Consumption CAPM and SDF Notes

1 Consumption CAPM

Assume you have a representative agent with utility at time t that is the following concave function of consumption at time t (C_t):

$$U(C_t) = C_t - \frac{a}{2}C_t^2 \tag{1}$$

This investor lives for two periods and consumes in both periods (C_t and C_{t+1}). The investor has wealth at time t of W_t. Each dollar the investor does not consume at time t is saved (total savings of S_t) and earns a (random) return, r_{t+1}. We will assume the investor generates random additional wealth at time $t+1$ of d_{t+1}, so $W_{t+1} = S_t(1 + r_{t+1}) + d_{t+1}$. They will consume all of their wealth at time $t+1$ (no point in saving if it's your last period), so $W_{t+1} = C_{t+1}$.

At time t, the investor maximizes expected utility over their life time by choosing consumption at time t:

$$\begin{aligned} \max_{C_t} U(C_t) + \frac{1}{1+p}E[U(C_{t+1})] \\ \text{s.t } S_t = W_t - C_t \\ W_{t+1} = C_{t+1} = S_t \times (1 + r_{t+1}) + d_{t+1}, \end{aligned}$$

where W_t is wealth at time t, C_t is consumption at time t and S_t is savings at time t. Consumption tomorrow is equal to the amount saved today times the rate of return. Note $\frac{1}{1+p} < 1$ is a time discount.

Utility will be maximized when marginal utility is equal to zero. Find the F.O.C. and re-arrange:

$$U'(C_t) - \frac{1}{1+p}E[(1 + r_{t+1})U'(C_{t+1})] = 0$$

$$\rightarrow U'(C_t) = \frac{1}{1+p}E[(1 + r_{t+1})U'(C_{t+1})] \tag{2}$$

This is called the Euler equation in economics. It says that at the optimum I'm indifferent between the increase in utility from an additional unit of consumption today and the expected increase in utility next period from delaying a unit of consumption today.

Using E[XY] = E[X]E[Y] + Cov(X,Y), where X and Y are random variables. Expand the Euler equation:

$$U'(C_t) = \frac{1}{1+p}\{E[1 + r_{t+1}]E[U'(C_{t+1})] + Cov(1 + r_{t+1}, U'(C_{t+1}))\}$$

Noting $U'(C_{t+1}) = 1 - aC_{t+1}$, gives us:

$$U'(C_t) = \frac{1}{1+p}\{E[1 + r_{t+1}]E[U'(C_{t+1})] - aCov(r_{t+1}, C_{t+1})\}$$

Solving for expected return:

$$E[1 + r_{t+1}] = \frac{1}{E[U'(C_{t+1})]}\{(1 + p)U'(C_t) + aCov(r_{t+1}, C_{t+1})\} \tag{3}$$

This equation should hold for any asset i in the economy:

$$E[1 + r_{i,t+1}] = \frac{1}{E[U'(C_{t+1})]}\{(1 + p)U'(C_t) + aCov(r_{i,t+1}, C_{t+1})\} \tag{4}$$

If the Euler equation did not hold for a particular asset, then the agent would have an incentive to buy or sell some of the asset, changing its price until the Euler equation held. For example, if the expected return was too low, the agent would sell the asset, pushing its price down and increasing its expected return. This would continue until the Euler equation held.

This equation tells us a lot. The expected return on an asset is increasing in our time discount (p), increasing in the marginal utility of current consumption (the more I value an extra unit of consumption today, the higher the expected return I demand), decreasing in my expected marginal utility tomorrow (if I think I'll value an additional unit of capital tomorrow more, I need a lower expected return to save today), and increasing in the covariance between the asset's return and next period's consumption (i.e., I like assets that hedge my consumption risk by having negative covariances). The last part is crucial. It tells us which risks we should care about as an investor (covariance with marginal utility which is determined by our level of consumption).

