

Adapting TP statistic to DIC model for Aids data

In DIC model the only choice for states j, l , and m , is $j = 1, l = 2$, and $m = 3$. The only null hypothesis for testing the Markov property against general alternative is $H_0 : \alpha_{23}(t|X(s) = 1) = \alpha_{23}(t|X(s) = 2), t > s$.

$$H_0 : \alpha_{23}(t|X(s) = 1) = \alpha_{23}(t|X(s) = 2), t > s.$$

Proper and improper MI rectangles

We divide MI rectangles obtained from M observation rectangles into two subsets according to whether the time of 2 to 3 transition is finite or is right censored in state 2:

$$\{R_u \equiv (l_u, r_u] \times (w_u, z_u], z_u < \infty, 1 \leq u \leq U_1\}$$

and

$$R_u \equiv (l_u, r_u] \times (w_u, \infty], 1 \leq u \leq U_2\}.$$

We refer to an MI rectangle in the first subset as proper and in the second as improper. Similarly an observation rectangle (OR) is proper or improper depending on whether its time of 2 to 3 transition is finite or right censored in state 2

1. Reduction step

If $(l_u, r_u] \times (w_u, z_u]$ is the MI rectangle for a proper OR, we reduce it to the proper pair (r_u, z_u) , thus assuming that 1 to 2 and 2 to 3 transitions occur at the end points of their respective intervals.

If $(l_u, r_u] \times (w_u, \infty]$ is the MI rectangle for an improper OR we reduce it to the improper pair (r_u, w_u) thus assuming that 1 to 2 transition occurs at the endpoint of its interval and keep the censoring time w_u in state 2 as is.

2. Specializing to Aids data. This dataset has $M = 96$ ORs of which $M_p = 27$ are proper and $M_{ip} = 69$ are improper rectangles. There are 25 MI rectangles, of which $m_p = 17$ are proper and $m_{ip} = 8$ are improper. The 17 proper MI rectangles are listed in Table 1 below and 8 improper MI rectangles are listed in Table 4 below. Among 17 proper MI rectangles, there are $V = 11$ unique times of the $2 \rightarrow 3$ transition. We denote these times by $t_1 < t_2 < \dots < t_V$. They are listed in Table 3.

The Aids data are such that each proper *OR* rectangle contains only one of 17 proper *MI* rectangles. For each proper *OR* we have to know which proper *MI* it contains, see Table 2 below.

.In Aids data, each improper *OR* is of the form

$$(L_m, R_m] \times (23, \infty],$$

where $(L_m, R_m]$ is an interval in which $1 \rightarrow 2$ transition occurs after which a patient is right censored in state 2 by time 23, which is the end of study time. In particular if improper *OR* is of the form

$$(L_m, L_m + 1] \times (23, \infty] \quad (1)$$

it coincides with one of the improper *MI* rectangles. There are 30 improper *ORs* that satisfy this condition, see Table 5. There are 9 other improper *ORs* which contain only one improper *MI* rectangle, see Table 6. We refer to the improper *OR* which contains only one improper *MI* as type 1 improper *OR*. We refer to the remaining $69 - 39 = 30$ improper *ORs* which contain more than one.improper *MI* rectangle as type 2 improper *ORs*.

Computation of the TP statistic for all observations using $s = 8$.
The logrank statistic is

$$\begin{aligned} \hat{U}_8 &= \sum_{v=1}^{11} U_8(t_v) \\ &= \sum_{v=1}^{11} \left\{ \sum_{i=1}^{27} I(X(8) = 1) dN_i(t_v) - \left(\sum_{i=1}^{96} I(X(8) = 1) Y_i(t_v) \right) \frac{\sum_{i=1}^{27} dN_i(t_v)}{\sum_{i=1}^{96} Y_i(t_v)} \right\}, \end{aligned} \quad (2)$$

where the integral with respect to t was replaced by the sum over t'_v s. $N(t)$ is a counting process for $2 \rightarrow 3$ transition by time t , and $Y(t)$ is the number at risk for 2 to 3 transition at time t . Here $s = 8$ was chosen so that for some proper reduced observations, $X(8) = 1$, and for the remaining reduced observations, $X(8) = 2$. Otherwise, (2) would be equal to zero.

