HL1 Assignment #6 Name: Jerry Jiang

1. Five people including Ali and Baba sit in a row. How many ways can this be done if Ali refuses to sit next to Baba?

100%

Excellent!!

① they sit together

② they don't sit together.

There's only 2 possibilities.

$^5P_5 - {}^4P_4 \cdot {}^2P_2$

$= 5! - 4! \cdot 2$

$= 120 - 24 \cdot 2$

$= 72$

2. The constant term in the expansion of $\left(2x - \frac{1}{x}\right)^6$ is an integer. Find its value.

$(2x)^3 \cdot (-\frac{1}{x})^3 \cdot \binom{6}{3}$

$= 8x^3 \cdot -\frac{1}{x^3} \cdot \cancel{\frac{6\times5\times4}{3\times2\times1}}$

$= -8 \cdot 20$

$= -160$

$2^0(1+2) + 2^4 \cdot 2 +$

3. Binary(base-2) is computer friendly but not very human friendly. Computer scientists therefore often find it convenient to use *hexadecimal*(base-16) instead of binary. Write the binary number 111100101100 in hexadecimal.

$(111100101100)_2 = (2^2 + 2^3 + 2^5 + 2^8 + 2^9 + 2^{10} + 2^{11})_{10}$

$= 2^0(4+8) + 2^4 \cdot 2 + 2^8(1+2+4+8)$

$= 2^0 \cdot 12 + 2^4 \cdot 2 + 2^8 \cdot 15 = 16^0 \cdot 12 + 16^1 \cdot 2 + 16^2 \cdot 15$

$\because$ a → 10

b → 11

c → 12

d → 13

e → 14

f → 15

$\therefore (111100101100)_2 = (f2c)_{16}$

$60+x \quad 100+x \quad 150+x$

4. By adding the same constant to each of the numbers 60, 100, and 150, a geometric sequence is obtained. Find the common ratio for the sequence.

$\frac{100+x}{60+x} = r$

$\frac{150+x}{60+x} = r^2$

$\frac{(100+x)^2}{(60+x)^2} = \frac{150+x}{60+x}$

$\frac{(100+x)^2}{60+x} = 150+x$

$\therefore 10000+200x+x^2 = 9000+210x+x^2$

$\therefore 10x = 1000$

$x = 100$

$\therefore 160, 200, 250.$

$\therefore r = \frac{200}{160} = \frac{250}{200} = \frac{5}{4}$

(I assume that 60, 100 & 150 are consecutive terms in the geometric sequence, or the answer could be $\frac{5}{4}$ or $\frac{4}{5}$ or $-\frac{4}{9}$ or $-\frac{9}{4}$ or $-\frac{5}{9}$ or $-\frac{9}{5}$)

5. Find the sum of the coefficients in the expansion of $(3-4x)^5$.

$S = \binom{5}{0}\cdot 3^5 + \binom{5}{1}\cdot 3^4\cdot(-4) + \binom{5}{2}\cdot 3^3\cdot(-4)^2 + \binom{5}{3}\cdot 3^2\cdot(-4)^3 + \binom{5}{4}\cdot 3\,(-4)^4 + \binom{5}{5}\cdot(-4)^5$

$= 3^5 - 5\cdot 4\cdot 3^4 + 10\cdot 3^3\cdot 16 - 10\cdot 9\cdot 64 + 5\cdot 3\cdot 4^4 - 4^5$

$= -1$ (I assume that 3^5 the constant coefficient is considered in the sum)

6. Solve $\log_2 x - \log_8 x = 4$.

$\log_2 x - \frac{1}{3}\log_2 x = 4$

$\frac{2}{3}\log_2 x = 4$

$\log_2 x = 6$

$x = 2^6$

$= 64$

7. Complete the row of Pascal's triangle that begins 1 7 21. Hence give the *exact* value of 1.01^7.

$\binom{7}{0}\binom{7}{1}\binom{7}{2}\binom{7}{3}\binom{7}{4}\binom{7}{5}\binom{7}{6}\binom{7}{7}$

1 7 21 35 35 21 7 1

(0)

(1)(0)(7)(21)(0)(35)(0)(35)(0)(21)(0)(7)(0)(1)

$= 1.07213535210701$

(similar rule when doing 11^n, add to overflow to the higher digits)

8. Four men and five women stand in a line. How many ways can this be done if no two men are adjacent?

women have become separation, providing 6 potential space for men.

