Econometric Methods, 2023 **Assignment 6**

Due: 10/18, 0:00am

Problem 1

Consider the following linear regression:

$$Y_i = X_i'\beta + e_i,$$

where Y_i is the equity premium of the ith month, and

$$X_i := (1, \mathsf{x\_dfy}, \mathsf{x\_infl}, \mathsf{x\_svar}, \mathsf{x\_tms}, \mathsf{x\_tbl}, \mathsf{x\_dfr}, \mathsf{x\_dp}, \mathsf{x\_ltr}, \mathsf{x\_ep}, \mathsf{x\_bmr}, \mathsf{x\_ntis})',$$

for $i = 1, 2, \cdots, 504$, based on the data Equity_Premium.csv. Let $\widehat{\beta}_j$ be the jth element of the LS estimator $\widehat{\beta}$ for β. Please check the null hypothesis:

$$H_o : \beta_j = 0$$

using the t test at the size $\alpha = 5\%$ for each j, based on the asymptotic normality of the LS estimator.

Problem 2

Let $\{(Y_i, X_i')'\}_{i=1}^n$ be an i.i.d. sequence of Gaussian random vectors. Assume that

$$Y_i = X_i'\beta + e_i,$$

where $e_i|\boldsymbol{X} \sim \mathcal{N}(0, \sigma^2)$ and $\boldsymbol{X} := (X_1, \cdots, X_n)'$, and $\sum_{i=1}^n X_i X_i'$ is positive definite. Denote $\widehat{\beta} := (\sum_{i=1}^n X_i X_i')^{-1}(\sum_{i=1}^n X_i Y_i)$, $\widehat{e}_i = Y_i - X_i'\widehat{\beta}$, and $\widehat{\sigma}^2 = \sum_{i=1}^n \widehat{e}_i^2/n$.

1. Please show the asymptotic distribution of $n^{1/2}(\widehat{\beta} - \beta)$, as $n \to \infty$.
2. Please show the asymptotic distribution of $n^{1/2}(\widehat{\sigma}^2 - \sigma^2)$, as $n \to \infty$.
3. Please show the asymptotic distribution of $n^{1/2}(\widehat{\sigma} - \sigma)$, as $n \to \infty$.

Solution:

1.

$$\hat{\beta} - \beta = \left[\sum_{i=1}^{n} X_i X_i'\right]^{-1} \left[\sum_{i=1}^{n} X_i e_i\right]$$

$$\sqrt{n}(\hat{\beta} - \beta) = \left[\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right]^{-1} \left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i e_i\right] \xrightarrow{d} \mathcal{N}(0, \sigma^2 \mathbb{E}[X_i X_i']^{-1}),$$

as $n \to \infty$.

2.

$$n^{1/2}(\hat{\sigma}^2 - \sigma^2) = n^{-1/2}\sum_{i=1}^{n}(\hat{e}_i^2 - \sigma^2) = n^{-1/2}\sum_{i=1}^{n}(e_i^2 - \sigma^2) + n^{-1/2}\sum_{i=1}^{n}(\hat{e}_i^2 - e_i^2)$$

and

$$\frac{1}{n}\sum_{i=1}^{n}(\hat{e}_i^2 - e_i^2) = -\frac{2}{n}\sum_{i=1}^{n} X_i e_i(\hat{\beta} - \beta) + (\hat{\beta} - \beta)'[\frac{1}{n}\sum_{i=1}^{n} X_i X_i'](\hat{\beta} - \beta)$$

and hence

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\hat{e}_i^2 - e_i^2) &= -2\left[\frac{1}{n}\sum_{i=1}^{n} X_i e_i\right]\sqrt{n}(\hat{\beta} - \beta) + (\hat{\beta} - \beta)'[\frac{1}{n}\sum_{i=1}^{n} X_i X_i']\sqrt{n}(\hat{\beta} - \beta) \\ &= o_p(1)O_p(1) + o_p(1)O_p(1)O_p(1) = o_p(1).\end{aligned}$$

Therefore,

$$\begin{aligned}n^{-1/2}\sum_{i=1}^{n}(\hat{e}_i^2 - \sigma^2) &= n^{-1/2}\sum_{i=1}^{n}(e_i^2 - \sigma^2) + o_p(1) \\ &\xrightarrow{d} \mathcal{N}(0, \mathbb{E}[e_1^4] - \sigma^4) \stackrel{d}{=} \mathcal{N}(0, 2\sigma^4), \text{ as } n \to \infty,\end{aligned}$$

provided by $\mathbb{E}[e_1^4] = 3\sigma^4$.

