FM2 Assignment #7 Name: Jerry Jiang

1. Suppose $f\colon G \to G'$ is a group homomorphism with identities e and e' respectively. Prove that $f(e) = e'$.

$f(e * e) = f(e) \circ f(e)$

$\because f(e * e) = f(e)$

$\therefore f(e) = f(e) \circ f(e)$

$\therefore f(e) \circ f(e)^{-1} = f(e) \circ f(e) \circ f(e)^{-1}$

$\therefore e' = f(e)$

$\frac{10}{10}$ Excellent!

2. Show that the improper integral $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$ converges and find its value.

Let $x = \tan(u)$, then $dx = [\tan(u)]'du = \frac{1}{\cos^2 u} du$

Therefore, $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \int_0^\infty \frac{1}{\frac{\cos^2 u}{\cos^2 u} + \frac{\sin^2 u}{\cos^2 u}} \cdot \frac{1}{\cos^2 u}\,du$

$\displaystyle = \int_0^\infty \frac{1}{\cos^2 u + \sin^2 u}\,du$

$\displaystyle = u \Big|_0^\infty$

$\displaystyle \therefore \int_0^\infty \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^\infty = \frac{\pi}{2}.$

3. Suppose $\phi\colon \mathbb{Z}_{30} \to \mathbb{Z}_{30}$ is a homomorphism with $\ker(\phi) = \{0, 10, 20\}$. If $\phi(23) = 9$ find all elements that map to 9.

- $\phi(a) = 3a \bmod 30$ for $\forall a \in \mathbb{Z}_{30}$.
- $\phi(a+b) = 3a + 3b \bmod 30 = 3a \bmod 30 + 3b \bmod 30 = \phi(a) + \phi(b)$.
- To verify, $\ker(\phi)$ is indeed $\{0, 10, 20\}$.
- For $\phi(x) = 9$, $3x \bmod 30 = 9$, $3x = 9 + 30k$ where $k \in \mathbb{Z}$.
- $x = 3 + 10k$. since x is integer from 0 to 29, the three possible x are 3, 13, and 23.

More simply $\phi(23+10) = \phi(23) + \phi(10)$ by homomorphism

so $\phi(13) = 9$, etc.

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4. By considering the permutations $\alpha = (12)$ and $\beta = (123)$ in S_4, show that $f\colon S_4 \to S_4$ defined by $f(p) = p \circ p$ is not a homomorphism.

$f(\alpha \circ \beta) = f\left((12)(123)\right) = f\left((12)(12)(231)\right) = f\left((231)\right) = (23)(23) = ()$.

$f(\alpha)\circ f(\beta) = (12)(12)(123)(123) = (12)(23)(23)(31) = (12)(31) = (21)(13) = (213)$.

$\because f(\alpha\circ\beta) \neq f(\alpha)\circ f(\beta)$

$\therefore f$ is not a homomorphism.

5. Consider the curve $y = x^3$. The tangent at a point P on the curve meets the curve again at Q. The tangent at Q meets the curve again at R. Denoting the x-coordinates of P, Q, R by x_1, x_2, x_3 respectively where $x_1 \neq 0$, show that x_1, x_2, x_3 form the first three terms of a divergent geometric sequence.

- Let P be (a, a^3).
- $f'(x) = 3x^2$, $f'(a) = 3a^2$.

$\therefore T_P: \begin{cases} y = 3a^2 x - 2a^3 \\ y = x^3 \end{cases}$

$x^3 - 3a^2 x + 2a^3 = 0$

$(x-a)(x^2+ax-2a^2) = 0$

$x_{1,2} = a$ (point P)

$x_3 = -2a$ (point Q)

$\therefore Q(-2a, -8a^3)$

- $f'(-2a) = 12a^2$

$\therefore T_Q: \begin{cases} y = 12a^2 x + 16a^3 \\ y = x^3 \end{cases}$

$x^3 - 12a^2 x - 16a^3 = 0$

$(x+2a)^2(x-4a) = 0$

$x_{1,2} = -2a$ (point Q)

$x_3 = 4a$ (point R).

Therefore, $x_1 = a$, $x_2 = -2a$, $x_3 = 4a$.

$r = -2$.

Since $|r| > 1$, the sequence diverge.

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