Algorithms (V)

Linear Programming

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Linear Programming

Linear Programming

- In a linear programming problem we are given a set of variables, and we want to assign real values to them so as to
  1. satisfy a set of linear equations and/or linear inequalities involving these variables, and
  2. maximize or minimize a given linear objective function.

Example: Profit Maximization

- A boutique chocolatier has two products:
    - its flagship assortment of triangular chocolates, called Pyramide,
    - and the more decadent and deluxe Pyramide Nuit.
- Q: How much of each should it produce to maximize profits?
    - Every box of Pyramide has a a profit of $1.
    - Every box of Nuit has a profit of $6.
    - The daily demand is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
    - The current workforce can produce a total of at most 400 boxes of chocolate per day.

LP Formulation

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Objective function
Constraints
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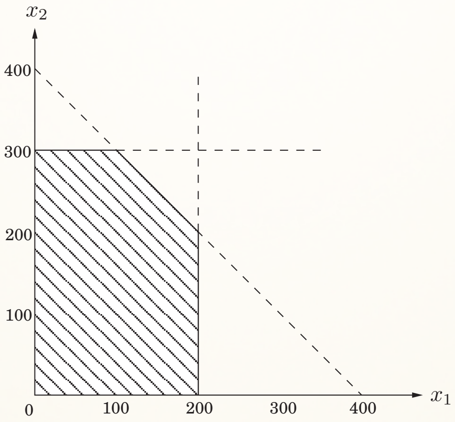
$$
\begin{aligned}
&\max x_1 + 6x_2 \\
&x_1 \leq 200 \\
&x_2 \leq 300 \\
&x_1 + x_2 \leq 400 \\
&x_1, x_2 \geq 0
\end{aligned}
$$

- A linear equation in x_1 and x_2 defines a line in the two-dimensional (2D) plane, and a linear inequality designates a half-space, the region on one side of the line.
- Thus the set of all feasible solutions of this linear program, that is, the points (x_1, x_2) which satisfy all constraints, is the intersection of five half-spaces.
- It is a convex polygon.

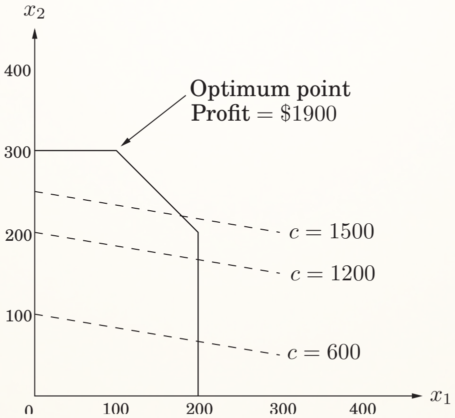
The Convex Polygon



The Optimal Solution

- We want to find the point in this polygon at which the objective function - the profit - is maximized.
- The points with a profit of c dollars lie on the line $x_1 + 6x_2 = c$, which has a slope of $-1/6$.
- As c increases, this "profit line" moves parallel to itself, up and to the right.
- Since the goal is to maximize c, we must move the line as far up as possible, while still touching the feasible region.
- The optimum solution will be the very last feasible point that the profit line sees and must therefore be a vertex of the polygon.

The Convex Polygon



The Optimal Solution

- It is a general rule of linear programs that the optimum is achieved at a vertex of the feasible region.
- The only exceptions are cases in which there is no optimum; this can happen in two ways:
  1. The linear program is infeasible; that is, the constraints are so tight that it is impossible to satisfy all of them.
     - For instance, $x \leq 1$, $x \geq 2$.
  2. The constraints are so loose that the feasible region is unbounded, and it is possible to achieve arbitrarily high objective values.
     - For instance, $\max x_1 + x_2$
     - $x_1, x_2 \geq 0$

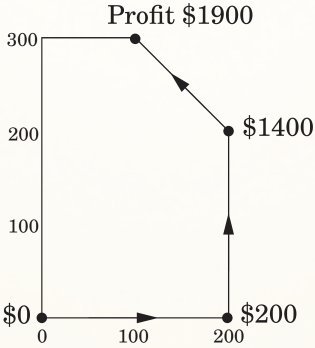
Solving Linear Programs

- Linear programs (LPs) can be solved by the simplex method, devised by George Dantzig in 1947.
- This algorithm starts at a vertex, and repeatedly looks for an adjacent vertex (connected by an edge of the feasible region) of better objective value.
- In this way it does hill-climbing on the vertices of the polygon, walking from neighbor to neighbor so as to steadily increase profit along the way.
- Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.

Solving Linear Programs

- Q: Why does this local test imply global optimality?
- By simple geometry - think of the profit line passing through this vertex. Since all the vertex's neighbors lie below the line, the rest of the feasible polygon must also lie below this line.

