An investigation of the Monty Hall Problem applied to playing cards

— To what extent do the experimental success probabilities values for the "switching strategy" in the Monty Hall problem match the theoretical values predicted by applying the Bayes' theorem?

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1 Introduction and rationale

For my investigation, I have studied and applied a famous problem from game theory to understand how close experimental probabilities can be to the theoretical probability values predicted for the outcomes of a mathematical event. This investigation is based on the Monty Hall problem, a mathematical problem well known for intriguing human intuition. It is named after the presenter of a TV game show that had a similar problem, and it consists of three random doors - two of them hiding a goat behind, while the remaining door hides a desirable car. At the beginning of the game show, the presenter asks the player to pick one door at random, to win whatever is behind that door. Then, the presenter looks at the other two doors and opens one of them, revealing a goat. Finally, the player is given the option of changing their selection to the remaining door - the door neither picked by the participant nor opened by the presenter. Is it more likely to win the car by switching to the remaining door or sticking with the original choice?

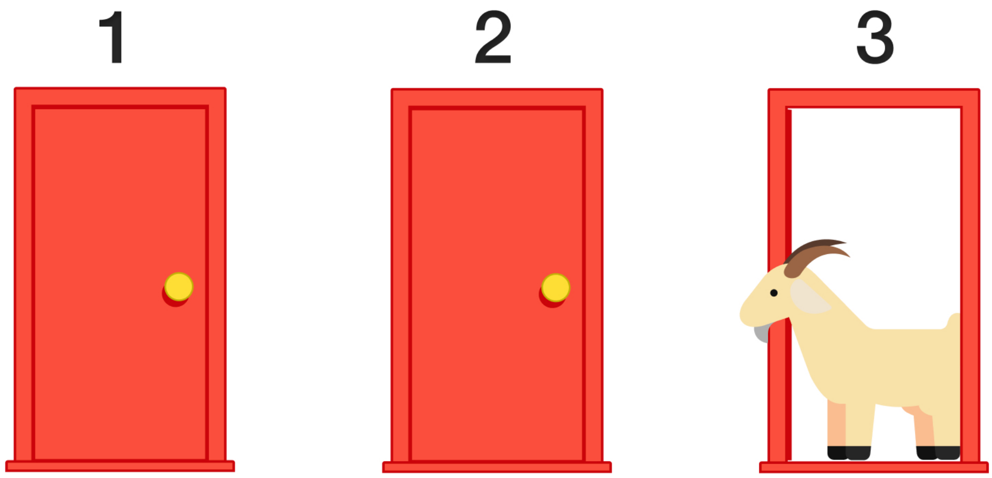
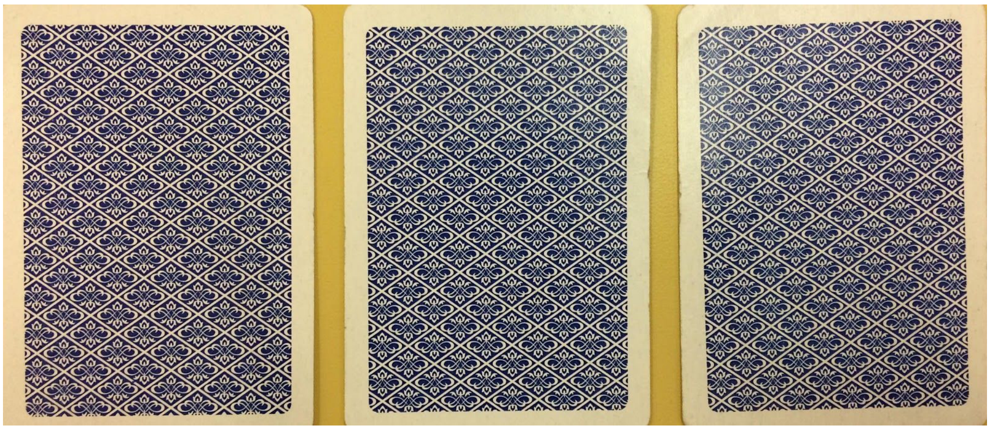


Figure 1: An illustration of the Monty Hall problem. [1]

Intuitively, we are tempted to think that the probability of winning the car by switching is the same as the probability of winning it by sticking since we are choosing one between two (intuitively) equally random doors. But the values for the probabilities of these two distinct events are not simply 50%/50% as we tend to think. However, we cannot forget about the door filtering done by the presenter when he revealed the goat behind one door. Instead, winning by switching is more likely than winning by sticking, with a probability value of $\frac{2}{3}$=0.666... compared to $\frac{1}{3}$, respectively.

