FM1 Assignment #8 Name: Jerry Jiang

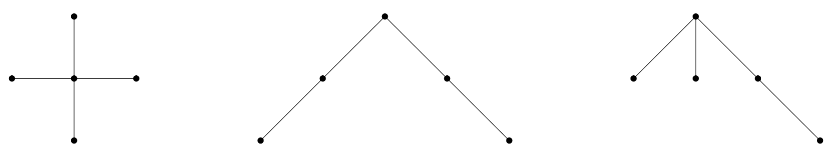
1. Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}$$
$$= \lim_{n\to\infty} \frac{2^n \cdot (n+1)^2}{2^{n+1} \cdot n^2}$$
$$= \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2}$$
$$= \frac{1}{2}$$

So according to the ratio test, since $L = \frac{1}{2} < 1$, the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

2. Define spanning tree. Draw all the non-isomorphic spanning trees of K_5.

Spanning tree is a subgraph of a connected graph that contains no cycles while containing every vertex in that connected graph.



These are the only three spanning trees for K_5, since $f = 1$ when there're no cycles, and any $e > 4$ makes $f > 1$ according to Euler's formula.

3. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

(a) What elementary row operation is needed to transform A into row echelon form?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) What is the corresponding elementary matrix for the above row operation?

$$R_3 - 2R_2 \to R_3 : \; E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

4. A permutation of the form $(a_1a_2\dots a_n)$ is called a *cycle of length* n or an n*-cycle*. A 2-cycle is called a *transposition*.

(a) Write the 5-cycle (12345) as a product of transpositions.

$(12)\circ(23)\circ(34)\circ(45)$

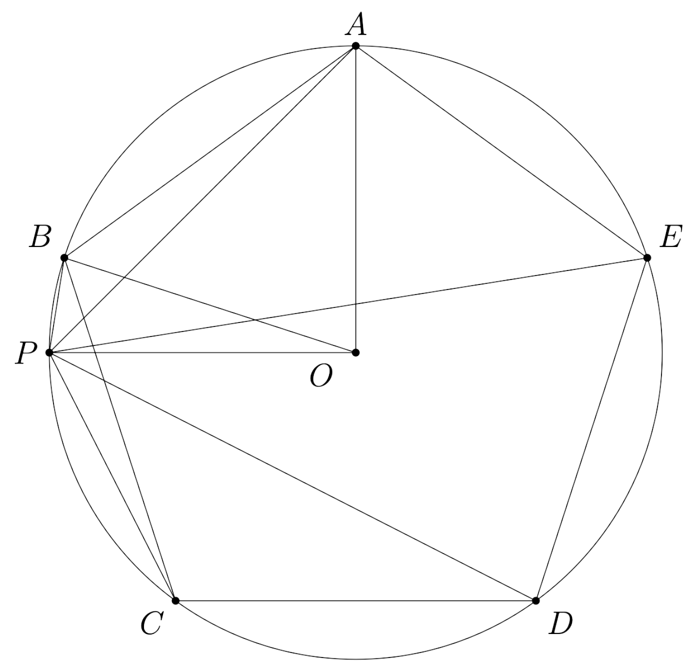
(b) What is the order of an n-cycle?

n, since the n^{th} product of the cycle with itself gets to the identity element.

(c) If α and β are disjoint cycles of length 180 and 216 respectively, what is the order of the product $\alpha\beta$?

The order for α is 180, and the order for β is 216, so the order for $\alpha\beta$ is $\text{lcm}(180, 216) = 1080$.

5. The regular pentagon $ABCDE$ is inscribed in a circle and point P is on $\overset{\frown}{BC}$. Prove $PA + PD = PB + PC + PE$.



Let $\angle BOP = 2\theta$.

Then $\angle POA = 72^\circ + 2\theta$.

$\therefore PA = 2\cdot r\cdot \sin\left(\frac{1}{2}\angle POA\right)$

$= 2r\sin(36^\circ+\theta)$

Similarly,

$PB = 2r\sin\theta$.

$PC = 2r\sin(36^\circ-\theta)$.

$PD = 2r\sin(72^\circ-\theta)$.

$PE = 2r\sin(72+\theta)$.

$$\begin{aligned}
\therefore\quad & PA+PD-PB-PC-PE \\
&= 2r\left[\sin(36^\circ+\theta)+\sin(72^\circ-\theta)-\sin\theta-\sin(36^\circ-\theta)-\sin(72^\circ+\theta)\right] \\
&= 2r[\sin 36^\circ\cos\theta+\cos 36^\circ\sin\theta+\sin 72^\circ\cos\theta-\cos 72^\circ\sin\theta-\sin\theta- \\
&\quad \sin 36^\circ\cos\theta+\cos 36^\circ\sin\theta-\sin 72^\circ\cos\theta-\cos 72^\circ\sin\theta] \\
&= 2r\sin\theta[\cos 36^\circ-\cos 72^\circ-1+\cos 36^\circ-\cos 72^\circ] \\
&= 2r\sin\theta\cdot 0\quad \text{(GDC)} \\
&= 0
\end{aligned}$$

Therefore, $PA+PD=PB+PC+PE$.