For a given t_v

$$\sum_{i=1}^{27} I(X(8) = 1) dN_i(t_v) = \sum_{i=1}^{27} I(8 \leq r_i) dN_i(t_v), \quad (3)$$

the number of subjects who were in state 1 at time 8 and made $2 \rightarrow 3$ transition at time t_v , while

$$\sum_{i=1}^{27} dN_i(t_v) \quad (4)$$

is the total number of subjects who made $2 \rightarrow 3$ transition at time t_v . Next

$$\begin{aligned} \sum_{i=1}^{96} Y_i(t_v) &= \sum_{i=1}^{27} I(r_i < t_v \leq z_i) + \sum_{i=28}^{66} I(r_i < t_v) \\ &+ \sum_{i=67}^{96} [I(R_i < t_v) + I(R_i \geq t_v) \frac{\sum_{k=1}^{K_i} I(r_{i,k} < t_v) P(MI_{i,k})}{P(OR_i)}] \end{aligned}$$

is the total number of patients who are at risk for $2 \rightarrow 3$ transition at time t_v . The first part of above expression is the contribution from 27 proper ORs to the risk set at time t_v , the second is the contribution of 39 type 1 improper ORs, and the last one is the contribution of type 2 improper ORs, where K_i is the number of improper MI rectangles contained in type 2 improper OR_i and $r_{i,k}$ is the right endpoint of the k'th improper MI rectangle belonging to improper OR_i .

Finally, the number of patients at risk of the $2 \rightarrow 3$ transition at t_v and who were in state 1 at time $s = 8$ is

$$\begin{aligned} \sum_{i=1}^{96} I(8 \leq r_i) Y_i(t_v) &= \sum_{i=1}^{27} I(8 \leq r_i < t_v \leq z_i) \\ &+ \sum_{i=28}^{66} [I(8 \leq r_i < t_v)] \\ &+ \sum_{i=67}^{96} [I(8 \leq R_i < t_v) + I(R_i > t_v) \frac{\sum_{k=1}^{K_i} I(8 \leq r_{i,k} < t_v) P(MI_{i,k})}{P(OR_i)}]. \end{aligned} \quad (5)$$

The four quantities (3)-(5) are computed using the information in Tables 1-10 in Section 4.

3 The standardized test statistic

Under H_0 , the standardized \widehat{U}_8 , namely

$$\widetilde{U}_8 = \frac{\widehat{U}_8}{\sqrt{V(\widehat{U}_8)}},$$

has asymptotically standard normal distribution, see Titman and Putter (2022), who also give the variance of \widehat{U}_8 as

$$V(\widehat{U}_8) = \sum_{v:t_v \geq 8} \frac{n_{v1}(n_v - n_{v1})}{(n_v)^2}, \quad (6)$$

where

$$n_v = \sum_{i=1}^{96} Y_i(t_v), \quad n_{v1} = \sum_{i=1}^{96} I(8 \leq r_i) Y_i(t_v).$$

4. Data

Table 1 Proper MI rectangles with their probabilities

	Proper MI	p
1	(5, 7) × (11, 12)	0.0208
2	(5, 7) × (13, 13)	0.0208
3	(1, 7) × (16, 16)	0.0104
4	(3, 7) × (17, 17)	0.0104
5	(7, 9) × (19, 19)	0.0104
6	(9, 10) × (15, 15)	0.0156
7	(10, 11) × (14, 15)	0.0156
8	(10, 11) × (15, 16)	0.0312
9	(9, 11) × (17, 18)	0.0312
10	(10, 12) × (16, 17)	0.0104
11	(9, 12) × (22, 22)	0.0104
12	(10, 12) × (23, 23)	0.0104
13	(12, 13) × (20, 20)	0.0104
14	(13, 14) × (17, 18)	0.0312
15	(13, 14) × (19, 20)	0.0104
16	(13, 14) × (20, 21)	0.0208
17	(14, 15) × (23, 23)	0.0104

Table 2 Proper OR(first column), with its proper MI rectangle(2nd column) and in reduced form (third column) There are 27 proper ORs listed below.

