Inversive Geometry and its Applications

What is inversive geometry and how can it be applied to simplify and provide new perspectives to geometry problems?

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1 Rationale

When I studied Ptolemy's theorem, I found the provided proof listed below:

In Figure 1, $ABCD$ is a cyclic quadrilateral. Connect AC, BD and show that $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

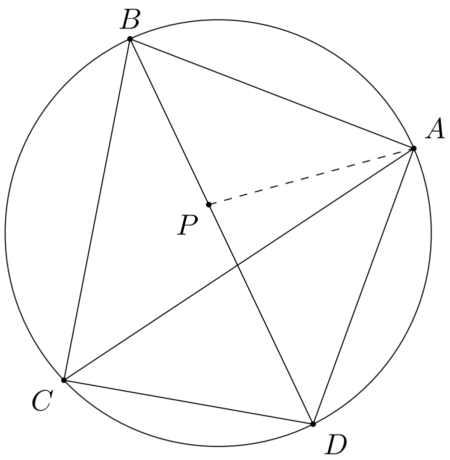

Figure 1: Provided Proof of Ptolemy's Theorem

Proof. Construct $\angle BAP = \angle CAD$ where P lies on line segment BD. Since $\angle ABD$ and $\angle ACD$ are inscribed angles of chord AD, we have:

$$\angle ABD = \angle ACD.$$

According to the construction, $\angle BAP$ and $\angle CAD$ are equiangular, therefore:

$$\triangle ABP \sim \triangle ACD.$$

Similarly,

$$\angle ABD = \angle ACD,$$

$$\angle BAC = \angle BAP + \angle PAC = \angle CAD + \angle PAC = \angle PAD.$$

Therefore,

$$\triangle ABC \sim \triangle APD$$

According to the property of similar triangles, in $\triangle ABP$ and $\triangle ACD$,

$$\frac{AB}{BP} = \frac{AC}{CD}.$$

And in $\triangle ABC$ and $\triangle APD$,

$$\frac{BC}{AC} = \frac{PD}{AD}.$$

Therefore,

$$AB \cdot CD = AC \cdot BP,$$

$$BC \cdot AD = AC \cdot PD.$$

Adding the two equations we get:

$$AB \cdot CD + BC \cdot AD = AC \cdot (BP + PD) = AC \cdot BD.$$

So now, we have $AC \cdot BD = AB \cdot CD + BC \cdot AD$ for cyclic quadrilateral $ABCD$. □

However, I thought that the constructions and side relationships in this approach were not intuitive at all. When I asked my teacher about other approaches to the proof, he introduced me to the concept of inversive geometry. Inversive geometry was invented by Ludwig Immanuel Magnus, a German mathematician, in 1831 [1], and involves the reflection of points in a circle. Realizing that the properties of circle inversions can simplify the proof of Ptolemy's theorem, I became interested in this idea and decided to explore it in greater depth [4].

2 Introduction to Inversions

2.1 Definition of Inversions

One of the most important ideas in inversive geometry is a transformation called inversion. A circle called the circle of inversion is used to carry out this transformation, where objects in the circle will be reflected outside and objects outside will be reflected in based on the following definition:

Definition. Let circle O be the circle of inversion with radius r. Inversion of object A produces image A' where A' lies on ray OA and $OA \cdot OA' = r^2$.

Remark 1. In this paper, the notation circle O or $\odot O$, unless otherwise specified, represents the circle with center O. When point A is said to be reflected in the circle O, A is transformed by inversion with $\odot O$ being the circle of inversion. Normal reflection in Euclidean geometry that preserves distance will be specified as Euclidean Reflection. For the purpose of easier recognition in diagrams, circles of inversions are dashed, objects are colored red, and reflected images are colored blue.

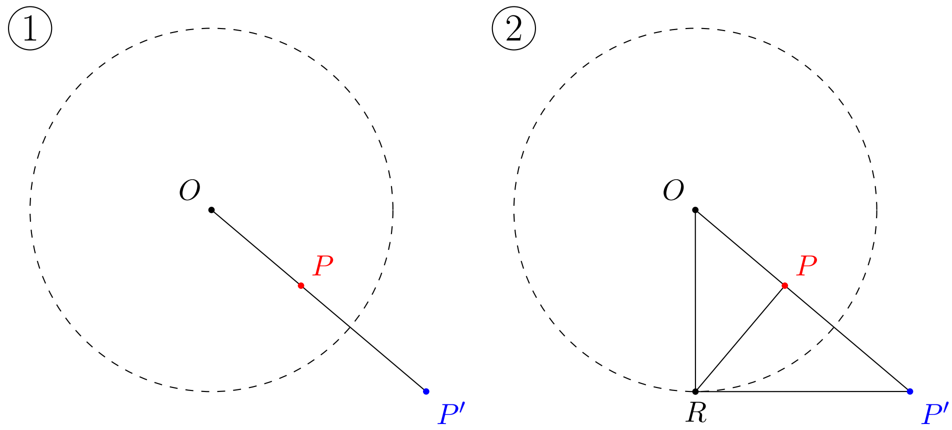

Figure 2: Locating the Image of a Point Reflected in the Circle

In Figure 2 ①, let $\odot O$ be the circle of inversion with radius r. When a point P is reflected in the circle, a point P', the image of P, is obtained outside the circle of inversion. Since $OP \cdot OP' = r^2$, we know that $OP' = \frac{r^2}{OP}$. In order to locate P', we find a point on ray OP whose distance to O is $\frac{r^2}{OP}$.

However, locating the image of an object after inversion by measuring and calculating lengths involves loss of accuracy and is not geometrically pure. Therefore, an alternative way of finding the location of reflected image with a straight edge and compass is provided:

In Figure 2 ②, draw ray OP. Construct line PR perpendicular to ray OP at P while point R is on $\odot O$. Connect OR and draw line RP' perpendicular to OR at R while line RP' intersects ray OP at P'.

Let us prove this approach of locating the image to be valid: in Figure 2 ②, given that $\angle OPR = \angle ORP' = 90°$, show that P' is the image of P after inversion.

Proof. In $\triangle OPR$ and $\triangle ORP'$, we have $\angle OPR = \angle ORP' = 90°$ and the common angle $\angle O$. Therefore,

$$\triangle OPR \sim \triangle ORP'.$$

According to the property of similar triangles, we have:

$$\frac{OP}{OR} = \frac{OR}{OP'}.$$

This is equivalent to:

$$OP \cdot OP' = OR^2.$$

Therefore, P' is the reflected image of P. □

2.2 Relationship between Euclidean Reflections and Inversions

Before we start our discussion around the relationship between Euclidean reflections and inversions, we need to define a concept called a point at infinity:

Definition. A point at infinity is an idealized limiting point at the end of each line that is infinitely far away from all other points, where two parallel lines intersect.

We claim that Euclidean reflection is a special type of inversion from the explanation below.

In Figure 3, let $\odot O$ be the circle of inversion where O is a point at infinity so that a point R on $\odot O$ is infinitely far away from O. When point P is reflected in circle O, the arc around R is infinitely close to a line. Name the reflected image of P as P' and let the radius of circle O be r, the length of $PR = a$, and the length of $P'R = a'$. We want to show that inversion of P in this case is the same as Euclidean reflection.

Figure 3: Inversion and Euclidean Reflection

Proof. According to the rule of inversion, we know that:

$$OP \cdot OP' = OR^2.$$

Since $OR = r$, $PR = a$, and $P'R = a'$,

$$OP = OR - PR = r - a,$$

$$OP' = OR + P'R = r + a'.$$

Substituting in the lengths of the segments, we have:

$$(r - a)(r + a') = r^2.$$

Simplifying the equation, we get:

$$r^2 + (a' - a)r - aa' = r^2,$$

$$(a' - a)r = aa'.$$

Because R is infinitely far away from O, we have $r \to \infty$. Therefore,

$$a' - a = \lim_{r \to \infty} \frac{aa'}{r} = 0.$$

Now since $a' - a = 0$, we finally get : $a = a'$. This indicates that P can be mapped to P' with Euclidean reflection. □

This brings us to the conclusion that Euclidean reflection is a special type of inversion with the center of inversion being a point at infinity.

