TA section 6

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Homework 3: part (I)

§7.2 #2, #9, #11, #13, #18; §7.3 #1

2. 設T_1, T_2皆爲θ之不偏估計量，且T_1與T_2獨立，變異數分別爲σ_1^2及σ_2^2。試求a, b，使得$aT_1 + bT_2$爲θ之不偏估計量中MSE最小者。(**解.** $a = \sigma_2^2/(\sigma_1^2+\sigma_2^2), b = 1-a$)

- WLOG, assume $\theta \neq 0$.
- $T_1 \perp\!\!\!\perp T_2$. Let $Z := aT_1 + bT_2$.
- $\mathbb{E}[T_1] = \mathbb{E}[T_2] = \theta$, $\mathbb{E}[Z] = a\,\mathbb{E}[T_1] + b\,\mathbb{E}[T_2] = \theta \Rightarrow (a+b)\theta = \theta$. Thus, $b = 1 - a$.
- MSE: $R(\theta, Z) = \mathbb{E}[(Z-\theta)^2] = \mathrm{Var}[Z] = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$ $(Bias(Z) = 0)$
- $\frac{dR}{da}\big|_{a=a^*} = 0$, $\frac{d^2R}{da^2}\big|_{a=a*} > 0$.
- Solve for $a^* = \sigma_2^2/(\sigma_1^2+\sigma_2^2)$ and $b^* = 1 - a^* = \sigma_1^2/(\sigma_1^2+\sigma_2^2)$.

9. 設$X_1, \cdots, X_n$爲一組由$\mathcal{U}[1, 1+\theta]$分佈所產生之隨機樣本, $\theta > 0$ 。

(i) 試求θ之MLE T_1, 並問T_1是否爲不偏的;

(ii) 試求θ之動差估計量T_2, 並問T_2是否爲不偏的;

(iii) 試比較T_1與T_2之MSE 。

- $L(\theta) = \theta^{-n} \prod_{i=1}^{n} \boldsymbol{I}(1 \leq x_i \leq 1+\theta)$, $i = 1, 2, \cdots, n$.
- $T_1 := \widehat{\theta}_{MLE} = X_{(n)} - 1$.
- $f_{X_{(n)}}(t) = n(t-1)^{n-1}/\theta^n, \ \theta \in [1, 1+\theta]$.
- $$\mathbb{E}[T_1] = \mathbb{E}[X_{(n)}] - 1 = n\theta^{-n} \int_1^{1+\theta} t(t-1)^{n-1} dt - 1 = 1 + \frac{n\theta}{n+1} - 1 = \frac{n\theta}{n+1}.$$

  So, T_1 is biased for θ, $Bias(T_1) = -\theta/(n+1)$.

- $\mathbb{E}[X] = (1+1+\theta)/2 = 1+\theta/2 \Rightarrow \widehat{\theta}_{MME} = 2(\bar{X}-1) =: T_2$.
- $\mathbb{E}[T_2] = 2(\mathbb{E}[\bar{X}]-1) = 2(1+\theta/2-1) = \theta$, i.e., T_2 is unbiased for θ.
- $\mathbb{E}[X_{(n)}^2] = 1 + 2n\theta/(n+1) + n\theta^2/(n+2)$,
  $\mathrm{Var}[X_{(n)}] = \mathbb{E}[X_{(n)}^2] - \mathbb{E}[X_{(n)}]^2 = n\theta^2/(n+1)^2(n+2)$.
- MSE:
$$R(\theta, T_1) = Bias(T_1)^2 + \mathrm{Var}[X_{(n)}] = \frac{2\theta^2}{(n+1)^2(n+2)}.$$
-
$$R(\theta, T_2) = \mathrm{Var}[T_2] = 4\,\mathrm{Var}[\bar{X}] = \frac{\theta^2}{3n} > R(\theta, T_1).$$

11. 設X有$\mathcal{P}(\lambda)$分佈, $\lambda > 0$。令$\theta = P(X=0) = e^{-\lambda}$。

(i) 試問$T_1 = e^{-X}$是否爲θ之不偏估計量;

(ii) 試證$T_2 = I_{\{X=0\}}$爲θ之不偏估計量;

(iii) 試分別求T_1及T_2之MSE。

- By MGF $M_X(t) = \mathbb{E}[e^{tX}] = e^{\lambda(e^t-1)}$.
- 令 $t=-1$, $\mathbb{E}[T_1] = \mathbb{E}[e^{-X}] = e^{-\lambda(1-1/e)}$, T_1 is biased for θ.
- $\mathbb{E}[T_2] = \mathbb{P}(X=0) = e^{-\lambda} = \theta$, T_2 is unbiased for θ.