Proper OR	Proper MI	Reduced (r,z)
1(5, 7) × (11, 12)	(5, 7) × (11, 12)	(7, 12)
2(1, 10) × (11, 12)	(5, 7) × (11, 12)	(7, 12)
3(5, 8) × (13, 13)	(5, 7) × (13, 13)	(7, 13)
4(1, 7) × (12, 13)	(5, 7) × (13, 13)	(7, 13)
5(9, 13) × (14, 15)	(10, 11) × (14, 15)	(11, 15)
6(8, 10) × (15, 15)	(9, 10) × (15, 15)	(10, 15)
7(10, 11) × (14, 15)	(10, 11) × (14, 15)	(11, 15)
8(10, 12) × (15, 16)	(10, 11) × (15, 16)	(11, 16)
9(10, 14) × (15, 16)	(10, 11) × (15, 16)	(11, 16)
10(1, 7) × (16, 16)	(1, 7) × (16, 16)	(7, 16)
11(10, 11) × (15, 16)	(10, 11) × (15, 16)	(11, 16)
12(10, 12) × (16, 17)	(10, 12) × (16, 17)	(12, 17)
13(3, 7) × (17, 17)	(3, 7) × (17, 17)	(7, 17)
14(9, 13) × (17, 18)	(9, 11) × (17, 18)	(11, 18)
15(13, 14) × (17, 18)	(13, 14) × (17, 18)	(14, 18)
16(13, 15) × (17, 18)	(13, 14) × (17, 18)	(14, 18)
17(9, 12) × (17, 18)	(9, 11) × (17, 18)	(11, 18)
18(13, 15) × (17, 18)	(13, 14) × (17, 18)	(14, 18)
19(9, 11) × (17, 18)	(9, 11) × (17, 18)	(11, 18)
20(7, 9) × (19, 19)	(7, 9) × (19, 19)	(9, 19)
21(12, 13) × (20, 20)	(12, 13) × (20, 20)	(13, 20)
22(13, 14) × (19, 20)	(13, 14) × (19, 20)	(14, 20)
23(1, 15) × (20, 21)	(13, 14) × (20, 21)	(14, 21)
24 (13, 14) × (20, 21)	(13, 14) × (20, 21)	(14, 21)
25(9, 12) × (22, 22)	(9, 12) × (22, 22)	(12, 22)
26(10, 12) × (23, 23)	(10, 12) × (23, 23)	(12, 23)
27(14, 15) × (23, 23)	(14, 15) × (23, 23)	(15, 23)

Table 3 (For proper ORs) First column lists 11 unique times of 2 → 3 transition, second the number of patients that made the 2 → 3 transition at t_v and were in state 1 at $s = 8$, third, the number of patients who are at risk for 2 → 3 at t_v , the last one gives the number of patients at risk at t_v who were in state 1 at $s=8$.

t_v	$\sum_{i=1}^{27} dN_i(t_v)$	$\sum_{i=1}^{27} I(8 \leq r_i) dN_i(t_v)$	$\sum_{i=1}^{27} Y_i(t_v)$	$\sum_{i=1}^{27} I(8 \leq r_i) Y_i(t_v)$
12	2	0	16	10
13	2	0	17	13
15	3	3	22	20
16	4	3	20	18
17	2	1	16	15
18	6	6	14	14
19	1	1	8	8
20	2	2	7	7
21	2	2	5	5
22	1	1	3	3
23	2	2	2	2

Note that multiplicities (entries in second column) add up to 27, which is the number of proper ORs. The entries in Table 3 are obtained from Table 2 in the following way.

