Assignment 1

2013-30068
최정훈
경제학부 박사과정

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Value Function Iteration and Guess and Verify

(1) Write down the Bellman equation

Answer)

The model in question can be written as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \; u(F(k_t - k_{t+1} + (1-\delta)k_t)$$

subject to

$$0 \leq k_{t+1} \leq \; \text{ and } \; k_0 > 0 \; \text{ given.}$$

Let $v(k_0)$ be the maximized level of utility associated with optimizing behavior given initial capital k_0.

$$v(k_0) \equiv \max_{\{k_{t+1}\}_{t=0}^{\infty}} \; \sum_{t=0}^{\infty} \beta^t u\Big(F(k_t) - k_{t+1} + (1-\delta)k_t \Big) = \sum_{t=0}^{\infty} \beta^t u\Big(F(k_t^*) - k_{t+1}^* + (1+\delta)k_t^* \Big)$$

where $\{k_{t+1}\}_{t=0}^{\infty}$ is the optimal sequence of capital stock.

$$\begin{aligned} v(k_0) &= u\Big(F(k_0) - k_1^* + (1-\delta)k_0 \Big) + \sum_{t=1}^{\infty} \beta^t u\Big(F(k_t^*) - k_{t+1}^* + (1-\delta)k_t^* \Big) \\ &= u\Big(F(k_0) - k_1^* + (1-\delta)k_0 \Big) + \beta \underbrace{\sum_{t=1}^{\infty} \beta^{t-1} u\Big(F(k_t^*) - k_{t+1}^* + (1-\delta)k_t^* \Big)}_{v(k_1)} \end{aligned}$$

Therefore

$$v(k_0) = u\Big(F(k_0) - k_1^* + (1-\delta)k_0 \Big) + \beta v(k_1^*)$$

or

$$v(k_0) = \max_{k_1} \; \Big\{ u\Big(F(k_0) - k_1 + (1-\delta)k_0 \Big) + \beta v(k_1) \Big\} \tag{1}$$

The date $t = 0$ and $t = 1$ in (1) are arbitrary, so we can simply denote the current capital stock by k and the next period's capital stock k'. Then,

$$v(k) = \max_{k'} \; \Big\{ u\Big(F(k) - k' + (1-\delta)k \Big) + \beta v(k') \Big\}$$

(2) Now assume that $u(c) = \log(c)$ $F(k) = Ak^{\alpha}$ where $A > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$ and $\delta = 1$. Derive $V(k)$ by hand

Answer)

I make a guess : $v(k) = E + F\log(k)$ where E and F are undetermined coefficients. Then, the value function in question can be expressed as

$$v(k) = \max_{k'} \ \log(Ak^{\alpha} - k') + \beta(E + F\log k')$$

The F.O.C with respect to k' is as follows.

$$k' = \frac{\beta F}{1+\beta F} Ak^{\alpha} \tag{2}$$

From the resource constraint, $c = Ak^{\alpha} - k'$ and (2), we have

$$c = Ak^{\alpha} - \frac{\beta F}{1+\beta F}Ak^{\alpha} = Ak^{\alpha}\left(1 - \frac{\beta F}{1+\beta F}\right) = Ak^{\alpha}\left(\frac{1}{1+\beta F}\right) \tag{3}$$

Using (3), the Bellman equation in question can be expressed as

$$\begin{aligned}
E + F\log k &= \log\left(\frac{Ak^{\alpha}}{1+\beta F}\right) + \beta\left[E + F\log\left(\frac{\beta F}{1+\beta F}Ak^{\alpha}\right)\right] \\
&= \log\left(\frac{A}{1+\beta F}\right) + \alpha\log k + \beta E + \beta F\log\left(\frac{\beta F}{1+\beta F}A\right) + \beta F\alpha\log k \\
&= \log\left(\frac{A}{1+\beta F}\right) + \beta E + \beta F\log\left(\frac{\beta F A}{1+\beta F}\right) + \alpha\log k + \beta F\alpha\log k
\end{aligned}$$

Comparing both sides, we have E and F respectively as

$$\begin{aligned}
E &= \log\left(\frac{A}{1+\frac{\alpha\beta}{1-\alpha\beta}}\right) + \beta E + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\frac{\frac{\alpha\beta A}{1-\alpha\beta}}{1+\frac{\alpha\beta}{1-\alpha\beta}}\right) \\
&= \log A(1-\alpha\beta) + \beta E + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta A \\
&= \frac{1}{1-\beta}\left[\log A(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta A\right]
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
F &= \alpha(1+\beta F) \\
&= \frac{\alpha}{1-\alpha\beta}
\end{aligned} \tag{5}$$

Eventually, from (4) and (5), the Bellman equation is written as

$$v(k) = E + F\log k$$

where

$$\begin{aligned}
E &= \log A(1-\alpha\beta) + \beta E + \frac{\alpha\beta}{1-\alpha\beta}\log\alpha\beta A \\
F &= \frac{\alpha}{1-\alpha\beta}
\end{aligned}$$

(3) Derive your saving function $g(k)$.

Answer)

As shown in the formula in (2), we can derive the saving function $g(k)$ because we already have the undetermined coefficients E and F in (4) and (5). Formally,

$$g(k) = k' = \frac{\beta F}{1+\beta F} A k^{\alpha} = \frac{\frac{\alpha\beta}{1-\alpha\beta}}{1+\frac{\alpha\beta}{1-\alpha\beta}} A k^{\alpha} = \alpha\beta A k^{\alpha} \tag{6}$$

(4) Write a code for a value function iteration (grid search). Report total running time as well as total number of iterations.

