CAPM Notes

Big Question: What will be the relationship(s) between expected return and risk in a CAPM world?

In a CAPM world (i.e., with all the assumptions made in the CAPM):

1. Everyone wants to hold the same portfolio of risky assets.
2. In equilibrium, the optimal portfolio of risky assets must be the market portfolio since everyone wants to hold the same proportion of each risky asset and the total holdings must equal the total value of all risky assets (i.e., supply must equal demand).
3. Investors with different risk aversions can increase or decrease their risk exposure by changing their weight in the risk-free asset versus the risky portfolio.

Let's set up the investor's mean-variance optimization problem and solve for the expected return and risk relationship that will keep the market in equilibrium.

The utility for a mean-variance investor, z, with risk aversion γ_z, is:

$$U = E[r_c] - \frac{\gamma_z}{2}\sigma_c^2, \tag{1}$$

where the expected return on their complete portfolio is:

$$E[r_c] = (\sum_{i=1}^{N} w_i E[r_i]) + (1 - \sum_{i=1}^{N} w_i) r_f$$

and the variance of the portfolio is:

$$\sigma_c^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{i,j}$$

Note: the risk-free asset is... risk-free, that's why there is no risk-free-related term in the variance formula. Also, note the overall weight in the risky portfolio is: $w_{risky} = \sum_{i=1}^{N} w_i$. The overall weight in the risk-free asset is: $w_f = 1 - \sum_{i=1}^{N} w_i$. The investor will solve the following optimization problem:

$$\max_{w_1, w_2, \ldots w_N} U = E[r_c] - \frac{\gamma_z}{2}\sigma_c^2 \tag{2}$$

If we are in equilibrium, no one will want to deviate or change their portfolio. The investors cannot improve their utility by shifting the portfolio weight in an asset just a little bit. For this to be the case, the investor's marginal utility must be zero at each equilibrium weight. We'll take the derivative of the utility function with respect to a weight in some asset A (w_A) and set it equal to zero:

$$\frac{\partial U}{\partial w_A} : E[r_A] - r_f - (\frac{\gamma_z}{2}) \times 2 \times (\sum_{i=1}^{N} w_i) \times \sigma_{A,m} = 0 \tag{3}$$

where $\sigma_{m,A}$ is the covariance of asset A with the market portfolio. Note that this relationship should hold for all assets.

The derivative of the variance component may not be obvious. Here is the solution. The components of variance related to the weight in asset A (w_A) are the weighted variance term and the $2 \times (N-1)$ weighted covariance terms:

$$w_A^2\sigma_A^2 + 2\sum_{i \neq A}^{N} w_A w_i \sigma_{i,A}$$

Take the derivative with respect to w_A and simplify:

$$\begin{aligned}
2w_A\sigma_A^2 + 2\sum_{i \neq A}^{N} w_i \sigma_{i,A} \\
\Rightarrow 2\sum_{i=1}^{N} w_i \sigma_{i,A} \\
\Rightarrow 2\sum_{i=1}^{N} w_i Cov(r_A, r_i) \\
\Rightarrow 2\sum_{i=1}^{N} Cov(r_A, w_i r_i) \\
\Rightarrow 2Cov(r_A, \sum_{i=1}^{N} w_i r_i)
\end{aligned}$$

In equilibrium, every investor holds the market portfolio as their portfolio of risky assets. Therefore, the weights are proportional to the market capitalization weights. Let's scale the weights, so they sum to one.

$$(\sum_{i=1}^{N} w_i) \times 2Cov(r_A, \sum_{i=1}^{N} \frac{w_i}{\sum_{i=1}^{N} w_i} r_i)$$

This scaled portfolio return is just the return on the market portfolio.

$$\begin{aligned}
\Rightarrow 2 \times (\sum_{i=1}^{N} w_i) \times Cov(r_A, r_m) \\
\Rightarrow 2 \times (\sum_{i=1}^{N} w_i) \times \sigma_{A,m}
\end{aligned}$$