$$\begin{aligned} \sum_{i=1}^{27} dN_i(t_v) &= \sum_{i=1}^{27} I(z_i = t_v), \\ \sum_{i=1}^{27} I(8 \leq r_i) dN_i(t_v) &= \sum_{i=1}^{27} I(8 \leq r_i < z_i = t_v), \\ \sum_{i=1}^{27} Y_i(t_v) &= \sum_{i=1}^{27} I(z_i \geq t_v), \\ \sum_{i=1}^{27} I(8 \leq r_i) Y_i(t_v) &= \sum_{i=1}^{27} I(8 \leq r_i, z_i \geq t_v). \end{aligned}$$

Table 4. Improper MI rectangles with their probabilities and in reduced form

	Improper MI rectangle (l,r)	probability	reduced (r, 23)
1	(5, 7] × (23, ∞)	0.0437	(7, 23)
2	(8, 9] × (23, ∞)	0.0235	(9, 23)
3	(9, 10] × (23, ∞)	0.0639	(10, 23)
4	(10, 11] × (23, ∞)	0.1649	(11, 23)
5	(11, 12] × (23, ∞)	0.0379	(12, 23)
6	(12, 13] × (23, ∞)	0.1682	(13, 23)
7	(14, 15] × (23, ∞)	0.1548	(15, 23)
8	(15, 16] × (23, ∞)	0.0619	(16, 23)

Table 5. Type 1 improper ORs which are of the form $OR_m = (L_m, L_m + 1] \times (23, \infty)$ with their probabilities and in reduced form.

Improper OR=Improper MI	multiplicity	reduced (r, 23)	p
(9, 10) × (23, ∞)	2	(10, 23)	0.0639
(10, 11) × (23, ∞)	7	(11, 23)	0.1649
(11, 12) × (23, ∞)	1	(12, 23)	0.0379
(12, 13) × (23, ∞)	7	(13, 23)	0.1682
(14, 15) × (23, ∞)	8	(15, 23)	0.1548
(15, 16) × (23, ∞)	5	(16, 23)	0.0619
total	30		

In "reduced" column, the entry (10, 23) means that a patient made $1 \rightarrow 2$ transition at $r = 10$ and stayed in state 2 until the end of the study denoted by 23

Table 6. Type 1 improper ORs which are not of the form $OR_m = (L_m, L_m + 1] \times (23, \infty]$ with their probabilities and in reduced form.

Improper OR	Improper MI	multiplicity	reduced $(r, 23)$
$(1, 7) \times (23, \infty)$	$(5, 7) \times (23, \infty)$	1	$(7, 23)$
$(5, 7) \times (23, \infty)$	$(5, 7) \times (23, \infty)$	2	$(7, 23)$
$(7, 9) \times (23, \infty)$	$(8, 9) \times (23, \infty)$	1	$(9, 23)$
$(12, 14) \times (23, \infty)$	$(12, 13) \times (23, \infty)$	2	$(13, 23)$
$(13, 15) \times (23, \infty)$	$(14, 15) \times (23, \infty)$	3	$(15, 23)$
total		9	

We now compute the test statistic based on 66 ORs which contain only one MI rectangle assuming that $s = 8$. This statistic is

$$\widehat{U}_8 = \sum_{v=1}^{11} U_8(t_v),$$

where

$$U_8(t_v) = \sum_{i=1}^{27} I(8 \leq r_i) dN_i(t_v) - [\sum_{i=1}^{66} I(8 \leq r_i) Y_i(t_v)] \frac{\sum_{i=1}^{27} dN_i(t_v)}{\sum_{i=1}^{66} Y_i(t_v)}.$$

and is to be computed based on **Table 6a** below.

