HL1 Assignment #18

Name: Jerry Jiang

100%

Excellent !!

1. Write the set $\{x \in \mathbb{R} \mid |x+2| < 3\}$ in interval notation.

$-3 < x+2 < 3$

$-5 < x < 1$

$x \in]-5, 1[.$

2. Without the calculator solve $\sin 2x = \cos x$ for $x \in [-2\pi, 2\pi]$.

$2 \sin x \cos x = \cos x$

① $\cos x = 0.$ ② $\cos x \neq 0.$

$x = \frac{\pi}{2}$ or $\frac{3}{2}\pi \pm 2k\pi$ $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ or $\frac{5}{6}\pi \pm 2k\pi$

$\therefore x = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{\pi}{6}, \frac{5}{6}\pi, -\frac{3}{2}\pi, -\frac{1}{2}\pi, -\frac{11}{6}\pi, -\frac{7}{6}\pi$

3. The stem and leaf diagram shows the IQ scores for a class of elementary students.

Stem	*Leaf*	
9	3 8	2
10	2 5 8	3
11	1 3 3 6 7 8	6
12	2 5 5 5 6 7 8	7
13	3 4 4 9	4
14	2 4	2
15	1	1

Scale: $9 \mid 3 = 93$

Calculate the mean IQ for the class.

$\bar{x} = \frac{2930 + 119}{25} = 121.96$

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4. Solve $3\log_2 x - 6\log_x 2 = 7$.

let $\log_2 x = a$.

$3a - \frac{6}{a} = 7$

$3a^2 - 7a - 6 = 0$

$(3a+2)(a-3) = 0$

$a_1 = -\frac{2}{3}$

$a_2 = 3$

$x_1 = 2^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{4}}$

$x_2 = 2^3 = 8$

✓

5. Find the equation of the normal to the curve $y = x^3 + x^2 + x + 1$ at the point where $x = -1$.

$f'(x) = 3x^2 + 2x + 1$

$f'(-1) = 3 - 2 + 1 = 2$

$\therefore l: y = -\frac{1}{2}x + b,$

$(-1, 0)$

$\frac{1}{2} + b = 0$

$b = -\frac{1}{2}$

$\therefore l: y = -\frac{1}{2}x - \frac{1}{2}$

✓

6. Write the series $1 \cdot 2 - 4 \cdot 5 + 7 \cdot 8 - 10 \cdot 11 + 13 \cdot 14 - \cdots - 100 \cdot 101$ in sigma notation.

$\sum_{i=0}^{33} (-1)^i (3n+1)(3n+2)$ ✓

7. Find the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$ from first principles.

$$f(x) = \frac{1}{\sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x} \cdot h} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \cdot \frac{\cancel{x} - \cancel{x} - \cancel{h}}{\sqrt{x}\sqrt{x+h} \cdot \cancel{h} \cdot (\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \cdot \frac{-1}{x\sqrt{x+h} + \sqrt{x}(x+h)}$$

$$= -\frac{1}{2\sqrt{x^3}}$$

$$= -\frac{1}{2} x^{-\frac{3}{2}}$$

✓

8. Prove $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

$$\tan 3\theta = \tan(\theta + 2\theta)$$

$$= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \cdot \tan 2\theta}$$

$$= \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta \cdot \frac{2\tan\theta}{1-\tan^2\theta}}$$

$$= \frac{\tan\theta - \tan^3\theta + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

✓

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9. Use the above result to show that the roots of the equation $t^3 - 3t^2 - 3t + 1 = 0$ are $\tan\alpha$, $\tan 5\alpha$ and $\tan 9\alpha$ where α is to be determined.

we have:

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

let $\tan 3\theta = b$,
$\tan\theta = a$,

we have:

$b - 3b\cdot a^2 = 3a - a^3$

$\therefore a^3 - 3b\cdot a^2 - 3a + b = 0$

if $b = 1$,

then $a^3 - 3a^2 - 3a + 1 = 0$

a is root of the equation

so $\tan 3\theta = 1$

$3\theta = \frac{\pi}{4} \pm \pi$

$\theta = \frac{\pi}{12} \pm \frac{1}{3}\pi$

$\therefore \theta = \frac{\pi}{12}$

and $\tan\frac{\pi}{12}$, $\tan\frac{5\pi}{12}$, $\tan\frac{9\pi}{12}$ are the three roots of the cubic equation.

✓

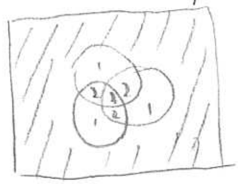
10. How many permutations of the word PEARSON neither fix the letter P, nor the letter R, nor the letter N?

$7! - (3\times 6! - 3\times 5! + 4!)$

$= 5040 - 2160 + 360 - 24$

$= 3216$

the principle of inclusion & exclusion PIE



P : 6!
R : 6!
N : 6!

$P\cap R$: 5!
$P\cap N$ 5!
$N\cap R$: 5!

$P\cap R\cap N = 4!$

$U - (P + R + N - P\cap R - P\cap N - N\cap R + P\cap R\cap N)$

✓

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Solutions to HL1 Assignment #18

1. $]-5, 1[$

2. $2\sin x \cos x - \cos x = 0$, whence $\cos x = 0$ or $\sin x = \frac{1}{2}$. Hence $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$.

3. 121.96

4. Letting $u = \log_2 x$ we have $3u - \frac{6}{u} = 7$, whence $3u^2 - 7u - 6 = 0$. So $\log_2 x = -\frac{2}{3}$ or $\log_2 x \doteq 3$. Hence $x = 0.630$ (3 s.f.) or $x = 8$.

5. Here $y' = 3x^2 + 2x + 1$. So $y'(-1) = 2$ and $m_N = -\frac{1}{2}$. Hence $N\colon y - 0 = -\frac{1}{2}(x+1)$.

6. $\displaystyle\sum_{n=0}^{33}(-1)^n(3n+1)(3n+2)$

7. $f'(x) = \displaystyle\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0}\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$. Rationalizing the numerator gives

$$f'(x) = \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})},$$

whence $f'(x) = \dfrac{-1}{2x\sqrt{x}} = -\frac{1}{2}x^{-3/2}$.

8. $\tan 3\theta = \tan(\theta + 2\theta) = \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}$. Letting $t = \tan\theta$ gives

$$\tan 3\theta = \frac{t + \frac{2t}{1-t^2}}{1 - \frac{2t^2}{1-t^2}} = \frac{3t - t^3}{1 - 3t^2},$$

as required.

9. Using the above result with $\tan 3\theta = 1$ gives $3t - t^3 = 1 - 3t^2$ or $t^3 - 3t^2 - 3t + 1 = 0$. So the solutions to the given equation are the solutions to $\tan 3\theta = 1$, whence $t = \tan\alpha, \tan 5\alpha, \tan 9\alpha$ where $\alpha = \frac{\pi}{12}$.

10. $n(P' \cap R' \cap N') = 7! - n(P \cup R \cup N) = 7! - (3\cdot 6! - 3\cdot 5! + 4!) = 3216$.