90% Very good

FURTHER MATHEMATICS
HIGHER LEVEL

August 2019

Name in block letters

Review Assignment

J	E	R	R	Y		J	I	A	N	G	

INSTRUCTIONS

- Do not use the calculator unless directed to do so in the question.
- There are 20 questions. Try to answer them all.
- All numerical answers must be given exactly or correct to three significant figures.

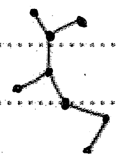
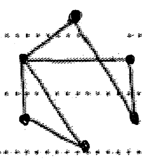
21 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working or explanations. Where an answer is incorrect, some marks may be given for a correct method provided this is shown by written working. You are therefore advised to show all working. Working may be continued below the lines, if necessary.

1. (a) Draw a tree that has no Hamiltonian path.

   (b) Draw a graph with an Eulerian circuit but no Hamiltonian cycle.

   (c) For what values of n does the complete graph K_n have an Eulerian circuit?

(a)  ✓ (b)  ✓

一遍认为 n为奇的 Kn为 Eulerian circuit

(c) • when n=1, there's no circuit. ← usually taken as there " a circuit

• when n>2, • n is even, then all vertices are odd degreed, so it's not possible to start and end at all vertices.

• n is odd, there are Eulerian circuits.

• when n=2, a line connecting the two vertices is the Eulerian circuit. ✗ doesn't start & end at the same vertex

Therefore, ~~n=2 or~~ all the odd numbers when n>2.

Make the vertices clearer

把端点标清楚

4

2. Consider the elementary matrices $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

(a) To what elementary row operations do E_1 and E_2 correspond?

(b) Write down $\det E_1$ and $\det E_2$.

(c) Write down E_1^{-1} and E_2^{-1}.

(a) E_1: $R_1 \to R_2$, $R_2 \to R_1$

E_2: $R_3 - 2R_1 \to R_3$

(b) $\det E_1 = -1$

$\det E_2 = 1$

(c) $E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

5

3. Consider the Abelian group $(\{2, 4, 6, 8\}, \otimes)$ where the operation $\otimes$ is multiplication modulo 10.

   (a) Construct the Cayley table for the group

   (b) List all the proper subgroups of the group.

   (c) Is this group cyclic? If so, name a generator.

(a)

$\otimes$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(b) $(\{4, 6\}, \otimes)$

(c) Yes. 2. $2 \otimes 2 = 4$

$2 \otimes 2 \otimes 2 = 8$

$2 \otimes 2 \otimes 2 \otimes 2 =$ identity.

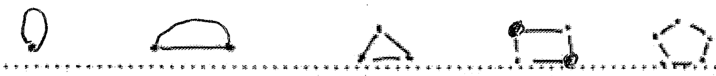
5

4. A *cycle graph* C_n is a graph on n vertices that is a cycle.

(a) Draw the first five cycle graphs C_1 through C_5

(b) For what values of n is C_n bipartite?

(c) Prove that a bipartite graph contains no cycle of odd length.

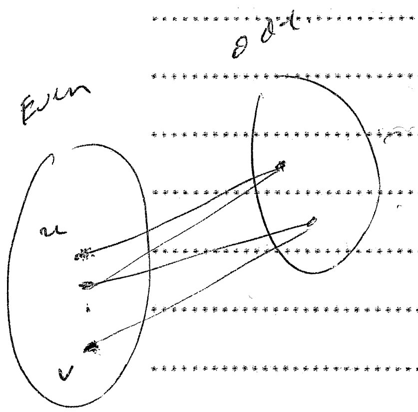
(a) 

C_1 C_2 C_3 C_4 C_5

(b) when $n=2$, C_n is bipartite × n is even

$\boxed{\text{n为偶其实都是 bipartite.}}$

(c) a cycle starts and ends at the same vertex. In a bipartite graph, getting back to the starting side requires even-number moves. so there's no cycle of odd length.



3

5. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

(a) Show that the series converges by comparing the series to a suitable p-series.

(b) Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

(c) Hence find the exact sum of the series.

(a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$\frac{1}{n^2+n} < \frac{1}{n^2}$ when n is postive integer

$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\because p = 2 > 1$ in $\sum_{n=1}^{\infty} \frac{1}{n^2}$, it converges.

Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges.

(b) $\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$

$= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\right) - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$

$= 1$

5/5

6. Consider the matrix $M = \begin{pmatrix} x & x+2 \\ x-5 & -x \end{pmatrix}$ where $\det M = 1$

(a) Find the two possible values of x.

(b) Let A be the matrix when x = 3. Find the smallest group of matrices that contains A and state another group to which this group is isomorphic.

(a) $\det M = x\cdot(-x) - (x-5)(x+2) = 1$

$\therefore \quad -2x^2 + 3x + 10 = 1$

$\therefore \quad 2x^2 - 3x - 9 = 0$

$\therefore \quad (x-3)(2x+3) = 0$

$\therefore \quad x = 3 \text{ or } -\frac{3}{2}$ ✓

(b) $A = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$

$B = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$

$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

×	A	B	C	D
A	C	D	B	A
B	D	C	A	B
C	B	A	D	C
D	A	B	C	D

✓

The group in question #3 is isomorphic.
(D ⟺ 6, A ⟺ 2, B ⟺ 8, C ⟺ 4). ✓

$\boxed{\text{邏辑符号.}}$

5

7. Consider the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \cdots$

(a) Show that the ratio test cannot be used to establish the convergence or divergence of the series.

(b) Use the integral test, clearly stating any necessary conditions for its use, to establish whether the series converges or diverges.

(a) $\lim_{n\to\infty} \left| \frac{\frac{1}{2n+1}}{\frac{1}{2n-1}} \right| = \lim_{n\to\infty} \frac{2n-1}{2n+1} = 1$, inconclusive.

(b) the series is continuous, positive and decreasing.

$$\int_1^\infty \frac{1}{2n-1} = \left. \frac{\ln(2n-1)}{2} \right|_1^\infty = \infty$$

Therefore, the series diverges.

5

8. Let ω be the cube root of unity which has smallest positive argument.

(a) Show that $1+\omega+\omega^2=0$

(b) Find the matrix product $\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}$ giving your answer in simplest form.

(c) Hence solve the following system giving your answers as real numbers.

$$\begin{aligned} x+\quad y+\quad z&=3\\ x+\ \omega y+\omega^2 z&=-3\\ x+\omega^2 y+\ \omega z&=-3\end{aligned}$$

(a) $w = [1, \frac{2}{3}\pi]$.

$w^2 = [1, \frac{4}{3}\pi]$.

$1+\frac{\sqrt{3}}{2}i-\frac{1}{2}-\frac{\sqrt{3}}{2}i-\frac{1}{2}=0$.

so $1+w+w^2=0$.

(b) product $=\begin{pmatrix}3 & 1+w+w^2 & 1+w+w^2\\ 1+w+w^2 & 3 & 1+w+w^2\\ 1+w+w^2 & 1+w+w^2 & 3\end{pmatrix}=\begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix}$.

(c). From (b), we have:

$$\begin{pmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{pmatrix}^{-1}=\frac{1}{3}\begin{pmatrix}1&1&1\\1&w^2&w\\1&w&w^2\end{pmatrix}.$$

$$\begin{pmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-3\\-3\end{pmatrix}.$$

$$\begin{pmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{pmatrix}^{-1}\cdot\begin{pmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1&1&1\\1&w^2&w\\1&w&w^2\end{pmatrix}\begin{pmatrix}3\\-3\\-3\end{pmatrix}$$

$$\therefore\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&1&1\\1&w^2&w\\1&w&w^2\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

5

Turn over

9. (a) State De Morgan's laws for sets.

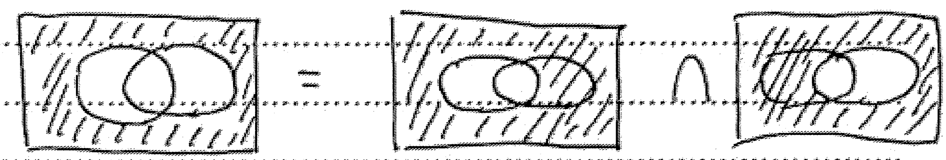
   (b) Use Venn diagrams to show that $(A \cup B)' = A' \cap B'$.