Table 6a The first two columns come from Table 3. Columns 3 and 4 use information from the same columns in Table 3 (first number). The second number is obtained by inspection of Tables 5 and 6.

t_v	$\sum_{i=1}^{27} dN_i(t_v)$	$\sum_{i=1}^{27} I(8 \leq r_i) dN_i(t_v)$	$n_v = \sum_{i=1}^{66} Y_i(t_v)$	$n_{v1} = \sum_{i=1}^{66} I(8 \leq r_i) Y_i(t_v)$	$U_8(t_v)$
12	2	0	$16 + 13 = 29$	$10 + 10 = 20$	-1.37931
13	2	0	$17 + 14 = 31$	$13 + 11 = 24$	-1.54839
15	3	3	$22 + 23 = 45$	$20 + 20 = 40$	0.33333
16	4	3	$20 + 34 = 54$	$18 + 31 = 49$	-0.62963
17	2	1	$16 + 39 = 55$	$15 + 36 = 51$	-0.85454
18	6	6	$14 + 39 = 53$	$14 + 36 = 50$	0.33962
19	1	1	$8 + 39 = 47$	$8 + 36 = 44$	0.06383
20	2	2	$7 + 39 = 46$	$7 + 36 = 43$	0.13043
21	2	2	$5 + 39 = 44$	$5 + 36 = 41$	0.13636
22	1	1	$3 + 39 = 42$	$3 + 36 = 39$	0.07143
23	2	2	$2 + 39 = 41$	$2 + 36 = 38$	0.14634

Adding up the entries in the last column gives $\widehat{U}_8 = -3.19052$

To compute the variance of the test statistic \widehat{U}_8 we use **Table 6**.

Table 6b The second and third columns are defined in **Table 6a**. The last column is obtained from expression for variance of the test statistic \widehat{U}_8 given in section 3 by (6).

t_v	n_V	n_{v1}	$n_{v1}(n_V - n_{v1})/n_V^2$
12	29	20	0.214031
13	31	24	0.174818
15	45	40	0.0987654
16	54	49	0.0840192
17	55	51	0.0674380
18	53	50	0.0533998
19	47	44	0.0597555
20	46	43	0.0609641
21	44	41	0.0635331
22	42	39	0.0663265
23	41	38	0.0678167

Adding up the entries in the last column gives $V(\widehat{U}_8) = 1.01087$. Then the standardized test statistic is

$$\frac{\widehat{U}_8}{\sqrt{V(\widehat{U}_8)}} = \frac{-3.19052}{\sqrt{1.01087}} = -3.1733,$$

which leads to the rejection of null hypothesis.

We next include type 2 improper ORs for the computation of the test statistic (there are such 30 such rectangles and they contain more than one improper MI rectangle). These ORs are listed in Table 7

Table 7 The 30 type 2 improper ORs together with their multiplicity and probabilities. The probabilities are obtained from Table 4 by adding the probabilities of improper MI rectangles that belong to a given improper OR as shown in the table. There are 18 different improper ORs in the table. The

multiplicity of each OR is shown in column 2.

Imp OR (L,R]	multiplicity	Total probability of improper OR of type 2
(1, 11]	1	$0.0437 + 0.0235 + 0.0639 + 0.1649 = 0.296$
(1, 12]	2	$0.0437 + 0.0235 + 0.0639 + 0.1649 + 0.0379 = 0.3339$
(1, 13)	2	$0.3339 + 0.1682 = 0.5021$
(1, 14)	3	0.4529
(1, 15)	2	$0.4529 + 0.1548 = 0.6077$
(1, 16)	1	$0.6077 + 0.0619 = 0.6696$
(3, 14)	1	0.4529
(3, 15)	1	0.6077
(7, 10)	1	$0.0235 + 0.0639 = 0.0874$
(7, 15)	1	$0.0874 + 0.1649 + 0.0379 + 0.1682 + 0.1548 = 0.6132$
(8, 10)	1	$0.0235 + 0.0639 = 0.0874$
(8, 15)	1	$0.6077 - 0.0437 = 0.564$
(14, 16)	2	$0.1548 + 0.0619 = 0.2167$
(9, 12)	3	$0.0639 + 0.1649 + 0.0379 = 0.2667$
(10, 12)	2	$0.1463 + 0.033 = 0.1766$
(10, 15)	1	$0.1649 + 0.0379 + 0.1682 + 0.1548 = 0.4258$
(11, 13)	4	$0.0379 + 0.1682 = 0.2061$
(13, 16)	1	$0.1548 + 0.0619 = 0.2167$

. The contribution of the observations in Table 7 to the computation of the test statistic. The observation may contribute "1" to the risk at time t_v .(column 3 in Table 6a) and if it does it may also contribute to column 4 in Table 6a These contributions are obtained through imputation.

