HL1 Assignment #14 Name: Jerry

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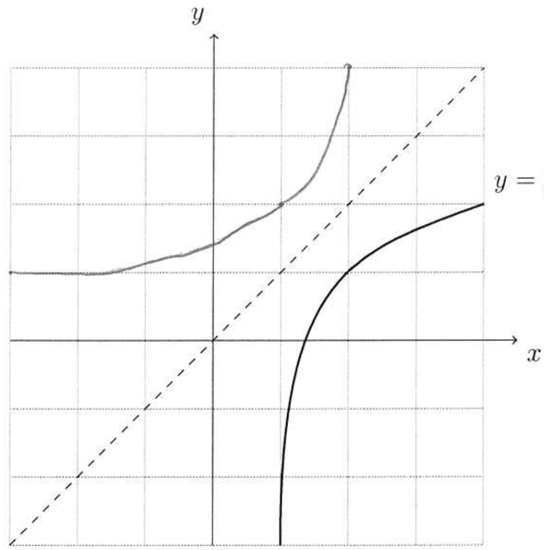
Very good.

1. The graph of $y = f(x)$ is drawn below. On the same grid draw the graph of $y = f^{-1}(x)$.



2. If $f(x) = x - 2$ and $g(x) = x^3$, solve $g \circ f(x) = 216$.

$(x-2)^3 = 216$

$x - 2 = 6$

$x = 8$

3. The graph of $f(x) = x^2 + 2x - 1$ is translated by the vector $\binom{3}{1}$. Sketch the resulting graph in the grid below.

$f'(x) = (x-3)^2 + 2(x-3) - 1 + 1$

$f'(x) = x^2 - 4x + 3 = (x-2)^2 - 1$

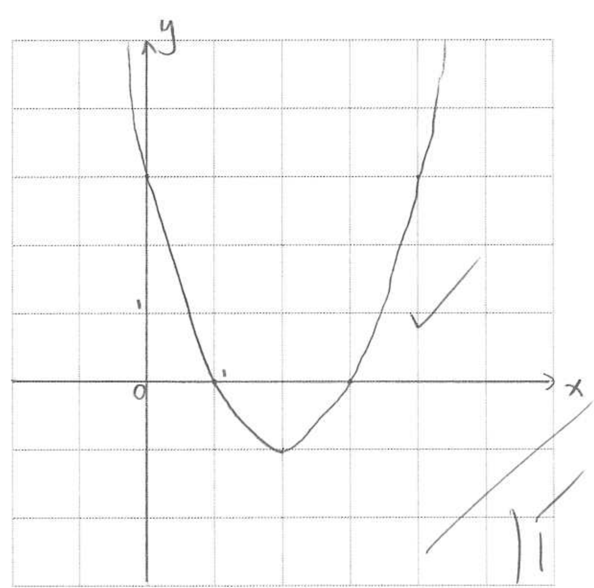
(2, -1)

(3, 0)

(1, 0)

(0, 3)

(4, 3)



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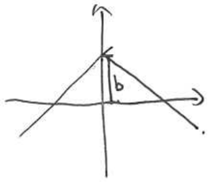
4. The area enclosed between the x-axis and the graph of $y = -|x| + b$ is 36. Given that $b > 0$ find its value.



$b^2 = 36$

$b = \pm 6$

$\because b > 0$

$\therefore b = 6$

5. One root of the quadratic equation $7x^2 - 8x + p = 0$ is three times the other. Find the value of p.

$4a = -\frac{-8}{7} = \frac{8}{7}$

$a = \frac{2}{7}$

$\therefore 3a = \frac{6}{7}$

$\therefore 3a^2 = \frac{12}{49}$

$\therefore \frac{12}{49} = \frac{p}{7}$

$\therefore p = \frac{12}{7}$

6. The factors of $x^4 + px^3 + qx^2 + rx + 6$ include $x - 1$, $x - 2$ and $x - 3$. Find the values of p, q and r.

$\frac{6}{(-1)\cdot(-2)\cdot(-3)x} = -1$

$\therefore (x-1)^2(x-2)(x-3)$

$= (x^2-2x+1)(x^2-5x+6)$

$= \underline{x^4} - \underline{5x^3} + \underline{6x^2} - \underline{2x^3} + \underline{10x^2} - \underline{12x} + \underline{x^2} - \underline{5x} + \underline{6}$