Let's introduce a risk-free asset which, by definition, doesn't covary with our consumption:

$$1 + r_{f,t+1} = \frac{1 + p}{E[U'(C_{t+1})]}\{U'(C_t)\} \tag{5}$$

Plug this into the expected return equation and simplify:

high marginal utility

$\Downarrow$

$$E[r_{i,t+1}] - r_{f,t+1} = \frac{aCov(r_{i,t+1}, C_{t+1})}{E[U'(C_{t+1})]} \tag{6}$$

weight $\uparrow$ $\Leftarrow$ bad time

$\Downarrow$

low risk premium

This equation tells us the risk premium on an asset is driven by its covariance with future consumption. Assets that payoff when consumption is low (i.e., returns are high when consumption is low) are preferred because they hedge our consumption risk.

Now, let's substitute $C_t(1 + g_{C,t+1})$ for C_{t+1}, where $g_{C,t+1}$ is consumption growth and simplify:

$$E[r_{i,t+1}] - r_{f,t+1} = \frac{aCov(r_{i,t+1}, C_t(1 + g_{C,t+1}))}{E[U'(C_{t+1})]}$$

$$E[r_{i,t+1}] - r_{f,t+1} = \frac{a \times C_t \times Cov(r_{i,t+1}, g_{C,t+1})}{E[U'(C_{t+1})]}$$

Divide and multiply by the variance of consumption growth, σ_C^2, and re-arrange:

$$E[r_{i,t+1}] - r_{f,t+1} = \frac{Cov(r_{i,t+1}, g_{C,t+1})}{\sigma_C^2} \frac{aC_t\sigma_C^2}{E[U'(C_{t+1})]} = \beta_{i,C}\lambda_C \tag{7}$$

Where $\beta_{i,C}$ is the loading of each asset on consumption risk and λ_C is the price of risk.

Empirically, estimating $\beta_{i,C}$ seems possible, but λ_C would be very difficult to estimate in its current form. Let's solve for a more intuitive price of risk. Assume that you can create a portfolio of assets with returns that mimic consumption growth. In other words, this portfolio's return, $r_{C,t+1}$, will track variation in $g_{C,t+1}$ perfectly (i.e., $Cov(r_{C,t+1}, g_{C,t+1}) = \sigma_C^2$), the expected excess return on that portfolio must be:

$$E[r_{C,t+1}] - r_{f,t+1} = \beta_{C,C}\lambda_C$$

and because $\beta_{C,C} = 1$, we get

$$E[r_{C,t+1}] - r_{f,t+1} = \lambda_C$$

Substitute for λ_C,

$$E[r_{i,t+1}] - r_{f,t+1} = \beta_{i,C}(E[r_{C,t+1}] - r_{f,t+1}) \tag{8}$$

This is a version of the consumption-CAPM. We dislike assets that covary more with our consumption and therefore demand a higher risk-premium on these assets.

2 Stochastic Discount Factor

2.1 Example

The intuition gathered from the CAPM and C-CAPM is that we value an additional dollar or additional unit of consumption more in bad times than in good times. In other words, we put more weight on payoffs in bad times than good times.

Let's look at a simple example of how weighting the payoffs based on the state of the world affects the price we are willing to pay. Assume we have two assets (A and B) that have similar expected payoffs, but they pay \$1 in different states of the world. Asset A pays \$1 in the good state of the world and zero otherwise, while B pays \$1 in the bad state of the world and zero otherwise. Let's pre-specify weights for each state of the world (i.e., how much we value \$1 in that state of the world). 0.6 for payouts in the good state and 0.9 for payouts in the bad state. You can think of these weights as the marginal utility of \$1 in that state of the world. We value \$1 more in the bad state than the good state.