Now back to Equation 3. Let's solve this for the marginal investor. The marginal investor is 100% invested in the market portfolio (the sum of weights in risky assets will be equal to one, i.e., $\sum_{i=1}^{N} w_i = 1$ and $w_f = 0$) and we'll label their risk aversion $\bar{\gamma}$.

$$E[r_A] - r_f - (\frac{\bar{\gamma}}{2}) \times 2 \times 1 \times \sigma_{A,m} = 0$$

Simplifying...

$$\frac{E[r_A] - r_f}{\sigma_{A,m}} = \bar{\gamma} \tag{4}$$

This is the reward-to-risk ratio. The risk we care about for each asset is it's covariance with the market portfolio. The intuition is thus: if we increase our weight in an asset, then the variance of our portfolio will change based on the asset's covariance with the market portfolio. The greater the asset's covariance with the market portfolio, the greater the increase in risk in our portfolio. We need to be compensated more for holding assets with a greater covariance with the market portfolio since they contribute more to the overall riskiness of the portfolio.

In equilibrium, this Equation 4 must hold for all assets and portfolios. If it did not hold, everyone would want to deviate and invest more in the assets or portfolios with higher reward-to-risk ratios and hold less of the assets or portfolios with lower reward-to-risk ratios. Prices of the high (low) reward-to-risk ratio assets and portfolios will rise (fall) until the ratio is the same for all assets and portfolios (i.e., we're back in equilibrium).

If all assets and portfolios must have the same ratio, then even the market portfolio will have the same reward-to-risk ratio:

$$\frac{E[r_A] - r_f}{\sigma_{m,A}} = \frac{E[r_B] - r_f}{\sigma_{m,B}} = \frac{E[r_m] - r_f}{\sigma_m^2} = \bar{\gamma}, \tag{5}$$

From here, we can derive the expected return equation for any asset A:

$$\begin{aligned} \frac{E[r_A] - r_f}{\sigma_{m,A}} &= \frac{E[r_m] - r_f}{\sigma_m^2} \\ \rightarrow E[r_A] - r_f &= \frac{\sigma_{m,A}}{\sigma_m^2}(E[r_m] - r_f) \\ \rightarrow E[r_A] - r_f &= \beta_{A,m}(E[r_m] - r_f) \end{aligned}$$

This is the CAPM expected return formula:

$$E[r_A] - r_f = \beta_{A,m}(E[r_m] - r_f) \tag{6}$$

In a CAPM world, every asset's expected return will depend on its beta with the market portfolio. The Security Market Line is the graphical representation of this formula with $E[r_A] - r_f$ on the y-axis and $\beta_{i,m}$ on the x-axis. The slope is the market risk premium.

The CAPM also predicts the expected return on the market portfolio (from Equation 5 and simplifying):

$$E[r_m] - r_f = \bar{\gamma}\sigma_m^2 \tag{7}$$

The Capital Market Line is the capital allocation line for the market portfolio:

$$E[r_c] = r_f + \frac{E[r_m] - r_f}{\sigma_m}\sigma_c = r_f + \bar{\gamma}\sigma_m\sigma_c \tag{8}$$

Lastly, we can solve for each investor's weight in the market portfolio ($w_{market} = \sum_{i=1}^N w_i$) based on their risk aversion and the risk aversion of the marginal investor. From Equation 5, we get $E[r_A] - r_f = \bar{\gamma}\sigma_{A,m}$. Plug this into Equation 3 for investor z and simplify:

$$\begin{aligned} \bar{\gamma}\sigma_{A,m} - \frac{\gamma_z}{2} \times 2 \times (w_{market,z}) \times \sigma_{A,m} &= 0 \\ \rightarrow \bar{\gamma} - \gamma_z \times (w_{market,z}) &= 0 \\ \rightarrow w_{market,z} &= \frac{\bar{\gamma}}{\gamma_z} \end{aligned}$$

To interpret: each investor's allocation to the market is the ratio of the marginal investor's risk aversion to their risk aversion. If the investor is more risk averse than the marginal investor, then they will hold some of the risk-free asset. If the investor is less risk averse than the marginal investor, then they will short the risk-free asset and leverage up their position in the market portfolio.