2.3 Algebraic Understanding of Inversions

One interesting way to view inversions is to look at points in the complex plane. The transformation of inversion I in the unit circle can, as we shall see, be represented as:

$$I(z) = \frac{1}{z^*}.$$

This means that a complex number $z = ae^{i\theta}$ in the complex plane, after being reflected in the unit circle, gives an image at:

$$\frac{1}{z^*} = \frac{1}{a}e^{i\theta}.$$

Remark 2. In this paper, the coordinates of a complex number in the complex plane will be presented in the form of $z = ae^{i\theta}$. The modulus of z, denoted $|z|$, equals a, and the argument of z, denoted $\arg(z)$, equals θ.

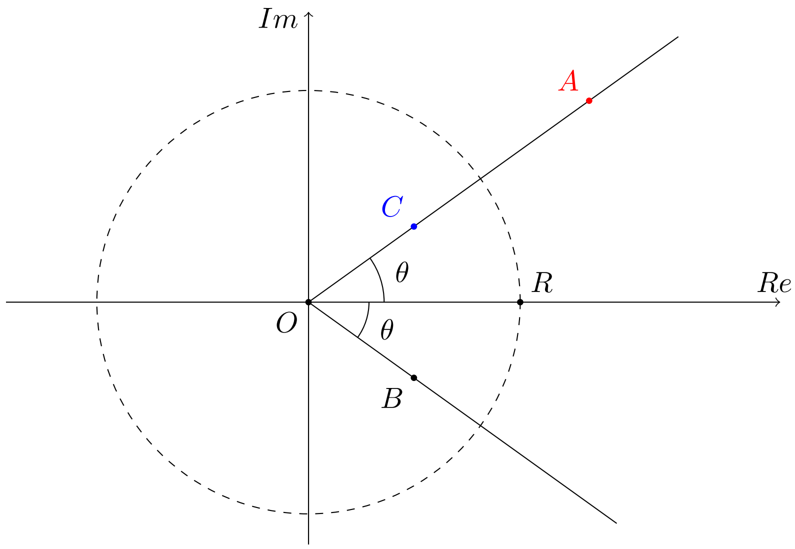

Figure 4: Inversion in the Complex Plane

Let us see why this is the case:

In Figure 4, the coordinates of A are $z = ae^{i\theta}$. Suppose point B is the inverse of A in the complex plane. Because multiplication of two complex numbers can be broken down into multiplication of the corresponding modulii and addition of the corresponding arguments, $|B|$ must have the reciprocal of $|A|$ and $\arg(B)$ should be the opposite of $\arg(A)$ so that the product of the two complex numbers comes to 1. Therefore, the coordinates of B are:

$$\frac{1}{z} = \frac{1}{a}e^{-i\theta}.$$

Then, we take the conjugate of B, which is to keep modulous same as $|B|$ and take the opposite of $\arg(B)$ as the new argument. The conjugate, point C, has coordinates of:

$$\frac{1}{z^*} = \frac{1}{a}e^{i\theta}.$$

Now if we look at $A : z = ae^{i\theta}$ and $C : \frac{1}{z^*} = \frac{1}{a}e^{i\theta}$, we notice that $|A| \cdot |C| = 1$ and $\arg(A) = \arg(C)$. This indicates that the origin O, A, and C lie on the same line and $OA \cdot OC = 1 = r^2$, where r is the radius of the unit circle. Therefore C is the reflected image of A, and hence we have the following theorem:

Theorem 1. *The inversion I that reflects an object in the complex plane in the unit circle can be expressed algebraically as:*

$$I(z) = \frac{1}{z^*}.$$

3 Properties of Inversion

In subsection 2.2, we have proved that Euclidean reflection is a special type of inversion in which the inversion center is a point at infinity. When Euclidean reflections are performed on objects, the corresponding images are exactly the same as the original objects. Let us explore how this can be different for inversions.

3.1 Inversion of a Line

When the inversion center is reflected in the circle of inversion, its image will be a point at infinity. Conversely, a point at infinity can be reflected to the center of inversion. As a line not passing through the circle of inversion extends to a point at infinity, we are claiming that, as we shall see, the image of this line is a circle that passes through the circle of inverison.

In Figure 5, O is the center of inversion and point R lies on the circle of inversion $\odot O$. Line AB intersects the circle at R while it is perpendicular to line OA at A. We name the image of A as A' and we will show that the image of line AB after inversion will be the circle with OA' as its diameter.

Proof. Draw the circle that has OA' as its diameter. Connect OB and it intersects the circle at B'. According to the construction, we know that:

$$\angle OAB = 90^\circ.$$

Since point B' is on the circle that has OA' as its diameter, we have:

$$\angle OB'A' = 90^\circ.$$

So,

$$\angle OAB = \angle OB'A'.$$

In triangle $\triangle OAB$ and $\triangle OB'A'$, there's a common angle $\angle AOB$. Therefore,

$$\triangle OAB \sim \triangle OB'A'.$$

According to the property of similar triangles,

$$\frac{OA}{OB} = \frac{OB'}{OA'}.$$

This is equivalent to:

$$OA \cdot OA' = OB \cdot OB'.$$

Becuase point A' is the image of A, we know that:

$$OA \cdot OA' = OR^2.$$

Therefore,

$$OB \cdot OB' = OR^2.$$

Since B can be any point on line AB, this indicates that the images of all the points on line AB will be on the circle that has OA' as its diameter. □

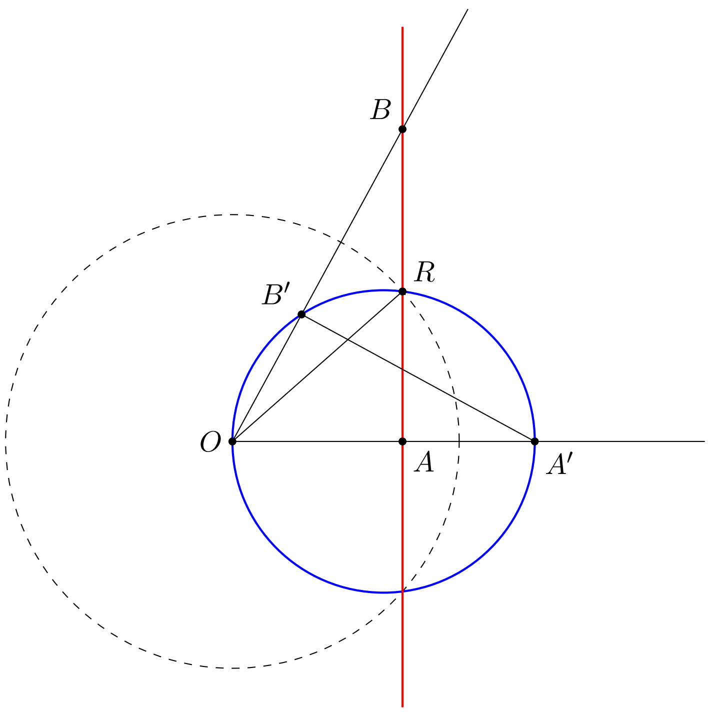

Figure 5: Reflected Image of a Line

We obtain Theorem 2 from the proof above.

Theorem 2. *The image of a line not passing through the inversion center is a circle that passes through the inversion center.*

When a point P is reflected in circle O, we have: $OP \cdot OP' = r^2$, where r is the radius of $\odot O$. When point P' is reflected again in the circle of inversion, we have: $OP' \cdot OP'' = r^2$. So we can conclude $OP = OP''$. As points $P, P', \text{and} P''$ lie on a single ray with vertex O, point P is

P''. Algebraically speaking, inversion in the complex plane can be represented as $I(z) = \frac{1}{z^*}$, so

$$I \circ I(z) = \frac{1}{\left(\dfrac{1}{z^*}\right)^*} = z.$$

We conclude accordingly that inversion is its own inverse. Thus, from Theorem 2, we know that the image of a circle passing through the inversion center is a line that does not pass through the inversion center.

