HL1 Assignment #15 Name: Jerry Jiang

1. Let $f\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-5, 3]$ be the function with rule $f(x) = 4\sin x - 1$. Give the full function definition for f^{-1}.

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Excellent!

$y = 4\sin x - 1$

$x = 4\sin y - 1$

$\sin y = \frac{x+1}{4}$

$y = \arccos\left(\frac{x+1}{4}\right)$

$\therefore f^{-1} : [-5,3] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $f^{-1}(x) = \arccos\left(\frac{x+1}{4}\right)$.

2. Let $z = 3 - 7i$ and $w = -4 + 6i$. Find real numbers p and q so that $pz + qw = 6.5 - 11i$.

$3p - 7pi + (-4q) + 6qi = 6.5 - 11i$

$(3p - 4q) + (-7p + 6q)i = 6.5 - 11i$

$$\begin{cases} 3p - 4q = 6.5 & \times 3 \\ -7p + 6q = -11 & \times 2 \end{cases}$$

$-5p = -2.5$

$$\begin{cases} p = \frac{1}{2} \\ q = -1.25 \end{cases}$$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function with rule $f(x) = 5x - 3$. Prove that f is one-to-one. injective.

for $x_1 \xrightarrow{f(x)} 5x_1 - 3$

if there's an x_2 that make $5x_1 - 3$

then $x_2 \xrightarrow{f(x)} 5x_2 - 3$, so $x_1 = x_2$.

So f must be injective.

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4. Solve $3\sin^2 x = 4\cos x - 1$ for $0 \leq x < 2\pi$.

$3(1-\cos^2 x) = 4\cos x - 1$

$3 - 3\cos^2 x = 4\cos x - 1$

$3\cos^2 x + 4\cos x - 4 = 0$

3   -2

1   2

$(3\cos x - 2)(\cos x + 2) = 0$

$\cos x = \frac{2}{3}$ or -2 (inadmissable)

$\therefore x = 0.841 / 5.44 \pm 2k\pi$

$\therefore x = \underline{0.841,\ 5.44}$ (3 s.f.)

5. The curve $y = \sin x$ is stretched scale factor a in the y-direction, then stretched scale factor b in the x-direction, then translated by $\binom{h}{k}$. The resulting curve has equation $y = 3\sin(2x + \pi) + 7$. Find the values of a, b, h and k.

$y = 3\sin[2(x + \frac{\pi}{2})] + 7$

$\frac{y-7}{3} = \sin\left(\frac{x + \frac{\pi}{2}}{\frac{1}{2}}\right)$

$a = 3$

$b = \frac{1}{2}$

$h = -\frac{\pi}{2}$

$k = 7$

6. One solution of the equation $3w^3 + aw^2 - 3w + 10 = 0$ where a is a constant is $w = -2$. Find the other two solutions.

$-24 + 4a + 6 + 10 = 0$

$a = 2$

$\therefore 3w^3 + 2w^2 - 3w + 10 = 0$

$$\begin{array}{r|l} & 3w^2 - 4w + 5 \\ \hline w+2 & 3w^3 + 2w^2 - 3w + 10 \\ & 3w^3 + 6w^2 \\ \hline & -4w^2 - 3w \\ & -4w^2 - 8w \\ \hline & 5w + 10 \\ & 5w + 10 \\ \hline & 0 \end{array}$$

$\therefore (w+2)(3w^2 - 4w + 5) = 0$

$\therefore 3w^2 - 4w + 5 = 0$

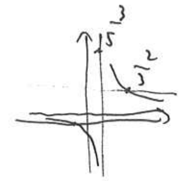
$\Delta = 16 - 60 = -44$

$w = \frac{4 \pm 2\sqrt{11}\,i}{6} = \frac{2 \pm \sqrt{11}\,i}{3}$

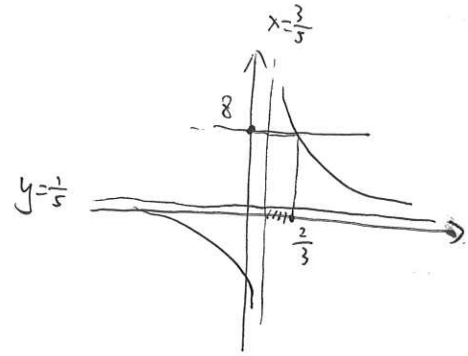
7. Solve the inequality $\dfrac{2+x}{5x-3} \geq 8$. Give your answer in interval notation. $f(x) = \dfrac{x+2}{5x-3}$

$x = \frac{3}{5}$

$y = \frac{1}{5}$



according to the graph on the right,



if $\dfrac{2+x}{5x-3} = 8$, $x = \frac{2}{3}$.

$\therefore \quad x \in]\frac{3}{5}, \frac{2}{3}]$.

8. Give a proof by contradiction to show that the sum of a rational number and an irrational number must be irrational.

if the sum of a radical & an irradical is radical,

we have $R_1 + IR \rightarrow R_2$

so we have $R_2 - R_1 = IR$.

we can say that $R_2 = \frac{a}{b}$, $\gcd(a,b) = 1$, $b \neq 0$.

$R_1 = \frac{c}{d}$, $\gcd(c,d) = 1$, $d \neq 0$.

$(a, b, c, d \in \mathbb{Q})$

$\therefore \frac{a}{b} - \frac{c}{d} = IR$

$= \dfrac{ad-bc}{bd}$

$\therefore IR = \dfrac{ad-bc}{bd}$

since IR number can't be written in fractional form,

hypothesis fails.

$\therefore$ the sum of a rational and an irrational number must be irrational.

