HL1 Assignment #23 Name: Jerry Jiang

100%

Excellent!!

1. Find two unit vectors perpendicular to $5\vec{i} - 12\vec{j}$.

$\begin{pmatrix} 5 \\ -12 \end{pmatrix} \quad \begin{pmatrix} a \\ b \end{pmatrix}$

$\frac{5a - 12b}{13 + 1} = 0$

$5a = 12b$

$\therefore a = \frac{12}{5} b$

$a^2 + b^2 = 1$

$\therefore b = \pm \frac{5}{13}$

$\therefore \begin{pmatrix} \frac{12}{13} \\ \frac{5}{13} \end{pmatrix}, \begin{pmatrix} -\frac{12}{13} \\ -\frac{5}{13} \end{pmatrix}$ ✓

2. Let $\vec{v} = 3\vec{i} + 4\vec{j}$ and $\vec{w} = 8\vec{i} - 15\vec{j}$. Find $(\vec{v} - \vec{w}) \cdot (\vec{v} + \vec{w})$.

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 8 \\ -15 \end{pmatrix}$

$\begin{pmatrix} -5 \\ 19 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -11 \end{pmatrix}$

$= -55 - 209$

$= -264$ ✓

3. Let $f(x) = e^x$. Solve $f^{-1} \circ f^{-1} \circ f^{-1}(x) = 1$ giving your value of x to three significant figures.

$f^{-1}(x) = \ln x$

$\ln(\ln(\ln x)) = 1$

$\therefore \ln(\ln x) = e$

$\therefore \ln x = e^e$

$\therefore \quad x = e^{e^e} = 3.81 \times 10^6$ (3s.f.) ✓

TT

4. The range of the function $f(x) = \dfrac{\ln x}{x^2}$ is $]-\infty, k]$. Find the *exact* value of k.

$$f'(x) = \frac{\frac{x^2}{x} - 2x\cdot\ln x}{x^4} = \frac{1-2\ln x}{x^3} = 0$$

$$f''(x) = -3x^4 - \frac{2-6\ln x}{x^4}$$

$f''(e^{\frac{1}{2}}) < 0,$

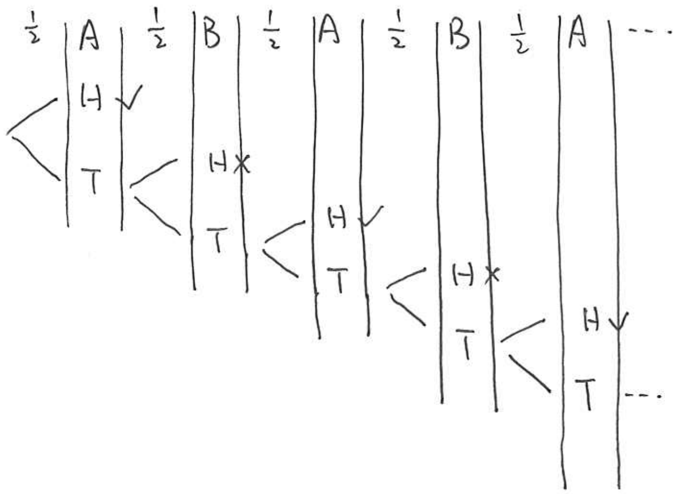
maximum.

$\ln x = \frac{1}{2}$

$x = e^{\frac{1}{2}}$

$$\therefore f(e^{\frac{1}{2}}) = k = \frac{\frac{1}{2}}{e} = \frac{1}{2e}$$

5. Alice and Bob take turns tossing a fair coin. The winner is the first person to toss a head. If Alice goes first, what is the probability she wins?



$$p = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots$$

$$= \frac{1}{2} \cdot \frac{1-\left(\frac{1}{4}\right)^{\infty}}{1-\frac{1}{4}}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{3}{4}}$$

$$= \frac{2}{3}$$

6. Write down the gradients $m(\vec{v})$ and $m(\vec{w})$ of the non-vertical vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Show that $m(\vec{v}) \times m(\vec{w}) = -1$ is equivalent to $\vec{v} \cdot \vec{w} = 0$.

