HL1 Test #2 Name: Jerry Jiang

1. The number of balls in figure 1 is the first triangular number T_1, the number of balls in figure 2 is the second triangular number T_2, and so on. Find the value of T_{100}.

100%

Bravo!!

$T_1 = 1$

$T_2 = 1+2$

$T_3 = 1+2+3$

$\cdots$

$T_n = 1+\cdots+n = \frac{(n+1)\cdot n}{2}$

$T_{100} = \frac{101 \cdot 100}{2} = 5050$

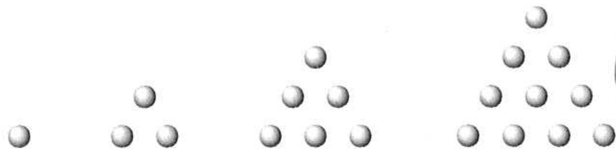


Fig. 1 Fig. 2 Fig. 3 Fig. 4

2. The coefficient of x^2y^3 in the expansion of $(x-2y)^5$ is an integer. Find its value.

$\binom{5}{2} \cdot x^2 \cdot (-2y)^3$

$= \frac{5\times 4}{2\times 1} \cdot x^2 \cdot (-8y^3)$

$= -80x^2y^3$

$\therefore$ the coefficient is -80.

3. In the street map below how many shortest paths are there from point A to point B?

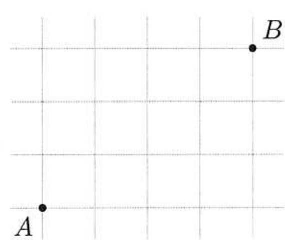
$\uparrow : 3 \quad \rightarrow : 4$

NNNEEEE

$\frac{7!}{3!\cdot 4!}$

$= \frac{7\times 6\times 5}{3\times 2\times 1}$

$= 35$



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4. Let $\log_a 2 = x$ and $\log_a 5 = y$. Find an expression for $\log_2 10$ in terms of x and y.

$$\frac{\log_a 5}{\log_a 2} = \frac{y}{x} = \log_2 5$$

$$\log_2 10 = \log_2 (2 \cdot 5) = \log_2 2 + \log_2 5$$
$$= 1 + \frac{y}{x}$$

$$\therefore \log_2 10 = \frac{x+y}{x}$$

5. The letters A, B, C, D and E are arranged in a row. In how many ways can this be done if each permutation begins and ends with a vowel?

① A _ _ _ E

② E _ _ _ A

$\therefore 2 \times {}^3P_3$

$= 2 \times 3!$

$= 12$

6. Solve $3^{2x+1} + 2 \times 3^x = 1$.

$$3 \cdot 3^{2x} + 2 \cdot 3^x - 1 = 0$$

$$(3 \cdot 3^x - 1)(3^x + 1) = 0$$

$\therefore 3^x = \frac{1}{3}$ or -1

$\because 3^x > 0$

$\therefore 3^x = \frac{1}{3}$

$\therefore x = -1$

real part.

7. If $z = 1 + i + i^2 + i^3 + \cdots + i^{2018}$, what is $\operatorname{Re} z$?

$$z = 1 + \underbrace{i - 1 - i + 1}_{\text{a cycle}} + \cdots + \underbrace{i - 1 - i + 1}_{2013-2016} + i - 1$$

$$= 1 + 0 \cdot n(\text{cycles}) + i - 1$$

$$= i$$

$$\therefore \operatorname{Re} z = 0$$

8. Prove Pascal's rule $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

$$\because \binom{n-1}{r-1} = \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$\binom{n-1}{r} = \frac{(n-1)!}{r!(n-r-1)!}$$

$$\therefore \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$= \frac{(n-1)! \cdot r + (n-1)! \cdot (n-r)}{r!\,(n-r)!}$$

$$= \frac{(n-1)!\,(r+n-r)}{r!\,(n-r)!}$$

$$= \frac{(n-1)! \cdot n}{r!\,(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

$$= \binom{n}{r}$$

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9. Given that $(1+x)^3(1+mx)^4 = 1 + nx + 93x^2 + \cdots + m^4x^7$, find the possible values of m and n.

