HL1 Test #5 Name: Jerry Jiang 98%

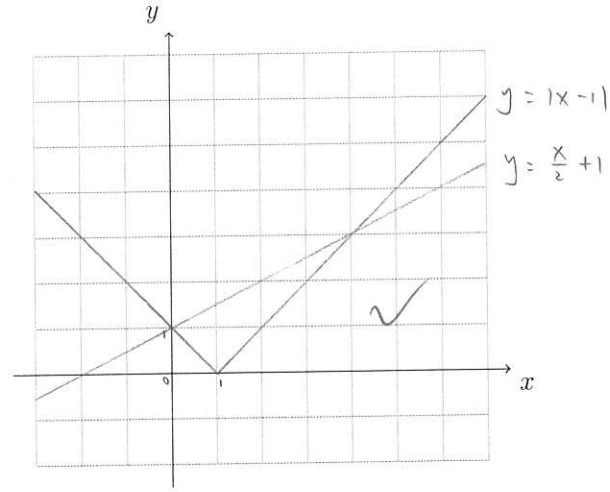
Excellent!

1. Draw the graphs of $y = \frac{x}{2} + 1$ and $y = |x - 1|$ on the grid below. Hence or otherwis solve $\frac{x}{2} + 1 = |x - 1|$.

the intersects have coordinate: $(0,1)$, $(4,3)$

$\therefore$ the two solutions to $\frac{x}{2} + 1 = |x-1|$ are

$x = 0$ or $x = 4$



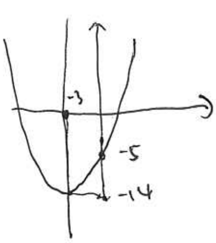
2. The function $f : x \mapsto x^2 + 6x - 5$ has domain $[m, \infty[$. Find the least value of m for which f is one-to-one.

let $y = x^2 + 6x - 5$

$= x^2 + 6x + 9 - 14$

$= (x+3)^2 - 14$

$\therefore$ vertex $(-3, -14)$



$\therefore$ $m_{min} = -3$, which take the right half of the original function.

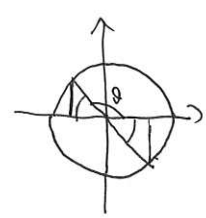
3. Given that θ is an obtuse angle measured in radians and that $\sin\theta = k$, find in terms of k expressions for

(a) $\sin(\theta + 2\pi)$;

$\sin(\theta + 2\pi) = \sin\theta = k$

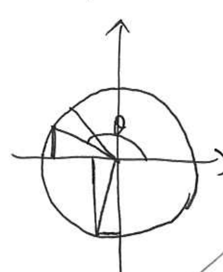
(b) $\sin(\theta + \pi)$;

$\sin(\theta + \pi) = -\sin\theta = -k$



(c) $\sin(\theta + \frac{\pi}{2})$.

$\sin(\theta + \frac{\pi}{2}) = -\cos\theta = -\sqrt{1 - \sin^2\theta} = -\sqrt{1-k^2}$



4. The line $y = 3x + c$ is tangent to the parabola $y = x^2 - x + 3$. Find the value of c.

$$\begin{cases} y = 3x + c \\ y = x^2 - x + 3 \end{cases}$$

$\therefore x^2 - 4x + 3 - c = 0$

$\therefore \Delta = 16 - 4(3 - c)$

$= 4 + 4c$

$= 0$

$\therefore c = -1$

5. The coefficients of x^2 and x^3 in the expansion of $\left(1 + \frac{x}{3}\right)^n$ are equal. Find the value of n.

$$1^{n-2} \cdot \left(\frac{1}{3}\right)^2 \cdot \binom{n}{2} = 1^{n-3} \cdot \left(\frac{1}{3}\right)^3 \cdot \binom{n}{3}$$

$$\frac{1}{9} \cdot \frac{n(n-1)}{2} = \frac{1}{27} \cdot \frac{n(n-1)(n-2)}{6}$$

$$9n(n-1) = n(n-1)(n-2)$$

① $n = 0$
② $n = 1$ (inadmissable, $n \geq 3$)

