HL1 Assignment #4

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100%

Excellent!

1. How many positive divisors does 1071 have?

$1071 = 3^2 \times 7 \times 17$

$(2+1) \times (1+1) \times (1+1)$

$= 3 \times 2 \times 2$

$= 12$

2. Find integers x and y such that $3^x 6^y = 24$.

$3^x \cdot (2 \cdot 3)^y = 24$

$= 3^{x+y} \cdot 2^y$

$24 = 2^3 \cdot 3$

$\begin{cases} x+y = 1 \\ y = 3 \end{cases}$

$\begin{cases} x = -2 \\ y = 3 \end{cases}$

3. Let $U = \{n \in \mathbb{Z} \mid 1 \leq n \leq 2018\}$, $A = \{n \in U \mid n \text{ is odd}\}$ and $B = \{n \in U \mid n \text{ is a multiple of } 3\}$. Find $n(A \cap B')$.

The question is equivalent to: In positive integers from 1 to 2018, inclusive, find the number of numbers which is neither multiple of 2 nor multiple of 3.

$n(A') = 2018 \div 2 = 1009$

$n(B) = \lfloor 2018 \div 3 \rfloor = 672$

$n(A' \cap B) = \lfloor 2018 \div 6 \rfloor = 336$

$n(A' \cup B) = 1009 + 672 - 336$

$= 1345$

$n(\text{Required}) = n(U) - n(A' \cup B)$

$= 2018 - 1345$

$= 673.$

4. Find the first term of the geometric sequence $6, 6\sqrt{2}, 12, 12\sqrt{2}, \ldots$ that exceeds 2018.

$r=\sqrt{2}$

$a_1=6.$

$6 \cdot \sqrt{2}^{n-1} \geq 2018.$

$n \geq 1+\log_{\sqrt{2}} \frac{2018}{6}$

$n \geq 17.8$

$\therefore n=18.$

$a_{18} = 6 \cdot \sqrt{2}^{17}$

$= 6 \cdot 2^{8} \cdot \sqrt{2}$

$= 1536\sqrt{2}$

5. Find the sum of the first 20 terms of the sequence defined recursively by $u_n = 9/u_{n-1}$ and $u_1 = 9$.

$U_2 = \frac{9}{9} = 1$

$U_3 = \frac{9}{1} = 9$

$U_4 = \frac{9}{9} = 1$

$U_5 = \frac{9}{1} = 9$

$(9+1) \times \frac{20}{2}$

$= 100$

6. What is the units digit of the sum $1! + 2! + 3! + 4! + \cdots + 2018!$?

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 4$

$5! = 0$

$6! = 0$

$7! = 0$

$8! = 0$

$9! = 0$

$10! = 0$

$11! = 0$

$12! = 0$

...

as long as there's a 2 and a 5 in the term, the unit digit will be 0.

so. $1+2+6+4=13$

so the units digit of the sum is 3.

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7. Solve $\log_2(x+1) - \log_4(3x-1) = 0.5$.

$\log_2(x+1) - \frac{1}{2}\log_2(3x-1) = 0.5$.

$\log_2(x+1) - \log_2(3x-1)^{\frac{1}{2}} = 0.5$

$\log_2 \frac{(x+1)}{(3x-1)^{\frac{1}{2}}} = 0.5$.

$\frac{(x+1)}{(3x-1)^{\frac{1}{2}}} = \sqrt{2}$

$\frac{(x+1)^2}{(3x-1)} = 2$

$x^2+2x+1 = 6x-2$

$x^2-4x+3=0$

$(x-3)(x-1)=0$

$x_1=3$

$x_2=1$

$\because x+1>0$

$3x-1>0$

$\therefore x>\frac{1}{3}$

$\therefore x=3$ or 1.

8. Is there a prime number p that satisfies the inequality $2018! + 2 \leq p \leq 2018! + 2018$? Justify your answer.

NO.

$2018!+2$ is the multiple of 2 since $2(1+1\times3\times4\times\cdots\times2018)$

$2018!+3$ is the multiple of 3 since $3(1+1\times2\times4\times\cdots\times2018)$

$2018!+4$ is the multiple of 4 since $4(1+1\times2\times3\times5\cdots\times2018)$

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$2018!+2017$ is the multiple of 2017 since $2017(1+1\times2\times3\times\cdots\times2016\times2018)$

$2018!+2018$ is the multiple of 2018 since $2018(1+2017!)$

So the 2017 integers, starting from $2018!+2$ and end with $2018!+2018$ are all composite numbers

So there is no possible p that satisfies the inequality above as a prime number.