$m(\vec{v}) = \frac{V_2}{V_1}$

$m(\vec{w}) = \frac{W_2}{W_1}$

$$m(\vec{v}) \cdot m(\vec{w}) = \frac{V_2 \cdot W_2}{V_1 \cdot W_1} = -1$$

$\therefore V_2 \cdot W_2 + V_1 \cdot W_1 = 0$

$\therefore \vec{v} \cdot \vec{w} = V_1 W_1 + V_2 W_2$

$= 0$

7. Find the point of intersection, if any, for the lines

$$\vec{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix} \quad \text{and} \quad \vec{r} = \begin{pmatrix}0\\-6\\-3\end{pmatrix} + t\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

① & ② $\begin{cases} 1+t = -u \\ 2+2t = -6+u \end{cases}$

$3+3t=-6$

$\begin{cases} t=-3 \\ u=2 \end{cases}$

③ $1+2t = -3-t$

satisfied

$\therefore \begin{pmatrix}-2\\-4\\-5\end{pmatrix}$ ✓

8. A miniature car moves in a straight line so that its position at time t seconds is given by the vector equation

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix}, \quad t \geq 0.$$

At the same time the car leaves the point $(2, 0)$, a miniature motorcycle leaves the point $(0, 2)$ travelling at constant speed on the line $y = x + 2$. If the two vehicles collide, what is the speed of the motorcycle?

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\cdot v$

$\begin{cases} 2+t = t\cdot v \\ 3t = 2+tv \end{cases}$

$3t = 2+2+t$

$2t=4$

$\begin{cases} t=2 \\ v=2 \end{cases}$

$\therefore V_{motorcycle} = 2\cdot\left|\begin{pmatrix}1\\1\end{pmatrix}\right|$

$= 2\sqrt{2} \cdot s^{-1}$ ✓

10

9. Express the vector $7\vec{i}+2\vec{k}$ as the sum of two vectors, one parallel to $\vec{i}+\vec{j}+\vec{k}$ and one perpendicular to it.

$$\begin{pmatrix}7\\0\\2\end{pmatrix} = \begin{pmatrix}a\\a\\a\end{pmatrix} + \begin{pmatrix}-b-c\\b\\c\end{pmatrix}$$

$$\therefore \begin{cases} a+b=0 \\ a+c=2 \\ a-b-c=7 \end{cases}$$

$$3a=9$$

$$\begin{cases} a=3 \\ b=-3 \\ c=-1 \end{cases}$$

$$\therefore \begin{pmatrix}7\\0\\2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} + \begin{pmatrix}4\\-3\\-1\end{pmatrix} \checkmark$$

10. Let $f(x) = x^2\ln(x+1)$. Find $f^{(2019)}(0)$ leaving your answer in factorial form.

$$f'(x) = \frac{x^2}{x+1} + 2x\ln(x+1)$$

$$f''(x) = \frac{x^2+2x}{(x+1)^2} + 2\ln(x+1) + \frac{2x}{x+1}$$

$$f'''(x) = \frac{2}{(x+1)^3} + \frac{2}{x+1} + \frac{2}{(x+1)^2}$$

$$= 2\left[(x+1)^{-3} + (x+1)^{-2} + (x+1)^{-1}\right]$$

$$f''''(x) = 2\left[(-3)(x+1)^{-4} + (-2)(x+1)^{-3} + (-1)(x+1)^{-2}\right]$$

$$f^{(n)}(x) = 2\left[(-3)\cdot(-4)\cdots(-n+1)(x+1)^{-n} + (-2)\cdots(-n+2)(x+1)^{-n+1} + (-1)\cdots(-n+3)(x+1)^{-n+2}\right]$$

$$= (-1)^{n+1}\cdot 2\left[\frac{(n-1)!}{2}(x+1)^{-n} + (n-2)!(x+1)^{-n+1} + (n-3)!(x+1)^{-n+2}\right]$$

