HL1 Assignment #13 Name: Jerry Jiang

Excellent!!

100%

1. Let $f(x) = \dfrac{-5-x}{3-2x}$. Find $f^{-1}(7)$ without finding $f^{-1}(x)$.

$\frac{-5-x}{3-2x} = 7$

$-5-x = 21-14x$

$13x = 26$

$x = 2 \qquad (2,7) \rightarrow (7,2).$

$f^{-1}(7) = 2$

2. Show that the function $f(x) = |x-2| - |x+2|$ is odd and sketch its graph in the grid below.

$-f(-x)$

$= -|-x-2| + |-x+2|$

$= |x-2| - |x+2|$

$= f(x)$

$x < -2$

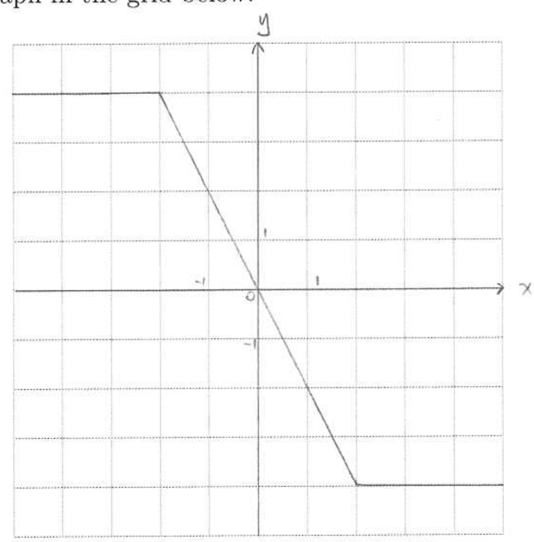
$f(x) = -x+2 \quad +x+2$

$= 4$

$-2 < x < 2$

$-x+2 \quad -x-2$

$= -2x$



3. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients and $f(2i) = f(2+i) = 0$. What is $a+b+c+d$?

according to the conjugate roots theorem.

roots are $2i,\ -2i,\ 2+i,\ 2-i$.

$d = -(2i)^2 \cdot (2^2 - i^2) = 4 \cdot (4+1) = 20$

$c = -[8 + \cancel{4i} + 8 - \cancel{4i} + \cancel{2i} - \cancel{2i}] = -16$

$b = 4 + \cancel{4i} \cancel{-2} \cancel{+4i} \cancel{+2} \cancel{-4i} \cancel{+2} \cancel{-4i} \cancel{-2} + 5 = 9.$

$a = -(2i - 2i + 2 + i + 2 - i) = -4$

$a+b+c+d = -4 + 9 - 16 + 20 = 9$

4. Solve $\sin 4x = \cos 2x$ for $0 \leq x < 2\pi$ without the GDC.

$2\sin 2x \cos 2x = \cos 2x$

① $\cos 2x = 0$ ② $\sin 2x = \frac{1}{2}$

OR $\begin{cases} 2x = \frac{1}{2}\pi \pm 2\pi \\ 2x = \frac{3}{2}\pi \pm 2\pi \end{cases}$ OR $\begin{cases} 2x = \frac{1}{6}\pi \pm 2\pi \\ 2x = \frac{5}{6}\pi \pm 2\pi \end{cases}$

OR $\begin{cases} x = \frac{1}{4}\pi \pm \pi \\ x = \frac{3}{4}\pi \pm \pi \end{cases}$ OR $\begin{cases} x = \frac{1}{12}\pi \pm \pi \\ x = \frac{5}{12}\pi \pm \pi. \end{cases}$

$\therefore x = \frac{1}{12}\pi, \frac{1}{4}\pi, \frac{5}{12}\pi, \frac{3}{4}\pi, \frac{13}{12}\pi, \frac{5}{4}\pi, \frac{17}{12}\pi, \frac{7}{4}\pi.$

5. A line of gradient 4 meets the parabola $y = 2x^2 + 3x$ at the points P and Q. Given that the x-value of P is 2, find the coordinates of Q.

$P(2, 14)$

$y = 4x + b$

$8 + b = 14$

$b = +6$

$\begin{cases} y = 4x + 6 \\ y = 2x^2 + 3x \end{cases}$

$2x^2 - x \cancel{+} 6 = 0$

$\begin{matrix} 2 & 3 \\ 1 & -2 \end{matrix}$

$(2x + 3)(x - 2) = 0$

$x_1 = 2$

$x_2 = -\frac{3}{2}$

$\therefore Q(-\frac{3}{2}, 0)$

6. Solve $|x - 1| + |x - 3| = 7$ without using the GDC.

① $x < 1$

$-x + 1 - x + 3 = 7$

$-2x = 3$

$x = -\frac{3}{2}$ (v)

