Technical Report of DeltaBoost: Gradient Boosting Decision Trees with Efficient Machine Unlearning

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1 PROOF OF THEOREM 3.1

For convenience, we restate Thoerem 3.1 in the main paper here as Theorem 1.1.

THEOREM 1.1. *Suppose* $\lambda \ll \sum_{j=1}^{k-1}\sum_{i\in B_j} h_i$, $\lambda \ll \sum_{j=k-1}^{m}\sum_{i\in B_j} h_i$, *and* $n_d \ll n$. *For* $\forall s_k (k \in [1,m])$,

$$\arg\max_k \phi_1(d_1, ..., d_m; s_k) \approx \arg\max_k \phi_0(s_k)$$

if $\zeta n_d \ll n$, *where* $\zeta = \frac{a_r c_l^2 + a_l c_r^2}{(b_r c_l - b_l c_r)^2}$,

$$a_l = \sum_{j=1}^{k-1}\sum_{i\in B_j} g_i^2,\ b_l = \sum_{j=1}^{k-1}\sum_{i\in B_j} g_i,\ c_l = \sum_{j=1}^{k-1}\sum_{i\in B_j} h_i$$
$$a_r = \sum_{j=k}^{m}\sum_{i\in B_j} g_i^2,\ b_r = \sum_{j=k}^{m}\sum_{i\in B_j} g_i,\ c_r = \sum_{j=k}^{m}\sum_{i\in B_j} h_i$$

Before proving Theorem 1.1, we first study the following problem, which is helpful for analyzing gain.

Sum of deleted values in a single bin: Given a list of values $\{v_i\}_{i\in B_j}$ in bin B_j, if we randomly choose n_d values from the list with replacement, what is the distribution of the sum of the chosen values that fall into the j-th bin, i.e., $\sum_{i\in d_j} v_i$?

Note that v_i may refer to either gradient g_i or Hessian h_i in GBDT. Choosing with replacement implies that each choice is independent; these chosen values are denoted as $Y_1, Y_2, ..., Y_k$. We define X_k as follows,

$$X_k = \begin{cases} Y_k, & \text{chosen value is in } B_j \\ 0, & \text{otherwise} \end{cases}.$$

Apparently, $\sum_{i\in d_j} v_i = \sum_{k=1}^{n_d} X_k$. We can therefore calculate $\mathbb{E}\left[\sum_{i\in d_j} v_i\right]$ by summing up $\mathbb{E}[X_k]$ according to the linearity of expectation.

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Proposition 1.1. $\forall j \in [1, m] \cup \mathbf{N}, \mathbb{E}\left[\sum_{i \in d_j} v_i\right] = \frac{n_d}{n} \sum_{i \in B_j} v_i$.

Proof. For the independence of Y_k, it holds $\forall i \in [1, n], \forall k \in [1, n_d]$,

$$\Pr[Y_k = v_i] = \frac{1}{n},$$

Therefore, $\forall k \in [1, n_d], \mathbb{E}[X_k]$ can be easily calculated by

$$\mathbb{E}[X_k] = \frac{1}{n}\left(\sum_{i \in B_j} v_i + \sum_{i \notin B_j} 0\right) = \frac{1}{n} \sum_{i \in B_j} v_i. \tag{1}$$

Due to the linearity of expectation, we have

$$\mathbb{E}\left[\sum_{i \in d_j} v_i\right] = \mathbb{E}\left[\sum_{k=1}^{n_d} X_k\right] = \sum_{k=1}^{n_d} \mathbb{E}\left[X_k\right] = \frac{n_d}{n} \sum_{i \in B_j} v_i. \tag{2}$$

□

Proposition 1.1 implies that the expected value of $\sum_{i \in d_j} v_i$ is simply the summation of all the values in the bin B_j multiplied by a constant. Meanwhile, a similar conclusion holds for multiple bins since the proof of Proposition 1.1 has no requirement on the set B_j.