Asset	State	Probability	Payoff	Weight (m_{t+1})
A	Good	0.50	1	0.6
	Bad	0.50	0	0.9
B	Good	0.50	0	0.6
	Bad	0.50	1	0.9

How much should we be willing to pay for each asset? Each asset has a 50% chance of paying out \$1 with an expected payoff of \$0.50, but this doesn't take into account the state of the world in which we receive the \$1 payout. The price we should be willing to pay is the weighted expected value:

$$Price_{i,t} = E[m_{t+1} \times Payoff_{i,t+1}] \tag{9}$$

For asset A, the price is \$0.30. For asset B, the price is \$0.45. The expected return on asset A is 66% and on asset B is 11.11%. Because asset B pays us more in bad times, we are willing to pay a higher price and demand a lower expected return.

2.2 SDF

More generally, if there are no arbitrage opportunities in the market, then it can be shown that there exists a stochastic discount factor (m_{t+1}, which we'll write as m for brevity) that prices all assets:

$$Price_{i,t} = E[m \times Payoff_{i,t+1}] \tag{10}$$

The stochastic discount factor gives a weight to the payoff in each possible state of the world. Typically, bad states will get more weight because we value an additional \$1 more in these states.

We can re-write Equation 10 as:

$$1 = E[m \times (1 + r_{i,t+1})] \tag{11}$$

This is a very general asset pricing equation. Most asset pricing models are captured by this general equation.

For example, take the very general Euler equation:

$$U'(C_t) = \frac{1}{1+p} E[(1 + r_{i,t+1})U'(C_{t+1})]$$

Reorganize:

$$1 = E[\{\frac{1}{1+p} \frac{U'(C_{t+1})}{U'(C_t)}\}(1 + r_{i,t+1})] \tag{12}$$

Re-write:

$$1 = E[m \times (1 + r_{i,t+1})], \tag{13}$$

where $m = \frac{1}{1+p}\frac{U'(C_t)}{U'(C_{t+1})}$ is the *stochastic discount factor*.

How about the CAPM? A special case where m is linear in the market return:

$$\boxed{m = a + b \times r_m,}$$

where

$$a = \frac{1}{1+r_f} + \frac{E[r_m](E[r_m] - r_f)}{(1+r_f)\sigma_m^2}$$

and

$$b = -\frac{E[r_m] - r_f}{(1+r_f)\sigma_m^2}$$

The CAPM is a single-factor model. We can extend the SDF to allow for multiple factors ($F = f_1, f_2, f_3, \ldots, f_k$):

$$m = a + b_1 f_1 + b_2 f_2 + b_3 f_3 + \ldots + b_k f_k \tag{14}$$

and allow for non-linearities. This is a very flexible approach to asset pricing.

2.3 Re-Arranging the SDF formula

We can write Equation 11 (here, $r_{i,t+1} = r_i$ for brevity) as :

$$1 = E[m]E[1 + r_i] + Cov(m, r_i)$$

$$\rightarrow E[1 + r_i] = \frac{1}{E[m]} - \frac{Cov(m, r_i)}{E[m]} \tag{15}$$

$1 = E[m(1+r_i)]$

$1 = E[m]E[1+r_i] + cov(m, r_i)$

$E[m]$

This must hold for the risk-free asset, which gives us:

$$1 + r_f = \frac{1}{E[m]}$$

$E[1+r_i] = 1+r_f - \frac{cov(m, r_i)}{E[m]}$

Substitute $1 + r_f$ for $\frac{1}{E[m]}$ in Equation 15:

$$E[1 + r_i] = 1 + r_f - \frac{Cov(m, r_i)}{E[m]}$$

$$\rightarrow E[r_i] - r_f = -\frac{Cov(m, r_i)}{E[m]}$$

$E[r_i] = r_f - \frac{cov(a+br_m, r_i)}{a+bE[r_m]}$

$= r_f - \frac{b\, cov(r_m, r_i)}{a+bE[r_m]}$

$= \frac{cov(m, r_i)}{\sigma_m^2} \frac{-\sigma_m^2}{E[m]}$

Multiply and divide by variance of the SDF:

$$E[r_i] - r_f = \frac{Cov(m, r_i)}{\sigma_m^2}\frac{-\sigma_m^2}{E[m]}$$

We can describe the risk premium of any asset as:

$$E[r_i] - r_f = \beta_{i,m}\lambda_m \tag{16}$$

where $\beta_{i.m}$ is the beta of the asset with respect to the SDF:

$$\beta_{i,m} = \frac{Cov(m, r_i)}{\sigma_m^2}$$

and λ_m is the price of risk:

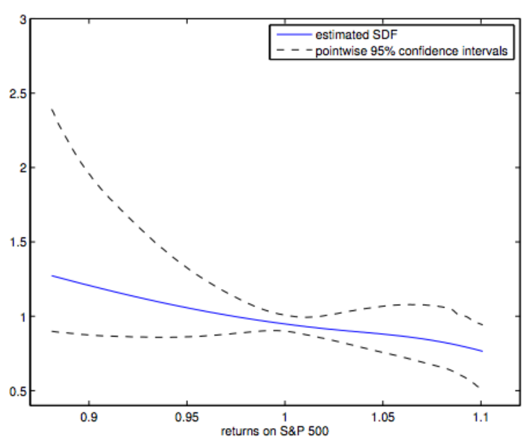
$$\lambda_m = \frac{-\sigma_m^2}{E[m]}$$

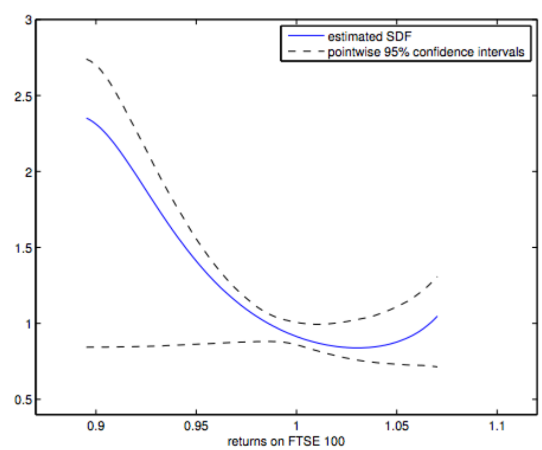
The SDF gives different weights to returns in different states of the world. High (low) m is bad (good) times. Assets that have higher payoffs when m is high ($Cov(m, r_i) > 0$) hedge against bad times, have higher betas, and have lower expected returns.

In the consumption-CAPM, bad times are when $\frac{U'(C_{t+1})}{U'(C_t)}$ (remember: marginal utility is high when consumption is low). In the CAPM, bad times are when the return on the market portfolio is low. Modern academic asset pricing is focused on coming up with the correct representation of the SDF (what are good and bad times for the marginal investor?). In industry, bad times may be when the financial sector is constrained like when the aggregate equity of primary dealers (e.g., large, important financial institutions) is too low and its difficult to get funding (see "Intermediary Asset Pricing: New Evidence from Many Asset Classes" by He, Kelly and Manela (*The Journal of Financial Economics*, 2017)). The stochastic discount factor approach may seem purely academic, but its not. Factor models that describe the SDF are now pervasive in industry (e.g., the CAPM, Fama-French 3 factor model, etc.).

2.4 The Shape of the SDF

How do investors actually weight different states of the world? What is the relationship between m and the market return? We can use option prices to estimate the SDF. Because option prices give us prices for payoffs in different states (returns on the index) we can back out implied weights (i.e., an implied m). (We must make an assumption about the underlying return distribution though.) Most theories would predict that states with lower market returns receive higher weights and that the weights are decreasing monotonically as returns increase. Most empirical studies do not find this predicted monotonicity, instead they find significant non-monotonicity. Here is a recent attempt by Linn, Shive and Shumway (2017) to estimate the SDF with respect to returns on major stock indices:





The shape of the S&P 500 SDF is in line with predictions. Here, the FTSE SDF displays some non-monotonicity although its insignificant.