HL1 Test #3 Name: Jerry Jiang

1. The perpendicular bisector of $[AB]$ where A is $(1, 0)$ and B is $(5, 2)$ meets the y-axis at $(0, c)$. Find the value of c.

$l_{AB}: y = kx + b \ (k \neq 0)$

$\begin{cases} x + b = 0 \\ 5x + b = 2 \end{cases}$

$4x = 2$

$\begin{cases} x = \frac{1}{2} \\ b = -\frac{1}{2} \end{cases}$

$\therefore l_{AB}: y = \frac{1}{2}x - \frac{1}{2}$

$\therefore l: y = -2x + c$

$\text{Mid}_{AB} = \left(\frac{1+5}{2}, \frac{0+2}{2}\right)$

$= (3, 1)$

$\therefore -6 + c = 1$

$\therefore c = 7.$

100%

Bravo!!

2. The parabola $y = ax^2 + bx + c$ passes through the origin and has its vertex at $(1, 2)$. Find the values of a, b and c.

$(0, 0).$

$\therefore c = 0$

$\therefore y = ax^2 + bx$

$= x(ax + b)$

$x_1 = 0$

$x_2 = -\frac{b}{a}$

$\therefore \frac{0 + (-\frac{b}{a})}{2} = 1$

$\therefore \frac{b}{a} = -2$

$\therefore b = -2a$

$\therefore y = ax^2 - 2ax$

$= a(x^2 - 2x + 1) - a$

$= a(x-1)^2 - a$

$\therefore -a = 2$

$\therefore a = -2.$

$\therefore \begin{cases} a = -2 \\ b = 4 \\ c = 0 \end{cases}$

3. The cubic polynomial $x^3 + bx^2 + cx + d$ has roots 2 and $\pm i$. Find the values of b, c and d.

$x^3 + bx^2 + cx + d$

$= (x-2)(x+i)(x-i)$

$= (x-2)(x^2+1)$

$= x^3 + x - 2x^2 - 2$

$= x^3 - 2x^2 + x - 2$

$\therefore \begin{cases} b = -2 \\ c = 1 \\ d = -2 \end{cases}$

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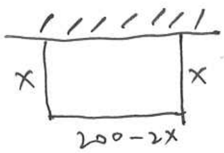
4. A rectangular chicken coop is built against an existing wall so that only three sides need be fenced. If 200 m of fencing is used, what are the dimensions of the coop that will maximize its area?



$S = x(200-2x)$

$= -2x^2 + 200x$

$= -2(x^2 - 100x + 2500) + 5000$

$= -2(x-50)^2 + 5000$

$\therefore$ length $= 100$ m,

width $= 50$ m.

$S_{max} = 5000\ m^2$

5. Solve $\dfrac{1}{\log_4 x} + \dfrac{1}{\log_5 x} = 1.$

$\log_x 4 + \log_x 5 = 1$

$\log_x 20 = 1$

$\therefore x^1 = 20$

$\therefore x = 20$

6. The constant term in the expansion of $\left(x - \dfrac{2}{x^2}\right)^{15}$ is an integer. Find its value.

x : to the 10

$\frac{2}{x^2}$: to the five

$\Big\}$ cancel out the x.

$\therefore \binom{15}{5} \cdot x^{10} \cdot \left(-\frac{2}{x^2}\right)^5$

$= \dfrac{\cancel{15}\cdot\cancel{14}^{7}\cdot 13\cdot\cancel{12}^{3}\cdot 11}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} \cdot \cancel{x^{10}} \cdot \left(-\dfrac{32}{\cancel{x^{10}}}\right)$

$= 7\cdot 13\cdot 3\cdot 11\cdot(-32)$

$= -96096$

7. Use the rational roots theorem to prove that $\sqrt{10}$ is irrational.

$\sqrt{10}$ is the root of $x^2 = 10$

$\therefore$ in the quadratic equation $x^2 - 10 = 0$,

the rational root candidates are:

$$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 5, \pm 10}{\pm 1} = \pm 1, \pm 2, \pm 5, \pm 10.$$

$\because$ clearly, $\sqrt{10}$ is not among them. (since $\sqrt{10}$ is between $\sqrt{9}$ and $\sqrt{16}$)

$\therefore$ $\sqrt{10}$ must be irrational to become the root of the equation.

