TA section 2

JERRY C.

Email: 108354501@nccu.edu.tw

Website: jerryc520.github.io/teach/MS.html

October 7, 2024

Review Exercises

§1.2-#10, #13, #14, #15; §1.3-#4

#14:
Q: A, B 不獨立, 問: $\mathbb{P}(A|B) > \mathbb{P}(A)$?
A: No.

- Two events A and B are independent if and only if: $\mathbb{P}(A|B) = \mathbb{P}(A)$.
- If A, B are dependent, then $\mathbb{P}(A|B) \neq \mathbb{P}(A)$ that we can only identify.
- Example: $\Omega = \{1, 2, 3, 4\}$,
  $\mathbb{P}(\{1\}) = 0.1, \mathbb{P}(\{2\}) = 0.2, \mathbb{P}(\{3\}) = 0.3, \mathbb{P}(\{4\}) = 0.4$.
- Let $A := \{3, 4\}, B := \{1, 2\}$. Then, $A \cap B = \phi$.
- $\mathbb{P}(A \cap B) = 0 \neq \mathbb{P}(A)\,\mathbb{P}(B) = 0.7 \times 0.3 = 0.21$. 即, A, B 不獨立, 但:
- $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\,\mathbb{P}(B) = 0/0.3 = 0 < \mathbb{P}(A) = 0.7$.

10. 設A與B獨立, A與C獨立, 且$B \cap C = \emptyset$。

(i) 試證A與$B \cup C$獨立;

(ii) 試舉一例說明若$B \cap C \neq \emptyset$, 則(i)之結論便不一定成立。

- (i). $(A \cap B) \cap (A \cap C) = A \cap (B \cap C) = A \cap \phi = \phi$,
- $\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}((A \cap B) \cup (A \cap C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) = \mathbb{P}(A)\,\mathbb{P}(B) + \mathbb{P}(A)\,\mathbb{P}(C) = \mathbb{P}(A)(\mathbb{P}(B) + \mathbb{P}(C)) = \mathbb{P}(A)\,\mathbb{P}(B \cup C)$. So, $A \perp\!\!\!\perp (B \cup C)$.

- (ii). $\Omega = \{1,2,3,4\}$. $\mathbb{P}(\{w\}) = 1/4, \forall\ w \in \Omega$. Let $A = \{1,2\}, B = \{1,3\}, C = \{1,4\}$, then $B \cap C = \{1\} \neq \phi$.
- $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/2$. $\mathbb{P}(B \cap C) = \mathbb{P}(\{1\}) = 1/4$.
- $B \cup C = \{1,3,4\}$, $\mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(\{1\}) = 1/4 \neq \mathbb{P}(A)\,\mathbb{P}(B \cup C) = 1/2 \times 3/4 = 3/8$.

13. 設$P(A), P(B) > 0$, 且$P(A|B) > P(A)$。試證$P(B|A) > P(B)$。

- Given $\mathbb{P}(A|B) > \mathbb{P}(A)$, it implies that $P(A \cap B) > \mathbb{P}(A)\,\mathbb{P}(B)$. Then,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} > \frac{\mathbb{P}(A)\,\mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(B).$$

15. 試證

(i) $P(A^c|B) = 1 - P(A|B)$;

(ii) $P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$。

- Hint:
  (i) by $\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$.
  (ii) by $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$.

- (i). $\mathbb{P}(A^c|B) = \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 1 - \mathbb{P}(A|B)$.
- (ii). $\mathbb{P}((A \cup B)|C) = \frac{\mathbb{P}((A \cup B) \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}((A \cap C) \cup (B \cap C))}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A|C) + \mathbb{P}(B|C) - \mathbb{P}((A \cap B)|C)$.

4. 投擲一公正的骰子一次，令X表所得之點數除以4之餘數。試求X之值域Ω_1，並對$\forall x \in \Omega_1$，給出$P_X(X=x)$。

- $(1 \mod 4) = 1$; $(2 \mod 4) = 2$; $(3 \mod 4) = 3$; $(4 \mod 4) = 0$; $(5 \mod 4) = 1$; $(6 \mod 4) = 2$.
- 令投擲骰子之點數為 Y, $Y \in \{1,2,3,4,5,6\}$, 點數除四餘數為 X, $X \in \{0,1,2,3\}$;
- $\Omega_1 := \{0,1,2,3\}$;
- $\mathbb{P}(X=0) = \mathbb{P}(Y=4) = 1/6$, $\mathbb{P}(X=1) = \mathbb{P}(Y \in \{1,5\}) = 2/6$, $\mathbb{P}(X=2) = \mathbb{P}(Y \in \{2,6\}) = 2/6$, $\mathbb{P}(X=3) = \mathbb{P}(Y=3) = 1/6$;
- pmf:

x	0	1	2	3
$\mathbb{P}_X(X=x)$	1/6	2/6	2/6	1/6