HL1 Assignment #24 Name: Jerry Jiang

1. Let $z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$, $z_2 = \sqrt{2}\,\text{cis}\,\frac{2\pi}{8}$ and $z_3 = 2e^{i\frac{3\pi}{8}}$. Find $z_1z_2z_3$ giving your answer in Cartesian form.

$z_1 : [1, \frac{\pi}{8}]$

$z_2 : [\sqrt{2}, \frac{2}{8}\pi]$

$z_3 : [2, \frac{3}{8}\pi]$

$z_1 \cdot z_2 \cdot z_3 = [2\sqrt{2}, \frac{3}{4}\pi] = (-2, 2)$ $-2+2i$ would be better form.

100 Excellent!!

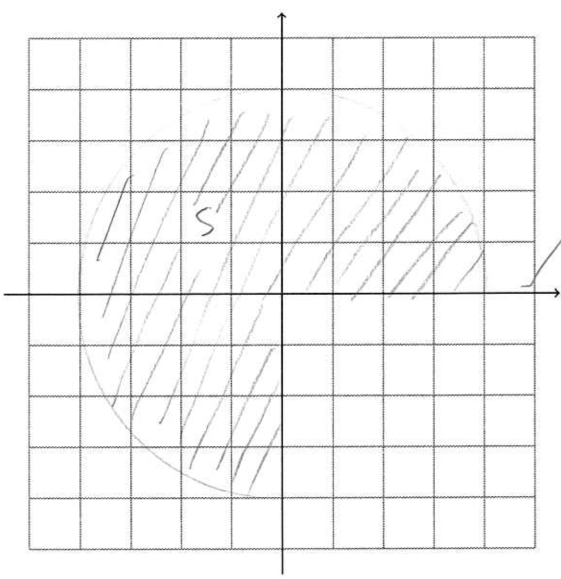
2. Without the calculator find $(\sqrt{3}+i)^9$ giving your answer in exponential form.

$[2, \frac{\pi}{6}]^9$

$= [2^9, \frac{3}{2}\pi]$

$= 512e^{\frac{3}{2}\pi \cdot i}$

3. Let $S = \{z \in \mathbb{C} \mid |z| \leq 4\} \cap \{z \in \mathbb{C} \mid 0 \leq \arg z \leq \frac{3\pi}{2}\}$. Sketch the set S in the Argand diagram (complex plane).



4. The derivative of $\sec x \tan x$ can be written in the form $\sec x(k \tan^2 x + 1)$. Find the value of k.

$$(\sec x \tan x)' = \sec x \tan^2 x + \sec^3 x$$
$$= \sec x(\tan^2 x + \sec^2 x)$$
$$= \sec x(\tan^2 x + \tan^2 x + 1)$$

$\therefore k = 2$ ✓

5. For what values of y and z is the vector $\begin{pmatrix} 6 \\ y \\ z \end{pmatrix}$ orthogonal to both $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}$?

$$\begin{cases} 18 - y + 4z = 0 \\ -24 + y + 2z = 0 \end{cases} \Rightarrow \begin{cases} y = 22 \\ z = 1 \end{cases}$$ ✓

6. Find the values of the real constant k for which the equation $k \cdot 2^x + 2^{-x} = 3$ has a single solution.

$$k \cdot (2^x)^2 - 3 \cdot 2^x + 1 = 0$$

① $k = 0$.

$3 \cdot 2^x = 1$.

$x = \log_2 \frac{1}{3}$

② $k \neq 0$.

$\Delta = 9 - 4k = 0$.

$k = \frac{9}{4}$

$\therefore k = 0$ or $\frac{9}{4}$ ✓

Also $k < 0$

1/1

$e^{i5\theta} = \cos 5\theta + i \sin 5\theta$

7. Use De Moivre's theorem and the binomial theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

$(e^{i\theta})^5 = e^{i5\theta}$ $\quad (e^{-i\theta})^5 = e^{-i5\theta}$

$(e^{i\theta})^5 = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta \, i\sin\theta + 10\cos^3\theta(-1)\sin^2\theta + 10\cos^2\theta(-i)\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$(e^{-i\theta})^5 = (\cos\theta - i\sin\theta)^5$

$= \cos^5\theta - 5\cos^4\theta \, i\sin\theta + 10\cos^3\theta(-1)\sin^2\theta - 10\cos^2\theta(-i)\sin^3\theta + 5\cos\theta\sin^4\theta - i\sin^5\theta$

$1 + \cos^4\theta - 2\cos^2\theta$

$(e^{i\theta})^5 + (e^{-i\theta})^5 = 2\cos 5\theta$

$= 2\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 10\cos\theta(1-\cos^2\theta)^2$

$\therefore \cos 5\theta = \cos^5\theta + 10\cos^5\theta - 10\cos^3\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ ✓

8. Solve $z^4 + z^3 + z^2 + z + 1 = 0$ for $z \in \mathbb{C}$ giving your answers in polar form.

$(z^4 + z^3 + z^2 + z + 1)(z-1) = z^5 - 1 = 0$

$\therefore z^5 = 1$

$\therefore z_1 = [1, 0]$. (x).

