HL1 Assignment #5 Name: Jerry Jiang

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Excellent!!

1. Solve the equation ${}^nP_2 = 9900$.

$$\frac{n!}{(n-2)!} = 9900$$

$$n(n-1) = 9900$$

$$n^2 - n - 9900 = 0$$

$$(n-100)(n+99) = 0$$

$$n_1 = 100$$

$$n_2 = -99 \text{ (inadmissible)}$$

$\therefore n = 100$

2. Find all values of x so that $3^{x^2-1} = (\sqrt{3})^{126}$.

$$3^{x^2-1} = 3^{\frac{126}{2}}$$

$$x^2 - 1 = \frac{126}{2}$$

$$x^2 = \frac{126+2}{2} = \frac{128}{2} = 64$$

$$x = \pm 8$$

3. If $A = 5^x + 5^{-x}$ and $B = 5^x - 5^{-x}$, find the value of $A^2 - B^2$.

$$A^2 - B^2$$

$$= (A+B)(A-B)$$

$$= (5^x + \cancel{5^{-x}} + 5^x - \cancel{5^{-x}})[\cancel{5^x} + 5^{-x} - (\cancel{5^x} - 5^{-x})]$$

$$= 2 \cdot 5^x \cdot 2 \cdot 5^{-x}$$

$$= 4 \cdot 5^0$$

$$= 4$$

4. Let $S = \{n \in \mathbb{Z} \mid 1 \leq n \leq 9\}$. How many four element subsets of S contain two odd and two even numbers?

There's 5 odd and 4 even numbers in integers 1-9.

so $^5C_2 \times {}^4C_2$

$= \frac{5\times4}{2\times1} \times \frac{4\times3}{2\times1}$

$= 60 \quad (= {}^5P_3)$

$\therefore$ 60 subsets meet the requirment.

5. What positive integer n satisfies $\log(225!) - \log(223!) = 1 + \log(n!)$?

$\log\left(\frac{225!}{223!}\right) = \log(10 \cdot n!)$

$\frac{225 \cdot 224}{10} = n!$

$n! = 5040$

$\therefore n = 7$.

6. Solve the equations $x + 2y = 5$ and $4^x = 8^y$ simultaneously.

$\begin{cases} x+2y=5 & ① \\ 4^x = 8^y & ② \end{cases}$

$②: 2^{2x} = 2^{3y}$

$\therefore 2x = 3y, \quad x = \frac{3}{2}y$

$\hookrightarrow ①: \quad \frac{3}{2}y + 2y = 5$

$\frac{7}{2}y = 5$

$y = \frac{10}{7}$

$\therefore \begin{cases} x = \frac{15}{7} \\ y = \frac{10}{7} \end{cases}$

7. Solve $\log_2(9x+5) - \log_2(x^2-1) = 2$.

$\log_2 \frac{9x+5}{x^2-1} = 2$

$\frac{9x+5}{x^2-1} = 4$

$9x+5 = 4x^2-4$

$4x^2-9x-9=0$

$(4x+3)(x-3)=0$

$x_1 = 3$

$x_2 = -\frac{3}{4}$

$\because \begin{cases} 9x+5>0 \\ x^2-1>0 \end{cases}$

$\therefore \quad x>1$

$\therefore \quad x=3$

8. The coefficient of x^2 in the expansion of $(1+2x)^n$ is 264. Find the value of n.

$\binom{n}{2} \cdot 1^{n-2} \cdot 4 = 264$

$= \frac{n(n-1)}{2} \cdot 4$

$= 2n(n-1)$

$= 264$

$\therefore n(n-1) = 132$

$\therefore n^2 - n - 132 = 0$

$(n-12)(n+11)=0$

$n_1 = 12 \quad n_2 = -11$ (inadmissible)

$\therefore n = 12$

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9. Solve $x\sqrt{x} = x^{\sqrt{x}}$ where $x > 0$.

let $\sqrt{x} = a$, $a > 0$

$\therefore x = a^2$

$a^2 \cdot a = (a^2)^a$

$a^3 = a^{2a}$

$\therefore 2a = 3$ or $a = 1$

$\therefore a = \frac{3}{2}$

$\therefore \sqrt{x} = \frac{3}{2}$

$\therefore x = \frac{9}{4}$ or $x = 1$ ✓

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10. Given that $(1+x)^6(1+mx)^5 = 1 + nx + 415x^2 + \cdots + m^5x.^{11}$, find the possible values of m and n.

$$\begin{cases} nx = 6x + 5mx \\ 415x^2 = 1 \cdot \binom{5}{2}(mx)^2 + 1 \cdot \binom{6}{2}x^2 + (6x)\cdot(5mx) \end{cases}$$

$$\begin{cases} n = 6 + 5m \quad ① \\ 415 = 10m^2 + 15 + 30m \quad ② \end{cases}$$

$\therefore$ ②: $10m^2 + 30m - 400 = 0$

$m^2 + 3m - 40 = 0$

$(m+8)(m-5) = 0$

$m_1 = -8$ (inadmissible).

$m_2 = 5$ ✓

$\therefore \begin{cases} m_1 = -8 \\ n_1 = -34 \end{cases} \quad \begin{cases} m_2 = 5 \\ n_2 = 31 \end{cases}$

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Solutions to HL1 Assignment #5

1. We must solve $n(n-1) = 9900$. By inspection $n = 100$.
2. We conclude $x^2 - 1 = 63$. So $x = \pm 8$.
3. We have $A^2 - B^2 = (A+B)(A-B) = (2 \times 5^x)(2 \times 5^{-x}) = 4$.
4. The required number is $\binom{5}{2} \times \binom{4}{2} = 10 \times 6 = 60$.
5. We have $\log(225 \times 224) = \log(10 \times n!)$, whence $n! = 5040$. So $n = 7$.
6. We must solve $x + 2y = 5$ and $2x = 3y$ simultaneously. We conclude $x = 15/7$ and $y = 10/7$.
7. We first note that any solution must satisfy $x > 1$. First we have $(9x+5)/(x^2-1) = 4$, from which we arrive at the quadratic equation $4x^2 - 9x - 9 = 0$. The admissible solution is $x = 3$.
8. The x^2 term is $\binom{n}{2}(2x)^2$. So we must solve $2n(n-1) = 264$, whence $n = 12$.
9. By inspection $x = 1$ is a solution. If $x \neq 1$, then we can take $\log_x$ of both sides giving $1 + \frac{1}{2} = \sqrt{x}$, whence $x = \frac{9}{4}$. We conclude $x = 1$ or $x = \frac{9}{4}$.
10. Expanding gives $(1 + 6x + 15x^2 + \cdots + x^6)(1 + 5mx + 10m^2x^2 + \cdots + m^5x^5) = 1 + (5m+6)x + (10m^2 + 30m + 15)x^2 + \cdots + m^5x^{11}$. Equating coefficients we conclude $5m + 6 = n$ and $10m^2 + 30m + 15 = 415$. Solving simultaneously gives $m = -8$ and $n = -34$ or $m = 5$ and $n = 31$.