sigma^2)^{-1/2}\boldsymbol{I}(-\infty < x < \infty)$, $c(\theta) = e^{-\mu^2/2\sigma^2}$, $w_1(\theta) = \mu/\sigma^2$, $w_2(\theta) = -1/\sigma^2$, $t_1(x) = x$, and $t_2(x) = x^2$.
- $C = \{(w_1(\theta), w_2(\theta)) = (\mu/\sigma^2, -1/\sigma^2) : (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+\}$, contains an open set on $\mathbb{R}^2$.
- By theorem, $T(\boldsymbol{X}) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a C.S.S. of (μ, σ^2), and so is $(\bar{X}, S_n^2)$ (equivalent statistics).

- σ^2 已知: let $T(\boldsymbol{X}) = \bar{X}$, and $S(\boldsymbol{X}) = \bar{X}^2 - \sigma^2/n$. Consider $T_1 = h(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X})$ ($\because S(\boldsymbol{X})$ is also a function of $T(\boldsymbol{X})$)
$\Rightarrow \mathbb{E}[T_1] = \mathbb{E}[\bar{X}^2 - \sigma^2/n] = \mathbb{E}[\bar{X}^2] - \sigma^2/n = \sigma^2/n + \mu^2 - \sigma^2/n = \mu^2$.
- So, T_1 is the UMVUE of μ^2 by R.-B. Theorem & L.-S. Thorem.
- σ^2 未知: let $T(\boldsymbol{X}) = (\bar{X}, S_n^2)$. Consider $T_2 = h(T(\boldsymbol{X})) = \bar{X}^2 - S_n^2/n \Rightarrow$ $\mathbb{E}[T_2] = \mathbb{E}[\bar{X}^2] - \mathbb{E}[S_n^2/n] = \sigma^2/n + \mu^2 - \sigma^2/n = \mu^2$.
- So, T_2 is the UMVUE of μ^2 by R.-B. Theorem & L.-S. Theorem.

10. 設$X_1, \cdots, X_n$爲一組由$\mathcal{P}(\lambda)$分佈所產生之隨機樣本, $\lambda > 0$。欲求$\theta = P(X_1 = 0) = e^{-\lambda}$之UMVUE。

(i) 先找出一完備充分統計量T;

(ii) 利用$I_{\{X_1=0\}}$爲θ之一不偏估計量, 以求出UMVUE;

(iii) 利用找一T的函數$h(T)$, 滿足$E(h(T)) = \theta$, 以求出UMVUE。

- (i)

$$
\begin{aligned}
f(x|\lambda) &= (x!)^{-1} e^{\lambda} \exp\{(\log \lambda) x\} \\
&=: h(x) c(\lambda) \exp\{w(\lambda) t(x)\},
\end{aligned}
$$

  belongs to the one-dimensional exponential family, where $h(x) = (x!)^{-1} \boldsymbol{I}(x = 0, 1, \cdots), c(\lambda) = \exp(\lambda), w(\lambda) = \log \lambda$, and $t(x) = x$.
- $C = \{w(\lambda) = \log \lambda : \lambda \in \mathbb{R}_+\}$ contains an open set in $\mathbb{R}$. So, $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is a C.S.S. of λ.
- (ii) Let $S(\boldsymbol{X}) = \boldsymbol{I}(X_1 = 0)$, such that $\mathbb{E}[S(\boldsymbol{X})] = e^{-\lambda} = \theta$. Then, $U(T(\boldsymbol{X})) := \mathbb{E}[S(\boldsymbol{X})|T = t]$ is an UMVUE of θ, where

- $$U(t) = \mathbb{E}[S(\boldsymbol{X})|T=t] = \frac{\mathbb{P}(X_1=0, T=t)}{\mathbb{P}(T=t)}$$
  $$= \frac{\mathbb{P}(X_1=0, \sum_{i=2}^n X_i = t-0)}{\mathbb{P}(T=t)}$$
  $$= \frac{(e^{-\lambda}\lambda^0/0!)(e^{-(n-1)\lambda}((n-1)\lambda)^{(t-0)}/(t-0)!)}{e^{-n\lambda}(n\lambda)^t/t!}$$
  $$= \left(\frac{n-1}{n}\right)^t.$$
  So, $U(\boldsymbol{X}) = [(n-1)/n]^{n\bar{X}}$ is the UMVUE of θ by R.-B. Theorem & L.-S. Theorem.
- Note: the conditional distribution $X_i|T=t \sim Bin(t, 1/n)$ (it's easy to check by yourself).
- verify:

$$\mathbb{E}[U] = \mathbb{E}[\left(\frac{n-1}{n}\right)^T] = \sum_{k=0}^{\infty}(\frac{n-1}{n})^k \frac{e^{-n\lambda}(n\lambda)^k}{k!}$$
$$= e^{-n\lambda}\sum_{k=0}^{\infty}\frac{(n\lambda(n-1)/n)^k}{k!} = e^{-n\lambda}e^{n\lambda((n-1)/n)} = e^{-\lambda} = \theta.$$

- Consider

$$U = h(T) = \left(\frac{n-1}{n}\right)^T,$$

such that $E[U] = \sum_{k=0}^{\infty} (\frac{n-1}{n})^k \frac{e^{-n\lambda}(n\lambda)^k}{k!} = e^{-n\lambda} \sum_{k=0}^{\infty} \frac{(n\lambda(n-1)/n)^k}{k!} = \theta$.
So, U is the UMVUE of θ.

