HL1 Assignment #0 Name: Jerry Jiang

1. The positive divisors of 8 are $\{1, 2, 4, 8\}$ and the positive divisors of 12 are $\{1, 2, 3, 4, 6, 12\}$. So the *greatest common divisor* of 8 and 12, denoted $\gcd(8, 12)$, is 4. Find $\gcd(180, 225)$.

$180 = 2^2 \times 3^2 \times 5$

$225 = 3^2 \times 5^2$

$\gcd(180, 225) = 3^2 \times 5 = 45$ ✓

100%

Excellent!

2. The positive multiples of 8 are $\{8, 16, 24, 32, \ldots\}$ and the positive multiples of 12 are $\{12, 24, 36, 48, \ldots\}$. So the *least common multiple* of 8 and 12, denoted $\text{lcm}(8, 12)$, is 24. Find $\text{lcm}(36, 45)$.

$36 = 2^2 \times 3^2$

$45 = 3^2 \times 5$ ✓

$\text{lcm}(36, 45) = 2^2 \times 3^2 \times 5 = 180$

3. Calculate 1×1, 11×11 and 111×111. Hence determine $111\,111\,111 \times 111\,111\,111$.

$1 \times 1 = 1$

$11 \times 11 = 121$

$111 \times 111 = 12321$

$111\,111\,111 \times 111\,111\,111 = 12345678987654321$ ✓

$1111111111 \times 1111111111 =$ ~~123456789~~ 1234567900987654321 ✓

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4. Digital computers use base-2 arithmetic. The number 5 in base-2 is 101 and the number 15 in base-2 is 1111. Write the number 45 in base-2.

$45 = 32 + 8 + 4 + 1$

$(45)_{10} = (101101)_2$ ✓

5. The set of *prime numbers* is $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$. Is 1001 in the set of prime numbers?

$1001 = 7 \times 11 \times 13$

NO. ✓

6. Two positive integers a and b are called *relatively prime* if $\gcd(a, b) = 1$. Are 1001 and 533 relatively prime?

$1001 = 7 \times 11 \times 13$

$533 = 13 \times 41$ ✓

NO. $\gcd(1001, 533) = 13$

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7. If x and y are integers, what is the smallest positive value of $30x + 18y$?

$30x+18y$

$=6(5x+3y)$

$5x+3y$ is a integer, so the smallest possible is 1

so the smallest positive value of $30x+18y$ is $6\times1=6$ as long as $5x+3y$ can be 1.

Obviously when $(x,y)=(5,-8)$, $5x+3y=1$, so the answer to the question is 6.

8. A *perfect square* is a number in the set $\{1, 4, 9, 16, 25, \ldots\}$. Which pair of numbers in the set $\{24, 45, 72, 75\}$ has a least common multiple that is a perfect square?

$24 = 2^3 \times 3$

$45 = 3^2 \times 5$

$72 = 2^3 \times 3^2$

$75 = 3 \times 5^2$

$\mathrm{lcm}(45, 75) = 225.$

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9. What is the smallest number greater than 1 that is a perfect square, a perfect cube, and a perfect fourth power?

$k = \cancel{a^2} = b^3 = c^4 \quad (a, b, c, k \in \mathbb{Z}^+)$

$b = d^4 \quad (d \in \mathbb{Z}^+)$

$c = e^3 \quad (e \in \mathbb{Z}^+)$

a is unnecessary to consider

$k = d^{12} = e^{12}$

$d = e = 2$ is the least possible value.

$\therefore k = 2^{12} = 4096$

10. Determine all integers x and y such that $x^4 - y^2 = 71$.

$(x^2+y)(x^2-y) = 71$

1×71

③ $y = 0$

$\emptyset$

① $y > 0$

$\begin{cases} x^2+y=71 \\ x^2-y=1 \end{cases}$

$2x^2 = 72$

$\begin{cases} x^2 = 36 \\ y = 35 \end{cases}$

$(x,y) = (6, 35), (-6, 35)$

② $y < 0$

$\begin{cases} x^2+y=1 \\ x^2-y=71 \end{cases}$

$x^2 = 36$

$y = -35$

$(x,y) = (-6,-35)(6,-35)$

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Solutions to HL1 Assignment #0

1. $\gcd(180, 225) = 45$

2. $\text{lcm}(36, 45) = 180$

3. $1 \times 1 = 1$, $11 \times 11 = 121$, $111 \times 111 = 12321$ and $111\,111\,111 \times 111\,111\,111 = 12\,345\,678\,987\,654\,321$.

4. $45 = 32 + 8 + 4 + 1 = 101101_2$

5. Since $1001 = 7 \times 11 \times 13$, 1001 is not prime.

6. Since $\gcd(1001, 533) = 13$, 1001 and 533 are not relatively prime.

7. Now $\{30x + 18y \mid x, y \in \mathbb{Z}\} = \{0, \pm 6, \pm 12, \pm 18, \ldots\}$. So the smallest positive value is 6. Notice that the greatest common divisor of 30 and 18 is 6. This is an example of a more general result that says that the least positive value in $\{ax + by \mid x, y \in \mathbb{Z}\}$ is $\gcd(a, b)$. In fact, $\{ax + by \mid x, y \in \mathbb{Z}\}$ is the set of all integer multiples of $\gcd(a, b)$.

8. Now $24 = 2^3 \cdot 3$, $45 = 3^2 \cdot 5$, $72 = 2^3 \cdot 3^2$ and $75 = 3 \cdot 5^2$. We conclude the pair $45, 75$ has a least common multiple that is a perfect square, namely $\text{lcm}(45, 75) = 225 = 15^2$.

9. Since $\text{lcm}(2, 3, 4) = 12$, the required number is $2^{12} = 4096$.

10. First observe $x^4 - y^2 = (x^2 - y)(x^2 + y)$. Now $x, y \in \mathbb{Z}$ and 71 is prime. So either $x^2 - y = 1$ and $x^2 + y = 71$, or $x^2 - y = 71$ and $x^2 + y = 1$. Solving simultaneously gives $x = \pm 6, y = -35$ or $x = \pm 6, y = 35$.