StockTrak

- **Purpose:** practice creating and implementing an investment strategy

- Form teams of 3-4
  - Set up one account on StockTrak per team

- We’ll track performance over the semester
  - Highest performing team receives extra credit

- Sign up on Google spreadsheet. Link in PDF.

Portfolio Optimization
Ch. 2: Preferences and Ch. 3: Mean-Variance Investing

Asset Management

- Over the next 8 weeks, we will learn how to make optimal investment decisions given our *preferences* and the distribution and dynamics of various return *factors*
  - Ch 2: Preferences
  - Ch 3: Mean-Variance Investing
  - Ch. 6: Factor Theory
  - Ch. 7: Factors
  - Ch. 10: Alpha
  - Ch. 14: Factor Investing

- Factor: common movement in returns across a set of assets
  - Fundamental, economy wide variables (e.g., inflation) or tradeable investment styles (e.g., value-growth)
  - These are the risks we should care about and be compensated for

Outline

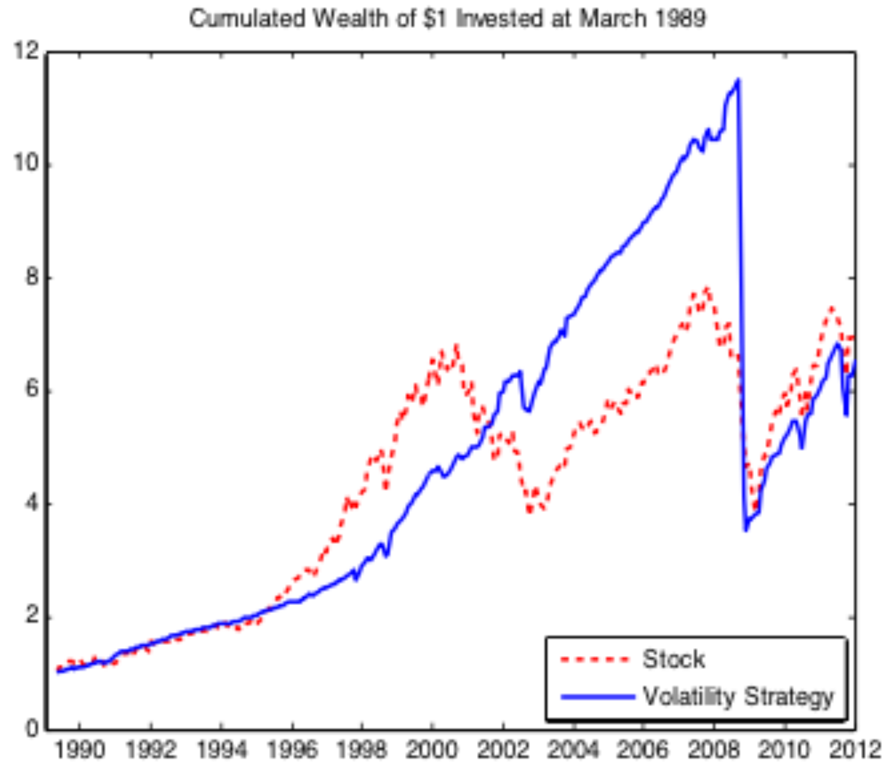
- Utility theory = characterizing bad times
  - Mean-variance utility (not very realistic, but the industry standard)
  - Other realistic utility functions
- Utility concepts:
  - Risk aversion
  - Certainty equivalent
- Optimal Portfolios
  - With risk-free asset
  - Without risk-free asset

Why Utility?

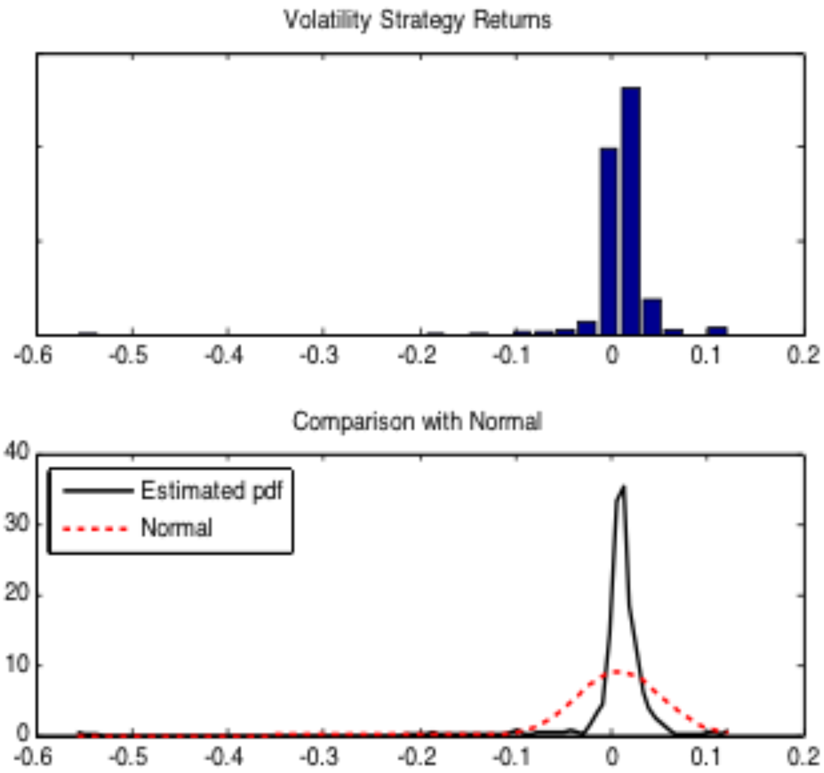
Compare:

- Short volatility strategy: investment strategy which nets a premium during stable periods (period after 2009), but has large losses during volatile times (financial crisis 2008-2009)
  - Technically, we can implement this by selling out-of-the-money put options
- S&P 500 equities

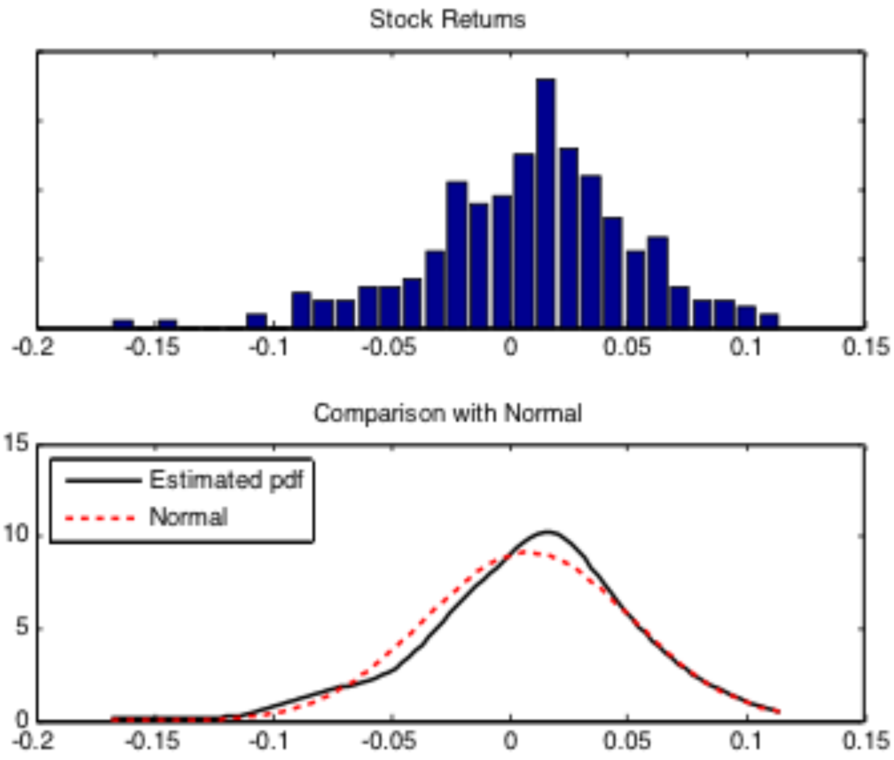
Volatility Strategy vs S&P 500



Volatility Strategy



S&P 500



Stock Returns
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
Comparison with Normal
15
10
5
0
Estimated pdf
Normal
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15

Volatility Strategy vs S&P 500

- Average return is approximately the same, at around 10%
- Dispersion, as measured by standard deviation = 15%, is also approximately the same
- But the volatility strategy exhibits large *negative skewness*, that is prone to occasional frightful losses

	Vol Strategy	S&P 500
Mean	9.9%	9.7%
Std. Deviation	15.2%	15.1%
Skewness	-8.3	-0.6
Kurtosis	104.4	4.0

Volatility Strategy vs S&P 500

- How does an investor judge each investment's "risk"
- How much of each, if any, should an investor hold?
- How do we take into account the asymmetric payoffs? Especially those associated with downside risk?

When we get paid matters

- Hypothetical lottery:
  - 0.0001% chance you win \$5M, \$0 otherwise
  - How much would you be willing to pay?
    - E[payoff] = \$5

- Hypothetical disability insurance contract:
  - 0.0001% chance you become disabled and get a payout (PV of payouts = \$5M), \$0 otherwise
  - How much would you be willing to pay?

- Should be willing to pay more for the disability insurance
  - Pays out when we need money the most
- In investing, we'll want to consider how the investment's payoffs correlate with our good and bad times ((i.e., low and high marginal utility)
  - In bad times, an extra dollar would make us feel a lot better relative to an extra dollar in good times

Utility Functions

Preferences

- Utility is how economists measure investors' *bad times*. The lower the utility from a specific outcome, the more it hurts.
- Utility = How you feel
- Utility is an index that measures bad times
  - High utility: good times. Usually low marginal utility.
  - Low utility: bad times. Usually high marginal utility.
- A utility function, U, measures the felicity or satisfaction of an individual. It assigns a numeric measure to an agent's desire.