This investigation simulates the Monty Hall problem, using a different, but simpler design. As materials for the game trials, I decided to use three cards of a modern deck of French playing cards: specifically, two jokers and an ace. The jokers and the ace are equivalent to the doors with a goat and the door with the car, respectively, of the original Monty Hall problem.

The three cards are shuffled randomly and then sorted out one beside another, just like shown below:



Card 1	Card 2	Card 3

I have been interested in testing the experimental probability of getting aces always following the strategy of switching, so in my investigation, I have divided the experimental events into three subsets of this game strategy:

1. **Choose** initially card 1 (in the left-hand side) and then **switch** to whichever card is left hidden, after one of the jokers is revealed.
2. **Choose** initially card 2 (in the middle) and then **switch** to whichever card is left hidden, after one of the jokers is revealed.
3. **Choose** initially card 3 (in the right-hand side) and then **switch** to whichever card is left hidden, after one of the jokers is revealed.

A sequence of events whose final card chosen is revealed to be an ace is called successful. If the final card chosen is a joker, the sequence of events is considered a failure. Experimental probabilities are tested quickly by adopting one of the sequence of events stated earlier (1, 2 or 3) and repeating it successively until there is a suitable number of trials. The outcome of a sequence of events can be easily known by revealing the two cards that were not chosen at first. If there are two jokers, it's a failure since switching from the original card would necessarily mean switching from an ace to a joker. If there is a joker and an ace, it constitutes a successful sequence of events because that joker would necessarily be discarded by the game presenter in the filtering process that precedes the switching option, so the card the participant switches to would necessarily be an ace.

For this investigation, I decided to keep track of the outcomes of 100 trials reproduced following each of the sequence of events to get a suitable range of data that would allow me to make safe conclusions about the closeness between experimental probability and theoretical probability, applied to this adaptation of the Monty Hall problem.

My interest for this investigation came after I first heard about the Monty Hall problem from a friend of mine, and I could not believe in the proposed solution for the problem, which was against my own intuition. Motivated and curious to understand that dilemma better, I decided to run several trials with the aim of investigating the trend involving the experimental probabilities associated to the problem. Moreover, I tried to redesign the problem, looking for something more appealing than a fictitious story of doors, cars, and goats, so I got the insight of using playing cards, a very common trinket, as the main tool of my game trials. Game theory is a field that plays around with randomness, but mathematics can help me trying to model some of its randomness trends through the study of probabilities.

2 Background Section

Before I elaborate further on the outcomes of my game trials, it is important to introduce the probability theory applied to the solution of the Monty Hall problem as well as to the enunciation of my game involving playing cards.

2.1 Proving that the theoretical probability of winning the original Monty Hall problem by switching is $\frac{2}{3}$

Let the set of three cards be designated by (A, B, C) from the left to the right. There are three possible arrangements for the jokers (J) and the ace

(K) hidden in these cards:

1 - K, J, J

2 - J, K, J

3 - J, J, K

Let's analyze the possible outcomes of always following the strategy of switching from the original choice of card to the card not revealed by the presenter.

Table 1: Outcomes of the "switching strategy" applied to the Monty Hall problem

	Door picked initially by the participant	**Outcome of the game**
Arrangement 1 (K, J, J)	A	Fail
	B	Success
	C	Success
Arrangement 2 (J, K, J)	A	Success
	B	Fail
	C	Success
Arrangement 3 (J, J, K)	A	Success
	B	Success
	C	Fail
Number of total outcomes		9
Number of favorables outcomes (success)		6
Probability of success		$\frac{6}{9} = \frac{2}{3}$

2.2 Application of conditional probability and Bayes' theorem to the game adapted with playing cards

From the theory of probability, for two events designated by letters A and B, the conditional probability of A given B (A|B) or of B given A (B|A) can be enunciated by the following expressions:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \tag{2}$$

Since intersection is a commutative operation, $P(A \cap B) = P(B \cap A) = P(B|A) \times P(A)$. Substituting in (1):

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)} \tag{3}$$

But:

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A) \times P(A) + P(B|A') \times P(A') \tag{4}$$

Substituting (4) in (3) leads to the general expression of the Bayes' theorem:

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B|A) * P(A) + P(B|A') * P(A')}$$

In the game designed for my investigation, let's define events A and B as:

- Event A $\rightarrow$ event of the player finding the ace hidden in card Y(Ace=Y)
- Event B $\rightarrow$ event of the game presenter flipping card X (Flip=X), which is necessarily not an ace, therefore it is not a random event.

