Lab 5: Rejection Sampling

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Agenda

We can often end up with posterior distributions that we only know up to a normalizing constant. For example, in practice, we may derive

$$p(\theta \mid x) \propto p(x \mid \theta)p(\theta)$$

and find that the normalizing constant $p(x)$ is very difficult to evaluate. Such examples occur when we start building non-conjugate models in Bayesian statistics.

Given such a posterior, how can we appropriate it's density? One way is using rejection sampling. As an example, let's suppose our resulting posterior distribution is

$$f(x) \propto sin^2(\pi x), x \in [0, 1].$$

In order to understand how to approximate the density (normalized) of f, we will investigate the following tasks:

1. Plot the densities of $f(x)$ and the Unif(0,1) on the same plot. According to the rejection sampling approach sample from $f(x)$ using the Unif(0,1) pdf as an enveloping function.
2. Plot a histogram of the points that fall in the acceptance region. Do this for a simulation size of 10^2 and 10^5 and report your acceptance ratio. Compare the ratios and histograms.
3. Repeat Tasks 1 - 3 for Beta(2,2) as an enveloping function.
4. Provide the four histograms from Tasks 2 and 3 using the Uniform(0,1) and the Beta(2,2) enveloping proposals. Provide the acceptance ratios. Provide commentary.
5. Do you recommend the Uniform or the Beta(2,2) as a better enveloping function (or are they about the same)? If you were to try and find an enveloping function that had a high acceptance ratio, which one would you try and why?

Task 1

Plot the densities of $f(x)$ and the Unif(0,1) on the same plot.

Let's first create a sequence of points from 0 to 1, so that we can have a grid of points for plotting both of the proposed functions.

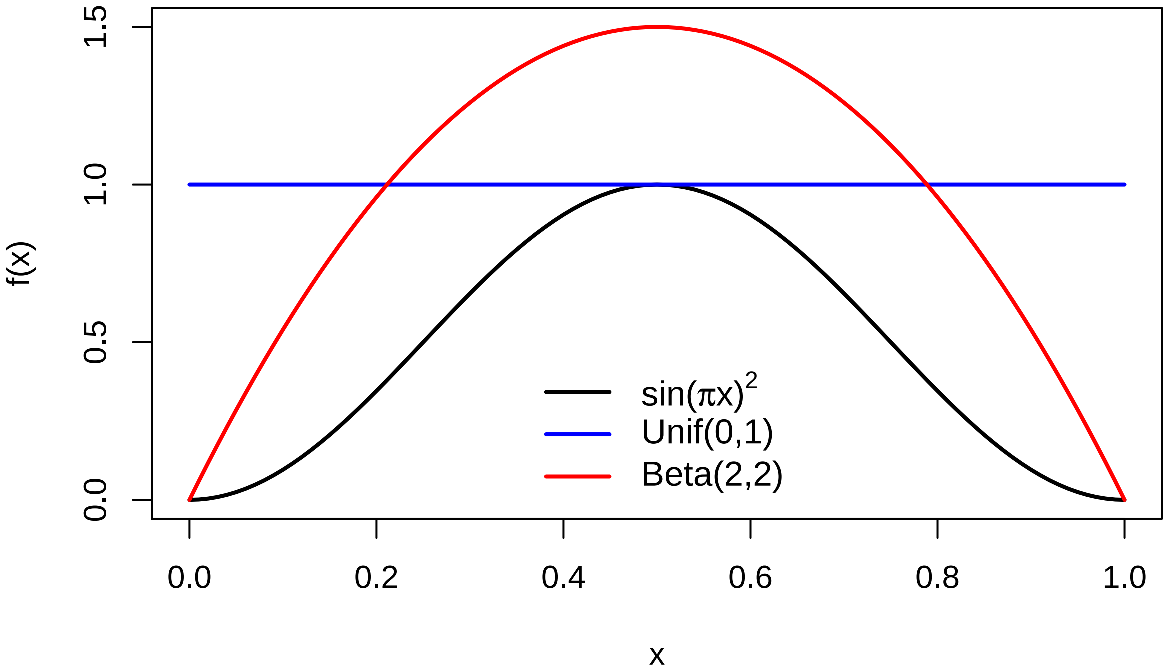


Figure 1: Comparision of the target function and the Unif(0,1) and the Beta(2,2) densities on the same plot.

Tasks 2 – 4

According to the rejection sampling approach sample from f(x) using the Unif(0,1) pdf as an enveloping function. In order to do this, we write a general rejection sampling function that also allows us to plot the historams for any simulation size. Finally, our function also allows us to look at task 4 quite easily.