Utility Functions

- We start with utility as a function of wealth, W, and then move to considering other variables which affect utility
  - Rich: good time
  - Poor: bad time
  - Utility increases with wealth, but the rate of increase is different for different investors
    - Utility functions increase with wealth: property of non-satiation

- Utility functions are typically concave: going from \$1 to \$2 is much more valuable than going from \$100,000 to \$100,001

- A utility function provides a systematic way to rank different choices. **The actual numerical value ("cardinality") has no meaning**; for wealth W, U(W) and a U(W) + b have exactly the same rankings. (See Appendix.)

Utility Functions

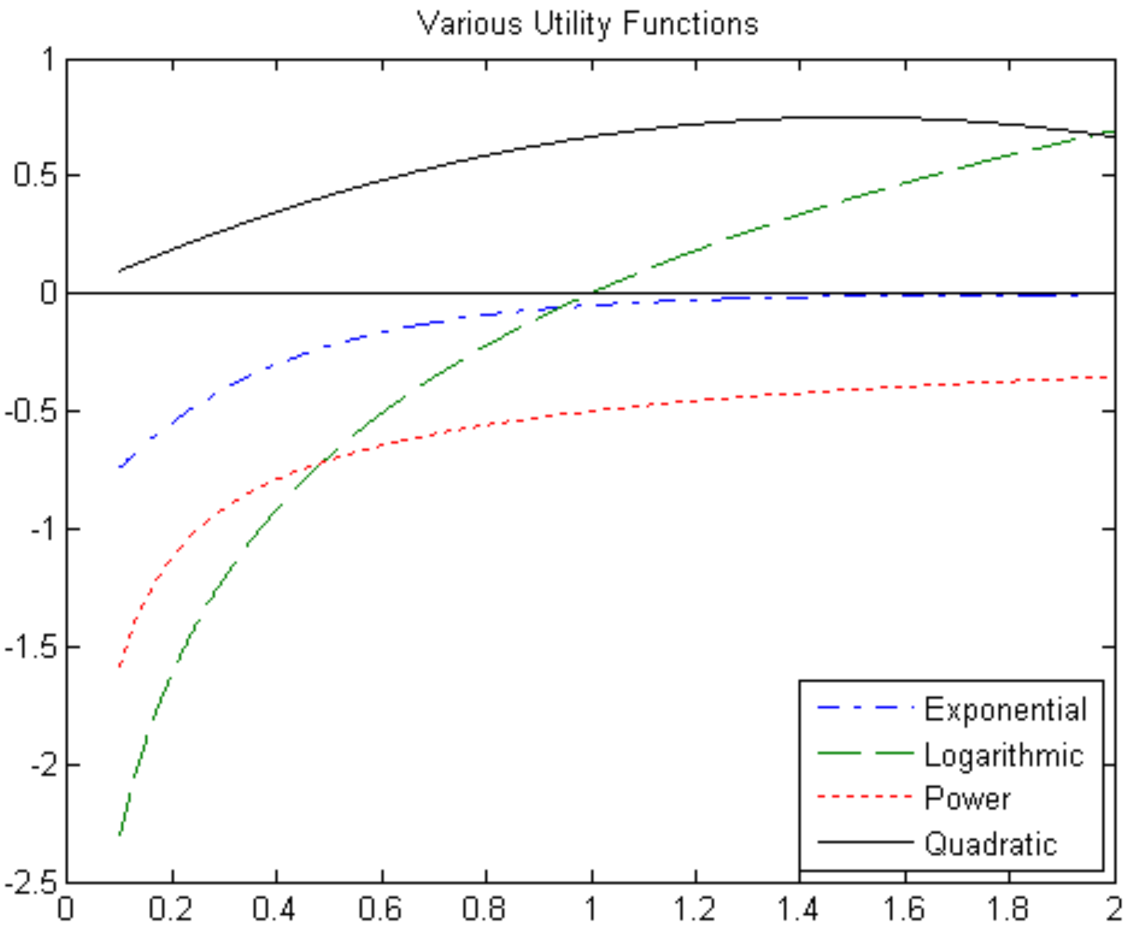
- Popular utility functions:

–Exponential (CARA): $U(W) = -e^{-aW}, a > 0$

–Log: $U(W) = \ln(W)$

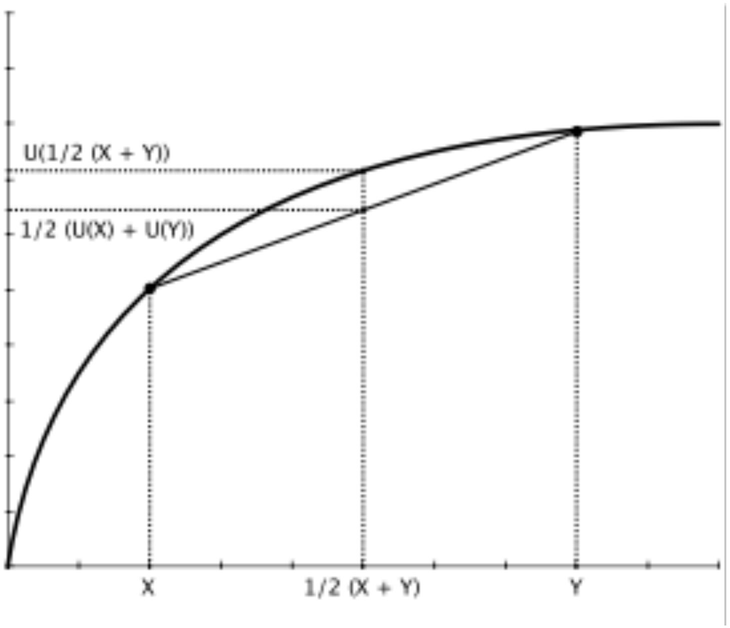
–Power (CRRA): $U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \gamma \geq 0, \gamma \neq 1$ [locally mean-variance]

–Quadratic: $U(x) = W - bW^2, b > 0$ [mean-variance utility]


Various Utility Functions
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Exponential
Logarithmic
Power
Quadratic

Utility Functions

- A sure value of (x+y)/2 is preferred to a 50-50 chance of x or y. The shape of *concavity* is a measure of *risk aversion*, which trades off risk and return.



Risk Aversion

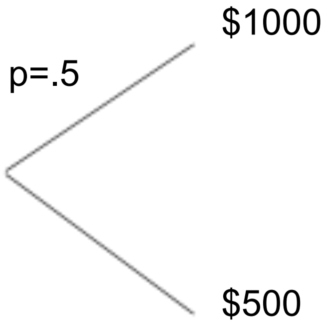
- *Risk aversion* is a key concept in utility functions. The opposite of risk aversion is *risk tolerance*
- The more risk averse the investor, the flatter the utility function
- A very *risk-averse investor* has a steep utility function, or *high marginal utility*
  - Moving from right to left (rich to poor), your utility changes a lot. You feel the pain of bad times horribly.
- A very *risk-tolerant investor* has a flat utility function, or *low marginal utility*
  - Moving from right to left (rich to poor), your utility doesn't change very much. Your bad times don't feel that bad.

Risk Aversion

- Risk aversion is a crucial parameter for utility functions. For mean-variance utility (and CRRA utility), it is the only parameter.
- Typically, most individuals would have risk aversions between 1 to 10 with it being very rare to have risk aversions greater than 10. This comes from a large body of experimental and survey evidence.
  - Metrick (1995), studies the game show Jeopardy, and finds low risk aversion levels
- There are a variety of ways to measure risk aversion
  - Questionnaires, especially those employed by financial planners
  - Certainty equivalent comparisons
  - Revealed preference: portfolio holdings or bets made

Risk Aversion

- What's your risk aversion?
- How much would you pay for the following bet? [Calibrated with CRRA utility]



γ	Amount you would pay
0	750
0.5	729
1	707
2	667
3	632
4	606
5	586
10	540

Example of certainty equivalent comparison

Certainty Equivalent

- The utility certainty equivalent is the sure amount of wealth that is equivalent, in utility terms, to a risky position. It is the amount C satisfying

$$U(C) = E[U(W)]$$

> Suppose that you are the only income earner in the family, and you have a good job guaranteed to give you your current (family) income every year for life. You are given the opportunity to take a new and equally good job, with a 50–50 chance it will double your (family) income and a 50–50 chance that it will cut your (family) income by a third. Would you take the new job?

If the answer to the first question is "yes," the interviewer continues:

> Suppose the chances were 50–50 that it would double your (family) income, and 50–50 that it would cut it in half. Would you still take the new job?

If the answer to the first question is "no," the interviewer continues:

> Suppose the chances were 50–50 that it would double your (family) income and 50–50 that it would cut it by 20 percent. Would you then take the new job?

Barsky, Juster, Kimball and Shapiro (QJE, 1997)

TABLE I

RISK PREFERENCE SURVEY DESIGN

	Gamble[a]	Relative risk aversion (1/θ) Upper bound	Relative risk aversion (1/θ) Lower bound	Relative risk aversion (1/θ) Mean[b]	Relative risk tolerance (θ) Lower bound	Relative risk tolerance (θ) Upper bound	Relative risk tolerance (θ) Mean[b]	Expectation conditional on survey response[c] 1/θ	Expectation conditional on survey response[c] θ
I.	Reject both one-third and one-fifth	∞	3.76	15.8	0	0.27	0.11	15.7	0.15
II.	Reject one-third but accept one-fifth	3.76	2	2.9	0.27	0.5	0.36	7.2	0.28
III.	Accept one-third but reject one-half	2	1	1.5	0.5	1	0.68	5.7	0.35
IV.	Accept both one-third and one-half	1	0	0.7	1	∞	1.61	3.8	0.57

a. Gambles all have a 50 percent probability of doubling lifetime income and a 50 percent probability of losing half, one-third, or one-fifth of lifetime income.

b. These columns report the mean if the *true* value is between the lower and upper bounds.

c. These columns give the expected value of relative risk tolerance and relative risk aversion conditional on *observing* response I, II, III, or IV. This conditional expectation takes into account measurement error in the survey response. This baseline case assumes lognormality, no status quo bias, and no persistent measurement error. (See text for details and Table XIV for other cases.)