MSE:

- $$
\begin{aligned}
R(\theta, T_1) &= Bias(T_1)^2 + \mathrm{Var}[T_1] \\
&= e^{-2\lambda}[e^{\lambda/e} - 1] + [e^{\lambda/e^2} - e^{2\lambda/e}]e^{-2\lambda} \\
&= (1 - 2e^{\lambda/e} + e^{\lambda/e^2})e^{-2\lambda}.
\end{aligned}
$$

- $R(\theta, T_2) = \mathrm{Var}[T_2] = \mathbb{E}[T_2^2] - \mathbb{E}[T_2]^2 = \mathbb{P}(X=0) - \mathbb{P}(X=0)^2 = e^{-\lambda}(1 - e^{-\lambda})$.

13. 設$X_1, \cdots, X_n$爲一組由某一期望值爲μ, 變異數爲σ^2之分佈所產生之隨機樣本, μ, σ^2皆設爲未知 。令$T(\boldsymbol{X}) = \sum_{i=1}^n c_i X_i$, 其中$c_1, \cdots, c_n$爲常數 。

(i) 試證T爲μ之不偏估計量, 若且唯若$\sum_{i=1}^n c_i = 1$;

(ii) 試證在型如$\sum_{i=1}^n c_i X_i$之μ的不偏估計量中, $\overline{X}_n$爲一致最小變異不偏估計量 。

- $\mathbb{E}[T] = \sum_{i=1}^n c_i \, \mathbb{E}[X_i] = \mu \sum_{i=1}^n c_i = \mu$ iff $\sum_{i=1}^n c_i = 1$.

- $\text{Var}[T] = \sum_{i=1}^n c_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^n c_i^2 =: Q(c)$.
- $$c_i^* = \arg\min_{c_i}\{Q(c) : \sum_{i=1}^n c_i = 1, i = 1, \cdots, n\}.$$
- By Lagrangian multiplier:
$$L(c, \lambda) = Q(c) + \lambda(\sum_{i=1}^n c_i - 1),$$
- $\partial L/\partial c_i = 0 \Rightarrow c_i = -\lambda/2\sigma^2$;
- $\partial L/\partial \lambda = 0 \Rightarrow \sum_{i=1}^n c_i = 1 \Rightarrow \lambda = -2\sigma^2/n$;
- So, $c_i^* = 1/n$,
  i.e., $T(\boldsymbol{X}) = n^{-1} \sum_{i=1}^n X_i = \bar{X}$ is an UMVUE of μ.

18. 設$X_1, \cdots, X_n$爲一組由p.d.f.$f(x|\theta) = \theta(1+x)^{-(1+\theta)}$, $x > 0$, $\theta > 0$, 所產生之隨機樣本。

(i) 試求θ之一不偏估計量;

(ii) 是否存在一$g(\theta) = \theta^{-1}$之不偏估計量? 若有則給出一個。

- $f(x|\theta) = \theta \exp[-(1+\theta)\log(1+x)] =: h(x)c(\theta)\exp(w(\theta)t(x))\boldsymbol{I}_A(x)$, where $h(x) = \boldsymbol{I}(x_i > 0)$, $c(\theta) = \theta$, $w(\theta) = -(1+\theta)$, and $t(x) = \log(1+x)$. So, $C = \{w(\theta) : \theta \in \Omega\}$ contains a nonempty open set, $T(\boldsymbol{X}) = \sum_{i=1}^n \log(1+X_i)$ is a C.S.S.
- Let $Y = \log(1+X) \Rightarrow X = e^Y - 1$,
- $$f_Y(y) = f_X(e^y - 1).e^y = \theta e^{-\theta y},$$
  i.e., $Y \sim \mathcal{E}(\theta) = \Gamma(1, 1/\theta)$.
- $T \sim \sum_{i=1}^n Y_i \sim \Gamma(n, 1/\theta)$.

