FM2 Test #3 Name: Jerry Jiang

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1. Is the group $(\mathbb{Z}_7^*, \otimes)$ cyclic? Justify your answer.

According to the Cayley table on the right,

$\mathbb{Z}_7^* = \langle 3 \rangle = \langle 5 \rangle$.

It's cyclic. ✓

$\otimes$	1	2	3	4	5	6
1	1	2	3	4	[illegible]	[illegible]
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Not

V. Good

2. Find $\lim\limits_{n\to\infty} \dfrac{\pi}{2n}\left(1+\cos\dfrac{\pi}{2n}+\cos\dfrac{2\pi}{2n}+\cdots+\cos\dfrac{(n-1)\pi}{2n}\right)$.

$$L = \lim_{n\to\infty} \frac{1}{n}\left[\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\cdot\frac{1}{n}+\cos\frac{\pi}{2}\cdot\frac{2}{n}+\cdots+\cos\frac{\pi}{2}\cdot\frac{n-1}{n}\right)\right]$$

$= \lim\limits_{n\to\infty} U_n$ for $f(x) = \frac{\pi}{2}\cos\frac{\pi}{2}x$ from 0 to 1.

$= \displaystyle\int_0^1 \frac{\pi}{2}\cos\frac{\pi}{2}x\,dx$ ✓ GDC

$= 1$ ✓

$\frac{1}{n}\cdot\left[\frac{\pi}{2}\left(1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\cdots+\cos\frac{(n-1)\pi}{2n}\right)\right]$

$\frac{\pi}{2}\cos\frac{\pi}{2}x$

$\frac{\pi}{2}\cdot\left(-\sin\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}$

$\frac{\pi}{2}\cdot\sin\frac{\pi}{2}x$

3. For each of the following either explain why the graph cannot exist or draw a graph with the given property.

(a) A graph with degree sequence 3, 2, 2, 1, 1.

$\sum \deg(v) = 2 \times \#$ of edges $=$ a even number.

But $3+2+2+1+1 = 9$ is a odd number. ✓

Therefore the graph doesn't exist.

(b) A complete bipartite graph on 5 vertices that has a Hamiltonian path and an Eulerian trail.

Case ①: $K_{2,3}$  not satisfied.

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Case ②: $K_{1,4}$  not satisfied.

Therefore, such graph doesn't exist.

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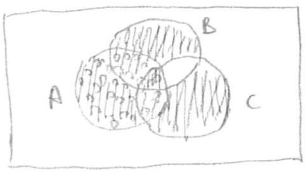
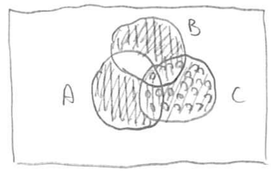
4. Draw Venn diagrams illustrating the sets $A \triangle (B \triangle C)$ and $(A \triangle B) \triangle C$. What is your conclusion?

$A \triangle (B \triangle C)$ $\qquad\qquad$ $(A \triangle B) \triangle C$





$A \triangle (B \triangle C) = (A \triangle B) \triangle C.$

So we can see that the operation $\triangle$ is associative.

5. The space $S = \langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rangle$ is a subspace of $\mathbb{R}^3$. Find a Cartesian equation for S.

$S = a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

$\therefore \begin{cases} x = a + 3b \\ y = 2a + 2b \\ z = 3a + b. \end{cases}$

$(a+3b) + (3a+b) - 2(2a+2b) = 0$

$\therefore \quad x - 2y + z = 0.$

$\frac{576}{576} + \frac{24}{576} - \frac{1}{576}$

6. Use the first three terms in the binomial expansion of $(1+\frac{1}{8})^{1/3}$ to find an approximation to $\sqrt[3]{9}$. Give your answer as a fraction in simplest terms.

$$\left(1+\frac{1}{8}\right)^{\frac{1}{3}} \approx 1 + \frac{\frac{1}{3}}{1}\cdot\frac{1}{8} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\cdot 2}\cdot\frac{1}{8^2} = 1 + \frac{1}{24} + \left(-\frac{1}{9}\right)\cdot\frac{1}{64} = \frac{576+24-1}{576} = \frac{599}{576}$$

$$\sqrt[3]{\frac{9}{8}} = \frac{\sqrt[3]{9}}{2} \approx \frac{599}{576}.$$

Therefore, $\sqrt[3]{9} \approx \frac{599}{576} \times 2 = \frac{599}{288}.$

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7. Determine the rank, nullity and null space of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}$.

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

row space $= \langle (1\ 0\ -1\ -2\ -3), (0\ 1\ 2\ 3\ 4) \rangle$,

column space $= \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle$.

$\therefore$ rank $(A) = 2$.