Theorem 3. *The image of a circle passing through the inversion center is a line that does not pass through the inversion center.*

3.2 Inversion of a Circle

In subsection 3.1, we looked at what a line transforms into after being reflected in the circle of inversion from a geometric prespective. Now we will be exploring what the image of a circle not passing through the inversion center becomes after inversion using an algebraic view.

The equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle centered at (a, b) with radius r. Expand the equation and we get a new equation to plot a circle in the Cartesian plane:

$$Ax^2 + Ay^2 + Bx + Cy + D = 0, A, B, C, D \in \mathbb{R}.$$

It is worth noting the equation only plots a circle if $A \neq 0$; when $A = 0$, the equation graphs a line.

Recall from Theorem 1, inversions in the complex plane can be seen algebraically as: $I(z) = \dfrac{1}{z^*}$. This means that if represent the equation of a circle in the complex plane, we will be able to operate the transformation of inversion through an algebraic mean.

In fact, the equation of a circle in the complex plane can be expressed in the form of:

$$Azz^* + wz + w^*z^* + D = 0, A \neq 0, A, D \in \mathbb{R}.$$

Proof. We can start with substituting some variables in the equation of circle in the Cartesian plane: $Ax^2 + Ay^2 + Bx + Cy + D = 0$. Let $z = x + iy$ and $z^* = x - iy$. We notice that:

$$\begin{aligned} zz^* &= (x + iy)(x - iy) \\ &= x^2 + y^2. \end{aligned}$$

Let $w = \dfrac{1}{2}(B - iC)$ and $w^* = \dfrac{1}{2}(B + iC)$. We notice that:

$$\begin{aligned} wz + w^*z^* =& \frac{1}{2}(B - iC)(x + iy) + \frac{1}{2}(B + iC)(x - iy) \\ =& \frac{1}{2}(Bx + iBy - iCx + Cy) + \frac{1}{2}(Bx - iBy + iCx + Cy) \\ =& \frac{1}{2}(2Bx + 2Cy) \\ =& Bx + Cy. \end{aligned}$$

Substituting $x^2 + y^2$ with zz^* and $Bx + Cy$ with $wz + w^*z^*$, we get:

$$Azz^* + wz + w^*z^* + D = 0,$$

where $z = x + iy$ and $w = \dfrac{1}{2}(B - iC)$. □

Let the circle in the complex plane be reflected in the unit circle, which means, algebraically, that z are all mapped to $\frac{1}{z^*}$. Therefore, the equation of the circle becomes:

$$A\left(\frac{1}{z^*}\right)\left(\frac{1}{(z^*)^*}\right)+w\left(\frac{1}{z^*}\right)+w^*\left(\frac{1}{(z^*)^*}\right)+D=0,$$

$$A+wz+w^*z^*+Dzz^*=0.$$

What's interesting is that the new equation obtained is almost exactly the same to the form of

$$Azz^*+wz+w^*z^*+D=0$$

in a way that only the first and last coefficient is switched. Let us look at the following four circumstances:

1. When $A \neq 0$ and $D \neq 0$, the object is a circle that does not pass through the center of inversion. The resulting image with equation

$$A+wz+w^*z^*+Dzz^*=0, A\cdot D\neq 0$$

plots another circle in the complex plane that does not pass through the inversion center.

Theorem 4. *The image of a circle that does not pass through the center of inversion is another circle that does not pass through the inversion center.*

2. When $A = 0$ and $D = 0$, the object is a line through the center of inversion. The resulting image with equation

$$wz+w^*z^*=0, A=D=0$$

plots the line itself. Therefore, a line passing through the inversion center will be reflected to itself.

Theorem 5. *The image of a line that passes through the center of inversion is itself.*

3. When $A \neq 0$ while $D = 0$, the object is a circle that passes through the center of inversion. The resulting image with equation

$$A+wz+w^*z^*=0, A\neq 0, D=0$$

plots a line not passing through the inversion center, verifying Theorem 3.

4. When $A = 0$ while $D \neq 0$, the object is a line that does not pass through the center of inversion. The resulting image with equation

$$wz+w^*z^*+Dzz^*=0, A=0, D\neq 0$$

plots a circle that passes through the center of inversion, verifying Theorem 2.

Summarizing theorem 2-5, we can get the table below:

Inversion Image	Line	Circle
Pass through inversion center	Itself	A line that does not pass through the inversion center
Does not pass through inversion center	A circle that pass through the inversion center	Another circle that does not pass through the inversion center

4 Proof of Ptolemy's Theorem Using Inversive Geometry

After looking at some basic properties of inversion, let us get back to where we started: proving the Ptolemy's Theorem.

So how is inversive geometry related to proving the Ptolemy's Theorem? As the the connection is not obvious, let us first look at what happens to two points in the circle of inversion after being reflected in the circle.

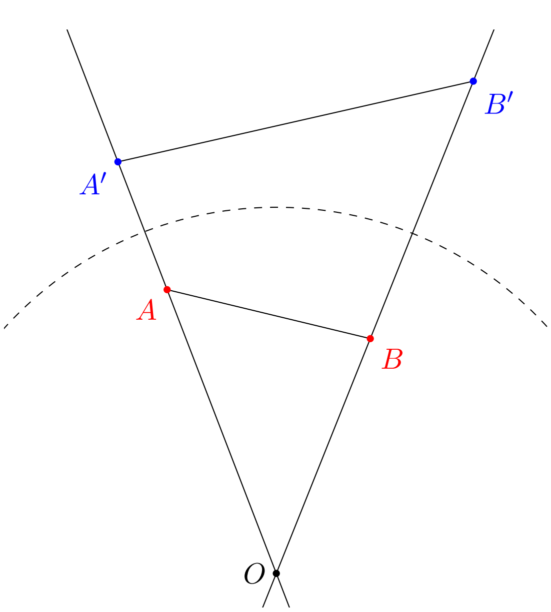

Figure 6: Lemma for the Ptolemy's Theorem Proof Using Inversive Geometry

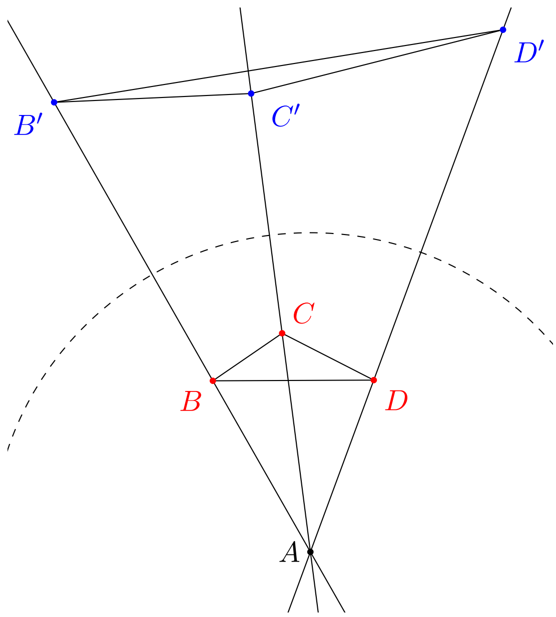

Figure 7: Proof of Ptolemy's Theorem Using Inversive Geometry

In Figure 6, let $\odot O$ be the circle of inversion with radius r. Point A and B are two points inside circle O and their reflected images are point A' and B' correspondingly. We claim that:

$$A'B' = r^2 \cdot \frac{AB}{OA \cdot OB}$$

.

