International Baccalaureate
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Bachillerato Internacional

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1 hour Groups & Relationship.. Sets, relations & groups.

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

55/60

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Excellent!

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

(a) The relation R is defined on $\mathbb{Z}^+$ by aRb if and only if ab is even. Show that only one of the conditions for R to be an equivalence relation is satisfied. *[5 marks]*

(b) The relation S is defined on $\mathbb{Z}^+$ by aSb if and only if $a^2 \equiv b^2 \pmod 6$.

(i) Show that S is an equivalence relation.

(ii) For each equivalence class, give the four smallest members. *[9 marks]*

2. *[Maximum mark: 13]*

The binary operations $\odot$ and $*$ are defined on $\mathbb{R}^+$ by

$$a \odot b = \sqrt{ab} \text{ and } a * b = a^2 b^2 .$$

Determine whether or not

(a) $\odot$ is commutative; *[2 marks]*

(b) $*$ is associative; *[4 marks]*

(c) $*$ is distributive over $\odot$; *[4 marks]*

(d) $\odot$ has an identity element. *[3 marks]*

3. *[Maximum mark: 16]*

The group $\{G, \times_7\}$ is defined on the set $\{1, 2, 3, 4, 5, 6\}$ where $\times_7$ denotes multiplication modulo 7.

(a) (i) Write down the Cayley table for $\{G, \times_7\}$.

(ii) Determine whether or not $\{G, \times_7\}$ is cyclic.

(iii) Find the subgroup of G of order 3, denoting it by H.

(iv) Identify the element of order 2 in G and find its coset with respect to H. *[10 marks]*

(b) The group $\{K, \circ\}$ is defined on the six permutations of the integers 1, 2, 3 and $\circ$ denotes composition of permutations.

(i) Show that $\{K, \circ\}$ is non-Abelian.

(ii) Giving a reason, state whether or not $\{G, \times_7\}$ and $\{K, \circ\}$ are isomorphic. *[6 marks]*

4. *[Maximum mark: 9]*

The groups $\{G, *\}$ and $\{H, \odot\}$ are defined by the following Cayley tables.

G

$*$	**E**	**A**	**B**	**C**
E	E	A	B	C
A	A	E	C	B
B	B	C	A	E
C	C	B	E	A

H

$\odot$	**e**	**a**
e	e	a
a	a	e

By considering a suitable function from G to H, show that a surjective homomorphism exists between these two groups. State the kernel of this homomorphism.

5. *[Maximum mark: 8]*

Let $\{G, *\}$ be a finite group and let H be a non-empty subset of G. Prove that $\{H, *\}$ is a group if H is closed under $*$.

1\. (a) • $3 \not R 3$, so not reflexive.

• xRy, which means $xy = 2k$, $k \in \mathbb{Z}^+$. so $yx = 2k$. yRx. it's symmetric

• xRy, yRz, if y is even and x and z are odd, then $x \not R z$. not transitive.

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(b) i. • $a^2 \equiv r \pmod 6$ for $0 \le r < 6$. $a^2 = 6k + r$, for $k \in \mathbb{Z}$.

so $a^2 \equiv r \equiv a^2 \pmod 6$. aSa.

• if aSb, then $a^2 \equiv b^2 \equiv r \pmod 6$. then $b^2 \equiv r \equiv a^2 \pmod 6$.

bSa. symmetric

• if aSb, bSc, then $a^2 \equiv b^2 \equiv r \pmod 6$, $b^2 \equiv c^2 \pmod 6$.

let $0 \le r < 6$, $a^2 = 6m + r$, $b^2 = 6n + r$ for $m, n \in \mathbb{Z}$.

Therefore $b^2 \equiv c^2 \equiv r \pmod 6$ $c^2 = 6q + r$.

So $a^2 \equiv c^2 \equiv r \pmod 6$. aSc. transitive.

Therefore, S is an equivalence relationship.

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ii. $[1]$: 1, 5, 7, 11.

$[2]$: 2, 4, 8, 10.

$[3]$: 3, 9, 15, 21

$[6]$: 6, 12, 18, 24

2\. (a) $a \odot b = \sqrt{ab} = \sqrt{ba} = b \odot a$. $\odot$ is commutative.

(b) $a * (b * c) = a * (b^2 c^2) = a^2 b^4 c^4$, $(a*b)*c = (a^2b^2)*c = a^4 b^4 c^2$. $a*(b*c) \neq (a*b)*c$. $*$ isn't associative.

(c) $a * (b \odot c) = a * (\sqrt{bc}) = a^2 \cdot bc = a^2 bc$.

$(a*b) \odot (a*c) = (a^2 b^2) \odot (a^2 c^2) = \sqrt{a^2 b^2 \cdot a^2 c^2} = \sqrt{a^4 b^2 c^2} = a^2 bc$.

$*$ is distributive over $\odot$.

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(d) $a \odot e = a$. $a \odot e = \sqrt{ae}$. when $a = \sqrt{ae}$, $e = a$.

as a is any element, there's no identity element for $\odot$.

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3. (a) i.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

✓

ii. $5^1 = 5$

$5^2 = 4$

$5^3 = 6$

$5^4 = 2$

$5^5 = 3$ ✓

$5^6 = 1 = e$.

$\therefore G = \langle 5 \rangle$.

iii. $G \cong \mathbb{Z}_6$.

$H = \{1, 2, 4\} = \langle 2 \rangle = \langle 4 \rangle$.

iv. $6^1 = 6$, $6^2 = 1 = e$. ✓ $|6| = 2$.

coset: $\{6, 5, 3\}$. ✓

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(b) i. $(12)\circ(123) = (12)(12)(23) = (23)$ ✗ Therefore, it's not abelian.

$(123)\circ(12) = (23)(31)(12) = (23)(312) = (1)$. (13) $(123)(12) = (312)(12) = (31)$ 5 −1

ii. $\{G, \times_7\}$ is cyclic, while $\{K, \circ\}$ isn't. Therefore, they're not isomorphic. ✓

4. $f: \{E, A, B, C\} \to \{e, a\}$ is defined by $f(E) = e$, $f(A) = e$, $f(B) = a$, $f(C) = a$. ✓

By construction, we know that f is surjective. 5

$\ker(f) = \{E, A\}$. ✓

you must show

$f(x * y) = f(x) \odot f(y)$ −4

5. • closure is given.

• Since $H \neq \emptyset$, there's $a \in H$. Due to closure, $a, a^2, a^3 \cdots$ all $\in H$ and they're not all different. Let $a^j = a^i$ where $j > 1$. $e = a^i \cdot (a^i)^{-1} = a^j \cdot a^{-i} = a^{j-i}$. $\because j > 1$.

$\therefore j - i > 0$, $j - i \geq 1$, $\therefore a^{j-i} \in H$. $\therefore e \in H$.

• if $a = e$, then $\{e, *\}$ is a group. if $a \neq e$, then $a^{j-i} \neq a$, $a^{j-i-1} \neq e$.

$j-i-1 > 0$, so $j-i-1 \geq -1$, $a^{j-i-1} \in H$.

$\because a^{j-i-1} \cdot a = a^{j-i} = e$.

$\therefore a^{j-i-1} = a^{-1}$.

$\therefore a^{-1} \in H$ ✓

According to the 3-step subgroup test, $H \leq G$. 5