Schwab Risk Tolerance Questionnaire

- https://www.schwab.com/public/file/P-778947/InvestorProfileQuestionnaire.pdf

Expected Utility

- Since wealth is random, the utilities over wealth are also random. We specify that agents care about *expected utility*. Expected utility is the main workhorse of modeling individual choices in economics.

$$\max_{\text{choice variables}} E[U(W)]$$

- The agent maximizes expected utility by choosing different
  - Spending/savings plans => optimal savings/consumption problem
  - Holding different assets => optimal asset allocation problem
  - Production plans => optimal firm investment problem

  etc.

- Agents do not, however, always behave in accordance with expected utility (see later)

Utility Functions

<u>How are Preferences Used?</u>

- Portfolio Choice Problem:

  Two assets: stocks and bonds with weights wS and wB

$$\max_{w_S, w_B} E[U(W)]$$

$$W = W_0(1 + w_S r_S + (1 - w_S) r_B) \quad \text{st } w_S + w_B = 1$$

- This set up allows us to incorporate our preferences into the optimal portfolio problem
- Answer depends on
  - Risk aversion (concavity of U) and other properties of U
  - Properties of the assets (investment universe)
- Each investor has a different U. They may have access to the same or different assets.

Mean-Variance Utility

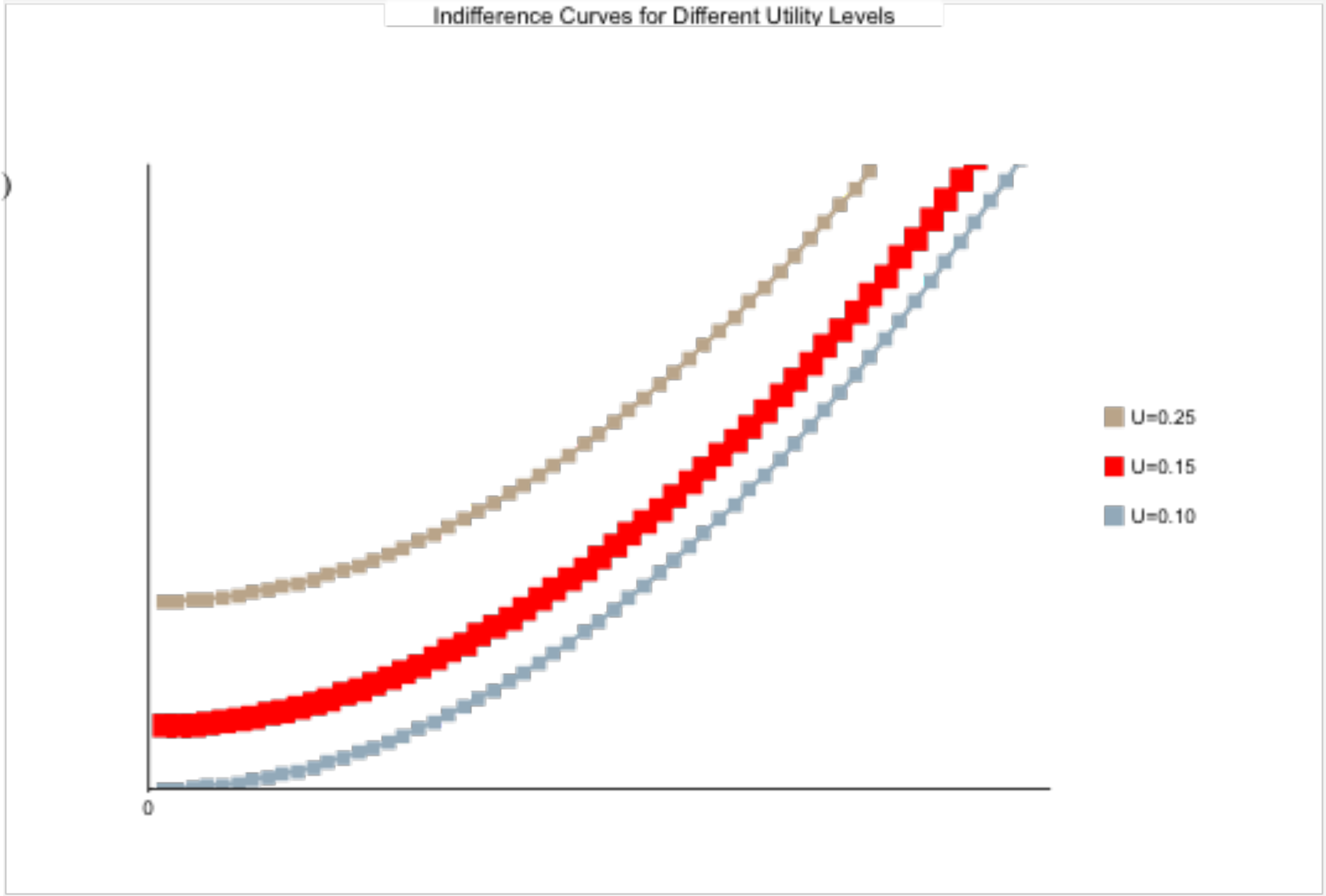
Mean-Variance Utility

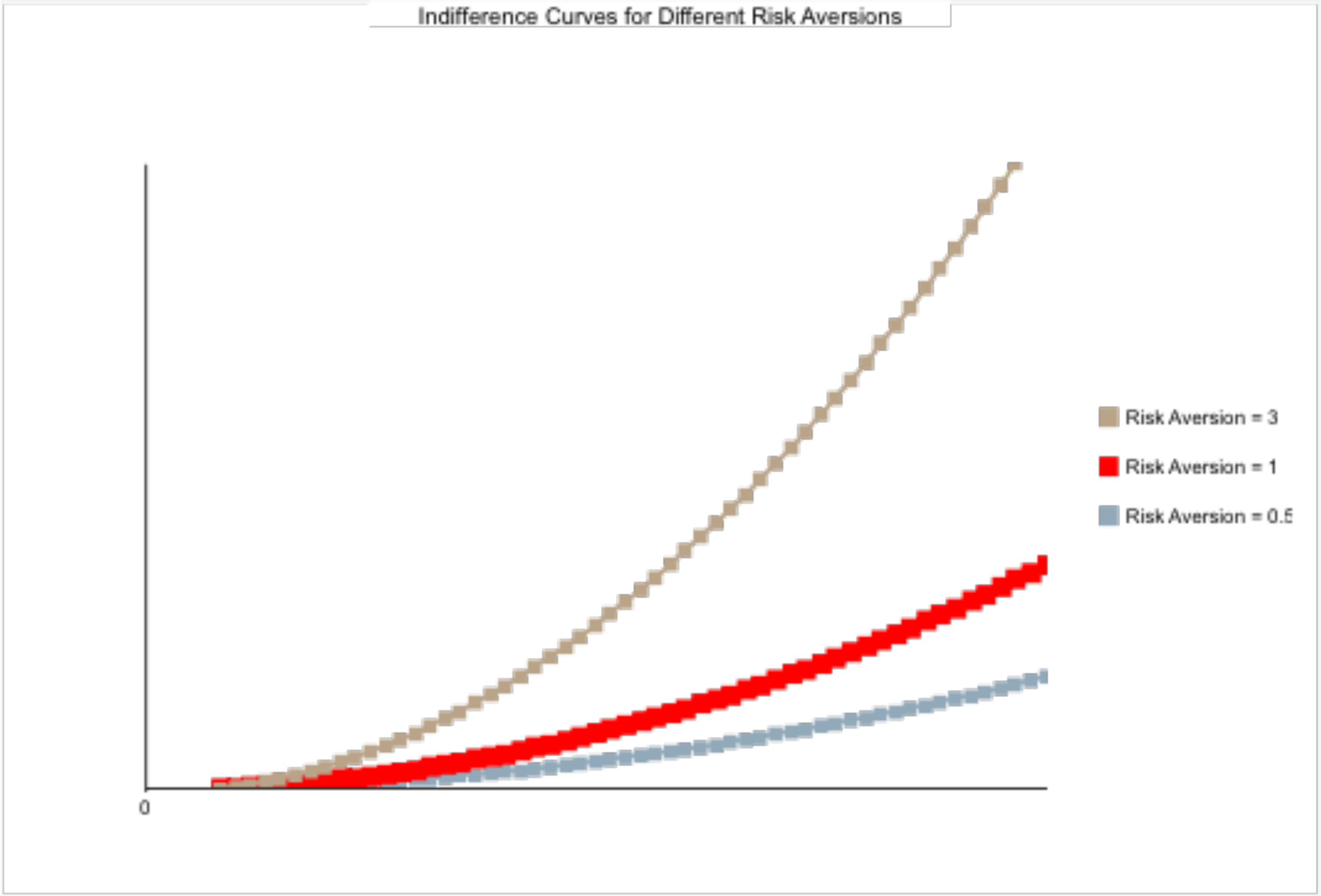
- The most commonly used utility function in industry is mean-variance utility.
  - Defined in terms of returns or initial wealth, does not matter.
    - Why is this a desirable property for use in asset management?
- Mean-Variance Utility:

$$U = E(r_p) - \frac{\gamma}{2}\text{var}(r_p)$$

where rp = portfolio return, g = risk aversion

$$E(r_p) = U + \frac{\gamma}{2}\operatorname{var}(r_p)$$




Indifference Curves for Different Risk Aversions
0
Risk Aversion = 3
Risk Aversion = 1
Risk Aversion = 0.5