Lemma 1.2. *For* $\forall S \subseteq [1, m] \cup \mathbb{N}$*, we have*

$$\mathbb{E}\left[\sum_{j \in S} \sum_{i \in d_j} v_i\right] = \frac{n_d}{n} \sum_{j \in S} \sum_{i \in B_j} v_i$$

Proof. Since Proposition 1.1 has no requirement on the set B_j, this corollary can be similarly derived by following the proof of Lemma 1.1 and substituting the set B_j to a larger set $\{B_j | j \in S\}$. □

Besides the expected value, we can calculate the second raw moment of $\sum_{i \in d_j} v_i$, i.e., $\mathbb{E}\left[\left(\sum_{i \in d_j} v_i\right)^2\right]$, which is very helpful for estimating the expected value of $\phi_1(d_1, ..., d_m; s_k)$.

Proposition 1.2. $\forall j \in [1, m] \cup \mathbf{N},\ \mathbb{E}\left[\left(\sum_{i \in d_j} v_i\right)^2\right] = \frac{n_d}{n} \sum_{i \in B_j} v_i^2 + \frac{n_d^2 - n_d}{n^2}\left(\sum_{i \in B_j} v_i\right)^2$

Proof.

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{i\in d_j} v_i\right)^2\right] &= \mathbb{E}\left[\left(\sum_{k=1}^{n_d} X_k\right)^2\right] \\
&= \mathrm{Var}\left(\sum_{k=1}^{n_d} X_k\right) + \mathbb{E}\left[\sum_{k=1}^{n_d} X_k\right]^2 \\
&= \sum_{k=1}^{n_d} \mathrm{Var}(X_k) + \frac{n_d^2}{n^2}\left(\sum_{i\in B_j} v_i\right)^2 \quad \text{(independence of } X_k\text{)} \\
&= \sum_{k=1}^{n_d} \left(\mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2\right) + \frac{n_d^2}{n^2}\left(\sum_{i\in B_j} v_i\right)^2 \\
&= \sum_{k=1}^{n_d} \left(\frac{1}{n}\sum_{i\in B_j} v_i^2 - \left(\frac{1}{n}\sum_{i\in B_j} v_i\right)^2\right) + \frac{n_d^2}{n^2}\left(\sum_{i\in B_j} v_i\right)^2 \\
&= \frac{n_d}{n}\sum_{i\in B_j} v_i^2 - \frac{n_d}{n^2}\left(\sum_{i\in B_j} v_i\right)^2 + \frac{n_d^2}{n^2}\left(\sum_{i\in B_j} v_i\right)^2 \\
&= \frac{n_d}{n}\sum_{i\in B_j} v_i^2 + \frac{n_d^2 - n_d}{n^2}\left(\sum_{i\in B_j} v_i\right)^2
\end{aligned}
\tag{3}
$$

□

Lemma 1.3. $\forall S \subseteq [1, m] \cup \mathbf{N}$, $\mathbb{E}\left[\left(\sum_{j\in S}\sum_{i\in d_j} v_i\right)^2\right] = \frac{n_d}{n}\sum_{j\in S}\sum_{i\in B_j} v_i^2 + \frac{n_d^2 - n_d}{n^2}\left(\sum_{j\in S}\sum_{i\in B_j} v_i\right)^2$

Proof. Since Proposition 1.2 has no requirement on the set B_j, this corollary can be similarly derived by following the proof of Proposition 1.2 and substituting the set B_j to a larger set $\{B_j | j \in S\}$. □

With Lemma 1.2 and Lemma 1.3, we can estimate the expected value of gain according to Theorem 1.4.

Theorem 1.4. *The expected value of the gain after removal on split value s_k can be estimated by*