$$f^{(2019)}(0) = 2\left(\frac{2018!}{2} + 2017! + 2016!\right) \checkmark$$

```
Last login: Tue Apr 23 23:25:52 on console
JerrydeMBP:~ jerry_jiang$ python
Python 2.7.10 (default, Feb 22 2019, 21:17:52)
[GCC 4.2.1 Compatible Apple LLVM 10.0.1 (clang-1001.0.37.14)] on darwin
Type "help", "copyright", "credits" or "license" for more information.
>>> import math
>>> print 2*(math.factorial(2018)/2+math.factorial(2017)+math.factorial(2016))
947630663682277061751173822776829694074752803556249593074137946273415435077629741860071430146086227273180933880406676114551225567702697292551998839797291846245731626204790882923701437616079633770487359200
842947044359743570044464884142693359503665266886868573168502069183616560076641735302954582631898057424251980666638414203282970939945576535953466165591468423275890824492606667647598361969549428389189813996
902978808176153996057637204873875869485528422898186163055578387268545952569773921091832492025052831456028246659149290065852535407842408010356747971904369193766767663222899424806912084916464071077721820423
128080290170005238335426359675638961296636577462859499247117066942121768289275123188683009811734008848313651289083132465782583216231947942014539003409532158574685083476897878054616538601239483277005876174
754149221673344744931689837981366724058970251923343539896960594932162368267839834417802754559665392990440227888129296843613257144509384677643302331447736941996228814788519446899191174888978582531722276465
936270227024268431942048444556865874415956272034997199735970755031652957211237983049365636919701661503835355330679797435095596100335273881187849266859580275211896407818428186174668810632754681507768129396
557055430193396073083876302784878746824101965134464566556299068887302755065717055480518408713033457287416880653233876711462553153707155343695773846378668798134756607862911839294101184545987463547333829946
611402164012906589537780937377106455771844587295705070639619077373387672946049062941947891651462426463167443441349225183453569568538256533835467867317518500229783158335098900490367636376272513924326977245
416018717410710146252832950019211625706205930278002219873226997978479292361443075673310882561991313901228012705999361854782229433625521095048500204786978767271986094513662205649518687551548503234108649123
750337547137014021192919415135675366709864041959123538359838998642847619483351530427441468223139126430150216446869976535239329655067392017967348742955330211508591352036936036809611511019561345848947748884 5
941889404017297299491115775809478309444800099229849110451861663330063781618985471655336656966208078032044282199879839081031087947654109448771583892985156776043207450309025205637451522271206696755966432311
259338901300816754064136571224654152452220433560533753411550961859536183596997259467031906142118908139351952992708471136425197375044167256136200450672123262214852036661342480938243575843294537121565715261
614744942629548619841203308594793018906404740699598260527912563946239027920199568016546739842670297202253230517830541821056389911671344559427111550882609505540046957445960965142054260922146792969884380921
566776911864873396122643523773767965454741824705958933785378543461051604594675704402343194087693486424557908698514656644430079532817566992028323120542390334892432991287585721017958922734494975416671381001
676977171521963852536786344462413839440982821465732464079152560015723759002350561479992356138313712721642480529274992323864337079957491912801210253150239510667227616699050647008593758059211107255380562198
380311682816047587860605898597190713550970620719132152732081633412837925216841006394692613278183411197944268081947035605240558928542614551288717739160596777896929798129081117224466449003833494377305103956 7
828511912859059725696029496003021714958228417104474054976216230398343889296384314201757977981703776984444923296604981299088388200376216633917435052692777081355107572094776898586160528238759441961427644321
673077894890394264878892464769809669623129346212134450391122192476841669135008122895834378072749330069576838653118517331055190360402111614045519651426575618003376686372762423458314677405646146719756430735
494700015594180374854520887214035995481675497978546377640637312883599051142812460795352601329071554494537233336782241621949965626912068297914138745772612881963166768573864653623239013964883826516770811476
009017692351585551926813435318689802486484907925000006004573073052029741561678768308362632807167763364781057261680565107741949551338435964075562217791222626380651475291557043673092765270593684802954750253 2
477914904417402476011164659017172488094874502480981905966613021466360720132313752530924676091506355306529384108722529886572944552313907723819355328672024412839106571302303952226286440885833994480196483383
675605602373994651903186808524763125318453464521273882977865122513676375333374327357164320711430979703109672208018931409679825149177346991179145228354727493280868286557486275604510125796916348242400490839
782055423398826798338985579960408603159501439429541935252852563273690714373417024585202155877072441669333923493611982841707723160787863372033528354612815889350588966028383776335277763397321040453610720037
854020851086996832057081381571261854178883610642477354438359972335078485667096722322889066178204111122735747976671617875256517894212415138429549799370264701048718054688687937259184155194353821365562001658
823616664871949075666743311437408983775555708636449405910120641208368666594275548298194101375627550392578699762088120448588918103255959065675075892920361084554904240943220598320250381820648732321319027890
268327922486960138402141855907119956066823277251886113610926492232073457207093632810525667307335556747685534727750606756085908674193827574627777385284237410804158791354505826327846783252483276800000000000 0
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000
>>> 
```