For event A, there is one restriction:

- The probability of finding the ace behind a card that was flipped before is 0 since the flipped card always reveals a joker.

Similarly, for event B, there are two restrictions:

- The probability of flipping a card that has the ace behind is 0 because the presenter of the game will only flip a card that has a joker behind.
- The probability of flipping the card that was initially picked by the player is 0 because of the game rules set.

When looking at the sequence of events B|A or A|B, it is worth reminding that:

- Finding the ace hidden in a card that was flipped before by the presenter is an impossible event since the flipped card always reveals a joker. In probability notation:

$$P(Ace = X|Flip = X) = 0$$

- It is impossible for the presenter to flip a card that hides an ace because the presenter of the game will always flip a card that reveals a joker.

$$P(Flip = Y|Ace = Y) = 0$$

- Obviously, it is impossible for the presenter to flip the card that was initially picked by the player.

Due to the symmetry of the problem, all cards appeal the same in the beginning of the game, so the Bayes' theorem can be applied to one combination of picked card/flipped card and then generalized for all other possible combinations.

For the sake of simplifying the mathematical work, let's assume that card 1 was initially picked by the player. With no noticeable differentiation between card 2 and card 3, we can assume that these cards are equally likely to be flipped by the presenter, so P(Flip=2) = P(Flip=3) =$\frac{1}{2}$. Let's say that card 3 was to be flipped by the game presenter.

Since event A is the event of finding the ace hidden in a card, it can only happen for card 1 (Ace=1) or card 2 (Ace=2). Since event B is the event of the presenter flipping a card, it can only happen for card 2 (Flip=2) or card 3 (Flip=3).

The value of P(Ace=Y | Flip=X) can be found by applying the Bayes' theorem, but it is important to find out P(Flip=X | Ace=Y) first. P(Flip=X | Ace=Y) is the notation of the probability of the presenter flipping card **given that** the ace is found hidden in card Y. Table 2 gathers some important probability values that can be substituted in the expression of the Bayes' theorem applied to the game of playing cards.

Table 2: Conditional probabilities of flipping a card X (event B) given that the ace lies behind a card Y (event A)

Event B	**Conditional A**	**Conditional probability notation**	**Probability value**
Flip card 3	Ace behind card 3	P(Flip=3|Ace=3)	0
Flip card 3	Ace behind card 2	P(Flip=3|Ace=2)	1
Flip card 3	Ace behind card 1	P(Flip=3|Ace=1)	1/2
Flip card 2	Ace behind card 2	P(Flip=2|Ace=2)	0
Flip card 2	Ace behind card 3	P(Flip=2|Ace=3)	1
Flip card 2	Ace behind card 1	P(Flip=2|Ace=1)	1/2

The probabilities substituted in the Bayes' theorem expression should be probabilities determined right before the correspondent event happened. So, P(Flip=3) is $\frac{1}{2}$ because in the moment before a card was flipped by the presenter, there were two equally likely cards that could have been flipped - cards 2 and 3.

Similarly, P(Ace=1)=$\frac{1}{3}$, because it was initially possible for the ace to be hidden in three equally likely cards.

Given that card 3 was flipped (Flip=3) by the presenter, what is the probability of the player to **win by sticking** to his/her original choice (card 1), i.e finding an ace hidden in card 1 (Ace=1)?

