TA section 2

JERRY C.

Email: 108354501@nccu.edu.tw

Website: jerryc520.github.io/teach/MS.html

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HW 1: Part (I)

§4.2#1, #9, #14, §4.3#6, #10

1. 設X_n有$P(n\lambda)$分佈，$n \geq 1$, $\lambda > 0$。試證$X_n/n \xrightarrow[n \to \infty]{p} \lambda$, 且$(X_n - n\lambda)/(\sqrt{n\lambda}) \xrightarrow[n \to \infty]{d} \mathcal{N}(0,1)$。

- $\mathbb{E}[X_n] = \mathrm{Var}[X_n] = n\lambda$; $\mathbb{E}[X_n/n] = \lambda$, $\mathrm{Var}[X_n/n] = \lambda/n$, and by Chebyshev's inequality,

$$\mathbb{P}(|X_n/n - \mathbb{E}[X_n/n]| > \epsilon) \leq \mathrm{Var}[X_n/n]/\epsilon^2, \ \forall \epsilon > 0.$$

Then,

$$\mathbb{P}(|X_n/n - \lambda| > \epsilon) \leq \lambda/(n\epsilon^2) \to 0$$

as $n \to \infty$. So, $X_n/n \xrightarrow{\text{p}} \lambda$.

- 另解: $X_n := \xi_1 + \cdots + \xi_n$ $\xi_i \sim i.i.d. P(\lambda)$ with $\mathbb{E}[\xi_i] = \lambda$. By LLN, $X_n/n \xrightarrow{\text{p}} \mathbb{E}[\xi_i] = \lambda$. So, $X_n/n \xrightarrow{\text{p}} \lambda$.

- Let $X_n := \xi_1 + \cdots + \xi_n,\ \xi_i \sim i.i.d.P(\lambda)$ with $\mathbb{E}[\xi_i] = \mathrm{Var}[\xi_i] = \lambda$.
- By CLT,

$$(X_n - n\lambda)/\sqrt{n\lambda} = (X_n/n - \lambda)/\sqrt{\lambda/n} \xrightarrow{\mathsf{d}} N(0,1)$$

as $n \to \infty$.

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9. 設X_n有$\Gamma(n,\lambda)$分佈, $n \geq 1$。試證$X_n/n \xrightarrow[n\to\infty]{p} \lambda$, 並利用中央極限定理, 估計$n$很大時, $P(X_n \leq x)$之值, $x \in R$。

- $\mathbb{E}[X_n] = n\lambda, \mathrm{Var}[X_n] = n\lambda^2$; $\mathbb{E}[X_n/n] = \lambda$, $\mathrm{Var}[X_n/n] = \lambda^2/n$, and by Chebyshev's inequality,

$$\mathbb{P}(|X_n/n - \mathbb{E}[X_n/n]| > \epsilon) \leq \mathrm{Var}[X_n/n]/\epsilon^2, \ \forall \epsilon > 0.$$

  Then,

$$\mathbb{P}(|X_n/n - \lambda| > \epsilon) \leq \lambda^2/(n\epsilon^2) \to 0$$

  as $n \to \infty$. So, $X_n/n \xrightarrow{\mathsf{p}} \lambda$.
- 另解: $X_n := \xi_1 + \cdots + \xi_n$ $\xi_i \sim i.i.d.\Gamma(1,\lambda)$ with $\mathbb{E}[\xi_i] = \lambda$. By LLN, $X_n/n \xrightarrow{\mathsf{p}} \mathbb{E}[\xi_i] = \lambda$. So, $X_n/n \xrightarrow{\mathsf{p}} \lambda$.

- $X_n := \xi_1 + \cdots + \xi_n \ \xi_i \sim i.i.d.\Gamma(1, \lambda)$ with $\mathbb{E}[\xi_i] = \lambda$ and $\mathrm{Var}[\xi_i] = \lambda^2$.
- $$\frac{X_n/n - \lambda}{\lambda/\sqrt{n}} = \frac{(X_n - n\lambda)}{\lambda\sqrt{n}}.$$
- By CLT,
$$\frac{X_n/n - \lambda}{\lambda/\sqrt{n}} \xrightarrow{\text{d}} N(0, 1)$$
as $n \to \infty$.
- So, $\mathbb{P}(X_n \leq x) \approx \Phi(\frac{x-n\lambda}{\lambda\sqrt{n}})$, as $n \to \infty$.

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14. 設 $X_1 \sim \mathcal{U}[0,1]$, 令 $X_n = X_1^n$, $n \geq 1$。試證 $X_n \xrightarrow[n \to \infty]{p} 0$。

- Given any $\epsilon > 0$,
  $\mathbb{P}(|X_n| > \epsilon) = \mathbb{P}(X_1 > \epsilon^{1/n}) = 1 - \epsilon^{1/n}$.
- $$\lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) = 1 - \lim_{n\to\infty} \epsilon^{1/n} = 0.$$

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6. 設X_1, X_2, X_3爲獨立的隨機變數，且X_i有$\mathcal{N}(i, i^2)$分佈，$i = 1, 2, 3$。試利用X_1, X_2, X_3的函數，分別造出有如下的分佈。
(i)χ_3^2, (ii)$\mathcal{T}_2$, (iii)$\mathcal{F}_{1,2}$。

(iii)

- Let $Z_i := (X_i - i)/i \sim N(0,1)$, for $i = 1, 2, 3$.
- $\because Z_1^2 \sim \chi^2(1), Z_2^2 + Z_3^2 \sim \chi^2(2), \therefore F = \frac{Z_1^2/1}{(Z_2^2+Z_3^2)/2} \sim F(1,2)$.
- So, let $U := Z_1^2 = (X_1 - 1)^2/1, V := Z_2^2 := (X_2 - 2)^2/2, W := (X_3 - 3)^2/3$, we have $\frac{U}{V+W/2} \sim F(1,2)$.

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10. 設X_1與X_2獨立, 且皆有$\mathcal{N}(0, 25)$分佈。令$D = \sqrt{X_1^2 + X_2^2}$。試求$P(D \leq 12.25)$。(**解.** 約0.95)

- $X_i = 5Z_i, Z_i \sim N(0,1)$ for $i = 1, 2$. Then,
  $D = (X_1^2 + X_2^2)^{1/2} = 5(Z_1^2 + Z_2^2)^{1/2} = 5(\chi^2(2))^{1/2}$
- $\mathbb{P}(D \leq 12.25) = \mathbb{P}(\chi^2(2) \leq (12.25/5)^2) = \mathbb{P}(\chi^2(2) \leq (2.45)^2) = 1 - \exp(-(2.45)^2/2) \approx 0.95.$
  (or 查卡方機率值表)
- Note 1: $W := \chi^2(2)$, $f_W(w) = (1/2)\exp(-w/2)$. Then,
  $\mathbb{P}(W \leq x) = \int_{w=0}^{x} f_W(w)dw = 1 - \exp(-x/2)$.
- Note 2: by CLT,

$$\mathbb{P}(\chi^2(2) \leq (2.45)^2) \approx \Phi(((2.45)^2 - 2)/\sqrt{4}) \approx 0.977.$$

不建議用 CLT 近似, 因為自由度只有 2.