HL1 Assignment #21 Name: Jerry Jiang. (100%) !! Excellent

1. Let $f(x) = \sin 2x$. Fill in the following table and hence determine $f^{(2019)}(x)$.

n	1	2	3	4
$f^{(n)}(x)$	$2-4\sin^2 x$	$-4\sin 2x$	$-8+16\sin^2 x$	$16\sin 2x$

n = odd. $(-1)^{\frac{n-1}{2}}[2^n - 2^{n+1}\sin^2 x]$

n = even. $(-1)^{\frac{n}{2}} \cdot 2^n \cdot \sin 2x$

$\therefore f^{(2019)}(x) = -2^{2019} + 2^{2020} \sin 2x$

2. The table shows some values of the functions f and g and their respective derivatives f' and g'.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	2	1	3	9
1	2	5	4	6

Calculate the value of $h'(0)$ where $h(x) = (f \circ g)(x)$.

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(0) = f'(g(0)) \cdot g'(0)$

$= 4 \cdot 9 = 36$

3. A balloon is being filled with air at the rate of $12\,\mathrm{cm}^3\,\mathrm{s}^{-1}$. At what rate is the radius increasing after three seconds?

$\frac{4}{3}\pi r^3 = 12t$

$r = \left(\frac{9t}{\pi}\right)^{\frac{1}{3}} = \left(\frac{9}{\pi}\right)^{\frac{1}{3}} \cdot t^{\frac{1}{3}}$

$\therefore r' = \left(\frac{9}{\pi}\right)^{\frac{1}{3}} \cdot \frac{1}{3} t^{-\frac{2}{3}}$

$t=3$, $r' = \frac{1}{\sqrt[3]{\pi} \cdot 3} = 0.228.$

4. In a room of four students find the probability that at least two students were born on the same day of the week.

no one.

$$\frac{7\times6\times5\times4}{7^4} = \frac{120}{7^3}$$

$$\therefore P = 1 - \frac{120}{7^3} = \frac{223}{343}$$

5. The curve $y = \cos^2 x$ has a point of inflection at $x = a$ where $0 < a < 1$. Find the *exact* value of a.

$$y' = -\sin 2x$$

$$y'' = -(2 - 4\sin^2 x)$$

$$= 4\sin^2 x - 2$$

$$\sin^2 x = \frac{1}{2}$$

$$\therefore \sin x = \pm\frac{\sqrt{2}}{2}$$

$$\text{OR} \begin{cases} x = \frac{\pi}{4} + 2k\pi \\ x = \frac{3}{4}\pi + 2k\pi \\ x = \frac{7}{4}\pi + 2k\pi \\ x = \frac{5}{4}\pi + 2k\pi \end{cases}$$

$\because$ $0 < a < 1$

$\therefore$ $a = \frac{\pi}{4}$

6. Without the calculator solve $6\cos^2 x + 5\sin x = 2$ for $0 \le x < 2\pi$.

$$6 - 6\sin^2 x + 5\sin x = 2$$

$$6\sin^2 x - 5\sin x - 4 = 0$$

$$(2\sin x + 1)(3\sin x - 4) = 0$$

$$\sin x = -\frac{1}{2} \text{ or } \frac{4}{3} (\times)$$

$$\therefore x = -\frac{\pi}{6} + 2k\pi \,/\, \frac{7}{6}\pi + 2k\pi$$

$$\therefore x = \frac{7}{6}\pi \text{ or } \frac{11}{6}\pi$$

7. The line $y = 4x - 2$ is tangent to the curve $y = mx^3 + nx^2 - 1$ at $x = 1$. Find the values of m and n.

$g(x) = 4x - 2$

$g'(1) = 4 = f'(1)$

$g(1) = 2$

$(1, 2)$

$f'(x) = 3mx^2 + 2nx$

$f'(1) = 3m + 2n = 4$

$f(1) = m + n - 1 = 2$

$\therefore \begin{cases} m = -2 \\ n = 5 \end{cases}$ ✓

8. The sum of the first two terms of a geometric series whose terms are all positive is 15 and the sum to infinity is 27. Find the first term.

$\begin{cases} a + ar = 15 \\ a \cdot \frac{1}{1-r} = 27 \end{cases}$

$a = 27 - 27r$

$27r^2 = 12$

$r^2 = \frac{4}{9}$

$\therefore r = \pm\frac{2}{3}$ (negative ×)

$\therefore a = 27 - 27 \cdot \frac{2}{3} = 9.$ ✓

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9. Find the *exact* coordinates of the stationary point on the curve $y = 6xe^{2x}$ and use the second derivative test to determine its nature.

