Arbitrage Pricing Theory Notes

1 Arbitrage Example

Assume you have created a well-diversified portfolio, Portfolio A, with return:

$$r_{A,t} = r_{f,t} + \beta_A(r_{m,t} - r_{f,t}) + \alpha_A \tag{1}$$

where $r_{A,t}$ is the return on portfolio A, $r_{f,t}$ is the risk-free rate, $r_{m,t}$ is the return on the market portfolio, and α_A is a constant greater than zero. Notice, the only uncertainty comes from the return on the market portfolio.

Let's create a pure arbitrage strategy to take advantage of the positive α. A **pure arbitrage strategy** is a strategy that earns a profit without investing any capital or taking on any risk. To eliminate risk, we are going to create a portfolio that mimics the risk exposure of Portfolio A. To do this, we will use the risk-free asset and the market portfolio. The mimicking portfolio will have a return of:

$$r_{mim,t} = w_f r_{f,t} + w_m r_{m,t} \tag{2}$$

where w_f and w_m are the weights in the risk-free and market portfolio, respectively. We need to solve for the appropriate weights to perfectly hedge our risk in portfolio A. Let's first re-arrange the mimicking portfolio's return:

$$\begin{aligned} r_{mim,t} &= w_f r_{f,t} + w_m r_{m,t} \\ &= (1 - w_m) r_{f,t} + w_m r_{m,t} \\ &= r_{f,t} + w_m(r_{m,t} - r_{f,t}) \end{aligned}$$

From here, it's easy to see that setting $w_m = \beta_A$, will create a portfolio with returns that perfectly mimic portfolio A (except for the α). The return on the mimicking portfolio becomes:

$$r_{mim,t} = r_{f,t} + \beta_A(r_{m,t} - r_{f,t}) \tag{3}$$

Portfolio A has identical returns to the mimicking portfolio except it earns an additional constant α. Now, let's create a zero net investment portfolio. A **zero net investment portfolio** is a portfolio in which the investor does not contribute any capital. The investor goes long and short the same amount (we're assuming there are no margin requirements or costs). The portfolio weights will sum to zero since we're not contributing any capital. Because portfolio A has a higher return due to the positive α we want to long portfolio A (weight equal to 100%) and short the mimicking portfolio (weight equal to -100%). The return on the zero net investment portfolio will be:

$$100\% \times r_{A,t} - 100\% \times r_{mim,t} = r_{f,t} + \beta_A(r_{m,t} - r_{f,t}) + \alpha_A - [r_{f,t} + \beta_A(r_{m,t} - r_{f,t})] = \alpha \tag{4}$$

We have created a pure arbitrage strategy that earns a riskless profit of α on every dollar long and short without investing our own capital. This is the basic idea underlying most active investor strategies. You find two assets or portfolios with similar risk exposures, go long the higher return asset/portfolio and short (i.e., take an off-setting position in) the lower return asset/portfolio. In reality, investors are unable to execute a pure arbitrage strategy due to margin requirements, fees, transaction costs, or because most portfolios will have some idiosyncratic risk as well that cannot be perfectly hedged. Most investors will instead be exposed to some risk. Most "arbitrage" strategies are really risk arbitrage strategies (we call these risk arbitrage strategies to distinguish them from pure arbitrage strategies).

The arbitrage pricing theory assumes that pure arbitrage opportunities will be competed away immediately, so there shouldn't be alpha on well-diversified portfolios. The return on portfolio A must then be:

$$r_{A,t} = r_{f,t} + \beta_A(r_{m,t} - r_{f,t}) \tag{5}$$

with expected return:

$$E[r_{A,t}] = r_{f,t} + \beta_A(E[r_{m,t}] - r_{f,t}) \tag{6}$$

The main implication is that two portfolios with identical risk exposures should have the same expected return. This seems to hold in most developed markets. If two identical portfolios were priced differently smart investors would take advantage of the mispricing immediately.