2. 設X與Y獨立, 皆有$\mathcal{N}(\mu,\sigma^2)$分佈。分別對參數爲$\mu$及$\sigma^2$時, 求(i) $X+Y$, (ii) $X-Y$之資訊數。(**解.** (i) $I_{X+Y}(\mu)=2/\sigma^2$, $I_{X+Y}(\sigma^2)=1/(2\sigma^4)$, (ii) $I_{X-Y}(\mu)=0$, $I_{X-Y}(\sigma^2)=1/(2\sigma^4)$)

- $T_1 := X+Y \sim \mathcal{N}(2\mu, 2\sigma^2)$, $T_2 := X-Y \sim \mathcal{N}(0, 2\sigma^2)$.
- $\log L(\mu,\sigma^2;T_1) = -2^{-1}\log(4\pi\sigma^2) - (t_1-2\mu)^2/(4\sigma^2)$.
- $\partial \log L(\mu)/\partial\mu = (t_1-2\mu)/(\sigma^2)$;
  $\partial \log L(\sigma^2)/\partial\sigma^2 = -1/(2\sigma^2) + (t_1-2\mu)^2/(4\sigma^4)$.
- $I_{T_1}(\mu) = \mathbb{E}[(\partial \log L(\mu)/\partial\mu)^2] = \mathbb{E}[(T_1-2\mu)^2]/\sigma^4 = 2/(\sigma^2)$;
- $I_{T_1}(\sigma^2) = \mathbb{E}[(\partial \log L(\sigma^2)/\partial\sigma^2)^2] = \mathbb{E}[(-1/(2\sigma^2)+(T_1-2\mu)^2/4\sigma^4)^2] = \mathbb{E}\left[1/(4\sigma^4)+(T_1-2\mu)^4/(16\sigma^8)-(T_1-2\mu)^2/(4\sigma^6)\right] = 1/(2\sigma^4)$.
- Note: $Z\sim\mathcal{N}(\mu,\sigma_Z^2) \Rightarrow \mathbb{E}[Z-\mu]=0, \mathbb{E}[(Z-\mu)^2]=\sigma_Z^2, \mathbb{E}[(Z-\mu)^3]=0, \mathbb{E}[(Z-\mu)^4]=3\sigma_Z^4$.

- $\log L(\mu, \sigma^2; T_2) = -2^{-1}\log(2\sigma^2) - y^2/(4\sigma^2)$.
- $\partial \log L(\mu)/\partial\mu = t_2/(\sigma^2)$;
  $\partial \log L(\sigma^2)/\partial\sigma^2 = -1/(2\sigma^2) + {t_2}^2/(4\sigma^4)$.
- $I_{T_2}(\mu) = \mathbb{E}[(\partial \log L(\mu)/\partial\mu)^2] = 0$;
  $I_{T_2}(\sigma^2) = \mathbb{E}[(\partial \log L(\sigma^2)/\partial\sigma^2)^2] = \mathbb{E}\left[(-1/(2\sigma^2) + T_2^2/(4\sigma^4))^2\right] =$
  $\mathbb{E}\left[1/(4\sigma^4) + T_2^4/(16\sigma^8) - T_2^2/(4\sigma^6)\right] = 1/(2\sigma^4)$.

11. 設$X_1, \cdots, X_n$爲一組由p.d.f.$f_i(x|\theta)$所產生之隨機樣本, $i = 1, 2$, 其中

(i) $f_1(x|\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0$;

(ii) $f_2(x|\theta) = \theta^x \log\theta/(\theta-1), 0 < x < 1, \theta > 1$。

試分別對此二p.d.f., 討論是否有一θ之函數$q(\theta)$, 使得存在一$q(\theta)$之不偏估計量, 且其變異數達到CRLB。若有則找出來, 若沒有亦證明之。

• Theorem (Efficiency Attainment)

假設 $\boldsymbol{X} = (X_1, \cdots, X_n)$ 有一 joint pdf $f(\boldsymbol{x}|\theta)$, 與其對應之 likelihood function $L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i|\theta)$, 則對 $q(\theta)$ 的任一不偏估計量 $U(\boldsymbol{X})$ 之變異數可達到 CRLB, 若且唯若存在一個 (θ, n) 的函數 $a(\theta, n)$ 使得以下等式成立:

$$a(\theta, n)\Big[U(\boldsymbol{X}) - q(\theta)\Big] = \frac{\partial}{\partial\theta}\log L(\theta|\boldsymbol{x}).$$

- (i)

$$\begin{aligned}
\partial \log L/\partial\theta &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}[\log\theta + (\theta-1)\log x_i] \\
&= \sum_{i=1}^{n}\frac{\partial}{\partial\theta}[\log\theta + (\theta-1)\log x_i] \quad \text{(why)} \\
&= \sum_{i=1}^{n}\left[\theta^{-1} + \log x_i\right] \\
&= -n\Big[-\sum_{i=1}^{n}\log x_i/n - \frac{1}{\theta}\Big] = a(\theta,n)\Big[U(\boldsymbol{X}) - q(\theta)\Big].
\end{aligned}$$

- Then, $a(\theta,n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n}\log X_i/n$ is the UMVUE of $1/\theta$, which attains the CRLB.

