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1. (a)

$\hat{Y} = w^T X \qquad \hat{Y} = R_{YX} R_X^{-1} X$

$\therefore w^* = R_{YX} R_X^{-1}$

$K = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad R_{YX} = 2$ (cross-covariance)

$R_X = 5 \qquad R_X^{-1} = \frac{1}{5}$

$w^* = 2 \cdot \frac{1}{5} = \boxed{\frac{2}{5}}$

(b)

$MMSE = E[(Y - \hat{Y})^2] = R_Y - R_{YX} \cdot R_X^{-1} \cdot R_{XY}$

$= 4 - 2^2 \cdot \frac{1}{5} = 4 - \frac{4}{5} = \boxed{\frac{16}{5}}$

(c)

$\hat{Y} = w^* X \rightarrow \hat{Y} = E[Y|X]$

$= \mu_Y + R_{YX} \cdot R_X^{-1} (X - \mu_X) \qquad$ let $\mu_X, \mu_Y = 0$

$= R_{YX} \cdot R_X^{-1} \cdot X$

$= w^* X \qquad \therefore$ QED

(d)

Since there are same covariance matrix K, $\vec{w}$ will remain as $\frac{2}{5}$ even if X and Y are not jointly gaussian. Although non-jointly Gaussian case have same w with jointly gaussian case, the linear MMSE estimator may not be optimal for non-jointly Gaussian case. Therefore, MMSE might be different from the jointly Gaussian case.

2-

(a) $K = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ $\det(K - \lambda I) = 0$

$(4-\lambda)(14 - 8\lambda + \lambda^2)$

$56 - 32\lambda + 4\lambda^2 - 14\lambda + 8\lambda^2 - \lambda^3$

$\det\begin{pmatrix} 4-\lambda & -1 & 2 \\ -1 & 5-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} = 0$

$\Rightarrow (4-\lambda)\left[(5-\lambda)(3-\lambda) - 1\right] - (-1)\left[(-1)(3-\lambda) - (-2)\right] + 2\left[1 - 2(5-\lambda)\right]$

$(4-\lambda)\left[(15 - 8\lambda + \lambda^2) - 1\right] + (-3 + \lambda + 2) + 2(1 - 10 + 2\lambda) = 0$

$\Rightarrow 56 - 32\lambda + 4\lambda^2 - 14\lambda + 8\lambda^2 - \lambda^3 + \lambda - 1 - 18 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 12\lambda^2 - 41\lambda + 37 = 0$

$\lambda_1 \approx 1.43 \quad \lambda_2 \approx 3.86 \quad \lambda_3 \approx 6.71$

$K - \lambda_1 I = \begin{pmatrix} 4-1.43 & -1 & 2 \\ -1 & 5-1.43 & -1 \\ 2 & -1 & 3-1.43 \end{pmatrix} = \begin{pmatrix} 2.57 & -1 & 2 \\ -1 & 3.57 & -1 \\ 2 & -1 & 1.57 \end{pmatrix}$

$\begin{pmatrix} 2.57 & -1 & 2 \\ -1 & 3.57 & -1 \\ 2 & -1 & 1.57 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$

$e_1 = \begin{bmatrix} -0.59 \\ 0.63 \\ -0.49 \end{bmatrix}$

$K - \lambda_2 I = \begin{pmatrix} 0.14 & -1 & 2 \\ -1 & 1.14 & -1 \\ 2 & -1 & -0.81 \end{pmatrix}$

$e_2 = \begin{bmatrix} 0.59 \\ -0.05 \\ -0.99 \end{bmatrix}$

$K - \lambda_3 I = \begin{pmatrix} -2.71 & -1 & 2 \\ -1 & -1.71 & -1 \\ 2 & -1 & -3.71 \end{pmatrix}$

$e_3 = \begin{bmatrix} 0.53 \\ 0.77 \\ 0.35 \end{bmatrix}$

2.(b)

$$K = \sum_{k=1}^{3} \lambda_k e_k e_k^T$$

$$= \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T + \lambda_3 e_3 e_3^T$$

$$= 1.43 \begin{bmatrix} -0.59 \\ 0.63 \\ -0.49 \end{bmatrix} \begin{bmatrix} -0.59 & 0.63 & -0.49 \end{bmatrix}$$

$$+ 3.86 \begin{bmatrix} 0.59 \\ -0.05 \\ -0.79 \end{bmatrix} \begin{bmatrix} 0.59 & -0.05 & -0.79 \end{bmatrix}$$

$$+ 6.71 \begin{bmatrix} 0.53 \\ 0.77 \\ 0.35 \end{bmatrix} \begin{bmatrix} 0.63 & 0.77 & 0.35 \end{bmatrix}$$

$$= \begin{bmatrix} 3.9 & -0.9 & 2.09 \\ -0.9 & 4.9 & -1 \\ -2.1 & -1.1 & 3.2 \end{bmatrix} \approx K$$

