HL1 Assignment #9

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100%

Excellent!!

1. Find the value of $\log_2\left((\log_{16} 2)^{(\log_5 125)}\right)$.

$\log_{16} 2 = \frac{1}{4}$

$\log_5 125 = 3$

$\therefore \log_2 \left(\frac{1}{4}\right)^3$

$= 3 \log_2 \frac{1}{4}$

$= 3 \cdot (-2)$

$= -6$

✓

2. Find the value of k so that the line containing $(k, 4)$ and $(-1, 7)$ is perpendicular to the line $4x + 2y = 3$.

$2y = -4x + 3$

$y = -2x + \frac{3}{2}$

$\Updownarrow$

$y = \frac{1}{2}x + b$

$-\frac{1}{2} + b = 7$

$b = \frac{15}{2}$

$\therefore y = \frac{1}{2}x + \frac{15}{2}$

$\therefore \frac{k+15}{2} = 4$

$\therefore k = -7$

✓

3. Find the quotient and remainder when $2x^4 + 3x^2 - 8x + 2$ is divided by $x^2 + x + 1$.

$$
\begin{array}{r|l}
 & 2x^2 - 2x + 3 \\
\hline
x^2 + x + 1 & 2x^4 + 0x^3 + 3x^2 - 8x + 2 \\
 & 2x^4 + 2x^3 + 2x^2 \\
\hline
 & -2x^3 + x^2 - 8x \\
 & -2x^3 - 2x^2 - 2x \\
\hline
 & 3x^2 - 6x + 2 \\
 & 3x^2 + 3x + 3 \\
\hline
 & -9x - 1
\end{array}
$$

✓

$\therefore$ quotient $2x^2 - 2x + 3$

remainder $-9x - 1$

4. When $z^{40} + kz - 3$ is divided by $z + i$ the remainder is 4. Find the value of k.

according to the remainder theorem,

$(-i)^{40} + k(-i) - 3 = 4$

$1 - ik = 7$

$k = -\frac{6}{i}$

$\therefore k = 6i$ ✓

5. An integer is said to be *fiveless* if it is written without using the digit 5. For example, 274 and 43 are fiveless whereas 252 is not. How many integers in the set $\{ n \in \mathbb{Z} \mid 1 \leq n < 1000 \}$ are fiveless?

1 - 100:

5. 15. 25. 35. 45. 65. 75. 85. 95.

50. 51. 52. 53. 54. 55. 56. 57. 58. 59.

$1 + 2 \times 9 = 19$

101 - 200: 19

201 - 300: 19

301 - 400: 19

401 - 499: 19

500 - 599: 100

600 - 699: 19

700 - 799: 19

800 - 899: 19

900 - 999: 19

$\therefore$ fiveful $= 19 \times 9 + 100$
$= 171 + 100$
$= 271$

$\therefore$ fiveless $= 999 - 271$ ✓
$= 728$

6. The cubic equation $x^3 + bx^2 + cx + d = 0$ has roots -3 and $-1 \pm i\sqrt{5}$. Find the values of b, c and d.

$-1 \pm \sqrt{5}\, i \iff \frac{-2 \pm \sqrt{-20}}{2}$

$a = 1 \quad b = 2 \quad \Delta = -20$

$\Delta = 4 - 4 \cdot 1 \cdot c = -20$

$c = 6$

$\therefore x^2 + 2x + 6 = 0$

$(x^2 + 2x + 6)(x + 3) = 0$

$x^3 + 2x^2 + 6x + 3x^2 + 6x + 18 = 0$

$x^3 + 5x^2 + 12x + 18 = 0$ ✓

$\therefore \begin{cases} b = 5 \\ c = 12 \\ d = 18 \end{cases}$

7. Find the value of k so that the line $y = -3x + k$ is tangent to the parabola $y = 2x^2 + x$.

$$\begin{cases} y = -3x+k \\ y = 2x^2+x \end{cases}$$

$2x^2 + x = -3x + k$

$2x^2 + 4x - k = 0$

$0 = 16 + 4 \cdot k \cdot 2$

$= 16 + 8k = 0$

$\therefore k = -2$

✓

8. Suppose $b^{0.9} = 2$ and $b^{1.4} = 3$. Find the value of $\log_b 4\sqrt{3}$.

$\log_b 2 = 0.9$

$\log_b 3 = 1.4$

$\log_b 2^2 \cdot 3^{\frac{1}{2}}$

$= 2\log_b 2 + \frac{1}{2}\log_b 3$

$= 2 \cdot 0.9 + \frac{1}{2} \cdot 1.4$

$= 1.8 + 0.7$

$= 2.5$

✓

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9. Find the coordinates of all points in the plane that are equidistant from the x-axis, the y-axis and the point $(2, 1)$.

the points are on $y=x$ or $y=-x$

$\therefore (a,a), (a,-a)$

$\therefore$ distance $= \sqrt{(2-a)^2+(1-a)^2} = |a|$ or $\sqrt{(2-a)^2+(1+a)^2} = |a|$

$a^2-2a+5=0$

$\Delta < 0$.

inadmissable.

