MATHEMATICS
HIGHER LEVEL

97%

Excellent!

Wednesday 22 May 2019

Name in block letters

2 hours

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INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Calculators are not permitted in this examination.
- There are 20 questions. Try to answer them all.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

$\boxed{7}$

16 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working or explanations. Where an answer is incorrect, some marks may be given for a correct method provided this is shown by written working. You are therefore advised to show all working. Working may be continued below the lines, if necessary.

1. Let $\vec{a} = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -3 \\ k+2 \\ k \end{pmatrix}$. If $\vec{a}$ and $\vec{b}$ are perpendicular find the possible values of k.

$\vec{a} \cdot \vec{b} = 0.$

$-6 + k(k+2) - k = 0$

$\therefore\ k^2 + k - 6 = 0$

$\therefore\ (k+3)(k-2) = 0$

$\therefore\ k_1 = -3,$

$k_2 = 2.$ ✓

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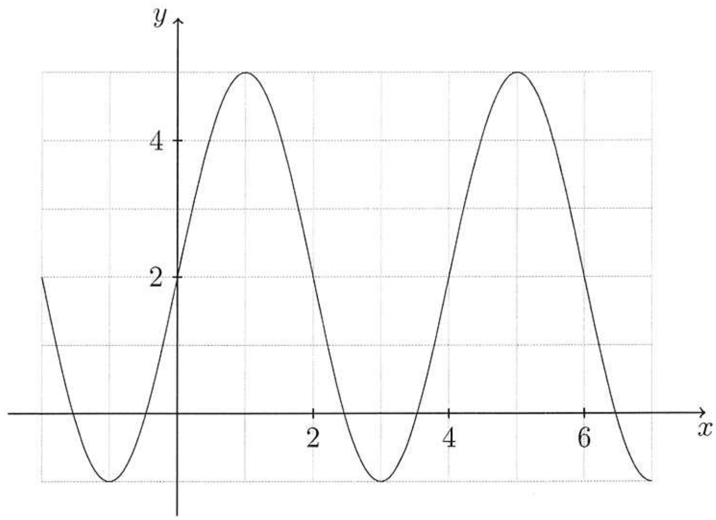
2. Part of the graph of the function $f(x) = a\cos(b(x+c)) + d$ is drawn below. The graph has a maximum at $(1, 5)$ and a minimum at $(3, -1)$.



(a) Find the values of a and d.

(b) Find the value of b.

(c) Find two possible values for c.

(a) $a = 3$, $d = 2$

(b) $b = \frac{2\pi}{4} = \frac{\pi}{2}$

(c) $-c = 1$ or -3

$\therefore c = -1$ or 3.

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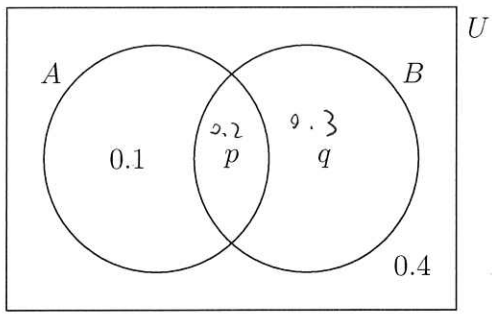
3. The Venn diagram shows the events A and B where $P(A) = 0.3$. The values shown are probabilities.



	A	A'	
B	0.2	0.3	0.5
B'	0.1	0.4	0.5
	0.3	0.7	1

(a) Write down the values of p and q.

(b) If $A \triangle B = (A \cap B') \cup (A' \cap B)$ find $P(A \triangle B)$.

(c) Find $P(A \triangle B \mid A \cup B)$.

(a) $p = 0.2$, $q = 0.3$

(b) $P(A \triangle B) = 0.1 + 0.3 = 0.4$

(c) $P = \dfrac{0.4}{0.1 + 0.2 + 0.3} = \dfrac{2}{3}$

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4. Let $f(x) = \dfrac{2x-1}{x+3}$.

(a) Write down the equation of the vertical asymptote for the graph of f.

(b) Find $f^{-1}(x)$.

(c) Find the equation of the horizontal asymptote for the graph of f^{-1}.

(a) $l: x = -3$ ✓

(b) $y = \frac{2x-1}{x+3}$,

$f^{-1}(x): x = \frac{2y-1}{y+3}$

$\therefore y = -\frac{3x+1}{x-2}$ ✓

(c) In $f^{-1}(x)$, vertical assymptote : $l: x=2$.

so In $f(x)$, there's assymptote : $l: y=2$. ✗

~~vert~~ for f^{-1} $l: y=-3$.

