HL1 Assignment #12 Name: Jerry 98%

Excellent!

1. Find 366 mod 7. Hence determine what day of the week Tuesday will be one leap year later.

$366 \equiv 2 \mod 7.$

Thursday

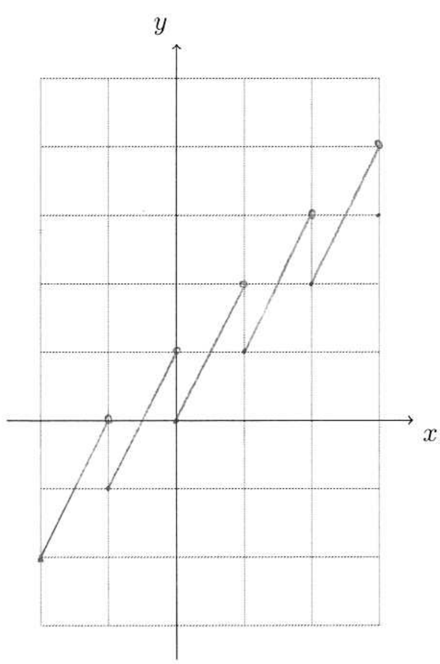
2. Find the domain of the function $f(x) = \dfrac{2019}{\sqrt{4-x^2}}$.

$4 - x^2 > 0$

$x^2 < 4$

$-2 < x < 2$

3. Let $f(x) = 2x - \lfloor x \rfloor,\ x \in [-2, 3]$. Draw the graph of f in the grid below.



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0.01 0.05

4. How many different sums of money can be made up by using one or more coins selected from a cent, a nickel, a dime, a quarter, a loonie and a toonie?

0.1 0.25 1 2

$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 6 + 15 + 20 + 15 + 6 + 1 = 63$

✓

5. Let $f(x) = x^3 + bx^2 + cx + d$ where $b, c, d \in \mathbb{R}$. If $f(2) = 0$ and $f(1+i) = 0$, what is the value of $f(3)$?.

according to the conjugate root theorem,

$f(1-i) = 0$

$\therefore (x-2)(x^2-2x+2) = 0$

$\therefore x^3 - 2x^2 + 2x - 2x^2 + 4x - 4 = 0$

$\therefore f(x) = x^3 - 4x^2 + 6x - 4$

$\therefore f(3) = 27 - 36 + 18 - 4$

$= 5$

✓

6. Let $z = 1 + 2i$ and $w = 2 + ai$ where $a \in \mathbb{R}$. If $\text{Re}(zw) = 2\,\text{Im}(zw)$, find the value of a.

$zw = 2 + ai + 4i - 2a$

$= (2-2a) + (4+a)i$

$\therefore 2 - 2a = 2(4+a)$

$\therefore 2 - 2a = 8 + 2a$

$\therefore a = -1.5$

✓

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7. A circular disc is cut into twelve sectors whose areas form an arithmetic sequence. If the angle of the largest sector is twice the angle of the smallest sector find the radian measure of the angle of the smallest sector.

let the radian of the smallest angle be θ.

$$\frac{(\theta + 2\theta) \cdot 12}{2} = 2\pi$$

$$3\theta = \frac{1}{3}\pi$$

$$\theta = \frac{1}{9}\pi$$

8. The family of parabolas $y = x^2 + 2kx + k$ shares a common point. Find the coordinates of that point.

let $k=1$ $\quad y = x^2 + 2x + 1$

let $k=0$ $\quad y = x^2$

$\therefore 2x + 1 = 0$

$\therefore x = -\frac{1}{2}$

$\therefore y = \frac{1}{4}$

$\therefore (-\frac{1}{2}, \frac{1}{4})$

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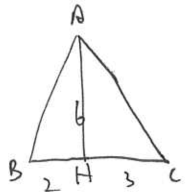
9. Without the calculator show that $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.



let $AH = 6$

if $BH = 2$, $CH = 3$.

then ~~tan~~ $\tan \angle BAH = \frac{1}{3}$

$\tan \angle CAH = \frac{1}{2}$

$\therefore \arctan\frac{1}{3} = \angle BAH$

$\arctan\frac{1}{2} = \angle CAH$

$\because \tan(\angle BAH + \angle CAH) = \dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}$

$= \dfrac{\frac{5}{6}}{\frac{5}{6}}$

$= 1$

Also $\angle BAC < 90°$

$\therefore \angle BAC = \frac{\pi}{4}$

$= \angle BAH + \angle CAH$

$\therefore \arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.

