FM2 Test #2 Name: Jerry Jiang 11

98% Excellent!

1. Find the order of $(3, 15)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{18}$.

In $\mathbb{Z}_4$, $|3| = 4$; in $\mathbb{Z}_{18}$, $|15| = 6$.

Therefore we take $\text{lcm}(4, 6) = 12$ so that both 3 and 15 can get to their identity.

$|(3, 15)| = 12$.

2. An equivalence relation on the set $\{1, 2, 3, 4, 5\}$ creates the partition $\{\{1, 2, 3\}, \{4\}, \{5\}\}$. Give the relation matrix.

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$v - e + f = 2$

3. For each of the following either explain why the graph cannot exist or draw a graph with the given property.

(a) A tree with seven vertices and seven edges.

A tree satisfies the euler's formula as it's simple, connected and planar.

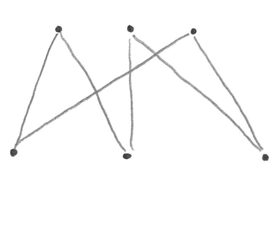
$v - e + f = 2$. $7 - 7 + \boxed{1} = 1 \neq 2$. $\rightarrow$ faces of trees are 1 as there're no cycles present.

Therefore, it doesn't exist.

(b) A simple bipartite graph on six vertices with an Eulerian circuit and a Hamiltonian cycle.

In order for a simple bipartite graph with m, n vertices on the two parts to have Hamiltonian cycle, m must equal n.

In fact, the graph below satifies the given property.



4. Prove that every cyclic group is Abelian.

Let $G = \langle a \rangle$.

For $a^i, a^j \in G$,

$a^i \circ a^j = a^{i+j} = a^{j+i} = a^j \circ a^i$. ✓

Therefore all cyclic groups are abelian.

5. Prove that the function $f : \mathbb{N} \to \mathbb{N}$ with rule $f(n) = n + 1$ is injective but not surjective.

- let $n_1, n_2 \in N$, $n_1 \neq n_2$, $f(n_1) = f(n_2)$.

  then $n_1 + 1 = n_2 + 1$, $n_1 = n_2$. ✓

  The contradiction proves the injectivity.

- $0 \in N$. But when $f(n) = n+1 = 0$, $n = -1$.

  $-1 \notin N$. So 0 is in the codomain but not in the range. ✓

  Therefore there's no surjectivity.

6. Determine the null space, nullity and rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

$\text{rref}(A) = S = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$.

For $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, let $x_2 = m$, $x_3 = n$.

- null space : $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = m \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + n \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$. ✓

- nullity $= 2$. ✓

- basis for row space $= \{ (1\ 2\ 3) \}$.

- basis for column space $= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

- rank $(A) = 1$. ✓

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7. The relation $\sim$ is defined on $\mathbb{Z}^+$ by $x \sim y$ if $x + y$ is even. Prove that $\sim$ is an equivalence relation and give the equivalence classes.

- For an odd $a \in \mathbb{Z}^+$, $a + a = 2a$ is even, $a \sim a$;
  For an even $b \in \mathbb{Z}^+$, $b + b = 2b$ is even, $b \sim b$. reflexive ✓.
- For $m, n \in \mathbb{Z}^+$, $m \sim n$, $m+n$ is even. Since addition is commutative, $n+m$ is even. Therefore $n \sim m$. symmetric ✓.
- For $x, y, z \in \mathbb{Z}^+$, $x \sim y$, $y \sim z$. So $x+y$ and $y+z$ are both even. We know that x & y have the same parity, y & z have the same parity, therefore x & z have the same parity, thus $x \sim z$. transitive ✓

Equivalence classes are odd numbers and even numbers.

✓

8. Use the mean value theorem to find $a, b \in \mathbb{Q}$ so that $a < \sqrt[3]{10} < b$.

$f(x) = x^{\frac{1}{3}}$ is continuous on $[8, 10]$ and differentiable on $]8, 10[$.

Apply the MVT so that there's a $c \in]8, 10[$ that satisfies:

$$f(10) = f(8) + f'(c)(10-8). \quad \text{so } \sqrt[3]{10} = 2 + 2f'(c).$$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}. \quad \text{so } \frac{1}{3\sqrt[3]{8^2}} > f'(c) > \frac{1}{3\sqrt[3]{10^2}} > \frac{1}{3\sqrt[3]{27^2}} = \frac{1}{27}$$

Therefore $\frac{1}{27} < f'(c) < \frac{1}{12}$, $2\frac{2}{27} < 2 + 2f'(c) < 2\frac{1}{6}$.

So $2\frac{2}{27} < \sqrt[3]{10} < 2\frac{1}{6}$. $a = 2\frac{2}{27}$, $b = 2\frac{1}{6}$.

✓

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9. Let G be the group of 2×2 matrices under addition and $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 \right\}$. Prove that $H < G$.

- $H = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$. For $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in H$.

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & -a-x \end{pmatrix}$. since $a+x-a-x=0$, closure ✓

- identity of G is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. since all entries are 0, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in H$. identity ✓.
- $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & a \end{pmatrix}$. since $(-a) + a = 0$, inverse $\in H$ ✓.

According to the 3-step subgroup test, $H \leq G$.
Since $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in G$ but $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin H$, $H \neq G$.
Therefore $H < G$.