The Example



More Products

- The chocolatier decides to introduce a third and even more exclusive line of chocolates, called Pyramide Luxe. One box of these will bring in a profit of $13.
- Let x_1, x_2, x_3 denote the number of boxes of each chocolate produced daily, with x_3 referring to Luxe.
- The old constraints on x_1 and x_2 persist, although the labor restriction now extends to x_3 as well: the sum of all three variables can be at most 400.
- What's more, it turns out that Nuit and Luxe require the same packaging machinery, except that Luxe uses it three times as much, which imposes another constraint $x_2 + 3x_3 \leq 600$.

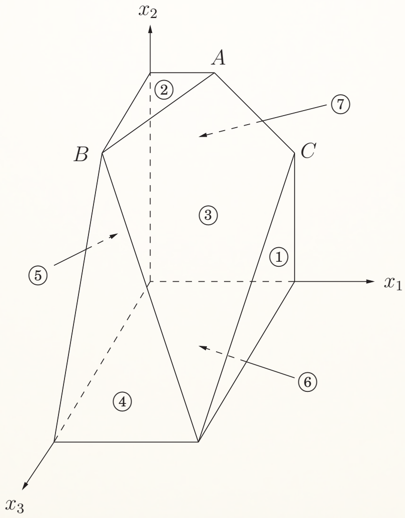
LP

$$\max x_1 + 6x_2 + 13x_3$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 + x_3 \leq 400$$
$$x_2 + 3x_3 \leq 600$$
$$x_1, x_2, x_3 \geq 0$$

LP

- The space of solutions is now three-dimensional.
- Each linear equation defines a 3D plane, and each inequality a half-space on one side of the plane.
- The feasible region is an intersection of seven half-spaces, a polyhedron.
- A profit of c corresponds to the plane $x_1 + 6x_2 + 13x_3 = c$.
- As c increases, this profit-plane moves parallel to itself, further and further into the positive orthant until it no longer touches the feasible region.

The Example

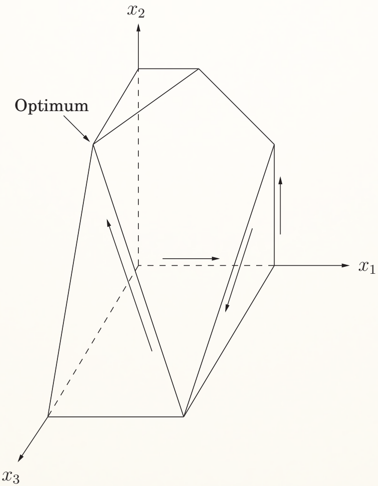


LP

- The point of final contact is the optimal vertex: $(0, 300, 100)$, with total profit $\$3100$.
- Q: How would the simplex algorithm behave on this modified problem?
- A possible trajectory

$$\frac{(0,0,0)}{\$0} \rightarrow \frac{(200,0,0)}{\$200} \rightarrow \frac{(200,200,0)}{\$1400} \rightarrow \frac{(200,0,200)}{\$2800} \rightarrow \frac{(0,300,100)}{\$3100}$$

The Example



A Proof to Optimum

- Here is why you should believe that $(0, 300, 100)$, with a total profit of \$3100, is the optimum.
- Recall

$$
\begin{aligned}
\max x_1 + 6x_2 + 13x_3 \\
x_1 \le 200 \\
x_2 \le 300 \\
x_1 + x_2 + x_3 \le 400 \\
x_2 + 3x_3 \le 600 \\
x_1, x_2, x_3 \ge 0
\end{aligned}
$$

- Add the second inequality to the third, and add to them the fourth multiplied by 4.
- The result is the inequality $x_1 + 6x_2 + 13x_3 \le 3100$.

ILP and Rounding

Example: Production Planning

- The company makes handwoven carpets, a product for which the demand is extremely seasonal.
- Our analyst has just obtained demand estimates for all months of the next calendar year: $d_1, d_2, \ldots, d_{12}$, ranging from 440 to 920.
- We currently have 30 employees, each of whom makes 20 carpets per month and gets a monthly salary of $2000.
- We have no initial surplus of carpets.

Example: Production Planning

- Q: How can we handle the fluctuations in demand? There are three ways:
  1. Overtime, but this is expensive since overtime pay is 80% more than regular pay. Also, workers can put in at most 30% overtime.
  2. Hiring and firing, but these cost $320 and $400, respectively, per worker.
  3. Storing surplus production, but this costs $8 per carpet per month. We currently have no stored carpets on hand, and we must end the year without any carpets stored.