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5. The magnitudes of the vectors $\vec{u}$ and $\vec{v}$ are 4 and $\sqrt{3}$ respectively. The angle between the vectors is $\frac{\pi}{6}$. If $\vec{w} = \vec{u} - \vec{v}$ find the magnitude of $\vec{w}$.

$\cos\frac{\pi}{6} = \frac{\vec{u}\cdot\vec{v}}{4\cdot\sqrt{3}} = \frac{\sqrt{3}}{2}$

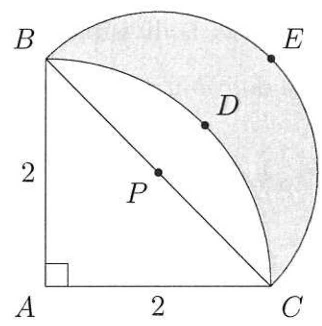
$\therefore \vec{u}\cdot\vec{v} = 6$

$\because |\vec{w}|^2 = (\vec{u})^2 + (\vec{v})^2 - 2\vec{u}\cdot\vec{v}$

$\therefore |\vec{w}|^2 = 16 + 3 - 2\times 6 = 7$

$\therefore |\vec{w}| = \sqrt{7}$

6. The triangle ABC is a right-angled isosceles triangle with $AB = AC = 2$ and point P is the midpoint of side BC. The arc BDC is part of a circle with centre A and the arc BEC is part of a circle with centre P.



(a) Calculate the area of the segment $BDCP$.

(b) Calculate the area of the shaded region $BECD$.

(a) $A_{ABDC} = \frac{1}{4}\cdot\pi 2^2 = \pi.$

$A_{\triangle ABC} = \frac{1}{2}\cdot 2\cdot 2 = 2.$

$\therefore A_{BDCP} = \pi - 2$

(b) $A_{BCEP} = \frac{1}{2}\cdot\pi\left[\left(\frac{2}{\sin\frac{\pi}{4}}\right)\cdot\frac{1}{2}\right]^2 = \pi$

$\therefore A_{BECD} = \pi - (\pi - 2) = 2$

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7. In this question, we signify that a number is written in base n by using the subscript n at the right end of the number. For example, 243_6 is a number written in base 6.

   (a) Write the number 1234_8 in base 10.

   (b) Find the value of the digit b if $123b_8$ is divisible by 7.

   (c) Find the possible values of the digit b if $123b_8 \bmod 4 = 2$.

(a) $(1234)_8 = 8^3 + 2\times 8^2 + 3\times 8 + 4 = 668$ ✓

(b) $(123b)_8 = 664 + b.$

$664 \equiv 6 \pmod 7.$ ✓

$\therefore b = 1.$

(c) $(123b)_8 = 664 + b.$

$664 \equiv 0 \pmod 4$ ✓

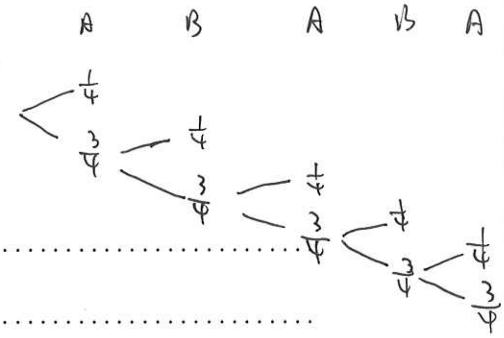
$\therefore b = 2$ or $6.$

8. Alice and Bob take turns throwing a fair tetrahedral die. The winner is the first person to throw a four. Alice goes first.

   (a) What is the probability that Alice wins on her first throw?

   (b) What is the probability that Alice wins on her second throw?

   (c) What is the probability that Alice wins?



(a) $p = \frac{1}{4}$

(b) $p = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$

(c) $p = \frac{1}{4} \cdot \frac{1-(\frac{9}{16})^{\infty}}{1-\frac{9}{16}} = \frac{1}{4} \cdot \frac{16}{7} = \frac{4}{7}$

9. The tangent to the curve $y = xe^{2x}$ at the point $(1, e^2)$ meets the x-axis at the point (a, b).

(a) Write down the value of b.

(b) Find $\dfrac{dy}{dx}$.

(c) Find the value of a.