2 Arbitrage Pricing Theory

The arbitrage pricing theory makes three assumptions:

1. Security returns can be described by a factor model
2. There are sufficient securities to diversify away idiosyncratic risk
3. No arbitrage

Assumption 1 means that the return on any asset i can be described as follows:

$$r_{i,t} = E[r_{i,t}] + \sum_{j=1}^{N} \beta_{i,j}\hat{f}_{j,t} + \epsilon_{i,t} \tag{7}$$

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where $r_{i,t}$ is the return on asset i, $E[r_{i,t}]$ is the expected return on asset i, $\beta_{i,j}$ is asset i's factor loading on systematic risk factor j, $\hat{f}_{j,t}$ is the mean zero deviation of the factor f from it's expected value (i.e., shock to the factor), and $\epsilon_{i,t}$ is the mean zero idiosyncratic risk of asset i, which is uncorrelated across assets.

To derive the formula for asset expected returns, we will set up a general arbitrage strategy, then assume that the profits on the arbitrage strategy must be zero (by assumption 3). We will create the arbitrage strategy in three steps:

1. Create a portfolio that diversifies away idiosyncratic risk
2. Create a factor mimicking portfolio that hedges the systematic risk in our portfolio
3. Create a zero net investment portfolio

2.1 Step 1: Diversify away idiosyncratic risk

Assumption 2 says that there are enough assets to combine together to make a well-diversified portfolio (a portfolio that is not exposed to idiosyncratic risk). The return on a portfolio P with a large number of M assets will be:

$$\begin{aligned} r_{p,t} &= \sum_{i=1}^{M} w_i(E[r_{i,t}] + \sum_{j=1}^{N} \beta_{i,j}\hat{f}_{j,t} + \epsilon_{i,t}) \\ &= \sum_{i=1}^{M} w_i(E[r_{i,t}]) + \sum_{i=1}^{M} w_i(\sum_{j=1}^{N} \beta_{i,j}\hat{f}_{j,t}) + \sum_{i=1}^{M} w_i(\epsilon_{i,t}) \end{aligned}$$

By the law of large numbers, the idiosyncratic risk term will go to zero. Leaving us with:

$$r_{p,t} = \sum_{i=1}^{M} w_i(E[r_{i,t}]) + \boxed{\sum_{i=1}^{M} w_i(\sum_{j=1}^{N} \beta_{i,j}\hat{f}_{j,t})}$$

Letting $\beta_j = \sum_{i=1}^{M} w_i \beta_{i,j}$ and $E[r_{p,t}] = \sum_{i=1}^{M} w_i E[r_{i,t}]$ we can simplify the portfolio return equation:

$$r_{p,t} = E[r_{p,t}] + \sum_{j=1}^{N} \beta_j \hat{f}_{j,t}$$

Now, assume that we can create a portfolio for each risk factor, $\hat{f}_{j,t}$, that perfectly tracks its realizations, each tracking portfolio will have return:

$$r_{j,t} = E[r_{j,t}] + \hat{f}_{j,t}$$

We can combine these factor tracking portfolios into a mimicking portfolio that has identical factor risk exposure as any portfolio P. The weight in each factor tracking portfolio will be $w_j = \beta_j$ with the weight in the risk-free rate of $w_f = 1 - \sum_{i=1}^{N} \beta_j$. The return on the mimicking portfolio will be:

$$r_{mim,t} = w_f r_{f,t} + \sum_{j=1}^{N} w_j (r_j)$$

$$r_{mim,t} = (1 - \sum_{j=1}^{N} \beta_j) r_{f,t} + \sum_{j=1}^{N} \beta_j (E[r_{j,t}] + \hat{f}_{j,t})$$

$$r_{mim,t} = r_{f,t} + \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t} + \hat{f}_{j,t})$$

$$r_{mim,t} = r_{f,t} + \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) + \sum_{j=1}^{N} \beta_j \hat{f}_{j,t}$$