$$x^2: \begin{cases} \binom{3}{2} + \binom{4}{2}m^2 + \binom{3}{1}\cdot\binom{4}{1}\cdot m = 93 \\ x \quad \binom{4}{1}\cdot m + \binom{3}{1} = n \end{cases}$$

$$\therefore \begin{cases} 3 + 6m^2 + 3\cdot 4\cdot m = 93 \quad ① \\ 4m + 3 = n \quad ② \end{cases}$$

$① : \; 6m^2 + 12m - 90 = 0$

$m^2 + 2m - 15 = 0$

$(m+5)(m-3) = 0$

$\therefore m_1 = -5$

$m_2 = 3$

$\therefore n_1 = -20 + 3 = -17$

$n_2 = 12 + 3 = 15$

$$\therefore \begin{cases} m_1 = -5 \\ n_1 = -17 \end{cases} \quad \begin{cases} m_2 = 3 \\ n_2 = 15 \end{cases}$$

10. By considering the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$ or otherwise, show that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

- suppose we have a group of 2n items, and we want to chose n items from them, then there's $\binom{2n}{n}$ types of choices.
- we divide the group into ~~to~~ two smaller ones, each consisting n items.
- ① if we want 0 item from group 1, then we have to take n items from group 2, making $\binom{n}{0}\cdot\binom{n}{n}$

  ② if we want 1 item from group 1, then we have to take n-1 items from group 2, making $\binom{n}{1}\cdot\binom{n}{n-1}$
- generalizing, if we want a items from group 1, we take n-a items from group 2, making $\binom{n}{a}\cdot\binom{n}{n-a}$
- so the first way, which is $\binom{2n}{n}$, is equivalent to the second way, $\sum_{a=0}^{n}\binom{n}{a}\binom{n}{n-a}$

$$\therefore \binom{2n}{n} = \sum_{a=0}^{n}\binom{n}{a}\binom{n}{n-a}$$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}$$

$\because \binom{x}{x-y} = \binom{x}{y}$ (chosing y element from a set of x elements can be done by chosing the complement, which is x-y)

$$\therefore \binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

Solutions to HL1 Test #2

1. The 100-th triangular number is $T_{100} = 1 + 2 + 3 + \cdots + 100 = 5050$.

2. The required term is $\binom{5}{3}x^2(-2y)^3 = -80x^2y^3$. So the required coefficient is -80.

3. There are $\binom{7}{4} = 35$ shortest paths from A to B.

4. By the change of base formula
$$\log_2 10 = \frac{\log_a 10}{\log_a 2} = \frac{x+y}{x}.$$

5. Let the first action be to arrange the vowels. There are 2 ways to do this. Let the second action be to arrange the consonants. There are $3! = 6$ ways to do this. By the rule of the product there are therefore 12 such permutations.

6. Let $y = 3^x$. Then the given equation reduces to $3y^2 + 2y - 1 = 0$, which has solutions $y = -1$ and $y = 1/3$. Hence $x = -1$.

7. Notice that $1 + i + i^2 + i^3 = 0$. So $1 + i + i^2 + i^3 + \cdots + i^{2018} = i^{2016} + i^{2017} + i^{2018} = 1 + i - 1 = i$. Hence $\operatorname{Re} z = 0$.

8. See our red book page 223.

9. Expanding gives $(1+3x+3x^2+x^3)(1+4mx+6m^2x^2+\text{others}) = 1+(4m+3)x+(6m^2+12m+3)x^2+\text{others}$. Equating coefficients we conclude $4m + 3 = n$ and $6m^2 + 12m + 3 = 93$. Solving simultaneously gives $m = -5$ and $n = -17$ or $m = 3$ and $n = 15$.

10. Using the LHS of the identity the coefficient of x^n is $\binom{2n}{n}$ while from the RHS the coefficient is
$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{r}\binom{n}{n-r} + \cdots + \binom{n}{n}\binom{n}{0},$$
whence the result since $\binom{n}{n-r} = \binom{n}{r}$.