98%

Excellent!!

MATHEMATICS
HIGHER LEVEL

Block Week December 2018

2 hours

Name in block letters

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INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Calculators are not permitted in this examination.
- There are 20 questions followed by an optional bonus question marked by a star.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

16 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working or explanations. Where an answer is incorrect, some marks may be given for a correct method provided this is shown by written working. You are therefore advised to show all working. Working may be continued below the lines, if necessary.

1. In an arithmetic sequence the first term is 33 and the second term is 26.

   (a) Write down the common difference.

   (b) Find the tenth term.

   (c) Find the sum of the first ten terms of the sequence.

(a) $d = 33 - 26 = 7$ -7 $d = 26 - 33 = -7$

(b) $u_n = 33 - 7\cdot(n-1) = 33 - 7n + 7 = 40 - 7n$

$\therefore u_{10} = 40 - 70 = -30$ ✓

(c) $S_{10} = \frac{(33-30)\times 10}{2} = 5 \times 3 = 15$ ✓

4

2. Consider the expansion of the expression $(2-x)^{11}$.

   (a) Write down the number of terms in the expansion.

   (b) Calculate the value of the binomial coefficient $\binom{11}{9}$.

   (c) The coefficient of the x^9 term is an integer. Find its value.

(a) 12

(b) $\binom{11}{9} = \binom{11}{2} = \frac{11\times10}{2\times1} = 55$

(c) $2^2 \cdot (-x)^9 \cdot \binom{11}{9} = 4 \cdot (-x^9) \cdot 55 = -220x^9$

$\therefore$ the coefficient is -220

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3. The points A and B have coordinates $(-2, 3)$ and $(4, 1)$ respectively.

   (a) Write down the midpoint of line segment $[AB]$.

   (b) Write down the gradient of line (AB).

   (c) Determine the y-intercept of the perpendicular bisector of $[AB]$.

(a) $\left(\frac{-2+4}{2}, \frac{3+1}{2}\right) \Rightarrow (1, 2)$ ✓

(b) gradient $(AB) = \frac{3-1}{-2-4} = \frac{2}{-6} = -\frac{1}{3}$ ✓

(c) pass through $(1,2)$ with gradient $\frac{-1}{-\frac{1}{3}} = 3$

$\therefore l: y = 3x + b$

$3 + b = 2$

$b = -1$

$\therefore y = 3x - 1$

$\therefore$ the y-intercept is $(0, -1)$ ✓

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4. A CVV number is a three digit security code found on the back of a credit card.

   (a) How many possible CVV numbers are there?

   (b) How many CVV numbers begin and end with an odd digit?

   (c) How many CVV numbers begin with a zero and have the second digit less than the third?

0 1 2 3 4 5 6 7 8 9
✓ ✓ ✓ ✓ ✓

(a) $10 \times 10 \times 10 = 1000$ ✓

(b) $5 \times 10 \times 5 = 250$ ✓

(c) 0 0 _ : 9 (1, 2, ---, 9)

0 1 _ : 8 (2, 3, --- , 9)

0 2 _ : 7 (3, 4, ---, 9)

0 3 _ : 6 (4, 5, ---, 9)

0 4 _ : 5 (5, 6, ---, 9)

0 5 _ : 4 (6, 7, 8, 9)

0 6 _ : 3 (7, 8, 9)

0 7 _ : 2 (8, 9)

0 8 _ : 1 (9)

0 9 _ : 0 ✓

$\therefore \frac{(1+9)\times 9}{2} = 45$

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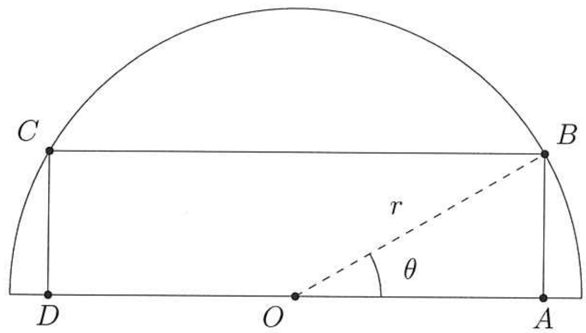
5. The rectangle $ABCD$ is inscribed in the semicircle with centre O and radius r.