2.(c)

$$X = \sum_{k=1}^{3} Z_k e_k$$

$$= Z_1 e_1 + Z_2 e_2 + Z_3 e_3$$

$$= Z_1 \begin{bmatrix} -0.59 \\ 0.63 \\ -0.49 \end{bmatrix} + Z_2 \begin{bmatrix} 0.59 \\ -0.05 \\ -0.79 \end{bmatrix} + Z_3 \begin{bmatrix} 0.53 \\ 0.77 \\ 0.35 \end{bmatrix}$$

Where $Var(Z_1) = 1.43$, $Var(Z_2) = 3.86$, $Var(Z_3) = 6.71$

2(d)

$$\tilde{x} = z_2 e_2 + z_3 e_3$$

$$= \sqrt{\lambda_2} e_2 + \sqrt{\lambda_3} e_3$$

$$= \sqrt{3.86} \begin{bmatrix} 0.53 \\ 0.77 \\ 0.35 \end{bmatrix} + \sqrt{6.71} \begin{bmatrix} -0.59 \\ 0.63 \\ -0.49 \end{bmatrix}$$

2(e)

$$\tilde{x} = z_2 e_2 + z_3 e_3$$

$\therefore$ MSE $E[\|x - \tilde{x}\|^2] = \lambda_1 =$ **1.43**

4. (a)

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Wavenumbers estimates for the eight largest spectral peak:
Wavenumber: 750.43, Intensity: 15275.06
Wavenumber: 1250.89, Intensity: 5203.31
Wavenumber: 2975.92, Intensity: 4767.23
Wavenumber: 1031.96, Intensity: 3351.59
Wavenumber: 1427.29, Intensity: 3219.41
Wavenumber: 3184.21, Intensity: 1359.35
Wavenumber: 579.54, Intensity: 1014.52
Wavenumber: 877.21, Intensity: 668.68
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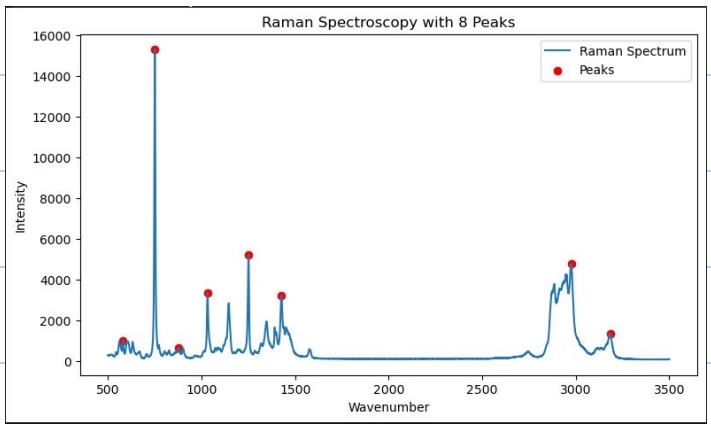
These 8 peaks correspond to the most prominent features in the Raman spectrum.

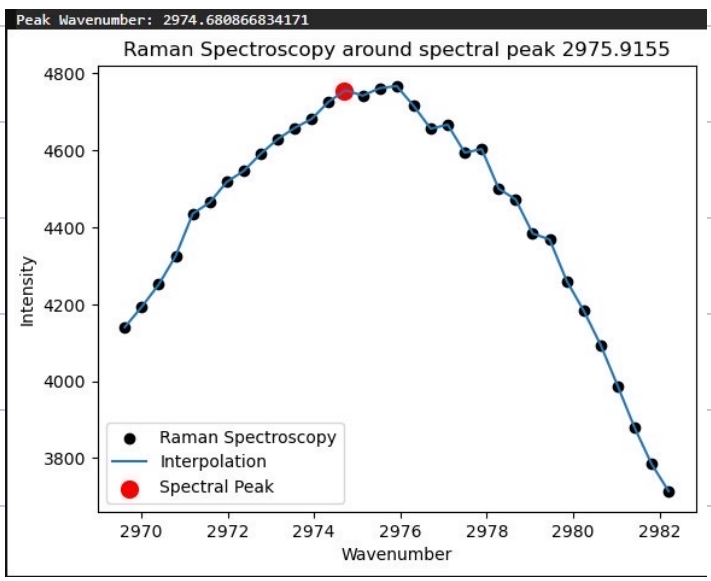
8 largest spectral peaks from largest to smallest.

