**FM2** Assignment #6 Name: Jerry $\frac{9}{10}$ V. good

1. Prove, without recourse to Venn diagrams, that $A \setminus B$ and $B \setminus A$ are disjoint sets.

$(A \setminus B) \cap (B \setminus A)$

$= (A \cap B') \cap (B \cap A')$ Therefore, the two sets disjoint.

$= A \cap (B' \cap B) \cap A'$

$= A \cap \emptyset \cap A'$

$= A \cap A'$ ✓

$= \emptyset$

2. Suppose that $K$ is a proper subgroup of $H$ and $H$ is a proper subgroup of $G$. If $|K| = 42$ and $|G| = 420$, what are the possible orders of $H$?

According to the Lagrange's Theorem, $|H|$ is a multiple of 42 and a divisor of 420.

The divisor of 420: ~~1~~, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~10~~, ~~12~~, ~~14~~, ~~15~~, ~~20~~.
~~420~~, 210, ~~140~~, ~~105~~, 84, ~~70~~, ~~60~~, ~~42~~, ~~35~~, ~~30~~, ~~28~~, ~~21~~.

Therefore, the possible orders of H are 84 and 210. ✓

3. Suppose $G$ is a finite group of order $n$ and $m$ is relatively prime to $n$. If $g \in G$ and $g^m = e$, prove that $g = e$.

According to the corollary of Lagrange's Theorem, $|g| \mid n$.

Since $g^m = e$, $|g| \mid m$.

$\gcd(m, n) = 1$, so $\gcd(|g|, n) = 1$.

Because $|g| \mid n$, $|g| = 1$.

Therefore $g = e$. /

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4. Determine the null space, nullity and rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

basis for column space: $\left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$.

basis for row space: $\{ (1 \ 0 \ -1 \ -2), (0 \ 1 \ 2 \ 3) \}$.

$\therefore$ $\mathrm{rank}(A) = 2$. ✓

let $x_3 = s$, $x_4 = t$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ nullity of A is 2, ✓

null space of A is $s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$. ✓

5. Let $T$ be a tree with $n > 1$ vertices. Use the handshaking lemma to prove that $T$ has at least two leaves.

- In tree T, we start from a leaf. How do we know a leaf exists?
- Choose any walk and proceed to the next vertex. If the degree of the vertex is no more than 2, proceed; if the degree is more than 2, remove all the excess branches so that the vertex's degree is reduced to 2.
- Continue this process until the walk can't proceed any more.
- Now except the first and last vertex, all other vertices have degree of 2. According to the handshaking lemma, since the degree of the first vertex is 1, the ending vertex has to have an odd degree. Since the degree of the last vertex is no more than 2, it has degree one, which is a leaf.
- Since the reduced graph is a subgraph of T, T must also has at least 2 leaves.

If deg = 1, how do we proceed unless this is the starting leaf?

Where does $n > 1$ come in your proof?

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