FM2 Test #1

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Bravo!!

1. The permutation $a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$. Find a^{10} giving your answer in cycle notation.

$a^{10} = (1243)(1243)(1243)(1243)(1243)(1243)(1243)(1243)\cancel{(1243)(1243)}$

$\therefore a^{10} = (14)(23) = a^2$ since $a^4 = ()$.

2. Use the inverse matrix method without the aid of the calculator to solve the system $\begin{cases} x + 2y = 19 \\ 3x - y = 15 \end{cases}$.

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 19 \\ 15 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 19 \\ 15 \end{pmatrix}$$

$$= \frac{1}{-1-6}\begin{pmatrix} -19-30 \\ -57+15 \end{pmatrix}$$

$$= -\frac{1}{7}\begin{pmatrix} -49 \\ -42 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

3. The group isomorphism $f\colon \mathbb{Z}_4 \to G$ is defined by $f(0) = a$, $f(1) = b$, $f(2) = c$ and $f(3) = d$. Construct the operation table for G.

Z_4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\xrightarrow{f}$

G	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

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4. The three complex numbers 1, w and z form a cyclic group under multiplication. Find w and z.

Let the group be $G = \langle w \rangle$ with $e = 1$.

$z = w^2$

$1 = e = w^3$

$\therefore w = [1, 0], [1, \frac{2}{3}\pi], [1, \frac{4}{3}\pi].$

$\because 1 = [1, 0]$

$\therefore$ either $\begin{cases} w = [1, \frac{2}{3}\pi] \\ z = [1, \frac{4}{3}\pi] \end{cases}$ or $\begin{cases} w = [1, \frac{4}{3}\pi] \\ z = [1, \frac{2}{3}\pi] \end{cases}$

5. Calculate the values of x for which the determinant $\begin{vmatrix} x & 5 & -1 \\ 1 & 3 & x \\ 1 & 4 & 7 \end{vmatrix}$ is zero.

$x(3\cdot 7 - 4\cdot x) - 5(1\cdot 7 - x) + (-1)(1\cdot 4 - 1\cdot 3) = 0$

$\therefore 21x - 4x^2 - 35 + 5x - 4 + 3 = 0$

$\therefore 4x^2 - 26x + 36 = 0$

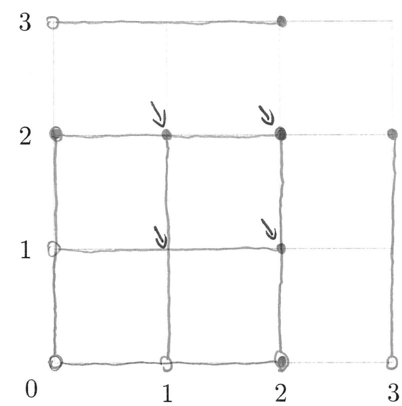
$\therefore x_1 = 2, \; x_2 = \frac{9}{2}.$

6. Let $X =]0, 2]$ and $Y = \{0, 1, 2, 3\}$. Sketch the set $X \times Y$ in the grid. Hence or otherwise determine $|(X \times Y) \cap (Y \times X)|$.

$(X \times Y) \cap (Y \times X)$

$= \{(1,1), (2,1), (1,2), (2,2)\}$

$\therefore |(X \times Y) \cap (Y \times X)| = 4$



$X \times Y$

$Y \times X$

7. The set $S = \{261x + 126y \mid x, y \in \mathbb{Z}\}$ forms a group under addition. Explain why this group must be cyclic and hence explain why 333 must be in S.

$\gcd(261, 126) = 9$

$\therefore S = \{9(29x + 14y) \mid x, y \in \mathbb{Z}\}$ and $\gcd(14, 29) = 1$.

$\therefore S = \langle 9 \rangle$.

when $x = 1, y = -2$, $9(29x + 14y) = 9$.

$x = -1, y = 2$, $9(29x + 14y) = -9$.

multiples of x and y will then turn out to be multiples of 9, thus 9 is the generator.

Since $333 = 9 \times 37$

$\therefore 333 \in S$.

8. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(a) Find a basis for the null space of A.

rref: $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad x_3 = s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ basis is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(b) Find all vectors $\vec{v} \in \mathbb{R}^4$ such that $A\vec{v} = \begin{pmatrix} 10 \\ 10 \\ 15 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 15 \end{pmatrix}.$$

$\downarrow$

$$\begin{cases} a + 2b + 3c + 4d = 10 & (1) \\ 4a + 3b + 2c + d = 10 & (2) \\ 5a + 5b + 5c = 15 & (3) \end{cases}$$

$(3)/5 \to a + b + c = 3$

$((1)+(2))/5 \to a + b + c + d = 4$

$\therefore \underline{\underline{d = 1}}$

$a = 3 - b - c$

Then in (1)

$3 - b - c + 2b + 3c + 4 = 10$

$\underline{\underline{b = 3 - 2c}}$

so $\underline{\underline{a = 3 - (3 - 2c) - c = c}}$

$\therefore \vec{v} = \begin{pmatrix} c \\ 3-2c \\ c \\ 1 \end{pmatrix}$ for all $c \in \mathbb{R}$.