Now, create a zero net investment arbitrage opportunity with the portfolio P and the mimicking portfolio that is long P ($w_p = 1$) and short the mimicking ($w_{mim} = -1$), and simplify:

$$\begin{aligned} 1 \times r_{p,t} - 1 \times r_{mim,t} &= \{E[r_{p,t}] + \sum_{j=1}^{N} \beta_j \hat{f}_{j,t}\} - \{r_{f,t} + \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) + \sum_{j=1}^{N} (\beta_j \hat{f}_{j,t})\} \\ &= E[r_{p,t}] - r_{f,t} - \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) \end{aligned}$$

The return on this strategy is a constant (there are no random variables (risk)). The difference in returns must be equal to zero otherwise we will have a pure arbitrage opportunity (assumption 3 rules out arbitrage). Therefore, we can describe the risk premium on any well-diversified portfolio P as:

$$E[r_{p,t}] - r_{f,t} = \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) \tag{8}$$

This equation tells us that the risk premium on any security is a function of the portfolio's loading on each factor (β_j) and the risk premium of each factor ($E[r_{j,t}] - r_{f,t}$).

2.2 Implications and Limitations of the APT

1. The expected return on a well-diversified portfolio is:

$$E[r_{p,t}] - r_{f,t} = \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) \tag{9}$$

2. The expected return on individual assets or not well-diversified portfolios may have α (i.e., differ from the expected return equation above).

$$E[r_{p,t}] - r_{f,t} = \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) + \alpha \tag{10}$$

This is because you cannot create a pure arbitrage opportunity for less well-diversified positions since you will still be exposed to idiosyncratic risk. For example, the return on the "arbitrage" strategy long asset i and short the factor mimicking portfolio will be:

$$\begin{aligned} r_{i,t} - r_{mim,t} = \{E[r_{i,t}] + \sum_{j=1}^{N} \beta_{i,j} \hat{f}_{j,t} + \epsilon_{i,t}\} - \{r_{f,t} + \sum_{j=1}^{N} \beta_{i,j}(E[r_{j,t}] - r_{f,t}) + \sum_{j=1}^{N} (\beta_{i,j} \hat{f}_{j,t})\} \\ = E[r_{p,t}] - r_{f,t} - \sum_{j=1}^{N} \beta_j (E[r_{j,t}] - r_{f,t}) + \epsilon_{i,t} \end{aligned}$$

The idiosyncratic risk $\epsilon_{i,t}$ remains, which means this is not a pure arbitrage strategy.

3. The α on individual assets should be unbiased, with αs equally likely to be positive or negative. Otherwise, we could create a well-diversified portfolio with α.

4. The APT doesn't provide any economic intuition about the systematic risk factors (e.g., value, growth, inflation?) or the magnitude of each factor's risk premium. Discovering which factors are priced and estimating their risk premiums is left to the empiricists.

5. How should we think about the long-short (zero net investment) factors like value-growth (HML) in the Fama-French 3 factor model in the context of APT? Note that the HML factor is $r_{value} - r_{growth}$, which can be re-written as $[r_{value} - r_f] - [r_{growth} - r_f]$. SMB can be expanded in a similar manner. Therefore, long-short zero net investment factors are special cases where we are long and short the same amount in two related systemic risk portfolios. We can re-write the Fama-French Model:

$$\begin{aligned} E[r_{p,t}] - r_{f,t} = \beta_{MKT}(E[r_{MKT,t}] - r_{f,t}) + s_i E[SMB] + h_i E[HML] \\ = \beta_{MKT}(E[r_{MKT,t}] - r_{f,t}) + s_i(E[r_{small}] - r_f) - s_i(E[r_{big}] - r_f) + h_i(E[r_{value}] - r_f) - h_i(E[r_{growth}] - r_f) \end{aligned}$$

The FF3 model is just one possible version of the more general APT formula.