- $\mathbb{E}[T^{-1}] = \theta/(n-1) \Rightarrow \mathbb{E}[(n-1)/T] = \theta$.
  (recall: $Z \sim \Gamma(\alpha, \beta) \Rightarrow \mathbb{E}[Z^k] = \beta^k \Gamma(\alpha + k)/\Gamma(\alpha)$).
- Let
  $$T_1 := \frac{(n-1)}{\sum_{i=1}^{n} \log(1 + X_i)},$$
  which is unbiased for θ, such that $\mathbb{E}[T_1] = \theta$.
- $\mathbb{E}[T] = n/\theta \Rightarrow \mathbb{E}[T/n] = 1/\theta$, i.e., let
  $$T_2 := \sum_{i=1}^{n} \log(1 + X_i)/n,$$
  which is unbiased for $1/\theta$, such that $\mathbb{E}[T_2] = 1/\theta$.

1. 設$X_1, \cdots, X_n$為一組由$\mathcal{B}er(\theta)$分佈所產生之隨機樣本, $0 \le \theta \le 1$。試分別求θ, θ^2, $\theta(1-\theta)$之UMVUE。

求 UMVUE 三招:

- 「充份 + 不偏」: 定理 3.1: Rao-Blackwell Theorem (非唯一解)
- 「完備充份 + 不偏」: 定理 3.2: Lehmann-Scheffé Theorem (唯一解)
- 「不偏 + CRLB」: (i) "Efficiency Attainment" (最效率可達性); (ii) 定理 4.3: 滿足 CRLB 的不偏之一個參數 (one-dimensional) 指數族。

使用 R-B Thm. & L-S Thm 原則:

- 給定 $T(\boldsymbol{X})$ is a C.S.S.,
- 設法找一個 $h(T(\boldsymbol{X}))$ 為 $q(\theta)$ 之不偏估計量, 則 $h(T(\boldsymbol{X})) = \mathbb{E}[h(T(\boldsymbol{X}))|T(\boldsymbol{X})]$ 為 $q(\theta)$ 之一 UMVUE。
- 若 $h(T(\boldsymbol{X}))$ 不易找出, 則設法造出:
  - (1): 找一個 $q(\theta)$ 之不偏估計量,$S(\boldsymbol{X})$ (不一定是 $T(\boldsymbol{X})$ 的函數, 若是 $T(\boldsymbol{X})$ 的函數, 則同上);
  - (2): 造出 $\mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})]$, 此即 $q(\theta)$ 之一 UMVUE。

定理 3.1.: 充份 $+$ 不偏 $\Rightarrow$ 有效

• Theorem

設 $T(\boldsymbol{X})$ 為 θ 之一充份統計量, 設 $S(\boldsymbol{X})$ 為 $q(\theta)$ 之一不偏估計量, 且 $\mathbb{E}\,|S(\boldsymbol{X})| < \infty$, $\forall \theta \in \Omega$。令 $T^*(\boldsymbol{X}) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})]$, 則 $\forall \theta \in \Omega$,

$$R(\theta, T^*) \leq R(\theta, S).$$

定理 3.2.: 完備充份 $+$ 不偏 $\Rightarrow$ 有效

• Theorem

設 $T(\boldsymbol{X})$ 為一完備充份統計量, 且 $S = S(\boldsymbol{X})$ 為 $q(\theta)$ 之一不偏估計量。則 $T^*(\boldsymbol{X}) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})]$ 為 $q(\theta)$ 之一 UMVUE; 若 $\mathrm{Var}[T^*] < \infty, \forall\theta \in \Omega$, 則 T^* 為 $q(\theta)$ 唯一之 UMVUE。

定理 4.3.

