FM2 Assignment #13 Name: Jerry Jiang

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1. Use l'Hôpital's rule to evaluate $\lim_{x \to 1} \dfrac{\arctan x - \pi/4}{x-1}$.

Apply L'Hôpital's rule since $\lim_{x\to 1} \arctan x - \frac{\pi}{4} = \lim_{x \to 1} x-1 = 0$.

$$\lim_{x\to 1} \frac{\arctan x - \frac{\pi}{4}}{x-1} = \lim_{x \to 1} \frac{\frac{1}{1+x^2}}{1} = \frac{1}{1+1} = \frac{1}{2}.$$

Therefore, $\lim_{x\to 1} \dfrac{\arctan x - \frac{\pi}{4}}{x-1} = \dfrac{1}{2}$.

2. Let $A = (-1, 0)$ and $B = (1, 0)$. Find the locus of a point P that moves so that $PA^2 + PB^2 = 10$.

Let P be (a, b).

$PA^2 = (a+1)^2 + b^2$, $PB^2 = (a-1)^2 + b^2$.

$2a^2 + 2 + 2b^2 = 10$.

$a^2 + b^2 = 4 = 2^2$.

Therefore, the loci of P is a circle centered at the origin with radius 2.

3. The third degree Taylor polynomial for the function f centred at 1 is $4 - (x-1) + 3(x-1)^2 - 5(x-1)^3$.

$f(x) \approx P_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$.

(a) Write down the value of $f''(1)$.

$\frac{f''(1)}{2!} = 3$, $f''(1) = 6$.

(b) Approximate $f'(1.2)$.

$f'(x) = -1 + 6(x-1) - 15(x-1)^2$

$f'(1.2) = -0.4$.

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4. The sequence $\{u_n\}$ is defined recursively by $u_1 = 2$ and $u_{n+1} = \frac{1}{2}(u_n + 4)$. Use mathematical induction to show that $\{u_n\}$ is an increasing sequence bounded above by 4. What is the limit of the sequence?

We need to prove $U_n < U_{n+1} < 4$ for all $n \in \mathbb{Z}^+$.

① for $n=1$, $U_1 = 2 < 4$, $U_2 = \frac{1}{2}(2+4) = 3$. $U_1 < U_2 < 4$.

② for $n = m$, $U_m < 4$. we claim that $U_m < U_{m+1} < 4$.

proof: $U_{m+1} = \frac{1}{2}(U_m + 4) < \frac{1}{2}(4+4) = 4$, so $U_{m+1} < 4$.

$U_{m+1} = \frac{1}{2}(U_m + 4) > \frac{1}{2}(U_m + U_m) = U_m$, so $U_{m+1} > U_m$.

Therefore, $U_m < U_{m+1} < 4$, the sequence is increasing and bounded above by 4.

• Let $U_{n+1} = f(x) = \frac{1}{2}(x+4)$ where $x = U_n$. the sequence has limit L.

$\lim_{n\to\infty} U_{n+1} = L$. $\lim_{n\to\infty} U_{n+1} = \lim_{n\to\infty} \frac{1}{2}(X_n + 4) = \frac{1}{2}(\lim_{n\to\infty} X_n + 4) = \frac{1}{2}(L+4)$

$\therefore L = \frac{1}{2}(L+4)$, $L = 4$.

Therefore, the limit of the sequence is 4.

5. The function f has derivatives of all orders for all real numbers. The third degree Taylor polynomial for f centred at 2 is $7 - 9(x-2)^2 - 3(x-2)^3$. If $|f^{(4)}(x)| \leq 6$ for all x in the open interval $]0, 2[$, show that $f(0)$ must be negative.

$f(x) = 7 - 9(x-2)^2 - 3(x-2)^3 + \frac{f^{(4)}(c)}{4!}(x-2)^4$ for some $c \in]0,2[$.

let $x=0$, $a=2$.

$f(0) = 7 - 36 + 24 + \frac{2}{3} f^{(4)}(c) = -5 + \frac{2}{3} f^{(4)}(c)$

Since $|f^{(4)}(x)| \leq 6$, $f(0) = -5 + \frac{2}{3} f^{(4)}(c) \leq -5 + \frac{2}{3} \times 6 = -5+4 = -1 < 0$.

Therefore, $f(0) < 0$.

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