   (c) With the help of De Morgan's laws prove that $[(A' \cup B) \cap (A \cup B')]' = A \triangle B$.

(a) $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

(b)



$(A \cup B)' \quad = \quad A' \quad \cap \quad B'$

(c) $[(A' \cup B) \cap (A \cup B')]' = (A' \cup B)' \cup (A \cup B')'$

$= (A \cap B') \cup (A' \cap B)$

$= (A \setminus B) \cup (B \setminus A)$

$= A \triangle B$

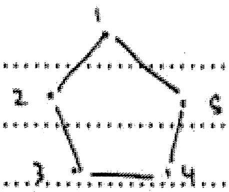
10. Consider the cycle graph C_5.

(a) Draw the complement C_5' of C_5.

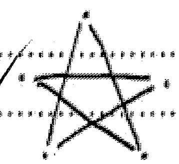
(b) Draw another graph with five vertices that is also isomorphic to its complement.

(c) If G is a simple graph with five vertices, find the sum of the adjacency matrices $A(G)$ and $A(G')$.

(a)



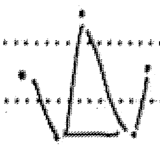
C_5

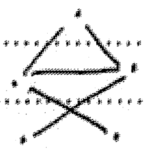


C_5'

✓ ✓

(b)





✓

(1, 1, 2, 3, 3). 1 possibility

(1, 2, 2, 2, 3) 3 possibilities.

(2, 2, 2, 2, 2) 1 possibility.

(c)

$$\begin{matrix} 0 & 1 & 5 & 8 & 10 \\ 1 & 0 & 2 & 6 & 9 \\ 5 & 2 & 0 & 3 & 7 \\ 8 & 6 & 3 & 0 & 4 \\ 10 & 9 & 7 & 4 & 0 \end{matrix}$$

$2^{10} = 1024.$

✗ 题目读错

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

3

Turn over

11. Consider the points $A(-3,9)$ and $B(1,5)$ in the Cartesian plane.

(a) Find the equation of the circle with diameter $[AB]$.

(b) The locus of the point P such that PA = 3PB is the circle $\mathcal{C}$. Find the centre and radius of $\mathcal{C}$.

(c) The tangents to $\mathcal{C}$ through A meet $\mathcal{C}$ at P_1 and P_2 respectively. Find the lengths AP_1 and AP_2.

(a) $O\left(\frac{-3+1}{2}, \frac{9+5}{2}\right) \Rightarrow O(-1,7)$.

$OB^2 = 2^2+2^2 = 8$

$\therefore (x+1)^2+(y-7)^2 = 8.$ ✓

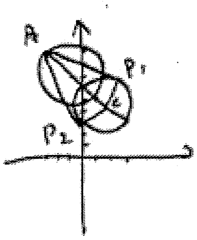
(b) $P_1(0,6)$.

$P_1A = 3\sqrt{2}$, $P_1B = \sqrt{2}$, satisfy $PA = 3PB$.

$P_2(3,3)$

$P_2A = 6\sqrt{2}$, $P_2B = 2\sqrt{2}$, satisfy $PA = 3PB$.

$\therefore$ center $C\left(\frac{3}{2}, \frac{9}{2}\right)$, radius $= \frac{3}{2}\sqrt{2}$ ✓



(c) Note: P_1 & P_2 in (c) is different from that in (b).

$C\left(\frac{3}{2}, \frac{9}{2}\right)$.

$\therefore AC = \frac{9}{2}\sqrt{2}$.

$\therefore AP_1 = \sqrt{\left(\frac{9}{2}\sqrt{2}\right)^2 - \left(\frac{3}{2}\sqrt{2}\right)^2} = AP_2$

$= 6$.

$\therefore AP_1 = AP_2 = 6$.

✓

12. The parametric equations of the hyperbola $\mathcal{H}$ are $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$.

(a) Find the Cartesian equation of $\mathcal{H}$.

(b) Find the coordinates of the foci of $\mathcal{H}$.

(c) Use parametric differentiation to find the gradient of $\mathcal{H}$ when $t = \ln 2$.