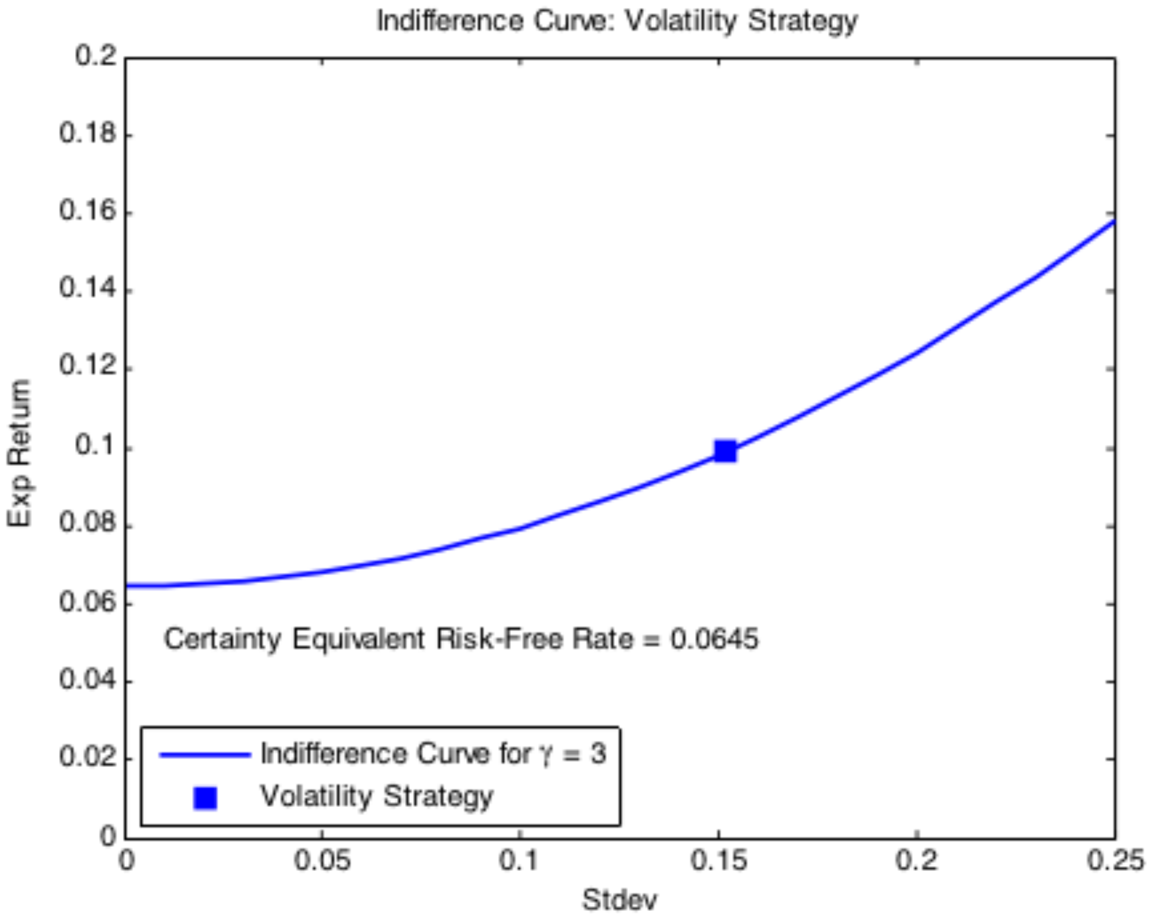
Mean-Variance Utility

- There are an infinite number of indifference curves. One particular indifference curve represents a particular level of utility.
- Along an indifference curve, an investor is indifferent to all mean-volatility (or mean-variance) combinations
- The more risk averse an investor, the steeper the slope of the indifference curve
- Mathematically:
  - Along an indifference curve, U is held fixed and E(r) and var(r) change to give the same value of U:

$$E(r_p) = U + \frac{\gamma}{2}\text{var}(r_p)$$

Certainty Equivalent

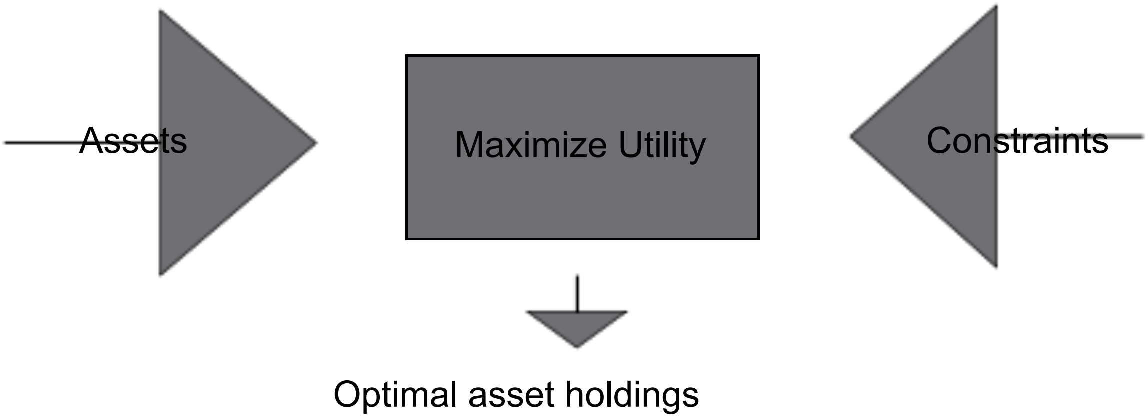
- For mean-variance utility, take the indifference curve going through the risky asset. [Why is there only one?] The certainty equivalent is the point where the indifference curve intersects the y-axis.
  - Note that the certainty equivalent risk-free rate is identical to the utility level! Thus, the level of utility is meaningful in mean-variance utility (unlike general utility functions)


Indifference Curve: Volatility Strategy
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Exp Return
Certainty Equivalent Risk-Free Rate = 0.0645
Indifference Curve for γ = 3
Volatility Strategy
0
0.05
0.1
0.15
0.2
0.25
Stdev

Optimal Portfolio of Risky Assets

Mean-Variance Utility

- Maximizing utility is the same as finding the highest possible indifference curve. But, there are an infinite number of indifference curves.
- The portfolio choice problem usually specifies constraints:



Mean-Variance Utility

Example

- Data 1926:01-2011:12 (Ibbotson data)

	Mean	Std. Dev.
Stocks	0.1119	0.1915
Bonds	0.0591	0.0833
Correlation(stocks,bonds)=0.1113		

- Investor can only hold stocks or bonds. For a given risk aversion, what holdings are optimal? Assume the investor cannot short.

- The asset space and constraints pin down the set of feasible indifference curves. In this case, it is the mean-variance frontier. For the solution, see Appendix B.

Mean-Variance Frontier

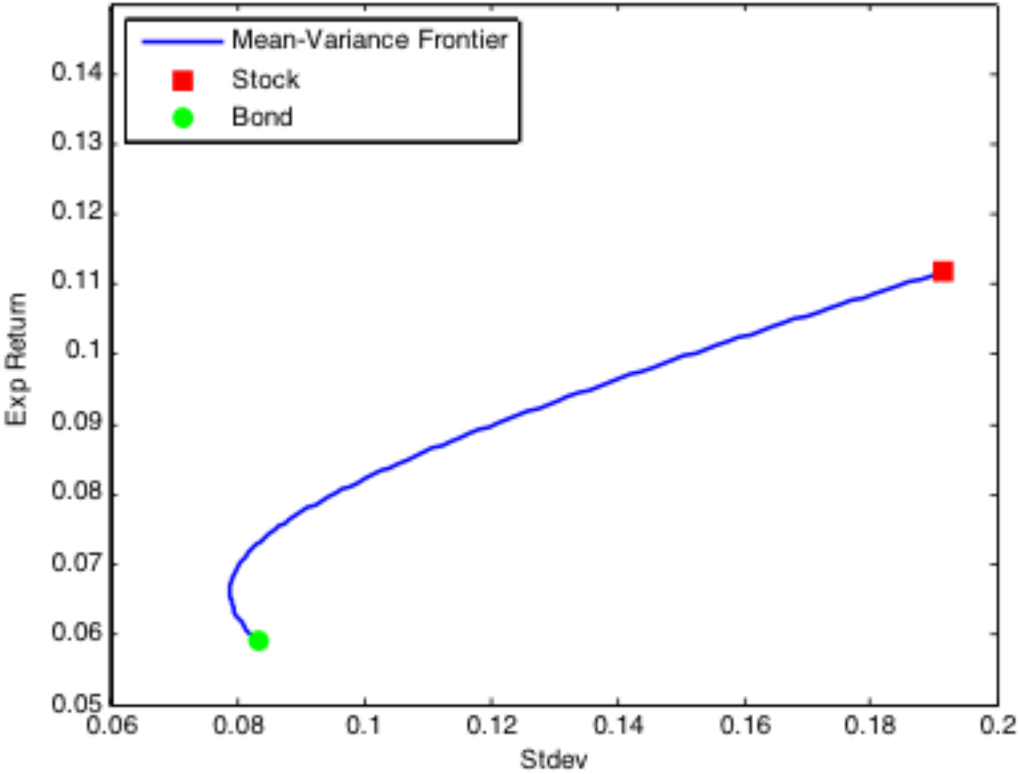
- For two assets, the mean-variance frontier is traced by taking all combinations of the two assets A and B:
- Asset A: E(rA) and sA, held with weight w
- Asset B: E(rB) and sB, held with weight 1-w
- Correlation between A and B: r
- Portfolio p: E(rp) and sp

