**FM2** Assignment #10 Name: Jerry Jiang $\frac{10}{10}$ Excellent!

1. The functions $i\colon x \to x$, $f\colon x \to 1/x$, $g\colon x \to -x$, $h\colon x \to -1/x$, form a group under composition. Construct the Cayley table for this group and state to which well-known group the given group is isomorphic.

| $\circ$ | $i$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|
| $i$ | $i$ | $f$ | $g$ | $h$ |
| $f$ | $f$ | $i$ | $h$ | $g$ |
| $g$ | $g$ | $h$ | $i$ | $f$ |
| $h$ | $h$ | $g$ | $f$ | $i$ |

$\cong V_4$.

2. Consider the function $f(x) = \begin{cases} |x-2|+1, & x<2 \\ ax^2+bx, & x \geq 2 \end{cases}$. If $f$ and $f'$ are both continuous at $x=2$, find $a$ and $b$.

$f(x) = \begin{cases} 3-x, & x<2 \\ ax^2+bx, & x \geqslant 2. \end{cases} \longrightarrow f'(x) = \begin{cases} -1, & x<2 \\ 2ax+b, & x \geqslant 2. \end{cases}$

$\lim_{x\to 2^-} f(x) = f(\lim_{x\to 2^-} x) = 3-2 = 1.$

$\lim_{x\to 2^-} f'(x) = f'(\lim_{x\to 2^-} x) = -1.$

$\therefore \lim_{x\to 2^+} f(x) = \underline{4a+2b=1.}$

$\therefore \lim_{x\to 2^+} f'(x) = \underline{4a+b=-1}$

$\longrightarrow \begin{cases} a = -\frac{3}{4} \\ b = 2 \end{cases}$

3. Suppose $f\colon G \to G'$ and $g\colon G' \to G''$ are group homomorphisms. Prove $g \circ f$ is a homomorphism from $G$ to $G''$.

Let $a, b \in (G, *)$, $a', b' \in (G', \circ)$.

$f(a * b) = f(a) \circ f(b) = a' \circ b'$

$g(a' \circ b') = g(a') \cdot g(b')$

Therefore, $g \circ f(a*b) = g(f(a*b)) = g(a' \circ b') = g(a') \cdot g(b')$

$g \circ f$ is the homomorphism.

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4. Suppose $f\colon G \to G'$ is a group homomorphism. Prove $\operatorname{ran}(f) \leq G'$.

- let $a, b \in G$, $f(a)=a', f(b)=b' \in G'$.
  since $f(a)=a'$, $f(b)=b'$, $a', b' \in \operatorname{ran}(f)$.
  $\because a * b \in G$, $\therefore f(a*b) = f(a) \circ f(b) = a' \circ b' \in \operatorname{ran}(f)$. closure ✓
- $f(e)=e'$. $\therefore e' \in \operatorname{ran}(f)$. identity ✓.
- let $x' \in \operatorname{ran}(f)$, $f(x)=x'$. since there's an inverse for $x$, $(x^{-1} \in G)$,
  we have $f(x^{-1}) = (x')^{-1} \in \operatorname{ran}(f)$. inverse ✓

According to the 3-step subgroup test, $\operatorname{ran}(f) \leq G'$.

✓

5. The relation $\sim$ on $\mathbb{R}^2$ is defined by $(a, b) \sim (c, d)$ if $d - b = 2(c - a)$. Show that $\sim$ is an equivalence relation and describe the equivalence classes geometrically.

- $(a,b) \sim (a,b)$, reflexive: $b-b = 2(a-a)=0$.
- $(a,b) \sim (c,d)$ then $d-b = 2(c-a)$, $(-1)(d-b) = (-2)(c-a)$, $b-d = 2(a-c)$.
  $\therefore (c,d) \sim (a,b)$. symmetric.
- $(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$, then $d-b = 2(c-a)$, $f-d = 2(e-c)$.
  $\therefore (f-d)+(d-b) = 2(e-c) + 2(c-a)$, $f-b = 2(e-a)$.
  $\therefore (a,b) \sim (e,f)$. transitive.

For every $c$ in $\mathbb{R}$, there's an equivalence class of the set of all points that lie on $y = 2x + c$ in the cartesian plane.

lines of gradient 2 !

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