Consider the first OR (OR1) in Table 7.namely (1,11]. Going back to Table 4, we see that

$OR1 = (1, 11] = (5, 7] \cup (8, 9] \cup (9, 10] \cup (10, 11]$ and numbering these MI rectangles by 1-4, we list them with their probabilities and conditional probabilities:

$$\begin{array}{lllll} 1 = (5, 7] & 2 = (8, 9] & 2 = (8, 9] & 4 = (10, 11] \\ 0.0437 & 0.0235 & 0.0639 & 0.1649 & 0.296 \\ 0.147635 & 0.079392 & 0.215878 & 0.557095 & 1 \end{array}$$

We impute an MI rectangle by drawing from the distribution with values in the first row and the probabilities in the third row. The contributions of the imputed MI rectangle to the entries in columns n_v and n_{v1} in Table 6a are stated in table I1

Table I1

draw\(<t_v></t_v>	12	13	15	16	17	18	19	20	21	22	23
if $1 \leq i \leq 4, Y_1(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 1, I(8 \leq 7)Y_1(t_v)$	0	0	0	0	0	0	0	0	0	0	0
if $2 \leq i \leq 4, I(8 \leq r_i)Y_1(t_v)$	1	1	1	1	1	1	1	1	1	1	1

$OR2 = (1, 12] = (5, 7] \cup (8, 9] \cup (9, 10] \cup (10, 11] \cup (11, 12]$. The unconditional probabilities of MI rectangles are in the second raw and their conditional probabilities in the third row

$$\begin{array}{llllll} 1 & 2 & 3 & 4 & 5 \\ 0.0437 & 0.0235 & 0.0639 & 0.1649 & 0.0379 & 0.3339 \\ 0.130878 & 0.0703804 & 0.191375 & 0.493860 & 0.113507 & 1 \end{array}$$

There are two OR2s so you have to draw twice from the distribution which has values in the first row and probabilities in the third row. Table I2 below shows how a draw increases or does not increase the entries in columns n_v and n_{v1} in Table 6a.

Table I2

draw\(<t_v></t_v>	12	13	15	16	17	18	19	20	21	22	23
if $1 \leq i \leq 4, Y_2(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, Y_2(t_v)$	0	1	1	1	1	1	1	1	1	1	1
:											
if $i = 1, I(8 \leq r_1)Y_2(t_v)$	0	0	0	0	0	0	0	0	0	0	0
if $2 \leq i \leq 4, I(8 \leq r_i)Y_2(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, I(8 \leq r_5)Y_2(t_v)$	0	1	1	1	1	1	1	1	1	1	1

OR3=(1,13] = (5, 7] \cup (8, 9] \cup (9, 10] \cup (10, 11] \cup (11, 12] \cup (12, 13]

1	2	3	4	5	6	
0.0437	0.0235	0.0639	0.1649	0.0379	0.1682	0.5021
0.0870345	0.0468034	0.127265	0.328421	0.0754830	0.334993	1

There are two OR3s so you have to draw twice from the distribution with values in the first row and probabilities in the third row