(a) $x^2 = e^{2t} + e^{-2t} + 2$

$y^2 = e^{2t} + e^{-2t} - 2$

$x^2 - y^2 = 4$

$\frac{x^2}{4} - \frac{y^2}{4} = 1$

(b) $\begin{cases} a^2 = 4 \\ a^2(1-E^2) = -4 \end{cases}$ E: eccentricity.

$\therefore a = 2,\ E = \sqrt{2}$

$\therefore F(aE, 0) \rightarrow F(2\sqrt{2}, 0)$.

4월 3.

(c) $4x^2 - y^2 = 4$.

$8x - 2y \cdot y' = 0$

$\therefore y' = \frac{4x}{y}$

when $t = \ln 2$, $\begin{cases} x = 2 + \frac{1}{2} = \frac{5}{2} \\ y = 2 - \frac{1}{2} = \frac{3}{2} \end{cases}$

$\therefore$ gradient $= \frac{10}{\frac{3}{2}} = \frac{20}{3}$.

not parametric differentiation.

This is implicit differentiation

이것은 implicit

주로 parametric

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$$

when $t = \ln 2$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 + \frac{1}{2}}{2 - \frac{1}{2}} = \frac{\frac{5}{2}}{\frac{3}{2}} = \frac{5}{3}$$

3

Turn over

13. Let S be the series $\sum_{n=0}^{\infty}\left(\frac{t}{t+1}\right)^n$ where $t \neq 0$

(a) Find the value to which S converges when t = 1.

(b) Determine the values of t for which S converges.

(c) Find all values of t for which the sum of the series is greater than 10.

(a) $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1+\frac{1}{2}+\frac{1}{4}+\cdots = 1\cdot\frac{1-(\frac{1}{2})^{\infty}}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$ ✓

(b) in order for S to converge, $\frac{t}{t+1}$ has to be less than 1. and > -1.

so $\frac{t}{t+1} < 1$, $\frac{t+1-1}{t+1} < 1$, $1 - \frac{1}{t+1} < 1$

$\boxed{-1 < \text{ratio} < 1}$.

$\therefore \frac{1}{t+1} > 0$. $\therefore t+1 > 0$. $\therefore t > -1$

(c) $1 \cdot \frac{1-(\frac{t}{t+1})^{\infty}}{1-\frac{t}{t+1}} > 10$

① $t = -1$, not defined

② $t < -1$, diverge. ✓

③ $t > -1$, $\frac{t}{t+1} < 1$

$\therefore 1 - (\frac{t}{t+1})^{\infty} > 10 - \frac{10t}{t+1}$

$\therefore \frac{10t}{t+1} > 9$

$\therefore 10t > 9t+9$

$\therefore t > 9$.

Therefore, $t < -1$ or $t > 9$. ✓

×

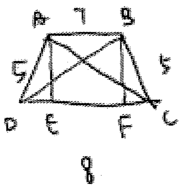
we say the sum does not exist in this case.

$\boxed{\text{diverge 时认为无 sum 而不是 } \infty \text{ ~~为~~ sum}}$

14. (a) Prove that the base angles of an isosceles trapezium are equal.

(b) Hence prove that an isosceles trapezium is cyclic.

(c) An isosceles trapezium has sides of length 5, 5, 7 and 8. Use Ptolemy's theorem to find the lengths of the diagonals.



(a) we have $AD = BC$, $AB \parallel CD$.

Draw two heights AE and BF.

Since $AB \parallel CD$, the distance between the two lines should be the same, so $AE = BF$.

Therefore, $\triangle ADE \cong \triangle BCF$, $\angle D = \angle C$.

(b) $\because AB \parallel CD$

$\therefore \angle C + \angle ABC = 180°$.

$\therefore \angle D + \angle ABC = 180°$

$\therefore$ ABCD is a cyclic quadrilateral.

(c) $\begin{cases} \angle D = \angle C, \\ CD = DC \\ AD = BC \end{cases} \Rightarrow \triangle ADC \cong \triangle BCD.$

$\therefore AC = BD$

$\because$ ABCD is cyclic

$\therefore AC \cdot BD = 7 \cdot 8 + 5^2 = 81$

$\therefore AC = BD = 9$.

$$3\times 4 \qquad 4\times 1 \qquad \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

15. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 6 \end{pmatrix}$ $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(a) Use your calculator to find the reduced row echelon form for A.

