FM2 Assignment #15 Name: Jerry Jiang $\frac{10}{10}$ Excellent!

1. Write down the coefficient of x^6 in the Maclaurin series for $\cos 2x$. Hence determine the coefficient of x^6 in the Maclaurin series for $\cos^2 x$ giving your answer as a fraction in lowest terms.

$f(x) = \cos 2x$, then $f^{(6)}(x) = -64\cos 2x$.

$$C_6 = \frac{f^{(6)}(0)}{6!} = \frac{-64}{720} = -\frac{4}{45}$$

$$\cos 2x = 2\cos^2 x - 1, \quad \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\therefore C_6' = -\frac{2}{45}.$$

✓

2. Find the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^{n-1}}$.

- Let $U_n = \frac{(x-3)^n}{n2^{n-1}}$

$$\left|\frac{U_{n+1}}{U_n}\right| = \frac{n}{n+1}\left|\frac{x-3}{2}\right| = \left|\frac{x-3}{2}\right| \text{ as } n \to \infty.$$

- By the Ratio Test, it converges absolutely when $\left|\frac{x-3}{2}\right| < 1$ and diverges when $\left|\frac{x-3}{2}\right| > 1$. The test fails when $\left|\frac{x-3}{2}\right| = 1$.
- Therefore the radius of convergence is 2. ✓

3. Use the substitution $y = 1/x$ and L'Hôpital's rule to evaluate $\lim_{y\to\infty} y - \sqrt{1+y^2}$. Confirm you answer using a series approach.

- $$\lim_{y\to\infty} y - \sqrt{1+y^2} = \lim_{x\to 0} \frac{1}{x} - \sqrt{1+\frac{1}{x^2}} = \lim_{x\to 0} \frac{1 - x\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{1-\sqrt{x^2+1}}{x}$$

Apply the L'Hôpital's rule, $= \lim_{x\to 0} -\frac{2x}{2\sqrt{x^2+1}} = -\frac{0}{1} = 0.$ ✓

- $$\lim_{y\to\infty} y - \sqrt{1+y^2} = \lim_{x\to 0} \frac{1}{x} - \sqrt{1+\frac{1}{x^2}} = \lim_{x\to 0} \frac{1-\sqrt{x^2+1}}{x} = A.$$

$$\sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}} = 1 + \binom{\frac{1}{2}}{1}x^2 + \binom{\frac{1}{2}}{2}x^4 + O(x^6) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + O(x^6)$$

$$\therefore A = \lim_{x\to 0} \frac{1 - 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - O(x^6)}{x} = \lim_{x\to 0} -\frac{1}{2}x + \frac{1}{8}x^3 - O(x^5) = 0.$$ ✓

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4. Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

- Let $u_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$.
- Apply ratio test, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n+1}}{\sqrt{n+2}}\right| \cdot |-3x| = 3|x|$ when $n \to \infty$.
- when $3|x|<1$, series converges absolutely; when $3|x|>1$ it diverges.
- For $3|x|=1$, $x=\pm\frac{1}{3}$.

$\to$ $x=-\frac{1}{3}$, $\sum_{n=1}^{\infty} \frac{(-3\cdot -\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$. Diverges according to the integeral test.

$\to$ $x=\frac{1}{3}$, $\sum_{n=1}^{\infty} \frac{(-3\cdot\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. Converges conditionally.

Therefore, the interval of convergence is $]-\frac{1}{3}, \frac{1}{3}]$.

5. Prove in a conditionally convergent series both the series of positive terms and the series of negative terms diverge.

$U_n = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges, $|U_n| = \sum_{n=1}^{\infty} a_n$ diverges.

$U_n = U_n^+ - U_n^-$, where $U_n^+ = \sum_{n=1}^{\infty} a_{2n-1}$, $U_n^- = \sum_{n=1}^{\infty} a_{2n}$.

so $|U_n| = U_n^+ + U_n^-$.

- $U_n + |U_n| = 2U_n^+$. Since $|U_n|$ is divergent, $U_n + |U_n|$ is divergent, so U_n^+ is divergent.
- $-U_n + |U_n| = 2U_n^-$. Since $|U_n|$ is divergent, subtracting the finite value that U_n converges to is still infinite, so $-U_n + |U_n|$ is divergent. so U_n^- is divergent.

Therefore both series diverge.

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