(a) $b = 0$ ✓

(b) $\frac{dy}{dx} = e^{2x} + x \cdot e^{2x} \cdot 2$

$= e^{2x}(1+2x)$

(c) at 1, $m = e^2 \cdot 3 = 3e^2$

$\therefore y - e^2 = 3e^2(x-1)$

$\therefore$ when $y=0$, $a = x = \frac{2}{3}$ ✓

10. The lengths of two sides of a triangle are 4 cm and 5 cm. The triangle has an area of $\frac{5\sqrt{15}}{2}$ cm^2. Let θ be the angle between the two given sides.

(a) Show that $\sin\theta = \frac{\sqrt{15}}{4}$.

(b) Find the two possible values for the length of the third side.

(a) $A_\Delta = 4 \cdot 5 \cdot \sin\theta \cdot \frac{1}{2} = \frac{5}{2}\sqrt{15}$

$\therefore \quad 4\sin\theta = \sqrt{15}$

$\therefore \quad \sin\theta = \frac{\sqrt{15}}{4}$

(b) $\cos\theta = \pm\sqrt{1 - \frac{15}{16}} = \pm\frac{1}{4}$

$c^2 = 16 + 25 - 2 \cdot 4 \cdot 5 \cdot (\pm\frac{1}{4})$

$= 41 \pm 10$

$= 31$ or 51

$\because c > 0$

$\therefore \quad c = \sqrt{31}$ or $\sqrt{51}$

11. Solve each of the following equations over the set of real numbers.

(a) $\log_3(x+17) - 2 = \log_3 2x$.

(b) $2^{2x+2} - 10 \times 2^x + 4 = 0$.

(a) $\log_3 \frac{(x+17)}{9 \cdot 2x} = 0$

$\frac{x+17}{9 \cdot 2x} = 1$

$\therefore x+17 = 18x$

$\therefore x = 1$

(b) let 2^x be a.

$4a^2 - 10a + 4 = 0$

$(2a-1)(a-2) = 0$

$\therefore a_1 = \frac{1}{2}$,

$a_2 = 2$.

$\therefore 2^x = \frac{1}{2}$ or 2.

$\therefore x = 1$ or -1

12. Solve $z^2 = 4e^{i\frac{\pi}{2}}$, giving your answers in the form

(a) $re^{i\theta}$ where $r, \theta \in \mathbb{R}$, $r \geq 0$;

(b) $a + bi$ where $a, b \in \mathbb{R}$.

(a) $z^2 = [4, \frac{\pi}{2}]$

$z = [2, \frac{\pi}{4}]$ or $[2, \frac{5}{4}\pi]$

$\therefore z = 2e^{i\frac{\pi}{4}}$ or $2e^{i\frac{5}{4}\pi}$ ✓

(b) $z = \sqrt{2} + \sqrt{2}\,i$ or $-\sqrt{2} - \sqrt{2}\,i$ ✓

13. The three numbers 1, a and b have mean 5 and variance 14.

(a) Write down the standard deviation of the three numbers.

(b) If $a < b$ find the values of a and b.

(a) $\sqrt{14}$

(b) $\begin{cases} \dfrac{4^2+(5-a)^2+(5-b)^2}{3} = 14 \\ 1+a+b=15 \end{cases}$

$a^2+b^2-10(a+b)+50=42$

$a^2+b^2=116$

$\therefore \begin{cases} a+b=14 \\ ab=40 \end{cases}$

$\therefore t^2-14t+40=0$

$\therefore t_1=10,\ t_2=4.$

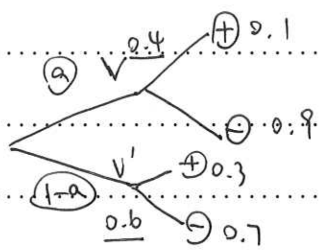
$\therefore \begin{cases} a=4 \\ b=10. \end{cases}$

14. A flu virus is spreading among the students at Pearson College. A vaccination is available to protect against the virus. If a student has had the vaccination the probability of catching the virus is 0.1; without the vaccination the probability is 0.3. The probability of a randomly selected student catching the virus is 0.22.

(a) Find the percentage of the students who have been vaccinated.

(b) A student catches the virus. Find the probability that this student was vaccinated.

(a)



$a \cdot 0.1 + (1-a) \cdot 0.3 = 0.22$

$\therefore a = 0.4$

$\therefore 40\%$

(b) $p = \dfrac{0.4 \cdot 0.1}{0.6 \cdot 0.3 + 0.4 \cdot 0.1} = \dfrac{2}{11}$

15. Consider the function $f(x) = \ln(x^4 + 1)$, $x \in \mathbb{R}$.

(a) Show that the graph of f has only one stationary point and determine its nature.

(b) Find the coordinates of any inflection points on the graph of f.