Proof. According to the rule of inversion, we have:

$$OA \cdot OA' = r^2,$$

$$OB \cdot OB' = r^2.$$

From the two squation, we can obtain:

$$\frac{OA}{OB} = \frac{OB'}{OA'}.$$

Considering the shared angle $\angle A$, we can see that:

$$\triangle OAB \sim \triangle OB'A'.$$

According to the property of similar triangles,

$$\frac{A'B'}{AB} = \frac{OA'}{OB}.$$

Since $OA' = \dfrac{r^2}{OA}$, we get:

$$\frac{A'B'}{AB} = \frac{r^2}{OA \cdot OB},$$

which is equivalent to:

$$A'B' = r^2 \cdot \frac{AB}{OA \cdot OB}.$$

□

Lemma. *When two points A and B are reflected in circle O with radius r,*

$$A'B' = r^2 \frac{AB}{OA \cdot OB},$$

where A' and B' are the corresponding images of points A and B.

With the lemma above in mind, let us look at the Ptolemy's Theorem. In Figure 7, $ABCD$ is a quadrilateral and let $\odot A$ with radius r be the circle of inversion. Points B', C', and D' are the corresponding images of points B, C, and D after the inversion.

Proof. According to the lemma, we can see that:

$$B'C' = r^2 \frac{BC}{AB \cdot AC},$$

$$C'D' = r^2 \frac{CD}{AC \cdot AD},$$

$$B'D' = r^2 \frac{BD}{AB \cdot AD}.$$

When $ABCD$ is a cyclic quadrilateral, we know that:

$$\angle ABC + \angle ADC = 180^\circ.$$

Also, by refering to the prove of the lemma, it is easy to see that:

$$\triangle ABC \sim \triangle AC'B',$$

$$\triangle ADC \sim \triangle AC'D'.$$

Therefore,

$$\angle ABC = \angle AC'B',$$

$$\angle ADC = \angle AC'D'.$$

So now we have:

$$\angle AC'B' + \angle AC'D' = 180^\circ,$$

which is saying that points B', C', and D' are collinear:

$$B'C' + C'D' = B'D'.$$

Substituting in $B'C'$, $C'D'$, and $B'D'$, we have:

$$r^2 \frac{BC}{AB \cdot AC} + r^2 \frac{CD}{AC \cdot AD} = r^2 \frac{BD}{AB \cdot AD}.$$

Dividing both sides by r^2 and multiplying by the common denominator, we get:

$$AD \cdot BC + AB \cdot CD = AC \cdot BD.$$

□

But notice that what we can get during this process is not only Ptolemy's theorem, we are also able to obtain Ptolemy's inequality:

Proof. As the shortest distance between two points is the line segment between them, we have:

$$B'C' + C'D' \geq B'D'.$$

Substitue using the equations obtained from the lemma, we get:

$$r^2 \frac{BC}{AB \cdot AC} + r^2 \frac{CD}{AC \cdot AD} \geq r^2 \frac{BD}{AB \cdot AD}.$$

We can obtain Ptolemy's inequality by dividing both sides by r^2 and multiplying by the common denominator:

$$AD \cdot BC + AB \cdot CD \geq AC \cdot BD.$$

□

Corollary. *For any given quadrilateral with verticies A, B, C, and D, we have:*

$$AD \cdot BC + AB \cdot CD \geq AC \cdot BD.$$

5 Futher Explorations of Inversion Properties

Apart from the inversive properties needed to prove Ptolemy's Theorem, there are some more that are really interesting to look at as well.

5.1 Preservation of Tangency

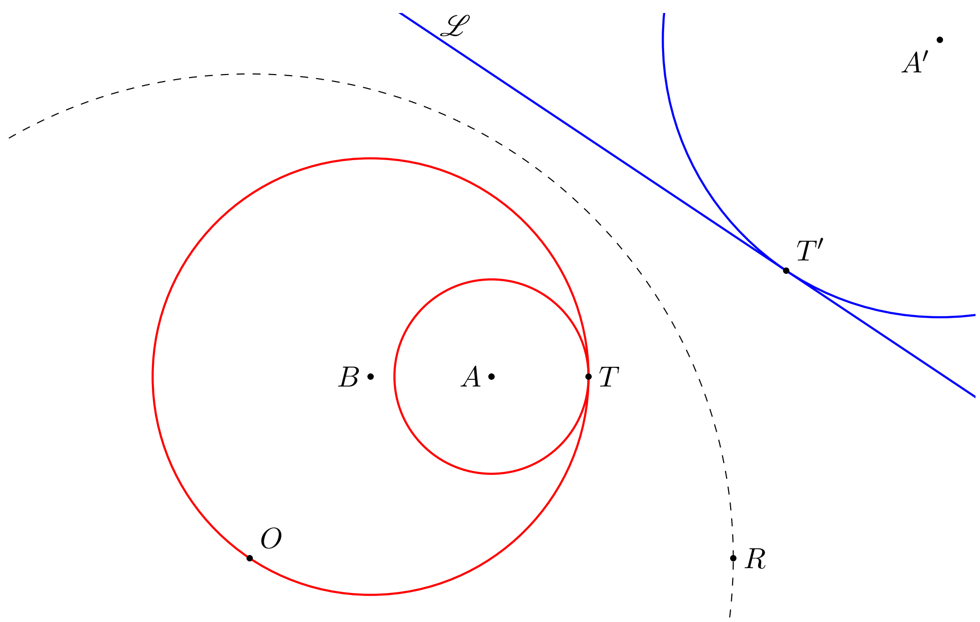

Figure 8: Preservation of Tangency

In Figure 8, we have two circles $\odot A$ and $\odot B$ tangent at T where $\odot B$ passes through the circle of inversion $\odot O$. Circle A', line $\mathscr{L}$, and point T' are the corresponding images of $\odot A$, $\odot B$, and T after being reflected in Cirlce O. We claim that circle A' is still tangent to line $\mathscr{L}$.

Proof. If we see T as a point on circle A, then its image T' must also lie on $\odot A'$; if we view T as a point on circle B, then its image T' must also lie on $\mathscr{L}$. Meanwhile, T as a single point will be reflected to another point T' according to the definition of inversion. Therefore, Cicle A' and $\mathscr{L}$ must have a common point at T'.

It is given that $\odot A$ and $\odot B$ are tangent, which means that there are no other common points between $\odot A$ and $\odot B$ other than T. This ensures that all points other than T will be reflected to different points and therefore no other common points other than T' in the corresponding images. □

Hence, we can say that the tangency of the two images are preserved after the inversion.

Theorem 6. *When two curves are tangent to each other, their reflected images are also tangent to each other.*

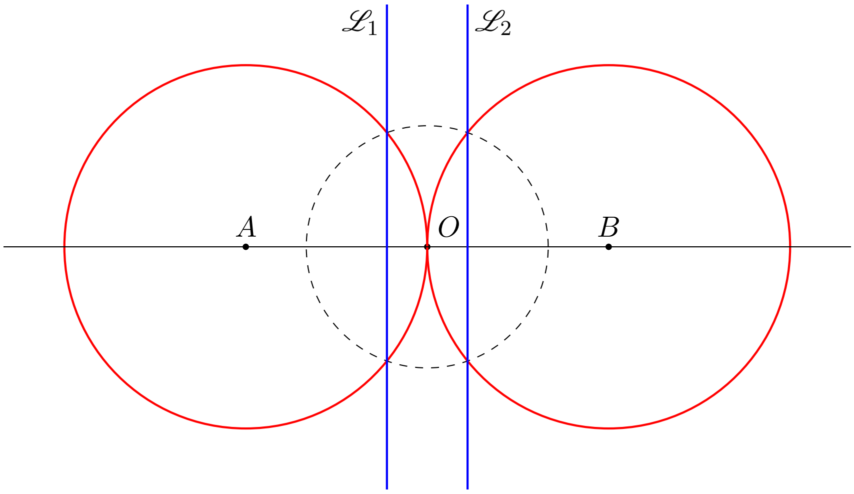

Figure 9: Inversion of two Circles That are Tangent at the Inversion Center

With Theorem 6 in mind, we look at an extreme case in Figure 9: Circle O is the circle of inversion. Two circles $\odot A$ and $\odot B$ are tangent at point O. Let $\odot A$ and $\odot B$ be reflected in $\odot O$. Point O will be reflected to a point at infinity after the inversion. Meanwhile, recall from Theorem 3 that circles passing through the inversion center will be reflected to lines. The only way to satisfy both requirements is to have the images to be two parallel lines $\mathscr{L}_1$ and $\mathscr{L}_2$ otherwise they will intersect at some point. In fact, we say that $\mathscr{L}_1$ and $\mathscr{L}_1$ are tangent at a point at infinity.