$= x^4 - 7x^3 + 17x^2 - 17x + 6$

$\therefore p = -7$

$q = 17$

$r = -17$

7. Show that the quadratic equation $x^2 - (5-k)x - (k+2) = 0$ has two distinct real roots for all real values of k.

$\Delta = (5-k)^2 + 4(k+2)$

$= k^2 - 10k + 25 + 4k + 8$

$= k^2 - 6k + 33$

$= k^2 - 6k + 9 + 24$

$= (k-3)^2 + 24 > 0$

$\therefore$ always has two distinct real roots.

✓

8. Let m and n be positive integers. If $m \leq n$ and $\gcd(m,n) = 1$, then we say m is a *totative* of n. For example, the totatives of 12 are 1, 5, 7 and 11. The *Euler totient* function is the function $\phi\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ with rule $\phi(n)$ is the number of totatives of n. For example, we have just seen that $\phi(12) = 4$. Show that the Euler totient function is neither injective nor surjective.

totatives of 5 are 1, 2, 3, 4.

$\therefore \phi(5) = 4 = \phi(12)$

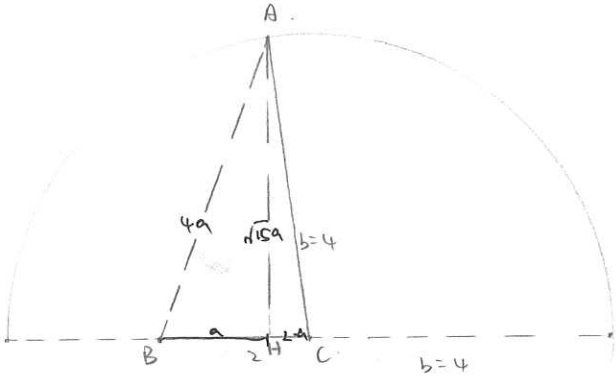
$\therefore$ not injective. ✓

$\because \phi(12) \neq n(\mathbb{Z}^+)$?

$\therefore$ not surjective.

2/7

9. In $\triangle ABC$, $a = 2$, $b = 4$ and $\cos B < \frac{1}{4}$. Find the range of possible values for c.



let $\angle ABH = \arccos\frac{1}{4}$

if $AB=4a$, $BH=a$

$\therefore AH = \sqrt{16a^2-a^2} = \sqrt{15}a$

$HC = 2-a$

$\therefore 4^2 = (2-a)^2 + 15a^2$

$\therefore 16 = 16a^2 - 4a + 4$

$\therefore 16a^2 - 4a - 12 = 0$

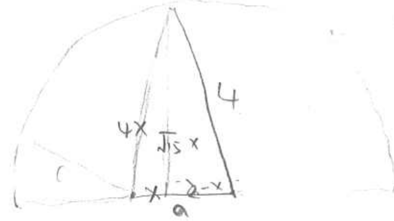
$\therefore 4a^2 - a - 3 = 0$

4 3

1 −1

$\therefore (4a+3)(a-1) = 0$

$\therefore a_1 = 1 \quad a_2 = -\frac{3}{4}(\times)$



$\because AB = c$

$\therefore c \in]2,4[$.

10. Let $f(n) = \dfrac{2n^2 - 10n - 4}{n^2 - 4n + 3}$, $n \in \mathbb{Z}$. For what values of n is $f(n)$ also an integer?

$f(n) = \dfrac{2n^2-8n+6-2n-10}{n^2-4n+3}$

$= 2 - \dfrac{2n+10}{n^2-4n+3}$

let $\dfrac{2n+10}{n^2-4n+3} = k$, $k \in \mathbb{Z}$,

① $k=0$

$\therefore n=-5$ (✓)

② $k \neq 0$.

