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1.

(a)

$$P_{x_j|w_i}(x_j|w_i) = \frac{1}{\Gamma(p_i)} \lambda_j^{p_i} x_j^{p_i-1} e^{-\lambda_j x_j}, \quad p_i, \lambda_j > 0$$

Bayes' rules : $P(w_i|x) = \dfrac{P(x|w_i)P(w_i)}{P(x)}$

$$P(w_i | x_1 = x_1, \cdots, x_k = x_k) = \frac{P(x_1 = x_1 \cdots x_k = x_k | w_i)\, P(w_i)}{P(x_1, \cdots, x_k)}$$

$$\propto p(x_1|w_i, \cdots, x_k|w_i)\, p(w_i)$$

$$= \prod_{j=1}^{k} \frac{1}{\Gamma(p_i)} \lambda_j^{p_i} x_j^{p_i-1} e^{-\lambda_j x_j} \cdot p(w_i)$$

$$= \frac{p(w_i)}{\Gamma(p_i)} \prod_{j=1}^{k} \lambda_j^{p_i} x_j^{p_i-1} e^{-\lambda_j x_j}$$

decide w_1 if

$$\frac{p(w_1)}{\Gamma(p_1)} \prod_{j=1}^{k} \lambda_j^{p_1} x_j^{p_1-1} \cancel{e^{-\lambda_j x_j}} > \frac{p(w_2)}{\Gamma(p_2)} \prod_{j=1}^{k} \lambda_j^{p_2} x_j^{p_2-1} \cancel{e^{-\lambda_j x_j}}$$

$$\Rightarrow \frac{p(w_1)}{\Gamma(p_1)} \prod_{j=1}^{k} \lambda_j^{p_1} x_j^{p_1-1} > \frac{p(w_2)}{\Gamma(p_2)} \prod_{j=1}^{k} \lambda_j^{p_2} x_j^{p_2-1}$$

otherwise, decide w_2 ■

(b)

when $|p_2 - p_1| = 1$, decision boundaries is linear. ■

(c)

$p_1 = 4$, $p_2 = 2$, $c = 2$, $k = 4$, $\lambda_1 = \lambda_3 = 1$. $\lambda_2 = \lambda_4 = 2$

$$\frac{P(\cancel{x}\omega_1)}{\gamma(p_1)} \prod_{j=1}^{4} \lambda_j^{p_1} x_j^{p_1-1} > \frac{P(\cancel{x}\omega_2)}{\gamma(p_2)} \prod_{j=1}^{4} \lambda_j^{p_2} x_j^{p_2-1}$$

$$\Rightarrow \frac{1}{3\times 2\times 1} \cdot \prod_{j=1}^{4} \lambda_j^{4} x_j^{3} > \frac{1}{1} \prod_{j=1}^{4} \lambda_j^{2} x_j$$

$$\Leftrightarrow \frac{1}{6} (1^4 \cdot 0.1^{\cancel{3}\,2})(2^4 \cdot 0.2^{\cancel{3}\,2})(1^4 \cdot 0.3^{\cancel{3}\,2})(\cancel{2}^4 \cdot 4^{\cancel{3}\,2}) > (1^2 \cdot \cancel{0.1})(\cancel{2}^2 \cdot \cancel{0.2})(1^2 \cdot \cancel{0.3})(\cancel{2}^2 \cdot \cancel{4})$$

$$\Rightarrow \frac{1}{6}(0.1^2)(2^4 \cdot 0.2^2)(0.3^2) \cdot 4^2 > 1$$

$$\Rightarrow \frac{16}{6}(0.1\times 0.2\times 0.3\times 4)^2 > 1$$

$$\Rightarrow 0.0015 < 1$$

$\Rightarrow$ W_2 is the decision $C(x) = 2$

(d)

$p_1 = 3.2$, $p_2 = 8$, $C = 2$, $K = 1$, $\lambda_1 = 1$

$$\frac{P(\cancel{x}\omega_1)}{\gamma(p_1)} \prod_{j=1}^{k} \lambda_j^{p_1} x_j^{p_1-1} > \frac{P(\cancel{x}\omega_2)}{\gamma(p_2)} \prod_{j=1}^{k} \lambda_j^{p_2} x_j^{p_2-1}$$

$$\Rightarrow \frac{1}{\gamma(3.2)} \cancel{1^{3.2}} x^{2.2} > \frac{1}{\gamma(8)} \cancel{1^{8}} \cdot x^{7}$$

$$\Rightarrow \frac{\gamma(8)}{\gamma(3.2)} = x^{4.8}$$

$$\Rightarrow \frac{5040}{2.423} = x^{4.8}$$

$$X = \sqrt[4.8]{\frac{5040}{2.423}} \doteqdot 4.912$$

$$x^* = 4.912$$ ▨

Type 1 error (False Positive Rate)

$$T_1 = P(x > x^* \mid w_1)$$

$$= \int_{x^*}^{\infty} p(x \mid w_1)\, dx$$

$$\approx 0.159$$ ▨

Type 2 error (False Negative Rate)

$$T_2 = P(x \leq x^* \mid w_2)$$

$$= \int_0^{x^*} p(x \mid w_2)\, dx$$

$$\approx 0.124$$ ▨

(e)