$$
\begin{aligned}
\mathbb{E}[\phi_1(d_1, ..., d_j; s_k)] = &\frac{a_l n_d + \frac{(n-n_d)^2 - n_d}{n} b_l^2}{(n-n_d)c_l + n\lambda} + \frac{a_r n_d + \frac{(n-n_d)^2 - n_d}{n} b_r^2}{(n-n_d)c_r + n\lambda} \\
&- \frac{(a_l + a_r)n_d + \frac{(n-n_d)^2 - n_d}{n}(b_l + b_r)^2}{(n-n_d)(c_l + c_r) + n\lambda} \\
&+ O\left(\frac{Var(S_1)\mathbb{E}[R_1^2]}{\mathbb{E}[S_1]^3} - \frac{Cov(R_1^2, S_1)}{\mathbb{E}[S_1]^2} + \frac{Var(S_2)\mathbb{E}[R_2^2]}{\mathbb{E}[S_2]^3}\right. \\
&\left. - \frac{Cov(R_2^2, S_2)}{\mathbb{E}[S_2]^2} - \frac{Var(S_3)\mathbb{E}[R_3^2]}{\mathbb{E}[S_3]^3} + \frac{Cov(R_3^2, S_3)}{\mathbb{E}[S_3]^2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&a_l = \sum_{j=1}^{k-1} \sum_{i \in B_j} g_i^2,\ b_l = \sum_{j=1}^{k-1} \sum_{i \in B_j} g_i,\ c_l = \sum_{j=1}^{k-1} \sum_{i \in B_j} h_i \\
&a_r = \sum_{j=k}^{m} \sum_{i \in B_j} g_i^2,\ b_r = \sum_{j=k}^{m} \sum_{i \in B_j} g_i,\ c_r = \sum_{j=k}^{m} \sum_{i \in B_j} h_i \\
&R_1 = \sum_{j=1}^{m} \sum_{i \in B_j} g_i - \sum_{j=1}^{m} \sum_{i \in d_j} g_i \\
&S_1 = \sum_{j=1}^{m} \sum_{i \in B_j} h_i - \sum_{j=1}^{m} \sum_{i \in d_j} h_i + \lambda \\
&R_2 = \sum_{j=1}^{k-1} \sum_{i \in B_j} g_i - \sum_{j=1}^{m} \sum_{i \in d_j} g_i \\
&S_2 = \sum_{j=1}^{k-1} \sum_{i \in B_j} h_i - \sum_{j=1}^{m} \sum_{i \in d_j} h_i + \lambda \\
&R_3 = \sum_{j=k}^{m} \sum_{i \in B_j} g_i - \sum_{j=1}^{m} \sum_{i \in d_j} g_i \\
&S_3 = \sum_{j=k}^{m} \sum_{i \in B_j} h_i - \sum_{j=1}^{m} \sum_{i \in d_j} h_i + \lambda
\end{aligned}
$$

Proof. The gain shifting can be written as

$$
\begin{aligned}
&\mathbb{E}[\phi_1(d_1, ..., d_j; s_k)] \\
=\ &\mathbb{E}\left[\frac{\left(\sum_{j=1}^{k-1} \sum_{i \in B_j} g_i - \sum_{j=1}^{k-1} \sum_{i \in d_j} g_i\right)^2}{\sum_{j=1}^{k-1} \sum_{i \in B_j} h_i - \sum_{j=1}^{k-1} \sum_{i \in d_j} h_i + \lambda}\right] + \\
&\mathbb{E}\left[\frac{\left(\sum_{j=k}^{m} \sum_{i \in B_j} g_i - \sum_{j=k}^{m} \sum_{i \in d_j} g_i\right)^2}{\sum_{j=k}^{m} \sum_{i \in B_j} h_i - \sum_{j=k}^{m} \sum_{i \in d_j} h_i + \lambda}\right] - \\
&\mathbb{E}\left[\frac{\left(\sum_{j=1}^{m} \sum_{i \in B_j} g_i - \sum_{j=1}^{m} \sum_{i \in d_j} g_i\right)^2}{\sum_{j=1}^{m} \sum_{i \in B_j} h_i - \sum_{j=1}^{m} \sum_{i \in d_j} h_i + \lambda}\right].
\end{aligned}
\tag{4}
$$

The estimation of the three expected values is similar; we take the third expected value as an example. For simplicity, we denote

$$
\begin{aligned}
&R_1 = \sum_{j=1}^{m} \sum_{i \in B_j} g_i - \sum_{j=1}^{m} \sum_{i \in d_j} g_i \\
&S_1 = \sum_{j=1}^{m} \sum_{i \in B_j} h_i - \sum_{j=1}^{m} \sum_{i \in d_j} h_i + \lambda
\end{aligned}
\tag{5}
$$

According to Lemma 1.2 and Lemma 1.3, we have

$$
\begin{aligned}
\mathbb{E}[R_1^2] &= \mathbb{E}\left[\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i - \sum_{j=1}^{m}\sum_{i\in d_j} g_i\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{j=1}^{m}\sum_{i\in d_j} g_i\right)^2\right] - 2\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)\left(\mathbb{E}\left[\sum_{j=1}^{m}\sum_{i\in d_j} g_i\right]\right) \\
&\quad + \left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)^2 \\
&= \frac{n_d}{n}\sum_{j=1}^{m}\sum_{i\in B_j} g_i^2 + \frac{n_d^2 - n_d}{n^2}\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)^2 \\
&\quad - \frac{2n_d}{n}\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)^2 + \left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)^2 \\
&= \frac{n_d}{n}\sum_{j=1}^{m}\sum_{i\in B_j} g_i^2 + \frac{(n-n_d)^2 - n_d}{n^2}\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)^2
\end{aligned}
\tag{6}
$$

