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HL1 Test #7

1. If $\log_a 2 = b$ and $\log_a 3 = c$, express $\log_a \sqrt{72}$ in terms of b and c.

$$\log_a \sqrt{72} = \log_a 6\sqrt{2}$$
$$= \log_a 2^{\frac{3}{2}} \cdot 3$$
$$= \frac{3}{2} b + c$$

$3 \cdot 6 \times 2$

$6\sqrt{2}$

$2 \cdot 3 \cdot 2^{\frac{1}{2}} = 2^{\frac{3}{2}} \cdot 3$

Bravo!!

2. Find the coordinates of the point on the parabola $y = x^2 - x$ where the tangent is parallel to the line $y = 9x$.

$$y' = 2x - 1 = 9$$
$$x = 5$$
$$\therefore (5, 20).$$

3. Let $f(x) = x^3 \cos x$ and $g(x) = (2x + 3)^5$.

(a) Find the derivative $f'(x)$.

$$f'(x) = 3x^2 \cdot \cos x + x^3 \cdot (-\sin x)$$
$$= 3x^2 \cos x - x^3 \cdot \sin x$$

-114.522

(b) Find the derivative $g'(x)$.

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$$g'(x) = 5(2x+3)^4 \cdot 2$$
$$= 10(2x+3)^4$$

4. Prove $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$. You may assume the result $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x \cdot \sin x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{1+\cos x}$$

$$= \frac{0}{1+1}$$

$$= 0$$

5. Without the calculator solve the equation $8\sin x\cos x = 2$ where $0 \le x \le 2\pi$.

$4\sin 2x = 2$

$\sin 2x = \frac{1}{2}$

$\therefore\ 2x = \text{OR} \begin{cases} \frac{\pi}{6} + 2k\pi \\ \frac{5\pi}{6} + 2k\pi \end{cases}$

$\therefore\ x = \text{OR} \begin{cases} \frac{\pi}{12} + k\pi \\ \frac{5\pi}{12} + k\pi \end{cases}$

$\therefore\ x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

6. Find the equation of the normal to the curve $y = \frac{2x-1}{x+1}$ at the point where $x = 2$.

$$y' = \frac{2(x+1)-(2x-1)}{(x+1)^2}$$

$$= \frac{2x+2-2x+1}{(x+1)^2}$$

$$= \frac{3}{(x+1)^2}$$

$f'(2) = \frac{3}{9} = \frac{1}{3}$

$\therefore\ l: y = -3x+b$

$f(2) = \frac{3}{3} = 1$

$\therefore\ (2, 1)$

$-6+b=1$

$\therefore\ b=7$

$\therefore\ l: y = -3x+7$

7. You have two dice: one is a standard die while the other has its six faces labelled 1, 3, 4, 6, 7, 9 respectively. You randomly choose one of the dice and throw it. The result is an even number. What is the probability that you chose the standard die?

$\frac{1}{2}$ normal: even $\frac{1}{2} \to \frac{1}{4}$ ✓; odd $\frac{1}{2} \to \frac{1}{4}$

$\frac{1}{2}$ biased: even $\frac{1}{3} \to \frac{1}{6}$ ✓; odd $\frac{2}{3} \to \frac{2}{6}$

$$\therefore\ p = \frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{6}} = \frac{3}{3+2} = \frac{3}{5}$$

✓

8. Find the values of k for which the curve $y = x^4 + 4x^3 + kx^2 + 4x + 1$ has no inflection points.

$$y'' = (4x^3 + 12x^2 + 2kx + 4)' = 12x^2 + 24x + 2k \neq 0.$$

$$\therefore\ \Delta = 24^2 - 4\cdot 12\cdot 2k < 0$$
$$= 576 - 96k < 0$$

$\therefore\ k > 6$

However at $k = 6$, y'' has no negative value, so there's no change of sign, thus no point of inflection.

$\therefore\ k \geq 6.$

✓

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9. A geometric series has first term 2 and common ratio 0.95. The sum of the first n terms of the series is denoted by S_n and the sum to infinity is denoted by S_∞. Calculate the least value of n for which $S_\infty - S_n < 1$.

$$S_n = \frac{2\cdot(1-0.95^n)}{1-0.95} = 40(1-0.95^n)$$

$40 \cdot 0.95^n < 1$

$$S_\infty = \frac{2\cdot(1-0.95^\infty)}{1-0.95} = \frac{2}{0.05} = 40$$

$$S_\infty - S_n = \frac{2-2(1-0.95^n)}{0.05}$$

$$= \frac{2\cdot 0.95^n}{0.05} < 1$$

$\therefore \quad 0.95^n < \frac{1}{40}$

$\therefore \quad n > \log_{0.95} 0.025$

$\therefore \quad n > 71.9$

$\therefore \quad n_{min} = 72$

✓

10. A piece of wire 2 m long is to be cut into two pieces. A circle is formed from one piece and a square from the other. Find the *exact* radius of the circle if the sum of the areas of the circle and the square is

(a) a minimum;

(b) a maximum.