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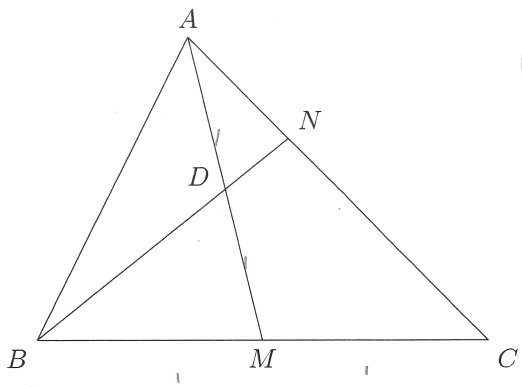
9. In $\triangle ABC$, median $[AM]$ has midpoint D. Prove that the cevian $[BN]$ trisects side $[AC]$.



According to the Mannelaus' Theorem,

$$\frac{AN}{NC} \cdot \frac{CB}{BM} \cdot \frac{MD}{DA} = -1$$

$$\frac{AN}{NC} \cdot \frac{2}{-1} \cdot \frac{1}{1} = -1$$

$$\therefore \frac{AN}{NC} = \frac{1}{2}$$

$$\therefore AN = \frac{1}{2} NC = \frac{1}{3} AC$$

Therefore, $[BN]$ trisects $[AC]$.

10. The centre of a group G, denoted $Z(G)$, is the set of elements in G that commute with every element of G. That is, $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$. Prove that $Z(G)$ is a subgroup of G.

- since for any $a \in G$, $ea = ae = a$, $e \in Z(G)$ identity ✓.
- for any $a \in Z(G)$, $ax = xa$ for all $x \in G$.
  since $a \in G$, there's an a^{-1}.

  $a^{-1} \cdot ax = a^{-1} \cdot xa \rightarrow x = a^{-1} \cdot x \cdot a$ (pre-multiply)

  $x \cdot a^{-1} = a^{-1} \cdot x \cdot a \cdot a^{-1} \rightarrow xa^{-1} = a^{-1} \cdot x$ (post-multiply)

  $\therefore x \cdot a^{-1} = a^{-1} \cdot x$. $a^{-1} \in Z(G)$ inverse ✓
- for any $a, b \in Z(G)$, $ax = xa$, $bx = xb$ for all $x \in G$.
  $ax = xa$. premultiplying by b.

  $b \cdot (ax) = b \cdot (xa) = (bx)a$ (associativity). since $bx = xb$

  $\therefore (ba)x = b(ax) = (bx)a = (xb)a = x(ba)$, ab comute with all $x \in G$.

  $\therefore ba \in Z(G)$ closure ✓

Therefore, according to the 3-step subgroup test, $Z(G) \leq G$.

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Solutions to FM1 Test #1

1. Since a is a 4-cycle, $a^4 = e$. So $a^{10} = a^2 = (14)(23)$.

2. We have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 19 \\ 15 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 19 \\ 15 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$. Hence $x = 7$, $y = 6$.

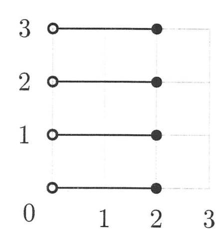
3. The operation table for G is

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

4. The three roots of unity form the required cyclic group. These roots are 1, $[1, 120°]$ and $[1, 240°]$. So $w = [1, 120°]$ and $z = [1, 240°]$ will do.

5. Expanding the determinant across the first row gives $x(21-4x)-5(7-x)-(4-3)$. Hence we solve $2x^2-13x+18 = 0$, whence $x = 2, \frac{9}{2}$.

6. The diagram illustrates $X \times Y$. The diagram for $Y \times X$ will be the reflection of the given diagram in the line $y = x$. We conclude $|(X \times Y) \cap (Y \times X)| = 4$.



7. We are given that $(S, +)$ is a group and clearly S is a proper subset of $\mathbb{Z}$. So $(S, +) \leq (\mathbb{Z}, +)$. Since $(\mathbb{Z}, +)$ is cyclic we conclude $(S, +)$ is cyclic since every subgroup of a cyclic group is also cyclic. A generator for $(S, +)$ is $\gcd(261, 126) = 9$. So $S = \langle 9 \rangle$. Since $9 \mid 333$, we conclude $333 \in S$.

8. (a) Using the calculator $\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. So a basis for null$(A)$ is $\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}\}$.

   (b) We spot $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ as a particular solution. Hence the full solution is $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$, $t \in \mathbb{R}$.

9. Menelaus's theorem with unsigned lengths gives

$$\frac{AN}{NC} \times \frac{CB}{BM} \times \frac{MD}{DA} = 1.$$

Solving for $AN : NC$, gives $AN : NC = 1 : 2$, which is to say cevian $[BN]$ trisects side $[AC]$.

10. We use the 3-step subgroup test.

    i. Suppose $a, b \in Z(G)$ and $x \in G$. Then $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$. Hence $ab \in Z(G)$. So $Z(G)$ is closed under the group operation.

    ii. Since $ex = xe$ for all $x \in G$, we have $e \in Z(G)$.

    iii. Suppose $a \in Z(G)$ and $x \in G$. Then $ax = xa$. So $axa^{-1} = x$, from which it follows that $xa^{-1} = a^{-1}x$. Thus $a^{-1} \in Z(G)$.

    Hence $Z(G)$ is a subgroup of G.