Solutions to HL1 Assignment #23

1. Notice $\vec{v} = 12\vec{i} + 5\vec{j}$ is perpendicular to $5\vec{i} - 12\vec{j}$ and that $|\vec{v}| = 13$. So the unit vectors are $\pm\frac{1}{13}\vec{v}$.

2. We have $(\vec{v} - \vec{w}) \cdot (\vec{v} + \vec{w}) = |\vec{v}|^2 - |\vec{w}|^2 = 5^2 - 13^2 = -264$.

3. We have $x = e^{e^e}$, whence $x = 3.81 \times 10^6$ (3 s.f.).

4. By the quotient rule $f'(x) = \dfrac{1 - 2\ln x}{x^3}$. So f has a maximum at $x = \sqrt{e}$. We conclude $k = \dfrac{1}{2e}$.

5. Let A be the event that Alice wins. Then $P(A) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots$, which is an infinite geometric series with first term $\frac{1}{2}$ and common ratio $\frac{1}{4}$, whence $P(A) = \frac{2}{3}$.

6. Here $m(\vec{v}) = \frac{v_2}{v_1}$ and $m(\vec{w}) = \frac{w_2}{w_1}$. Hence $m(\vec{v}) \times m(\vec{w}) = -1 \Leftrightarrow v_2 w_2 = -v_1 w_1 \Leftrightarrow v_1 w_1 + v_2 w_2 = 0 \Leftrightarrow \vec{v} \cdot \vec{w} = 0$.

7. Solving simultaneously and remembering to use different parameters gives $t = -3$ and $u = 2$. Hence the point of intersection is $(-2, -4, -5)$.

8. Let the time of collision be t and the point of collision be (x, y). Then $x = 2 + t, y = 3t$ and $y = x + 2$. Solving simultaneously gives $t = 2$ and the point of collision as $(4, 6)$. So the motorcycle travelled from $(0, 2)$ to $(4, 6)$ in 2 seconds. Hence the speed of the motorcycle is $2\sqrt{2}\,$u/s.

9. Let

$$\begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} = m \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^{\perp}$$

where m and n are constants to be determined. Taking the scalar product of both sides of this equation with the vector $\vec{i} + \vec{j} + \vec{k}$ gives $9 = 3m$, whence $m = 3$. So the required sum is

$$\begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}.$$

10. After differentiating three times we find

$$f'''(x) = 2(x+1)^{-1} + 2(x+1)^{-2} + 2(x+1)^{-3},$$

from which we foresee that

$$f^{(2019)}(x) = 2 \cdot 2016! \cdot (x+1)^{-2017} + 2 \cdot 2017! \cdot (x+1)^{-2018} + 2018! \cdot (x+1)^{-2019},$$

whence $f^{(2019)}(0) = 2 \cdot 2016! + 2 \cdot 2017! + 2018!$.