Let $\angle AOB = \theta$.

(a) Show that the area of rectangle $ABCD$ is $2r^2 \sin\theta \cos\theta$.

(b) Hence find the largest possible area for a rectangle inscribed in a semicircle of radius 10.

(a) $AB = r \cdot \sin\theta$

$OA = r \cdot \cos\theta$, $AD = 2OA = 2r\cos\theta$

$\therefore$ Area $= r \cdot \sin\theta \cdot 2r\cos\theta = 2r^2 \sin\theta\cos\theta$

(b) Area $= r^2 \cdot 2\sin\theta\cos\theta = r^2 \cdot \sin 2\theta$

$\sin 2\theta_{max} = 1$

$\therefore$ Area$_{max}$ $= 10^2 \cdot 1 = 100$

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6. It is given that $(a + bi)^2 = -15 + 8i$ for $a, b \in \mathbb{R}$.

(a) Obtain a pair of simultaneous equations involving a and b.

(b) Hence find the two square roots of $-15 + 8i$.

(a) $a^2 - b^2 + 2ab\,i = -15 + 8i$

$$\begin{cases} a^2 - b^2 = -15 \\ 2ab = 8 \end{cases}$$

(b) $b = \frac{4}{a}$

$\therefore\ a^2 - \frac{16}{a^2} + 15 = 0$

$\therefore\ a^4 + 15a^2 - 16 = 0$

$\therefore\ (a^2 + 16)(a^2 - 1) = 0$

$\therefore\ a^2 = 1$

$\therefore\ a = \pm 1$

$\therefore\ b = \pm 4$

$\therefore$ the square roots are $\pm(1 + 4i)$

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7. Let $p(x) = x^{20} - x^{15} + x^{10} - x^5 + 5$.

(a) Find the remainder when $p(x)$ is divided by $x - 1$.

(b) Find the remainder when $p(x)$ is divided by $x + 1$.

(c) Find the remainder when $p(x)$ is divided by $x^2 - 1$.

(a) $p(1) = 1-1+1-1+5 = 5$

(b) $p(-1) = 1-(-1)+1-(-1)+5 = 9$

(c) let the remainder be $ax+b$

$p(x) = (x^2-1)\cdot q(x) + ax+b$

$p(1) = a+b = 5$

$p(-1) = -a+b = 9$

$\therefore \begin{cases} a+b=5 \\ -a+b=9 \end{cases}$

$\therefore 2b=14$

$\therefore b=7,\ a=-2$

$\therefore$ the remainder is $-2x+7$

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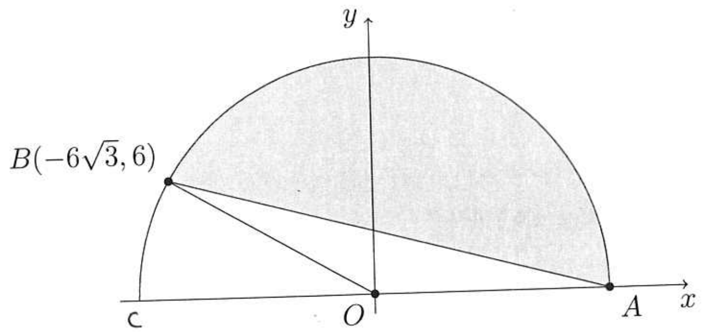
8. In the diagram the points A and B lie on the semicircle with centre O.



(a) Find the coordinates of point A.

(b) Find the radian measure of $\angle AOB$.

(c) Find the area of the shaded segment.

