FM1 Assignment #2 Name: Jerry Jiang

1. Let $A = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \le a+b \le 5\}$ and $B = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid b = a^2\}$. List the elements of $A \cap B$.

$B = \{(0,0), (\pm 1, 1), (\pm 2, 4), (\pm 3, 9), (\pm 4, 16) \cdots\}$

what does the $\times$ in $\mathbb{Z} \times \mathbb{Z}$ mean?

$A \cap B = \{(0,0), (1,1), (-1,1), (-2,4)\}$.

2. List the four possible reduced row echelon forms for a 2×2 matrix.

$\left(\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right)$

$\left(\begin{array}{cc|c} 1 & 0 & m \\ 0 & 0 & n \end{array}\right)$ what about $\left(\begin{array}{cc|c} 1 & a & m \\ 0 & 0 & n \end{array}\right)$?

that also counts right?

$\left(\begin{array}{cc|c} 0 & 1 & m \\ 0 & 0 & n \end{array}\right)$

$\left(\begin{array}{cc|c} 0 & 0 & m \\ 0 & 0 & n \end{array}\right)$

3. Let the radius of the circumcircle of $\triangle ABC$ be R. Prove $\dfrac{a}{\sin A} = 2R$ using the diagram below. (Can you see how this result can be used to prove the sine rule?)

$\angle BAC = \angle BA'C$

$\therefore \sin A = \sin A'$

when BA' pass O,

$\angle BCA' = 90°$

$\therefore \sin A' = \dfrac{BC}{2R} = \dfrac{a}{2R}$

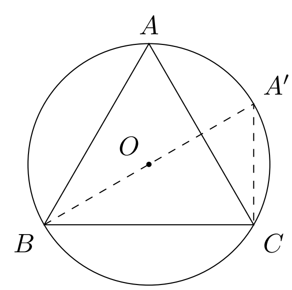
$\therefore \sin A = \dfrac{a}{2R}$

$\therefore \dfrac{a}{\sin A} = 2R$.

Similarly,

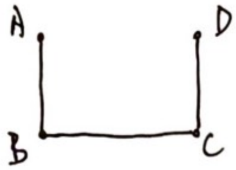
$2R = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$,

and that's the sine rule.



4. The *degree sequence* of a graph is the non-increasing list of its vertex degrees. A sequence is called *graphic* if there is a simple graph whose degree sequence is that sequence.

(a) Draw a simple graph to show that the sequence $2, 2, 1, 1$ is graphic.



$\deg(B) = \deg(C) = 2.$

$\deg(A) = \deg(D) = 1.$

(b) Explain why the sequence $4, 3, 2, 1, 1$ is not graphic.

there're 5 vertices, if it's simple, which means all vertices are connected to each other, there're at most $\binom{5}{2}$ edges. since $4+3+2+1+1 = 11 > \binom{5}{2}$, the graph is not simple, thus the sequence is not graphic.

5. (a) Explain why $(1 + x)^n > 1 + nx$ for $x > 0$ and $n > 1$.

- $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-2}x^{n-2} + \binom{n}{n-1}x^{n-1} + x^n$
  $= 1 + nx + \cdots + nx^{n-1} + x^n$
- since $x>0$, $n>1$, $nx^{n-1} + x^n > 0$,
- so $(1+x)^n > 1+nx$.

(b) Hence deduce that $r^n \to \infty$ as $n \to \infty$ when $r > 1$.

- let $r = 1+x$, then $x = r-1$.
- from (a), we have $1+nx < (1+x)^n$, so we have:
  $1+n(r-1) < r^n < \infty$.
- $\lim\limits_{n\to\infty} 1+n(r-1) = \infty$ when $r>1$; $\infty \le \lim\limits_{n\to\infty} r^n \le \infty$, $\lim\limits_{n\to\infty} r^n = \infty$.
- therefore when $r>1$, as $n\to\infty$, $r^n \to \infty$.

(c) Hence show that $r^n \to 0$ as $n \to \infty$ when $0 < r < 1$.

from (b), we have $\lim\limits_{n\to\infty} k^n = \infty$ when $k>1$.

we know $0 < \frac{1}{k} < 1$. let $r = \frac{1}{k}$, $0<r<1$.

$$\lim_{n\to\infty} \left(\frac{1}{k}\right)^n = \frac{\lim\limits_{n\to\infty} 1^n}{\lim\limits_{n\to\infty} k^n} = \frac{1}{\infty} = 0.$$

$\therefore \lim\limits_{n\to\infty} r^n = 0$, which means, as $n\to\infty$, $r^n \to 0$.