HL1 Assignment #11 Name: Jerry Jiang

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Excellent!!

1. What is the remainder when $2x^3 - 3x^2 + 4x + 3$ is divided by $x - 2$?

$f(2) = 2\cdot 8 - 3\cdot 4 + 4\cdot 2 + 3$

$= 16 - 12 + 8 + 3$

$= 15$ ✓

2. Solve $\cos(\theta + 70°) = 0.5$ for $0° \le \theta < 360°$.

$\because \cos(\theta + 70°) = \frac{1}{2}$

$\therefore \theta + 70° = 60°$ or $300°$ or $420°$ ...

$\therefore \theta = -10°$ or $230°$ or $350°$

$\because 0° \le \theta < 360°$

$\therefore \theta = 230°$ or $350°$ ✓

3. If $z = a + bi$ and $\frac{z}{z^*} = c + di$, prove $c^2 + d^2 = 1$.

$$\frac{z}{z^*} = \frac{a+bi}{a-bi} = \frac{(a+bi)^2}{a^2+b^2} = c + di$$

$$\therefore c = \frac{a^2-b^2}{a^2+b^2} \quad d = \frac{2ab}{a^2+b^2}$$

$$\because c^2 = \frac{a^4+b^4-2a^2b^2}{(a^2+b^2)^2} \quad d^2 = \frac{4a^2b^2}{(a^2+b^2)^2}$$

$$\therefore c^2 + d^2 = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$ ✓

4. Expand and simplify $(3-\sqrt{2})^4$. Give your answer in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

$(3-\sqrt{2})^4$

$= (11-6\sqrt{2})^2$

$= 121+72-132\sqrt{2}$

$= 193-132\sqrt{2}$

✓

5. Find the value of $(1^2+3^2+5^2+\cdots+99^2)-(2^2+4^2+6^2+\cdots+100^2)+(4+8+12+\cdots+200)$.

$S = -[(100^2-99^2)+\cdots+(2^2-1^2)] + \frac{204 \cdot 50}{2}$

$= -[100+99+\cdots+2+1]+5100$

$= 5100-5050$

$= 50$

✓

6. For what values of m is the line $y = mx+5$ tangent to the parabola $y = 4-x^2$?

$\begin{cases} y = mx+5 \\ y = 4-x^2 \end{cases}$

$\therefore x^2+mx+1=0$

$\Delta = m^2-4=0$

$\therefore m = \pm 2.$

✓

7. One solution of $z^3 + bz^2 + 34z - 40 = 0$ is $z = 3 + i$. If $b \in \mathbb{R}$ find the value of b.

according to the conjugate root theorem,

$z_1 = 3+i$, then $z_2 = 3-i$

$\because z_1 + z_2 = 6, \quad z_1 \cdot z_2 = 10$

$\therefore z^2 - 6z + 10 = 0$ have root $3 \pm i$.

$\because (z^2 - 6z + 10)(z-4) = 0$

$\therefore z^3 - 10z^2 + 34z - 40 = 0$

$\therefore b = -10$

✓

8. The quadratic equation $x^2 - 5x - 3 = 0$ has roots α and β. Find a quadratic equation with roots $\dfrac{1}{\alpha+1}$ and $\dfrac{1}{\beta+1}$.

$\alpha + \beta = 5$

$\alpha\beta = -3$

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} = \frac{\alpha+\beta+1+1}{(\alpha+1)(\beta+1)} = \frac{5+2}{\alpha\beta+\alpha+\beta+1} = \frac{7}{-3+5+1} = \frac{7}{3}$$

$$\frac{1}{\alpha+1} \cdot \frac{1}{\beta+1} = \frac{1}{(\alpha+1)(\beta+1)} = \frac{1}{\alpha\beta+\alpha+\beta+1} = \frac{1}{-3+5+1} = \frac{1}{3}$$

$\therefore 3x^2 - 7x + 1 = 0$ ✓

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9. The point (a, b) is the point on the curve $y = x^2$ that is closest to $(6, 0)$. Calculate the value of a.

$A(a, a^2)$

$D = \sqrt{a^4 + a^2 - 12a + 36}$

$(D^2)' = 4a^3 + 2a - 12$

$D^2_{min} \Rightarrow D_{min}$

$\therefore$ take a when $(D^2)' = 0$.

$\therefore 4a^3 + 2a - 12 = 0$

let $a = b + c$

$\therefore a^3 = b^3 + c^3 + 3bc(b+c)$

$\therefore a^3 = b^3 + c^3 + 3abc$

$\therefore a^3 - 3bc \cdot a - (b^3 + c^3) = 0$

$\therefore \begin{cases} bc = -\frac{1}{6} \\ b^2 + c^2 = 3 \end{cases}$

$\therefore b^3c^3 = -\frac{1}{216}$

$\therefore b^3, c^3$ are roots of

$t^2 - (b^3 + c^3)t + b^3c^3 = 0.$

$\therefore t^2 - 3t - \frac{1}{216} = 0$

$\therefore 216t^2 - 648t - 1 = 0$

$\therefore \Delta = 420768$

$= 487 \times 2^5 \times 3^3$

$\therefore t = \frac{648 \pm 12\sqrt{2922}}{432}$

$\therefore b = \sqrt[3]{\frac{648 + 12\sqrt{2922}}{432}}$

$c = \sqrt[3]{\frac{648 - 12\sqrt{2922}}{432}}$

$\because a = b + c$

$\therefore a = \frac{\sqrt[3]{-} + \sqrt[3]{-}}{\sqrt[3]{432}}$

$\therefore a = \frac{\sqrt[3]{648 + 12\sqrt{2922}} + \sqrt[3]{648 - 12\sqrt{2922}}}{6\sqrt[3]{2}}$

