HL1 Test #1 Name: Jerry Jiang

1. An arithmetic sequence has third term 12 and seventh term 32, find the first term and the common difference.

let first term a, common difference d.

96%

Excellent!

$$\begin{cases} a+(3-1)d=12 \\ a+(7-1)d=32 \end{cases}$$

$4d=20$

$$\begin{cases} d=5 \\ a=2 \end{cases}$$

✓

2. 7. 12. 17. 22. 27. 32
1 2 3 4 5 6 7.

So the first term is 2 and the common difference is 5.

2. Solve $\log_2(x+2)+\log_2(x-2)=5$.

$\log_2(x+2)(x-2)=5$

$(x+2)(x-2)=25$ × 32

$x^2=29$

3

$x=\pm\sqrt{29}$

$\because x+2>0,\ x-2>0$

$\therefore x>2$

$\therefore x=\sqrt{29}$

3. A geometric sequence has third term 11 and sixth term 297, find the first term and the common ratio.

let the first term be a and the common ratio be r.

$$\begin{cases} a\cdot r^{3-1}=11 & ① \\ a\cdot r^{6-1}=297 & ② \end{cases}$$

$\frac{11}{9}$ $\frac{11}{3}$ 11 33 99 297
1 2 3 4 5 6

$\frac{②}{①}: \frac{r^5}{r^2}=27$

$\therefore r^3=27$

$$\therefore\begin{cases} r=3 \\ a=\frac{11}{9} \end{cases}$$

✓

$\therefore$ the first term is $\frac{11}{9}$ while the common ratio is 3.

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4. The sum of the first n positive integers is 210. Find n.

$$\frac{(1+n)\times n}{2} = 210$$

$$n^2+n-420=0$$

$$(n+21)(n-20)=0$$

$n_1 = -21$ (inadmissable)

$n_2 = 20$

$\therefore n = 20$

5. Solve $\log_2 x + \log_4 9 = \log_2 12$.

$$\log_2 x + \log_{2^2} 9 = \log_2 12$$

$$\log_2 x + \frac{1}{2}\log_2 9 = \log_2 12$$

$$\log_2 x + \log_2 9^{\frac{1}{2}} = \log_2 12$$

$\therefore \log_2 3x = \log_2 12$

$\therefore x = 4$

6. Solve $9^x - 3^x = 20$.

$$(3^x)^2 - 3^x - 20 = 0$$

$$(3^x-5)(3^x+4)=0$$

$3^x = 5$ or -4.

$\because 3^x > 0$

$\therefore 3^x = 5$

$\therefore x = \log_3 5$

$= 1.46$ (3sf).

7. Let $U = \{ n \in \mathbb{Z}^+ \mid 1 \leq n \leq 500 \}$.How many numbers in U are neither multiples of 9 nor multiples of 15?

9. 18. 27. 36. 45. ... 495. 55
15. 30. 45. 60. ... 495 33
45. 90. 135 ... 495. 11

$n(M_9) = \lfloor 500 \div 9 \rfloor = 55$

$n(M_{15}) = \lfloor 500 \div 15 \rfloor = 33$

$n(U) = 500$

$\mathrm{lcm}(9, 15) = 45$

$n(M_{45}) = \lfloor 500 \div 45 \rfloor = 11$

$n(M_9 \cup M_{15}) = n(M_9) + n(M_{15}) - n(M_{45})$

$= 55 + 33 - 11$

$= 77$

$n(M_9 \cup M_{15})' = 500 - 77 = 423$

$\therefore$ There are 423 numbers in U that are neither multiples of 9 or multiples of 15.

8. Prove the second log law. That is, prove $\log_a(x \div y) = \log_a x - \log_a y$.

let $\log_a x$ be p and $\log_a y$ be q.

then, $a^p = x$, $a^q = y$

$\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$

$\therefore \log_a (x \div y) = \log_a a^{p-q} = p - q$

$= \log_a x - \log_a y$, as required.

