HL1 Test #8 Name: Jerry Jiang

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Excellent!

1. Find a unit vector in the direction of the vector $\vec{v} = 2\vec{i} - \vec{j} + 2\vec{k}$.

$\left|\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\right| = 3$ ✓

$\therefore \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \frac{1}{3} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

2. Let $f(x) = e^x \cos 3x$ and $g(x) = \sqrt{1-x^2}$. $= (1-x^2)^{\frac{1}{2}}$

(a) Find the derivative $f'(x)$.

$f'(x) = e^x \cdot \cos 3x + e^x \cdot (-\sin 3x \cdot 3)$

$= e^x (\cos 3x - 3\sin 3x)$ ✓

(b) Find the derivative $g'(x)$.

$g'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$= -\dfrac{x}{\sqrt{1-x^2}}$ ✓

3. Let $f(x) = \ln 2x$. Find the value of $f^{(8)}(1)$.

$f'(x) = \frac{2}{2x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = -(-2) \cdot x^{-3}$

$f''''(x) = -(-2)\cdot(-3) \cdot x^{-4}$

$f^{n}(x) = (-1)^{n+1} (n-1)! \cdot x^{-n}$

$\therefore f^{(8)}(1) = (-1)^9 \cdot (8)! \cdot 1^{-8}$

$= -8!$

$= -40320$

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4. Without the calculator solve $\sec^2 2x + 2\tan 2x = 0$ for $0 \le x < \pi$.

$$\frac{1}{\cos^2 2x} + \frac{2\sin 2x}{\cos 2x} = 0$$

$$1 + 2\sin 2x \cos 2x = 0$$

$$\sin 4x = 1$$

$$4x = \frac{3}{2}\pi + 2k\pi$$

$$x = \frac{3}{8}\pi + \frac{k}{2}\pi$$

$$\therefore x = \frac{3}{8}\pi \text{ or } \frac{7}{8}\pi$$

$$\frac{1}{\cos^2 2x} + 2\frac{\sin 2x}{\cos 2x} = 0$$

$$1 + 2\sin 2x\cos 2x = 0$$

$$1 + \sin 4x = 0$$

$$\sin 4x = -1$$

1.18

$$4x = \frac{3}{2}\pi + 2k\pi$$

2.74

$$x = \frac{3}{8}\pi + \frac{k}{2}\pi$$

$\frac{5}{8}\pi$

5. The equation $z^4 + bz^3 + cz^2 + d = 0$ has real coefficients. Two of the roots are $\log_2 6$ and $i\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$. If $d = \log_2 k$ find the value of k.

$\log_2 6, \; +i\sqrt{3}, \; -i\sqrt{3}. \; x_4 \to 2$

$$x_4 = 3 + \log_2 3 - 1 - \log_2 3 = 2$$

$$\therefore d = \prod_{i=1}^{4} x_i$$

$$= 2 \cdot \log_2 6 \cdot (\sqrt{3})^2 \cdot (-i^2)$$

$$= 6\log_2 6$$

$$= \log_2 6^6$$

$\therefore k = 46656$

6. Find the point of intersection, if any, for the lines $\vec{r} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{r} = \begin{pmatrix} -1 \\ -5 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

① & ②:

$$\begin{cases} 2 + t = -1 - u \\ 4 + 2t = -5 + u \end{cases}$$

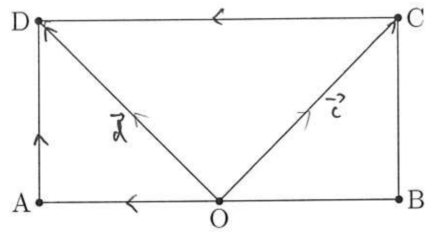
$$\therefore \begin{cases} t = -4 \\ u = 1 \end{cases}$$

③: $3 + 2t = -4 - u$

Satisfied.

$$\therefore \begin{pmatrix} -2 \\ -4 \\ -5 \end{pmatrix}$$

7. The side AB of rectangle $ABCD$ has midpoint O. Let $\vec{c} = \overrightarrow{OC}$ and $\vec{d} = \overrightarrow{OD}$.



Express each of the following vectors in terms of $\vec{c}$ and $\vec{d}$.

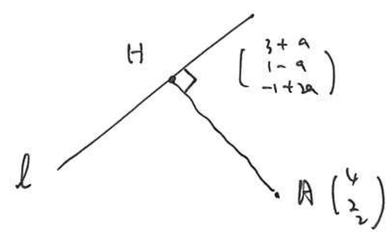
(a) $\overrightarrow{CD}$ $= \vec{d} - \vec{c}$ ✓

(b) $\overrightarrow{OA}$ $= \frac{1}{2}\overrightarrow{CD} = \frac{1}{2}\vec{d} - \frac{1}{2}\vec{c}$ ✓

(c) $\overrightarrow{AD}$ $= \vec{d} - \frac{1}{2}\vec{d} + \frac{1}{2}\vec{c} = \frac{1}{2}\vec{d} + \frac{1}{2}\vec{c}$ ✓

8. Find the distance from the point $A(4, 2, 2)$ to the line ℓ with vector equation $\vec{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

let H be $\begin{pmatrix} 3+a \\ 1-a \\ -1+2a \end{pmatrix}$.



