FM2 Assignment #9 Name: Jerry Jiang

10/10 Excellent!

1. Give two reasons why the set of odd integers under addition is not a group.

① no identity : 0 is not an odd number.

② no closure : $1+3=4$, 4 is not an odd number.

2. Construct the Cayley table for $(\mathbb{Z}_{12}^*, \otimes)$.

$\mathbb{Z}_{12}^* = \{ n \mid n \in [1, 11], \gcd(n, 12) = 1 \} = \{1, 5, 7, 11\}$

⊗	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

3. The space $S = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rangle$ is a subspace of $\mathbb{R}^3$. Find a Cartesian equation for S.

①: $\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (3-2)i - (3-1)j + (2-1)k = i - 2j + k.$

$\therefore x - 2y + z + c = 0$

$\because (1,1,1)$ is on the plane,

$\therefore 1 - 2 + 1 + c = 0, \quad c = 0.$

$\therefore S: x - 2y + z = 0.$

② $\begin{cases} x = a + b \\ y = a + 2b \\ z = a + 3b \end{cases}$ $\because (a+b) - 2(a+2b) + (a+3b) = 0$

$\therefore x - 2y + z = 0.$

6

4. A sequence is defined recursively by $u_1 = 1$ and $u_{n+1} = \dfrac{1}{1+u_n}$. Assuming the sequence is convergent find its limit.

$U_{n+1} = \dfrac{1}{1+\dfrac{1}{1+U_{n-1}}}$, writing this recursively, we get:

$$U_{n+1} = \cfrac{1}{1+\boxed{\cfrac{1}{1+\cfrac{1}{1+\cdots\cfrac{}{1+\cfrac{1}{1+1}}}}}\,a}$$, let the blue part be a.

when $n \to \infty$, $a = \dfrac{1}{1+a} = U_n$. $a^2+a-1=0$, $a = \dfrac{-1 \pm \sqrt{5}}{2}$ or perhaps $u_\infty =$

$\because U_n > 0$

$\therefore U_n = \dfrac{\sqrt{5}-1}{2}$ Better to write $\lim_{n\to\infty} u_n = \dfrac{\sqrt{5}-1}{2}$

Hint: $\left|\dfrac{1-\sqrt{5}}{2}\right| < 1$

Question: By observation, $U_n = \dfrac{\text{fibo}(n)}{\text{fibo}(n+1)}$.

How to show $U_\infty = \lim_{n\to\infty} \dfrac{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]}$ equals to the $\dfrac{\sqrt{5}-1}{2}$ obtained above?

5. Prove that the set of 3×3 matrices with real entries of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is a group under matrix multiplication.

① $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d+a & e+af+b \\ 0 & 1 & f+c \\ 0 & 0 & 1 \end{pmatrix}$, closure ✓.

② $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$, identity ✓ which belongs to the set taking $a=b=c=0$

③ matrix multiplication is associative. ✓

in set?

④ $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$ inverse ✓.

4

Therefore, it's a group.