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9. Solve $x^{\log x} = \dfrac{x^3}{100}$.

$$\log_x\left(x^{\log x}\right) = \log_x\left(\frac{x^3}{100}\right)$$

$$\log x = \log_x x^3 - \log_x 100$$

$$\log x = 3 - \frac{\log 100}{\log x}$$

$$\log x = 3 - \frac{2}{\log x}$$

$$(\log x)^2 - 3\log x + 2 = 0$$

$$(\log x - 2)(\log x - 1) = 0$$

$$\log x = 1 \text{ or } 2$$

$$x = 10 \text{ or } 100.$$

10. The number 1071 is equal to $a060$ in base b. Find the values of a and b.

$$6 \times b + a \cdot b^3 = 1071, \quad (a, b \in \mathbb{Z}^+, 1 \le a, b \le 9)$$

$$b(6 + ab^2) = 1071$$

$\because$ the integer divisor of 1071 in range 1 to 9 is 3, 7, 9.

$\therefore$ since there's a '6' in 'a060',

the base be have to be no less than 7.

① $b = 7$.

$$6 + a \cdot 49 = 153$$

$$a \cdot 49 = 147$$

$$a = 3$$

② $b = 9$

$$6 + 81a = 119$$

$$a = 1.40$$

(inadmissable)

$$\therefore \begin{cases} a = 3 \\ b = 7. \end{cases}$$

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Solutions to HL1 Assignment #4

1. Since $1071 = 3^2 7^1 17^1$, we have $\tau(1071) = 3 \cdot 2 \cdot 2 = 12$.

2. We have $3^x 6^y = 3^x 2^y 3^y = 2^y 3^{x+y} = 24 = 2^3 3^1$. So $y = 3$ and $x + y = 1$. We conclude $x = -2$, $y = 3$.

3. Observe $A \cap B = \{3, 9, 15, \ldots, 2013\}$. So $n(A \cap B) = 336$. Since $n(A) = 1009$, we conclude $n(A \cap B') = n(A) - n(A \cap B) = 1009 - 336 = 673$.

4. Here $u_n = 6 \times (\sqrt{2})^{n-1}$. Solving $u_n > 2018$ gives $n_{\min} = 18$. So the first term to exceed 2018 is $u_{18} = 1536\sqrt{2}$.

5. The sequence to 20 terms is $1, 9, 1, 9, 1, 9, \ldots, 9$. So $S_{20} = 10 \times (1 + 9) = 100$.

6. Observe that the units digit for $n!$ when $n > 4$ is 0. So the units digit of this sum is the same as the units digit of $1! + 2! + 3! + 4!$, which is 3.

7. We first note that any solution must satisfy $x > 1/3$. Next changing base gives $\log_2(x+1) - \frac{1}{2}\log_2(3x - 1) = 0.5$, which gives in turn
$$\log_2 \frac{(x+1)^2}{3x-1} = 1.$$
So $(x+1)^2 = 2(3x-1) \Leftrightarrow x^2 - 4x + 3 = 0 \Leftrightarrow x = 1$ or $x = 3$, and both of these solutions are valid.

8. There is no such prime number since $2018! + k$ where $2 \leq k \leq 2018$ is divisible by k.

9. Taking logs of both sides we have $\log x \times \log x = 3\log x - 2$. Letting $y = \log x$ gives the quadratic equation $y^2 - 3y + 2 = 0$, whence $y = 1$ or $y = 2$. So $x = 10$ or $x = 100$.

10. Recall $1071 = 3^2 \times 7 \times 17$. Now 1071 in base b has 0 as its units digit, so b must be a divisor of 1071. Next $a060$ contains a 6, so the base b cannot be 3. Also a base of 17 or more would not give a four digit representation for 1071 as $17^3 > 1017$. So our remaining candidates for b are 7 and 9. Since $1071 = 1420_9$, we conclude $b = 7$. Lastly $1071 = 3060_7$, so $a = 3$.