✓

10. The three roots of $x^3 + kx^2 - x - 3 = 0$ are rational numbers in arithmetic progression. Find the value of k.

according to the rational roots theorem.

$RRC = \dfrac{\pm 1, \pm 3}{\pm 1} = 1, -1, 3, -3.$

$-3, -1, 1, 3$ forms a AP with $a = -3$ and $d = 2$.

Since the costant term is -3, the three roots have to be $-3, -1, 1$ instead of $-1, 1, 3$.

$\therefore (x+3)(x+1)(x-1) = 0$

$\therefore x^3 - x + 3x^2 - 3 = 0$

$\therefore x^3 - 3x^2 - x - 3 = 0$

$\therefore k = 3.$

What about rejected roots and $d=0$?

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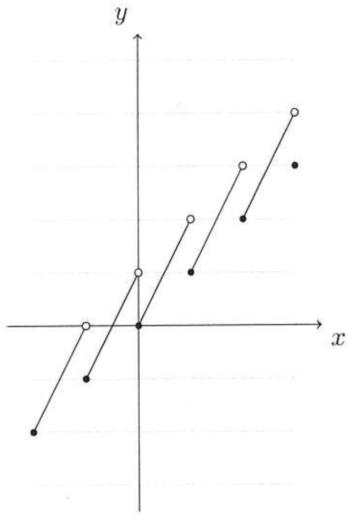
Solutions to HL1 Assignment #12

1. Since 366 mod $7 = 2$, the day of the week is Thursday.

2. We want $4 - x^2 > 0$, or equivalently $-2 < x < 2$. So $D_f =]-2, 2[$.

3. The graph of $f(x) = 2x - \lfloor x \rfloor$ for $x \in [-2, 3]$ is drawn below.



4. This is the same counting problem as counting the number of bit strings of length 6 excluding the 000000 string. Hence the required number is $2^6 - 1 = 63$.

5. By the conjugate roots theorem another root is $1 - i$. Hence $f(x) = (x-2)(x^2 - 2x + 2)$. So $f(3) = 5$.

6. First $zw = (2 - 2a) + (a+4)i$, whence $2 - 2a = 2(a+4) = 2a + 8$. Hence $a = -\frac{3}{2}$.

7. We have $A_1 + A_2 + \cdots + A_{12} = 6(A_1 + 2A_1) = 18A_1 = \pi r^2$. Next by the area of a sector formula $A_1 = \frac{1}{2}r^2\theta_1$, so we conclude $\theta_1 = \frac{\pi}{9}$.

8. We want the point that is independent of the value of k. So we want $2kx + k = 0$, whence $x = -\frac{1}{2}$. So the common point is $(-\frac{1}{2}, \frac{1}{4})$.

9. First note that since $\tan\frac{\pi}{4} = 1$, we must have $0 < \arctan\frac{1}{2} + \arctan\frac{1}{3} < \frac{\pi}{2}$. Next by the sum formula

$$\tan(\arctan\frac{1}{2} + \arctan\frac{1}{3}) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = 1.$$

Given this result and the above inequality, we conclude $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.

10. Let the roots be $a - d, a, a + d$. Then we know $x_1x_2x_3 = a(a^2 - d^2) = 3$ and $x_1x_2 + x_2x_3 + x_3x_1 = (a^2 - ad) + (a^2 + ad) + (a^2 - d^2) = 3a^2 - d^2 = -1$. Solving simultaneously gives the equation $2a^3 + a + 3 = 0$ whose only rational root is -1. Next $k = -(x_1 + x_2 + x_3) = -3a$, whence $k = 3$.