FM1 Assignment #6 Name: Jerry Jiang

1. In $\triangle ABC$, $a = 8$, $b = 12$ and $c = 10$. Point L is on side $[BC]$ such that $\angle BAL = \angle CAL$. Find BL.

   Since AL is the angle bisector of $\angle BAC$, we have: $\frac{AB}{BL} = \frac{AC}{CL}$.
   So $\frac{10}{BL} = \frac{12}{8-BL}$. $BL = \frac{40}{11}$.

2. Calculate the values of x for which the determinant $\begin{vmatrix} x & 5 & -1 \\ 1 & 3 & x \\ 1 & 4 & 7 \end{vmatrix}$ is zero.

   $$\begin{aligned} \det A &= x(21-4x) - 5(7-x) + (-1)(4-3) \\ &= -4x^2 + 26x - 36 \\ &= 0 \end{aligned}$$

   So $2x^2 - 13x + 18 = (2x-9)(x-2) = 0$, $x = 4.5$ or 2.

3. Consider the group $(G, *)$ with identity e. If $x * x = e$ for all $x \in G$, show that $(G, *)$ is Abelian.

   Let x and y be elements in G.
   Since $e = (x * y) * (x * y)$, we have: $(x * y)^{-1} = x * y$.
   Also, $(x * y) * (y * x) = x * y * y * x(\text{associativity}) = x * e * x = e$.
   So $(x * y)^{-1} = (y * x)$.
   We now have: $(x * y)^{-1} = x * y = y * x$, which proves the group to be abelian.

4. Use the integral test, clearly stating the conditions for its use, to show that the harmonic series diverges.

   Since $f(x) = \frac{1}{x}$ is continuous, decreasing and positive for all $x \geqslant 1$, we can use the integral test.
   We are looking at the convergence of the series $\{\frac{1}{n}\}$.
   Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx = [\ln x]_1^\infty = \infty$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

5. The simple graph G has the adjacency matrix below. Find the maximum number of edges that can be added to G so that it remains simple and planar. Be sure to justify your answer.

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

In the diagram below, you can find graph G drawn in black. There're three more lines available to add in red. Now, only node 2 and 4 are not connected compared to a complete K_5 graph. Since K_5 is not planar (proved in page 1620, example 26), we know that this is all we can do and the maximum number of edges that can be added is 3.