P(Ace≠1) is the complement of P(Ace=1), so:

$$P(Ace \neq 1) = 1 - P(Ace = 1) = 1 - \frac{1}{3} = \frac{2}{3}$$

P(Flip=3 | Ace≠1) is the probability of the presenter flipping card 3 given that the ace is **not** in card 1. If the ace is not in card 1, it is either in card 2 or card 3. If it is in card 2, card 3 will be the one flipped by the presenter. If it is in card 3, card 2 will be the one flipped instead. Since there are only two possible outcomes of Ace≠1, P(Flip=3 | Ace≠1) is equal to 1 in 2, so:

$$P(Flip = 3|Ace \neq 1) = \frac{1}{2}$$

Applying the Bayes' theorem and substituting relevant probability values (found in table 2 and above):

$$\begin{aligned} P(Ace = 1|Flip = 3) &= \frac{P(Flip = 3|Ace = 1) \times P(Ace = 1)}{P(Flip = 3|Ace = 1) \times P(Ace = 1) + P(Flip = 3|Ace \neq 1) \times P(Ace \neq 1)} \\ &= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3}} \\ &= \frac{1}{3} \end{aligned}$$

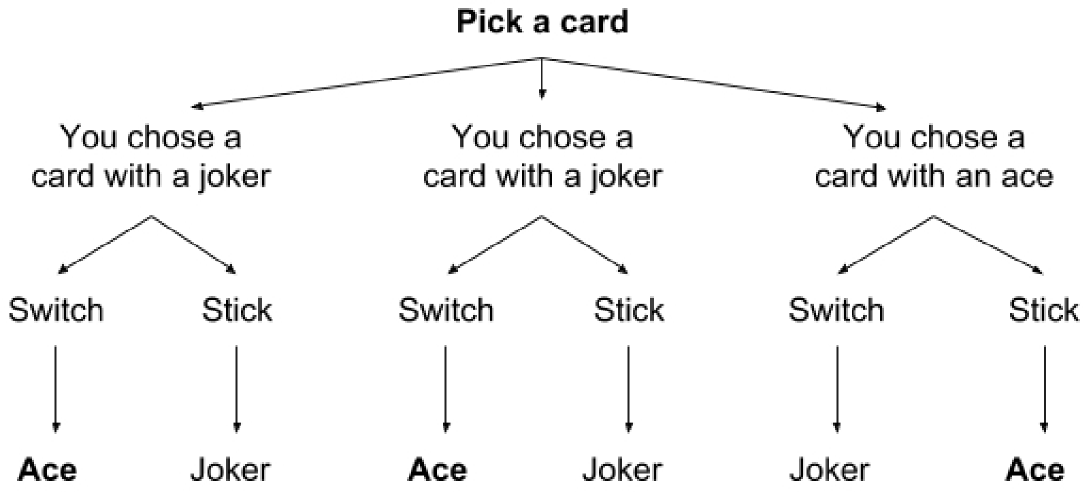
Given that card 3 (Flip=3) was flipped by the presenter, what is the probability of the presenter to **win by switching**, i.e finding an ace hidden in card 2 (Ace=2)?

$$P(Ace = 2|Flip = 3) = 1 - P(Ace = 1|Flip = 3) = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus, it has been shown that the (theoretical) probability of getting the ace by switching is twice the probability of getting it by sticking to the original choice, if the original choice is card 1, and card 3 is the one flipped by the presenter. Due to symmetry, this conclusion can be expanded to the 5 other possible combinations of picked card/flipped card, without changing the final probability values.

To better visualize the outcomes of the game and the solution found by applying the Bayes' theorem, I have drawn the following game decision tree:

Figure 2: Decision tree for the game adapted with playing cards



Notice that for 2 out of 3 trials in which the "switching strategy" is adopted, the player wins an ace whereas the ace is only won 1 out of 3 trials if the player sticks with their original choice.

3 Experimental Probability

3.1 Raw and Processed Data

I conducted 100 trials of the game with playing cards, following each of the different kinds of "switching strategies" stated in the introduction. I'm aware that all strategies are expected to be equivalent to each other, but I wanted to see if there is a bias associated to the location of the first card chosen. The number of successful outcomes and the experimental probabilities of success were processed for every set of trials multiple of 10, and the data is presented in tables 3, 4, and 5, each of these for a different kind of "switching strategy."