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9. Find the smallest positive integer x for which the sum $x + 2x + 3x + 4x + \cdots + 100x$ is a perfect square.

$$S = x + 2x + \cdots + 100x$$
$$= x(1 + 2 + \cdots + 100)$$
$$= x \cdot \frac{(1+100)\cdot 100}{2}$$
$$= 5050x$$
$$= 2 \times 5^2 \times 101 \times x$$

$\therefore x = 2 \times 101$
$= 202$

since it must make all the prime elements of S have even powers.

$\therefore x = 202.$

1, 2. 3, 4, 5 6, 7, 8 9, 10, 11 12, 13, 14 48, 49.
0 1 2 3. 4 16.

10. Calculate the sum of the series $\sum_{r=1}^{49} \left\lfloor \frac{17r}{50} \right\rfloor$ where $\lfloor x \rfloor$ is the floor of x.

$f(r) = \left\lfloor \frac{17r}{50} \right\rfloor$

$f(1) = f(2) = 0$

$f(3) = f(4) = f(5) = 1$

$f(6) = f(7) = f(8) = 2$

$\cdots\cdots$

$f(45) = f(46) = f(47) = 15$

$f(48) = f(49) = \cancel{15}\ 16.$

in fact, when $3n \in [3, 49]$

$f(3n) = f(3n+1) = f(3n+2) = n$

$$\therefore \sum_{r=1}^{49} \left\lfloor \frac{17r}{50} \right\rfloor = 0 \times 2 + 1 \times 3 + 2 \times 3 + \cdots\cdots + 15 \times 3 + 16 \times 2$$
$$= (1 + 2 + \cdots + 15) \times 3 + 32$$
$$= \frac{(1+15) \times 15}{2} \times 3 + 32$$
$$= 392$$

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Solutions to HL1 Test #1

1. We have the system

$$\begin{cases} a + 2d = 12 \\ a + 6d = 32. \end{cases}$$

   Solving simultaneously give $d = 5$ and $a = 2$.

2. We first note that $x > 2$. Next the first law of logs gives $\log_2(x+2)(x-2) = 5$, whence $x^2 - 4 = 32$ and we conclude $x = 6$.

3. We have the system

$$\begin{cases} ar^2 = 11 \\ ar^5 = 297. \end{cases}$$

   Dividing the second equation by the first gives $r^3 = 27$. So $r = 3$ and we conclude $a = 11/9$.

4. The sum of the first n positive integers is $\frac{n}{2}(1+n)$. So we must solve

$$\frac{n}{2}(1+n) = 210 \Leftrightarrow n(n+1) = 420$$

   whence $n = 20$.

5. Choosing a base of 2, the equation becomes $\log_2 x + \frac{1}{2}\log_2 9 = \log_2 12$, which becomes $\log_2 3x = \log_2 12$. We conclude $x = 4$.

6. Letting $y = 3^x$ gives the quadratic equation $y^2 - y - 20 = 0$, which has solutions $y = 5$ and $y = -4$. We conclude $3^x = 5$, whence $x = 1.46$ (3 s.f.).

7. $n(M_9 \cup M_{15}) = n(M_9) + n(M_{15}) - n(M_{45})$, from which we conclude $n(M_9 \cup M_{15}) = 55 + 33 - 11 = 77$. Hence $n(M_9' \cap M_{15}') = 500 - 77 = 423$.

8. See our red book page 136.

9. The sum of this series is $x(1 + 2 + 3 + \cdots + 100) = x \times 5050 = 2 \times 5^2 \times 101 \times x$. So the smallest value of x for this sum to be a perfect square is $2 \times 101 = 202$.

10. Some calculation, most likley using the GDC, gives the series in expanded form as

$$0 + 0 + (1+1+1) + (2+2+2) + (3+3+3) + \cdots + (15+15+15) + 16 + 16,$$

   and hence the sum is $3 \times (1 + 2 + 3 + \cdots + 15) + 32 = 392$.