Table I3

draw \ t_v	12	13	15	16	17	18	19	20	21	22	23
if $1 \leq i \leq 4, Y_3(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, Y_3(t_v)$	0	1	1	1	1	1	1	1	1	1	1
if $i = 6, Y_3(t_v)$	0	0	1	1	1	1	1	1	1	1	1
if $i = 1, I(8 \leq r_1)Y_3(t_v)$	0	0	0	0	0	0	0	0	0	0	0
if $2 \leq i \leq 4, I(8 \leq r_i)Y_3(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, I(8 \leq r_i)Y_3(t_v)$	0	1	1	1	1	1	1	1	1	1	1
if $i = 6, I(8 \leq r_i)Y_3(t_v)$	0	0	1	1	1	1	1	1	1	1	1

OR4=(3,14]=[5, 7] \cup (8, 9] \cup (9, 10] \cup (10, 11] \cup (11, 12] \cup (12, 13]

Table I4

draw \ t_v	12	13	15	16	17	18	19	20	21	22	23
if $1 \leq i \leq 4, Y_7(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, Y_7(t_v)$	0	1	1	1	1	1	1	1	1	1	1
if $i = 6, Y_7(t_v)$	0	0	1	1	1	1	1	1	1	1	1
if $i = 1, I(8 \leq r_1)Y_7(t_v)$	0	0	0	0	0	0	0	0	0	0	0
if $2 \leq i \leq 4, I(8 \leq r_i)Y_7(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, I(8 \leq r_i)Y_7(t_v)$	0	1	1	1	1	1	1	1	1	1	1
if $i = 6, I(8 \leq r_i)Y_7(t_v)$	0	0	1	1	1	1	1	1	1	1	1

OR5=(3,15]=[5, 7] \cup (8, 9] \cup (9, 10] \cup (10, 11] \cup (11, 12] \cup (12, 13] \cup (14, 15]

Table I5

draw\(t_v	12	13	15	16	17	18	19	20	21	22	23
if $1 \leq i \leq 4, Y_5(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, Y_5(t_v)$	0	1	1	1	1	1	1	1	1	1	1
if $i = 6, Y_5(t_v)$	0	0	1	1	1	1	1	1	1	1	1
if $i = 1, I(8 \leq r_1)Y_5(t_v)$	0	0	0	0	0	0	0	0	0	0	0
if $2 \leq i \leq 4, I(8 \leq r_i)Y_5(t_v)$	1	1	1	1	1	1	1	1	1	1	1
if $i = 5, I(8 \leq r_i)Y_5(t_v)$	0	1	1	1	1	1	1	1	1	1	1
if $i = 6, I(8 \leq r_i)Y_5(t_v)$	0	0	1	1	1	1	1	1	1	1	1
if $i = 7, I(8 \leq r_i)Y_5(t_v)$	0	0	0	1	1	1	1	1	1	1	1

There is a row missing in Table I5, The missing row should read

$$\text{if } i = 7, Y_5(t_v) \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$$

$$\text{OR6}=(1,16]$$

$$\text{OR7}=(3,14]$$

$$\text{OR8}=(3,15]$$

$$\text{OR9}=(7, 10) = (8, 9] \cup (9, 10)$$

draw\(t_v	12	13	15	16	17	18	19	20	21	22	23
$i = 1 \text{ or } 2, Y_9(t_v)$	1	1	1	1	1	1	1	1	1	1	1
$i = 1 \text{ or } 2, I(8 \leq r_i)Y_9(t_v)$	1	1	1	1	1	1	1	1	1	1	1

$$\text{OR10}=(7, 15) = (8, 9] \cup (9, 10] \cup (10, 11] \cup (11, 12] \cup (12, 13] \cup (14, 15]$$

draw\(t_v	12	13	15	16	17	18	19	20	21	22	23
$1 \leq i \leq 3, Y_{10}(t_v)$	1	1	1	1	1	1	1	1	1	1	1
$i = 4, Y_{10}(t_v)$	0	1	1	1	1	1	1	1	1	1	1
$i = 5, Y_{10}(t_v)$	0	0	1	1	1	1	1	1	1	1	1
$i = 6, Y_{10}(t_v)$	0	0	0	1	1	1	1	1	1	1	1

There is exactly the same table for $I(8 \leq r_i)Y_{10}(t_v)$.