TA section 8

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Homework 4 (II):
§2.3-#21, #22, #23, #29

§2.3-#21

21. 設X有$\mathcal{B}e(\alpha,\beta)$分佈，試求$Y=1-X$之分佈 。

- $f_Y(y) = f_X(x = 1-y)|dx/dy|$

$$f_Y(y) = \frac{1}{B(\alpha,\beta)}[1-y]^{\alpha-1}[1-(1-y)]^{\beta-1}$$
$$= \frac{1}{B(\alpha,\beta)}[1-y]^{\alpha-1}y^{\beta-1} = \frac{1}{B(\beta,\alpha)}y^{\beta-1}(1-y)^{\alpha-1}, y \in (0,1).$$

since $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta) = B(\beta,\alpha)$ (symmetry).

- $\therefore\ Y \sim \mathcal{B}e(\beta,\alpha)$.

§2.3 #22

22. 設X有$\mathcal{U}(0,1)$分佈。令$Y = X^r$, $r > 0$, 試求Y之p.d.f., 並指出此爲那一常見的分佈, 參數爲何。

- $Y = X^r, r > 0$. $x \in (0,1) \Rightarrow y \in (0,1)$.
- $f_Y(y) = f_X(x = y^{1/r})|dx/dy| = r^{-1}y^{1/r-1}, y \in (0,1)$.
- Let $\alpha = 1/r, \beta = 1$, so that $B(\alpha,1) = \Gamma(\alpha)\Gamma(1)/\Gamma(\alpha+1) = \Gamma(\alpha)\Gamma(1)/\alpha\Gamma(\alpha) = 1/\alpha = r$. Then,

$$f_Y(y) = \alpha y^{\alpha-1} = \frac{y^{\alpha-1}(1-y)^{1-1}}{B(\alpha,1)},$$

providing that $Y \sim \mathcal{B}e(\alpha = 1/r, \beta = 1)$.

§2.3 #23

23. 設X有$\mathcal{B}e(\alpha, \beta)$分佈。試求下述各隨機變數之p.d.f.。

(i) $Y = rX$, 其中$r > 1$爲一常數;

(ii) $Z = rX/(1 - X)$, 其中$r > 0$爲一常數。

- $Y = rX \Rightarrow y \in (0, r)$.
- (i) $f_Y(y) = f_X(x = y/r)|dx/dy|$.

$$f_Y(y) = \frac{(y/r)^{\alpha-1}(1 - y/r)^{\beta-1}}{B(\alpha, \beta)} \cdot \frac{1}{r} = \frac{y^{\alpha-1}(r - y)^{\beta-1}}{B(\alpha, \beta) r^{\alpha+\beta-1}}, \ y \in (0, r).$$

§2.3 #23

- (ii) $Z = rX/(1-X)$, so that $x \in (0,1) \Rightarrow z > 0$.
- $$f_Z(z) = f_X(x = z/(r+z))|dx/dz| = f_X(x = \frac{z}{(r+z)}) \cdot \frac{r}{(r+z)^2}.$$
- $$\begin{aligned} f_Z(z) &= \frac{(z/(r+z))^{\alpha-1}(1-z/(r+z))^{\beta-1}}{B(\alpha,\beta)} \cdot \frac{r}{(r+z)^2} \\ &= \frac{(z/(r+z))^{\alpha-1}(r/(r+z))^{\beta-1}}{B(\alpha,\beta)} \cdot \frac{r}{(r+z)^2} \\ &= \frac{z^{\alpha-1}r^{\beta}}{B(\alpha,\beta)(r+z)^{\alpha+\beta}},\ z > 0. \end{aligned}$$

29. 設X有$\mathcal{U}(-\pi/2, \pi/2)$分佈。試求$Y = \tan X$之分佈。

- $f_Y(y) = f_X(x = \tan^{-1} y)|dx/dy|$, and $x \in (-\pi/2, \pi/2) \Rightarrow y \in (-\infty, \infty)$.
- $$f_Y(y) = \pi^{-1} \frac{1}{1+y^2} = \frac{1}{\pi(1+y^2)}, \; y \in \mathbb{R}.$$
  i.e., $Y \sim C(0, 1)$ (standard Cauchy).
- Note:
  $$dx/dy = \frac{1}{(dy/dx)} = \frac{1}{\sec^2 x} = \frac{1}{(1+\tan^2 x)} = \frac{1}{(1+y^2)}.$$
  $$dy/dx = d\tan x/dx = d[\sin x/\cos x]/dx = \cdots = 1/\cos^2 x = \sec^2 x.$$