HL1 Assignment #10

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Excellent!!

1. What can be said about the complex number z if

   (a) $z = z^*$;

   $z \in \mathbb{R}$

   (b) $z = -z^*$?

   $Re(z) = 0$

2. In $\triangle ABC$, $A = 120°$, $B = 45°$ and $a = 15$. If $b = k\sqrt{6}$, find the value of k.

according to the law of sine,

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \frac{\frac{\sqrt{3}}{2}}{15} = \frac{\frac{\sqrt{2}}{2}}{k\sqrt{6}}$$

$\therefore 3k\sqrt{2} = 15\sqrt{2}$

$\therefore k = 5$

3. The point $P(5, 5\sqrt{3})$ is rotated 75° anticlockwise about the origin to the point P'. Find the coordinates of P'.

$P: (5, 5\sqrt{3}) \Leftrightarrow (10, \frac{\pi}{3})$

$\therefore$ when $\arg(P) + = \frac{5}{12}\pi$

we get:

$(10, \frac{3}{4}\pi)$

$\therefore P(10 \cdot \sin\frac{3}{4}\pi, 10 \cdot \cos\frac{3}{4}\pi)$.

$\therefore P(5\sqrt{2}, -5\sqrt{2})$

4. Show that $x - c$ is a factor of $(x-b)^3 + (b-c)^3 + (c-x)^3$.

let $f(x) = (x-b)^3 + (b-c)^3 + (c-x)^3$

$f(c) = (c-b)^3 + (b-c)^3 + 0$

$= -(b-c)^3 + (b-c)^3$

$= 0$

$\therefore$ c is a root of $f(x)$.

according to the factor theorem,

$x - c$ is a factor of $f(x) = (x-b)^3 + (b-c)^3 + (c-x)^3$.

5. Given that $5 + 2i$ is a root of $2x^3 - 15x^2 + 8x + 145 = 0$, find the other roots without using a calculator.

according to the conjugate root theorem,

$x_1 = 5 + 2i$, then $x_2 = 5 - 2i$.

x_1 & x_2 are root of $x^2 - 10x + 29 = 0$

$\because (2x+5)(x^2 - 10x + 29) = 2x^3 - 15x^2 + 8x + 145$

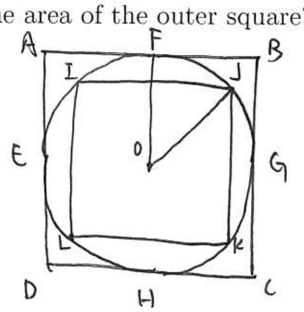
$\therefore x_3 = -\frac{5}{2}$

$\therefore$ In conclusion, $x_1 = 5 + 2i$

$x_2 = 5 - 2i$

$x_3 = -\frac{5}{2}$

6. The largest possible circle is inscribed into a square. The largest possible square that will fit is then inscribed in that circle. What is the ratio of the area of the inner square to the area of the outer square?



let $OF = OJ = r$

$IJ = \frac{r}{\sin 45} = \sqrt{2}r$

$AB = 2OF = 2r$

$\therefore S_{ABCD} = (2r)^2 = 4r^2$

$S_{IJKL} = (\sqrt{2}r)^2 = 2r^2$

$\therefore \frac{S_{IJKL}}{S_{ABCD}} = \frac{2r^2}{4r^2} = \frac{1}{2}$

$\therefore$ the ratio of the inner square to the outer square is $\frac{1}{2}$.

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7. Find a cubic equation with integer coefficients that has $3-\sqrt[3]{2}$ as a root. Answer in the form $ax^3+bx^2+cx+d=0$.

$f(x)=ax^3+bx^2+cx+d$

$f(3-\sqrt[3]{2})=(25-27\cdot 2^{\frac{1}{3}}+9\cdot 2^{\frac{2}{3}})a+(9+2^{\frac{2}{3}}-6\cdot 2^{\frac{1}{3}})b+(3-2^{\frac{1}{3}})c+d=0$

$\therefore \begin{cases} 25a+9b+3c+d=0 \\ -27a+(-6b)-c=0 \\ 9a+b=0 \end{cases}$

$\Downarrow$

$\begin{cases} b=-9a \\ c=27a \\ d=-25a \end{cases}$

$\therefore\ x^3-9x^2+27x-25=0$

$\therefore$ if $a=1$

$\begin{cases} b=-9 \\ c=27 \\ d=-25 \end{cases}$

✓

8. Let $f(x)=x^3-18x^2+72x+k$ where k is a constant. If the roots of $f(x)=0$ form a geometric sequence, what is the value of k?

let $x_1 r^2 = x_2\cdot r = x_3$

according to Vieta's Theorem,

$\begin{cases} x_1+x_2+x_3=18 \\ x_1x_2+x_2x_3+x_3x_1=72 \\ x_1x_2x_3=-k \end{cases}$

$\therefore \begin{cases} x_1(1+r+r^2)=18 \\ r\cdot x_1^2(1+r+r^2)=72 \end{cases}$

$\therefore\ rx_1=4$

$\because\ -k=x_1\cdot x_1r\cdot x_1r^2$

$=x_1^3\cdot r^3$

$\therefore\ -k=4^3=64$

$\therefore\ k=-64.$

✓

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9. Find the sum of the series $\sum_{n=1}^{100} ni^n$ where $i^2 = -1$.

$$\sum_{n=1}^{100} ni^n$$

$$= (1+5+9+\cdots+97)\cdot i + (2+6+10+\cdots+98)\cdot(-1) + (3+7+11+\cdots+99)\cdot(-i) + (4+8+12+\cdots+100)\cdot 1$$

$$= \frac{98\cdot 25}{2} i + \frac{100\cdot 25}{2}\cdot(-1) + \frac{102\cdot 25}{2}\cdot(-i) + \frac{104\cdot 25}{2}$$

$$= 1225i - 1250 - 1275i + 1300$$

$$= 50 - 50i$$

10. Prove that $2^{\sin x} + 2^{\cos x} \geq 2^{1-1/\sqrt{2}}$ for all $x \in \mathbb{R}$.

$\because (a-b)^2 \geqslant 0$

$a^2+b^2-2ab \geqslant 0$

$\therefore a^2+b^2+2ab \geqslant 4ab$

$\therefore (a+b)^2 \geqslant 4ab$

let $a = 2^{\sin x}$,

$b = 2^{\cos x}$.