3. Let $g(t) := t^{1/2}$, such that $dg/dt = (1/2)t^{-1/2}$ with $t \neq 0$, then

$$\sigma = (\sigma^2)^{1/2}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\sigma^2)} = \frac{1}{2}(\sigma^2)^{-1/2} = \frac{1}{2\sigma} \neq 0,$$

for $\sigma > 0$. By the 1st-order delta method,

$$\sqrt{n}(\hat{\sigma} - \sigma) = \frac{1}{2\sigma}\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4\sigma^2}(\mathbb{E}[e_1^4] - \sigma^4)\right) \stackrel{d}{=} \mathcal{N}\left(0, \frac{\sigma^2}{2}\right),$$

as $n \to \infty$.

Remark. *Recall the delta-method theorem:*
(a) (1st-order δ-method) If $g \in C^1(\mathbb{R})$ and $a_n(X_n - \mu) \xrightarrow{d} Z$, and let $G(\mu) := \nabla g(\mu) \neq 0$, then

$$a_n(g(X_n) - g(\mu)) \xrightarrow{d} G(\mu)^\top Z.$$

In particular, if $Z \sim \mathcal{N}(0, V)$ and $\sqrt{n}(X_n - \mu) \xrightarrow{d} \mathcal{N}(0, V)$, then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \mathcal{N}(0, G(\mu)^\top V G(\mu)).$$

(b) (2nd-order δ-method) If $G(\mu) = 0$, but $\nabla G(\mu) \neq 0$, then

$$a_n(g(X_n) - g(\mu)) = a_n \frac{\nabla G(\mu)}{2}(X_n - \mu)^2 + o_p(1).$$

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Problem 3

Following #2, let $\widehat{\beta}_{ML}$ and $\widehat{\sigma}^2_{ML}$ be the Gaussian MLEs of β and σ^2, respectively.

1. Please derive $\widehat{\beta}_{ML}$ and $\widehat{\sigma}^2_{ML}$.
2. Please show the asymptotic distribution of

$$\begin{bmatrix} n^{1/2}(\widehat{\beta}_{ML} - \beta) \\ n^{1/2}(\widehat{\sigma}^2_{ML} - \sigma^2) \end{bmatrix},$$

as $n \to \infty$.

3. Please verify the information matrix equality: $B + H = 0$.

Solution:

1.

$$(\hat{\boldsymbol{\beta}}'_{ML}, \hat{\sigma}^2_{ML})' = \arg \max_{\boldsymbol{\beta} \in \mathbb{R}^k, \sigma^2 > 0} \log L(\boldsymbol{\beta}, \sigma^2).$$

The Gaussian log-likelihood function is

$$\log L(\boldsymbol{\theta}) \propto -\frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}),$$

where $\boldsymbol{\theta} := (\boldsymbol{\beta}', \sigma^2)'$. Denote $\widehat{\boldsymbol{\theta}} := (\widehat{\boldsymbol{\beta}}'_{ML}, \widehat{\sigma}^2_{ML})'$. Via the first order conditions (F.O.C.)