LP Formulations

$$
\begin{aligned}
w_i &= \text{number of workers during } i\text{-th month; } w_0 = 30. \\
x_i &= \text{number of carpets made during } i\text{-th month.} \\
o_i &= \text{number of carpets made by overtime in month } i. \\
h_i, f_i &= \text{number of workers hired and fired, respectively,} \\
& \text{ at beginning of month } i. \\
s_i &= \text{number of carpets stored at end of month } i\text{; } s_0 = 0.
\end{aligned}
$$

LP Formulation

- First, all variables must be nonnegative:

$$w_i, x_i, o_i, h_i, f_i, s_i \geq 0, i = 1, \ldots, 12$$

- The total number of carpets made per month consists of regular production plus overtime:

$$x_i = 20w_i + o_i$$

(one constraint for each $i = 1, \ldots, 12$).

- The number of workers can potentially change at the start of each month:

$$w_i = w_{i-1} + h_i - f_i$$

LP Formulation

- The number of carpets stored at the end of each month is what we started with, plus the number we made, minus the demand for the month:

$$s_i = s_{i-1} + x_i - d_i$$

- And overtime is limited:

$$o_i \leq 6w_i$$

LP Formulation

- The objective function is to minimize the total cost:

$$\min 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + 180 \sum_i o_i$$

Integer Linear Programming

- The optimum solution might turn out to be fractional; for instance, it might involve hiring 10.6 workers in the month of March.
- This number would have to be rounded to either 10 or 11 in order to make sense, and the overall cost would then increase correspondingly.
- In the present example, most of the variables take on fairly large (double-digit) values, and thus rounding is unlikely to affect things too much.

Integer Linear Programming

- There are other LPs, however, in which rounding decisions have to be made very carefully in order to end up with an integer solution of reasonable quality.
- In general, there is a tension in linear programming between the ease of obtaining fractional solutions and the desirability of integer ones.
- As we shall see in NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.

Standard Linear Programming

Variants of Linear Programming

- A general linear program has many degrees of freedom:
  1. It can be either a maximization or a minimization problem.
  2. Its constraints can be equations and/or inequalities.
  3. The variables are often restricted to be nonnegative, but they can also be unrestricted in sign.
- We will now show that these various LP options can all be reduced to one another via simple transformations.

Variants of Linear Programming

1. To turn a maximization problem into a minimization (or vice versa), just multiply the coefficients of the objective function by -1.

Variants of Linear Programming

❶ To turn an inequality constraint like $\sum_{i=1}^{n} a_i x_i \leq b$ into an equation, introduce a new variable s and use

$$\sum_{i=1}^{n} a_i x_i + s = b$$

$$s \geq 0$$

This s is called the slack variable for the inequality.

❷ To change an equality constraint into inequalities is easy: rewrite $ax = b$ as the equivalent pair of constraints $ax \leq b$ and $ax \geq b$.

Variants of Linear Programming

- Finally, to deal with a variable x that is unrestricted in sign, do the following:
  - Introduce two nonnegative variables, $x^+, x^- \geq 0$.
  - Replace x, wherever it occurs in the constraints or the objective function, by $x^+ - x^-$.

Standard Form

Therefore, we can reduce any LP (maximization or minimization, with both inequalities and equations, and with both nonnegative and unrestricted variables) into an LP of a much more constrained kind that we call the standard form:

- the variables are all nonnegative,
- the constraints are all equations,
- and the objective function is to be minimized.

$$\begin{array}{c} \max x_1 + 6x_2 \\ x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} \min -x_1 - 6x_2 \\ x_1 + s_1 = 200 \\ x_2 + s_2 = 300 \\ x_1 + x_2 + s_3 = 400 \\ x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array}$$

Quiz

In the shortest path problem, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the weight of a shortest path from s to t.

Shortest Path in LP

$$\max d_t$$

$$\begin{aligned} d_v &\leq d_u + w(u,v) && (u,v) \in E \\ d_s &= 0 \\ d_i &\geq 0 && i \in V \end{aligned}$$

Q: Another formalization?

Shortest Path in LP

Let $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$; that is, $\mathcal{S}$ is the set of all s-t cuts in the graph. Then we can model the shortest s-t path problem with the following integer program,

$$\min \sum_{e \in E} w_e x_e$$

$$\begin{aligned} \sum_{e \in \delta(S)} x_e \geq 1 & \quad S \in \mathcal{S} \\ x_e \in \{0, 1\} & \quad e \in E \end{aligned}$$

where $\delta(S)$ is the set of all edges that have one endpoint in S and the other endpoint not in S.

- Can we relax the restriction $x_e \in \{0, 1\}$ to $0 \leq x_e \leq 1$?
- How about $x_e \geq 0$?

Referred Materials

- Content of this lecture comes from Section 7.1, in [DPV07], Section 29.2 in [CLRS09], and Section 7.3 in [WS11].