HL1 Assignment #17

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Excellent!

1. Let $f(x) = \dfrac{\sqrt{1-x^2}}{\arccos x}$. Give the domain of the function f expressing your answer in interval notation.

$$\begin{cases} 1-x^2 \geqslant 0 \\ x \in [-1,1] \\ \arccos x \neq 0. \end{cases}$$

$\therefore\ x \in [-1,1] \setminus 0.$

$x^2 \leq 1$

$-1 \leq x \leq 1$

$\cdot\ x \neq 0.$

$\arccos x \neq 0 \Rightarrow x \neq 1$

$\Downarrow$

$x \in [-1, 1[.$

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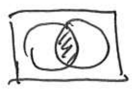
2. Complete the table of outcomes and hence find $P(A \cap B \mid A \cup B)$.

$$P = \frac{n[(A\cap B)\cap(A\cup B)]}{n(A\cup B)}$$



$$= \frac{n(A\cap B)}{n(A\cup B)}$$

$$= \frac{11}{30+50-11}$$

$$= \frac{11}{69}$$

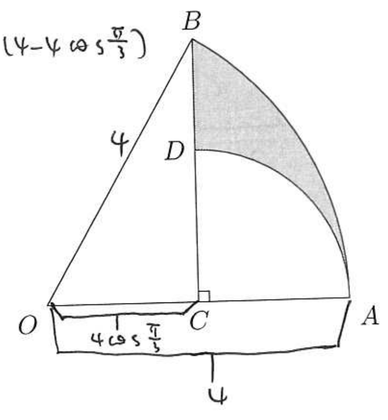
	A	A'	
B	11	39	50
B'	19	21	40
	30	60	90

3. The sector OAB has radius 4 and the arc AD has centre C. If $\angle AOB = \frac{1}{3}\pi$ find the perimeter of the shaded region.

$$P = \frac{1}{6}\cdot 2\pi\cdot 4 + 4\cdot \sin\frac{\pi}{3} - (4 - 4\cdot\cos\frac{\pi}{3}) + \frac{1}{4}\cdot\pi\cdot 2\cdot(4-4\cos\frac{\pi}{3})$$

$$= \frac{4}{3}\pi + 2\sqrt{3} - 2 + \pi$$

$$= \frac{7}{3}\pi + 2\sqrt{3} - 2$$



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4. Find the coefficient of x^8 in the expansion of $(2+x)(2x-x^2)^6$.

$(2x-x^2)^6$

x^8 : $(-x^2)^2 \cdot (2x)^4 \cdot \binom{6}{2} = x^4 \cdot 16 \cdot x^4 \cdot 15 = 240x^8$

x^7 : $(-x^2)^1 \cdot (2x)^5 \cdot \binom{6}{1} = -x^2 \cdot 32x^5 \cdot 6 = -192x^7$

x^8 : $240 \times 2 + (-192) \times 1 = 480 - 192 = 288$

5. Solve $\log_2(x+1) - \log_4(3x-1) = 0.5$ without a calculator.

$\log_4 \frac{(x+1)^2}{(3x-1)} = \frac{1}{2}$

$x^2+2x+1 = 6x-2$

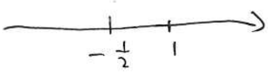
$x^2-4x+3=0$

$(x-3)(x-1)=0$

$x_1=3$

$x_2=1$

6. Solve the inequality $3|x-1| < |2x+1|$.



① $x \le -\frac{1}{2}$

$3-3x < -2x-1$

$x > 4$

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② $-\frac{1}{2} < x < 1$

$3-3x < 2x+1$

$5x > 2$

$x > 0.4$

$0.4 < x < 1$

③ $x \geq 1$

$3x-3 < 2x+1$

$x < 4$

$1 \le x < 4$

$\therefore$ $0.4 < x < 4$

7. The smallest positive solution of the equation $2\cos^2(n\theta) = 3\sin(n\theta)$, where n is a positive integer, is $10°$. Find the value of n and hence find the largest solution of this equation in the interval $0° \leq \theta \leq 360°$.

$2 - 2\sin^2(n\theta) - 3\sin(n\theta) = 0.$

$\sin(n\theta) = \frac{1}{2}$ or (-2) (X).

$\therefore n\theta = 30°/150° \pm 360°k$

$\therefore \theta = \dfrac{30° \pm 360°k}{n}$ or $\dfrac{150° \pm 360°k}{n}$

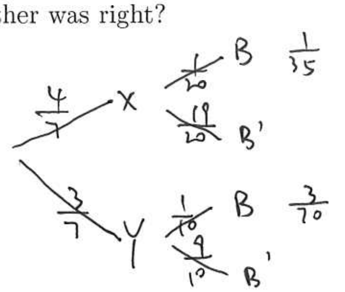
$\therefore n = 3.$

$\therefore \theta = 10°/50° \pm 120°k.$

$\therefore \theta = 10°, 50°, 130°, 170°, 250°, 290°.$

$\therefore \theta_{max} = 290°.$

8. Xiaohong washes the dishes after dinner four times a week and her brother Yang washes them the other three times. The probability of a breakage while Xiaohong is doing the dishes is 0.05 while Yang's probability is 0.1. One day after dinner, Father hears a crash and says: "This must be Yang's day for doing the dishes." What is the probability that Father was right?