$z_2 = [1, \frac{2}{5}\pi]$

$z_3 = [1, \frac{4}{5}\pi]$

$z_4 = [1, \frac{6}{5}\pi]$

$z_5 = [1, \frac{8}{5}\pi]$

$\therefore z = [1, \frac{2}{5}\pi], [1, \frac{4}{5}\pi], [1, \frac{6}{5}\pi], [1, \frac{8}{5}\pi]$. ✓

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9. Let $p(x) = x^4 + 1$. By solving $p(x) = 0$ for $x \in \mathbb{C}$, write $p(x)$ as the product of four linear polynomials. Hence write $p(x)$ as the product of two quadratic polynomials with real coefficients.

$$x^4 = -1 = [1, \pi]$$

$$x = [1, \tfrac{1}{4}\pi], [1, \tfrac{3}{4}\pi], [1, \tfrac{5}{4}\pi], [1, \tfrac{7}{4}\pi]$$

$$p(x) = [x - (\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i)] \cdot [x - (\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i)] \cdot [x - (-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i)] \cdot [x - (-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i)].$$

$$\therefore p(x) = (x^2 - \sqrt{2}x + 1) \cdot (x^2 + \sqrt{2}x + 1)$$

✓

10. Find the sum of the series $\displaystyle\sum_{n=0}^{1009} (-1)^n \binom{2019}{2n}$.

$$(1+i)^{2019} = \binom{2019}{0} + \binom{2019}{1}i - \binom{2019}{2} - \binom{2019}{3}i + \binom{2019}{4} \cdots$$

$$(1-i)^{2019} = \binom{2019}{0} - \binom{2019}{1}i - \binom{2019}{2} + \binom{2019}{3}i + \binom{2019}{4} \cdots$$

$$(1+i)^{2019} + (1-i)^{2019} = 2\left[\binom{2019}{0} - \binom{2019}{2} + \binom{2019}{4} - \cdots\right]$$

$$\therefore \sum_{n=0}^{1009} (-1)^n \binom{2019}{2n} = \frac{(1+i)^{2019} + (1-i)^{2019}}{2}$$

$$[\sqrt{2}, \tfrac{\pi}{4}]^{2019} = [2^{\frac{2019}{2}}, \tfrac{2019}{4}\pi] = [2^{1009} \cdot \sqrt{2}, \tfrac{3}{4}\pi]$$  $= -2^{1009} + 2^{1009}i$

$$[\sqrt{2}, -\tfrac{\pi}{4}]^{2019} = [2^{\frac{2019}{2}}, -\tfrac{2019}{4}\pi] = [2^{1009} \cdot \sqrt{2}, \tfrac{5}{4}\pi]$$  $= -2^{1009} - 2^{1009}i$

$$\therefore \sum_{n=0}^{1009} (-1)^n \binom{2019}{2n} = \frac{-2 \cdot 2^{1009}}{2} = -2^{1009}.$$

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Solutions to HL1 Assignment #24

1. $z_1 z_2 z_3 = [1, \frac{\pi}{8}] \cdot [\sqrt{2}, \frac{2\pi}{8}] \cdot [2, \frac{3\pi}{8}] = [2\sqrt{2}, \frac{6\pi}{8}] = -2 + 2i$.

2. $(\sqrt{3} + i)^9 = [2, \frac{\pi}{6}]^9 = 512e^{i\frac{3\pi}{2}}$.

3. Three quarters of a disc with centre the origin and radius 4. The quarter in quadrant IV is omitted.

4. $(\sec x \tan x)' = \sec x \tan^2 x + \sec^3 x = \sec x(2\tan^2 x + 1)$. So $k = 2$.

5. Taking scalar products we have the equations $18 - y + 4z = 0$ and $-2x + y + 2z = 0$. Solving simultaneously gives $y = 22$ and $z = 1$.

6. We first note $k = 0$ gives a single solution for x. Next let $t = 2^x$. Then we have $kt + \frac{1}{t} = 3$, or equivalently $kt^2 - 3t + 1 = 0$. We will have one solution for x if this quadratic in t has a single solution which is positive, or one positive and one negative solution. The one positive solution occurs when the discriminant $9 - 4k = 0$, which gives $k = \frac{9}{4}$. To have one positive and one negative solution the product of the roots $\frac{1}{k}$ should be less than zero, or equivalently $k < 0$. Altogether, $k \leq 0$ or $k = \frac{9}{4}$.

7. We have $\cos 5\theta = \mathrm{Re}[(\cos\theta + i\sin\theta)^5] = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$. Remembering that $\sin^2\theta = 1 - \cos^2\theta$ and simplifying gives the required result.

8. Notice $(z-1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1$. So we solve $z^5 - 1 = 0$ omitting the solution $z = 1$. That is $z = [1, (\frac{2\pi}{5})k]$, $k = 1, 2, 3, 4$.

9. Solving $x^4 = -1 = [1, \pi]$ gives $x = [1, (\frac{2\pi}{4})k]$, $k = 0, 1, 2, 3$. Denoting these four roots by x_0, x_1, x_2, x_3, we have $x^4 + 1 = (x - x_0)(x - x_1)(x - x_2)(x - x_3)$. Notice $x_0 = x_3^*$ and $x_1 = x_2^*$, so we have
$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1).$$

10. Consider the expansion of $(1 + i)^{2019}$. The real part of this expansion is the required sum. Now $(1 + i)^{2019} = [\sqrt{2}, \frac{\pi}{4}]^{2019}$ has real part -2^{1009}. Hence the sum of the series is -2^{1009}.