FM1 Assignment #8 Name: Jerry Jiang

1. Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges.

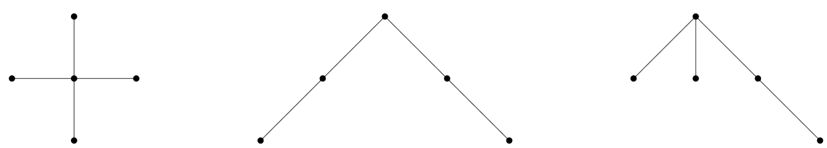
$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}$$
$$= \lim_{n\to\infty} \frac{2^n \cdot (n+1)^2}{2^{n+1} \cdot n^2}$$
$$= \lim_{n\to\infty} \frac{n^2 + 2n + 1}{2n^2}$$
$$= \frac{1}{2}$$

Excellent work 10/10. Perhaps an easier argument for your postscript in question 2 is that the number of edges in a tree is always one less than the number of vertices, so 5−1=4 edges in this case. For question 5, using a GDC for this trigonometric result may not be considered a complete and correct proof by an IB examiner; perhaps in a question worth 5 marks this would score 4 because of this. I have attached my solution using Ptolemy's theorem (Three times: BPCA, BPCE, BPCD and set the side length to unit can obtain the result easily).

So according to the ratio test, since $L = \frac{1}{2} < 1$, the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

2. Define spanning tree. Draw all the non-isomorphic spanning trees of K_5.

Spanning tree is a subgraph of a connected graph that contains no cycles while containing every vertex in that connected graph.



These are the only three spanning trees for K_5, since $f = 1$ when there're no cycles, and any $e > 4$ makes $f > 1$ according to Euler's formula.

3. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

(a) What elementary row operation is needed to transform A into row echelon form?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2 \to R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

(b) What is the corresponding elementary matrix for the above row operation?

$$R_3 - 2R_2 \to R_3 : \; E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

4. A permutation of the form $(a_1 a_2 \dots a_n)$ is called a *cycle of length* n or an n*-cycle*. A 2-cycle is called a *transposition*.

(a) Write the 5-cycle (12345) as a product of transpositions.

$(12) \circ (23) \circ (34) \circ (45)$

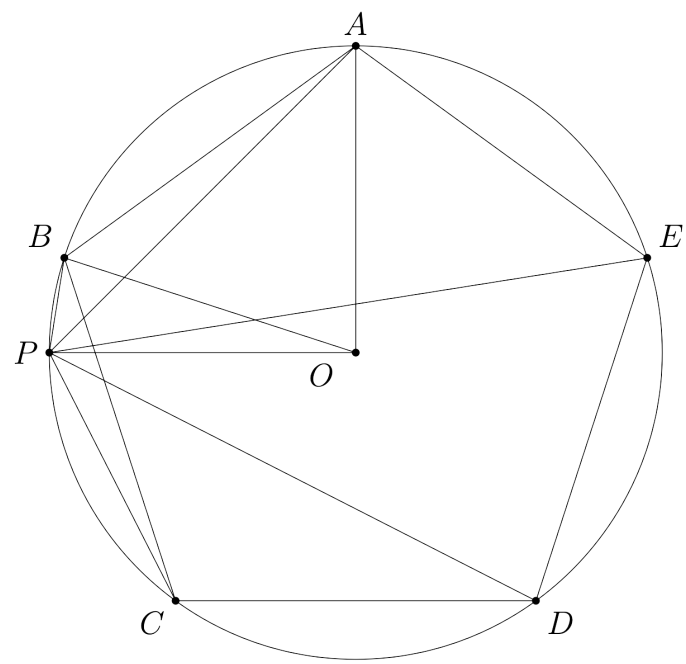
(b) What is the order of an n-cycle?

n, since the n^{th} product of the cycle with itself gets to the identity element.

(c) If α and β are disjoint cycles of length 180 and 216 respectively, what is the order of the product $\alpha\beta$?

The order for α is 180, and the order for β is 216, so the order for $\alpha\beta$ is $\operatorname{lcm}(180, 216) = 1080$.

5. The regular pentagon $ABCDE$ is inscribed in a circle and point P is on $\overset{\frown}{BC}$. Prove $PA + PD = PB + PC + PE$.



Let $\angle BOP = 2\theta$.

Then $\angle POA = 72^\circ + 2\theta$.

$$\therefore PA = 2 \cdot r \cdot \sin\left(\frac{1}{2}\angle POA\right)$$

$$= 2r\sin\left(36^\circ + \theta\right)$$

Similarly,

$PB = 2r\sin\theta$.

$PC = 2r\sin\left(36^\circ - \theta\right)$.

$PD = 2r\sin\left(72^\circ - \theta\right)$.

$PE = 2r\sin\left(72 + \theta\right)$.

$$\therefore \quad PA + PD - PB - PC - PE$$

$$= 2r\left[\sin(36^\circ + \theta) + \sin(72^\circ - \theta) - \sin\theta - \sin(36^\circ - \theta) - \sin(72^\circ + \theta)\right]$$

$$= 2r[\sin 36^\circ \cos\theta + \cos 36^\circ \sin\theta + \sin 72^\circ \cos\theta - \cos 72^\circ \sin\theta - \sin\theta -$$

$$\sin 36^\circ \cos\theta + \cos 36^\circ \sin\theta - \sin 72^\circ \cos\theta - \cos 72^\circ \sin\theta]$$

$$= 2r\sin\theta[\cos 36^\circ - \cos 72^\circ - 1 + \cos 36^\circ - \cos 72^\circ]$$

$$= 2r\sin\theta \cdot 0 \quad \text{(GDC)}$$

$$= 0$$

Therefore, $PA + PD = PB + PC + PE$.