TA section 1

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Homework 1

§1.1 #5, #13, #20, #25, #26

5. 設$\Omega = \{1,2,3,4\}$。試寫出包含$\{2\}$及$\{1,4\}$之最小的σ-體。

• Definition

A class of subsets of Ω, denoted as $\mathcal{F}$, is a σ-algebra (information set) if
(i) $\Omega \in \mathcal{F}$;
(ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
(iii) $A_i \in \mathcal{F}$, $\forall i$, then $\cup_i A_i \in \mathcal{F}$.
A σ-algebra is closed under countable unions and complements.

- The σ-algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$, is the smallest σ-algebra in $\mathcal{F}$, which includes all elements of $\mathcal{C}$, i.e., $\mathcal{C} \in \mathcal{F}$.
- In words, a $\sigma(\mathcal{C})$ is the smallest set of subsets, which contains every element which is in the intersection and in the union, by De Morgan's Law.

- De Morgan's Law:

$$(A \cap B)^c = A^c \cup B^c, \;\; (A \cup B)^c = A^c \cap B^c.$$

- $\Omega := \{1, 2, 3, 4\}.$
- $\sigma(\{2\}, \{1, 4\}) = \{\phi, \Omega, \{2\}, \{1, 4\}, \{1, 3, 4\}, \{2, 3\}, \{1, 2, 4\}, \{3\}\}.$
- $\{2\}^c = \{1, 3, 4\}$, $\{1, 4\}^c = \{2, 3\}$, $\{2\} \cap \{1, 4\} = \phi$, $\{2\} \cup \{1, 4\} = \{1, 2, 4\}$, $\{1, 3, 4\} \cap \{2, 3\} = \{3\}$, $\{1, 3, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\} = \Omega$.
- total $\# = 2^{1+2} = 8$.

• Proposition

設 $A_1, A_2, \cdots, A_k$ 為樣本空間 Ω 中之互斥非空子集。
(i) 若 $\cup_{i=1}^k A_i = \Omega$, 則包含 $A_1, A_2, \cdots, A_k$ 之最小 σ-algebra, 共有 2^k 個元素;
(ii) 若 $\cup_{i=1}^k A_i \neq \Omega$, 則包含 $A_1, A_2, \cdots, A_k$ 之最小 σ-algebra, 共有 2^{1+k} 個元素。

13. 投擲一骰子一次，並觀測所得之點數。試給出兩個不同的機率空間。

- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- σ-algebra: $\mathcal{F} = \{\mathcal{A} : \mathcal{A} \subseteq \Omega\}$, 包含樣本空間 Ω 的所有可能事件 (子集) 的集合。
- Probability function: consider $\mathbb{P}_1(\omega) := 1/6$, $\forall\, \omega \in \{1, 2, 3, 4, 5, 6\}$, and

$$\mathbb{P}_2(\omega) := \begin{cases} 0.1, & \omega \in \{1, 2, 3, 4, 5\} \\ 0.5, & \omega \in \{6\}. \end{cases}$$

  $\mathbb{P}(\Omega) = 1$.
- $(\Omega, \mathcal{F}, \mathbb{P}_1)$ and $(\Omega, \mathcal{F}, \mathbb{P}_2)$ are two possible probability spaces.

20. 設$P(A)=1/3$, $P(B^c)=1/4$, 試問A,B可否爲互斥事件。

- No. If yes, $\mathbb{P}(A\cap B)=0$. But,
  $\mathbb{P}(A\cup B)=\mathbb{P}(A)+\mathbb{P}(B)=13/12>1(\rightarrow\leftarrow)$.
  So, the two events cannot be disjoint.

25. 某家庭有10位成員，試求其生日皆相異的機率。

26. 設事件$A_1 \subset A_2 \subset A_3$，且$P(A_1) = 1/4$, $P(A_2) = 5/12$, $P(A_3) = 7/12$。試求下述各事件之機率: $A_1^c \cap A_2$, $A_1^c \cap A_3$, $A_2^c \cap A_3$, $A_1 \cap A_2^c \cap A_3^c$, $A_1^c \cap A_2^c \cap A_3^c$。

25

- $$\mathbb{P}(\text{所有人生日相異}) = \frac{365}{365} \times \frac{364}{365} \times \cdots \frac{356}{365}.$$

- 或
$$\mathbb{P}(\text{所有人生日相異}) = \binom{365}{10} / 365^{10}.$$

26

- Key: $A_1 \subset A_2 \subset A_3$.
- $\mathbb{P}(A_2) = \mathbb{P}(\{A_1 \cup A_1^c\} \cap A_2) = \mathbb{P}(\{A_1 \cap A_2\} \cup \{A_1^c \cap A_2\}) = \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1^c \cap A_2)$
  $\Rightarrow \mathbb{P}(A_1^c \cap A_2) = \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) = 5/12 - 1/4 = 1/6.$
  Similary,
- $\mathbb{P}(A_1^c \cap A_3) = \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_3) = 7/12 - 1/4 = 1/3.$
- $\mathbb{P}(A_2^c \cap A_3) = \mathbb{P}(A_3) - \mathbb{P}(A_2 \cap A_3) = 7/12 - 5/12 = 1/6.$
- $\mathbb{P}(A_1 \cap A_2^c \cap A_3^c) = 0.$
- $\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) = \mathbb{P}(A_2^c \cap A_3^c) - \mathbb{P}(A_1 \cap A_2^c \cap A_3^c) = \mathbb{P}(A_2^c \cap A_3^c)$
  $= \mathbb{P}((A_2 \cup A_3)^c) = 1 - \mathbb{P}(A_2 \cup A_3) = 1 - \mathbb{P}(A_3) = 1 - 7/12 = 5/12.$