So basically, although the order of H = order of $G = \infty$, G includes all 2×2 matrices while H is more selective since "$a+d=0$". Therefore they can't be isomorphic or the same and H has to be a proper subgroup.

✓

10. Suppose $f\colon G \to G'$ is a group homomorphism with $a \in G$. If a has finite order prove $|f(a)|$ divides $|a|$.

Let $|a| = m$, then $a^m = e$.

$$f(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ times}}) = \underbrace{f(a) \circ f(a) \circ \cdots f(a)}_{m \text{ times}}$$

since $f(\underbrace{a \cdot a \cdots a}_{m \text{ times}}) = f(a^m) = f(e) = e'$

so $[f(a)]^m = e'$. ✓

let $|f(a)| = m'$, so $[f(a)]^{m'} = e'$
$[f(a)]^m = \{[f(a)]^{m'}\}^{\frac{m}{m'}} = (e')^{\frac{m}{m'}} = e'$
in order to satisfy the equation, $\frac{m}{m'} \in \mathbb{Z}$
therefore $m' \mid m$

therefore $|f(a)| \,\Big|\, |a|$.

Since $e^3 = e^2$, couldn't we say $e^{\frac{3}{2}} = e$, but $\frac{3}{2} \notin \mathbb{Z}$.

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Solutions to FM2 Test #2

1. The order of the element in the direct product is the least common multiple of the component orders. Now the order of 3 in $\mathbb{Z}_4$ is 4 and the order of 15 in $\mathbb{Z}_{18}$ is 6. We conclude that the order of $(3, 15)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{18}$ is $\operatorname{lcm}(4, 6) = 12$.

2. The relation matrix is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is the 3×3 matrix of ones, B is the 3×2 zero matrix, C is the 2×3 zero matrix and D is the 2×2 identity matrix.

3. (a) In a tree $|E| = |V| - 1$. Since $7 \neq 7 - 1$, there is no such tree. (b) The cycle graph C_6 satisfies the criteria.

4. See your notes.

5. Suppose $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$ or equivalently $n_1 = n_2$. Hence f is not injective. There is no $n \in \mathbb{N}$ satisfying $f(n) = 0$. Hence f is not surjective.

6. Observe matrix A has only one independent row vector, so $\operatorname{rank}(A) = 1$, and so $\operatorname{nullity}(A) = 3 - 1 = 2$. The null space is the span of the vectors $-2\vec{\imath} + \vec{\jmath}$ and $-3\vec{\jmath} + \vec{k}$.

7. We wish to show that $\sim$ is reflexive, symmetric and transitive.

   i For any $x \in \mathbb{Z}^+$, $x + x = 2x$, which is even. So $\sim$ is reflexive.

   ii If $x \sim y$ then $x + y$ is even but then $y + x$ is even, which implies $y \sim x$. So $\sim$ is symmetric.

   iii Suppose $x \sim y$ and $y \sim z$. Then $x + y$ is even and $y + z$ is even. So their sum $x + 2y + z$ is also even, from which it follows that $x + z$ is even. So $\sim$ is transitive.

   We conclude that $\sim$ is an equivalence relation. There are two equivalence classes [1] and [2], namely the odd and even positive integers.

8. Using MVT in the form $f(b) = f(a) + f'(c)(b - a)$ with $a = 8$, $b = 10$ and $f(x) = \sqrt[3]{10}$ gives $\sqrt[3]{10} = 2 + 2f'(c)$ where $c \in \,]8, 10[$. Next $f'(x) = \frac{1}{3}x^{-2/3}$. Now observe

   $$\frac{1}{3} \cdot 27^{-2/3} < \frac{1}{3} \cdot 10^{-2/3} < f'(c) < \frac{1}{3} \cdot 8^{-2/3},$$

   whence $1/27 < f'(c) < 1/12$. Hence $56/27 < \sqrt[3]{10} < 13/6$.

9. The *trace* of a matrix is the sum of the elements on the main diagonal. So H is the set of 2×2 matrices with zero trace.

   We use the 3-step subgroup test.

   i Since $\operatorname{trace}(A + B) = \operatorname{trace}(A) + \operatorname{trace}(B)$, we conclude H is closed under addition.

   ii The trace of the zero matrix is zero. So H contains the zero matrix, which is the additive identity.

   iii The additive inverse of A is $-A$. Since $\operatorname{trace}(-A) = -\operatorname{trace}(A)$, we conclude $-A \in H$.

   Now note that $I \notin H$ since $\operatorname{trace}(I) = 2$. So H is a proper subset of G and $H < G$ as required.

10. Let $f(a) = a'$, $f(e) = e'$ and let the order of a be n. By homomorphism $f(a^n) = (a')^n$. But $f(a^n) = f(e) = e'$ as the image of the identity in G is the identity in G' by a standard homomorphism result. So $f(a')^n = e'$. This means the order of a' in G' is finite, say m. Now by the division algorithm $n = mq + r$ where $0 \leq r < m$. So $(a')^n = [(a')^m]^q\,(a')^r$, whence $(a')^r = e'$. Since m was the order of a', it is the least positive integer m for which $(a')^m = e'$, so $r = 0$. Hence m divides n, and we are done.