TA section 10

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Homework 4: (part II)

§8.4 #6, #7, #8

6. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}[0,\theta]$分佈所產生之隨機樣本。欲檢定H_0 : $\theta=\theta_0$, vs. $H_a : \theta \neq \theta_0$。試給一$\alpha$下之LRT。(**解.** 拒絕域爲$\{X_{(n)} < \theta_0\alpha^{1/n}\}$)

- likelihood $L(\theta) = \theta^{-n}\boldsymbol{I}(0 < x_i < \theta),\ \forall i = 1, 2, \cdots, n.$
- Under $\Omega := \{\theta : \theta > 0\}$, $\widehat{\theta}_{ML} = X_{(n)} \Rightarrow L(\widehat{\theta}) = (1/X_{(n)})^n$.
- Under $\Omega_o = \{\theta : \theta = \theta_0\}$, $\ddot{\theta}_{ML} = \theta_0 \Rightarrow L(\ddot{\theta}) = (1/\theta_0)^n$.
- LRT statistic: $\lambda(\boldsymbol{X}) = L(\ddot{\theta})/L(\widehat{\theta}) = (X_{(n)}/\theta_0)^n \in (0, 1]$, for $X_{(n)} \leq \theta_0$,
- a testing rule: $\phi(\boldsymbol{X}) = \boldsymbol{I}(\lambda(\boldsymbol{X}) \leq c) \iff$

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(X_{(n)} \leq c' = c^{1/n}\theta_0),$$

  such that $c \in (0, 1)$ satisfying $\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\lambda(\boldsymbol{X}) \leq c | H_o : \theta = \theta_0)$.
- $\alpha = \theta_0^{-n} \int_0^{c^{1/n}\theta_0} nx^{n-1}dx \Rightarrow c = \alpha$ and $c' = \alpha^{1/n}\theta_0$.

7. 設$X_1, \cdots, X_n$爲一組由$\mathcal{E}(\theta)$分佈所產生之隨機樣本。

(i) 試給一α下之$H_0: \theta = \theta_0$, vs. $H_a: \theta \neq \theta_0$的LRT, 又當$n$很大時,

利用定理4.2, 得到近似的拒絕域;

(ii) 試給一α下之$H_0: \theta = \theta_0$, vs. $H_a: \theta > \theta_0$的LRT。

- (i) Under Ω, $\widehat{\theta} = 1/\bar{X}$; under Ω_o, $\ddot{\theta} = \theta_0$.
- Consider a sufficient statistic $T(\boldsymbol{X}) = \bar{X}$,
  LRT statistic:

$$\lambda(T(\boldsymbol{x})) = (\theta_0 \bar{X})^n \exp(n - n\theta_0 \bar{x}) \in (0, 1]$$

- Note that:

$$d \log \lambda(T(\boldsymbol{x}))/d\bar{x} \begin{cases} > 0, & [\bar{x} < 1/\theta_0] \\ = 0, & [\bar{x} = 1/\theta_0] \\ < 0, & [\bar{x} > 1/\theta_0]. \end{cases}$$

  So, a testing rule: $\phi(\boldsymbol{X}) = \boldsymbol{I}(\lambda(\boldsymbol{X}) \leq c) \iff$

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\{\bar{X} < c_1\} \cup \{\bar{X} > c_2\}),$$

  such that $c_1, c_2 \in (0, 1)$ with
  $\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\{\bar{X} < c_1\} \cup \{\bar{X} > c_2\} | H_o : \theta = \theta_0)$.