Corollary 6.1. *Two circles that are tangent at the center of inversion are reflected to two parallel lines; conversely, two parallel lines are reflected to circles that are tangent at the center of inversion.*

5.2 Image of an Orthogonal Circle

Is there an object that has itself as the reflected image after inversion? An obvious one is a point on the circle of inversion as its distance to the inversion center equals to the circle's radius. According to Theorem 2, another answer can be a line passing through the circle of inversion. In fact, a circle that's orthogonal to the circle of inversion does satisfy the question as well.

Definition. Let $\odot A$ and $\odot B$ be two circles intersecting at X and Y. We call circle A and circle B orthogonal cirlces if both the tangents of the cirlces at X and the tangents of the circles at Y are perpendicular to each other.

We claim that if a circle is orthogonal to the circle of inversion, then the circle will be reflected to itself.

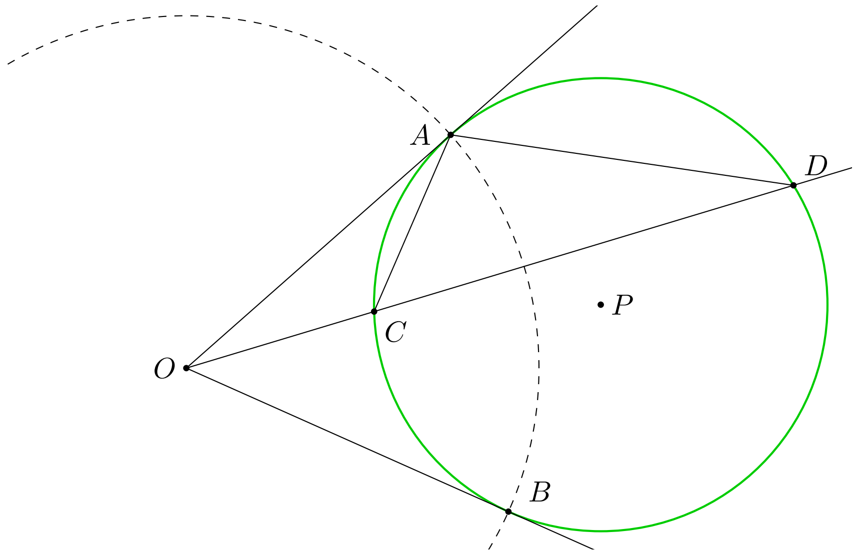

Figure 10: Image of an Orthogonal Circle

Remark 3. For the purpose of easier recognition in diagrams, circles orthogonal to the circles of inversions are colored green.

Proof. In Figure 10, let $\odot O$ be the circle of inversion with radius r and $\odot O$ and $\odot P$ are orthogonal circles. The two circles intersect at A and B.

Construct ray OA, ray OB, and a random ray between OA and OB starting from O that intersects the circle at C and D.

Since circle O and circle P are orthogonal circles, we know that OA and OB are the tangents of $\odot P$. Hence, ray OA only intersect circle P at A and ray OB only intersect circle P at B.

Now let us look at ray OC. According to the tangent-secant theorem, we know that:

$$OA^2 = OC \cdot OD.$$

But we also have:

$$OA = r^2.$$

Therefore,

$$OC \cdot OD = r^2.$$

This indicates that C will be reflected to D and D will be in return reflected to C. Since ray OC is any ray between ray OA and ray OB, we know that any point M except points A and B on circle P will be reflected in circle O to the other intersection of ray OM and circle P. Since it is evident that points A and B will be reflected to itself. we can conclude that $\odot P$ will be reflected in $\odot O$ to itself. $\square$

Theorem 7. *The image of a circle after inversion is itself if it is orthogonal to the circle of inversion.*

5.3 Conformal Property of Inversions

Definition. When a transformation preserves angles between curves or lines, the trasnformation is considered as a conformal map and it is said to be conformal (or angle-preserving).

Remark 4. When we talk about "angles between curves or lines" in conformal mapping, the angle between two lines refers to the acute angle (angle less than 180°) at the intersection of the two lines, the angle between a line and a curve refers to the acute angle at the intersection of the line and the tangent of the curve, while the angle between two curves refers to the acute angle at the intersection of the tangents of the two curves.

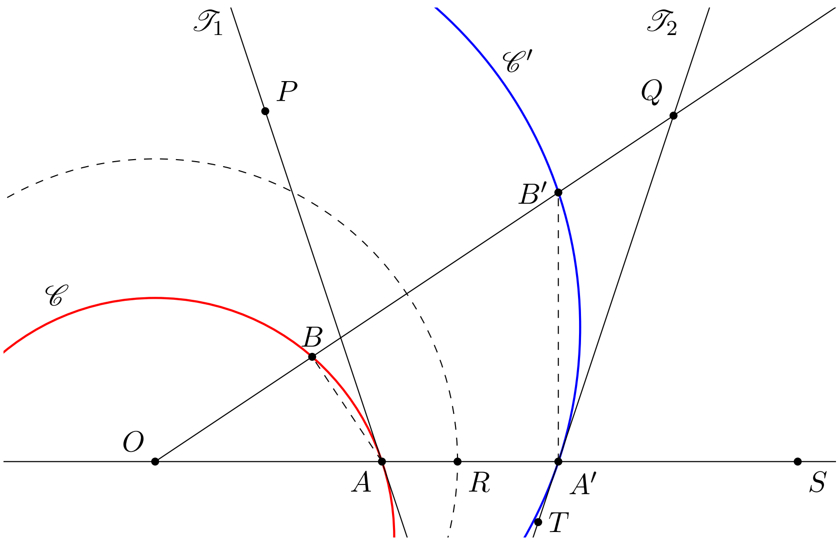

Figure 11: Conformal Property

To illustrate the conformal property of inversions, let us first look at the conformal property of a relatively special case. In Figure 11, circle O is the circle of inversion with radius r. The two objects we are looking at are line OS that passes through the inversion center and curve $\mathscr{C}$. Theorem 5 states that a line passing through the center of inversion will be reflected to itself, so line OS is unchanged after the inversion. Let curve $\mathscr{C}'$ be the image of curve $\mathscr{C}$. Curve $\mathscr{C}$ and $\mathscr{C}'$ intersects line OS at A and A' correspondingly. Point P is a point on $\mathscr{T}_1$, the tangent of curve $\mathscr{C}$ at A, and the tangent of curve $\mathscr{C}'$ at A' is $\mathscr{T}_2$. Ray OQ is ray starting from O that intersects curve $\mathscr{C}$ at B, curve $\mathscr{C}'$ at B', and line $\mathscr{T}_2$ at Q. The angle between line OS and curve $\mathscr{C}$ is $\angle OAP$ and the angle between line OS and curve $A'B'$ is $\angle QA'S$. We claim that $\angle OAP = \angle QA'S$.

Proof. It is easy to show that A' is the image of A and B' is the image of B, so we have:

$$OB \cdot OB' = r^2 = OA \cdot OA'.$$

Therefore,

$$\frac{OB}{OA'} = \frac{OA}{OB'}.$$

Considering the common angle $\angle BOA$, we get:

$$\triangle OBA \sim \triangle OA'B'.$$

So we have:

$$\angle OAB = \angle OB'A'.$$

If we rotate ray OQ so that it is infinitely close to line OS, we find:

$$\angle OAB \to \angle OAP,$$

$$\angle OB'A' \to \angle OA'T.$$

As opposite angles, $\angle OA'T = \angle QA'S$. So,

$$\angle OAP = \angle QA'S.$$

□

After proving the inversion of a line and an arc of a circle to be conformal, we can generalize the case and approach the problem using a different method: complex analysis.