$$E(r_p) = wE(r_A) + (1-w)E(r_B)$$

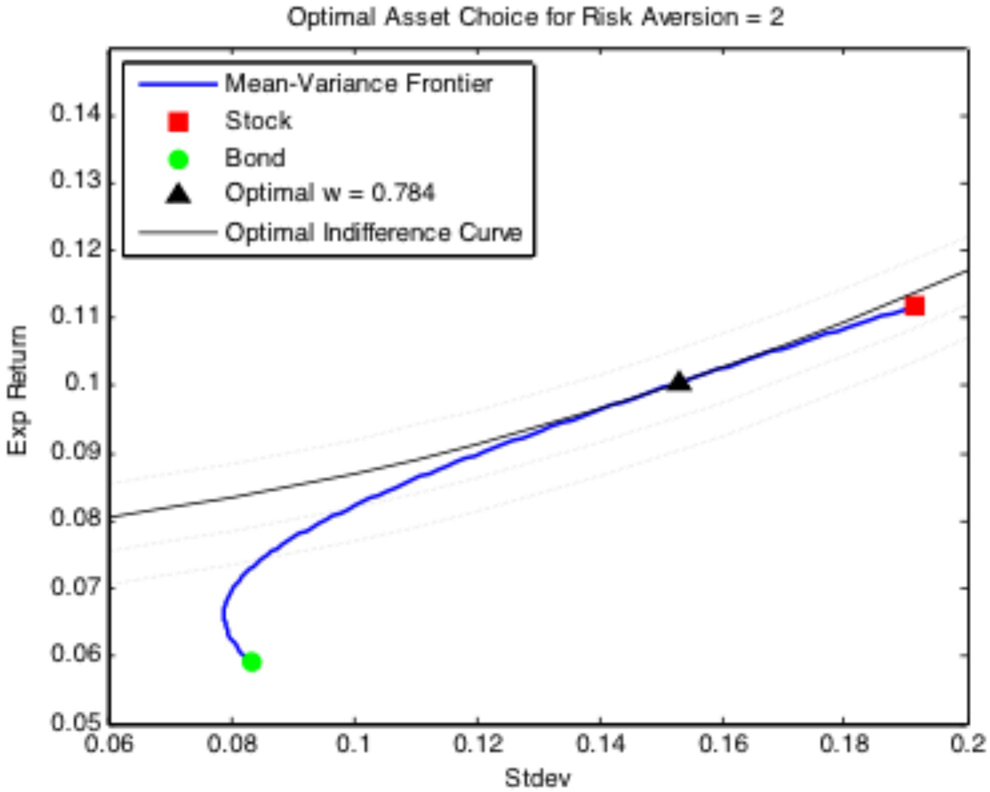
$$\sigma_p = \sqrt{w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\rho\sigma_A\sigma_B}$$

[What does it mean for w<0 or w>1?]