$f'(x) = 6(x\cdot e^{2x}\cdot 2 + e^{2x}) = 0 \Rightarrow x = -\frac{1}{2}$

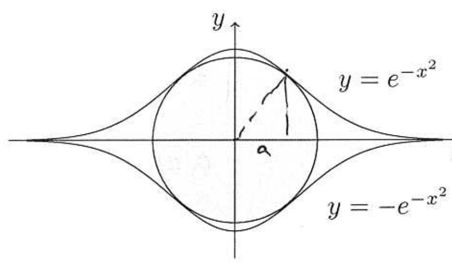
$f''(x) = 6[e^{2x}(2x+1)\cdot 2 + e^{2x}\cdot 2]$

$f''(-\frac{1}{2}) = \frac{12}{e} > 0$, concave up.

$\therefore$ minimum at $(-\frac{1}{2}, -\frac{3}{e})$ ✓

10. Find the *exact* area of the circle which is inscribed between the bell curves $y = e^{-x^2}$ and $y = -e^{-x^2}$.

(a, e^{-a^2}).



$r = (a^2 + e^{-2a^2})^{\frac{1}{2}}$

$r' = \frac{1}{2}(a^2+e^{-2a^2})^{-\frac{1}{2}}\cdot[2a + e^{-2a^2}\cdot(-4a)]$

$= \dfrac{2a + e^{-2a^2}(-4a)}{2\sqrt{a^2+e^{-2a^2}}} = 0$

$e^{-2a^2}\cdot\cancel{(-4a)} = \cancel{-2a}\ \frac{1}{2}$

$-2a^2 = \ln\frac{1}{2}$

$a^2 = -\dfrac{\ln\frac{1}{2}}{2} = \dfrac{\ln 2 - \ln 1}{2} = \dfrac{\ln 2}{2}$

$\therefore a^2 = \dfrac{\ln 2}{2}$, $a = \pm\sqrt{\dfrac{\ln 2}{2}}$.

$e^{-\frac{\ln 2}{2}}$

$(e^{\ln 2})^{-\frac{1}{2}} = 2^{-\frac{1}{2}} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$

$\therefore r^2 = \dfrac{\ln 2}{2} + \dfrac{1}{2} = \dfrac{\ln 2 + 1}{2}$

$\therefore A_{circ} = \pi r^2 = \dfrac{\pi(\ln 2 + 1)}{2}$. ✓

$\dfrac{\ln 2 + 1}{2}$

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Solutions to HL1 Assignment #21

1. $f^{(2019)}(x) = -2^{2019}\cos 2x$

2. By the chain rule $h'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot 9 = 36$.

3. When $t = 3$, $V = 36$. So $\frac{4}{3}\pi r^3 = 36$, whence $r = 3\pi^{-1/3}$. Next by the chain rule $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, so $12 = 4\pi \cdot 9\pi^{-2/3} \cdot \frac{dr}{dt}$, whence $\frac{dr}{dt} = \frac{1}{3}\pi^{-1/3}$.

4. Let E be the event of at least two students were born on the same day of the week. Then $P(E') = \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} = \frac{120}{343}$. Hence $P(E) = \frac{223}{243}$.

5. Here $y' = -2\cos x \sin x = -\sin 2x$. So $y'' = -2\cos 2x$. Solving $y'' = 0$ gives $x = \frac{\pi}{4}$ for $x \in]0, 1[$. We also observe that the second derivative is changing in sign through this value of x. Hence $a = \frac{\pi}{4}$.

6. Let $s = \sin x$. Then we have $6(1 - s^2) + 5s = 2$, whence $s = -0.5$. Hence $x = \frac{7\pi}{6}, \frac{11\pi}{6}$.

7. We know $y(1) = 2$ and $y'(1) = 4$. Hence $m + n = 3$ and $3m + 2n = 4$. Solving simultaneously gives $m = -2$, $n = 5$.

8. We have $a + ar = 15$ and $a = 27(1 - r)$. Solving simultaneously and remembering that $r > 0$ gives $a = 9$.

9. Here $y' = 6e^{2x} + 12xe^{2x} = 6e^{2x}(1 + 2x)$. Solving $y' = 0$ gives $x = -0.5$. So our stationary point is $(-0.5, -3e^{-1})$. Next $y'' = 24e^{2x}(1 + x)$, which is positive at $x = -0.5$. So our stationary point is a minimum.

10. We can reason from the radius of the inscribed circle is the minimum distance of the curve from the origin or from the normal to the curve must pass through the centre of the circle. Using the former approach and denoting the square of the radius by s we have
$$s = x^2 + e^{-2x^2},$$
whence $s' = 2x - 4xe^{-2x^2}$. Solving $s' = 0$ gives $x^2 = \frac{1}{2}\ln 2$, which is easily checked by the first derivative test to give a minimum value of s. Hence the area of the circle is $\pi s = \frac{1}{2}\pi(1 + \ln 2)$.