$\overrightarrow{AH} = \begin{pmatrix} 3+a-4 \\ 1-a-2 \\ -1+2a-2 \end{pmatrix} = \begin{pmatrix} -1+a \\ -1-a \\ -3+2a \end{pmatrix}$.

$\overrightarrow{AH} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$

$\therefore$ $-1 + a + 1 + a - 6 + 4a = 0$

$\therefore$ $6a = 6$

$\therefore$ $a = 1$

$\therefore$ $\overrightarrow{AH} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$

$\therefore$ $|\overrightarrow{AH}| = \sqrt{5}$

$\begin{pmatrix} 1-a \\ 1+a \\ 3-2a \end{pmatrix}$

$1 - a - 1 - a$
$+ 6 - 4a = 0$

$6a = 6$
$a = 1$

$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

✓ ✓

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9. Find the values of k for which the function $f(x) = \dfrac{e^{kx}}{x^2+1}$ has both a maximum and a minimum.

$$f'(x) = \frac{k \cdot e^{kx}(x^2+1) - e^{kx} \cdot 2x}{(x^2+1)^2}$$

$$\frac{k e^{kx} \cdot (x^2+1) - e^{kx} \cdot 2x}{(x^2+1)^2} = 0$$

$$k \cdot e^{kx}(x^2+1) - e^{kx} \cdot 2x = 0$$

$$\therefore kx^2 - 2x + k = 0$$

$$k(x^2+1) = 2x$$

$$\therefore \Delta = 4 - 4k^2 > 0$$

$$kx^2 - 2x + k = 0$$

$$\Delta = 4 - 4k^2 > 0$$

$$\therefore -1 < k < 1$$

$$4k^2 < 4$$

$$k^2 < 1$$

$$f'(x) = \frac{e^{kx}(kx^2 + k - 2x)}{(x^2+1)^2} \Rightarrow \boxed{-1<k<1}$$

according to GDC, when $-1<k<1$,

all values of k gives a max and a min to the function except when $k=0$, which only produces a maximum at 0.

$$\therefore k \in \,]-1, 1[\setminus 0.$$

10. Show that $f(x) = \dfrac{\ln x}{x}$ is decreasing when $x > e$. Hence determine which is bigger 2019^{2020} or 2020^{2019}.

$$f'(x) = \frac{\frac{x}{x} - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

when $1 = \ln x$, $x = e$.

$$f''(x) = \frac{\frac{x^2}{x} - (1-\ln x) \cdot 4x^3}{x^4}$$

$f''(e) < 0$, concave down, maximum at e.

$\therefore f(x) = \dfrac{\ln x}{x}$ is decreasing when $x > e$.

$$f(2019^{2020}) = \frac{2020 \ln 2019}{2019^{2020}}$$

$$f(2020^{2019}) = \frac{2019 \ln 2020}{2020^{2019}}$$

$$\therefore f(2019^{2020}) < f(2020^{2019})$$

more explanation needed.

$$\therefore 2019^{2020} > 2020^{2019}$$

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Solutions to HL1 Test #8

1. Here $|\vec{v}| = 3$. So $\hat{v} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

2. (a) $f'(x) = e^x(\cos 3x - 3\sin 3x)$ (b) $g'(x) = \dfrac{-x}{\sqrt{1-x^2}}$

3. Since $f^{(8)}(x) = -7!\,x^{-8}$, $f^{(8)}(1) = -5040$.

4. Using $\sec^2 2x = \tan^2 2x + 1$, gives $(\tan^2 2x + 1)^2 = 0$. So $\tan 2x = -1$, whence $x = \frac{3\pi}{8}, \frac{7\pi}{8}$ for $x \in [0, \pi[$.

5. The roots are $\log_2 6$, $\pm i\sqrt{3}$, 2. So $d = \log_2 6 \cdot i\sqrt{3} \cdot -i\sqrt{3} \cdot 2 = 6\log_2 6$, whence $k = 6^6 = 46\,656$.

6. Solving simultaneously and remembering to use different parameters gives $t = -4$ and $u = 1$. Hence the point of intersection is $(-2, -4, -5)$.

7. (a) $\vec{d} - \vec{c}$ (b) $\frac{1}{2}(\vec{d} - \vec{c})$ (c) $\frac{1}{2}(\vec{d} + \vec{c})$

8. $d(A, \ell) = \sqrt{5}$

9. By the quotient rule $f'(x) = \dfrac{e^{kx}(kx^2 - 2x + k)}{(x^2+1)^2}$. For $f'(x) = 0$ we must have $kx^2 - 2x + k = 0$. Since $\Delta = 4 - 4k^2$, we have two roots to the quadratic when $-1 < k < 1$ and $k \neq 0$. The sign of the first derivative also changes appropriately through these roots, so $-1 < k < 1$ and $k \neq 0$ is also the requirement for f to have both a maximum and a minimum.

10. First recall that a function is decreasing when its derivative is negative. Here $f'(x) = \dfrac{1 - \ln x}{x^2}$, so f is decreasing when $1 - \ln x < 0$, whence $x > e$. It follows that $f(2020) < f(2019)$ or

$$\frac{\ln 2020}{2020} < \frac{\ln 2019}{2019},$$

whence $\ln 2020^{2019} < \ln 2019^{2020}$, from which we conclude $2020^{2019} < 2019^{2020}$ since the natural logarithm function is everywhere increasing.