Table 3: Number of successful outcomes and the experimental probabilities of success when the first card to be chosen is card 1 (left-hand side)

Total number of outcomes	Number of successful outcomes	Experimental probability of getting the ace by switching	Decimal probability value with 3 d.p.
10	7	7/10	0.700
20	12	12/20	0.600
30	18	18/30	0.600
40	26	26/40	0.650
50	32	32/50	0.640
60	39	39/60	0.650
70	46	46/70	0.657
80	53	53/80	0.662
90	62	62/90	0.689
100	68	68/100	0.680

Table 4: Number of successful outcomes and the experimental probabilities of success when the first card to be chosen is card 2 (middle)

Total number of outcomes	Number of successful outcomes	Experimental probability of getting the ace by switching	Decimal probability value with 3 d.p.
10	9	9/10	0.900
20	16	16/20	0.800
30	24	24/30	0.800
40	32	32/40	0.800
50	40	40/50	0.800
60	45	45/60	0.750
70	51	51/70	0.729
80	57	57/80	0.713
90	64	64/90	0.711
100	70	70/100	0.700

Table 5: Number of successful outcomes and the experimental probabilities of success when the first card to be chosen is card 3 (right-hand side)

Total number of outcomes	Number of successful outcomes	Experimental probability of getting the ace by switching	Decimal probability value with 3 d.p.
10	8	8/10	0.800
20	15	15/20	0.750
30	22	22/30	0.733
40	29	29/40	0.725
50	36	36/50	0.720
60	41	41/60	0.683
70	49	49/70	0.700
80	57	57/80	0.713
90	60	60/90	0.667
100	67	67/100	0.670

3.2 Presentation of Data

The first and fourth columns of tables 3, 4, and 5 were used to make discrete plots of experimental probability of success against the total number of trials run, as shown in the three figures below.

Figure 3: Trend for the experimental probability of success given that card 1 (left-hand side) is the first to be chosen

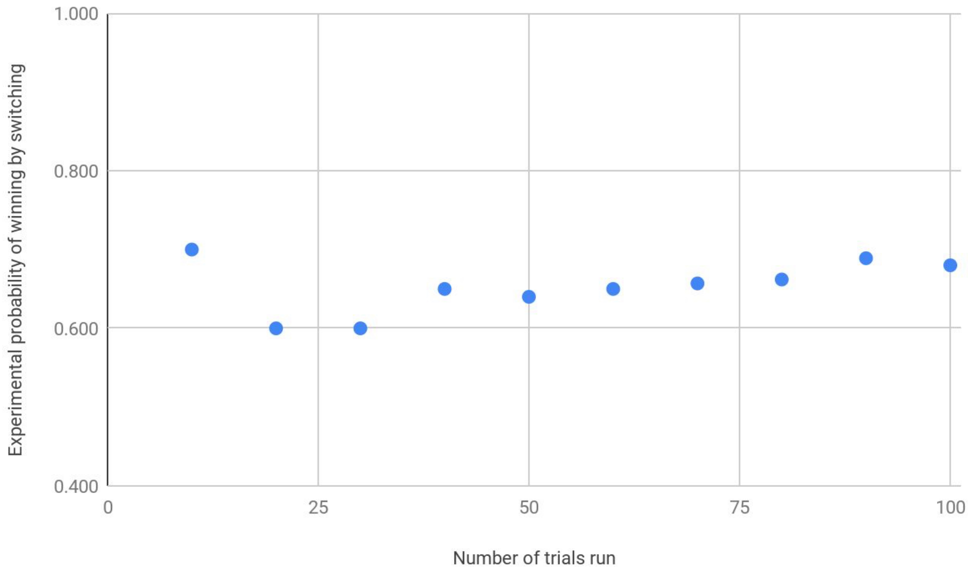


Figure 4: Trend for the experimental probability of success given that card 2 (middle) is the first to be chosen