$2n+10 = kn^2 - 4kn + 3k$

$kn^2 - (4k+2)n + 3k - 10 = 0$

$4k - 8k - 4 + 3k - 10 = 0$

$-k = 14$

$k = -14$

$n_1 + n_2 = \dfrac{4k+2}{k} = 4 + \dfrac{2}{k} \Rightarrow k = \pm1, \pm2.$

$n_1 \cdot n_2 = \dfrac{3k-10}{k} = 3 - \dfrac{10}{k} \Rightarrow k = \pm1, \pm2, \pm5, \pm10.$

$\Delta = (4k+2)^2 - 4k(3k-10) \geq 0$

$16k^2 + 16k + 4 - 12k^2 + 40k \geq 0$

$4k^2 + 56k + 4 \geq 0$

$\therefore k \leq -13.9 \,/\, k \geq -0.0718$

$\therefore k = 1$ or 2 or 5 or 10.

(1) $k=1$

$n = 7$ or -1 (✓)

(2) $k=2$

$n^2 - 5n - 2 = 0$ (×)

(3) $k=5$

$5n^2 - 22n + 5 = 0$ (×)

(4) $k=10$

$5n^2 - 21n + 10 = 0$ (×)

$\therefore n = -5$ or 7 or -1

Mw 2, 4

4

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Solutions to HL1 Assignment #14

1. The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$. In this particular question, the graph of f^{-1} also contains the points $(1, 2)$, $(2, 4)$ and has an asymptote at $y = 1$.

2. We must solve $(x-2)^3 = 216$, whence $x = 8$.

3. The translated graph has equation $y - 1 = (x-3)^2 + 2(x-3) - 1$, or equivalently $y = x^2 - 4x + 3$. This is the graph that should be drawn.

4. The enclosed region is a triangle with height b and base $2b$. Hence the area is b^2; whence $b = 6$.

5. Let the roots be α and 3α. By Vieta's formulas $4\alpha = 8/7$ and $3\alpha^2 = p/7$, whence $\alpha = 2/7$ and $p = 12/7$.

6. By the factor theorem $p(x) = (x-1)(x-2)(x-3)(x-a)$. By the product of the roots $a = 1$. By expansion $p(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$, whence $p = -7$, $q = 17$ and $r = -17$.

7. Here $\Delta = (k-5)^2 + 4(k+2) = k^2 - 6k + 33 = (k-3)^2 + 24$, which is positive for all $k \in \mathbb{R}$. Hence the quadratic equation has two distinct real roots for all real values of k.

8. Since $\phi(3) = 2$ and $\phi(4) = 2$, ϕ is not injective. Next, notice that if d is a totative of $n > 1$ then so is $n - d$. And these totatives are distinct unless $d = n - d$, or equivalently $n = 2d$. But then d would not be a totative of n unless $d = 1$, whence $n = 2$. We conclude that $\phi(n)$ is even unless $n = 1$ or $n = 2$. Hence ϕ is not surjective, as for example there is no n for which $\phi(n) = 3$.

9. First note that by the triangle inequality $2 < c < 6$. The fact that $\cos B < \frac{1}{4}$ will allow us to sharpen this inequality. By the cosine rule
$$\cos B = \frac{4 + c^2 - 16}{4c} = \frac{c^2 - 12}{4c}.$$
Since $\cos B < \frac{1}{4}$, we have $c^2 - 12 < c$, or equivalently $c^2 - c - 12 < 0$, whence $-3 < c < 4$. Respecting the triangle inequality we conclude $2 < c < 4$.

10. Observe
$$f(n) = \frac{(2n^2 - 8n + 6) - (2n + 10)}{n^2 - 4n + 3} = 2 - \frac{2n + 10}{n^2 - 4n + 3}.$$
So $f(n)$ could only be an integer if $2n + 10 = 0$ or $|2n + 10| \geq |n^2 - 4n + 3|$. Solving we find $n = -5$ or $-1 \leq n \leq 7$. We can now test these candidates, most easily using the TABLE feature of the GDC, to find $n = -5, -1, 2, 4, 7$.