(a) $OA = OB = \sqrt{(-6\sqrt{3})^2 + 6^2} = 12 = r$

$\therefore A(12, 0)$

(b) $\angle BOC = \arctan\left(\frac{6}{6\sqrt{3}}\right) = 30°$

$\therefore \angle AOB = 180° - 30° = 150° \quad \therefore \angle AOB = \frac{5}{6}\pi$

(c) Area $= \frac{1}{2} \cdot \angle AOB \cdot r^2 - \frac{1}{2} \cdot \sin\angle AOB \cdot AO \cdot OB$

$= \frac{1}{2} r^2 (\angle AOB - \sin\angle AOB)$

$= \frac{1}{2} \cdot 144 \left(\frac{5}{6}\pi - \sin\frac{5}{6}\pi\right)$

$= \frac{1}{2} \cdot 144 \left(\frac{5}{6}\pi - \frac{1}{2}\right)$

$= 60\pi - 36$

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9. Consider the parabola $y = 4x - x^2$ and the line $y = x + c$.

   (a) Find the coordinates of the vertex for the parabola.

   (b) Find the value of c for the line to be tangent to the parabola.

(a) $y = -x^2 + 4x$

$= -(x^2 - 4x + 4) + 4$

$= -(x-2)^2 + 4$

vertex $(2, 4)$

(b) $\begin{cases} y = 4x - x^2 \\ y = x + c \end{cases}$

$\therefore \quad x^2 - 3x + c = 0$

$\therefore \quad \Delta = 9 - 4c = 0$

$\therefore \quad c = \frac{9}{4}$

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10. The numbers a and b are such that $\log_9 a = 11$ and $\log_9 b = 6$.

(a) Find the value of $\log_9 a^2 b$.

(b) Find the value of $\log_9 3\sqrt{a}$.

(c) Find the value of $\log_b 27$.

(a) $\log_9 a^2 b = 2\log_9 a + \log_9 b = 22 + 6 = 28$

(b) $\log_9 3a^{\frac{1}{2}} = \frac{1}{2}\log_9 9 + \frac{1}{2}\log_9 a = \frac{1}{2} + \frac{1}{2}\cdot 11 = 6$

(c) $\log_b 27 = \dfrac{\log_9 27}{\log_9 b} = \dfrac{\frac{3}{2}}{6} = \dfrac{1}{4}$

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11. The obtuse angle A is such that $\sec A = -\frac{5}{4}$.

(a) Find the value of $\cos A$.

(b) Find the value of $\sin 2A$.

(c) Find the value of $\tan 2A$.

(a) $\sec A = \frac{1}{\cos A} = -\frac{5}{4}$

$\therefore \cos A = -\frac{4}{5}$

(b) $\sin A = \pm\sqrt{1-\left(-\frac{4}{5}\right)^2} = \pm\frac{3}{5}$

since it's obtuse,
negative is inadmissable

$\sin A = \frac{3}{5}$

$\sin 2A = 2\sin A\cos A = 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right) = -\frac{24}{25}$

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{\sin 2A}{2\cos^2 A-1} = \frac{-\frac{24}{25}}{2\cdot\frac{16}{25}-1} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$

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12. Rows 0, 1, 2 and 3 of Pascal's triangle are given below.

$$
\begin{array}{ccccccccccc}
0 & & & & & 1 & & & & & \\
1 & & & & 1 & & 1 & & & & \\
2 & & & 1 & & 2 & & 1 & & & \\
3 & & 1 & & 3 & & 3 & & 1 & & \\
4 & 1 & & 4 & & 6 & & 4 & & 1 &
\end{array}
$$

(a) Write down the numbers in row 6 of Pascal's triangle.

(b) Find the value of $\sum_{r=0}^{6}\binom{6}{r}$.

(c) Find the sum of all the numbers in Pascal's triangle that lie above row 10.

(a) ~~1 5 10 10 5 1~~ 1 6 15 20 15 6 1

(b) $\sum_{r=0}^{6}\binom{6}{r} = \binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}$

$= 1 + 6 + 15 + 20 + 15 + 6 + 1$

$= 7\times 2 + 15\times 2 + 20$

$= 64$

(c) $\sum_{r=0}^{n}\binom{n}{r} = 2^n$ if n is the row number.

$\therefore$ sum $= \sum_{r=0}^{9} 2^r = 1+2+4+8+16+32+64+128+256+512$

$= \frac{1\,[1-2^{10}]}{(1-2)}$

$= \frac{2^{10}-1}{2-1}$

$= 1023$

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13. Let $\alpha = \dfrac{\pi}{6}$.