$P_1 = P_2 = 4$, $C = 2$, $k = 2$, $\lambda_1 = 8$, $\lambda_2 = 0.3$ $P(w_1) = \frac{1}{4}$ $P(w_2) = \frac{3}{4}$

$$\frac{P(w_1)}{\cancel{\gamma(p_1)}} \prod_{j=1}^{k} (\lambda_j)^{p_1} (x_j)^{p_1 - 1} = \frac{P(w_2)}{\cancel{\gamma(p_2)}} \prod_{j=1}^{k} (\lambda_j)^{p_2} (x_j)^{p_2 - 1}$$

$$\frac{1}{4} [\cancel{8^4} x_1^3\ 0.\cancel{3^4} x_2^3] = \frac{3}{4} [\cancel{8^4} \cdot x_1^3\ 0.\cancel{3^4} x_2^3]$$

$$(x_1 x_2)^3 = 3 (x_1 x_2)^3$$

$2(x_1 x_2)^3 = 0$

$f(x_1, x_2) = 0$ when $x_1 x_2 = 0$

2.

$X_i | w_j \sim Lap(m_{ij}, \lambda_i)$ $\quad i \in \{1, 2, \dots, k\}$

$P_{x_i | w_j}(x_i | w_j) = \frac{\lambda_i}{2} e^{-\lambda_i |x_i - m_{ij}|}, \quad \lambda_i > 0 \quad j \in \{1, 2, \dots, c\}$

$p(w_1) = p(w_2) = \dots = p(w_c)$

decide w_i if

$p(w_i | x) > p(w_j | x) \quad \forall i \neq j$

$$\frac{p(x|w_i)\, p(w_i)}{p(\cancel{x})} > \frac{p(x|w_j)\, p(w_j)}{p(\cancel{x})} \quad \forall i \neq j$$

$$p(w_i) \prod_{j=1}^{k} p(x_j | w_i) > p(w_j) \prod_{j=1}^{k} p(x_j | w_j) \quad \forall i \neq j$$

$$p(w_i) \prod_{j=1}^{k} \left[\frac{\lambda_j}{2} e^{-\lambda_j |x_j - m} \right.$$

2

$X_i | w_j \sim Lap(m_{ij}, \lambda_i)$ $\quad i \in \{1, 2, \dots, k\}$

$p_{x_i|w_j}(x_i|w_j) = \frac{\lambda_i}{2} e^{-\lambda_i |x_i - m_{ij}|}, \quad \lambda_i > 0$ $\quad j \in \{1, 2, \dots, c\}$

$p(w_1) = p(w_2) = \dots = p(w_c)$

$p(w_j|x) \propto p(x|w_j) \cdot p(w_j)$

$p(x|w_j) = \prod_{i=1}^{k} p_{x_i|w_j}(x_i|w_j)$

$p(x|w_j) = \prod_{i=1}^{k} \frac{\lambda_i}{2} e^{-\lambda_i |x_i - m_{ij}|}$

$\log p(x|w_j) = \sum_{i=1}^{k} \log\left(\frac{\lambda_i}{2} e^{-\lambda_i |x_i - m_{ij}|}\right)$

$= \sum_{i=1}^{k} \left(\log \frac{\lambda_i}{2} - \lambda_i |x_i - m_{ij}|\right)$

$= \sum_{i=1}^{k} -\lambda_i |x_i - m_{ij}|$

weighted Manhattan distance classifier

$= \sum_{i=1}^{k} \lambda_i |x_i - m_{ij}|$ ∎

The weighted Manhattan distance classifier becomes the minimum Manhattan distance classifier when all weights λ_j are equal. ∎

3. $p(w_1) = \frac{1}{10}$ $\quad p(w_2) = \frac{1}{5}$ $\quad p(w_3) = \frac{1}{2}$ $\quad p(w_4) = \frac{1}{5}$

$$R(\alpha_i | x) = \sum_{j=1}^{4} \lambda(\alpha_i | w_j)\, p(w_j | x)$$

$$P(x_1) = \frac{1}{3} \times \frac{1}{10} + \frac{1}{2} \times \frac{1}{5} + \frac{1}{6} \times \frac{1}{2} + \frac{2}{5} \times \frac{1}{5}$$

$$= 0.2967$$

$$P(x_2) = \frac{1}{3} \times \frac{1}{10} + \frac{1}{4} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{2} + \frac{2}{5} \times \frac{1}{5}$$

$$= 0.33$$

$$P(x_3) = \frac{1}{3} \times \frac{1}{10} + \frac{1}{4} \times \frac{1}{5} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{5}$$