- circle: radius a.
- square: $\underline{2-2a\pi}$ perimeter.
- $A_{sum} = \pi a^2 + \left(\frac{2-2a\pi}{4}\right)^2$

$$= \pi a^2 + \frac{4+4a^2\pi^2-8a\pi}{16}$$

$$= \frac{(4\pi+\pi^2)a^2-2\pi a+1}{4}$$

- $A'_{sum} = \frac{1}{4}\cdot\left[2(4\pi+\pi^2)a - 2\pi\right]$
- $A'_{sum} = 0$, then:

$\cancel{2}(4\pi+\pi^2)a = \cancel{2}\pi$

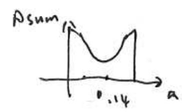
$$a = \frac{1}{4+\pi}$$

- $A''_{sum} = \frac{1}{4}\cdot 2(4\pi+\pi^2) > 0.$

$\therefore$ when $a = \frac{1}{4+\pi}$, it's a minimum

(a) minimum, radius $= \frac{1}{4+\pi}$

(b) maximum, according to A_{sum}, a is as big as possible to get A_{sum} max.



$\therefore$ radus $= \frac{2}{2\pi} = \frac{1}{\pi}$ ✓

✓

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Solutions to HL1 Test #7

1. $\log_a \sqrt{72} = \frac{1}{2}\log_a(2^3 \cdot 3^2) = \frac{3}{2}b + c$.

2. Here $y' = 2x - 1$. Solving $2x - 1 = 9$ gives $x = 5$. Hence the point is $(5, 20)$.

3. (a) $f'(x) = 3x^2 \cos x - x^3 \sin x$. (b) $g'(x) = 5(2x+3)^4 \cdot 2 = 10(2x+3)^4$.

4. $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x =} = \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = 1 \cdot 0 = 0.$

5. $8 \sin x \cos x = 2$ becomes $4 \sin 2x = 2$ or equivalently $\sin 2x = 0.5$. Hence $2x = \frac{\pi}{6} + 2n\pi$ or $2x = \frac{5\pi}{6} + 2n\pi$, whence $x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}$.

6. Here $y' = \dfrac{2(x+1) - 1(2x-1)}{(x+1)^2} = \dfrac{3}{(x+1)^2}$. So at $x = 2$, $m_N = -3$. Hence $N : y - 1 = -3(x - 2)$.

7. Let S be the event of choosing the standard die and E the event of an even number. Then using a tree diagram, a Venn diagram or Bayes' theorem, we have

$$P(S \mid E) = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3}} = \frac{3}{5}.$$

8. Here $y' = 4x^3 + 12x^2 + 2kx + 4$ and $y'' = 12x^2 + 24x + 2k$. Notice y'' is a quadratic. Since y'' is zero and changing in sign at an inflection point, we require the discriminant of the quadratic, namely $24^2 - 96k \leq 0$, to be less than or equal to zero for no inflection points. Solving $\Delta \leq 0$ gives $k \geq 6$.

9. Now $S_\infty - S_n = ar^n + ar^{n+1} + ar^{n+2} + \cdots = \dfrac{2(0.95)^n}{1 - 0.95} = 40(0.95)^n$. So $S_\infty - S_n < 1$ requires $n \geq 72$. Hence $n_{\min} = 72$.

10. Denote the radius of the circle by r and the sum of the areas by A. Then we have

$$A = \pi r^2 + \frac{1}{4}(1 - \pi r)^2, \quad 0 \leq r \leq \frac{1}{\pi},$$

and so

$$A' = 2\pi r + \frac{1}{2} \cdot -\pi(1 - \pi r).$$

Solving $A' = 0$ gives $r = \frac{1}{4+\pi}$. Next $A'' = \frac{5}{2}\pi > 0$ for all r. Hence $r = \frac{1}{4+\pi}$ gives $A_{\min}$ and $A_{\max}$ must occur at an end point of the domain. Checking the areas at $r = 0$ and $r = \frac{1}{\pi}$ gives $A_{\max}$ at $r = \frac{1}{\pi}$, which means the maximum area occurs when all the wire is used to make a circle. This accords with the geometry as the circle contains the most area for a given perimeter.