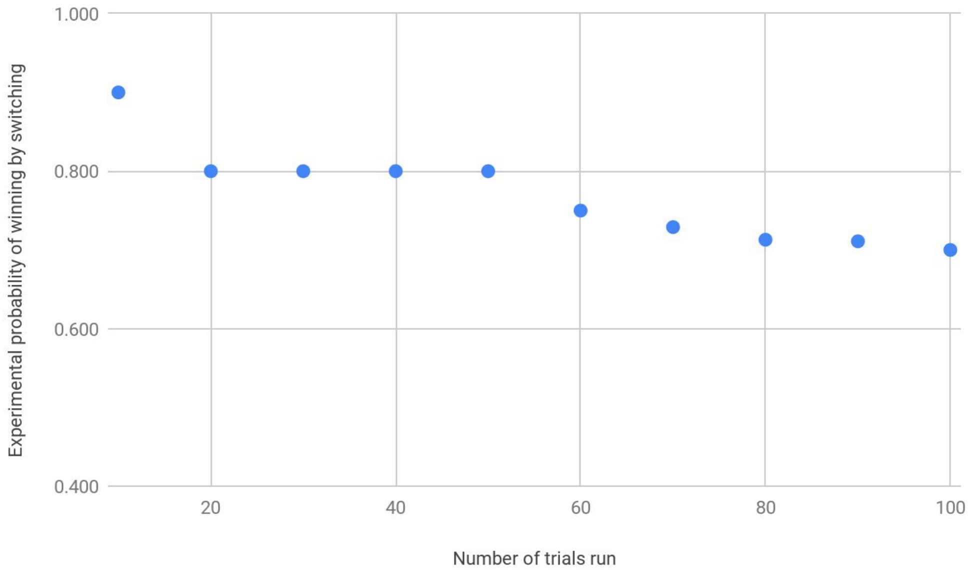
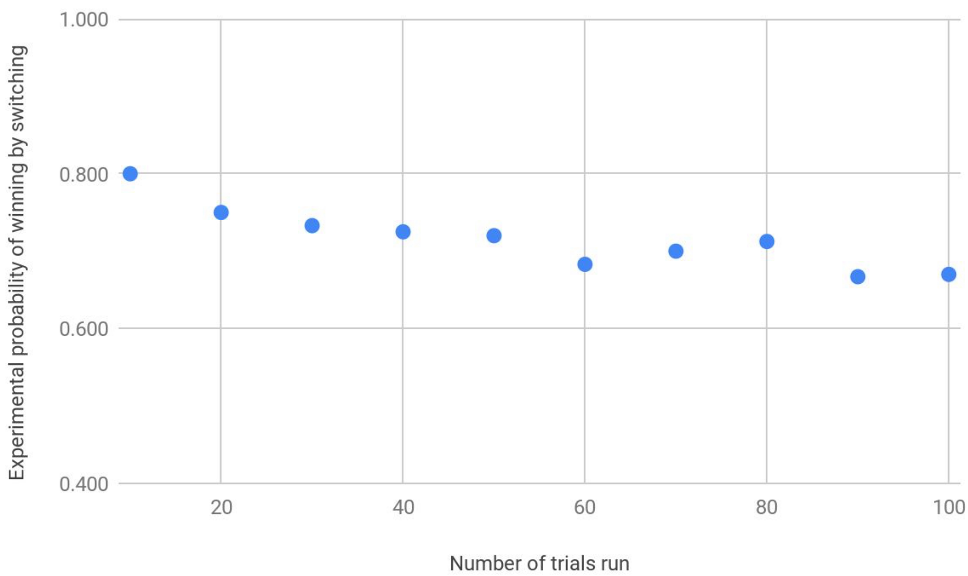


Figure 5: Trend for the experimental probability of success given that card 3 (right-hand side) is the first to be chosen



4 Hypothetical case when number of playing cards=100

As an extension of this investigation, it seems easier to see the solution for the Monty Hall adapted problem, when the number of starting cards is 100 (99 jokers and 1 ace), instead of only 3. Each card holds 1/100 probability of hiding the ace.

In the first turn, after the initial choice is picked:

- The first joker (j_1) is revealed
- There are 98 remaining cards (not counting with the initial choice), which together hold the probability 1-$\frac{1}{100} = \frac{99}{100}$ of hiding the ace.
- Probability of getting the ace by switching = $\frac{99}{100}$, distributed across 98 equally likely cards

Second turn:

- The second joker (j_2) is revealed
- There are 97 remaining cards
- Probability of winning by switching = $\frac{99}{100}$, distributed across 97 equally likely cards

... N^{th} turn ($1 \leq N \leq 98$)

- N^{th} joker (j_N) is revealed
- There are 100-(N+1)remaining cards.
- Probability of winning by switching = $\frac{99}{100}$, distributed across 100-(N+1) equally likely cards

Last turn (N=98)

- 98^{th} joker (j_{98}) is revealed.
- There is 100-(98+1)=1 remaining card
- Probability of winning by switching: $\frac{99}{100}$ concentrated in 1 card

The best strategy for the game with 100 cards is to stick with the original choice until only 1 card is remaining (not counting with the original choice), and then switch to the undiscarded card, which holds a probability 99/100 of hiding the ace. This scenario is similar to the scenario involving only 3 cards, which shows that the reasoning applied to both is the same: switching is better than sticking!