According to Lemma 1.2, we have

$$
\mathbb{E}[S_1] = \mathbb{E}\left[\sum_{j=1}^{m}\sum_{i\in B_j} h_i - \sum_{j=1}^{m}\sum_{i\in d_j} h_i + \lambda\right] = \left(1 - \frac{n_d}{n}\right)\sum_{j=1}^{m}\sum_{i\in B_j} h_i + \lambda. \tag{7}
$$

Combining Equation 6 and 7, it holds

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i - \sum_{j=1}^{m}\sum_{i\in d_j} g_i\right)^2}{\sum_{j=1}^{m}\sum_{i\in B_j} h_i - \sum_{j=1}^{m}\sum_{i\in d_j} h_i + \lambda}\right] \\
=&\frac{\mathbb{E}[R_1^2]}{\mathbb{E}[S_1]} + O\left(\frac{\mathrm{Var}(S_1)\mathbb{E}[R_1^2]}{\mathbb{E}[S_1]^3} - \frac{Cov(R_1^2, S_1)}{\mathbb{E}[S_1]^2}\right) \\
=&\frac{n_d\sum_{j=1}^{m}\sum_{i\in B_j} g_i^2 + \frac{(n-n_d)^2 - n_d}{n}\left(\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right)^2}{(n-n_d)\sum_{j=1}^{m}\sum_{i\in B_j} h_i + n\lambda} \\
&+ O\left(\frac{\mathrm{Var}(S_1)\mathbb{E}[R_1^2]}{\mathbb{E}[S_1]^3} - \frac{Cov(R_1^2, S_1)}{\mathbb{E}[S_1]^2}\right).
\end{aligned}
\tag{8}
$$

We can similarly derive that

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\left(\sum_{j=1}^{k-1}\sum_{i\in B_j} g_i - \sum_{j=1}^{k-1}\sum_{i\in d_j} g_i\right)^2}{\sum_{j=1}^{k-1}\sum_{i\in B_j} h_i - \sum_{j=1}^{k-1}\sum_{i\in d_j} h_i + \lambda}\right] \\
&= \frac{n_d \sum_{j=1}^{k-1}\sum_{i\in B_j} g_i^2 + \frac{(n-n_d)^2-n_d}{n}\left(\sum_{j=1}^{k-1}\sum_{i\in B_j} g_i\right)^2}{(n-n_d)\sum_{j=1}^{k-1}\sum_{i\in B_j} h_i + n\lambda} \\
&\quad + O\left(\frac{\mathrm{Var}(S_2)\mathbb{E}[R_2^2]}{\mathbb{E}[S_2]^3} - \frac{Cov(R_2^2, S_2)}{\mathbb{E}[S_2]^2}\right) \\
&\mathbb{E}\left[\frac{\left(\sum_{j=k}^{m}\sum_{i\in B_j} g_i - \sum_{j=k}^{m}\sum_{i\in d_j} g_i\right)^2}{\sum_{j=k}^{m}\sum_{i\in B_j} h_i - \sum_{j=k}^{m}\sum_{i\in d_j} h_i + \lambda}\right] \\
&= \frac{n_d \sum_{j=k}^{m}\sum_{i\in B_j} g_i^2 + \frac{(n-n_d)^2-n_d}{n}\left(\sum_{j=k}^{m}\sum_{i\in B_j} g_i\right)^2}{(n-n_d)\sum_{j=k}^{m}\sum_{i\in B_j} h_i + n\lambda} \\
&\quad + O\left(\frac{\mathrm{Var}(S_3)\mathbb{E}[R_3^2]}{\mathbb{E}[S_3]^3} - \frac{Cov(R_3^2, S_3)}{\mathbb{E}[S_3]^2}\right).
\end{aligned} \tag{9}
$$