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9. The first three terms of an arithmetic sequence are $2\sin\theta$, $3\cos\theta$ and $(\sin\theta + 2\cos\theta)$ respectively, where θ is an acute angle. The sum of the first twenty terms of this sequence is an integer. Find its value.

$a = 2\sin\theta$

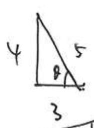
$d = 3\cos\theta - 2\sin\theta$

$\therefore\ 3\cos\theta + 3\cos\theta - 2\sin\theta = \sin\theta + 2\cos\theta$

$\therefore\ 4\cos\theta = 3\sin\theta$ ✓

$\therefore\ \tan\theta = \frac{4}{3}$.

θ is acute



$$S_{20} = \frac{[4\sin\theta + 19\cdot(3\cos\theta - 2\sin\theta)]\cdot 20}{2}$$

$= 10\,[57\cos\theta - 34\sin\theta]$

$= 570\cos\theta - 340\sin\theta$

$\because\ \cos\theta = \frac{3}{5},\ \sin\theta = \frac{4}{5}$ ✓

$\therefore\ S_{20} = 342 - 272 = 70$

10. Solve the simultaneous equations $z^2 + w^2 + 3z + 3w = 8$ and $zw + 4z + 4w = 2$ for $z, w \in \mathbb{C} \setminus \mathbb{R}$.

$z(w+4)+4(w+4)=18$

$(z+4)(w+4)=18$

1 -1

1 12

$z^2 + w^2 + 3z + 3w + 2wz + 8z + 8w = 8 + 4 = 12$

$(z+w)^2 + 11(z+w) - 12 = 0$

$\therefore\ (z+w) = 1$ or -12

$\therefore$ ① $z = 1 - w$ ② $z = -12 - w$

① $(1-w)w + 2 = 0$

$w^2 - w - 2 = 0$

$(w-2)(w+1) = 0$

$w_1 = 2$

$w_2 = -1$

(inadmissible).

② $-12w - w^2 - 48 - \cancel{4w} + \cancel{4w} = 2$

$w^2 + 12w + \cancel{46}\,50 = 0$

$\Delta = 144 - 184\ \ 200$.

$= -40\ \ -56$.

$w = \frac{-12 \pm 2\sqrt{14}i}{2} = -6 \pm \sqrt{14}\,i$

$\therefore \begin{cases} z_1 = -6 - \sqrt{14}\,i \\ w_1 = -6 + \sqrt{14}\,i \end{cases}$

$\begin{cases} z_2 = -6 + \sqrt{14}\,i \\ w_2 = -6 - \sqrt{14}\,i \end{cases}$

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Solutions to HL1 Assignment #15

1. The required inverse function is $f^{-1}\colon [-5,3] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ with rule $f^{-1}(x) = \arcsin(\frac{x+1}{4})$.

2. Equating real and imaginary parts gives the simultaneous equations $3p-4q = 6.5$ and $-7p+6q = -11$, whence $p = 0.5$ and $q = -1.25$.

3. Let $f(x_1) = f(x_2)$. So $5x_1 - 3 = 5x_2 - 3$, whence $x_1 = x_2$. Hence f is injective.

4. Let $c = \cos x$. Then we have the equation $3 - 3c^2 = 4c - 1$, or equivalently $3c^2 + 4c - 4 = 0$, whence $c = -2$ or $c = \frac{2}{3}$. Hence $x = 0.841, 5.44$.

5. $a = 3$, $b = \frac{1}{2}$, $h = -\frac{\pi}{2}$, $k = 7$.

6. By the factor theorem $p(-2) = 0$, hence $-24 + 4a + 6 + 10 = 0$, whence $a = 2$. Dividing $p(x)$ by $x - 2$ gives the quadratic factor $3w^2 - 4w + 5$, whose roots $(2 \pm i\sqrt{11})/3$ are the required solutions.

7. One approach that builds on our knowledge of the bilinear function is to draw the graph of $y = \frac{2+x}{5x-3}$ and see where this graph intersects or lies above the line $y = 8$. Doing so gives $x \in \,]\frac{3}{5}, \frac{2}{3}]$.

8. Suppose to the contrary that the sum of a rational number r_1 and an irrational number x is a rational number r_2. Then $r_1 + x = r_2$, or equivalently $x = r_2 - r_1$. But the set of rational numbers is closed under subtraction, so x must also be rational. This contradiction, namely x is both rational and irrational, means that what we supposed is false, which completes the proof.

9. Let $c = \cos\theta$ and $s = \sin\theta$. Then $3c - 2s = s - c$, or equivalently $4c = 3s$, whence $\tan\theta = \frac{4}{3}$. Since θ is acute we conclude $c = \frac{3}{5}$ and $s = \frac{4}{5}$. Hence our arithmetic sequence has first term $\frac{8}{5}$ and common difference $\frac{1}{5}$. So
$$S_{20} = 10\left(\frac{16}{5} + \frac{19}{5}\right) = 70.$$

10. Add twice the second equation to the first to give $(z+w)^2 + 11(z+w) - 12 = 0$, whence $z + w = 1$ or $z + w = -12$, or equivalently $w = 1 - z$ or $w = -12 - z$. Substituting the first of these in $zw + 4z + 4w = 2$ gives only real solutions. Substituting the second gives $-z(12+z) - 48 = 2$, or equivalently $z^2 + 12z + 50 = 0$, whence
$$z = \frac{-12 \pm \sqrt{-56}}{2} = -6 \pm i\sqrt{14}.$$
Finally, $(z,w) = (-6 + i\sqrt{14}, -6 - i\sqrt{14})$ or $(z,w) = (-6 - i\sqrt{14}, -6 + i\sqrt{14})$.