Computational Complexity **Autumn, 2018**

Solution : Homework 3

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Problem 5.5.

Solution:

1. PSPACE $\subseteq$ AP. Since TQBF can be solved in polynomial-time in AP (just "guess" the values), TQBF $\in$ AP. Since eave PSPACE language reduces to PSPACE, PSPACE $\subseteq$ AP.
2. AP $\subseteq$ PSPACE. The traversal of the configuration tree can be done in PSPACE.

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Problem 5.9.

Solution:

(a) As we known INDSET=$\{< G, k >$ | Graph G has an independent set of size $\geq$ k$\}$ $\in$ NP since INDSET admits a short certificate (i.e., $L \in$ INDSET can be done in P). $< G, k >\in$ EXACT INDSET iff $< G, k >\in$ INDSET and $\forall k' > k :< G, k' >\in \overline{INDEST}$. Therefore EXACT INDSET $\in \pi_2^p$.

(b) $< G, k >\in$ EXACT INDSET iff $< G, k >\in$ INDSET and $< G, k >\notin$ INDSET. Letting $L = \{< G, k > |G$ doesnt have an INDSET of size $k + 1\}$, we see from the above that EXACT INDSET = INDSET $\cap$ L. Its clear that INDSET $\in$ NP and L $\in$ coNP; hence EXACT INDSET $\in$ DP.

(c) Pick any $L \in DP$ we have $L = L1 \cap L2$, where $L1 \in NP$, L2$\in$ coNP. Let ϕ_1 be the formula obtained from reducing L1 to 3SAT, and ϕ_2 the formula obtained from reducing L2 to $\overline{\text{3SAT}}$. We augment the reduction by adding $k1$ nodes and add edges from each of them to every other node in the graph. Then the reduction will always give a maximum ind set of size k if the formula of k clauses is satisfiable, and a maximum INDSET of size $k1$ if its not. Let (G1, k1) and (G2, k2) be the results of applying this reduction to ϕ_1 and ϕ_2, respectively. If k1 and k2 are equal, then duplicate an arbitrary clause in ϕ_1 and recompute the reduction this will result in different values for k1 and k2.

Thus we have: $x \in L1 \Leftrightarrow \phi_1 \in 3SAT \Leftrightarrow (G1, k1) \in$ EXACT INDSET and $x \in L2 \Leftrightarrow 2 \notin 3SAT \Leftrightarrow (G2, k2) \notin$ INDSET$\Leftrightarrow (G2, k21) \in$ EXACT INDSET. Now define G = G1 G2, where V (G) = V (G1) V (G2), and theres an edge in G from (u1, u2) to (v1, v2) iff theres an edge from u1 to v1 in G1 or an edge from u2 to v2 in G2. Its not hard to see that, for any a and b, $(G1, a) \in$EXACT INDSET $\wedge(G2, b) \in$ EXACT INDSET $\Leftrightarrow (G, ab) \in$ EXACT INDSET. We then can proof $x \in L \Leftrightarrow (G, k1(k2-1)) \in$ EXACT INDSET.

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Problem 6.3.

Solution: The time hierarchy theorem implies that there exists a language $L' \in$ DTIME($2^{|x|^2}$ but $L' \in$ DTIME(T') for some larger T'. In particular, L' is decidable. Construct the unary variant of L': $L := \{1^n : n = x, x \in L'\}$. Obviously L is decidable since L' is decidable.

1. $L \in \mathrm{P}_{poly}$. Let $C_n(x) := x_1 \wedge ... \wedge x_n$ iff $n \in L'$, and $C_n(x) = 0$ otherwise. Then C_n form a polynomial-size circuit family that decides L.

2. Assume that $L \in P$. Then there is a polynomial p and a TM M that decides L in time $p(|x|)$. We construct the Turing machine M' : Given input x, it runs $b := M(1^n)$ with $n := x$ and returns b. Since M decides L, M' decides L'.

   The running time of M' is: $O(p(2^{|x|}))$ because it runs M with an input 1^n of lengths $2^{|x|}$. Since p is a polynomial, $O(p(2^{|x|})) = O((2^{|x|})^c) = O(2^{c|x|}) \subseteq O(2^{|x|^2})$ for some constant $c > 0$. Thus M' runs in time $O(2^{|x|^2})$ and decides L', in contradiction to the assumption that $L' \notin \mathrm{DTIME}(2^{|x|^2})$.

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Problem 6.4.

Solution:

1. Assume L $\in$ P. We further assume M is a one tape TM bounded in time by $T(n) = cn^c$. We can reduce TM to Circuit as follows T_i are configurations. The symbols in T_i only relies on at most 3 symbols in

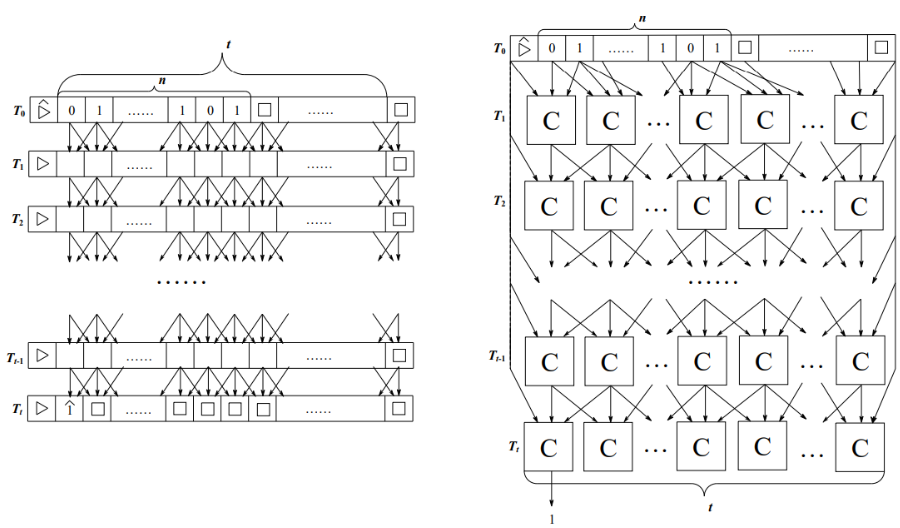

Figure 1.1: Reduction from TM to Circuit

   T_{i-1}. Therefore, the reduction can is computable in logspace.

2. If a language has logspace-uniform circuits, then there is an implicitly logspace computable function mapping 1^n to C_n. Therefore the language is in P.

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