- (i)

$$\begin{aligned}
\partial \log L/\partial\theta &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}[\log\theta + (\theta-1)\log x_i] \\
&= \sum_{i=1}^{n}\frac{\partial}{\partial\theta}[\log\theta + (\theta-1)\log x_i] \quad \textcolor{red}{\text{(why)}} \\
&= \sum_{i=1}^{n}\left[\theta^{-1} + \log x_i\right] \\
&= -n\Big[-\sum_{i=1}^{n}\log x_i/n - \frac{1}{\theta}\Big] = a(\theta,n)\Big[U(\boldsymbol{X}) - \textcolor{red}{q(\theta)}\Big].
\end{aligned}$$

- Then, $a(\theta,n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n}\log X_i/n$ is the UMVUE of $1/\theta$, which attains the CRLB.
- Note: $\nexists\ a(.)$ such that $\partial\log L/\partial\theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.

- (i)

$$\begin{aligned}
\partial \log L/\partial\theta &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}[\log\theta + (\theta-1)\log x_i] \\
&= \sum_{i=1}^{n}\frac{\partial}{\partial\theta}[\log\theta + (\theta-1)\log x_i] \quad \text{(why)} \\
&= \sum_{i=1}^{n}\left[\theta^{-1} + \log x_i\right] \\
&= -n\left[-\sum_{i=1}^{n}\log x_i/n - \frac{1}{\theta}\right] = a(\theta,n)\Big[U(\boldsymbol{X}) - q(\theta)\Big].
\end{aligned}$$

- Then, $a(\theta,n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n}\log X_i/n$ is the UMVUE of $1/\theta$, which attains the CRLB.
- Note: $\nexists\ a(.)$ such that $\partial\log L/\partial\theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.
- Note:「有限項部份和」與微分 (或極限) 可以直接交換符號。

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&= \sum_{i=1}^{n}\left[\theta^{-1} + \log x_i\right] \\
&= -n\left[-\sum_{i=1}^{n}\log x_i/n - \frac{1}{\theta}\right] = a(\theta,n)\left[U(\boldsymbol{X}) - q(\theta)\right].
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- Then, $a(\theta,n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n}\log X_i/n$ is the UMVUE of $1/\theta$, which attains the CRLB.
- Note: $\nexists\ a(.)$ such that $\partial\log L/\partial\theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.
- Note:「有限項部份和」與微分 (或極限) 可以直接交換符號。
- Note:「無窮項和」與微分 (或極限) 若可交換符號, 當此級數是均勻收斂 (uniform convergence) 或 控制收斂 (dominated (bounded) convergence)。

- (i)

$$\begin{aligned}
\partial \log L/\partial\theta &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}[\log\theta + (\theta-1)\log x_i] \\
&= \sum_{i=1}^{n}\frac{\partial}{\partial\theta}[\log\theta + (\theta-1)\log x_i] \quad \text{(why)} \\
&= \sum_{i=1}^{n}\left[\theta^{-1} + \log x_i\right] \\
&= -n\left[-\sum_{i=1}^{n}\log x_i/n - \frac{1}{\theta}\right] = a(\theta,n)\left[U(\boldsymbol{X}) - q(\theta)\right].
\end{aligned}$$

- Then, $a(\theta,n) = -n$, $q(\theta) = 1/\theta$, and $U(\boldsymbol{X}) = -\sum_{i=1}^{n}\log X_i/n$ is the UMVUE of $1/\theta$, which attains the CRLB.
- Note: $\nexists\ a(.)$ such that $\partial\log L/\partial\theta = a(.)[U(\boldsymbol{X}) - \theta]$, i.e., no UMVUE of θ that can attain the CRLB.
- Note:「有限項部份和」與微分 (或極限) 可以直接交換符號。
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- (ii)

$$\begin{aligned}\partial \log L/\partial\theta &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}[\log\log\theta - \log(\theta-1) + x_i\log\theta] \\ &= (\frac{n}{\theta\log\theta} - \frac{n}{\theta-1}) + \theta^{-1}\sum_{i=1}^{n}x_i \\ &= \frac{n}{\theta}\Big[\bar{x} - (\frac{\theta}{\theta-1} - \frac{1}{\log\theta})\Big] = a(\theta,n)\Big[U(\boldsymbol{X}) - q(\theta)\Big].\end{aligned}$$

- Then, $a(\theta, n) = n/\theta$, and $q(\theta) = \theta/(\theta-1) - 1/(\log\theta)$;
- $U(\boldsymbol{X}) = \bar{X}$ is the UMVUE of $q(\theta)$, which attains the CRLB.
- Note: No UMVUE of θ that attains the CRLB.