Answer)

There are two m-files in the folder named $assignment1$. One is $main.m$ which is a run file and the other is $vfunc_iter.m$ that is a function file to implement value function iteration for different number of grids. The domain of k, $\mathcal{K} = [\underline{k}, \overline{k}]$ is that

$$\mathcal{K} = [0.1, 0.9]$$

Total running time and total number of iterations are reported as the table below.

	N=30	N=90
Total Running Time	4.7119 sec	14.331 sec
Total Number of Iterations	1389	1389

* Tolerance level : 1e-07

(5) Plot a $true$ value function from (2) and your approximation from (4) in one graph. Also plot a distance between two in $\mathcal{K}$.

Answer)

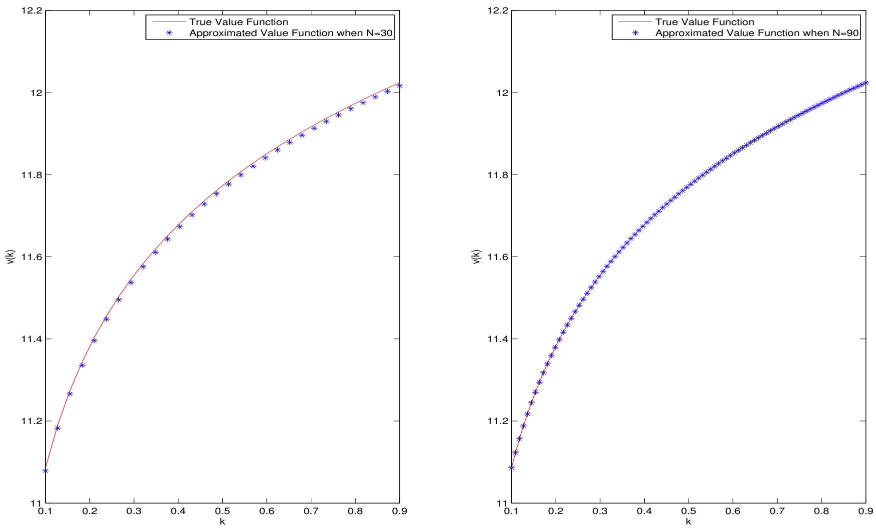

Figure 1: True and Approximated Value Function

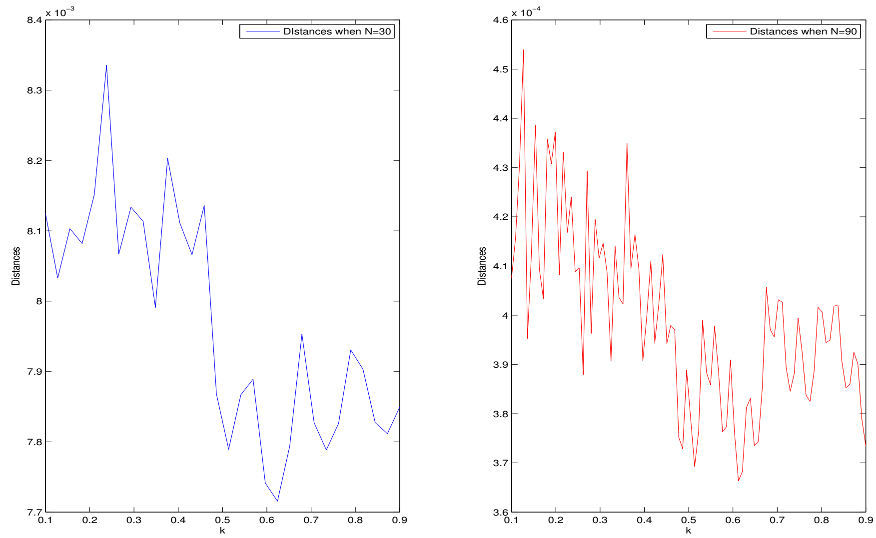

Figure 2: The Distance Between True and Approximated Value Functions

(6) Plot the approximation error of policy functions

Answer)

The approximation error is depicted in the figure below where $\hat{g}(k_i) - g(k_i)$ is the one in the left and $\tilde{g}(k_i) - g(k_i)$ is the one in the right.

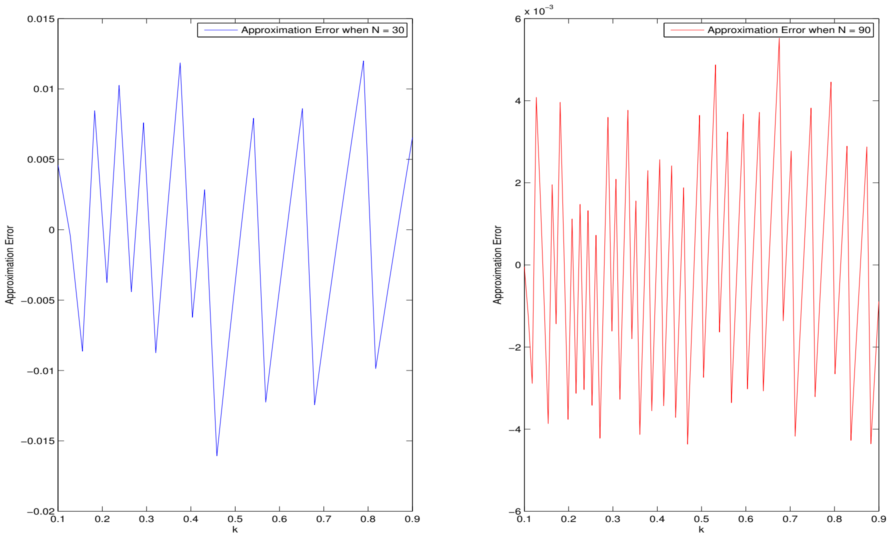

Figure 3: The Approximation Error of Policy Functions