③ $9 = n - 2$,
$n = 11$

$\therefore n = 11$

6. The graph of $y = a\sin(b(x + c)) + d$ is shown below. Find the values of a, b, c and d.

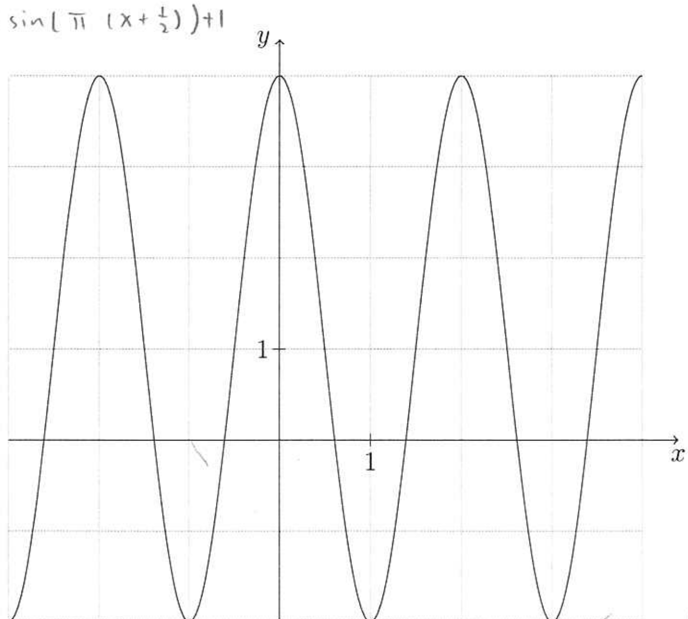
$y = 3\sin(\pi(x + \frac{1}{2})) + 1$

$a = \frac{4 - (-2)}{2} = 3$

$b = \frac{2\pi}{2} = \pi$

$c = -(-\frac{1}{2}) = \frac{1}{2}$

$d = 1$



1 2 3 4 5 6 7 8 9 10 11 12 X

7. How many subsets of size five chosen from $\{n \in \mathbb{Z} \mid 1 \leq n < 12\}$ contain at least two even numbers?

all subsets: $\binom{12}{5} = \frac{\cancel{12}\times 11\times \cancel{10}\times 9\times 8}{\cancel{5\times 4\times 3\times 2}\times 1} = 792$ $\binom{6}{2}\cdot\binom{10}{3}$.

no even: $\binom{6}{5} = \binom{6}{1} = 6$

one even: $6\times\binom{6}{4} = 6\times\binom{6}{2} = 6\times\frac{6\times 5}{2\times 1} = 90$

∴ at least two: $792 - 6 - 90 = 696$

无12.

All: $\binom{11}{5} = 462$

no even: $\binom{6}{5} = 6$

one even: $5\times\binom{6}{4} = 75$

$462 - 6 - 75 = 381$

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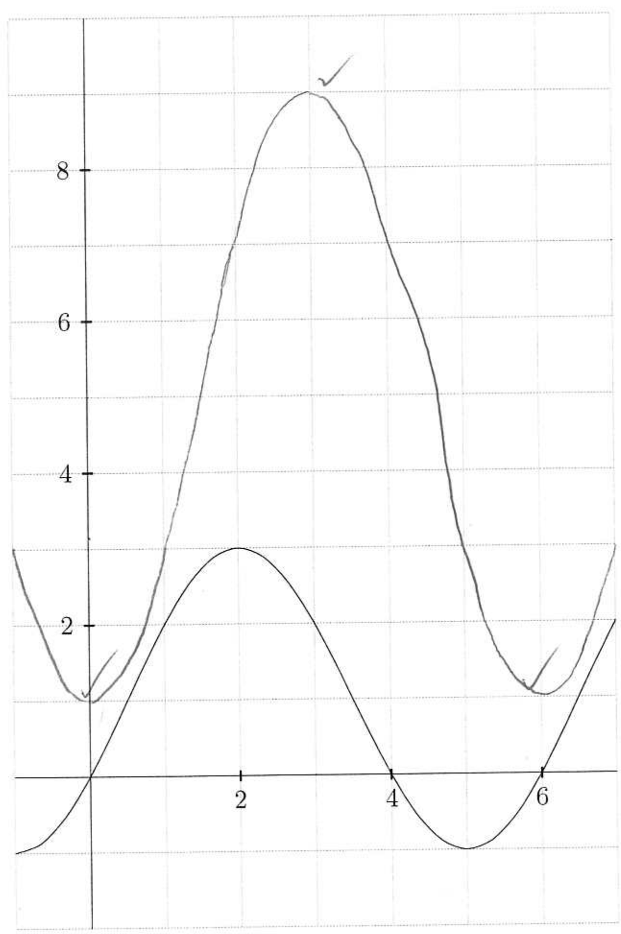
8. Part of the periodic function f is graphed below. Draw the graph of $y = 2f(x-1)+3$ on the same grid.

$\frac{y-3}{2} = f\left(\frac{x-1}{1}\right)$.

$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, \binom{1}{3}$.



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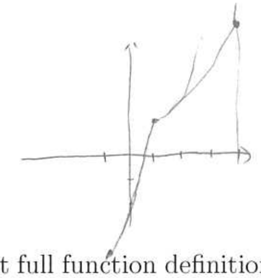
9. If the codomain of the piecewise function $f\colon [-1,4] \to \mathbb{R}$ with rule

$$f(x) = \begin{cases} 3x-2 & \text{for } -1 \le x \le 1, \\ 4/(5-x) & \text{for } 1 < x \le 4. \end{cases}$$



is suitably restricted a bijection results. Find the restriction and give the consequent full function definition for f^{-1}.