✓

10. If $\log_{(\tan\theta + \cot\theta)} \cos\theta = k$ where $k \in \mathbb{R}$, find an expression for $\log_{\tan\theta} \sin\theta$ in terms of k.

let $\tan\theta = a$, $\cos\theta = b$

$\because \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\therefore a = \frac{\sqrt{1 - b^2}}{b}$

$\therefore a^2b^2 = 1 - b^2$

$\therefore b^2(a + \frac{1}{a}) = \frac{1}{a}$

$\therefore a + \frac{1}{a} = \frac{1}{ab^2}$

$\because \log_{\frac{1}{ab^2}} b = k$

$\therefore \left(\frac{1}{ab^2}\right)^k = b$

$\therefore a^{-k} b^{-2k} = b$

$\therefore a^{-k} = b^{1+2k}$

$\therefore a^{\frac{-k}{1+2k}} = b$

$\because \log_{\tan\theta} \sin\theta$

$= \log_a ab$

$\therefore = \log_a a + \log_a b$

$= 1 + \frac{-k}{1+2k}$

$= \frac{k+1}{2k+1}$

✓

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Solutions to HL1 Assignment #11

1. Let $p(x) = 2x^3 - 3x^2 + 4x + 3$. By the remainder theorem $R = p(2) = 15$.

2. Here $\cos(\theta + 70°) = \cos(60°)$. So $\theta + 70° = 60° + n360°$ or $\theta + 70° = -60° + n360°$, whence $\theta = -10° + n360°$ or $\theta = -130° + n360°$. For the given interval we conclude $\theta = 230°, 350°$.

3. Notice $c^2 + d^2 = \dfrac{z}{z^*} \cdot \left(\dfrac{z}{z^*}\right)^* = \dfrac{z}{z^*} \cdot \dfrac{z^*}{z} = 1$.

4. By the binomial theorem $(3 - \sqrt{2})^4 = 3^4 - \binom{4}{1} \cdot 3^3 \cdot \sqrt{2} + \cdots + (\sqrt{2})^4$, which simplifies to $193 - 132\sqrt{2}$.

5. Notice $(1^2+3^2+5^2+\cdots+99^2)-(2^2+4^2+6^2+\cdots+100^2)+(4+8+12+\cdots+200) = (4+8+12+\cdots+200)-[(2^2-1^2)+(4^2-3^2)+(6^2-5^2)+\cdots+(100^2-99^2)]$, which is $(4+8+12+\cdots+200)-(3+7+11+\cdots+199)$, and this is simply $\underbrace{1+1+1+1+\cdots+1}_{50 \text{ times}} = 50$.

6. The line intersects the parablola when $mx + 5 = 4 - x^2$, or equivalently $x^2 + mx + 1 = 0$. Tangency occurs when $\Delta = 0$. That is when $m^2 - 4 = 0$, giving $m = \pm 2$.

7. We have the roots as $z_{1,2} = 3 \pm i$ and z_3. We are given $z_1 \cdot z_2 \cdot z_3 = 40$, so $z_3 = 4$. Lastly, $b = -(z_1 + z_2 + z_3) = -10$.

8. A quadratic equation with roots $\alpha + 1$ and $\beta + 1$ is $(x-1)^2 - 5(x-1) - 3 = 0$, or equivalently $x^2 - 7x + 3 = 0$. Hence the quadratic equation $3x^2 - 7x + 1 = 0$ solves our problem.

9. Notice that $b = a^2$. Next the square of the distance between the points is $d^2 = (a-6)^2 + (a^2-0)^2 = a^4 + a^2 - 12a + 36$. Since d is non-negative the value of a which gives the minimum value of d^2 is also the value of a that gives the minimum value of d. Using the minimum tool on the GDC gives $a = 1.33$ (3 s.f.).

10. In exponential form we are given $(\tan\theta + \cot\theta)^k = \cos\theta$, from which we obtain

$$\left(\frac{\sec^2\theta}{\tan\theta}\right)^k = \cos\theta \quad \Leftrightarrow \quad \left(\frac{1}{\tan\theta}\right)^k = \cos^{2k+1}\theta.$$

Multiplying both sides of the second form by $\tan^{2k+1}\theta$ gives $\tan^{k+1}\theta = \sin^{2k+1}\theta$, whence

$$\log_{\tan\theta}\sin\theta = \frac{k+1}{2k+1}.$$