From Equation 4, 8, and 9, with the definitions of $a_l, a_r, b_l, b_r, c_l, c_r$ in Theorem 1.4, it holds

$$
\begin{aligned}
\mathbb{E}[\phi_1(d_1, ..., d_j; s_k)] &= \frac{a_l n_d + \frac{(n-n_d)^2-n_d}{n} b_l^2}{(n-n_d)c_l + n\lambda} + \frac{a_r n_d + \frac{(n-n_d)^2-n_d}{n} b_r^2}{(n-n_d)c_r + n\lambda} \\
&\quad - \frac{(a_l + a_r)n_d + \frac{(n-n_d)^2-n_d}{n}(b_l+b_r)^2}{(n-n_d)(c_l+c_r) + n\lambda} \\
&\quad + O\left(\frac{\mathrm{Var}(S_1)\mathbb{E}[R_1^2]}{\mathbb{E}[S_1]^3} - \frac{Cov(R_1^2, S_1)}{\mathbb{E}[S_1]^2} + \frac{\mathrm{Var}(S_2)\mathbb{E}[R_2^2]}{\mathbb{E}[S_2]^3}\right. \\
&\quad \left. - \frac{Cov(R_2^2, S_2)}{\mathbb{E}[S_2]^2} - \frac{\mathrm{Var}(S_3)\mathbb{E}[R_3^2]}{\mathbb{E}[S_3]^3} + \frac{Cov(R_3^2, S_3)}{\mathbb{E}[S_3]^2}\right),
\end{aligned} \tag{10}
$$

□

We can further investigate how random removal affects the gain by estimating the expected value. Theorem 1.5 demonstrates that, when $n_d \ll n$, as n_d increases, the gain of each split candidate changes at an almost constant rate determined by histogram.

Theorem 1.5. *Suppose $\lambda \ll \sum_{j=1}^{k-1}\sum_{i\in B_j} h_i$, $\lambda \ll \sum_{j=k-1}^{m}\sum_{i\in B_j} h_i$, and $n_d \ll n$. For $\forall s_k (k \in [1, m])$,*

$$
\frac{\partial \mathbb{E}[\phi_1(d_1, ..., d_j; s_k)]}{\partial n_d} \approx \alpha(s_k, \{g_i\}_{i=1}^n, \{h_i\}_{i=1}^n).
$$

where α is unrelated to n_d, which is determined by the histogram and s_k.

Proof. From Theorem 1.4, we denote $f(n_d)$ as the approximation of $\mathbb{E}[\phi_1(d_1,...,d_j;s_k)]$.

$$\begin{aligned} f(n_d) = & \frac{a_l n_d + \frac{(n-n_d)^2-n_d}{n} b_l^2}{(n-n_d)c_l + n\lambda} + \frac{a_r n_d + \frac{(n-n_d)^2-n_d}{n} b_r^2}{(n-n_d)c_r + n\lambda} \\ & - \frac{(a_l+a_r)n_d + \frac{(n-n_d)^2-n_d}{n}(b_l+b_r)^2}{(n-n_d)(c_l+c_r)+n\lambda} \end{aligned} \tag{11}$$

Since $n_d^2 \ll n^2$, we also have $n_d \ll n^2$, therefore

$$\frac{(n-n_d)^2-n_d}{n^2} = 1 - \frac{2n_d}{n} + \frac{n_d^2 - n_d}{n^2} = 1 - \frac{2n_d}{n} + O\left(\frac{n_d^2}{n^2}\right). \tag{12}$$

With Equation 12 and $\lambda \ll \sum_{j=1}^{k-1}\sum_{i\in B_j} h_i,\ \lambda \ll \sum_{j=k-1}^{m}\sum_{i\in B_j} h_i$, Equation 11 can be approximated by

$$\begin{aligned} f(n_d) \approx & \frac{a_l \frac{n_d}{n} + (1-\frac{2n_d}{n})b_l^2}{(1-\frac{n_d}{n})c_l} + \frac{a_r \frac{n_d}{n} + (1-\frac{2n_d}{n})b_r^2}{(1-\frac{n_d}{n})c_r} \\ & - \frac{(a_l+a_r)\frac{n_d}{n} + (1-\frac{2n_d}{n})(b_l+b_r)^2}{(1-\frac{n_d}{n})(c_l+c_r)}. \end{aligned} \tag{13}$$