(a) Write down the value of $\cos\alpha$.

(b) Find the value of $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \cos 4\alpha + \cos 5\alpha$.

(c) Find the value of $\displaystyle\sum_{n=0}^{2018} \cos n\alpha$.

(a) $\cos\alpha = \frac{\sqrt{3}}{2}$

(b) $\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

$\cos 3\alpha = \cos\frac{\pi}{2} = 0$

$\cos 4\alpha = \cos\frac{2}{3}\pi = -\frac{1}{2}$

$\cos 5\alpha = \cos\frac{5}{6}\pi = -\frac{\sqrt{3}}{2}$

$\therefore$ sum $= \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 - \frac{1}{2} - \frac{\sqrt{3}}{2} = 0$

(c)

$\cos 0\alpha = 1$
$\cos 1\alpha = \frac{\sqrt{3}}{2}$
— — — —
$\cos 5\alpha = -\frac{\sqrt{3}}{2}$
$\cos 6\alpha = -1$

$\xrightarrow{\text{sum}} 0$

$\cos 7\alpha = -\frac{\sqrt{3}}{2}$
$\cos 8\alpha = -\frac{1}{2}$
$\cos 9\alpha = 0$
$\cos 10\alpha = \frac{1}{2}$
$\cos 11\alpha = \frac{\sqrt{3}}{2}$

$\xrightarrow{\text{sum}} 0$

$\cos 12\alpha = 1$
— — — —
$\cos 23\alpha = \frac{\sqrt{3}}{2}$

$\xrightarrow{\text{sum}} 0$

$\therefore \sum_{n=0}^{2018} = \cos 2016\alpha + \cos 2017\alpha + \cos 2018\alpha$

$= 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}$

$= \frac{3+\sqrt{3}}{2}$

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14. Let α and β be the roots of the quadratic equation $2x^2 - 6x + 1 = 0$. Without solving the equation for α and β find the values of

(a) $\alpha + \beta$;

(b) $\alpha^2 + \beta^2$;

(c) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$.

(a) $\alpha + \beta = -\frac{-6}{2} = 3$ ✓ $\alpha\beta = \frac{1}{2}$

(b) $(\alpha^2 + \beta^2) = (\alpha+\beta)^2 - 2\alpha\beta$

$= 9 - 1$ ✓

$= 8$

(c) $\cancel{\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\left(\frac{1}{\alpha^2} - \frac{1}{\alpha\beta} + \frac{1}{\beta^2}\right)}$

$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{\alpha^3\beta^3}$

$= \frac{(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)}{(\frac{1}{2})^3}$

$= \frac{3 \cdot (8 - \frac{1}{2})}{\frac{1}{8}}$

$= 24 \cdot \frac{15}{2}$

$= 12 \cdot 15$

$= 180$ ✓

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15. Let $z = a + bi$ where $a, b \in \mathbb{R}$.

(a) Show that $zz^* = a^2 + b^2$.

(b) Solve the equation $zz^* + 3z = 56 + 12i$.

(a) $z^* = a - bi$

$z \cdot z^* = (a+bi)(a-bi) = a^2 - b^2 \cdot (-1) = a^2 + b^2$

(b) $(a^2 + b^2 + 3a) + 3bi = 56 + 12i$

$\begin{cases} a^2 + b^2 + 3a = 56 \\ b = 4 \end{cases}$

$a^2 + 3a + 16 - 56 = 0$

$a^2 + 3a - 40 = 0$

$(a+8)(a-5) = 0$

$\therefore a_1 = -8, \ a_2 = 5$

$\therefore z = -8 + 4i$ or $5 + 4i$

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16. A sequence of positive integers is defined recursively by $u_n = u_{n-1} + u_{n-2}$ and $u_1 = a$, $u_2 = b$.

(a) Find an expression for u_7 in terms of a and b.

(b) If $u_7 = 120$ and $a < b$, find the values of a and b.

