TA section 4

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Homework 3

§6.2#2(i)(ii); §6.3#8(i)(ii), #11, #21; §6.4#3(i)(ii)

2. 設$X_1, \cdots, X_n$爲一組由$\mathcal{P}(\theta)$分佈所產生之隨機樣本, $\theta > 0$。

   (i) 試以動差法求兩種θ之估計量;

   (ii) 試利用(i)給出$P(X \neq 0)$之兩種動差估計量。

- Method of Moment Estimator Idea: "Matching the moments".
- $\mathbb{E}[X] = \mathrm{Var}[X] = \lambda$. Let $m_k = n^{-1}\sum_{i=1}^n X_i^k$, for $k \geq 1$.
- (1) $m_1 = \mathbb{E}[X] \Rightarrow \widehat{E}[X] = m_1$, i.e., $\widehat{\lambda}_1 = \overline{X}_n$.
  (2) matching: $m_1 = \mathbb{E}[X]$ and $m_2 = \mathbb{E}[X^2] \Rightarrow \widehat{E}[X] = m_1$ and $m_2 = \widehat{\mathbb{E}}[X^2] = \widehat{\mathrm{Var}}[X] + \widehat{\mathbb{E}}[X]^2 = \widehat{\lambda} + m_1^2$. Thus, $\widehat{\lambda}_2 = m_2 - m_1^2$.
- $\mathbb{P}\widehat{(X \neq 0)} = 1 - \exp(-\widehat{\lambda})$, with plugging in (1) and (2).

§6.3#8

8. 設$X_1, \cdots, X_n$為一組由p.d.f.$f(x|\boldsymbol{\theta})$所產生之隨機樣本, 其中$f(x|\boldsymbol{\theta}) = \sigma^{-1}\exp\{-(x-\mu)/\sigma\}$, $x \geq \mu$, $\boldsymbol{\theta} = (\mu, \sigma)$, $\mu \in R$, $\sigma > 0$。試求

   (i) μ, σ之MLE;

   (ii) $P(X \geq t)$之MLE, 其中$t > \mu$;

- (i) log-likelihood function:
  $\log L(\theta) = -n \log \sigma - \sigma^{-1} \sum_{i=1}^{n}(X_i - \mu)$, $\mu \leq X_{(1)}$.
- F.O.C.:
  fixed σ, $\partial \log L(\theta)/\partial\mu = 1/\sigma > 0$ for $\mu \leq X_{(1)} \Rightarrow \hat{\mu}_{MLE} = X_{(1)}$.
  $\partial \log L(\theta)/\partial\sigma = -n/\sigma + \sigma^{-2} \sum_{i=1}^{n}(X_i - \mu) \stackrel{\triangle}{=} 0$
  $\Rightarrow \hat{\sigma}_{ML} = n^{-1} \sum_{i=1}^{n}(X_i - \hat{\mu}_{MLE}) = n^{-1} \sum_{i=1}^{n}(X_i - X_{(1)})$.

- S.O.C.:

$$\partial^2 \log L(\theta)/\partial(\sigma)^2 \mid_{\theta=(\hat{\mu},\hat{\sigma})} = -\sum_{i=1}^{n}(x_i - \widehat{\mu})/\widehat{\sigma}^3 < 0.$$

- By invariance principle,

$$\mathbb{P}\widehat{(X > t)} = e^{-\widehat{(t-\mu)}/\sigma} = e^{-(t-\hat{\mu})/\hat{\sigma}},$$

$t > \hat{\mu}$.

§6.3#11

11. 設$X_1, \cdots, X_n$爲一組由$\mathcal{P}(\lambda)$分佈所產生之隨機樣本, $\lambda > 0$。試求$P(X = 0)$之MLE。

- log-likelihood function:

$$\log L(\lambda) = -n\lambda + \log \lambda \left(\sum_{i=1}^{n} X_i\right) - \log \left(\prod_{i=1}^{n} X_i!\right).$$

F.O.C.:
$d \log L(\lambda)/d\lambda = -n + \sum_{i=1}^{n} X_i/\lambda \overset{\triangle}{=} 0 \Rightarrow \hat{\lambda}_{MLE} = \sum_{i=1}^{n} X_i/n =: \bar{X}.$
S.O.C.:
$d^2 \log L(\lambda)/d\lambda^2 \mid_{\lambda=\hat{\lambda}} = -\sum_{i=1}^{n} X_i/\hat{\lambda}^2 = -n/\hat{\lambda} < 0.$

- $\mathbb{P}\widehat{(X = 0)} = e^{-\hat{\lambda}_{MLE}} = e^{-\bar{X}}.$

§6.3#21

21. 設$X_1, \cdots, X_n$爲一組由p.d.f.$f(x|\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, $0 \le \theta \le 1/2$, 所產生之隨機樣本。試分別求θ之動差估計量及MLE。

- Matching: $m_1 = \mathbb{E}[X] = \theta$. So, $\widehat{\theta}_{MME} = m_1 = \bar{X}_n$.
- log-likelihood function:
  $\log L(\theta) = \sum_{i=1}^n x_i \log\theta + (n - \sum_{i=1}^n x_i)\log(1-\theta), 0 \le \theta \le 1/2$. Then, F.O.C.:

$$d\log L/d\theta = \sum_{i=1}^n x_i/\theta - (n - \sum_{i=1}^n x_i)/(1-\theta) \stackrel{\triangle}{=} 0,\ \ 0 \le \theta \le 1/2,$$

  so

$$\widehat{\theta}_{MLE} = \bar{X}_n \boldsymbol{I}(0 \le \widehat{\theta} \le 1/2) = (\bar{X}_n \wedge 1/2),$$

  or $\min\{\bar{X}_n, 1/2\}$, i.e., when $\bar{X}_n \le 1/2$, $\widehat{\theta}_{MLE} = \bar{X}_n$; when $\bar{X}_n > 1/2$, $\widehat{\theta}_{MLE} = 1/2$.
- S.O.C.: $d^2\log L/d\theta^2|_{\theta=\widehat{\theta}} < 0$.

§6.4#3

3. 設X有$\mathcal{U}(0,\theta)$分佈, θ之事前分佈爲$\mathcal{E}(1)$。試求

(i) 在給定$X=x$之下, θ之事後分佈;

(ii) θ之貝氏估計量。

- Bayes Estimator in mean squares risk: "Finding the posterior mean" $\mathbb{E}[\theta|\boldsymbol{x}]$, where the posterior pdf:

$$\pi(\theta|\boldsymbol{x}) = \frac{f(\boldsymbol{x}|\theta)\pi(\theta)}{m(\boldsymbol{x})},$$

$\pi(\theta)$ is a prior distribution, and $m(\boldsymbol{x})$ is the marginal pdf of $\boldsymbol{X}$.

- $X|\theta \sim f(x|\theta) = \theta^{-1}\boldsymbol{I}(0 < x < \theta)$.
- 題目有誤: prior distribution $\theta \sim \pi(\theta) = \theta e^{-\theta}, \theta > x$.
  則 $f(x,\theta) = e^{-\theta}\boldsymbol{I}(0 < x < \theta)$,

$$m(x) = \int_x^{\infty} f(x,\theta)d\theta = \int_x^{\infty} e^{-\theta}d\theta = e^{-x} \Rightarrow \pi(\theta|x) = e^{x-\theta}\boldsymbol{I}(0 < x < \theta).$$

故, $\widehat{\theta}_{BE} = \mathbb{E}[\theta|x] = e^x \int_x^{\infty} \theta e^{-\theta} d\theta = x^x \left(xe^{-x} + e^{-x}\right) = x + 1.$