**FM2** Assignment #5 Name: Jerry Jiang $9\frac{1}{2}/10$ Very Good.

1. The adjacency matrix of graph $G$ is $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. What information do the diagonal elements of $A^2$ give?

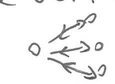
$$A^2 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

In graph $G$,

There're 3 walks of length 2 from A to A and from C to C each.

There're 2 walks of length 2 from B to B and from D to D each. $1\frac{1}{2}$

the degree of the vertex.



2. The circle group $T = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ is a subgroup of $\mathbb{C}^*$. Give a geometric description of the coset $(3+4i)T$.

$(3+4i) = 5e^{i\arctan(\frac{4}{3})}$

$\therefore (3+4i)T = 5e^{i[\theta+\arctan(\frac{4}{3})]} = 5e^{i\alpha}$ where $\alpha \in \mathbb{R}$.

Therefore, the coset is a circle with radius 5 centered at the origin of the complex plane.

3. A quadrilateral has vertices $A(-1,5)$, $B(4,7)$, $C(7,-1)$ and $D(-2,1)$. Find the coordinates of the point $P$ such that $PA = PC$ and $PB = PD$.

- $l_{AC}$: $y = -\frac{3}{4}x + \frac{17}{4}$, $M_{AC}(3,2)$;
- $l_{BD}$: $y = x+3$, $M_{BD}(1,4)$.
- perpendicular bisector of AC: $l_{\perp AC}$: $y = \frac{4}{3}x + c$.

  $\frac{4}{3}\cdot 3 + c = 2$, $c = -2$.

  $\therefore l_{\perp AC}$: $y = \frac{4}{3}x - 2$.
- perpendicular bisector of BD: $l_{\perp BD}$: $y = -x + d$.

  $-1 + d = 4$, $d = 5$.

  $\therefore l_{\perp BD}$: $y = -x+5$.

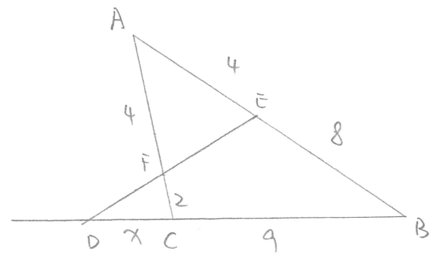
$\begin{cases} y = \frac{4}{3}x - 2 \\ y = -x+5 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 2 \end{cases}$

Since $(3,2)$ pass through the perpendicular bisectors of AC and BD, it satisfies PA=PC & PB=PD.

$5\frac{1}{2}$

4. In $\triangle ABC$, $a = 9$, $b = 6$ and $c = 12$. A circle with centre $A$ and radius 4 meets sides $[AB]$ and $[AC]$ at $E$ and $F$ respectively. The secant $(EF)$ meets $(BC)$ at $D$. Use Menelaus's theorem to calculate the length $CD$.

$$\frac{AE}{EB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA} = -1$$




$$\frac{4}{8} \cdot \frac{9+x}{-x} \cdot \frac{2}{4} = -1$$

$$\therefore \frac{9+x}{x} = 4, \quad x = 3$$

Therefore $CD = 3$.

5. Let $G = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \text{ and } a^2 + b^2 \neq 0\}$. Show that the groups $(G, \times)$ and $(\mathbb{C}^*, \times)$ are isomorphic.

$$G \xrightarrow{f} C^*,$$

$$f\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) = a+bi.$$

$$f\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \times \begin{pmatrix} c & -d \\ d & a \end{pmatrix}\right) = f\left(\begin{pmatrix} ac-bd & -bc-ad \\ bc+ad & ac-bd \end{pmatrix}\right) = (ac-bd)+(bc+ad)i = (a+bi)\cdot(c+di) = f\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) \times f\left(\begin{pmatrix} c & -d \\ d & c \end{pmatrix}\right).$$

① surjection.

$\because a^2+b^2 \neq 0$

$\therefore$ at least one of $a$ and $b$ is not $0$.

Therefore for any $w \in G$, $f(w) \neq 0+0i$.

$\because C^* = C \setminus \{0+0i\}$

$\therefore$ Surjection.

For any $z = m+ni \in C^*$, there's a $w = \begin{pmatrix} m & -n \\ n & m \end{pmatrix} \in G$ that maps to $z$, given that $m, n \in R$, $m^2+n^2 \neq 0$.

② injection.

Suppose $f(w_1) = f(w_2) = z = m+ni$, where $w_1 \neq w_2$.

then $w_1 = \begin{pmatrix} m & -n \\ n & m \end{pmatrix}$, $w_2 = \begin{pmatrix} m & -n \\ n & m \end{pmatrix}$.

$w_1 = w_2$. contradiction.

Therefore $f$ has injection.

In conclusion, $(G, \times) \cong (C^*, \times)$

4