HL1 Test #4

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1. The polynomial $x^3 + 3x^2 + ax - 1$ leaves remainder 5 on division by $x - 2$. Find the value of a.

$f(2) = 8 + 12 + 2a - 1 = 5$

$2a = 5 - 19 = -14$

$a = -7$

100%

Bravo!!

2. The polynomial $7x^4 + bx^3 + cx^2 + dx + e$ has real coefficients and roots $2 + i$ and $1 - 3i$. Find the value of e.

roots are $2+i$, $2-i$.

$1-3i$, $1+3i$.

$2+i + 2-i = 4$ $\quad$ $1-3i + 1+3i = 2$

$(2+i)(2-i) = 5$ $\quad$ $(1-3i)(1+3i) = 10$

$x^2 - 4x + 5 = 0$ $\quad$ $x^2 - 2x + 10 = 0$

$\therefore (x^2 - 4x + 5)(x^2 - 2x + 10)$

$= x^4 - 2x^3 + 10x^2 - 4x^3 + 8x^2 - 40x + 5x^2 - 10x + 50$

$= x^4 - 6x^3 + 23x^2 - 50x + 50$

$x^4 - 6x^3 + 23x^2 - 50x + 50$

$= 7x^4 - 42x^3 + (23 \cdot 7)x^2 - 350x + 350$

$\therefore e = 350$

3. How many bit strings of length 8 contain five 0's and three 1's with no two 1's adjacent?

The five 0's becomes the isolater between the 1's, leaving 6 potential spots.

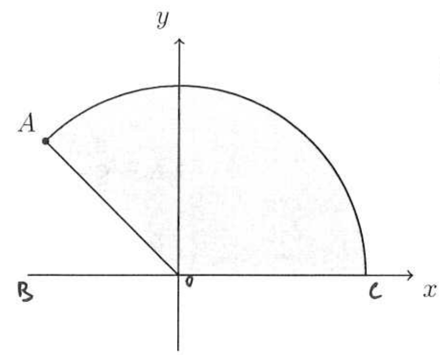
_ 0 _ 0 _ 0 _ 0 _ 0 _

$^6C_3 = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$

4. Find the area of the shaded sector if the point A has coordinates $(-6\sqrt{2}, 6\sqrt{2})$.

$\angle AOB = \arctan\left(\frac{|Y_A|}{|X_A|}\right) = \arctan(1) = 45°$

$\therefore \angle AOC = 180° - 45° = 135°$



$S = \frac{135°}{360°} \cdot \pi r^2$

$= \frac{3}{8} \cdot \pi AO^2$

$AO^2 = (-6\sqrt{2})^2 + (6\sqrt{2})^2 = 12^2 = 144$

$\therefore S = 54\pi$

$= 170.$ (3sf)

5. The constant term in the expansion of $\left(x - \frac{2}{x^2}\right)^9$ is an integer. Find its value.

$\binom{9}{3} \cdot x^6 \cdot \left(-\frac{2}{x^2}\right)^3 = \binom{9}{3} \cdot x^6 \cdot \left(-\frac{8}{x^6}\right) = -8 \cdot \binom{9}{3} = -8 \cdot \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = -672$

6. Solve $\tan\theta - 2\cot\theta = 1$ for $0° < \theta < 360°$.

let $\tan\theta = a$.

$a - \frac{2}{a} = 1$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$a_1 = 2,$

$a_2 = -1$

$\therefore \tan\theta = 2$ or -1

① $\tan\theta = 2$

$\theta^* = 63.4°$ $(\pm 180n°)$

$\therefore \theta = 63.4°$ or $243°$ (3sf).

② $\tan\theta = -1$

$\theta^* = 135°$ $(\pm 180n°)$

$\therefore \theta = 135°$ or $315°$

$\therefore \theta = 63.4°$ or $243°$ or $135°$ or $315°$.

7. Solve $4^x = 2^{x+2} - 3$.

$2^{2x} - 4\cdot 2^x + 3 = 0$

$(2^x - 3)(2^x - 1) = 0$

$\therefore\ 2^x = 3$ or 1

$\therefore\ x = \log_2 3$ or $\log_2 1$

$\therefore\ x = 1.58$ or 0
(3 s.f.).

✓

8. Solve $\cos 5\theta = \cos(\theta + 60°)$ for $0° \le \theta < 180°$.

or $\begin{cases} 5\theta = \theta + 60° \pm 360n° \\ 5\theta = -\theta - 60° \pm 360n° \end{cases}$

$\therefore$ or $\begin{cases} 4\theta = 60° \pm 360n° \\ 6\theta = -60° \pm 360n° \end{cases}$

$\therefore$ or $\begin{cases} \theta = 15° \pm 90n° \\ \theta = -10° \pm 60n° \end{cases}$

✓

$\therefore\ \theta = 15°, 105°, 50°, 110°, 170°.$

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9. The roots α and β of the quadratic equation $x^2 - 2kx + k - 1 = 0$ satisfy $\alpha^2 + \beta^2 = 4$. Find the values of k.

