FM1 Assignment #5 Name: Jerry Jiang

1. Use the n^{th} term divergence test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{5n+1}$ diverges.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n \ln n}{5n+1} \\
&= \lim_{n\to\infty} \frac{\ln n + 1}{5} \quad \text{(L'Hôpital's Rule)} \\
&= \frac{\infty}{5} \text{ diverges.}
\end{aligned}$$

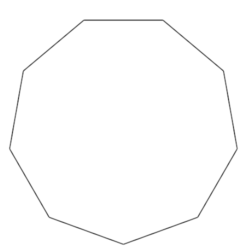
According to the n^{th} term divergence test, the series diverges.

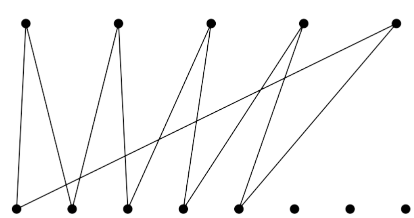
2. A square matrix Q is *orthogonal* if $Q^T Q = I$. Show that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

$$\begin{aligned}
Q &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
Q^T &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
Q^T Q &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \sin^2\theta + \cos^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= I
\end{aligned}$$

3. The *circumference* of a graph is the length of its longest cycle. Find the circumference of the complete graph K_9 and the circumference of the complete bipartite graph $K_{5,8}$.

Since the circumference of a graph is the length of its longest cycle, and a cycle is a walk where no vertex is visited more than once and begins and ends at the same vertex, the circumference of K_9 can be the outer boundary of the the nonagon formed by the nine vertices, which is 9, as the diagram shows below. The circumference of the complete bipartite graph $K_{5,8}$ can reach all the 5 vertices on the 5's side but only 5 vertices on the 8's side as well because according to the definition of the cycle, we have to return to the beginning vertex on the 8's side in the end, which gives a circumference of 10, as the diagram shows below.

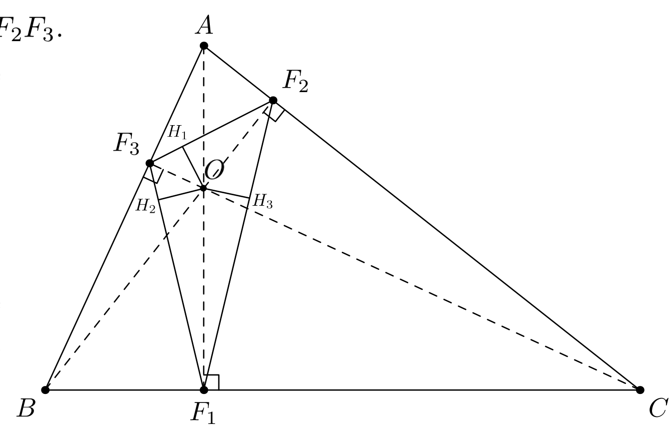




4. The set of $n \times n$ orthogonal matrices is denoted $O(n)$. Prove that $(O(n), \times)$ is a group.

- The product of orthogonal matrices with size $n \times n$ is still an orthogonal matrix with size $n \times n$.
  $(A \times B) \times (A \times B)^T = A \times B \times B^T \times A^T = A \times I \times A^T = I$.
- Matrix multiplication has associativity.
- The identity element is I.
- The inverse of an element Q is Q^T or Q^{-1}.

5. In the diagram below, the points F_1, F_2 and F_3 are the feet of the altitudes of $\triangle ABC$. Prove that the orthocentre of $\triangle ABC$ is the incentre of $\triangle F_1F_2F_3$.



We draw three altitudes OH_1, OH_2, OH_3 inside $\triangle F_1F_2F_3$.
Since O is the orthocentre of $\triangle ABC$, we have three cyclic quadrilaterals: $AF_2OF_3, BF_1OF_3, CF_1OF_2$.
Therefore, $\angle BF_3F_1 = \angle BOF_1, \angle AF_3F_2 = \angle AOF_2$.
Since $\angle BO_1 = \angle AOF_2$, $\angle BF_3F_1 = \angle AF_3F_2$.
So we have $\angle H_2F_3O = \angle H_1F_3O$.
Together with $\angle F_3H_2O = \angle F_3H_1O = 90°$ and the common side OF_3, we get: $\triangle F_3H_2O \cong \triangle F_3H_1O$, so side $OH_1 = OH_2$.
Similarly, we can get: $\triangle F_2H_1O \cong \triangle F_2H_3O$ and $\triangle F_1H_3O \cong \triangle F_1H_2O$.
We have $OH_1 = OH_2 = OH_3$.
Therefore, O is the center of the inscribed circle in $\triangle F_1F_2F_3$, which means it's the incentre.