

1. An arithmetic sequence has third term 12 and seventh term 32, find the first term and the common difference.

let first term a , common difference d .

$$\begin{cases} a + (3-1)d = 12 \\ a + (7-1)d = 32 \end{cases}$$

$$4d = 20$$

$$\begin{cases} d = 5 \\ a = 2 \end{cases}$$

$$\begin{array}{ccccccc} 2 & 7 & 12 & 17 & 22 & 27 & 32 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

So the first term is 2 and the common difference is 5.

2. Solve $\log_2(x+2) + \log_2(x-2) = 5$.

$$\log_2(x+2)(x-2) = 5$$

$$(x+2)(x-2) = 2^5 = 32$$

$$x^2 = 29$$

$$x = \pm\sqrt{29}$$

$$\therefore x+2 > 0, x-2 > 0$$

$$\therefore x > 2$$

$$\therefore x = \sqrt{29}$$

3. A geometric sequence has third term 11 and sixth term 297, find the first term and the common ratio.

let the first term be a and the common ratio be r .

$$\begin{cases} a \cdot r^{3-1} = 11 & \textcircled{1} \\ a \cdot r^{6-1} = 297 & \textcircled{2} \end{cases}$$

$$\frac{\textcircled{2}}{\textcircled{1}}: \frac{r^5}{r^2} = 27$$

$$\begin{array}{cccccc} \frac{11}{9} & \frac{11}{3} & 11 & 33 & 99 & 297 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$\therefore r^3 = 27$$

$$\therefore \begin{cases} r = 3 \\ a = \frac{11}{9} \end{cases}$$

\therefore the first term is $\frac{11}{9}$ while the common ratio is 3.

4. The sum of the first n positive integers is 210. Find n .

$$\frac{(1+n) \times n}{2} = 210$$

$$n^2 + n - 420 = 0$$

$$(n+21)(n-20) = 0$$

$$n_1 = -21 \text{ (inadmissible)}$$

$$n_2 = 20$$

$$\therefore n = 20$$

5. Solve $\log_2 x + \log_4 9 = \log_2 12$.

$$\log_2 x + \log_2 9 = \log_2 12$$

$$\log_2 x + \frac{1}{2} \log_2 9 = \log_2 12$$

$$\log_2 x + \log_2 9^{\frac{1}{2}} = \log_2 12$$

$$\therefore \log_2 3x = \log_2 12$$

$$\therefore x = 4$$

6. Solve $9^x - 3^x = 20$.

$$(3^x)^2 - 3^x - 20 = 0$$

$$(3^x - 5)(3^x + 4) = 0$$

$$3^x = 5 \text{ or } -4$$

$$\therefore 3^x > 0$$

$$\therefore 3^x = 5$$

$$\therefore x = \log_3 5$$

$$= 1.46 \text{ (3sf)}$$

7. Let $U = \{n \in \mathbb{Z}^+ \mid 1 \leq n \leq 500\}$. How many numbers in U are neither multiples of 9 nor multiples of 15?

$$n(M_9) = \lfloor 500 \div 9 \rfloor = 55$$

$$n(M_{15}) = \lfloor 500 \div 15 \rfloor = 33$$

$$n(U) = 500$$

$$\text{lcm}(9, 15) = 45$$

$$n(M_{45}) = \lfloor 500 \div 45 \rfloor = 11$$

$$\begin{aligned} n(M_9 \cup M_{15}) &= n(M_9) + n(M_{15}) - n(M_{45}) \\ &= 55 + 33 - 11 \\ &= 77 \end{aligned}$$

$$n(M_9 \cup M_{15})' = 500 - 77 = 423$$

9. 18. 27. 36. 45. ~~54~~ ... 495. 55.
15. 30. 45. 60. ... 495. 33.
45. 90. 135. ... 495. 11.

\therefore There are 423 numbers in U that are neither multiples of 9 nor multiples of 15.