(a) $f'(x) = \frac{1}{x^4+1} \cdot 4x^3$

when $f'(x) = 0$, $x = 0$, only one.

$f''(x) = \frac{12x^2(x^4+1) - 4x^3 \cdot 4x^3}{(x^4+1)^2} = \frac{4x^2(3-x^4)}{(x^4+1)^2}$

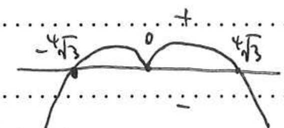
why not max? (reason)

$f''(0) = \frac{0}{1} = 0$, but $f''(0)$ doesn't change sign. so it's a minimum.

I'm not sure... how do you call the point with $f' = f'' = 0$? Can you put the correct answer if "min" is wrong?

(b) $f''(x) = 0$, $x = 0$ or $\pm\sqrt[4]{3}$

$\because$ at $x = \pm\sqrt[4]{3}$, $f'' = 0$, f'' change sign.



$\therefore$ inflection points: $(\sqrt[4]{3}, \ln 4)$, $(-\sqrt[4]{3}, \ln 4)$

explain.

in f'', as $x \to 0^+$, $x \to 0^-$,

f'' both > 0. so concave up,

minimum.

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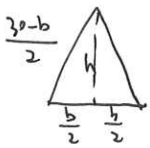
16. The isosceles triangle T has base b and perimeter 30.

(a) Show that the area of T is $\dfrac{b}{2}\sqrt{225 - 15b}$.

(b) Use calculus to show that the area of T is largest when T is equilateral.



(a) $h = \sqrt{\dfrac{-60b + 30^2}{4}} = \sqrt{225-15b}$

$\therefore A_\Delta = \frac{1}{2}\cdot b \cdot h = \frac{b}{2}\sqrt{225-15b}$

(b) $A_\Delta' = \frac{1}{2}\sqrt{225-15b} + \frac{b}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{225-15b}}\cdot(-15)$

when $A_\Delta' = 0$, $b = 10$

$A_\Delta'' = \frac{1}{4}\cdot(225-15b)^{-\frac{1}{2}} + \left(-\frac{15}{4}\right)\cdot\dfrac{15\sqrt{225-15b} - 15b\cdot\frac{1}{2}\cdot(225-15b)^{-\frac{1}{2}}}{225-15b}$

at $b=10$, $A_\Delta'' < 0$, maximum.

$\therefore$ when $b=10$, which is equilateral triangle, A_Δ is maximized.

17. The points A, B and C have coordinates $(4,4,6)$, $(1,1,0)$ and $(3,3,1)$ respectively.

(a) Find a vector equation of the line (BC).

(b) The distance from point A to the line (BC) is $a\sqrt{2}$ where $a \in \mathbb{Z}^+$. Find the value of a.

(c) Hence find the area of triangle ABC.

(a) $BC: \vec{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\2\\1\end{pmatrix}$

(b) $P\begin{pmatrix}1+2t\\1+2t\\t\end{pmatrix}$ $\quad \overrightarrow{AP} = \begin{pmatrix}2t-3\\2t-3\\t-6\end{pmatrix}$.

$2(2t-3)\times 2 + t-6 = 0$

$\therefore t=2$

$\therefore |\overrightarrow{AP}| = \sqrt{1+1+16} = 3\sqrt{2}$ ✓

$\therefore a=3$.

(c) $|\overrightarrow{BC}| = \sqrt{4+4+1} = 3$

$\therefore A_{ABC} = \frac{1}{2}\cdot 3\cdot 3\sqrt{2} = \frac{9}{2}\sqrt{2}$. ✓

18. Consider the binomial expansion $(1+x)^n = 1 + ax + bx^2 + cx^3 + \cdots + x^n$ where $n > 3$.

(a) Write down expressions for a, b and c in terms of n.

(b) If a, b, c are consecutive terms in an arithmetic sequence calculate the value of n.

(a) $a = \binom{n}{1}$, $b = \binom{n}{2}$, $c = \binom{n}{3}$

(b) $a = \frac{n}{1}$, $b = \frac{n(n-1)}{2}$, $c = \frac{n(n-1)(n-2)}{3}$ 6

$\frac{n(n-1)}{2} - \frac{n}{1} = \frac{n(n-1)(n-2)}{3} - \frac{n(n-1)}{2}$

$\therefore 2n^3 - 12n^2 + 10n = 0$

$\therefore n^3 - 6n^2 + 5n = 0$

$\because n \neq 0$

$\therefore n^2 - 6n + 5 = 0$

$\therefore (n-5)(n-1) = 0$

$\therefore n = 1$ or 5.

