HL1 Assignment #22

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Excellent!!

1. Simplify $a^{1/\ln a}$.

$a = e^{\ln a}$

$\therefore a^{\frac{1}{\ln a}} = e^{\ln a \cdot \frac{1}{\ln a}} = e$

2. Suppose that the function f is even and everywhere differentiable. Show that the derivative f' is odd.

$f(x) = f(-x)$

$f'(-x) = (f(x))' \cdot (-x)'$

$= f'(x) \cdot -1$

$= -f'(x)$

$\therefore$ odd function.

3. Fill in the table giving exact answers or answers correct to four decimal places.

n	1	100	10 000	1 000 000
$(1+\frac{2}{n})^n$	3	7.2446	7.3876	7.3890

Give the *exact* value of the limit $\lim_{n\to\infty}(1+\frac{2}{n})^n$ and justify your answer.

There's $f(x) = \ln x$, $f'(x) = \frac{1}{x}$. $f'(1) = 1$

$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$

$f'(1) = \lim_{h\to 0} \frac{\ln(1+h) - \cancel{\ln 1}}{h}$

$= \lim_{h\to 0} \ln(1+h)^{\frac{1}{h}}$

let $n = \frac{2}{h}$ $\frac{2}{n} \to 0 \Rightarrow n \to \infty$

$\therefore f'(1) = \lim_{n\to\infty} \cdot \ln(1+\frac{2}{n})^{\frac{1}{2}\cdot n} = 1$

$\therefore \lim_{n\to\infty} (1+\frac{2}{n})^{\frac{1}{2}\cdot n} = e$

$\therefore \lim_{n\to\infty} (1+\frac{2}{n})^{n} = e^2$

4. Find the derivative of the function $f(x) = x^x$.

$f(x) = (e^{\ln x})^x$

$= e^{\ln x \cdot x}$

$f'(x) = (e^{\ln x \cdot x})'$

$= e^{\ln x \cdot x} \cdot (\ln x \cdot x)'$

$= e^{\ln x \cdot x} \cdot (\frac{1}{x} \cdot x + \ln x)$ ✓

$= x^x (1 + \ln x)$

5. The normal to $y = \ln x$ at the point P has y-intercept $1 + e^2$. Find the coordinates of P.

$y' = \frac{1}{x} = f'(x)$

$P(a, \ln a)$

$f'(a) = \frac{1}{a}$

$\therefore l: y = -ax + 1 + e^2$

$\ln a = -a^2 + 1 + e^2$ ✓

$a = e$

$\therefore P(e, 1)$

6. Each side of an equilateral triangle is increasing at a rate of $3\,\text{cm}\,\text{s}^{-1}$. At what rate is the area of the triangle increasing when each side is $10\,\text{cm}$ long?

$A = \frac{\sqrt{3}}{4}(a)^2$ $\quad a = 3t$ $\quad \frac{da}{dt} = 3$

$\frac{dA}{da} = A' = \frac{\sqrt{3}}{2} a$

$\frac{dA}{da} \cdot \frac{da}{dt} = \cancel{A} = \frac{\sqrt{3}}{2} a \cdot 3 = \frac{dA}{dt}$.

when $a = 10$, $A' = 15 \cdot \sqrt{3}$ cm^2/s ✓

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7. Find the coordinates of any stationary points on the curve $y = \sin(\pi \sin x)$ given $x \in [0, \pi]$.

$y' = \cos(\pi . \sin x) \cdot \pi . \cos x = 0$

① $\cos x = 0$

$x = \frac{\pi}{2}$

② $\cos(\pi \sin x) = 0$

$\pi \sin x = \frac{\pi}{2} + k\pi$

$\sin x = \frac{1}{2} + k$

$\therefore \sin x = \frac{1}{2} / -\frac{1}{2}$

$x = \frac{\pi}{6} / \frac{5}{6}\pi$

$\therefore (\frac{\pi}{2}, 0)$

$(\frac{\pi}{6}, 1)$

$(\frac{5}{6}\pi, 1)$

8. Prove $\lim_{x\to\infty} \frac{\lfloor x \rfloor}{x} = 1$ where $\lfloor x \rfloor$ is the floor of x.

$\lim_{x\to\infty} \frac{x-a}{x}$ $(a \in [0, 1[)$.