• Theorem

設 $T(\boldsymbol{X})$ 為 $q(\theta)$ 一不偏估計量,$\mathbb{E}[T(\boldsymbol{X})] = q(\theta)$。設一分佈族 $\{P_\theta; \theta \in \Omega\}$ 滿足正規條件, 且為一個參數之指數族, 有 pdf 如下式:

$$f(\boldsymbol{x}|\theta) = h(\boldsymbol{x}) \exp(w(\theta)T(\boldsymbol{x}))\boldsymbol{I}_A(\boldsymbol{x}), \;\; \theta \in \Omega,$$

其中 $w(\theta)$ 有一連續且不為零之導數, $\forall \theta \in \Omega$, 若且唯若 $\mathrm{Var}[T(\boldsymbol{X})]$ 達到 CRLB, 且 $T(\boldsymbol{X})$ 為 $q(\theta)$ 之一 UMVUE。

最效率可達性

Theorem (Efficiency Attainment)

假設 $\boldsymbol{X} = (X_1, \cdots, X_n)$ 有一 joint pdf $f(\boldsymbol{x}|\theta)$, 與其對應之 likelihood function $L(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i|\theta)$, 則對 $q(\theta)$ 的任一不偏估計量 $U(\boldsymbol{X})$ 之變異數可達到 CRLB, 若且唯若存在一個 (θ, n) 的函數 $a(\theta, n)$ 使得以下等式成立:

$$a(\theta, n)\Big[U(\boldsymbol{X}) - q(\theta)\Big] = \frac{\partial}{\partial\theta} \log L(\theta|\boldsymbol{x}).$$

- $T(\boldsymbol{X}) = \sum_{i=1}^{n} X_i$ is a C.S.S.
- Let $S(\boldsymbol{X}) = \bar{X}$, such that $\mathbb{E}[S(\boldsymbol{X})] = \theta$.
- Then, let
$$h_1(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X}),$$
  $\mathbb{E}[h_1(T(\boldsymbol{X}))] = \mathbb{E}[\bar{X}] = \theta$. 故, $S(\boldsymbol{X}) = \bar{X}$ 為 θ 之一不偏估計量, 且為 $T(\boldsymbol{X})$ 的函數。So, $h_1(T(\boldsymbol{X})) = \bar{X}$ is an UMVUE of θ by R-B Thm & L-S Thm.

- Let $S(\boldsymbol{X}) = (n/(n-1))\bar{X}(1-\bar{X})$, such that $\mathbb{E}[S(\boldsymbol{X})] = \theta(1-\theta)$.
- Let $h_2(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X})$ ($\because S(\boldsymbol{X})$ is a function of $T(\boldsymbol{X})$),

$$\mathbb{E}[h_2(T(\boldsymbol{X}))] = \mathbb{E}[S(\boldsymbol{X})] = \mathbb{E}[(\frac{n}{n-1})\bar{X}(1-\bar{X})] = \theta(1-\theta).$$

Since that $S(\boldsymbol{X})$ is unbiased for $\theta(1-\theta)$ and is a function of $T(\boldsymbol{X})$. So, $h_2(T(\boldsymbol{X})) = (n/(n-1))\bar{X}(1-\bar{X})$ is an UMVUE of $\theta(1-\theta)$ by R-B Thm & L-S Them.

- Let $S(\boldsymbol{X}) = T(\boldsymbol{X})(T(\boldsymbol{X}) - 1)/n(n-1)$, such that $\mathbb{E}[S(\boldsymbol{X})] = \theta^2$.
- Let $h_3(T(\boldsymbol{X})) = \mathbb{E}[S(\boldsymbol{X})|T(\boldsymbol{X})] = S(\boldsymbol{X})$.
- Since $\mathbb{E}[T^2(\boldsymbol{X})] - \mathbb{E}[T(\boldsymbol{X})] = \mathrm{Var}[T(\boldsymbol{X})] + \mathbb{E}[T(\boldsymbol{X})]^2 - \mathbb{E}[T(\boldsymbol{X})] = n\theta(1-\theta) + n^2\theta^2 - n\theta = n(n-1)\theta^2$, then

$$\mathbb{E}[h_3(T(\boldsymbol{X}))] = \mathbb{E}[S(\boldsymbol{X})] = \mathbb{E}[\frac{T(\boldsymbol{X})(T(\boldsymbol{X}) - 1)}{n(n-1)}] = \theta^2.$$

So, $h_3(T(\boldsymbol{X})) = \sum_{i=1}^n X_i(\sum_{i=1}^n X_i - 1)/n(n-1)$ is an UMVUE of θ^2 by R-B Thm & L-S Thm.