$\therefore P(Y|B) = \dfrac{\frac{3}{70}}{\frac{1}{35} + \frac{3}{70}}$

$= \dfrac{3}{2+3}$

$= \dfrac{3}{5}$

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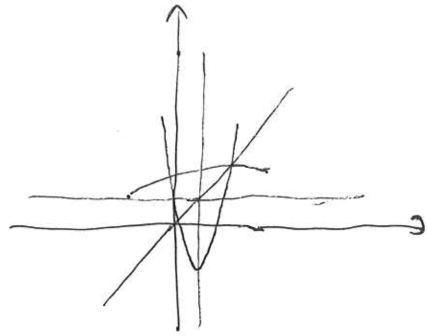
9. If the codomain of the function $f\colon [3,\infty[\to \mathbb{R}$ with rule $f(x) = 4x^2 - 24x + 11$ is suitably restricted a bijection results. Find the restriction and give the consequent full function definition for f^{-1}.

$f(3) = 36 - 72 + 11 = -25$.

$f : [3,\infty[\to [-25,\infty[$.

$f^{-1} : [-25,\infty[\to [3,\infty[$.



$4y^2 - 24y + 11 = x$

$y = \pm\sqrt{\frac{x+25}{4}} + 3$.

$\because y \geq 3$

$\therefore y = \sqrt{\frac{x+25}{4}} + 3$

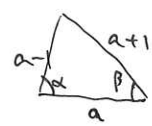
$\therefore f^{-1} : [-25,\infty[\to [3,\infty[,\ f^{-1}(x) = y = \frac{\sqrt{x+25}}{2} + 3$

10. The lengths of the sides of a triangle are consecutive integers and the largest angle is twice the smallest angle. Find the $\boxed{\text{degree}}$ measure of the smallest angle giving your answer to three significant figures.

let the 3 sides be $a-1$, a, $a+1$.

and $\alpha = 2\beta$.

$a \in \mathbb{Z}^+$



according to the law of sine,

$$\frac{\sin\beta}{a-1} = \frac{\sin 2\beta}{a+1} = \frac{\sin(180-3\beta)}{a}$$

$$\frac{\sin\beta}{a-1} = \frac{\sin\beta\cdot 2\cos\beta}{a+1} = \frac{\sin\beta(3-4\sin^2\beta)}{a}.$$

① $\sin\beta = 0$

$\beta = 0$ (x)

② $\sin\beta \neq 0$.

$$\frac{1}{a-1} = \frac{2\cos\beta}{a+1} = \frac{4\cos^2\beta - 1}{a}$$

$$2\cos\beta = \frac{a+1}{a-1}$$

$$\frac{\left(\frac{a+1}{a-1}\right)^2 - 1}{a} = \frac{1}{a-1}$$

$a = 5$

$\therefore \cos\beta = \frac{3}{4}$

$\therefore \beta = 41.4°$

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Solutions to HL1 Assignment #17

1. $D_f = [-1, 1[$

2. $P(A \cap B \mid A \cup B) = \frac{11}{69}$

3. Perimeter is $(2\sqrt{3} - 2) + (4 \cdot \frac{\pi}{3}) + (2 \cdot \frac{\pi}{2}) = \frac{7\pi}{3} + 2\sqrt{3} - 2$.

4. We want the coefficient of x^8 in the expansion of $x^6(2+x)(2-x)^6$, which is the same as the coefficient of x^2 in the expansion of $(2+x)(2-x)^6$, which is $2 \cdot \binom{6}{2} \cdot 2^4 - \binom{6}{1} \cdot 2^5 = 288$.

5. Changing to base 2 gives the equation $2\log_2(x+1) - \log_2(3x-1) = 1$. After some algebra we arrive at the quadratic equation $x^2 - 4x + 3 = 0$, whence $x = 1$ or $x = 3$.

6. We could use a graphical approach or we might recognize that this inequality is equivalent to $9(x-1)^2 < (2x+1)^2$, whence $5x^2 - 22x + 8 < 0$. Hence $0.4 < x < 4$.

7. Let $s = \sin n\theta$. Then we have $2(1 - s^2) = 3s$, or equivalently $2s^2 + 3s - 2 = 0$. The only admissible solution is $s = \frac{1}{2}$. Hence $n\theta = 30° + k \cdot 360°$ or $n\theta = 150° + k \cdot 360°$. If the smallest solution is $10°$ then $n = 3$. So $\theta = 10° + k \cdot 120°$ or $\theta = 50° + k \cdot 120°$. The largest angle in the required interval is therefore $290°$.

8. Let X be the event that the day is one on which Xiaohong washes the dishes with a similar notation for Y. Let B be the event that a dish is broken. Using a tree diagram, a Venn diagram, or Bayes' theorem, we find
$$P(Y \mid B) = \frac{P(Y \cap B)}{P(B)} = \frac{\frac{3}{70}}{\frac{2}{70} + \frac{3}{70}} = \frac{3}{5}.$$

9. Notice $f(x) = (2x-6)^2 - 25$, $x \geq 3$. So we restrict the codomain of f to $[-25, \infty[$ to make f a bijection. To find the inverse rule we let $y = (2x-6)^2 - 25$. Interchanging x and y and making y the subject we find $y = 3 \pm \frac{1}{2}\sqrt{x+25}$. Since $f^{-1}\colon [-25, \infty[\to [3, \infty[$, the required inverse rule is $f^{-1} = 3 + \frac{1}{2}\sqrt{x+25}$.

10. Let the lengths of the sides of the triangle be n, $n+1$, $n+2$ and the smallest be α. By the cosine rule we find
$$\cos\alpha = \frac{n+5}{2(n+2)} \quad \text{and} \quad \cos 2\alpha = \frac{n-3}{2n}.$$
Since $\cos 2\alpha = 2\cos^2\alpha - 1$, we have after some algebra $2n^3 - n^2 - 25n - 12 = 0$, whose only admissible solution is $n = 4$. Hence $\cos\alpha = \frac{3}{4}$ and $\alpha = 41.4°$ to three significant figures.