TA section 6

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Review Exercises:
§3.1-#2, #5, #8, #10, §3.2-#13

§3.1-#2

2. 設(X,Y)之聯合p.d.f.爲$f(x,y)=6xy^2, 0<x,y<1$。

   (i) 試驗證f爲一p.d.f.;

   (ii) 試求$P(X+Y\geq 1)$;

   (iii) 試求$P(1/2<X<3/4)$。

- (i). check pdf: (a): $6xy^2 \geq 0, \forall x,y \in (0,1)$; (b) $\int_0^1 \int_0^1 6xy^2 dxdy = 1$.
- (ii). $\mathbb{P}(X+Y\geq 1) = \int\int_{\{x+y\geq 1; 0<x,y<1\}} f(x,y)dxdy$

$$= \int_0^1 \int_{1-y}^1 6xy^2 dxdy = \int_0^1 (3x^2y^2)\Big]_{x=1-y}^{x=1} dy = \int_0^1 (6y^3 - 3y^4)dy = 9/10.$$

(Or, $= \int_0^1 \int_{1-x}^1 6xy^2 dydx$)

- (iii).

$$\begin{aligned}\mathbb{P}(1/2 < X < 3/4) &= \mathbb{P}(1/2 < X < 3/4, 0 < Y < 1)\\ &= \int_0^1 \int_{1/2}^{3/4} 6xy^2 dxdy\\ &= \int_0^1 3y^2 (x^2\Big]_{x=1/2}^{x=3/4}) dy\\ &= (5/16)\int_0^1 3y^2 dy = 5/16.\end{aligned}$$

5. 設(X,Y)之聯合p.d.f.為

$$f(x,y)=c(x+2y), 0<x<2, 0<y<1\text{。}$$

(i) 試決定常數c之值;

(ii) 試求X之邊際p.d.f.;

(iii) 試求$Z=9/(X+1)^2$之p.d.f.;

(iv) 試求(X,Y)之聯合分佈函數。

- (i). $\int_0^1\int_0^2 c(x+2y)dxdy = 1 \Rightarrow c\int_0^1 (x^2/2+2xy)\Big]_{x=0}^{x=2} dy = c\int_0^1 (2+4y)dy = c(2y+2y^2)\Big]_{y=0}^{y=1} = 4c = 1 \Rightarrow c = 1/4.$

- (ii). For $0 < x < 2$,

$$f_X(x) = \frac{1}{4}\int_0^1 (x+2y)dy = (x+1)/4,\ 0 < x < 2.$$

- (iii). For $0 < x < 2 \Rightarrow 1 < z < 9$.
  $\mathbb{P}(Z \leq z) = \mathbb{P}(\frac{9}{(X+1)^2} \leq z) = \mathbb{P}(3z^{-1/2} - 1 \geq X) = 1 - \mathbb{P}(X \leq 3z^{-1/2} - 1)$, then

$$\begin{aligned} f_Z(z) &= f_X(3z^{-1/2} - 1)(3/2)z^{-3/2} \\ &= (1/4)(3z^{-1/2})(3/2)z^{-3/2} \\ &= (9/8)z^{-2},\ 1 < z < 9. \end{aligned}$$

- check: $\int_1^9 (9/8)z^{-2}dz = 1$.

- (iv). $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$.

$$F_{X,Y}(x,y) = \frac{1}{4}\int_0^y \int_0^x (t+2s)dtds = \cdots = \frac{1}{8}x^2y + \frac{1}{4}xy^2,\ 0 < x,y < 1.$$

8. 設(X,Y)之聯合p.d.f.爲$f(x,y)=x+y, 0\le x,y\le 1$。試求$P(X>\sqrt{Y})$。

- $\mathbb{P}(X>\sqrt{Y}) = \int_0^1\int_{\sqrt{y}}^1 (x+y)dxdy = \int_0^1 [= (x^2/2+xy)\Big]_{\sqrt{y}}^1 dy = \int_0^1 (1/2+y/2-y^{3/2})dy = 7/20.$
  (or, $= \mathbb{P}(X^2>Y) = \int_0^1\int_0^{x^2}(x+y)dydx = \int_0^1 (xy+y^2/2)\Big]_0^{x^2} dx = \int_0^1 (x^3+x^4/2)dx = 7/20.$)

