FM2 Assignment #12 Name: Jerry Jiang $\frac{19}{10}$ Excellent!

1. What value should be assigned to k to make the function $f(x) = \begin{cases} x^2 - 1, & x < 3, \\ 2kx, & x \geq 3, \end{cases}$ continuous at $x = 3$.

$\lim_{x \to 3^-} f(x) = 9 - 1 = 8.$

$\lim_{x \to 3^+} f(x) = 6k.$

$\therefore 6k = 8, \ k = \frac{4}{3}.$

2. Construct a function that is continuous on $\mathbb{R}$ but fails to be differentiable at the four numbers $0, 1, 2, 3$.

$$f(x) = \begin{cases} x, & x < 0 \\ -x, & 0 \leq x < 1 \\ x - 2, & 1 \leq x < 2 \\ -x + 2, & 2 \leq x < 3 \\ x - 4, & x \geq 3. \end{cases}$$

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f'(x) > 0$ for all $x \in \mathbb{R}$. Prove that if $a < b$ then $f(a) < f(b)$.

Since f is differentiable, it's continuous on $[a, b]$ and differentiable in $]a, b[$.
Apply MVT. Suppose $a < b$, $f(a) \geq f(b)$. there's a c in $]a, b[$ that satisfies:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $b > a$, $b - a > 0$; $f(a) \geq f(b)$, $f(b) - f(a) \leq 0$.
Therefore in this case $f'(c) \leq 0$. This contradicts $f'(x) > 0$ for all $x \in \mathbb{R}$.
So $f(a)$ must be smaller than $f(b)$.

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4. The third degree Taylor polynomial of $\ln x$ about $x = 1$ is $a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$. Find the values of a_0, a_1, a_2 and a_3 and hence estimate $\ln 1.2$.

$f(x) = \ln x$. $f(1) = \ln 1 = 0$.

$p(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$.

$$\begin{cases} f'(x) = x^{-1},\ f''(x) = -x^{-2},\ f'''(x) = 2x^{-3}. \\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2. \end{cases}$$

$\therefore a_0 = 0, \quad a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{2}{6} = \frac{1}{3}$.

$p(x) = (x-1) + (-\frac{1}{2})(x-1)^2 + \frac{1}{3}(x-1)^3$.

$f(1.2) \approx p(1.2) = 0.183$ (3 s.f.).

(c.f. $f(1.2) = 0.182$ (3 s.f.)).

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5. In the trapezium $ABCD$, the midpoints of the parallel sides $[AB]$ and $[CD]$ are M and N respectively. The sides $[BC]$ and $[AD]$ are not parallel. Show that the diagonals and the line segment $[MN]$ are concurrent.

Let $A(a,m)$, $B(b,m)$, $C(c,n)$, $D(d,n)$, $M(\frac{a+b}{2}, m)$, $N(\frac{c+d}{2}, n)$, P intersection of $[BD]$ and $[AC]$. $m \neq n$.

$$\begin{cases} l_{BD}: & y - m = \frac{m-n}{b-d}(x-b) \\ l_{AC}: & y - n = \frac{n-m}{c-a}(x-c) \end{cases} \Rightarrow P\left(\frac{ad-bc}{a-b-c+d};\ \frac{an-bn-mc+md}{a-b-c+d}\right).$$

$$l_{MN}: \ y - m = \frac{2(m-n)}{(a+b)-(c+d)}\left(x - \frac{a+b}{2}\right) = \frac{m-n}{a+b-c-d}(2x-a-b).$$

$$\text{LHS}: \ y_P - m = \frac{(a-b)(n-m)}{a-b-c+d}$$

$$\text{RHS}: \ \frac{m-n}{a+b-c-d}\cdot(2x_P - a - b) = \frac{m-n}{a+\cancel{b-c-d}}\cdot\frac{(b-a)(a+\cancel{b-c-d})}{a-b-c+d}$$

$$= \frac{(a-b)(n-m)}{a-b-c+d}.$$

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$\therefore$ LHS = RHS.

$\therefore$ P is on l_{MN}.

Therefore, $[BD]$, $[AC]$, and $[MN]$ are concurrent.

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