8. Prove the second log law. That is, prove $\log_a(x \div y) = \log_a x - \log_a y$.

let $\log_a x$ be p and $\log_a y$ be q .

then, $a^p = x$, $a^q = y$

$$\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$$

$$\therefore \log_a(x \div y) = \log_a a^{p-q} = p - q$$

$= \log_a x - \log_a y$, as required.

10

9. Find the smallest positive integer x for which the sum $x + 2x + 3x + 4x + \dots + 100x$ is a perfect square.

$$S = x + 2x + \dots + 100x$$

$$= x(1 + 2 + \dots + 100)$$

$$= x \cdot \frac{(1+100) \cdot 100}{2}$$

$$= 5050x$$

$$= 2 \times 5^2 \times 101 \times x$$

$$\therefore x = 2 \times 101$$

$$= 202$$

since it must make all the prime elements of S have even powers.

$$\therefore x = 202.$$

10. Calculate the sum of the series $\sum_{r=1}^{49} \left\lfloor \frac{17r}{50} \right\rfloor$ where $\lfloor x \rfloor$ is the floor of x .

$$f(r) = \left\lfloor \frac{17r}{50} \right\rfloor$$

$$f(1) = f(2) = 0$$

$$f(3) = f(4) = f(5) = 1$$

$$f(6) = f(7) = f(8) = 2$$

$$f(45) = f(46) = f(47) = 15$$

$$f(48) = f(49) = 16.$$

in fact, when $3n \in [3, 49]$

$$f(3n) = f(3n+1) = f(3n+2) = n$$

$$\begin{aligned} \sum_{r=1}^{49} \left\lfloor \frac{17r}{50} \right\rfloor &= 0 \times 2 + 1 \times 3 + 2 \times 3 + \dots + 15 \times 3 + 16 \times 2 \\ &= (1+2+\dots+15) \times 3 + 32 \\ &= \frac{(1+15) \times 15}{2} \times 3 + 32 \\ &= 392 \end{aligned}$$

10

Solutions to HL1 Test #1

1. We have the system

$$\begin{cases} a + 2d = 12 \\ a + 6d = 32. \end{cases}$$

Solving simultaneously give $d = 5$ and $a = 2$.

2. We first note that $x > 2$. Next the first law of logs gives $\log_2(x+2)(x-2) = 5$, whence $x^2 - 4 = 32$ and we conclude $x = 6$.

3. We have the system

$$\begin{cases} ar^2 = 11 \\ ar^5 = 297. \end{cases}$$

Dividing the second equation by the first gives $r^3 = 27$. So $r = 3$ and we conclude $a = 11/9$.

4. The sum of the first n positive integers is $\frac{n}{2}(1+n)$. So we must solve

$$\frac{n}{2}(1+n) = 210 \Leftrightarrow n(n+1) = 420$$

whence $n = 20$.

5. Choosing a base of 2, the equation becomes $\log_2 x + \frac{1}{2} \log_2 9 = \log_2 12$, which becomes $\log_2 3x = \log_2 12$. We conclude $x = 4$.

6. Letting $y = 3^x$ gives the quadratic equation $y^2 - y - 20 = 0$, which has solutions $y = 5$ and $y = -4$. We conclude $3^x = 5$, whence $x = 1.46$ (3 s.f.).

7. $n(M_9 \cup M_{15}) = n(M_9) + n(M_{15}) - n(M_{45})$, from which we conclude $n(M_9 \cup M_{15}) = 55 + 33 - 11 = 77$. Hence $n(M'_9 \cap M'_{15}) = 500 - 77 = 423$.

8. See our red book page 136.

9. The sum of this series is $x(1 + 2 + 3 + \dots + 100) = x \times 5050 = 2 \times 5^2 \times 101 \times x$. So the smallest value of x for this sum to be a perfect square is $2 \times 101 = 202$.

10. Some calculation, most likley using the GDC, gives the series in expanded form as

$$0 + 0 + (1 + 1 + 1) + (2 + 2 + 2) + (3 + 3 + 3) + \dots + (15 + 15 + 15) + 16 + 16,$$

and hence the sum is $3 \times (1 + 2 + 3 + \dots + 15) + 32 = 392$.