Let $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. and let $x_3 = r$, $x_4 = s$, $x_5 = t$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = r \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

null space $= \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \right\rangle$ ✗ null space $\subseteq \mathbb{R}^5$ not $\mathbb{R}^3$

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$\therefore$ nullity $(A) = 3$.

$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ $\quad 1 - \frac{\frac{\pi^2}{9}}{2} + \frac{\frac{\pi^4}{81}}{24}$

8. Use the fourth degree Maclaurin polynomial for $\cos x$ to show that $\pi/3$ approximately satisfies the equation $x^4 - 12x^2 + 12 = 0$. Hence calculate an approximate value for π expressing your answer as a surd.

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, $\cos\frac{\pi}{3} = \frac{1}{2} \approx P_4\left(\frac{\pi}{3}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{3}\right)^2 + \frac{1}{24}\left(\frac{\pi}{3}\right)^4$

$\therefore \frac{1}{2} = 1 - \frac{1}{2}\left(\frac{\pi}{3}\right)^2 + \frac{1}{24}\left(\frac{\pi}{3}\right)^4$

$\therefore \left(\frac{\pi}{3}\right)^4 - 12\left(\frac{\pi}{3}\right)^2 + 12 = 0$,

Therefore $x = \frac{\pi}{3}$ is an approximate solution of $x^4 - 12x^2 + 12 = 0$.

Solving gives $x = \pm\sqrt{6 \pm 2\sqrt{6}}$.

The value in range of $0 - 2$ is $\sqrt{6 - 2\sqrt{6}}$.

Thus a good approximation of $\frac{\pi}{3}$ is $\sqrt{6 - 2\sqrt{6}}$. ✓

$(x^2 - 6)^2 = 24$

$x^2 - 6 = \pm 2\sqrt{6}$

$x^4 - 12x^2 + 36 = 24$

$x^2 = 6 \pm 2\sqrt{6}$

$x = \pm\sqrt{6 \pm 2\sqrt{6}}$

$\Delta = 144 - 48 = 96 = 16 \times 6$

$x = \frac{12 \pm 4\sqrt{6}}{2} = 6 \pm 2\sqrt{6}$

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9. Suppose $f\colon G \to H$ is a group homomorphism. Prove $\ker(f) \le G$.

- let $a, b \in \ker(f)$.
  According to homomorphism, $f(a*b) = f(a)\circ f(b) = e'\circ e' = e'$.
  Thus $a*b \in \ker(f)$. closure ✓
- According to the definition of homomorphism, $f(e) = e'$, ← This is a theorem.
  Hence $e \in \ker(f)$. identity ✓
- let $a \in \ker(f)$ and $a^{-1} \in G$.
  According to the characteristic of homomorphism saying that if $f(x) = x'$, then $f(x^{-1}) = (x')^{-1}$, we know that: $f(a^{-1}) = (e')^{-1} = e'$.
  So $a^{-1} \in \ker(f)$. inverse ✓

According to the 3-step subgroup test, $\ker(f) \le G$. ✓

10. Evaluate $\lim_{x\to 0}(1+\sin x)^{1/x}$.

$(1+\sin x)^{\frac{1}{x}} = 1 + \frac{1}{x}\cdot\sin x + \frac{\frac{1}{x}(\frac{1}{x}-1)}{2!}\sin^2 x$

According to Binomial Expansion:

$$(1+\sin x)^{\frac{1}{x}} = 1 + \frac{1}{x}\sin x + \frac{\frac{1}{x}(\frac{1}{x}-1)}{2!}\sin^2 x + \cdots$$

$$= 1 + \sum_{k=1}^{x}\binom{\frac{1}{x}}{k}\sin^k x, \quad \text{Let } U_n = \binom{\frac{1}{x}}{n}\sin^n x$$

Ratio Test for the summation:

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{\frac{\frac{1}{x}\cdots(\frac{1}{x}-n+1+1)}{(n+1)!}\sin^{n+1}x}{\frac{\frac{1}{x}\cdots(\frac{1}{x}-n+1)}{n!}\sin^n x}\right| = \left|\frac{(\frac{1}{x}-n)}{(n+1)}\cdot\sin x\right| = |-\sin x| \text{ as } n\to\infty.$$

so $\left|\frac{U_{n+1}}{U_n}\right| < 1$, which means convergence for all x.

$$\therefore \lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = 1 + \lim_{x\to 0}\sum_{k=1}^{x}\binom{\frac{1}{x}}{k}\sin^k x = 1 + 0 = 1.$$

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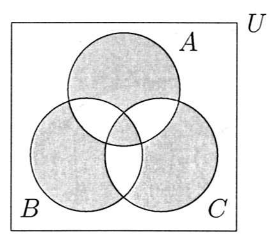
Solutions to FM2 Test #3

1. Since $\mathbb{Z}_7^* = \langle 3 \rangle$, $\mathbb{Z}_7^*$ is cyclic. (The only other generator of $\mathbb{Z}_7^*$ is 5.)