$\alpha + \beta = 2k$

$\alpha\beta = k - 1$

$(\alpha+\beta)^2 - 2\alpha\beta = 4$

$= 4k^2 - 2(k-1)$

$= 4k^2 - 2k + 2$

$\therefore 4k^2 - 2k - 2 = 0$

$\therefore 2k^2 - k - 1 = 0$

$(2k+1)(k-1) = 0$

$\therefore k_1 = 1$

$k_2 = -\frac{1}{2}$

$\Delta = 4k^2 - 4k + 4$

$= 4(k^2 - k + \frac{1}{4}) + 4 - 1$

$= 4(k - \frac{1}{2})^2 + 3$

-1.8229

0.8229

$(a^2 - b^2 + 2abi)(a + bi) - 107i$

$= a^3 + a^2bi - ab^2 - b^3i + 2a^2bi - 2ab^2 - 107i$

$= (a^3 - 3ab^2) + (-b^3 + 3a^2b - 107)i$

$a^3 = 3ab^2$

$b^3 = 3a^2b - 107$

10. Find c if a, b and c are positive integers which satisfy $c = (a + bi)^3 - 107i$ where $i^2 = -1$.

$c = (a^2 - b^2 + 2abi)(a + bi) - 107i$

$= a^3 + a^2bi - ab^2 - b^3i + 2a^2bi - 2ab^2 - 107i$

$= (a^3 - 3ab^2) + (-b^3 + 3a^2b - 107)i$

$\therefore \begin{cases} c = a^3 - 3ab^2 & (1) \\ b^3 = 3a^2b - 107 & (2) \end{cases}$

(2): $3a^2b - b^3 = 107$

$b(3a^2 - b^2) = 107$

$\therefore$ (1) $\begin{cases} b = 1 \\ 3a^2 - b^2 = 107 \end{cases}$

(2) $\begin{cases} b = 107 \\ 3a^2 - b^2 = 1 \end{cases}$

(1) $3a^2 = 108$

$a^2 = 36$

$a = \pm 6$

(2) $3a^2 = 11450$

$\because 3 \nmid 11450$

$\therefore$ inadmissable

$\therefore (a, b) = (-6, 1), (6, 1)$

$\therefore c = a^3 - 3ab^2$

(1) $(-6, 1)$

$c = -216 + 18 \cdot 1 < 0$

(X)

(2) $(6, 1)$

$c = 216 - 18 \cdot 1$

$= 216 - 18$

$= 198$

$\therefore c = 198$

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Solutions to HL1 Test #4

1. By the remainder theorem $8 + 12 + 2a - 1 = 5$, whence $a = -7$.

2. By the conjugate roots theorem, the complete set of roots is $2 \pm i$, $1 \pm 3i$. By the product of the roots $e/7 = 50$, whence $e = 350$.

3. The required number is $\binom{6}{3} = 20$.

4. By Pythagoras's theorem $OA = 12$. By trigonometry the central angle of the sector is $145°$. So the area of the circle is $\frac{3}{8} \times \pi \times 12^2 = 54\pi$.

5. The required term is $\binom{9}{3} x^6 (-2/x^2)^3 = -672$.

6. Equivalently $\tan^2\theta - \tan\theta - 1 = 0$, whence $\tan\theta = -1$ or $\tan\theta = 2$. Hence $\theta = 135°, 315°, 63.4°, 243°$ (3 s.f.).

7. Equivalently $2^{2x} - 4 \cdot 2^x + 3 = 0$, whence $2^x = 1$ or $2^x = 3$. Hence $x = 0$ or $x = 1.58$ (3 s.f).

8. Observe $5\theta = \theta + 60° + 360°n$ or $5\theta = -(\theta + 60°) + 360°n$, whence $4\theta = 60° + 360°n$ or $6\theta = -60° + 360°n$. Hence $\theta = 15° + 90°n$ or $\theta = -10° + 60°n$. We conclude $\theta = 15°, 105°, 50°, 110°, 170°$.

9. Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (2k)^2 - 2(k - 1)$. So $4k^2 - 2k + 2 = 4$, whence $k = -1/2$ or $k = 1$.

10. On expanding both sides of the given equation we have $c + 107i = (a^3 - 3ab^2) + (3a^2b - b^3)i$. Equating real parts and imaginary parts gives $c = a^3 - 3ab^2$ and $107 = 3a^2b - b^3 = (3a^2 - b^2)b$. Since a, b are integers, this means b is a divisor of 107, which is a prime number. Thus either $b = 1$ or $b = 107$. If $b = 107$, $3a^2 - 107^2 = 1$ so $3a^2 = 107^2 + 1$, but $107^2 + 1$ is not divisible by 3, a contradiction. Thus we must have $b = 1$, $3a^2 = 108$ so $a^2 = 36$ and $a = 6$ (since we know a is positive). Thus $c = 6^3 - 3 \cdot 6 = 198$.