HL2 Assignment #16 Name: Jerry Jiang $\frac{10}{10}$ Excellent!

1. Find the radius of convergence and interval of convergence for the series $\sum_{n=0}^{\infty} n!x^n$.

- let $u_n = n!\,x^n$.
- Ratio Test: $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x|$

when $n \to \infty$, $(n+1)|x|$ only has a limit when $x=0$.

Therefore, $R=0$, interval of convergence: $x=0$.

2. Is it possible to find a power series whose interval of convergence is $]0,\infty[$? Explain.

For power series with $R \neq 0$, $R \neq \infty$, there's a center that the interval of convergence is symmetric about. However, the end points 0, ∞ don't have a mid-point, thus it's impossible to find such power series.

3. Find the domain of the function $f(x) = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \ldots$.

$f(x) = 1 + \sum_{n=0}^{\infty} \frac{x^{3(n+1)}}{\prod_{i=0}^{n}(3i+2)(3i+3)}$, let $u_n = \frac{x^{3(n+1)}}{\prod_{i=0}^{n}(3i+2)(3i+3)}$.

Ratio Test: $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{x^{3(n+2)}}{\prod_{i=0}^{n+1}(3i+2)(3i+3)}}{\frac{x^{3(n+1)}}{\prod_{i=0}^{n}(3i+2)(3i+3)}}\right| = \left|\frac{x^3}{(3n+5)(3n+6)}\right| \to 0$ as $n \to \infty$.

Therefore the domain is all real numbers.

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4. Show that the function $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ is a solution to the differential equation $f''(x) = f(x)$.

$$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

$$f'(x) = \frac{2}{2!}x + \frac{4}{4!}x^3 + \frac{6}{6!}x^5 + \cdots + \frac{2n}{(2n)!}x^{2n-1} + \cdots$$

$$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$f''(x) = \frac{1}{1!}x^0 + \frac{3}{3!}x^2 + \frac{5}{5!}x^4 + \cdots + \frac{(2n-1)}{(2n-1)!}x^{2n-2} + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n-2}}{(2n-2)!} + \cdots$$

$$= f(x).$$

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5. Prove that every group of even order contains an odd number of elements of order 2.

Suppose a group of even order contains an even number of elements of order 2.

- identity element has order 1.
- elements in the group with order more than 2 come in pairs, as they and their inverse has the same order, and they can't be their own inverse as the order is bigger than 2. Even number
- 1 + even + even = odd. The contradiction proves the result.

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