② $1 \leq x \leq 3$

$x - 1 - x + 3 = 7$

$2 \neq 7$

③ $x > 3$

$2x - 4 = 7$

$x = \frac{11}{2}$ (v)

$\therefore x = -\frac{3}{2}$ or $\frac{11}{2}$

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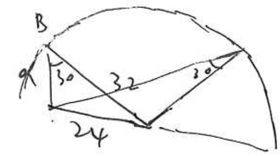
7. Find the sum of the series $\sum_{n=1}^{2020} ni^n$ where $i^2 = -1$.

$$\text{Sum} = \frac{(1+2017)\cdot 505}{2}\cdot i + \frac{(2+2018)\cdot 505}{2}\cdot(-1) + \frac{(3+2019)\cdot 505}{2}\cdot(-i) + \frac{(4+2020)\cdot 505}{2}\cdot 1$$

$$= -1010i + 1010$$

8. Alice claims there are two triangles ABC consistent with the data $b = 24$, $c = 32$ and $B = 30°$ while Bob claims there is only one and Carol insists there are none. Who is right?



$$24^2 = a^2 + 32^2 - 2\cdot a\cdot 32\cdot \cos 30°$$

$$576 = a^2 + 1024 - 32\sqrt{3}a$$

$$a^2 - 32\sqrt{3}a + 448 = 0$$

$$\Delta = 3072 - 1792 > 0$$

$\therefore$ there's 2 a, indicating 2 triangles possible.

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9. Six students, two of whom are Alice and Bob, line up for ice cream. How many ways can this be done if Alice is not first and Bob is not last?

Alice last: $5! = 120$.

Not last: $4 \times 4 \times 4! = 384$
last first

$120 + 384 = 504$

✓

10. Research the Euclidean algorithm. Use this algorithm to find gcd(1441, 1001).

$1441 = 1001 \times 1 + 440$

$1001 = 2 \times 440 + 121$

$440 = 121 \times 3 + 77$

$121 = 77 \times 1 + 44$

$77 = 44 \times 1 + 33$

$44 = 33 \times 1 + 11$

$33 = 11 \times 3 + 0$

$\therefore \gcd(1441, 1001) = 11$

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Solutions to HL1 Assignment #13

1. Finding $f^{-1}(7)$ is equivalent to solving $f(x) = 7$, from which we conclude $x = 2$.

2. $f(-x) = |-x-2| - |-x+2| = |x+2| - |x-2| = -f(x)$. Hence f is an odd function. The graph of f, which you should draw, therefore has symmetry with respect to the origin.

3. By the conjugate roots theorem $-2i$ and $2-i$ are also roots. Hence by the factor theorem the polynomial is $f(x) = (x-2i)(x+2i)(x-2-i)(x-2+i) = (x^2+4)(x^2-4x+5)$. Next observe $f(1) = 1+a+b+c+d = 10$. Hence $a+b+c+d = 9$.

4. Notice $\sin 4x = 2\sin 2x \cos 2x$. So $\sin 4x = \cos 2x$ is equivalent to $2\sin 2x\cos 2x - \cos 2x = 0$, which in turn is equivalent to $\cos 2x(\sin 2x - 1) = 0$. Hence $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

5. Since P has coordinates $(2, 14)$, the line has equation $y = 4x + 6$. This line meets the parabola when $4x + 6 = 2x^2 + 3x$, whence $x = 2$ or $x = -\frac{3}{2}$. So Q has coordinates $(-\frac{3}{2}, 0)$.

6. Consider the three cases: $x < 1$, $1 \le x < 3$ and $x \ge 3$. Solving accordingly gives $x = -\frac{3}{2}$ or $x = \frac{11}{2}$.

7. The series is $(-2+4-6+8-\cdots+2020)+(1-3+5-7+\cdots-2019)i$, which has sum $2\times 505 - 2\times 505i = 1010 - 1010i$.

8. The sine rule is one approach. Another approach is the cosine rule, where we obtain $24^2 = a^2 + 32^2 - 2\cdot a\cdot 32\cdot\cos 30°$ or equivalently $a^2 - 32\sqrt{3}a + 448 = 0$. Since the discriminant of this quadratic equation is positive, there are two solutions for a, and hence two triangles. So Alice is right.

9. Let A be the event that Alice is first and B be the event that Bob is last. We want $n(A' \cap B')$, which is $n(U) - n(A \cup B)$. Now $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5! + 5! - 4!$. So the required number is $6! - 2 \times 5! + 4! = 504$.

10. By the Euclidean algorithm $\gcd(1441, 1001) = \gcd(1001, 440) = \gcd(440, 121) = \cdots = \gcd(44, 33) = \gcd(33, 11) = \gcd(11, 0) = 11$.