$\because a>0,\ b>0$

$\therefore 2^{\sin x} + 2^{\cos x} \geqslant 2\sqrt{2^{\sin x+\cos x}}$

$\therefore 2^{\sin x} + 2^{\cos x} \geqslant \sqrt{2^{2+\sin x+\cos x}}$

$\boxed{\therefore 2^{\sin x} + 2^{\cos x} \geqslant 2^{\frac{2+\sin x+\cos x}{2}}}$

① let $y = \dfrac{2+\sin x+\cos x}{2}$

$\therefore \dfrac{dy}{dx} = \dfrac{\cos x}{2} - \dfrac{\sin x}{2}$

y_{max} & y_{min} lies at x when it makes $\frac{dy}{dx}=0$.

$\therefore \dfrac{\cos x}{2} = \dfrac{\sin x}{2}$

$\therefore x = \frac{\pi}{4} \pm 2k\pi$ or $\frac{5}{4}\pi \pm 2k\pi$

$\because f(\frac{\pi}{4} \pm 2k\pi) = \dfrac{2+\sqrt{2}}{2}$

$f(\frac{5}{4}\pi \pm 2k\pi) = \dfrac{2-\sqrt{2}}{2}$

$\therefore y_{max} = \dfrac{2+\sqrt{2}}{2}$

$y_{min} = \dfrac{2-\sqrt{2}}{2}$

$\therefore \dfrac{2+\sin x+\cos x}{2} \geqslant \dfrac{2-\sqrt{2}}{2}$

② $\because y = \sin x + \cos x = \sqrt{2}\sin(x+\frac{\pi}{4})$

$\therefore y_{min} = \sqrt{2}\cdot(-1) = -\sqrt{2}$

$\therefore \sin x + \cos x \geqslant -\sqrt{2}$

since $f(x) = 2^x$ is monotonically increasing.

we have:

$$2^{\sin x} + 2^{\cos x} \geqslant 2^{\frac{2+\sin x+\cos x}{2}} \geqslant 2^{\frac{2-\sqrt{2}}{2}}$$

which is equivalent to:

$$2^{\sin x} + 2^{\cos x} \geqslant 2^{1-\frac{1}{\sqrt{2}}}$$

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Solutions to HL1 Assignment #10

1. (a) z is real, that is $\operatorname{Im} z = 0$. (b) z is purely imaginary, that is $\operatorname{Re} z = 0$.

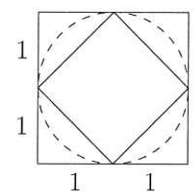
2. By the sine rule $\dfrac{b}{\sin 45^\circ} = \dfrac{15}{\sin 120^\circ}$. We conclude that $b = 5\sqrt{6}$. So $k = 5$.

3. By Pythagoras's theorem $OP^2 = 5^2 + (5\sqrt{3})^2$. So $OP = 10$. Hence $P' = (10\cos 135^\circ, 10\sin 135^\circ) = (-5\sqrt{2}, 5\sqrt{2})$.

4. By the remainder theorem $f(c) = 0$. So $x - c$ is a factor of this polynomial.

5. By the conjugate roots theorem a second root is $5 - 2i$. Since the sum of the roots is 7.5. We conclude the third root is -2.5.

6. Without loss of generality choose the outside square to have sides of length 2 as shown in the diagram.



Now let the area of the outer square be A and the area of the inner square be B. So $A = 2 \times 2 = 4$, and $B = 4 - 4 \times \frac{1}{2} \times 1 \times 1 = 2$. So $B : A = 1 : 2$.

7. Let $x = 3 - \sqrt[3]{2}$. Then $(x-3)^3 = -2$. Therefore such an equation is $x^3 - 9x^2 + 27x - 25 = 0$.

8. Let the geometric sequence of roots be a, ar, ar^2. By the factor theorem $f(x) = (x-a)(x-ar)(x-ar^2)$. Expanding and equating coefficients gives us $a + ar + ar^2 = 18$ and $a^2r + a^2r^2 + a^2r^3 = 72$. Solving simultaneously gives $ar = 4$. Next the product of the roots is $a^3r^3 = (ar)^3 = -k$, whence $k = -64$.

9. Let the sum be S. Then $S = (-2+4-6+8-10+-\cdots-98+100)+(1-3+5-7+-\cdots+97-99)i = 25 \times 2 - 25 \times 2i = 50 - 50i$.

10. Since the arithmetic mean of two positive numbers is greater than or equal to their geometric mean, we conclude
$$2^{\sin x} + 2^{\cos x} \geq 2\sqrt{2^{\sin x} \times 2^{\cos x}} = 2\sqrt{2^{\sin x + \cos x}}.$$
Now the minimum value of $\sin x + \cos x$ is $-\sqrt{2}$ occurring when $x = 225^\circ$. So we must have
$$2^{\sin x} + 2^{\cos x} \geq 2\sqrt{2^{-\sqrt{2}}} = 2 \times 2^{-1/\sqrt{2}} = 2^{1-1/\sqrt{2}},$$
as required.