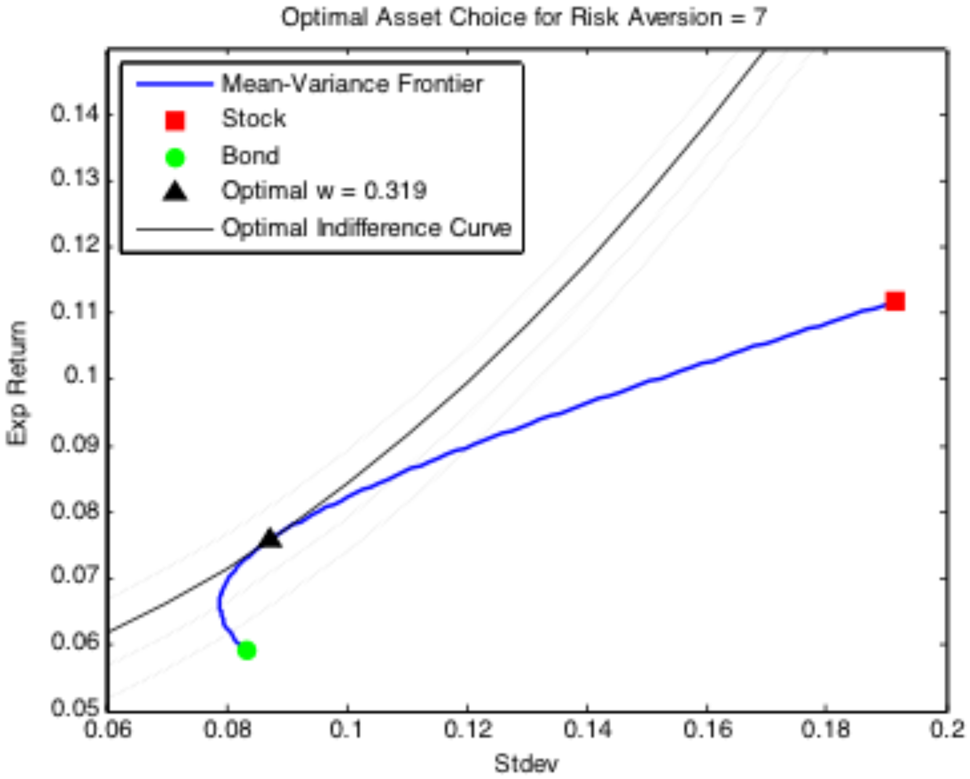
Mean-Variance Utility



Mean-Variance Utility



Mean-Variance Utility



Capital Allocation Line

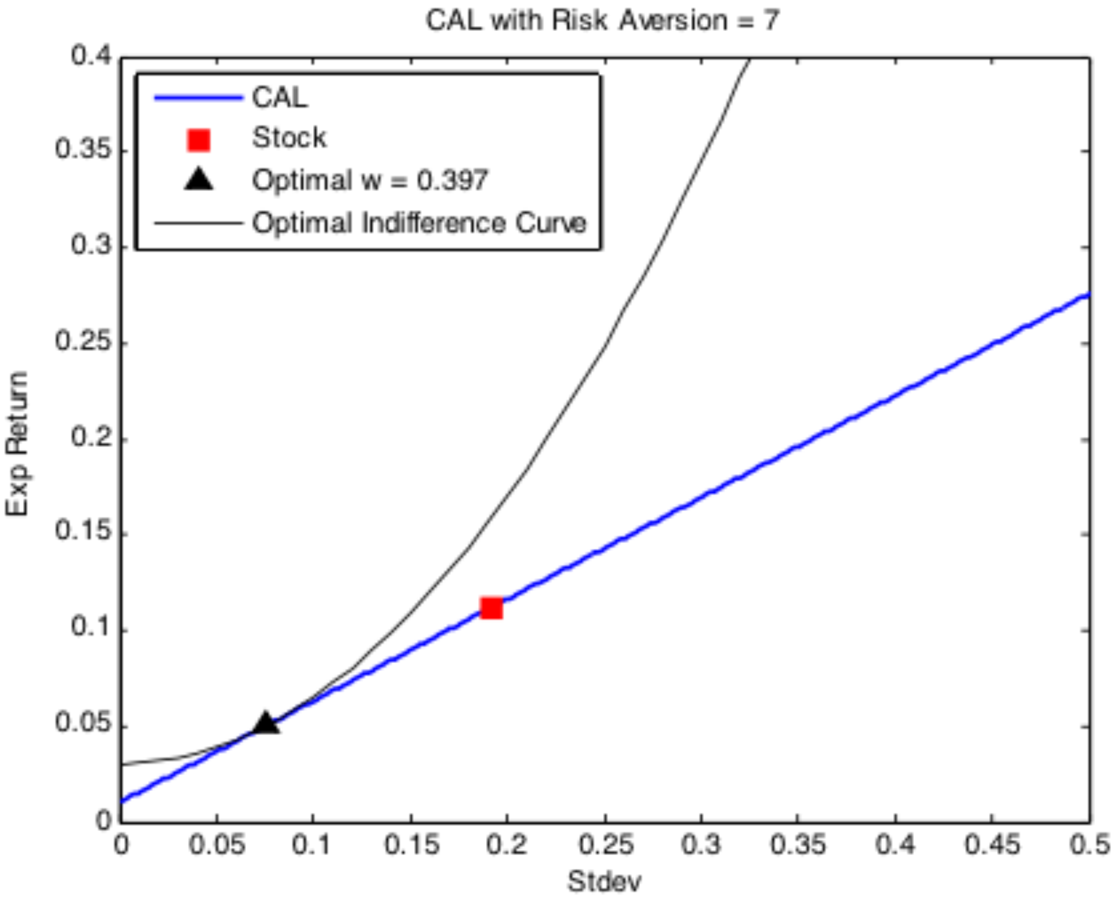
Capital Allocation Line

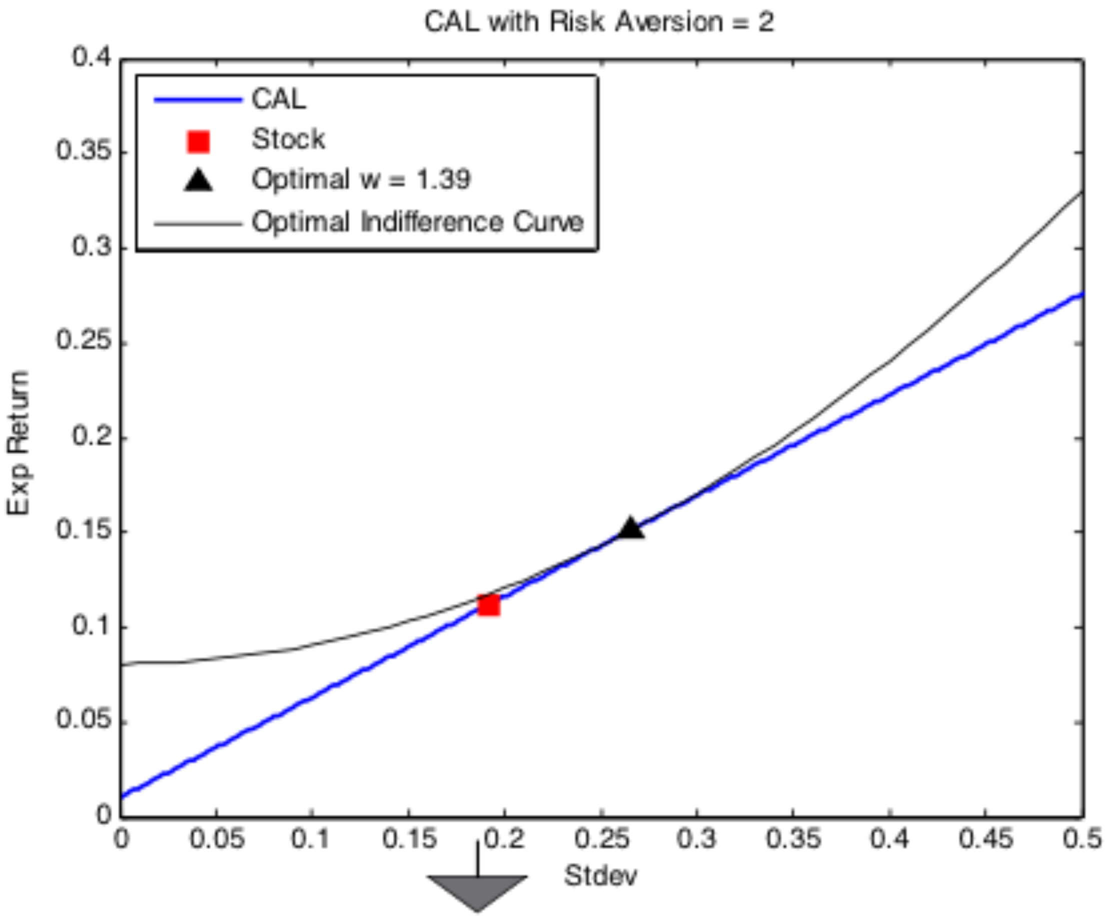
- Suppose now we have just one risky asset, equities, and a risk-free asset (say T-bills). Equities have expected return *E(rp)* and volatility *sp*. The risk-free rate is *rf*.
- The set of possible portfolios of equities and the T-bills is described by the Capital Allocation Line (CAL)
- The CAL is described in portfolio mean-standard deviation space by:

$$E[r_c] = r_f + \frac{E[r_p] - r_f}{\sigma_p}\sigma_c$$

- The Sharpe ratio is the reward per unit of risk:

$$\text{Sharpe ratio} = \frac{E[r_p] - r_f}{\sigma_p}$$


CAL with Risk Aversion = 7
CAL
Stock
Optimal w = 0.397
Optimal Indifference Curve
Exp Return
Stdev
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5


CAL with Risk Aversion = 2
CAL
Stock
Optimal w = 1.39
Optimal Indifference Curve
Exp Return
Stdev
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Utility

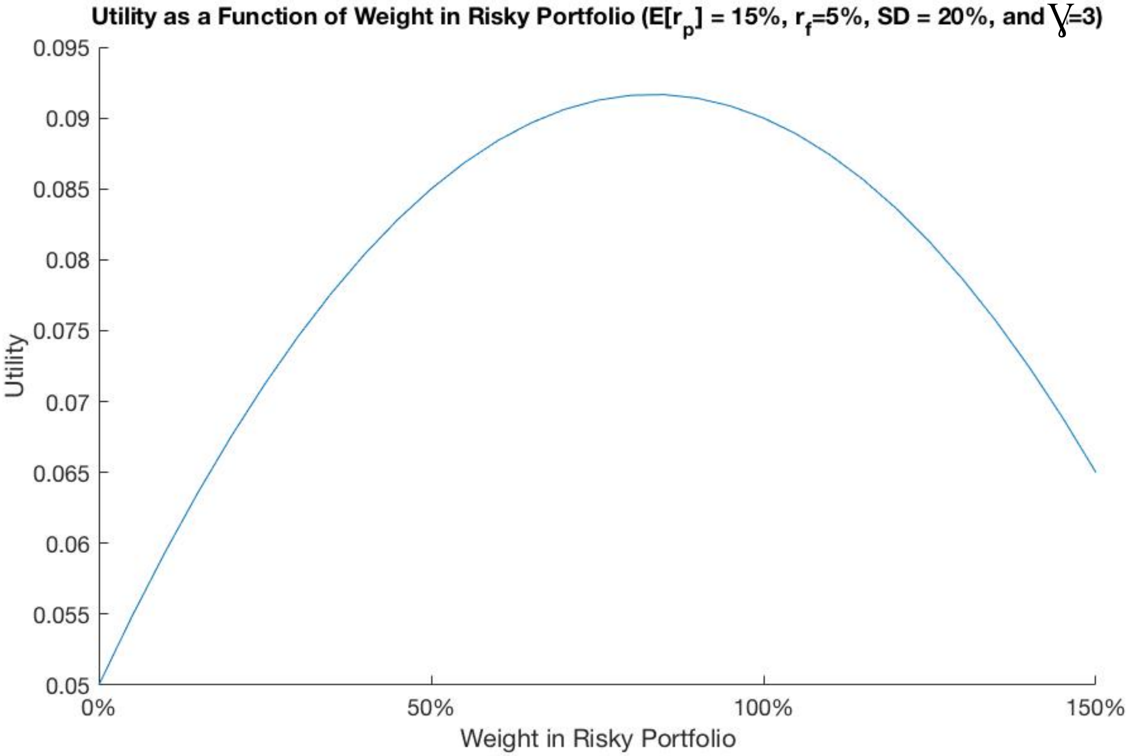
$$U = E[r_c] - \frac{\gamma}{2}\sigma_c^2$$

- Substitute for w:

$$U = E[r_f + w(r_p - r_f)] - \frac{\gamma}{2}(w\sigma_p)^2$$

- Utility is increasing in the risk-free rate, the return on the risky portfolio and decreasing in the standard deviation and risk aversion
- U-shaped in terms of the weight in the risky portfolio

Utility Example



Investor's Problem:

- Find the weight that maximizes investor's utility

$$\max_w r_f + w(E[r_p] - r_f) - \frac{\gamma}{2}(w\sigma_p)^2$$

To Solve

- We need to take the derivative of our utility with respect to the weight in the risky portfolio and set the derivative to zero (first order condition)
- This will find where the slope of the utility function is zero ("top of the arch")