$f: \boxed{[-1,4] \to [-5,4]}$

$$\begin{cases} f_1(x)\ [-1,1] \to [-5,1],\ f_1(x) = 3x-2 \\ f_2(x)\ ]1,4] \to]1,4],\ f_2(x) = \frac{4}{5-x} \end{cases}$$

$y = 3x-2$
$x = 3y-2$
$y = \frac{x+2}{3}$

$y = \frac{4}{5-x}$
$x = \frac{4}{5-y}$
$5x - xy = 4$
$xy = 5x-4$
$y = \frac{5x-4}{x} = 5 - \frac{4}{x}$

$$\begin{cases} f_1^{-1}(x)\ [-5,1] \to [-1,1],\ f_1^{-1}(x) = \frac{x+2}{3} \\ f_2^{-1}(x)\ ]1,4] \to]1,4].\ f_2^{-1}(x) = 5 - \frac{4}{x} \end{cases}$$

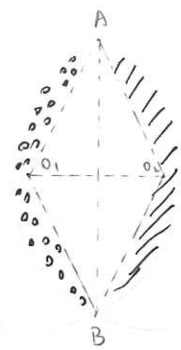
so. $f^{-1}\cancel{(x)}: [-5,4] \to [-1,4],\ f(x) = \begin{cases} \frac{x+2}{3} & \text{for } -5 \le x \le 1 \\ 5 - \frac{4}{x} & \text{for. } 1 < x \le 4 \end{cases}$

10. Two circles each of unit radius overlap. If the area of the overlapping region is $\frac{\pi}{2}$ how far apart are the centres?

let $\angle AO_1B = \theta$

then $A_{\triangle AO_1B} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\theta$

$A_{\widehat{AO_1B}} = \pi \cdot 1^2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{2}$



$\because A_{AO_1BO_2} = 2A_{\triangle AO_1B} = \sin\theta$

$\therefore A_{shaded} = \frac{\theta}{2} - \sin\theta$

$\therefore A_{circled} = A_{shaded} = \frac{\theta}{2} - \sin\theta$

$\therefore A_{total} = A_{\widehat{AO_1B}} + A_{circled}$

$= \frac{\theta}{2} + \frac{\theta}{2} - \sin\theta$

$= \theta - \sin\theta = \frac{\pi}{2}$

* $\sin\frac{\pi}{2}$ is half of ~~the~~ one circle's area, the two centers have to be inside the overlapping region.

$\therefore$ according to GDC, $\theta = 2.31$ is the solution

$\because \angle AO_1B = \angle AO_1O_2 + \angle BO_1O_2 = \angle AO_1O_2 + \angle AO_2O_1 = \theta$

$\therefore \angle O_1AO_2 = \pi - \theta = 0.832$

$\therefore O_1O_2 = \sqrt{1^2+1^2 - 2 \cdot 1 \cdot 1 \cdot \cos\angle O_1AO_2}$

$= 0.808$

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Solutions to HL1 Test #5

1. From the graph we read $x = 0$ or $x = 4$.

2. Since $f(x) = (x+3)^2 - 14$, we conclude $m = -3$.

3. (a) k (b) $-k$ (c) $-\sqrt{1-k^2}$

4. Solving $x^2 - x - 3 = 3x + c$ gives $x^2 - 4k + (3-c) = 0$. Next we have $\Delta = 16 - 4(3-c)$. We want $\Delta = 0$, whence $c = 1$.

5. We want $\frac{1}{2}n(n-1) \cdot \frac{1}{9} = \frac{1}{6}n(n-1)(n-2) \cdot \frac{1}{27}$, whence $n = 11$.

6. $a = 3$, $b = \pi$, $c = \frac{1}{2}$, $d = 1$.

7. Let A be the event of obtaining at least two even numbers. Then A' is the complementary event of obtaining at most one even number and this is easier to count. Next $n(A') = \binom{6}{5} + \binom{6}{4}\binom{5}{1} = 81$. So $n(A) = \binom{11}{5} - 81 = 381$.

8. The graph should show a periodic function with minima at $(0,1)$ and $(6,1)$, and a maximum at $(3,9)$.

9. The required codomain restriction for the function f is $[-5,4]$. The full function definition for f^{-1} is $f^{-1}\colon [-5,4] \to [-1,4]$ with rule

$$f^{-1}(x) = \begin{cases} (x+2)/3 & \text{for } -5 \le x < 1, \\ 5 - 4/x & \text{for } 1 \le x \le 4. \end{cases}$$

10. The overlapping area consists of two segments each of area $\frac{1}{2} \cdot 1^2(\theta - \sin\theta)$ where θ is the central angle for the segment measured in radians. Hence we must solve the equation

$$\theta - \sin\theta = \frac{\pi}{2}.$$

Using the CDC we find $\theta = 2.31$ to 3 significant figures. The distance between the centres is therefore $2\cos\frac{\theta}{2} = 0.808$ to 3 significant figures.