Let $t = \frac{n_d}{n}$. Consider

$$\begin{aligned} g(t) = & \frac{a_l t + (1-2t)b_l^2}{(1-t)c_l} + \frac{a_r t + (1-2t)b_r^2}{(1-t)c_r} \\ & - \frac{(a_l+a_r)t + (1-2t)(b_l+b_r)^2}{(1-t)(c_l+c_r)} \end{aligned} \tag{14}$$

The derivative of $g(t)$ is

$$g'(t) = \frac{a_r c_l^2 + a_l c_r^2 - (b_r c_l - b_l c_r)^2}{c_l c_r (c_l+c_r)(1-t)^2}. \tag{15}$$

Since $n_d \ll n$, we have $\frac{1}{(1-t)^2} \approx 1$. From Equation 15, it holds

$$\begin{aligned} \frac{\partial \mathbb{E}[\phi_1(d_1,...,d_j;s_k)]}{\partial n_d} = g'(t)\cdot\frac{\partial t}{n_d} &= \frac{a_r c_l^2 + a_l c_r^2 - (b_r c_l - b_l c_r)^2}{n c_l c_r (c_l+c_r)(1-t)^2} \\ &\approx \frac{a_r c_l^2 + a_l c_r^2 - (b_r c_l - b_l c_r)^2}{n c_l c_r (c_l+c_r)}, \end{aligned} \tag{16}$$

which is unrelated to n_d. Therefore, Theorem 1.5 holds with

$$\alpha(s_k, \{g_i\}_{i=1}^n, \{h_i\}_{i=1}^n) = \frac{a_r c_l^2 + a_l c_r^2 - (b_r c_l - b_l c_r)^2}{n c_l c_r (c_l+c_r)}. \tag{17}$$

□

Then, our main observation (Theorem 1.1) can be naturally proved according to the constant gradient.

Proof. Given $\zeta = \frac{a_r c_l^2 + a_l c_r^2}{(b_r c_l - b_l c_r)^2}$, it holds

$$\begin{aligned}\alpha(s_k, \{g_i\}_{i=1}^n, \{h_i\}_{i=1}^n) =& \frac{\zeta - 1}{n} \cdot \frac{(b_r c_l - b_l c_r)^2}{c_l c_r (c_l + c_r)} \\ =& \frac{\zeta - 1}{n} \cdot \left(\frac{b_l^2}{c_l} + \frac{b_r^2}{c_r} - \frac{(b_l + b_r)^2}{c_l + c_r} \right).\end{aligned} \tag{18}$$

Note that if regularization term $\lambda \ll c_l, c_r$ (λ can also be set to 0 with only a small effect on the accuracy [26]), it holds

$$\phi_0(s_k) = \frac{b_l^2}{c_l + \lambda} + \frac{b_r^2}{c_r + \lambda} - \frac{(b_l + b_r)^2}{c_l + c_r + \lambda} \approx \frac{b_l^2}{c_l} + \frac{b_r^2}{c_r} - \frac{(b_l + b_r)^2}{c_l + c_r}, \tag{19}$$

where $\phi_0(s_k)$ is the gain before removal. Combining Equation 18 and 19, with a sufficiently small ζ, it holds

$$\alpha(s_k, \{g_i\}_{i=1}^n, \{h_i\}_{i=1}^n) = \frac{\zeta - 1}{n} \phi_0(s_k) \tag{20}$$

If $\zeta n_d \ll n$, $\zeta n_d / n$ is negligible. Thus, $\phi_1(d_1, ..., d_m; s_k)$ can be estimated by the first-order Taylor expansion as below.

$$\begin{aligned} & \arg\max_k \phi_1(d_1, ..., d_m; s_k) \\ \approx\ & \arg\max_k \left(\phi_0(s_k) + n_d \alpha(s_k, \{g_i\}_{i=1}^n, \{h_i\}_{i=1}^n) \right) \\ =\ & \arg\max_k \left(\phi_0(s_k) - \frac{n_d}{n} \phi_0(s_k) + \frac{\zeta n_d}{n} \right) \\ \approx\ & \arg\max_k \left(\phi_0(s_k) - \frac{n_d}{n} \phi_0(s_k) \right) \\ =\ & \arg\max_k \left(\left(1 - \frac{n_d}{n}\right) \phi_0(s_k) \right) \\ =\ & \arg\max_k \phi_0(s_k) \end{aligned} \tag{21}$$

□