$$= 0.3733$$

$$R(\alpha_1 | x_1) = \sum_{j=1}^{4} \lambda(\alpha_1 | w_j) \frac{p(x_1 | w_j) \cdot p(w_j)}{p(x_1)}$$

$$= \frac{(0 \times \frac{1}{3} \times \frac{1}{10}) + (2 \times \frac{1}{2} \times \frac{1}{5}) + (3 \times \frac{1}{6} \times \frac{1}{2}) + (4 \times \frac{2}{5} \times \frac{1}{5})}{0.2967}$$

$$= \frac{0.77}{0.2967} = 2.5952$$

$$R(\alpha_2 | x_1) = \frac{(1 \times \frac{1}{10} \times \frac{1}{3}) + (0 \times \frac{1}{2} \times \frac{1}{5}) + (1 \times \frac{1}{6} \times \frac{1}{2}) + (8 \times \frac{2}{5} \times \frac{1}{5})}{0.2967}$$

$$= 2.5506$$

$$R(\alpha_3 | x_1) = \frac{(3 \times \frac{1}{10} \times \frac{1}{3}) + (2 \times \frac{1}{2} \times \frac{1}{5}) + (0 \times \frac{1}{6} \times \frac{1}{2}) + (2 \times \frac{2}{5} \times \frac{1}{5})}{0.2967}$$

$$= 1.5506$$

$$R(\alpha_4|x_1) = \frac{(5\times\frac{1}{10}\times\frac{1}{5}) + (3\times\frac{1}{2}\times\frac{1}{5}) + (1\times\frac{1}{6}\times\frac{1}{2}) + (0\times\frac{2}{5}\times\frac{1}{5})}{0.2967}$$

$$= 1.8539$$

$$R(\alpha_1|x_2) = \frac{(0\times\frac{1}{10}\times\frac{1}{5}) + (2\times\frac{1}{4}\times\frac{1}{5}) + (3\times\frac{1}{3}\times\frac{1}{2}) + (4\times\frac{2}{5}\times\frac{1}{5})}{0.33}$$

$$= 2.7879$$

$$R(\alpha_2|x_2) = \frac{(1\times\frac{1}{10}\times\frac{1}{5}) + (0\times\frac{1}{4}\times\frac{1}{5}) + (1\times\frac{1}{3}\times\frac{1}{2}) + (8\times\frac{2}{5}\times\frac{1}{5})}{0.33}$$

$$= 2.5455$$

$$R(\alpha_3|x_2) = \frac{(3\times\frac{1}{10}\times\frac{1}{5}) + (2\times\frac{1}{4}\times\frac{1}{5}) + (6\times\frac{1}{3}\times\frac{1}{2}) + (2\times\frac{2}{5}\times\frac{1}{5})}{0.33}$$

$$= 1.0909$$

$$R(\alpha_4|x_2) = \frac{(5\times\frac{1}{10}\times\frac{1}{5}) + (3\times\frac{1}{4}\times\frac{1}{5}) + (1\times\frac{1}{3}\times\frac{1}{2}) + (0\times\frac{2}{5}\times\frac{1}{5})}{0.33}$$

$$= 1.4646$$

$$R(\alpha_1|x_3) = \frac{(0\times\frac{1}{10}\times\frac{1}{5}) + (2\times\frac{1}{4}\times\frac{1}{5}) + (3\times\frac{1}{2}\times\frac{1}{2}) + (4\times\frac{1}{5}\times\frac{1}{5})}{0.3733}$$

$$= 2.7054$$

$$R(\alpha_2|x_3) = \frac{(1\times\frac{1}{10}\times\frac{1}{5}) + (0\times\frac{1}{4}\times\frac{1}{5}) + (1\times\frac{1}{2}\times\frac{1}{2}) + (8\times\frac{1}{5}\times\frac{1}{5})}{0.3733}$$

$$= 1.6161$$

$$R(\alpha_3|x_3) = \frac{(3\times\frac{1}{10}\times\frac{1}{5}) + (2\times\frac{1}{4}\times\frac{1}{5}) + (0\times\frac{1}{2}\times\frac{1}{2}) + (2\times\frac{1}{5}\times\frac{1}{5})}{0.3733}$$

$$= 0.7500$$

$$R(\alpha_4|x_3) = \frac{(5\times\frac{1}{10}\times\frac{1}{3}) + (3\times\frac{1}{4}\times\frac{1}{5}) + (1\times\frac{1}{2}\times\frac{1}{2}) + (0\times\frac{1}{5}\times\frac{1}{5})}{0.3733}$$

$$= 1.5179$$

(b)

$$R = R(\alpha_3|x_1)\cdot p(x_1) + R(\alpha_3|x_2)\cdot p(x_2) + R(\alpha_3|x_3)\cdot p(x_3)$$

$$= 1.5506(0.2967) + 1.0909(0.33) + 0.75(0.3733)$$

$$\doteqdot 1.1$$

4.