$\because n > 3$.

$\therefore n = 5$.

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19. The polynomial $p(x) = x^3 - 3x^2 + kx + 24$ has three distinct real roots, which can be written as $\log_2 a$, $\log_2 b$ and $\log_2 c$ where a, b, c are consecutive terms in a geometric sequence.

(a) Show that one of the roots is equal to 1.

(b) Find the other two roots.

(a) $\begin{cases} x_1 x_2 x_3 = -24 \\ x_1 + x_2 + x_3 = 3 \end{cases}$ $\quad \log_2 a = \log_2 a, \ \log_2 b = \log_2 ar, \ \log_2 c = \log_2 ar^2$

$\therefore \log_2 a^3 r^3 = 3$

$\therefore a^3 r^3 = 8$

$\therefore ar = 2.$

$\therefore \log_2 ar = \log_2 2 = 1 = b.$

(b) $1 - 3 + k + 24 = 0$

$\therefore k = -22$

$\therefore p(x) = x^3 - 3x^2 - 22x + 24$

$\log_2 a \cdot 1 \cdot \log_2 ar^2 = -24$

$\log_2 a \cdot (\log_2 \frac{4}{a}) = -24$

$\log_2 a (\log_2 4 - \log_2 a) = -24$

$\log_2 a \cdot (2 - \log_2 a) + 24 = 0$

$t(2-t) + 24 = 0$

$-t^2 + 2t + 24 = 0$

$t^2 - 2t - 24 = 0$

$(t-6)(t+4) = 0$

$\therefore t = 6$ or -4.

$\because t = \log_2 a$

$\therefore t > 0$

$\therefore \log_2 a = 6$

$\therefore a = 2^6$

$\therefore r = \frac{2}{2^6} = 2^{-5}$

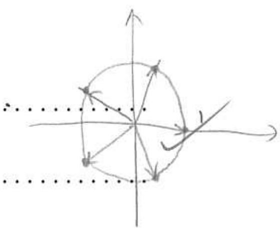
$\therefore \log_2 a = 6,$

$\log_2 c = \log_2 \frac{4}{a} = -4.$

20. (a) Find the fifth roots of unity and sketch them as position vectors in the complex plane.

(b) Hence write $z^4 + z^3 + z^2 + z + 1$ as the product of two quadratic factors with real coefficients.

(c) Hence find the value of the product $\cos \frac{2\pi}{5} \cos \frac{4\pi}{5}$.



(a) $[1,0]^{\frac{1}{5}} \Rightarrow [1,0],\ [1, \frac{2}{5}\pi],\ [1, \frac{4}{5}\pi],\ [1, \frac{6}{5}\pi],\ [1, \frac{8}{5}\pi]$.

(b) $z^5 - 1 = 0$, $z =$ the five roots above

$z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$

$\because z \neq 1$.

$\therefore$ root of $z^4+z^3+z^2+z+1$ is $[1, \frac{2}{5}\pi], [1, \frac{4}{5}\pi], [1, \frac{6}{5}\pi], [1, \frac{8}{5}\pi]$.

$\because [1, \frac{2}{5}\pi] + [1, \frac{8}{5}\pi] = 2\cos\frac{2}{5}\pi$, $[1, \frac{2}{5}\pi] \cdot [1, \frac{8}{5}\pi] = [1, 2\pi] = [1,0] = 1$.

$[1, \frac{4}{5}\pi] + [1, \frac{6}{5}\pi] = 2\cos\frac{4}{5}\pi$, $[1, \frac{4}{5}\pi] \cdot [1, \frac{6}{5}\pi] = [1, 2\pi] = [1,0] = 1$

$\therefore \{ z^4+z^3+z^2+z+1 = (z^2 - 2\cos\frac{2}{5}\pi z + 1)(z^2 - 2\cos\frac{4}{5}\pi z + 1)$

(c) let $z = i$

$z^4+z^3+z^2+z+1 = (-1+1-2\cos\frac{2}{5}\pi i)(-1+1-2\cos\frac{4}{5}\pi i) = 4\cos\frac{2}{5}\pi \cos\frac{4}{5}\pi i^2$

$= 1 - \cancel{i} - \cancel{1} + \cancel{i} + \cancel{1}$

$= 1$

$\therefore -4\cos\frac{2}{5}\pi \cos\frac{4}{5}\pi = 1$

$\therefore \cos\frac{2}{5}\pi \cos\frac{4}{5}\pi = -\frac{1}{4}$