TA section 4

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Review: Point Estimation

估計量的優劣評估

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    – $q(\theta)$ 之均方差 (MSE):
    $$MSE_\theta(T) = R(\theta, T) = \mathbb{E}[(T(\boldsymbol{X}) - q(\theta))^2] = \mathrm{Var}[T(\boldsymbol{X})] + Bias_\theta^2(T(\boldsymbol{X})).$$
    – $\mathrm{Var}[T(\boldsymbol{X})] = \mathbb{E}[(T(\boldsymbol{X}) - \mathbb{E}\,T(\boldsymbol{X}))^2]$, 且 $Bias_\theta(T(\boldsymbol{X})) = \mathbb{E}[T(\boldsymbol{X})] - q(\theta)$.
  - 最佳不偏估計量 (best unbiased estimator, BUE): CRLB (Cramer-Rao lower bound): $(q'(\theta))^2/I(\theta)$.
  - 一致最小變異不偏估計量 (uniformly minimum variance unbiased estimator, UMVUE):
    – [Rao-Blackwell Theorem]: (充份 $+$ 不偏 $\Rightarrow$ 效率)
    – [Lehmann-Scheffé (-Rao-Blackwell) Theorem]: (完備 $+$ 充份 $+$ 不偏 $\Rightarrow$ 效率)
  - BUE $\Rightarrow$ UMVUE. ,i.e., $\mathrm{Var}[\text{BUE}] \le \mathrm{Var}[\text{UMVUE}]$.

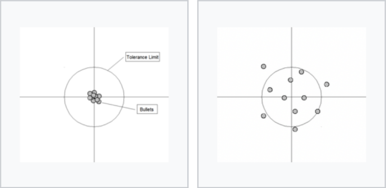
大樣本性質

- 一致性:
  - (weak consistency) $T_n \xrightarrow{\text{p}} q(\theta)$ as $n \to \infty$ if
  $$\lim_{n\to\infty} \mathbb{P}(|T_n - q(\theta)| < \epsilon) = 1, \forall \epsilon > 0.$$
  - (mean-squares consistency/L_2-norm consistency) $T_n \xrightarrow{\text{m.s.}} q(\theta)$ as $n \to \infty$ if
  $$\mathbb{E}[(T_n - q(\theta))^2] \to 0, \text{ as } n \to \infty.$$
  或 $\lim_{n\to\infty} \mathrm{Var}[T_n] = 0$, 且 $\lim_{n\to\infty} Bias_\theta(T_n) = 0$.
- 漸近不偏性: $\lim_{n\to\infty} \mathbb{E}[T_n] = q(\theta)$.
- 漸近效率性: $are(T_{2n}, T_{1n}) = \lim_{n\to\infty}[\mathrm{Var}[T_{1n}]/\mathrm{Var}[T_{2n}]] \leq 1$ for T_{1n}, T_{2n} two (asymptotic) unbiased estimators, 則 T_{1n} 具漸近有效。
- 漸近常態性: 給定一估計量, 其漸近變異數 $\mathrm{Avar}[T_n] = \sigma_n^2(\theta)$ 與漸近期望值 $\mathrm{Asy.}\,\mathbb{E}[T_n] = \mu_n(\theta)$, 則 $(T_n - \mu_n(\theta))/\sigma_n(\theta) \xrightarrow{\text{d}} \mathcal{N}(0,1)$.
- 最佳漸近常態性: 若 $n\sigma_n^2(\theta) \to v^2(\theta)$ 且 $n^{1/2}(\mu_n(\theta) - q(\theta)) \to 0$, 其中 $v^2(\theta) > 0$ 為某一 θ 之函數, 則稱 T_n 具有最佳漸近常態性。即, 此時漸近變異數為 $v^2(\theta)/n$。

Admissible Estimator

- (inadmissible, 不可行的/不可採用的): 對二估計量 S, T, 若
  $$R(\theta,T) \le R(\theta,S),\ \forall\theta\in\Omega$$
  且
  $$R(\theta,T) < R(\theta,S),\ \text{for some } \theta,$$
  則 T 較 S 為佳, 且 S 為不可行的/不可採用的。
- 若一個估計量 T, 找不到較其為佳的估計量 (找不到均方差更小), 則稱 T 可行的/可採用的。
- 若 S 為不可行的, 不代表 T 是可行的。

bias-variance tradeoff



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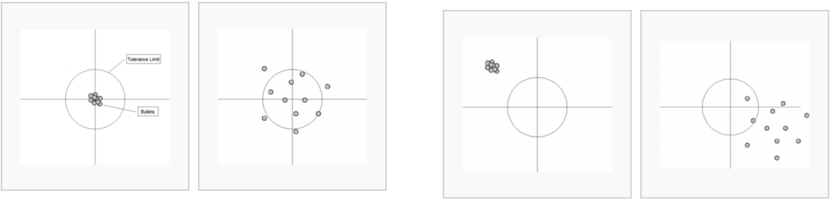

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Example

- $X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Let $q(\sigma^2) := \sigma^2$, consider two estimators of $q(\sigma^2)$, $T_1 = \sum_{i=1}^n (X_i - \bar{X})^2/n$, $T_2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1) = nT_1/(n-1)$.
- $\mathbb{E}[T_1] = (n-1)\sigma^2/n \neq \sigma^2$ (biased for σ^2), $\mathrm{Var}[T_1] = 2(n-1)\sigma^4/n^2$. So,

$$R(\theta, T_1) = \mathrm{Var}[T_1] + Bias^2(\theta, T_1) = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} = \frac{(2n-1)\sigma^4}{n^2}.$$

- $\mathbb{E}[T_2] = \sigma^2$ (unbiased for σ^2), $\mathrm{Var}[T_2] = 2\sigma^4/(n-1) > \mathrm{Var}[T_1]$. So,

$$R(\theta, T_2) = 2\sigma^4/(n-1) > R(\theta, T_1).$$

- T_2 不偏, 但 MSE 卻較大 (因變異數大的幅度超過 T_1 偏誤的幅度)
- $\lim_{n\to\infty} \mathbb{E}[T_1] = \lim_{n\to\infty} \mathbb{E}[T_2] = \sigma^2$ (漸近不偏), $\lim_{n\to\infty} \mathrm{Var}[T_1] = \lim_{n\to\infty} \mathrm{Var}[T_2] = 0$ (均方差一致性);

$$are(T_2, T_1) = \lim_{n\to\infty} \mathrm{Var}[T_1]/\mathrm{Var}[T_2] = \lim_{n\to\infty} \frac{2(n-1)\sigma^4/n^2}{2\sigma^4/(n-1)} = 1,$$

  且 T_2 是 UMVUE, 故 T_2 具漸近有效性。
- (若不考慮不偏性)

$$\mathrm{Var}[T_1] < CRLB(\sigma^2) = 2\sigma^4/n < \mathrm{Var}[T_2].$$