$\therefore a^2-4a+4+a^2-2a+1=a^2$

$\therefore a^2-6a+5=0$

$\therefore (a-5)(a-1)=0$

$\therefore a_1=5$

$a_2=1$

$\therefore (5,5)\ (1,1)$

✓

10. The three solutions of the equation $z^3 - 3z^2 + 3z + i = 1$ are the vertices of a triangle in the complex plane. What is the area of the triangle?

$z^3-3z^2+3z-1=-i$

$\therefore (z-1)^3=-i$

$\therefore z=\sqrt[3]{-i}+1$

let $(a+bi)^3=-i$

$\therefore a^3+3a^2bi-3ab^2-b^3i=-i$

$\therefore a^3-3ab^2+(3a^2b-b^3)i=-i$

$\therefore \begin{cases} a(a^2-3b^2)=0 \\ b(3a^2-b^2)=-1 \end{cases}$

① $a=0$, $-b^3=-1$

$\therefore b=1$

② $a\neq 0$, $a^2=3b^2$

$\therefore 8b^3=-1$, $b=-\frac{1}{2}$

$\therefore a=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2}$

$\because z=a+bi+1$

$\therefore z_1=i+1$

$z_2=\frac{\sqrt{3}}{2}-\frac{1}{2}i+1$

$z_3=-\frac{\sqrt{3}}{2}-\frac{1}{2}i+1$

$\therefore (1,1)\ \left(\frac{2-\sqrt{3}}{2}, -\frac{1}{2}\right)\ \left(\frac{2+\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\therefore S=\frac{1}{2}\cdot\left[1-\left(-\frac{1}{2}\right)\right]\cdot\left(\frac{2+\sqrt{3}}{2}-\frac{2-\sqrt{3}}{2}\right)$

$=\frac{\sqrt{3}}{2}\cdot\frac{3}{2}$

$=\frac{3}{4}\sqrt{3}$

✓

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Solutions to HL1 Assignment #9

1. $\log_2\left((\log_{16}2)^{(\log_5 125)}\right) = \log_2[(\frac{1}{4})^3] = -6$.

2. The slope of the given line is -2. So we must solve $\dfrac{7-4}{-1-k} = \dfrac{1}{2}$, whence $k = -7$.

3. Division gives $q(x) = 2x^2 - 2x + 3$ and $r(x) = -9x - 1$.

4. By the remainder theorem $f(-i) = 4$. So we have $(-i)^{40} - ki - 3 = 4$, whence $k = 6i$.

5. There are $8\times 9\times 9$ such 3-digit integers, 8×9 such 2-digit integers and 8 such 1-digit integers. Therefore in total there are 728 such integers.

   Alternatively, consider the related set S of decimal strings of length 3. The number of fiveless strings in S is $9\times 9\times 9 = 729$. So our given set must have $729-1 = 728$ fiveless integers since 0, corresponding to 000 in S, is absent.

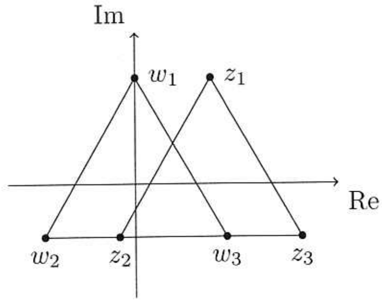
6. By the factor theorem the required cubic equation is $(x+3)(x+1-i\sqrt{5})(x+1+i\sqrt{5}) = x^3+5x^2+12x+18$. Hence $b = 5$, $c = 12$ and $d = 18$.

7. Substitution gives $-3x+k = 2x^2+x$, whence $2x^2+4x-k = 0$. For tangency we want just one solution for x, so the discriminant $\Delta = 16+8k = 0$, whence $k = -2$.

8. First we have $\log_b 2 = 0.9$ and $\log_b 3 = 1.4$. Now $\log_b 4\sqrt{3} = 2\log_b 2 + \frac{1}{2}\log_b 3 = 2\times 0.9 + \frac{1}{2}\times 1.4 = 2.5$.

9. Points that are equidistant from the two axes either lie on the line $y = x$ or the line $y = -x$. However, for these points to be also the same distance from the point $(2,1)$ they must lie on the line $y = x$ and be in the first quadrant. So such a point must have the form (x,x) where $x > 0$. Hence we must have $(x-2)^2 + (x-1)^2 = x^2$, whence $x = 1$ or $x = 5$. Hence the required points are $(1,1)$ and $(5,5)$.

10. Rewrite the equation as $z^3 - 3z^2 + 3z - 1 = -i$, which is equivalent to $(z-1)^3 = -i$. Letting $w = z-1$ we now have the equation $w^3 = -i$, which can be written as $w^3 - i^3 = 0$. Next $w^3 - i^3$ factors as $(w-i)(w^2+iw+i^2) = (w-i)(w^2+iw-1)$ giving the roots $w_1 = i$, $w_2 = (-\sqrt{3}-i)/2$ and $w_3 = (\sqrt{3}+i)/2$. Since $z_i = w_i + 1$, the z-triangle will be shifted one unit to the right of the w-triangle.



This means the two triangles are congruent and hence the z-triangle has the same area as the w-triangle, which has base $\sqrt{3}$ and height $3/2$. So the required area is $1/2\times\sqrt{3}\times 3/2 = 3\sqrt{3}/4$.