4. (b)

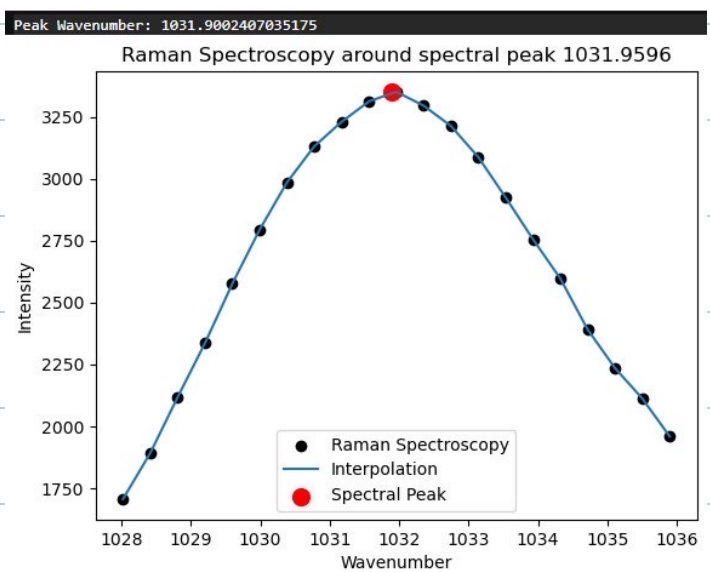


This image shows a plot of the Raman spectrum with the eight peaks marked. X axis is wavenumber and Y axis is Intensity, the peaks highlighted in red.

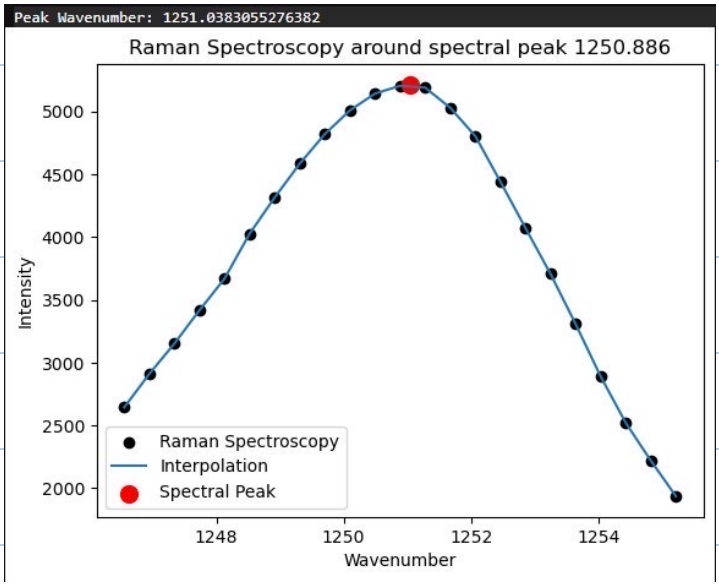
4. (c)



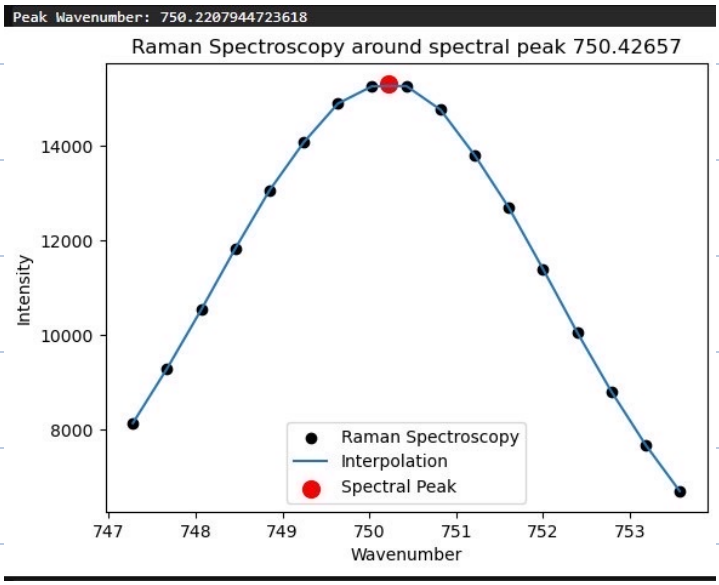
A peak with moderate intensity, indicating a significant molecular vibration.



A peak with moderate intensity, indicating a different vibrational mode.



Another significant peak, though less than the right, indicating a prominent vibrational mode.



The most intense peak, showing a very strong and well-defined vibration mode.