$= \lim_{x\to\infty} \left(\frac{x}{x} - \frac{a}{x}\right)$

$= 1 - 0$

$= 1$

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9. Find the values of k so that $y = e^x \cos kx$ satisfies the differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$ for all values of x.

$y' = e^x \cos kx - k \cdot e^x \sin kx$

$y'' = e^x \cos kx - k \cdot e^x \sin kx - k(e^x \cos kx \cdot k + e^x \cdot \sin kx)$

$\therefore \; y'' - 2y' + 2y = 0$

$\therefore \; e^x \cos kx (1 - k^2 - 2 + 2) + e^x \sin kx (\cancel{-2k + 2k}) = 0$

$\therefore \; k^2 = 1$

$\therefore \; k = \pm 1.$

10. For what value of k does $y = \sin(\ln x)$ satisfy the differential equation $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = k$? Hence find a solution to the differential equation

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = x\frac{dy}{dx} - y + 5.$$

① $-\sin\cancel{(\ln x)} - \cos\cancel{(\ln x)} + \cos\cancel{(\ln x)} + \sin\cancel{(\ln x)} = k$

$k = 0$.

② $x^2 \cdot \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = x\dfrac{dy}{dx} - y + 5$

$x^2 \cdot \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y - 5 = 0$

$\therefore \; y = \sin(\ln x) + 5.$

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Solutions to HL1 Assignment #22

1. $a^{1/\ln a} = (e^{\ln a})^{1/lna} = e^1 = e$.

2. If f is even then $f(x) = f(-x)$. Since f is differentiable we have by the chain rule $f'(x) = f'(-x) \cdot (-1) = -f(-x)$, whence f' is odd.

3. The table entries are 3, 7.2446, 7.3876 and 7.3890. Now $\lim_{n\to\infty}(1+\frac{2}{n})^n = \lim_{n\to\infty}[(1+\frac{2}{n})^{n/2}]^2 = e^2$.

4. First $x^x = e^{x\ln x}$. So $f'(x) = e^{x\ln x}(\ln x + 1) = x^x(\ln x + 1)$.

5. Let the coordinates of P be $(a, \ln a)$. Then $m_N = -a$. So $N\colon y - \ln a = -a(x-a)$. Solving when $x = 0$ and $y = 1 + e^2$ gives $a = e$. Hence P has coordinates $(e, 1)$.

6. Let the side length of the triangle be x cm and the area be $A\,\text{cm}^2$. Then $A = \frac{1}{2}x^2\sin 60° = \frac{\sqrt{3}}{4}x^2$. By the chain rule $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$, so
$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}x \times \frac{dx}{dt} = 15\sqrt{3}.$$
So the area is increasing at a rate of $15\sqrt{3}\,\text{cm}^2$ when the side is 10 cm.

7. Here $y' = \cos(\pi\sin x)\cdot\pi\cos x$. Solving $y' = 0$ gives $\cos x = 0$ or $\pi\sin x = \frac{\pi}{2}$. Hence the stationary points For $x \in [0, \pi]$ are $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{6}, 1)$ and $(\frac{5\pi}{6}, 1)$.

8. First observe that $x - 1 < \lfloor x\rfloor \le x$. So for $x > 0$
$$\frac{x-1}{x} < \frac{\lfloor x\rfloor}{x} \le \frac{x}{x}.$$
Since $\lim_{x\to\infty}\frac{x-1}{x} = 1$ and $\lim_{x\to\infty}\frac{x}{x} = 1$, we conclude by the squeeze theorem that $\lim_{x\to\infty}\frac{\lfloor x\rfloor}{x} = 1$ also.

9. To simplify the algebra let $C = \cos kx$ and $S = \sin kx$. Then $y' = e^x(C - kS)$ and $y'' = e^x(C(1-k^2) - 2kS)$. So
$$y'' - 2y' + 2y = e^x[C(1-k^2) - 2kS - 2(C - kS) + 2C] = e^xC(1-k^2).$$
We conclude $k = \pm 1$.

10. Let $C = \cos(\ln x)$ and $S = \sin(\ln x)$. Then $y' = C\cdot\frac{1}{x}$ and $y'' = -\frac{S+C}{x^2}$. So $x^2y'' + xy' + y = -S - C + C + S = 0$, whence $k = 0$. Next notice
$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}.$$
So our differential equation can be written as $x^2 + xy' + y = 5$. We conclude $y = \sin(\ln x) + 5$ is a solution.