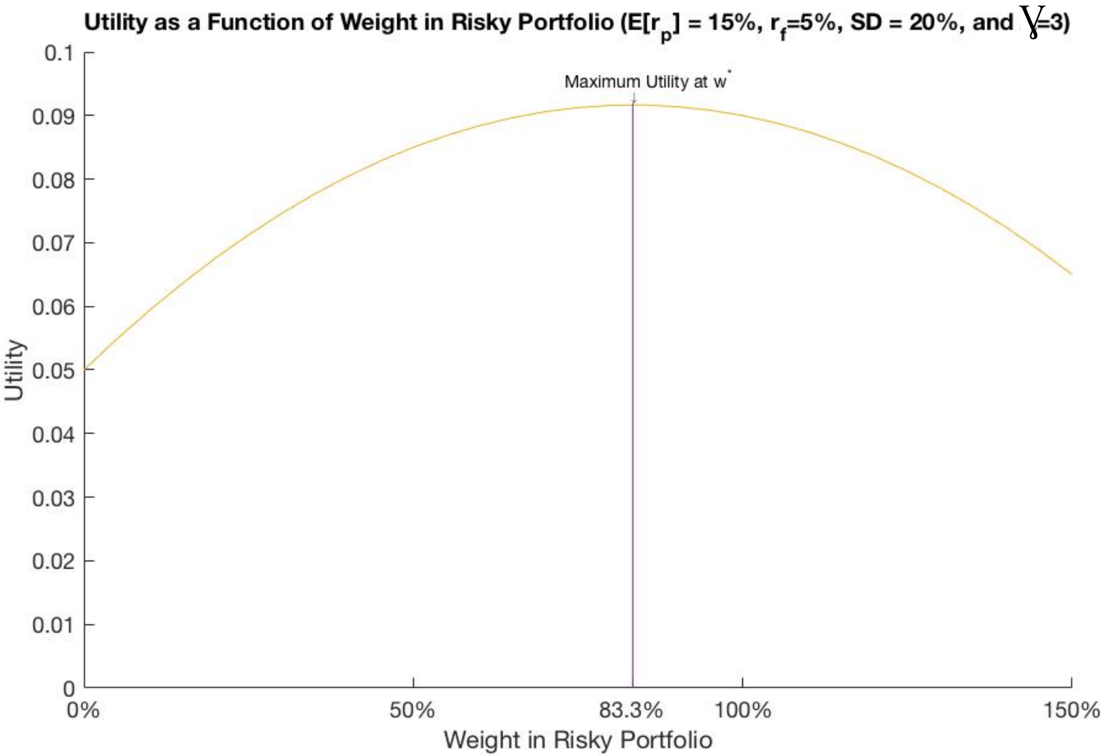
Solution

$$(E[r_p] - r_f) - \gamma w \sigma_p^2 = 0$$

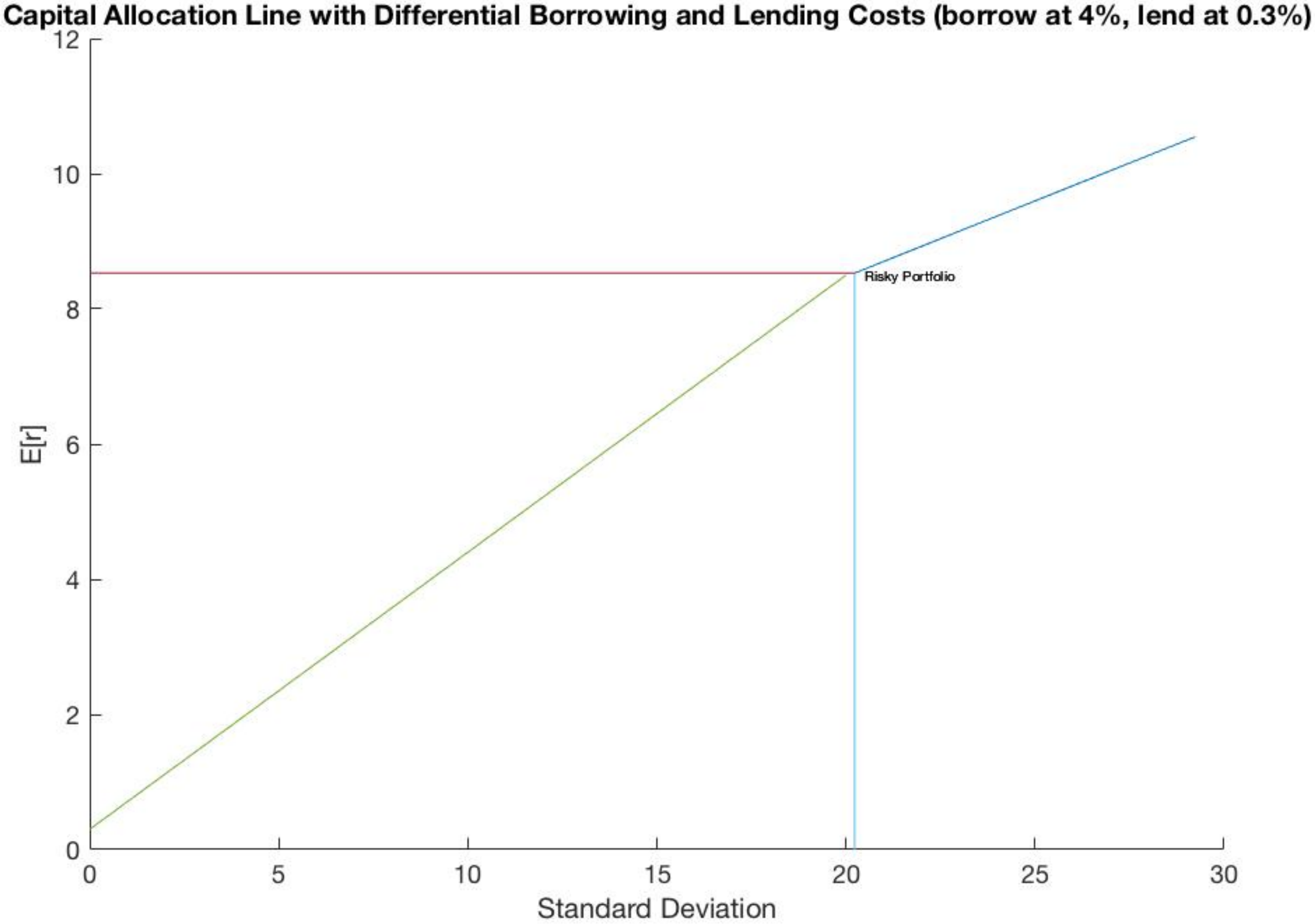
$$\Rightarrow w^* = \frac{E[r_p] - r_f}{\gamma \sigma_p^2}$$

How do Y, E[r], σ2 affect the weight in risky ?

Utility Example



Differential Borrowing and Lending Rates



Beyond Mean-Variance

Mean-Variance Utility Drawbacks

- Symmetry of upside and downside movements
  - Variance treats upside and downside returns the same, relative to the mean. People are generally more risk averse over downside losses than over upside gains.
- Only first two moments matter
  - People prefer positive skewness than negative skewness. People dislike fat tails, especially on the downside.
- Real-world vs. subjective probabilities
  - The actual return distributions of assets are different to those perceived by agents
- More realistic utility functions (see appendix for more details):
  - Safety first
  - Loss aversion or prospect theory
  - Disappointment aversion
  - Habit
  - Keeping up with the Jones
  - Uncertainty Aversion

Ellsberg Experiment

- You have an urn containing 30 red balls and 60 other balls either white or black.
- Choose between these two gambles
  - A: Receive $1,000 if you draw a red ball
  - B: Receive $1,000 if you draw a black ball
  - Choose A if and only if you believe # of white > # of black
- Different choice: choose between these two gambles
  - C: $1,000 if you draw a red or white ball
  - D: $1,000 if you draw a white or black ball
  - Choose C if and only if you believe # of white > # of black
- Most people choose A (# white># black) and D (# black< # white)
- Ellsberg paradox – evidence of ambiguity aversion
  - Prefer gambles with known odds to gambles with ambiguous odds (uncertainty about the probability distribution)

Summary

Summary

- Utility functions quantitatively rank good to bad outcomes
- Incorporating utility functions allows us to account for our preferences in the portfolio optimization problem
- The more risk averse an investor, the lower the utility during bad times, and the greater the difference in utility between good and bad times. Risk aversion can be measured by questionnaires, certainty equivalents, and revealed portfolio holdings.
- In mean-variance utility, there is only one preference parameter: risk aversion. Investors care only about means (which they like) and volatility (which they dislike). It is the most widely used utility function in industry, sadly.

Appendix A: More on Utility Functions

Further Details on Utility Functions

- The economic properties of a utility function are related to its derivatives
  - U(W) is increasing in wealth if U'(W) > 0
  - U(W) is strictly concave in wealth U''(W) < 0

- The degree of risk aversion is formally defined by the Arrow (1971) and Pratt (1964) risk aversion coefficients:

$$\text{Risk Aversion} = -\frac{U''(W)}{U'(W)} \qquad \text{Relative Risk Aversion} = -\frac{WU''(W)}{U'(W)}$$

- For CRRA utility, relative risk aversion is given by g:

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma} \qquad -\frac{WU''(W)}{U'(W)} = -\frac{-\gamma WW^{-\gamma-1}}{W^{-\gamma}} = \gamma$$

Further Details on Utility Functions

- The expected utility formulation was first provided by von Neumann and Morgenstern (1947) and axiomatized by Savage (1954). An important axiom is the independence axiom: that wealth in one state does not interact with wealth in another state so as to reverse the preference. This is violated in practice.

- Another important assumption is that expected utility assumes the probabilities and payoffs are known. But, often agents do not know exact probabilities, and can only specify vague sets of different probability models. This setting is described by "Knightian uncertainty" (see Gilboa and Schmeidler, 1989).

Realistic Utility Functions

Safety First

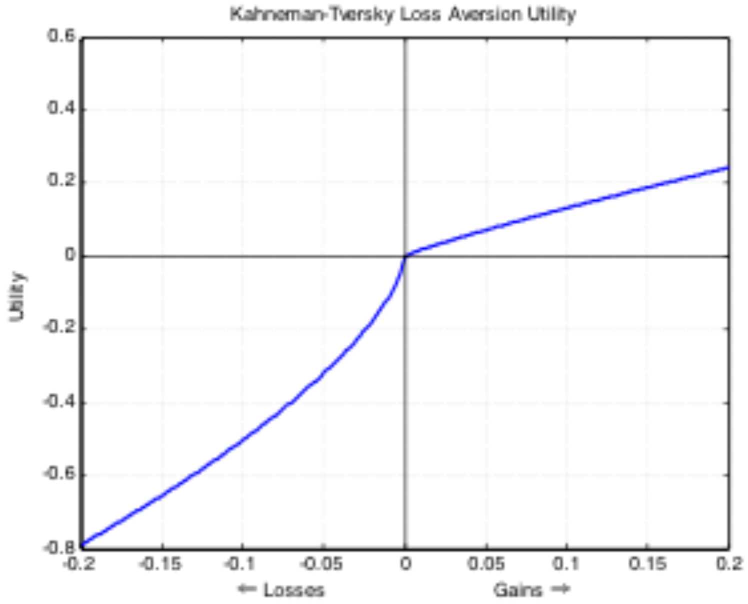
- Developed by Roy (1952)
- Minimizes the chance of a disaster. Establish a level d: if the portfolio return is less than d, then a disaster results and if greater than d, no disaster. The utility function is 0-1 depending on whether the return exceeds d or not.
- Value-at-Risk is similar
- Natural application is when meeting a liability is crucial
- Related utility functions include quantile utility, formulated by Manski (1988) and Rostek (2010). Agents maximize a given quantile of the distribution (the quantile is a utility parameter).