(a) $u_3 = a+b$ $u_4 = a+b+b$ $u_5 = 2a+3b$ $u_6 = 3a+5b$ $\boxed{u_7 = 5a+8b}$

(b) $5a+8b=120$

$b = \frac{120-5a}{8}$

$a=0$, $b=15$ (X) (possitive integers)

$a=1$, $b=\frac{115}{8}$ (x)

$a=2$, $b=\frac{110}{8}$ (x)

$a=3$, $b=\frac{105}{8}$ (x)

$a=4$, $b=\frac{100}{8}$ (x)

$a=5$ $b=\frac{95}{8}$ (x)

$a=6$ $b=\frac{90}{8}$ (x)

$a=7$, $b=\frac{85}{8}$ (x)

(integers)

$\therefore \begin{cases} a=8 \\ b=10 \end{cases}$

$a=8$, $b=\frac{80}{8}=10$ (v)

— — ~ — — -

$a=16$, $b=\frac{40}{8}=5$ (x) ($a<b$)

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17. The expression $\sin\theta + \sin 5\theta$ can be written as $a \sin b\theta \cos 2\theta$ for $a, b \in \mathbb{Z}^+$.

(a) Find the values of a and b.

(b) Hence or otherwise solve the equation $\sin\theta + \sin 5\theta = \cos 2\theta$ for $0° \leq \theta < 180°$.

(a) ~~$\sin\theta + \sin 5\theta = \sin\theta\cos 5\theta + \cos\theta \sin 5\theta$~~

$\sin\theta = \sin(3\theta - 2\theta) = \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta$

$\sin 5\theta = \sin(3\theta + 2\theta) = \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta$

$\therefore \sin\theta + \sin 5\theta = 2\sin 3\theta \cos 2\theta$

$\therefore a = 2, \quad b = 3.$

(b) $2\sin 3\theta \cos 2\theta = \cos 2\theta$

① $\cos 2\theta = 0$ ② $\sin 3\theta = \frac{1}{2}$

$2\theta = 90°$ or $270°$ OR $\begin{cases} 3\theta = 30° \pm 360n° \\ 3\theta = 150° \pm 360n° \end{cases}$

$\theta = 45°$ or $135°$

$\Downarrow$

OR $\begin{cases} \theta = 10° \pm 120n° \\ \theta = 50° \pm 120n° \end{cases}$

$\therefore \theta = 10°, 45°, 50°, 130°, 135°, 170°.$

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18. In this question, we signify that a number is written in base n by using the subscript n at the right end of the number. For example, 243_6 is a number written in base 6.

(a) Write the number 243_6 in base 7.

(b) If $23_n \times 14_n = 344_n$ find the value of n.

(c) Show that there are no possible values of n satisfying $34_n \times 135_n = 5152_n$.

(a) $2 \cdot 6^2 + 4 \cdot 6 + 3 = 72 + 24 + 3 = 99$

$99 - 49 \times 2 = 1$

$\therefore 243_6 = 201_7$

(b) $(2n+3)(n+4) = (3n^2 + 4n + 4)$

$= 2n^2 + 8n + 3n + 12$

$= 2n^2 + 11n + 12$

$\therefore n^2 - 7n - 8 = 0$ $\therefore n = 8$

$(n-8)(n+1) = 0$

$\therefore n = -1$ or 8 ($n = -1$ inadmissable)

(c) $(3n+4)(n^2+3n+5)$

$= 3n^3 + 9n^2 + 15n + 4n^2 + 12n + 20$

$= 3n^3 + 13n^2 + 27n + 20$

if it equals to $5152_n = 5n^3 + n^2 + 5n + 2$

then $2n^3 - 12n^2 - 22n - 18 = 0$

$n^3 - 6n^2 - 11n - 9 = 0$

rational root candidates are

$n = \dfrac{\pm 9, \pm 3, \pm 1}{\pm 1} = \pm 9, \pm 3, \pm 1.$

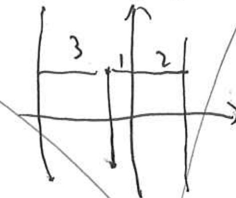
none of them satisfy the equation.