(b) Write down a basis for the row space of A.

(c) State the rank of A.

(d) State the nullity of A.

(e) Find a basis for the null space of A.

(a) $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ✓ ✓

(b) $\{(1\ 0\ -1\ 0), (0\ 1\ 2\ 0), (0\ 0\ 0\ 1)\}$. ✓

column vectors in I.B.

(c) rank = 3.

(d) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$. nullity (A) = 1. ✓

$x_3 = r$.

(e) basis for null space: $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$. ✓

5

16. Consider the simple connected planar graph G with v vertices, e edges and f faces.

(a) State Euler's formula for G

(b) If $v \geqslant 3$ prove that $e \leqslant 3v - 6$.

(c) Hence prove that K_n is not planar when $n \geqslant 5$.

(a) $v - e + f = 2$.

(b) three edges are the minimum required to form 2 faces.

$\therefore \begin{cases} 2e \geq 3f \\ v - e + f = 2 \end{cases} \Rightarrow f = 2 - v + e. \quad 2e \geq 6 - 3v + 3e.$

$\therefore \quad e \leq 3v - 6$

(c) $v = 5 > 3$.

$\therefore e \leq 15 - 6 = 9$.

$e(K_5) = \frac{5 \times 4}{2} = 10 > 9$.

$\therefore$ it's not planar.

$n > 5$?

$\boxed{n>5 \text{ 没讨论}}$ $n > 5$ 时，K_n 包含了 K_5，所以一定不是 planar.

4

17. A matrix A is called *skew symmetric* if $A^T = -A$.

(a) Calculate the product $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(b) Prove that if A is an $n \times n$ skew symmetric matrix and $\vec{x} \in \mathbb{R}^n$, then $\vec{x}^T A \vec{x} = 0$.

(a) $(1\ 2\ 3) \cdot \begin{pmatrix} 8 \\ -10 \\ 4 \end{pmatrix}$

$= 0$ ✓

(b) A has dimension: $n \times n$.

$p, q < n$.

In A, $a_{11}, a_{22} \cdots, a_{nn} = 0$.

$a_{pq} = -a_{qp}$

• For a_{pq}, after $\vec{x}^T A \vec{x}$,

the product is $\vec{x}^T_{1p} \cdot a_{pq} \cdot \vec{x}_{q1} = \vec{x}_{p1} \cdot a_{pq} \cdot \vec{x}_{q1}$

• For a_{qp}, after $\vec{x}^T A \vec{x}$,

the product is $\vec{x}^T_{1q} \cdot a_{qp} \cdot \vec{x}_{p1} = \vec{x}_{q1} \cdot a_{qp} \cdot \vec{x}_{p1}$

Sum $= \vec{x}_{q1} \cdot \vec{x}_{p1} \cdot (a_{pq} + a_{qp}) = 0$.

This is the same for all p and q, so $\vec{x}^T A \vec{x} = 0$.

✓

18. (a) Find $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

(b) Show that $\int_1^\infty xe^{-x}\,dx = \frac{2}{e}$

(c) Find $\lim_{x\to 0^+} \frac{e^{-1/x}}{x}$.

(a) $\lim_{x\to 0} e^x - 1 - x = \lim_{x\to 0} x^2 = 0.$

Apply L'Hôpital's Rule.

$\lim_{x\to 0} \frac{e^x - 1}{2x}$

Apply L'Hôpital's Rule again.

$\lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$ ✓

(b) $\int_1^\infty xe^{-x}\,dx$

$= -xe^{-x} - e^{-x} \Big|_1^\infty$

$= \lim_{x\to\infty} [-xe^{-x} - e^{-x}] - [-e^{-1} - e^{-1}]$

$= \lim_{x\to\infty} [-x \cdot 0 - 0] - [-\frac{2}{e}]$ ✓

$= \frac{2}{e}$

(c) $\lim_{x\to 0^+} e^{-\frac{1}{x}} = 0 = \lim_{x\to 0^+} x.$ $y = \frac{1}{x}$

Apply L'Hôpital's Rule.