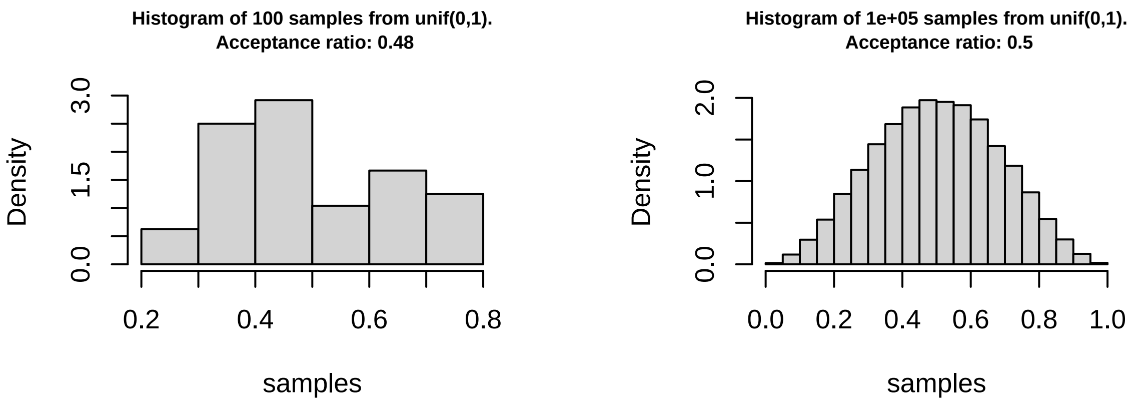
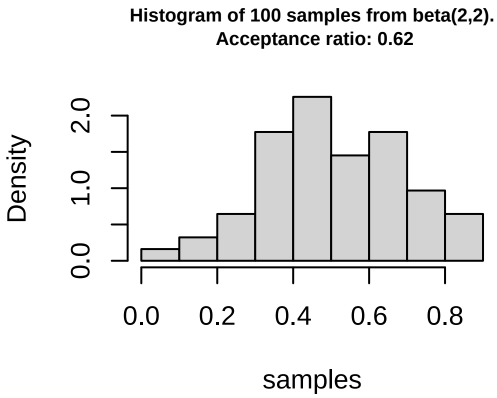


Figure 2: Comparision of the output of the rejection sampling for 100 versus 100,000 simulations with the uniform distribution as the envelope function.



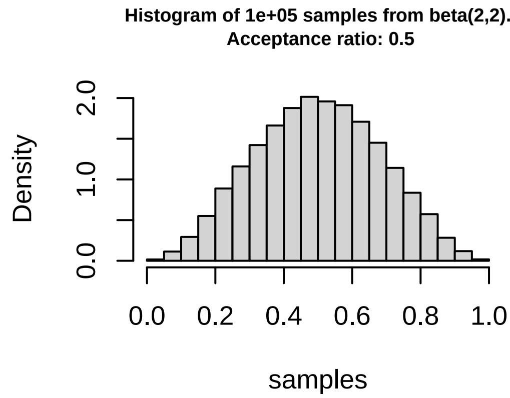


Figure 3: Comparision of the output of the rejection sampling for 100 versus 100,000 simulations with the beta distribution as the envelope function.

Results: The acceptance ratio for sampling 100 samples from a beta(2,2) function is 0.62. The acceptance ratio for sampling 10ˆ5 samples from a beta(2,2) function is 0.5. The acceptance ratio for sampling 100 samples from a uniform(0,1) function is 0.48. The acceptance ratio for sampling 10ˆ5 samples from a uniform(0,1) function is 0.5.

The histogram of sampling 100 samples from a beta(2,2) seems very roughly in the shape of a normal distribution. It seems that sampling more, in this case sampling 10ˆ5 compared to 100, will cause the histogram of samples from a beta(2,2) to roughly approach a normal distribution.

The histogram of sampling 100 samples from a uniform(0,1) seems very roughly skewed right, or a bimodal distribution. It seems that sampling more, in this case sampling 10ˆ5 compared to 100, will cause the histogram of samples from a uniform(0,1) to roughly approach a normal distribution.

Task 5

Comparing the acceptance ratios of 100 samples of the beta(2,2) and uniform(0,1) initally might indicated that the beta(2,2) function is better, since the acceptance ratio of 0.62 is greater than 0.48, which means that overall more samples got accepted. However, when expanding to 10ˆ5 samples and looking at the bigger picture in the long run, the acceptance ratios of the beta(2,2) an the uniform(0,1) function become the same at 0.5 each, which suggests that the beta(2,2) function is no better than a uniform(0,1) function in rejection sampling.

Overall, there seems to be no statisitically signifigant difference between the beta(2,2) and the uniform(0,1) as an enveloping functions, so I would not be able to recommend one over the other since they are both roughly the same when looking at the acceptance ratio over a large quantity such as 10ˆ5.(both have acceptance ratios of approximately 0.5)

If I were trying to find an enveloping function that had an high acceptance ratio, I would personally try the Gaussian Distribution(Normal Distribution). The shape of

$$f(x) = sin^2(\pi x)$$

is approximately a bellcurve shape, and it has a maximum as well as inflection points which indicate concavity. This is really similar to the Gaussian distribution, so I would try that.