evaluated at the $\widehat{\boldsymbol{\theta}}$, we have the estimating functions for $\hat{\boldsymbol{\theta}}$ are

$$\boldsymbol{s}(\widehat{\boldsymbol{\theta}}) := \begin{pmatrix} \nabla_{\boldsymbol{\beta}} \log L(\widehat{\boldsymbol{\theta}}) \\ \\ \nabla_{\sigma^2} \log L(\widehat{\boldsymbol{\theta}}) \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} \frac{X_i \hat{e}_i}{\widehat{\sigma}^2_{ML}} \\ \\ -\frac{1}{2\widehat{\sigma}^2_{ML}} + \frac{\hat{e}_i^2}{2\widehat{\sigma}^4_{ML}} \end{pmatrix} \triangleq 0,$$

where $\hat{e}_i = (Y_i - X_i{}'\widehat{\boldsymbol{\beta}}_{ML})$. Also, the second-order conditions (S.O.C.) evaluated at the $\widehat{\boldsymbol{\theta}}$ are held, $\nabla^2 \log L(\widehat{\boldsymbol{\theta}}) < 0$. Then, we can solve for

$$\hat{\beta}_{ML} = [\sum_{i=1}^{n} X_i X_i']^{-1} \sum_{i=1}^{n} X_i Y_i = \hat{\beta}_{LS}$$

and

$$\hat{\sigma}^2_{ML} = \frac{1}{n} \sum_{i=1}^{n} \hat{e}_i^2 = \hat{\sigma}^2_{LS}.$$

2. We first denote the correct specification of the log-likelihood function at $\boldsymbol{\theta}_0$ by,

$$\log L(\boldsymbol{\theta}_0) = \sum_{i=1}^{n} \log f_i(\boldsymbol{\theta}_0),$$

where $\boldsymbol{\theta}_0 := (\boldsymbol{\beta}', \sigma^2)'$. Let $s_i(\boldsymbol{\theta}) := \nabla \log f_i(\boldsymbol{\theta})$, such that $\boldsymbol{s}(\boldsymbol{\theta}) := \nabla \log L(\boldsymbol{\theta}) = \sum_{i=1}^{n} s_i(\boldsymbol{\theta})$. When the true value $\boldsymbol{\theta}_0$ is in the interior of a compact parameter space Θ, the mean-value expansion of $\boldsymbol{s}(\widehat{\boldsymbol{\theta}})$ about $\boldsymbol{\theta}_0$ gives that

$$\boldsymbol{s}(\widehat{\boldsymbol{\theta}}) = \boldsymbol{s}(\boldsymbol{\theta}_0) + \nabla \boldsymbol{s}(\boldsymbol{\theta}^+)(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o),$$

for some $\boldsymbol{\theta}^+ \in (\boldsymbol{\theta}_0, \widehat{\boldsymbol{\theta}})$. Note that requiring $\boldsymbol{\theta}_0$ in the interior of Θ is to ensure that the mean-value expansion is valid in the parameter space, and the left hand side of the equality is zero since $\widehat{\boldsymbol{\theta}}$ satisfies the first-order conditions under maximization problem. Given $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = o_p(1)$ (consistency), we have that $0 < |\boldsymbol{\theta}^+ - \boldsymbol{\theta}_0| < |\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0|$, such that $\boldsymbol{\theta}^+ - \boldsymbol{\theta}_0 = o_p(1)$, and by uniformly weak law of large numbers,

$$\sup_{\boldsymbol{\theta} \in \Theta} |n^{-1} \sum_{i=1}^{n} \nabla s_i(\boldsymbol{\theta}) - \mathbb{E}[\nabla s_i(\boldsymbol{\theta})]| = o_p(1),$$

then $\nabla \boldsymbol{s}(\boldsymbol{\theta}^+)/n \xrightarrow{p} \mathbb{E}[\nabla s_i(\boldsymbol{\theta}_0)] =: H_1$. In addition, assume that $\nabla \boldsymbol{s}(\boldsymbol{\theta}^+)$ is invertible a.s., and

$$\boldsymbol{s}(\boldsymbol{\theta}_0)/\sqrt{n} \xrightarrow{d} \mathcal{N}(0, B_1), \text{ as } n \to \infty,$$

where $B_1 := \text{Var}[\boldsymbol{s}(\boldsymbol{\theta}_0)/\sqrt{n}] = \mathbb{E}[s_i(\boldsymbol{\theta}_0) s_i(\boldsymbol{\theta}_0)']$. Then,

$$\begin{aligned} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= -[\nabla \boldsymbol{s}(\boldsymbol{\theta}^+)/n]^{-1} [\boldsymbol{s}(\boldsymbol{\theta}_0)/\sqrt{n}] \\ &= -H_1^{-1} [\boldsymbol{s}(\boldsymbol{\theta}_0)/\sqrt{n}] + o_p(1) \\ &= -H_1^{-1} B_1^{1/2} [B_1^{-1/2} \boldsymbol{s}(\boldsymbol{\theta}_0)/\sqrt{n}] + o_p(1). \end{aligned}$$