So there are no possible values of n satisfying $34_n \times 135_n = 5152_n$

/5

Turn over

19. Consider the cubic equation $2z^3 + bz^2 + cz + d = 0$ where $b, c, d \in \mathbb{R}^+$. The three roots of the equation, one of which is $-1 + 2i$, are the vertices of a triangle in the complex plane whose area is 6.



(a) Find the other two roots.

(b) Find the values of b, c and d.

(a) $z_1 = -1+2i \Rightarrow (-1, 2)$

$z_2 = -1-2i \Rightarrow (-1,-2)$

$\therefore\ 2-(-2) = 4, \quad \frac{6\times 2}{4} = 3$

$\therefore\ z_3$ has ~~$+2i$ or $-4i$~~ 2 or -4 as real part

$z_1 + z_2 = -2$

$z_1 z_2 = 5$

$\therefore\ z^2+2z+5=0$

since a cubic equation has to have a real root and z_1, z_2 is complex, then $z_3 = 2$ or -4

$\therefore\ (z^2+2z+5)(z-2)=0$ or $(z^2+2z+5)(z+4)=0$

$z_1 = -1+2i,\ z_2 = -1-2i,\ z_3 = 2$ or -4. Correction: $z_3 = -4$

(b) $z^3+2z^2+5z-2z^2-4z-10=0$ or $z^3+2z^2+5z+4z^2+8z+20=0$

$\Downarrow \qquad\qquad \Downarrow$

$z^3+z-10=0 \qquad z^3+6z^2+13z+20=0$

$\therefore \begin{cases} b=0 \\ c=1 \\ d=-10 \end{cases}$ or $\begin{cases} b=6 \\ c=13 \\ d=20 \end{cases}$.

Correction

$\begin{cases} b=12 \\ c=26 \\ d=40 \end{cases}$

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20. The first term and common ratio of the geometric sequence $u_1, u_2, u_3, \ldots$ are positive integers. If $\log_8 u_1 + \log_8 u_2 + \cdots + \log_8 u_{12} = 2018$ find the minimum possible value of u_1.

$2018 \times 3 = 6054$

$u_1 = a \quad u_2 = ar \quad u_3 = ar^2 \quad \cdots \quad u_{12} = ar^{11}$

$\log_8 u_1 + \cdots + \log_8 u_{12} = 2018$

$\therefore \log_8 a + \log_8 ar + \cdots + \log_8 ar^{11} = 2018$

$\therefore 12 \log_8 a + (1+2+\cdots+11) \log_8 r = 2018$

$\therefore 12 \log_8 a + 66 \log_8 r = 2018$

$\therefore \log_8 a^{12} r^{66} = 2018$

$\therefore a^{12} r^{66} = 8^{2018} = 2^{6054}$

$u_1 = a \Rightarrow u_{1\,min} = a_{min} \Rightarrow r_{max}$

$6054 \div 66 = 91 \cdots\cdots 48$

$\therefore r_{max} = 2^{91}$

$\therefore a = 2^{\frac{48}{12}} = 2^4 = 16$

$\therefore u_{1\,min} = 16$

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*21. Consider the expression $\cos 70°(2\cos 40° + 1)$.

(a) Show that this expression equals $\cos 30°$.

(b) Hence show that $x = \cos 70°$ is a solution of the equation $6x - 8x^3 = \sqrt{3}$.

(a) $\cos 30 = \cos(70-40) = \cos 70 \cos 40 + \sin 70 \sin 40$

Correction

(a) $\cos 70 \cos 40 + \cos 70 \cos 40 + \sin 70 \sin 40 + \sin 70 \sin 40 + \cos 70$

$= \cos(70+40) + \cos(70-40) + \cos 70$

$= \cos 110 + \cos 30 - \cos 110$

$= \cos 30$

(b) $6\cos 70 - 8\cos^3 70 = 2\cos 70(2\cos 40 + 1)$

$3 - 4\cos^2 70 = \cancel{4\cos 40+4}\ 2\cos 40 + 1$

$3 - 2(2\cos^2 70 - 1) - 2 = 2\cos 40 + 1$

$1 - 2\cos 140 = 2\cos 40 + 1$

$1 + 2\cos 40 = 2\cos 40 + 1$

✓