In Figure 12, curve $\mathscr{C}$ is the locus of point z in the $z-$plane. Curve $\mathscr{C}'$, the locus of point w, is the image of $\mathscr{C}$ after application of the function f. We claim that f is conformal at z_0, as long as f is differentiable at z_0.

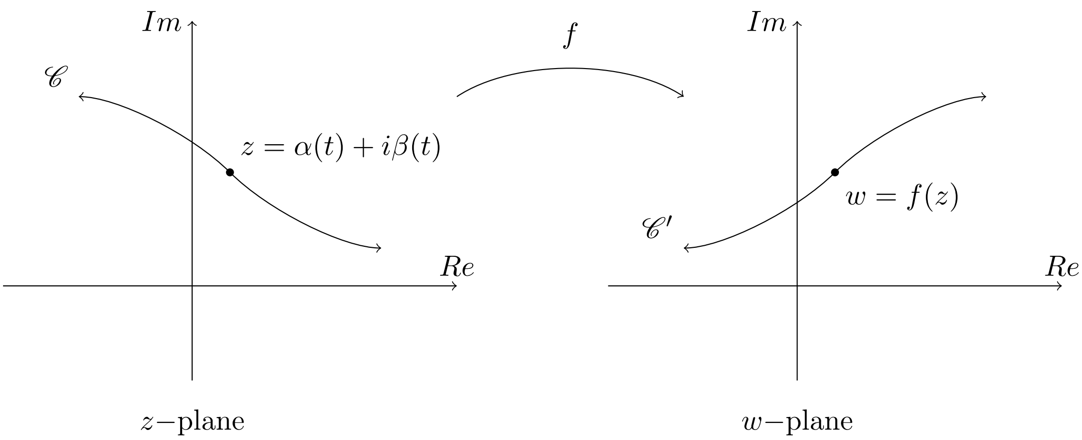

Figure 12: Conformal Property via Complex Analysis

Proof. Parameterize z and w so that $z = \alpha(t) + i\beta(t)$ and $w = \gamma(t) + i\theta(t)$. This way, with a fixed t, we have a fix point z_0, where

$$z_0 = \alpha(t_0) + i\beta(t_0) = z.$$

Accordingly,

$$w_0 = f(z_0) = \gamma(t_0) + i\theta(t_0).$$

The derivative of w with respect to t, by the chain rule, is:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = f'(z) \cdot \frac{\mathrm{d}z}{\mathrm{d}t}.$$

Therefore, the derivative of w with respect to t at $t = t_0$, as long as $f'(z_0)$ exists, is:

$$\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=t_0} = f'(z_0) \cdot \left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=t_0}. \tag{1}$$

As $\frac{\mathrm{d}z}{\mathrm{d}t} = \lim_{\Delta \to 0} \frac{\Delta z}{\Delta t}$, $\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=t_0}$ indicates how z changes with respect to t at t_0. The derivative becomes a direction vector of $\mathscr{C}$ at t_0. Similarly, $\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=t_0}$ is the direction vector of $\mathscr{C}'$ at t_0.

Since z and w are both complex numbers, we know that $\frac{\mathrm{d}z}{\mathrm{d}t}$ and $\frac{\mathrm{d}z}{\mathrm{d}t}$ are complex numbers. Thus, $\arg\left(\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=t_0}\right)$ and $\arg\left(\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=t_0}\right)$ are the angles of inclination for the direction vectors of the respective curves at t_0 in the complex plane. From (1), we get:

$$\arg\left(\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=t_0}\right) = \arg(f'(z_0)) + \arg\left(\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=t_0}\right). \tag{2}$$

Let smooth curve $\tilde{\mathscr{C}}$ be the loci of another point $\tilde{z}$ where it intersects $\mathscr{C}$ at z_0 in the $z-$plane. Smooth curve $\tilde{\mathscr{C}}'$, the loci of point $\tilde{w}$, is the image of $\tilde{\mathscr{C}}$ after transformation f and intersects $\mathscr{C}'$ at w_0 in the $w-$plane. Parameterize $\tilde{z}$ and $\tilde{w}$ so that $\tilde{z} = \tilde{\alpha}(t) + i\tilde{\beta}(t)$ and $\tilde{w} = \tilde{\gamma}(t) + i\tilde{\theta}(t)$. As before, we have:

$$\arg\left(\left.\frac{\mathrm{d}\tilde{w}}{\mathrm{d}t}\right|_{t=t_0}\right) = \arg(f'(z_0)) + \arg\left(\left.\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t}\right|_{t=t_0}\right). \tag{3}$$

Subtracting equation (2) from (3), we get:

$$\arg\left(\left.\frac{\mathrm{d}\tilde{w}}{\mathrm{d}t}\right|_{t=t_0}\right) - \arg\left(\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=t_0}\right) = \arg\left(\left.\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t}\right|_{t=t_0}\right) - \arg\left(\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=t_0}\right). \tag{4}$$

On the left hand side, we have the difference of the slope of the direction vectors of w and $\tilde{w}$ at t_0 equals the right hand side, the difference of the slope of the direction vectors of z and $\tilde{z}$ at t_0. This is to say that the orientation between the curves at $t = t_0$ remains unchanged before and after applying the function f, as long as $f'(z_0)$ exists. □

Lemma. *Transformation f is conformal at z_0, as long as f is differentiable at z_0.*

Let us look at inversion in the complex plane $I(z) = \frac{1}{z^*}$, which can be separated into two steps: taking the conjugate of z and then taking the reciprocal. The nature of conjugation in the complex plane is reflection across the real axis, a conformal transformation. Taking the reciprocal $R(z) = \frac{1}{z}$ is conformal except for the center of inversion according to the lemma, as $R'(z)$ exists for all z except when $z = 0$. Hence the combination of the two conformal transformation leads us to the theorem:

Theorem 8. *Inversion is a conformal transformation.*

6 Applications of Inversion

In section 4, we have seen how inversive geometry provides a natural setting for the proof of Ptolemy's theorem. Now let us look at some other cases where inversions make things simpler.

6.1 The Shoemaker's Knife Problem

The Shoemaker's Knife problem is a very typical geometry question with complicated solution through a normal approach. In Figure 13, AB is the diameter of semicircle $\mathscr{S}$ with center C and radius R. Two smaller semicircles are constructed using AC and BC as diameters and are named semicircle $\mathscr{S}_1$ and semicircle 0 correspondingly. Now a circle that is tangent to semicircle

$\mathscr{S}$, semicircle $\mathscr{S}_1$, and semicircle 0 is drawn and named circle 1. Another circle that is tangent to semicircle $\mathscr{S}$, semicircle $\mathscr{S}_1$, and circle 1 is drawn and named circle 2. Similarly, a circle that is tangent to semicircle $\mathscr{S}$, semicircle $\mathscr{S}_1$, and circle $(n-1)$ is drawn and named circle n.

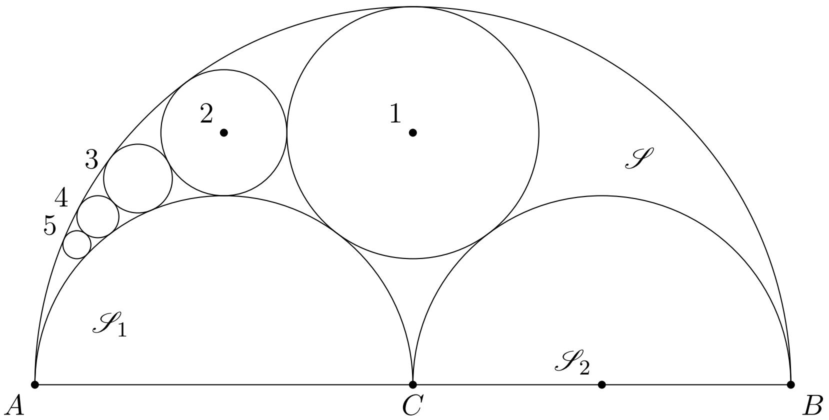

Figure 13: The Shoemaker's Knife Problem

The classical problem of The Shoemaker's Knife asks for an expression of the distance from the center of circle n to the diameter AB using r_n, the radius of $\odot n$, and R, the radius of semicircle $\mathscr{S}$ [2]. However, I am going to solve the problem in a great depth, namely, finding an expression of the distance soely in terms of R and n, the ordering of the circle.