Thus,

$$H_1^{1/2} B_1^{-1/2} H_1^{1/2} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{I}_k), \text{ as } n \to \infty.$$

Now,

$$s_i(\boldsymbol{\theta}_0) = \nabla \log f_i(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \begin{pmatrix} X_i'(Y_i - X_i'\beta)/\sigma^2 \\ -1/2\sigma^2 + (Y_i - X_i'\beta)^2/2\sigma^4 \end{pmatrix}.$$

Since that $\{Y_i, X_i'\}'$ are i.i.d., and the model is correctly specified,

$$\begin{aligned} H_1 := \mathbb{E}[\nabla s_i(\boldsymbol{\theta}_0)] &= \begin{pmatrix} -\mathbb{E}[X_i X_i']/\sigma^2 & -\mathbb{E}[X_i e_i]/\sigma^4 \\ -\mathbb{E}[e_i X_i']/\sigma^4 & -1/2\sigma^4 \end{pmatrix} = \begin{pmatrix} -\mathbb{E}[X_i X_i']/\sigma^2 & 0 \\ 0 & -1/2\sigma^4 \end{pmatrix} \\ &= -\mathbb{E}[s_i(\boldsymbol{\theta}_0) s_i(\boldsymbol{\theta}_0)'] =: -B_1. \end{aligned}$$

We thus have that

$$\begin{aligned} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = \begin{pmatrix} n^{1/2}(\widehat{\beta}_{ML} - \beta) \\ n^{1/2}(\widehat{\sigma}^2_{ML} - \sigma^2) \end{pmatrix} &\xrightarrow{d} \mathcal{N}(0, B_1^{-1}) \\ &\stackrel{d}{=} \mathcal{N}(0, \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1}\sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}), \end{aligned}$$

as $n \to \infty$.

Remark. Alternatively, one can utilize the results in Problem 2.1 and 2.2, since $\hat{\beta}_{ML} = \hat{\beta}_{LS}$ and $\hat{\sigma}^2_{ML} = \hat{\sigma}^2_{LS}$. Also, the off-diagonal elements of variance-covariance matrix of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$ are zeros from the fact that $\mathbb{E}[(\hat{\beta} - \beta)\hat{e}'|\boldsymbol{X}] = \mathbb{E}[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'ee'\boldsymbol{M}|\boldsymbol{X}] = \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{M} = 0$, in which $\hat{e} = \boldsymbol{M}e$ with $\boldsymbol{M} = \boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$.
Or, in a direct way,

$$\begin{aligned} \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &= \begin{pmatrix} (\boldsymbol{X}'\boldsymbol{X}/n)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{X}'e/\sqrt{n} \\ \widehat{e}'\widehat{e}/\sqrt{n} - \sqrt{n}\sigma^2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{X}'e/\sqrt{n} \\ e'e/\sqrt{n} - \sqrt{n}\sigma^2 \end{pmatrix} + o_p(1) \\ &\xrightarrow{d} \mathcal{N}(0, (\begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_i X_i']\sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1} & 0 \\ 0 & 1 \end{pmatrix}') \\ &\stackrel{d}{=} \mathcal{N}(0, \begin{pmatrix} \mathbb{E}[X_i X_i']^{-1}\sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}), \text{ as } n \to \infty, \end{aligned}$$

since (do it by yourself)

$$\begin{pmatrix} \boldsymbol{X}'e/\sqrt{n} \\ e'e/\sqrt{n} - \sqrt{n}\sigma^2 \end{pmatrix} = \sqrt{n} \begin{pmatrix} \boldsymbol{X}'e/n \\ e'e/n - \sigma^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \begin{pmatrix} \mathbb{E}[X_i X_i']\sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}), \text{ as } n \to \infty.$$