- $\because 2n\theta_0\bar{X} \sim \chi^2_{(2n)}$ under H_o,
  $\therefore$ choose $\alpha/2 = \mathbb{P}(2n\theta_0\bar{X} < c_1' = \chi^2_{\alpha/2,2n})$ and
  $\alpha/2 = \mathbb{P}(2n\theta_0\bar{X} > c_2' = \chi^2_{1-\alpha/2,2n})$.
- Thus,

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\{\bar{X} < \chi^2_{\alpha/2,2n}/(2n\theta_0)\} \cup \{\bar{X} > \chi^2_{1-\alpha/2,2n}/(2n\theta_0)\}),$$

  is a size α LRT for $H_o : \theta = \theta_0$.
- $\because -2\log\lambda \xrightarrow{\text{d}} \chi^2_1$, as $n \to \infty$. So, the rejection region of an approximate size α LRT can be: $C^* = \{-2\log\lambda > \chi^2_{1-\alpha/2,1} \vee -2\log\lambda < \chi^2_{\alpha/2,1}\}$.
- (ii) for $H_1 : \theta > \theta_0$,

$$\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\bar{X} > c | H_o : \theta = \theta_0) = \mathbb{P}(2n\theta_0\bar{X} > 2n\theta_0 c | H_o : \theta = \theta_0),$$

  $\Rightarrow c = \chi^2_{1-\alpha,2n}/(2n\theta_0)$, then a size α LRT is

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\bar{X} > \chi^2_{1-\alpha,2n}/(2n\theta_0)).$$

8. 設$X_1, \cdots, X_n$爲一組由$\mathcal{B}e(\theta, 1)$分佈所產生之隨機樣本。欲檢定H_0 : $\theta = \theta_0$, vs. $H_a : \theta \neq \theta_0$。試給一$\alpha$下之LRT。又當$n$很大時, 利用定理4.2, 得到近似的拒絕域。

- Under Ω, $\widehat{\theta} = -n/\sum_{i=1}^n \log X_i$; under Ω_o, $\ddot{\theta} = \theta_0$.
- Let $T(\boldsymbol{X}) = -\sum_{i=1}^n \log X_i$, a S.S. of θ,
- LRT statistic:

$$\lambda(T(\boldsymbol{x})) := L(\ddot{\theta})/L(\widehat{\theta}) = (\theta_0 T/n)^n \exp(n - \theta_0 T)$$

- $\log \lambda(T(\boldsymbol{x})) = n \log \theta + n \log T - n \log n + n - \theta_0 T$,

$$d \log \lambda(T(\boldsymbol{x}))/dT = n/T - \theta_0 \begin{cases} > 0, & [T < n/\theta_0] \\ = 0, & [T = n/\theta_0] \\ < 0, & [T > n/\theta_0], \end{cases}$$

so, a testing rule: $\phi(\boldsymbol{X}) = \boldsymbol{I}(\lambda(\boldsymbol{X}) \leq c) \iff$

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\{T < c_1\} \cup \{T > c_2\}),$$

such that $c_1, c_2 \in (0, 1)$ with
$\alpha = \sup_{\theta=\theta_0} \mathbb{P}(\{T < c_1\} \cup \{T > c_2\} | H_o : \theta = \theta_0)$.

- $\because 2\theta_0 T \sim \Gamma(n, 2) = \chi^2_{(2n)}$ under H_o,
  choose $\alpha/2 = \mathbb{P}(2\theta_0 T < c_1' = \chi^2_{\alpha/2,2n})$ and
  $\alpha/2 = \mathbb{P}(2\theta_0 T > c_2' = \chi^2_{1-\alpha/2,2n})$. Thus, a testing rule:

$$\phi(\boldsymbol{X}) = \boldsymbol{I}(\{T < \chi^2_{\alpha/2,2n}/(2\theta_0)\} \cup \{T > \chi^2_{1-\alpha/2,2n}/(2\theta_0)\})$$

  is a size α LRT for $H_o : \theta = \theta_0$.

- Since that $-2\log\lambda \xrightarrow{\text{d}} \chi^2_1$, as $n \to \infty$, the rejection of region of an approximate size α LRT can be:

$$C^* = \{-2\log\lambda > \chi^2_{1-\alpha/2,1} \vee -2\log\lambda < \chi^2_{\alpha/2,1}\}.$$