FM2 Assignment #17 Name: Jerry Jiang $\frac{10}{10}$ Excellent!

1. Let $f(x) = \cos(x^2)$. Use a series approach to find $f^{(8)}(0)$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - O(x^6)$

$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - O(x^{12})$

$f^{(8)}(0) = C_8 \cdot 8! = \frac{8!}{4!} = 1680.$

2. Find $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$.

It's the upper Reimann sum of $\int_0^1 \sqrt{x}\, dx$ as $n \to \infty$.

$\int_0^1 \sqrt{x}\, dx = \left[\frac{2}{3} x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}.$

Therefore, the value of the expression is $\frac{2}{3}$.

3. Find $\int_0^a x\, dx$ from first principles by taking the limit of a lower Riemann sum and the limit of an upper Riemann sum.

- $L_n = \frac{a}{n}\left(\frac{0\cdot a}{n} + \frac{1\cdot a}{n} + \frac{2\cdot a}{n} + \cdots + \frac{(n-1)\cdot a}{n}\right).$

$= \frac{a^2}{n^2}(1 + 2 + \cdots + (n-1))$

$= \frac{n(n-1)a^2}{2n^2}$

$= \frac{(n-1)a^2}{2n}$

- $U_n = \frac{a}{n}\left(\frac{1\cdot a}{n} + \cdots + \frac{n\cdot a}{n}\right)$

$= \frac{a^2}{n^2}(1 + \cdots + n)$

$= \frac{a^2(n+1)}{2n}$

- $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \frac{a^2}{2} = \int_0^a x\, dx.$

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4. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3n^2+2k^2}{n^3}$.

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left[3+2\left(\frac{k}{n}\right)^2\right]$$

$$= \lim_{n\to\infty} U_n \text{ for } 3+2x^2$$

$$= \int_0^1 3+2x^2\,dx$$

$$= \left[3x+\frac{2}{3}x^3\right]_0^1$$

$$= 3+\frac{2}{3}$$

$$= \frac{11}{3}$$

✓

5. Find $\int_0^1 \frac{1}{1+x}\,dx$ and deduce that

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}\right)=\ln 2.$$

Use $n = 100$ with the **sum** and **seq** functions of your calculator to estimate $\ln 2$. Why is your estimate too large?

$$\int_0^1 \frac{1}{1+x}\,dx = \left[\ln(x+1)\right]_0^1 = \ln 2 - \ln 1 = \ln 2.$$ ✓

$$L = \lim_{n\to\infty}\left[\frac{1}{n}\left(\frac{1}{1+\frac{0}{n}}+\frac{1}{1+\frac{1}{n}}+\cdots+\frac{1}{1+\frac{n-1}{n}}\right)\right] = \lim_{n\to\infty} U_n = \int_0^1 \frac{1}{1+x}\,dx = \ln 2,$$

as $\frac{1}{1+x}$ is decreasing from 0 to 1.

$U_{100} = 0.696$ (3 s.f.), $\ln 2 = 0.693$ (3 s.f.). This is because it's the upper Reimann Sum and $n \not\to \infty$.

✓

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