$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}} \cdot [-(-x^{-2})]}{1} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^2}$

Apply again. ?

✗ $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}} \cdot x^2}{2x} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}} \cdot x}{2} = 0$ ✓

Question: How is the "+" in $\lim_{x\to 0^+}$ presented?

$y = \frac{1}{x}$. $\lim_{x\to 0^+} e^{-\frac{1}{x}} = \lim_{y\to\infty} e^{-y} = 0$. from above, as a one-sided limit. $x = \frac{1}{y}$

原式 $= \lim_{y\to\infty} \frac{e^{-y}}{\frac{1}{y}} = \frac{y}{e^y}$

洛必达后: $\lim_{y\to\infty} \frac{1}{e^y} = 0$

19. Consider the structure $(\mathbb{R}\setminus\{-1\}, \circ)$ where the operation $\circ$ is defined by $a \circ b = a + ab + b$.

(a) Prove that the structure is an Abelian group.

(b) Solve the equation $2 \circ (x \circ (-3)) = 5$ where $x \in \mathbb{R}\setminus\{-1\}$.

(a) when $a=0$ or $b=0$,

$a \circ b = 0 + 0 \cdot b + b = b$.

① So 0 is the identity.

$a \circ b = a + ab + b + 1 - 1 = (a+1)(b+1) - 1$

if $a \circ b = -1$, then $(a+1)(b+1) = 0$,

either $a = -1$ or $b = -1$ and that's not possible.

② so it's closed within $R \setminus \{-1\}$.

$a \circ b = a + ab + b$, $b \circ a = b + ba + b = a + ab + b$.

③ so it's abelian.

$(a \circ b) \circ c = (a + ab + b) \circ c = a + ab + b + c + ac + abc + bc$

$= a + b + c + ab + ac + bc + abc$

$a \circ (b \circ c) = a \circ (b + bc + c) = a + b + bc + c + ab + abc + ac$

④ so it's associative.

For $a \circ b = 0$, $(a+1)(b+1) = 1$, then $a = \frac{-b}{b+1}$

For every b, its inverse $b^{-1} = \frac{b}{b+1}$.

⑤ has inverse.

Therefore, from ① − ⑤, we know that it's an Abelian group.

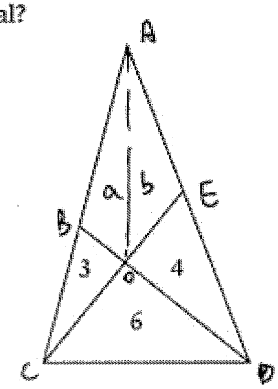
(b) $2 \circ (x - 3 - 3x) = 2 + x - 3 - 3x + 2x - 6 - 6x = 5$

$x = -2$

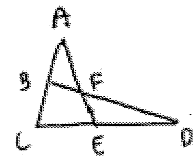
20. (a) State Menelaus's theorem.

(b) Use Menelaus's theorem to prove Ceva's theorem.

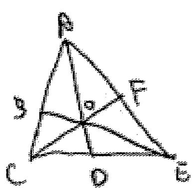
(c) In the diagram, the numbers 3, 4 and 6 are the areas of their respective triangles. What is the area of the unmarked quadrilateral?



有方向性的话积为-1



(a) Menelau's theorem: $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$ ✓ or sometimes -1 if we consider signed lengths.



(b) $\frac{AB}{BC} \cdot \frac{CE}{ED} \cdot \frac{DO}{OA} = 1$ ① $\frac{①}{②} \Rightarrow \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$

$\frac{AF}{FE} \cdot \frac{EC}{CD} \cdot \frac{DO}{OA} = 1$ ② ✓

(c). $\frac{BO}{OD} = \frac{1}{2}$, $\frac{CO}{OE} = \frac{3}{2}$.

$$\begin{cases} \frac{S_{\triangle AOC}}{S_{\triangle AOE}} = \frac{3+a}{b} = \frac{3}{2} \\ \frac{S_{\triangle AOB}}{S_{\triangle AOD}} = \frac{a}{b+4} = \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{9}{2} \\ b = 5 \end{cases}$$

$\therefore S_{qua} = \frac{19}{2}$ ✓