2. We recognize this limit as $\lim_{n\to\infty} L_n$ for the integral $\int_0^{\pi/2} \cos x\, dx$, which evaluates to 1.

3. (a) Not possible as a graph has an even number of vertices of odd degree. (b) $K_{3,2}$ fulfills the criteria.

4. The Venn diagrams for $A \triangle (B \triangle C)$ and $(A \triangle B) \triangle C$ are the same. The common result is illustrated below. Hence the operation of symmetric difference is associative on sets.



5. $x - 2y + z = 0$.

6. First observe $\sqrt[3]{9} = (8+1)^{1/3} = 2(1+\frac{1}{8})^{1/3}$. The first three terms of this binomial expansion give

$$(1+\tfrac{1}{8})^{1/3} \approx 1 + \frac{1}{3}\cdot\frac{1}{8} + \frac{\frac{1}{3}\cdot\frac{-2}{3}}{2!}\cdot\frac{1}{64} = \frac{599}{576}.$$

We conclude $\sqrt[3]{9} \approx \dfrac{599}{288}$.

7. Using the GDC to find rref(A) we conclude rank$(A) = 2$ and therefore by the rank-nullity theorem nullity$(A) = 3$. The null space is

$$\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \rangle.$$

8. The fourth degree Maclaurin polynomial for $\cos x$ is $P_4(x) = 1 - x^2/2! + x^4/4!$. Now $\cos(\pi/3) = 0.5$. So $P_4(\pi/3) \approx 0.5$. That is $\pi/3$ approximately satisfies the equation

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} = 0.5 \Leftrightarrow x^4 - 12x^2 + 12 = 0.$$

Calculation gives the appropriate root as $\sqrt{6-2\sqrt{6}}$. Hence $\pi \approx 3\sqrt{6-2\sqrt{6}}$.

9. See class notes.

10. Notice that this limit has the indeterminate form 1^∞. The standard approach for such a limit is to use logarithms. Letting $y = (1+\sin x)^{1/x}$ gives $\ln y = \dfrac{\ln(1+\sin x)}{x}$. Now using l'Hôpital's rule we have

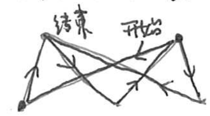
$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{1} = 1.$$

So our required limit is e.

Further Math Test 3 纠错.

3 (b). 读题: Hamiltonian Path 和 Eulerian Trail 而不是 Hamiltonian Cycle 和 Eulerian Circuit. 所以不用覆盖所有的点和边后回到出发点.

Hamiltonian Path



Eulerian Trail.

7. $$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\times(-1)$ $\times(-1)$ $\times(-1)$

$\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 3$.

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = r\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ as } x_3 = r,\ x_4 = s,\ x_5 = t.$$

$$\text{null space} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

✲: null space 求的方法不熟, 以后每次考前都要动笔做一次不能光想.

10. ✲: 本来就不会, 但考试时把 $x\to 0$ 看成 $x\to\infty$ 是不应该的.

Let $y = \lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}}$,

Then $\ln y = \lim\limits_{x\to 0} \frac{1}{x}\ln(1+\sin x) = \lim\limits_{x\to 0} \frac{\ln(1+\sin x)}{x} \overset{\text{L'Hô}}{=} \lim\limits_{x\to 0} \frac{\frac{1}{1+\sin x}}{1} = \frac{1}{1+0} = 1.$

$\therefore\ y = e^1 = e.$

反思: 这次考出了来到Pearson后的数学最低分, 心里不爽的同时也意识到这里考试频率太低, 学生水平太差导致自己无意识地漂起来了。现在这种关键的时候有空时应去思考文书, 有考试时应该认真复习而不是低效率地度过时间。

另外这次 3(b) 和 7 都属于会做的题错了的那种, 以后考试一定要注意看题, 动笔复习, 错这种乱七八糟的东西真的不应该。这让我想到了在华育拿到数学卷子发现考得一塌糊涂的原因是错了一堆不该错的题的心塞的感觉。发现初中毕业之后因为考试少了, 简单了, 老师批得松了, 粗心的影响就没以前这么大了, 但仔细一想在 CSC 和 Pearson 数学没拿满分的情况大多也是因为这个, 但因为分数还是很好看所以没放在心上。现在知道了那以后简单考试的目标不应是 95+ 而是 100 分整, 否则真的不能说自己数学好, 只能算是一个简单的卷子考得马马虎虎但没有小心谨慎的人罢了。 19/11/8日.