3. Under the correct model specification, the log-likelihood function at $\boldsymbol{\theta}_0$ is

$$\log L(\boldsymbol{\theta}_0) \propto -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(\boldsymbol{Y}-\boldsymbol{X\beta})'(\boldsymbol{Y}-\boldsymbol{X\beta}),$$

where $\boldsymbol{\theta}_0 := (\boldsymbol{\beta}', \sigma^2)'$. Via the first order derivatives evaluated at the true value $\boldsymbol{\theta}_0$, we have the score functions evaluated at $\boldsymbol{\theta}_0$ as

$$\boldsymbol{s}(\boldsymbol{\theta}_0) := \begin{pmatrix} \nabla_{\boldsymbol{\beta}} \log L(\boldsymbol{\theta}) \\ \\ \nabla_{\sigma^2} \log L(\boldsymbol{\theta}) \end{pmatrix}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = \sum_{i=1}^{n} \begin{pmatrix} \frac{X_i e_i}{\sigma^2} \\ \\ -\frac{1}{2\sigma^2} + \frac{e_i{}^2}{2\sigma^4} \end{pmatrix}, \ e_i = (Y_i - X_i{}'\boldsymbol{\beta}), \ i = 1, \ldots, n,$$

so that the conditional Information Matrix under i.i.d. assumption can be given by

$$B := \mathbb{E}\left[\boldsymbol{s}(\boldsymbol{\theta}_0)\boldsymbol{s}(\boldsymbol{\theta}_0)'|\boldsymbol{X}\right] = \sum_{i=1}^{n} \begin{pmatrix} \frac{\mathbb{E}(e_i{}^2|\boldsymbol{X})X_iX_i{}'}{\sigma^4} & -\frac{\mathbb{E}(X_ie_i|\boldsymbol{X})}{2\sigma^4} + \frac{\mathbb{E}(X_ie_i{}^3|\boldsymbol{X})}{2\sigma^6} \\ \\ -\frac{\mathbb{E}(X_ie_i|\boldsymbol{X})}{2\sigma^4} + \frac{\mathbb{E}(X_ie_i{}^3|\boldsymbol{X})}{2\sigma^6} & \frac{1}{4\sigma^4} - \frac{\mathbb{E}(e_i{}^2|\boldsymbol{X})}{2\sigma^6} + \frac{\mathbb{E}(e_i{}^4|\boldsymbol{X})}{4\sigma^8} \end{pmatrix}.$$

Since that $\mathbb{E}(e_i|\boldsymbol{X}) = 0$, $\mathbb{E}(e_i{}^2|\boldsymbol{X}) = \sigma^2$, $\mathbb{E}(e_i{}^3|\boldsymbol{X}) = 0$ and $\mathbb{E}(e_i{}^4|\boldsymbol{X}) = 3\sigma^4$, then the off-diagonal entries are zero and we have that

$$B = \begin{pmatrix} \frac{\boldsymbol{X}'\boldsymbol{X}}{\sigma^2} & 0 \\ \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}.$$

Also, the corresponding negative of conditional expected Hessian matrix can be obtained by

$$\begin{aligned} -H := -\mathbb{E}\left[\nabla \boldsymbol{s}(\boldsymbol{\theta}_0)|\boldsymbol{X}\right] &= -\sum_{i=1}^{n} \begin{pmatrix} -\frac{X_iX_i{}'}{\sigma^2} & -\frac{\mathbb{E}(X_ie_i|\boldsymbol{X})}{\sigma^4} \\ \\ -\frac{\mathbb{E}(e_iX_i{}'|\boldsymbol{X})}{\sigma^4} & \frac{1}{2\sigma^4} - \frac{\mathbb{E}(e_i{}^2|\boldsymbol{X})}{\sigma^6} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\boldsymbol{X}'\boldsymbol{X}}{\sigma^2} & 0 \\ 0 & -\frac{n}{2\sigma^4} + \frac{n}{\sigma^4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\boldsymbol{X}'\boldsymbol{X}}{\sigma^2} & 0 \\ \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix} \\ &= B, \end{aligned}$$

then the desired result follows.

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