FM2 Assignment #4 Name: Jerry Jiang. $\frac{9\frac{1}{2}}{10}$ Very Good

1. (a) Is $(3\mathbb{Z} \cap 4\mathbb{Z}, +)$ a group? If so describe it, if not explain why.

Yes. $3\mathbb{Z} \cap 4\mathbb{Z} = 12\mathbb{Z}$.

$(12\mathbb{Z}, +)$ is a group closed in $12\mathbb{Z}$, has e as 0, and has inverse as its own opposite number.

(b) Is $(3\mathbb{Z} \cup 4\mathbb{Z}, +)$ a group? If so describe it, if not explain why.

NO. $3+4=7$, which is not in $3\mathbb{Z} \cup 4\mathbb{Z}$. ✓ 2

Therefore, no closure thus not a group.

2. Let $A = (0, -1)$ and $B = (0, 2)$. Describe the locus of a point P that moves so that $PA = 2PB$.

$P(a,b)$

$\because PA = 2PB$

$\therefore PA^2 = 4PB^2$

$\therefore a^2 + b^2 + 2b + 1 = 4(a^2 + b^2 - 4b + 4)$

$\therefore 3a^2 + 3b^2 - 18b + 15 = 0$ ✓ 2

$\therefore a^2 + b^2 - 6b + 9 = -5 + 9$

$\therefore a^2 + (b-3)^2 = 4 = 2^2$

$\therefore$ The locus of P is a circle with radius 2 at $(0,3)$.

3. Describe the symmetry group of the graph of $x^2 + 4y^2 = 1$.

when $x=0$, $y^2 = \frac{1}{4}$, $y = \pm\frac{1}{2}$;

$y=0$, $x^2 = 1$, $x = \pm 1$.

Therefore, the graph is an ellipse passing through $(0, \pm\frac{1}{2})$ and $(\pm 1, 0)$.

The symmetry group is V4 as its comprises X, Y, H, and e.

✓ 2

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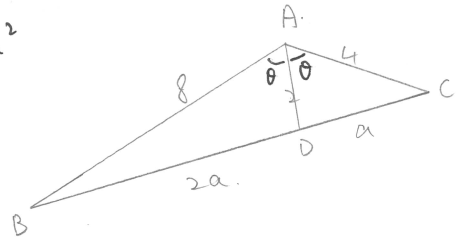
4. A triangle has sides of length 4 and 8. If the bisector of the angle between the sides has length 2, find the length of the third side, giving your answer in the form $\sqrt{a}$ where $a \in \mathbb{Z}^+$.

According to the angle bisector theorem, let $CD = a$, then $BD = 2a$.

According to the law of cosine,



$$\begin{cases} (2a)^2 = 8^2 + 2^2 - 2\cdot 8 \cdot 2 \cdot \cos\theta = 68 - 32\cos\theta = 4a^2 \\ a^2 = 4^2 + 2^2 - 2\cdot 4 \cdot 2 \cdot \cos\theta = 20 - 16\cos\theta \end{cases}$$

$\therefore 4(20 - 16\cos\theta) = 68 - 32\cos\theta$

$\therefore \cos\theta = \frac{3}{8}$

$\therefore a^2 = 20 - 16 \cdot \frac{3}{8} = 20 - 6 = 14$

$\therefore a = \sqrt{14}$ $(a > 0)$

$\therefore BC = 3a = 3\sqrt{14} = \sqrt{126}$ ✓

5. Let G be a group and H a non-empty subset of G. Show that $H \leq G$ if H is closed under division; by this we mean xy^{-1} is in H whenever x and y are in H.

- For any $x \in H$, we have $x \cdot x^{-1} = e \in H$. identity ✓.
- For any $x \in H$, we have $e \cdot x^{-1} = x^{-1} \in H$. inverse ✓. ✓
- For any $x, y \in H$, y^{-1} is also in H. we have $xy^{-1} = x \cdot (y^{-1})$.

Since xy^{-1} is closed, $x \cdot (y^{-1})$ is also in H, indicating closure for multiplication.

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According to the 3-step subgroup test, $H \leq G$.

$3\frac{1}{2}$

we should show

if $x, y \in H$ then $xy \in H$.

$y^{-1} \in H$.

$x \cdot (y^{-1})^{-1} = x \cdot y \in H$.