5 Hypothetical case for n cards

Expanding the analysis of my investigation even further to a hypothetical game with n cards, (n-1) jokers and 1 ace, we have the following scenario:

Each card: 1/n at the beginning

First turn:

- First joker (j_1) is revealed
- There are n-2 remaining cards (not counting with the initial choice), which together hold the probability 1-$\frac{1}{n} = \frac{n-1}{n}$ of hiding the ace
- Probability of winning by switching: $\frac{n-1}{n}$ distributed across n-2 cards

Second turn:

- Second joker (j_2) is revealed
- There are n-3 remaining cards
- Probability of winning by switching = $\frac{n-1}{n}$ distributed across n-3 cards

… N^{th} turn ($1 \leq N \leq n-2$)

- N^{th} joker (j_N) is revealed
- There are n-(N+1) remaining cards
- Probability of winning by switching = $\frac{n-1}{n}$ distributed across n-(N+1) cards

Last turn (N=n-2)

- $(n-2)^{th}$ joker (j_{n-2}) is revealed
- There is n-(n-2+1) = 1 remaining card
- Probability of winning by switching: $\frac{n-1}{n}$ concentrated in only 1 card

What happens when n tends to infinite ($n \rightarrow \infty$)?

$$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \frac{\frac{n}{n} - \frac{1}{n}}{\frac{n}{n}} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{1}$$

But $\lim_{n\to\infty} \frac{1}{n} = 0$, so:

$$\lim_{n\to\infty} \frac{n-1}{n} = 1 - 0 = 0$$

This means that as the number of the cards (n) gets larger, the probability of getting the car by using the best strategy (sticking until there is only 1 undiscarded card and then switching) gets similarly larger, closer and closer to 1.

6 Conclusion

The data presented in figure 3, 4, 5 is consistent with the mathematical theory developed in the background section (section 2) with regards to conditional probability and Bayes' theorem. The phenomenon of winning by switching to the undiscarded card was predicted to have a probability of $\frac{2}{3}$ of happening. In figure 3, the range of probabilities of success was found to be between 0.600 and 0.800. As the number of trials increased, the probability of success generally increased, approaching the predicted value of 0.666…, and reaching, finally, the value of 0.680 at the end of 100 completed trials.

In figure 4, the probabilities values started very high (0.900), but as trials went on, there was an overall decrease in the probability values, reaching a final value of 0.700 at the end of 100 completed trials. Figure 5, similarly to figure 3, kept the range of probabilities between 0.600 and 0.800, presenting an overall decreasing trend that stabilizes around 0.670 at the end of 100 completed trials.

The values of 0.680, 0.700, and 0.670 are close enough to themselves and to the predicted probability value of 0.666..., which points out to two main conclusions:

1. The probability of getting the ace by switching can safely be concluded to be higher than the probability of getting the ace by sticking to the original choice, according to both theoretical and experimental probability values, which happened to be close in the scenario of this investigation.

2. There was not enough information to assume the existence of correlation between the original choice of card (left, middle, or right) and the magnitude of the probability value calculated. Thus, the choice of the original card seemed to be an unbiased event.

In this investigation, the experimental probability values of winning the game by following each one of the 3 sequence of events introduced in section 1 were surprisingly closer to the predicted value of $\frac{2}{3}$ (obtained from the Bayes' theorem) than to the value we expected intuitively ($\frac{1}{2}$). This demonstrates that the experimental probabilities were to a high extent connected to the theoretical probabilities.

To find a stronger connection between experimental probabilities and theoretical probabilities, my game should be run for a higher number of trials. As the number of trials increases, the level of confidence of the experimental probability values will be greater, so more reliable conclusions could be drawn about the connection between experimental probabilities and theoretical probabilities in the scope of this investigation.

References

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