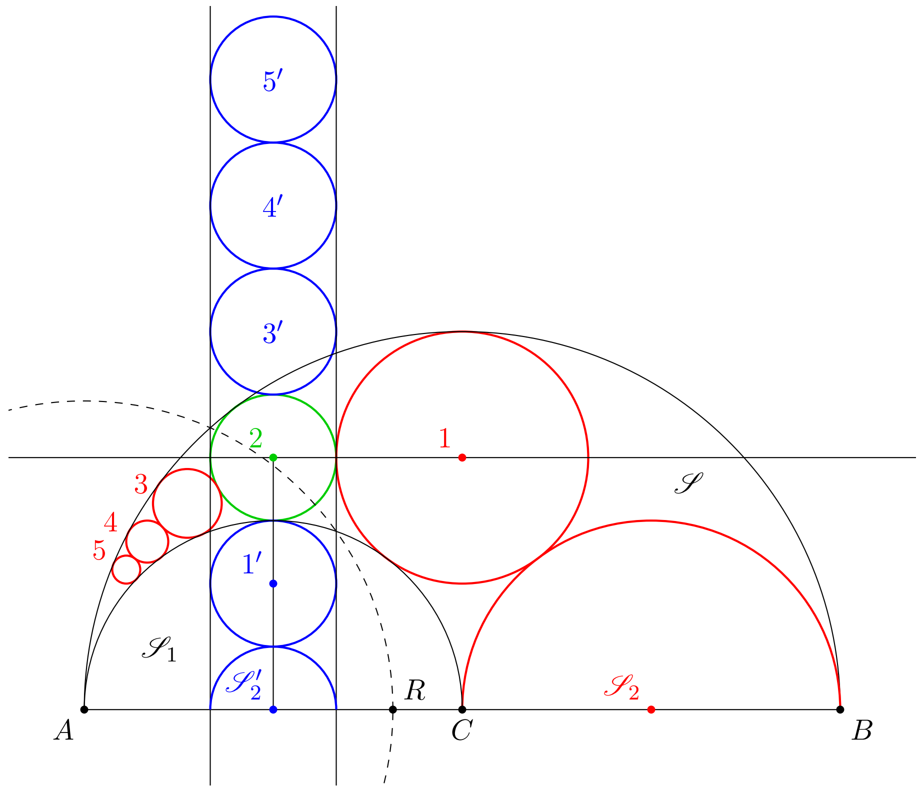

Figure 14: The Shoemaker's Knife Problem - Step 1

It is definitely doable to connect the centers of circles and look for the relationships between sides of triangles using Pythagoras' Theorem, but the amount of work required through this approach is tremendous. Instead, I am going to separate the solution into the two following steps:

Step 1. Find an expression of the distance from the center of circle n to the diameter AB using n, the ordering of circle n, and r_n, the radius of circle n.

Without the loss of generality, we use $\odot 2$ in Figure 14 to illustrate the approach. Let circle A be the circle of inversion and orthogonal to circle 2. According to Theorem 7, the image of a circle is itself if it is orthogonal to the circle of inversion, so we know that circle 2 is unchanged. Recall from Theorem 6 and 6.1 that tangency is preserved during inversion and circles tangent at the inversion center will be reflected to two parallel lines, we have semicircles $\mathscr{S}$ and $\mathscr{S}_1$ reflected to two parallel lines that are both tangent to $\odot 2$, while the image of circle 1 is stacked between the two parallel lines, on top of $\mathscr{S}_2'$, and below circle 2.

Since all images are between the parallel lines, they are of the same size. Therefore we know the distance of $\odot 2$ from AB is $4 \cdot r_2$. Generalizing the case, the distance of $\odot n$ from AB is $2n \cdot r_n$.

Step 2. Find an expression of r_n in terms of n and R, the radius of semicircle $\mathscr{S}$.

In Figure 15, let circle A with radius R be the circle of inversion. Without the loss of generality, $n = 5$ is used in the figure to illustrate the approach. Similar to step one, we know that the images after inversion are circles stacking on top of each other and bounded by two parallel lines on the sides. We know that $AB = 2R$, so

$$AB' = \frac{R^2}{2R} = \frac{R}{2}.$$

This shows that the diameters of the images are $\frac{R}{2}$, and consequently, the radii of the images are $\frac{R}{4}$.

Now in $\triangle AHO'$, the length of AH equals $R - \frac{R}{4} = \frac{3R}{4}$, while the length of HO' equals $10 \cdot \frac{R}{4} = \frac{5R}{2}$,the sum of 10 radii of the stacked circles. Therefore, according to the Pythagorean theorem, we obtain:

$$AO' = \sqrt{AH^2 + HO'^2} = \sqrt{\left(\frac{3R}{4}\right)^2 + \left(\frac{5R}{2}\right)^2} = \frac{\sqrt{109}}{4}R.$$

Generalizing the case gives:

$$AO' = \frac{\sqrt{9 + 4n^2}}{4}R.$$

Next, we know that the lengths of $O'Q'$ and $O'P'$ equal $\frac{R}{4}$. We then get:

$$AP' = AO' + O'P' = \frac{\sqrt{9 + 4n^2} + 1}{4}R,$$

$$AQ' = AO' - O'Q' = \frac{\sqrt{9 + 4n^2} - 1}{4}R.$$

Recall that points P' and Q' are the corresponding images of points P and Q, we have the relationships:

$$AP \cdot AP' = R^2,$$

$$AQ \cdot AQ' = R^2.$$

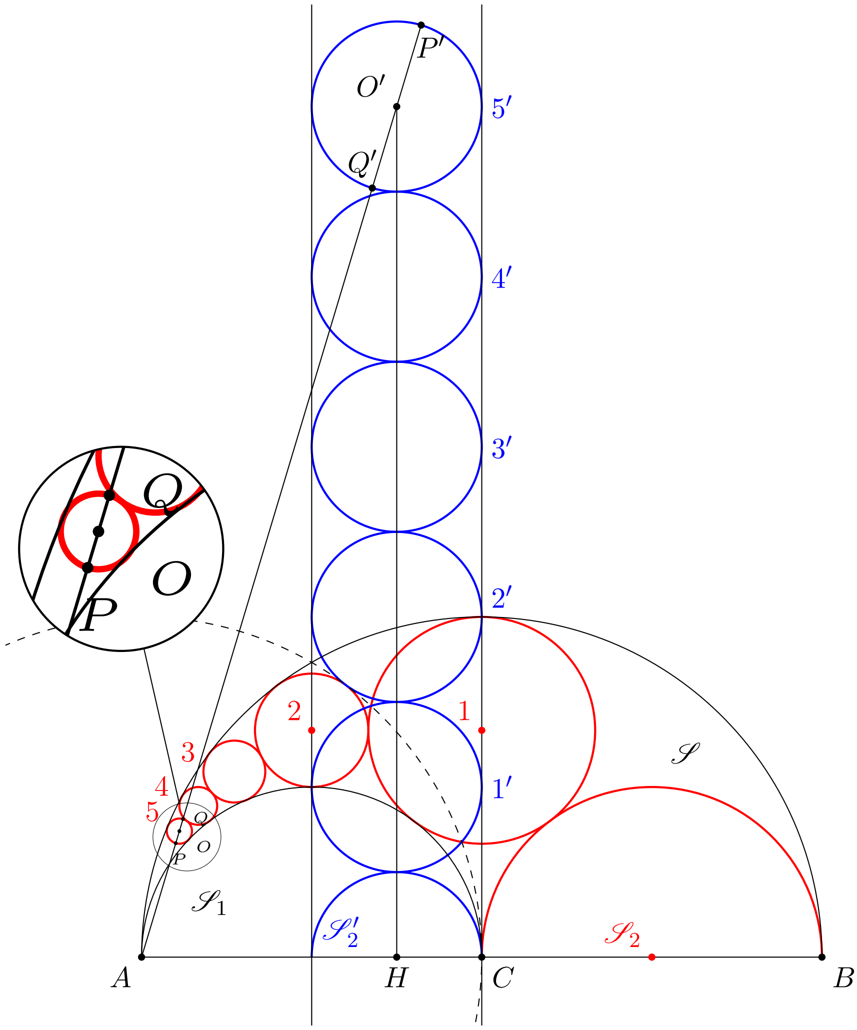

Figure 15: The Shoemaker's Knife Problem - Step 2

Substituting AP' and AQ' into the equations, we obtain:

$$AP = \frac{R^2}{\frac{\sqrt{9+4n^2}+1}{4}R} = \frac{4R}{\sqrt{9+4n^2}+1} = \frac{4(\sqrt{9+4n^2}-1)}{9+4n^2-1}R = \frac{\sqrt{9+4n^2}-1}{2+n^2}R,$$

$$AQ = \frac{R^2}{\frac{\sqrt{9+4n^2}-1}{4}R} = \frac{4R}{\sqrt{9+4n^2}-1} = \frac{4(\sqrt{9+4n^2}+1)}{9+4n^2-1}R = \frac{\sqrt{9+4n^2}+1}{2+n^2}R.$$

Hence, we can represent the length of PQ:

$$PQ = AQ - AP = \frac{\sqrt{9+4n^2}-1}{2+n^2}R - \frac{\sqrt{9+4n^2}+1}{2+n^2}R = \frac{2R}{2+n^2}.$$

Since PQ is the diameter of circle n, we conclude:

$$r_n = \frac{1}{2}PQ = \frac{R}{2+n^2}.$$

Finally, we combine the two steps by substituting r_n into the expression obtained in step 1. The distance of circle n from AB is:

$$\frac{2n}{2+n^2}R.$$

6.2 Inversive Approach to a Geometric Problem

"Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic."

This is the second problem of USAMO in year 1993 [3] and I manage to come up with a different approach that uses inversive geometry inspired by a suggestion from my teacher.

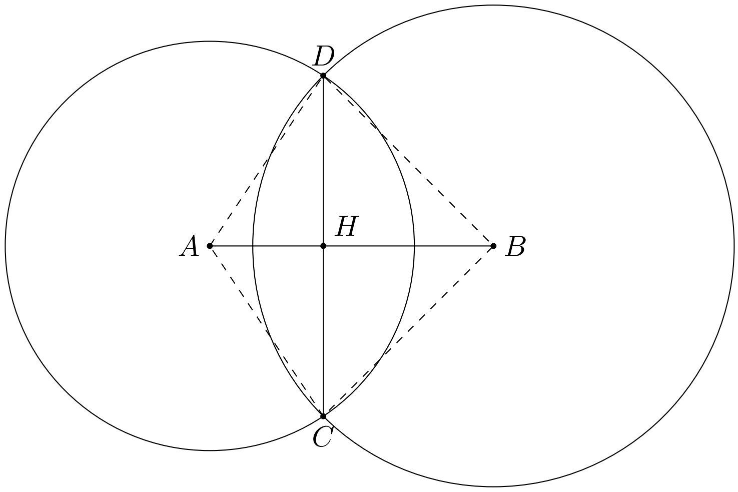

Figure 16: Lemma for Euclidean Reflection

Lemma. *Let circles A and B intersect at P and Q, then Q is the image of P after the Euclidean reflection across AB.*

Proof. In Figure 16, $\odot A$ and $\odot B$ intersect at points C and D. Connect AC, BC, AD, and BD. Construct segment CD so that it intersects AB at H. It is evident that:

$$AC = AD, BC = BD.$$

Together with the shared side AB, we have:

$$\triangle ACB \cong \triangle ADB.$$

Therefore,

$$\angle CAH = \angle DAH.$$

Now becuase $AC = AD$ and AH as the common side,

$$\triangle ACH \cong \triangle ADH.$$

Thus,

$$CH = DH,$$

$$\angle AHC = \angle AHD = 90°.$$

Accordingly, D is the image of point C after euclidean reflection across AB. □

Now we try to prove that the four points asked by the problem are concyclic.

Proof. In Figure 17, we construct circles A, B, C, and D so that they all pass through point E and they intersect at points E_1, E_2, E_3, and E_4. According to the lemma, we know that the points are the images of E after being reflected across the sides of quadrilateral $ABCD$.

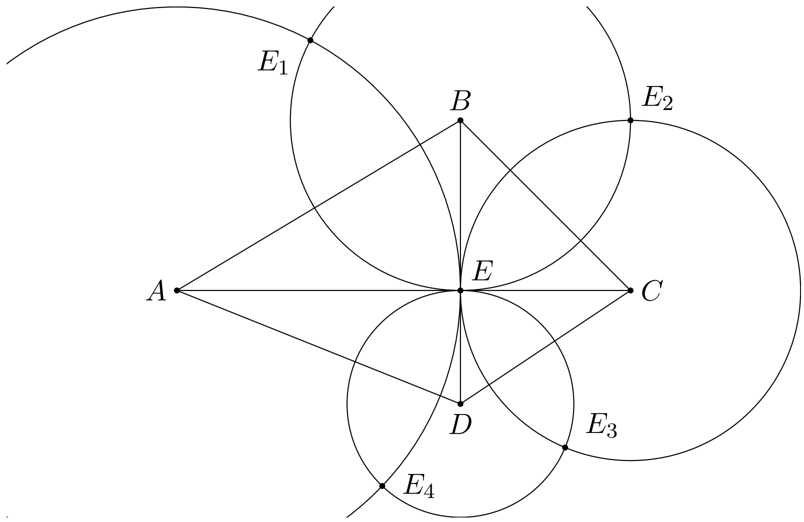

Figure 17: Images of Euclidean Reflections

It is given that AC and BD intersect at right angles, therefore $\odot A$ is orthogonal to both $\odot B$ and $\odot D$. Similarly, $\odot B$, $\odot C$, and $\odot D$ are orthogonal to the two adjacent circles. Now let point E be the center of inversion. Recall from Theorem 3 that a circle passing through the inversion center is reflected to a line that does not pass through the center of inversion. Also in Theorem 8, we've proved that inversion is a conformal map. Since the four circles intersect each other at right angles at points E_1, E_2, E_3, and E_4 due to the orthogonality, $\odot A$, $\odot B$, $\odot C$, and $\odot D$ can only be reflected to lines that form the sides of a rectangle where points E_1', E_2', E_3', and E_4', images of E_1, E_2, E_3, and E_4, are the vertices of the rectangle.

As the opposite angles in a rectangle are supplement to each other, E_1', E_2', E_3', and E_4' are concyclic. Let them be on a circle $\mathscr{C}'$. Since circle $\mathscr{C}'$ does not pass through point E, according to Theorem 5, the original object before the inversion must also be a circle, denoted $\mathscr{C}$. Since points E_1', E_2', E_3', and E_4' all lie on $\mathscr{C}'$, all four points E_1, E_2, E_3, and E_4 must also be lying on $\mathscr{C}$ before the inversion. This way, we can finally conclude that E_1, E_2, E_3, and E_4 are concyclic. □

7 Conclusion

In this extended essay, we have discussed how inversive geometry can help simplify and illustrate complex ideas in Euclidean geometry. We looked at properties of inversions and saw how inversions provide a more natural setting to the proof of Ptolemy's Theorem, solve The Shoemaker's Knife problem, and give elegant solutions to geometric problems as challenging as those appeared in USAMO. There are a lot more that is be explored about inversive geometry, and it has also become part of other more powerful transformations, for instance, inversion in a sphere, which can extend the techniques mentioned in this essay to problems in three-dimensions. These ideas, when applied properly, can inspire great minds and produce fascinating works of mathematics.

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