✓

8. A polynomial has remainder 3 when divided by $x - 1$ and remainder -7 when divided by $x + 1$. Find the remainder when the polynomial is divided by $x^2 - 1$.

according to the remainder theorem,

$p(x) = d(x) \cdot q(x) + r(x)$

$p(1) = 3$

$p(-1) = -7$

let $r(x) = ax + b$

$\therefore$ $p(x) = (x^2 - 1) \cdot q(x) + ax + b$

$p(1) = 3 = a + b$

$p(-1) = -7 = -a + b$

$$\therefore \begin{cases} a + b = 3 \\ -a + b = -7 \end{cases}$$

$2a = 10$

$$\begin{cases} a = 5 \\ b = -2 \end{cases}$$

$\therefore$ $r(x) = 5x - 2$

$\therefore$ the remainder is $5x - 2$

$ax^3 + bx^2 + cx + d$: $\cancel{(x-1)}$

$a + b + c + d = 3$

$-a + b - c + d = -7$

$b + d = -2$

$a + c = 5$

$$\begin{array}{r} ax + b \\ x^2 + 0x - 1 \overline{)\, ax^3 + bx^2 + cx + d} \\ ax^3 + 0x^2 - ax \\ \hline bx^2 + (c+a)x + d \\ bx^2 + 0x - b \\ \hline (c+a)x + b + d \end{array}$$

$5x - 2$

✓

9. Given that $(1+x)^5(1+ax)^6 = 1 + bx + 10x^2 + \cdots + a^6x^{11}$ and $a \neq 0$, find the values of a and b.

$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

$(ax+1)^6 = a^6x^6 + 6a^5x^5 + 15a^4x^4 + 20a^3x^3 + 15a^2x^2 + 6ax + 1$

$\therefore\ x^2:\quad 15a^2x^2 + 6ax \cdot 5x + 10x^2$

$= (15a^2 + 30a + 10)x^2$

$\therefore\ 15a^2 + 30a + 10 = 10$

$\therefore\ a^2 + 2a = 0$

$a(a+2) = 0$

$\therefore\ a_1 = 0$, (inadmissable)

$a_2 = -2.$

$\because x:\quad 6ax + 5x$

$= (6a+5)x$

$\therefore\ b = 6a + 5$

$= -12 + 5$

$= -7$

$\therefore \begin{cases} a = -2 \\ b = -7 \end{cases}$

10. The roots of $kx^2 + (k-3)x + 1 = 0$ are distinct, real and positive. Find the possible values of k.

① $k = 0.$

$-3x + 1 = 0$

$3x = 1$

$x = \frac{1}{3}$ (inadmissable).

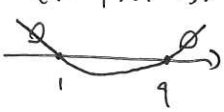
② $k \neq 0.$

(1) $\Delta = (k-3)^2 - 4k$

$= k^2 - 6k + 9 - 4k$

$= k^2 - 10k + 9 > 0$

$(k-9)(k-1) > 0$



$\therefore\ k < 1$ or $k > 9.$

(2) positive

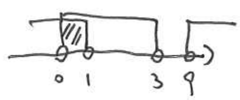
$x_1 + x_2 = \dfrac{3-k}{k} > 0$

$x_1 x_2 = \dfrac{1}{k} > 0$

$\therefore\ k > 0.$

$3 - k > 0$

$\therefore 0 < k < 3$



$\therefore\ 0 < k < 1$

In conclusion, $0 < k < 1$

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Solutions to HL1 Test #3

1. Here $m = 1/2$, so $m_{\perp} = -2$. Also $M = (3, 1)$. So the equation of the perpendicular bisector is $y - 1 = -2(x - 3)$ or $y = -2x + 7$. Hence $c = 7$.

2. Here $y = a(x - 1)^2 + 2$. Substituting $(0, 0)$ gives $a = -2$. So $y = -2x^2 + 4x$, whence $a = -2$, $b = 4$ and $c = 0$.

3. By the factor theorem the polynomial is $(x - 2)(x - i)(x + i)$, which expands to $x^3 - 2x^2 + x - 2$, whence $b = -2$, $c = 1$ and $d = -2$.

4. Here $A = x(200 - 2x)$. So the maximum area occurs when $x = 50$. Hence the dimensions of the coop for maximum area are 50 m by 100 m.

5. The equation is equivalent to $\log_x 4 + \log_x 5 = 1$, whence $\log_x 20 = 1$. Hence $x = 20$.

6. The required term is $\binom{15}{10} x^{10} \left(\frac{-2}{x^2}\right)^5$, which evaluates to -96096.

7. Consider the equation $x^2 - 10 = 0$. We have RRC $= \{\pm 1, \pm 2, \pm 5, \pm 10\}$. Since $\sqrt{10}$ is a root of the given equation but not in $\{\pm 1, \pm 2, \pm 5, \pm 10\}$ we must conclude $\sqrt{10}$ is irrational.

8. See exercise #10 in Polynomials #1.

9. Equating coefficients we have $1 \cdot \binom{6}{1}a + \binom{5}{1} \cdot 1 = b$ and $1 \cdot \binom{6}{2}a^2 + \binom{5}{1}\binom{6}{1}a + \binom{5}{2} \cdot 1 = 10$, which is equivalent to $6a + 5 = b$ and $15a^2 + 30a = 0$. Solving simultaneously and remembering $a \neq 0$ gives $a = -2$ and $b = -7$.

10. Since the roots are distinct and real $k \neq 0$ and $\Delta = (k - 3)^2 - 4k > 0$. Since the roots are positive the sum of the roots $\frac{3-k}{k} > 0$ and the product of the roots $\frac{1}{k} > 0$. Solving the three inequalities simultaneously gives $0 < k < 1$.