§3.1 #10

10. 設(X, Y)之聯合p.d.f.爲$f(x, y) = \lambda^2 e^{-\lambda(x+y)}, x, y \geq 0$。試求$P(X \geq 2Y)$。

- $\mathbb{P}(X \geq 2Y) = \mathbb{P}(X > 0, Y \leq X/2) =$

$$
\begin{aligned}
\int_0^\infty \int_0^{x/2} \lambda^2 e^{-\lambda(x+y)} dy dx &= \int_0^\infty \lambda e^{-\lambda x} \Big(\int_0^{x/2} \lambda e^{-\lambda y} dy \Big) dx \\
&= \int_0^\infty \lambda e^{-\lambda x} \mathbb{P}(Y < x/2) dx, \ Y \sim \epsilon(\lambda), \\
& \int_0^\infty \lambda e^{-\lambda x} \Big(1 - e^{-\lambda x/2}\Big) dx \\
& \int_0^\infty \lambda e^{-\lambda x} dx - \lambda \int_0^\infty e^{-3\lambda x/2} dx \\
&= 1 - 2/3 = 1/3.
\end{aligned}
$$

- or,

$$\int_0^\infty \int_{2y}^\infty \lambda^2 e^{-\lambda(x+y)} dxdy = \lambda^2 \int_0^\infty (-\lambda^{-1} e^{-\lambda(x+y)})\Big]_{2y}^\infty dy$$
$$= \lambda \int_0^\infty e^{-3\lambda y} dy = 1/3.$$

- NOTE: useful tool by the definition of pdf:

$$\int f(z)dz = 1,$$

e.g., $\int_0^\infty (3\lambda/2) e^{-3\lambda x/2} dx = 1 \Rightarrow \int_0^\infty e^{-3\lambda x/2} dx = 2/(3\lambda)$.

§3.2 #13

13. 設X有$\mathcal{N}(0,1)$分佈，令$Y = X^2$。試證Y有χ_1^2分佈。又問其逆是否成立？證明或否證之。

- $Y = X^2 \Rightarrow X = \pm\sqrt{Y}$.
- $J = \det(\partial \boldsymbol{x}/\partial \boldsymbol{y}) = dx/dy = \pm 1/(2\sqrt{y})$.
- $$f_Y(y) = f_X(x = \sqrt{y})|J| + f_X(x = -\sqrt{y})|J| = \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{-y/2} = \frac{y^{1/2-1}e^{-y/2}}{\Gamma(1/2)2^{1/2}}.$$
- Thus, $Y \sim \chi_1^2, \quad y > 0.$
- Or, using CDF, $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{(}y)) \Rightarrow f_Y(y) = (f_X(\sqrt{(}y)) + f_X(-\sqrt{y}))/(2\sqrt{y}) = \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{-y/2}$.

- The converse statement:
  "*$Y = X^2$ with $Y \sim \chi^2_1$, then $X \sim \mathcal{N}(0,1)$*" may NOT be true.
- $\because x = \sqrt{y} > 0$ or $x = -\sqrt{y} < 0$, the support of X is not in $\mathbb{R}$.

• Proposition

Let X have pdf $f_X(x)$ and $Y := g(X)$, where $g : \mathcal{X} \to \mathcal{Y}$, is a monotone function. Suppose that $f_X(x)$ is continuous on the support $\mathcal{X} := \{x : f_X(x) > 0\}$ and that $g^{-1}(y)$ has a continuous derivative on a subset $\mathcal{Y}$, where $g^{-1}(y) := \{x \in \mathcal{X} : g(x) = y\}$. Then the pdf of Y is given by:

$$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|, \; y \in \mathcal{Y}.$$