FURTHER MATHEMATICS
HIGHER LEVEL

Wednesday 22 May 2019

Name in block letters

J	E	R	R	Y		J	I	A	N	G	

45 minutes

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Calculators are not permitted in this examination.
- There are 4 questions. Try to answer them all.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

100%

Bravo!!

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working or explanations. Where an answer is incorrect, some marks may be given for a correct method provided this is shown by written working. You are therefore advised to show all working. Working may be continued below the lines, if necessary.

1. The line ℓ is the tangent to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$.

   (a) Use parametric differentiation to find the gradient of ℓ.

   (b) Show that the equation of ℓ is $x - yt + at^2 = 0$.

(a) $\frac{dy}{dt} = 2a$, $\frac{dx}{dt} = 2at$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

(b) $y - 2at = \frac{1}{t}(x - at^2)$

$yt - 2at^2 = x - at^2$

$\therefore x - yt + at^2 = 0$

5

2. The line ℓ is the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $P(x_1, y_1)$.

   (a) Use implicit differentiation to find the gradient of ℓ.

   (b) Show that the equation of ℓ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$.

(a) $b^2x^2 + a^2y^2 = a^2b^2$

implicit differentiation: $2b^2x + 2a^2y \cdot y' = 0$, $y' = -\dfrac{2b^2x}{2a^2y} = -\dfrac{b^2x}{a^2y}$

$\therefore$ y' at P is $-\dfrac{b^2x_1}{a^2y_1}$

(b) $y - y_1 = -\dfrac{b^2x_1}{a^2y_1}(x - x_1)$

$\therefore y \cdot a^2y_1 - a^2y_1^2 = -b^2x_1x + b^2x_1^2$

$\therefore a^2yy_1 + b^2xx_1 = a^2y_1^2 + b^2x_1^2$

$\because$ P is on ellipse,

$\therefore \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1.$

$\therefore x_1^2 \cdot b^2 + y_1^2 \cdot a^2 = a^2b^2$

$\therefore l: a^2yy_1 + b^2xx_1 = a^2b^2$

$\therefore l: \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$

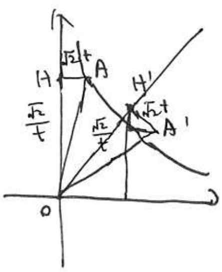
5

3. The hyperbola $\mathcal{H}$ has equation $xy = 2$.

   (a) Show that if $\mathcal{H}$ is rotated clockwise $45°$ about the origin its equation becomes $x^2 - y^2 = 4$.

   (b) Determine the coordinates of the foci of $\mathcal{H}$.

   (c) Determine the equations of the directrices of $\mathcal{H}$.



(a) A is a point on $xy=2$ with coordinate $(\sqrt{2}t, \frac{\sqrt{2}}{t})$

when $\triangle AHO$ is rotated, it gets to $\triangle A'H'O$, where H' is on $y=x$.

$\therefore H'(\frac{1}{t}, \frac{1}{t})$, $A'(\frac{1}{t} + t, \frac{1}{t} - t)$

$(x_A)^2 = \frac{1}{t^2} + t^2 + 2$

$(y_A)^2 = \frac{1}{t^2} + t^2 - 2$

$\therefore x^2 - y^2 = 4$.

(b) In $x^2-y^2=4$, $\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1$

$\therefore a = 2$, $b^2 = -a^2(1-e^2) = -4(1-e^2) = 4$

$\therefore 1-e^2 = -1$, $e^2 = 2$

$\therefore e = \sqrt{2}$.

$\therefore$ In $x^2-y^2=4$, $F(2\sqrt{2}, 0)$, $(-2\sqrt{2}, 0)$

$\therefore$ before rotation, $F(2,2)$, $(-2,-2)$

(c) In $x^2-y^2=4$, $a=2$, $e=\sqrt{2}$

$x_D = \frac{2}{\sqrt{2}} = \sqrt{2}$. l: $x=\sqrt{2}$, $x=-\sqrt{2}$

$\therefore$ before rotation, directrices: l: $y=-x+2$, $y=-x-2$.

/5

4. The parabola $y^2 = 4ax$ and the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ meet in the first quadrant at the point P. The tangents ℓ_1 and ℓ_2 to the parabola and ellipse respectively at P are perpendicular.

(a) Show that $b^2 = 2a^2$.

(b) If ℓ_1 and ℓ_2 have x-intercepts M and N respectively, show that $MN = 2\sqrt{2}a$.

(a) $P(at^2, 2at)$

$\dfrac{a^2t^4}{a^2} + \dfrac{4a^2t^2}{b^2} = 1$

$\therefore b^2t^4 + 4a^2t^2 = b^2$ ①

m parabola $T_P = \dfrac{1}{t}$, $m$parabola $N_P = -t$

$m$ellipse $T_P = -\dfrac{b^2}{a^2} \cdot \dfrac{x_P}{y_P} = -\dfrac{b^2}{a^2} \cdot \dfrac{at^2}{2at}$

$\therefore \quad -t = -\dfrac{b^2}{a^2} \cdot \dfrac{t}{2}$

$\therefore \quad 2a^2 = b^2$ ✓

(b) $l_1: x - yt + at^2 = 0$

$l_2: \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1 \Rightarrow \dfrac{x \cdot at^2}{a^2} + \dfrac{y \cdot 2at}{2a^2} = 1$

l_1: when $y=0$, $x = -at^2$, | l_2: $y=0$, $x = \dfrac{a}{t^2}$

$MN = \left|\dfrac{a}{t^2} - (-at^2)\right| = \left|\dfrac{a}{t^2} + at^2\right|$ ②

insert $(at^2, 2at)$ into $\dfrac{x^2}{a^2} + \dfrac{y^2}{2a^2} = 1$,

$2a^2t^4 + 4a^2t^2 - 2a^2 = 0.$

$\therefore t^2 + 2 - \dfrac{1}{t^2} = 0.$ ✓

$\therefore t^2 - \dfrac{1}{t^2} = -2$

$\therefore t^4 + \dfrac{1}{t^4} - 2 = 4$

$\therefore t^4 + \dfrac{1}{t^4} + 2 = 8$

$\therefore \left(t^2 + \dfrac{1}{t^2}\right)^2 = 8$

$\because t^2 + \dfrac{1}{t^2} > 0$

$\therefore t^2 + \dfrac{1}{t^2} = 2\sqrt{2}$

Back to ②

$MN = \left|\dfrac{a}{t^2} + at^2\right|$

$= |2\sqrt{2}a|$

$\because a > 0$

$\therefore MN = 2\sqrt{2}a$

✓

5