Loss Aversion or Prospect Theory

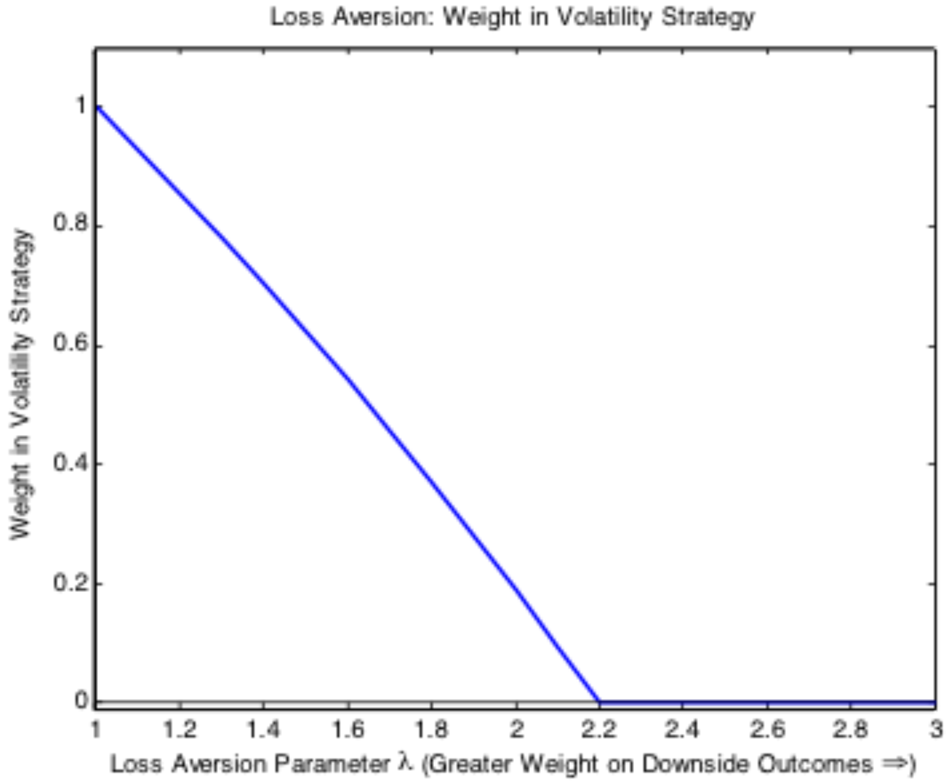
- Developed by Kahneman and Tversky (1979)
- Downside losses are more painful than the joy coming from upside gains
- "Behavioral" or non-rational utility
- Two parts to prospect theory:
  - Loss aversion utility. This specifies how investors treat losses and gains. Slope of the function over losses is steeper than the slope over gains.
  - Subjective rather than objective (actual or real-world) probabilities. Investors can over-weight the probabilities of disasters, for example.

Loss Aversion or Prospect Theory

- Investors are risk-seeking over losses, but loss averse over gains
  - People are willing to take on some risk to avoid sure losses



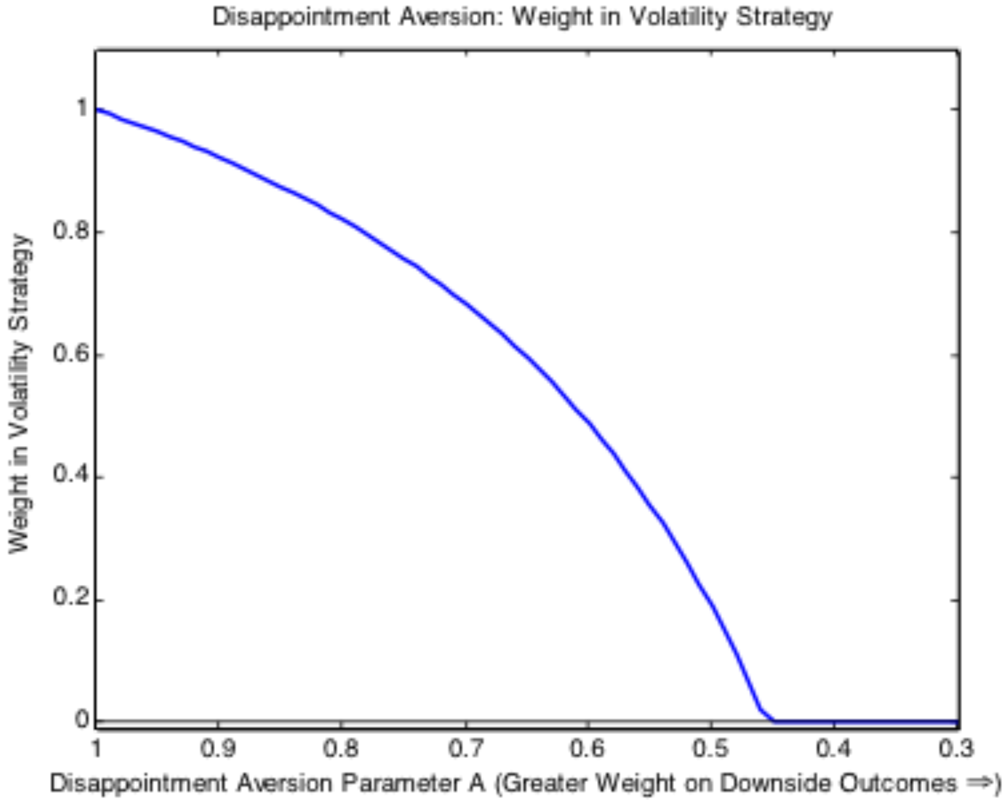
Loss Aversion or Prospect Theory



Disappointment Aversion

- "Rational" version of prospect theory, with the same motivation that agents care more about the downside vs the upside
- Originally formulated by Gul (1991) with an asset allocation application by Ang, Bekaert and Liu (2005)
- Like prospect theory, downside outcomes are penalized more ("disappointing outcomes") vs. upside outcomes ("elating outcomes"). There is a kink point between disappointing vs. upside outcomes, which is specified endogenously (the certainty equivalent) or is a preference parameter.

Disappointment Aversion

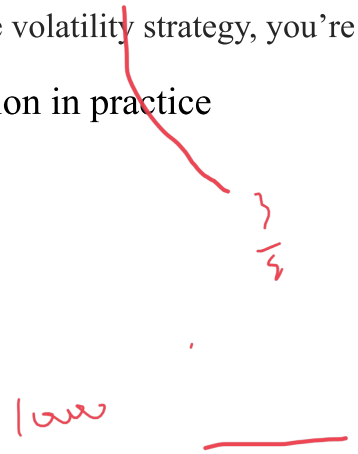


Habit

- It is not your wealth per se which gives utility, it is wealth relative to a habit level that is important. An agent gets used to a particular level of consumption – recent changes relative to that level contribute to consumers' well being.
- The habit level could also be interpreted as a required "subsistence" level. As consumption comes close to subsistence, the agent acts in a much more risk averse manner. However, the risk aversion is "endogenous" in that it results from the agent's actions, rather than from the risk aversion parameter.
- Habit can vary over time and be both "internal" (depends on past consumption levels) or "external" (is driven by external factors)
- Volatility strategies are generally unattractive for habit utility investors because when volatility spikes, equities tend to crash, and these are times when wealth creeps closer to habits

Catching/Keeping Up with the Jones

- Utility is set relative to your peers
  - Endowment and mutual fund managers are benchmarked against their peers
  - You want to outperform, or consume more than, your peers. Thus, your utility will be high when you are poor, but everyone is even poorer.
- In asset management, keeping up with the Joneses causes herding
  - If everyone else is doing the volatility strategy, you're going to do it, too.
- Very important utility function in practice



Uncertainty Aversion

- Distinguish between
  - Risk: the probability of a given outcome for a given distribution
  - Uncertainty: there are multiple possible distributions
- Risk aversion measures the pain of bad outcomes over the probability distribution; uncertainty aversion, or ambiguity aversion, measures how an investor does not like the range of possible distributions
- Expected utility is uncertainty aversion with only one probability distribution

Appendix B: Portfolio Calculations

Two-Asset Portfolio Problem with Mean-Variance Utility

- Problem: Two assets r1 and r2 with means m1 and m2 and volatilities s1 and s2, respectively with covariance s12:

$$\max_{w_S, w_B} Er_p - \frac{\gamma}{2}\operatorname{var}(r_p)$$

$$r_p = wr_1 + (1-w)r_2$$

- Setting the FOC equal to zero results in the optimal portfolio weight on the first asset being

$$w^* = \frac{(\mu_1 - \mu_2)/\gamma + \sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

Optimal Portfolios with Risk-Free and Risky Asset

- Math: Maximize utility subject to the CAL constraint

$$\max_{w} U(E[r_p], {}^{2}_{p}) = {}_{w}\left[{}_{p} - \tfrac{1}{2} A \, {}^{2}_{p} \right]$$

$$\text{CAL: } E[r_p] = r_f + w \cdot (E[r_1] - r_f) \text{ and } {}^{2}_{p} = {}^{2} \cdot {}^{2}_{1}$$

- We solve for the optimal weight by finding the derivative with respect to w, set it equal to zero, and solve for w* = optimal weight

- Assume risky asset has mean 11.19% and volatility 19.15% and risk-free rate is 1%

Optimal Portfolios with Risk-Free and Risky Asset

- Optimal holding of risky asset:

$$w^* = \frac{E[r_1] - r_f}{A\sigma_1^2}$$

- Solution for A = 7

$$w^* = \frac{0.1119 - 0.01}{7 \times (0.1915)^2} = 0.397$$

- Solution for A = 2

$$w^* = \frac{0.1119 - 0.01}{2 \times (0.1915)^2} = 1.39$$

- What is the intuition for the solution in terms of
  - Expected returns?
  - Volatility?
  - Risk aversion?

Capital Allocation Line

The Capital Allocation Line

$$\sigma_c^2 = (1-\omega)^2\sigma_f^2 + \omega^2\sigma_p^2 + 2\omega(1-\omega)\sigma_{f,p} \quad = \omega^2\sigma_p^2$$

$$\sigma_c = \omega\sigma_p \qquad \rightarrow \omega = \frac{\sigma_c}{\sigma_p}$$

$$E[r_c] = \omega \times r_f + (1-\omega) \times E[r_p]$$

$$= r_f + \omega(E[r_p] - r_f)$$

$$= r_f + \frac{\sigma_c}{\sigma_p}(E[r_p] - r_f)$$

$$= r_f + \frac{(E[r_p] - r_f)}{\sigma_p}\sigma_c$$