${}^6C_4 \cdot {}^4P_4 \cdot {}^5P_5$

$= \frac{6 \times 5 \times 4 \times 3}{\cancel{4 \times 3 \times 2 \times 1}} \cdot \cancel{4!} \cdot 5!$

$= 360 \cdot 120$

$= 43200$

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9. Find the sum of the series $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + 99 \times 99!$.

$$S = (1\times1! + 1!) + (2\times2! + 2!) + \cdots + (99\times99! + 99!) - (1! + 2! + \cdots + 99!)$$

$$= (2! + 3! + \cdots + 100!) - (1! + 2! + \cdots + 99!)$$

$$= 100! - 1$$

$= 9332621544394415268169923 8856266700 490715968$
$26438162146859296389521759999322991560894146$
$39761565182862536979208272237582511852109168$
$63999999999999999999999999$

10. Prove that $\sqrt{3}$ is irrational.

let's assume that $\sqrt{3}$ ~~can~~ is rational,

so we can have $\sqrt{3} = \frac{p}{q}$ $(p, q \in \mathbb{Z}, q \neq 0, \gcd(p,q)=1)$

$$\sqrt{3} = \frac{p}{q}$$

$$3 = \frac{p^2}{q^2}$$

$$3q^2 = p^2$$

$\because$ ~~p~~ $\gcd(p,q)=1$

$\therefore$ p is multiple of 3.

$\therefore$ p^2 is multiple of 9.

$\therefore$ let $p^2 = 9n$ $(n \in \mathbb{Z})$

$\therefore$ $3q^2 = 9n$

$\therefore$ $q^2 = 3n$

$\because$ $\gcd(p,q)=1$, $p^2 = 9n$,

$\therefore$ $\gcd(q, n) = 1$.

$\therefore$ q is the multiple of 3.

$\therefore$ $\gcd(q,p) = 3$.

so our hypothesis fails

$\therefore$ $\sqrt{3}$ is irrational

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Solutions to HL1 Assignment #6

1. The required number is $5! - 2 \times 4! = 72$.
2. The required term is $\binom{6}{3}(2x)^3(-1/x)^3 = -160$.
3. The simplest approach is to start from the right and break the binary number into blocks of 4 digits filling the final block with leading zeros if necessary. We then convert each block into its hexadecimal equivalent. Proceeding in this way gives $1111\,0010\,1100_2 = \mathrm{F2C}_{16}$.
4. Let the constant be k. Then we know $(100 + k)^2 = (60 + k)(150 + k)$. A little algebra gives $k = 100$. So the common ratio is $200/160 = 1.25$.
5. The sum of the coefficients is most easily found by substituting $x = 1$, giving the sum as $(3-4)^5 = -1$.
6. Letting $y = \log_2 x$, gives $y - \frac{1}{3}y = 4$. Hence $y = 6$ and so $x = 64$.
7. The row is 1 7 21 35 35 21 7 1. So the exact value of 1.01^7 is 1.07213535210701.
8. Consider the string $\_W_1\_W_2\_W_3\_W_4\_W_5\_$, where the W_i are the women and each $\_$ is a possible space for a man. We then have the required number as $\binom{6}{4} \times 4! \times 5! = 43\,200$.
9. Notice $(n+1)! - n! = n \times n!$. So our series is the same as

$$(2! - 1!) + (3! - 2!) + (4! - 3!) + (5! - 4!) + \cdots + (99! - 98!) + (100! - 99!),$$

which collapses to $100! - 1!$ and this is the required sum.

10. Suppose to the contrary that $\sqrt{3}$ is rational. Then $\sqrt{3} = p/q$ for integers p and q with $q \neq 0$. It follows that $p^2 = 3q^2$, but this is a contradiction since p^2 will contain the factor 3 an even number of times in its prime factorization while $3q^2$ will contain the factor 3 an odd number of times in its prime factorization. Hence what we supposed is false and $\sqrt{3}$ must therefore be irrational.