Mean-Variance Investing Notes

An investor's optimal portfolio is the portfolio that maximizes their expected utility. For a mean-variance investor, the optimal portfolio solves:

$$\max_{[w_1\ w_2\ \dots\ w_N]} E[r_p] - \frac{\gamma}{2}\sigma_p^2$$
$$\text{subject to} \sum_{i=1}^{N} w_i = 1$$

where w_i is the weight in asset i. The expected return of the portfolio is:

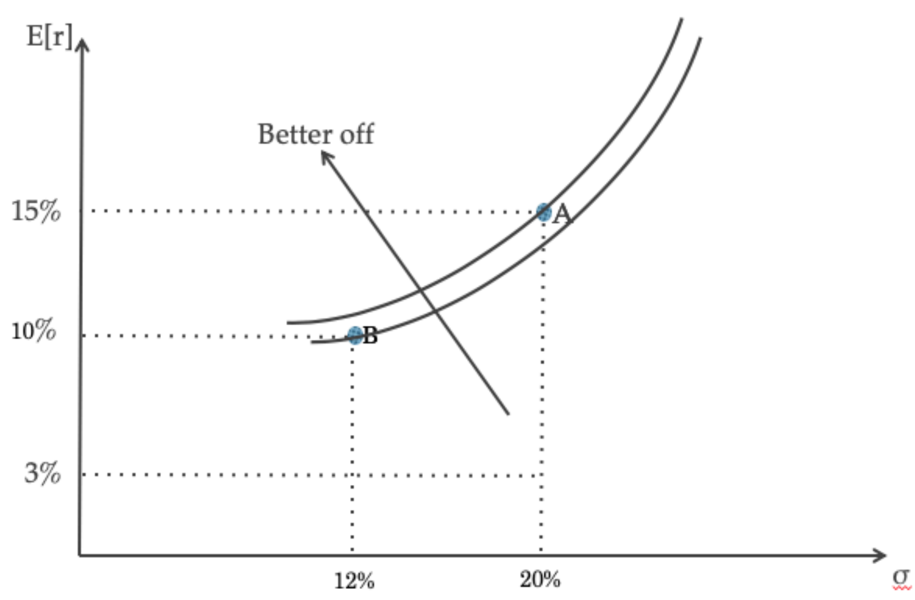
$$E[r_P] = \sum_{i=1}^{N} w_i E[r_i] = w_1 E[r_1] + w_2 E[r_2] + w_3 E[r_3] + \ldots + w_N E[r_N]$$

and the variance of portfolio P is:

$$\begin{aligned}
\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j Cov(r_i, r_j) = \\
w_1 w_1 Cov(r_1, r_1) + w_1 w_2 Cov(r_1, r_2) + \ldots + w_1 w_N Cov(r_1, r_N) \\
w_2 w_1 Cov(r_2, r_1) + w_2 w_2 Cov(r_2, r_2) + \ldots + w_2 w_N Cov(r_2, r_N) \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
w_N w_1 Cov(r_N, r_1) + w_N w_2 Cov(r_N, r_2) + \ldots + w_N w_N Cov(r_N, r_N)
\end{aligned}$$

0.1 Choosing between risky portfolios

No risk-free asset:
If we are deciding between two risky portfolios, say portfolio A and portfolio B, **as our only investment** we will select the risky portfolio that maximizes are utility. Choose A if $U_A = E[r_A] - \frac{\gamma}{2}\sigma_A^2 > U_B = E[r_B] - \frac{\gamma}{2}\sigma_B^2$. Choose B if $U_B > U_A$.



With a risk-free asset:
Typically, we assume we'll have access to a risk-free asset to combine with a risky portfolio. The combination of a risk-free asset and risky portfolio is called the **complete portfolio**. The expected return and variance of the complete portfolio are:

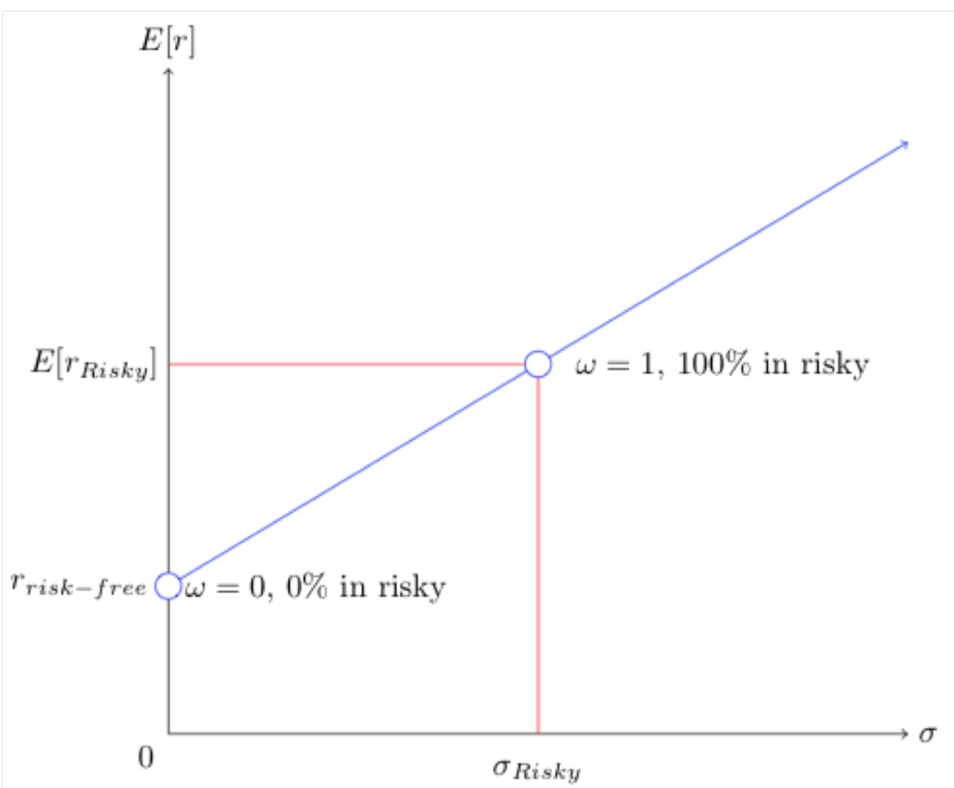
$$E[r_c] = w_p E[r_p] + w_f r_f = w_p E[r_p] + (1 - w_p) r_f = r_f + w_p(E[r_p] - r_f)$$
$$\sigma_c^2 = w_p^2 \sigma_p^2 + w_f^2 \sigma_f^2 + 2 w_p w_f \sigma_{p,f} = w_p^2 \sigma_p^2$$

Since the risk-free asset is not risky, its variance is zero and its covariance with other assets is zero. The **capital allocation line** for a risky portfolio P describes the combinations of expected return and risk we can achieve by combining the risk-free asset with portfolio P. The formula for the capital allocation is:

$$E[r_c] = r_f + \frac{E[r_p] - r_f}{\sigma_p} \sigma_c = r_f + Sharpe_p \times \sigma_c$$

Solution:

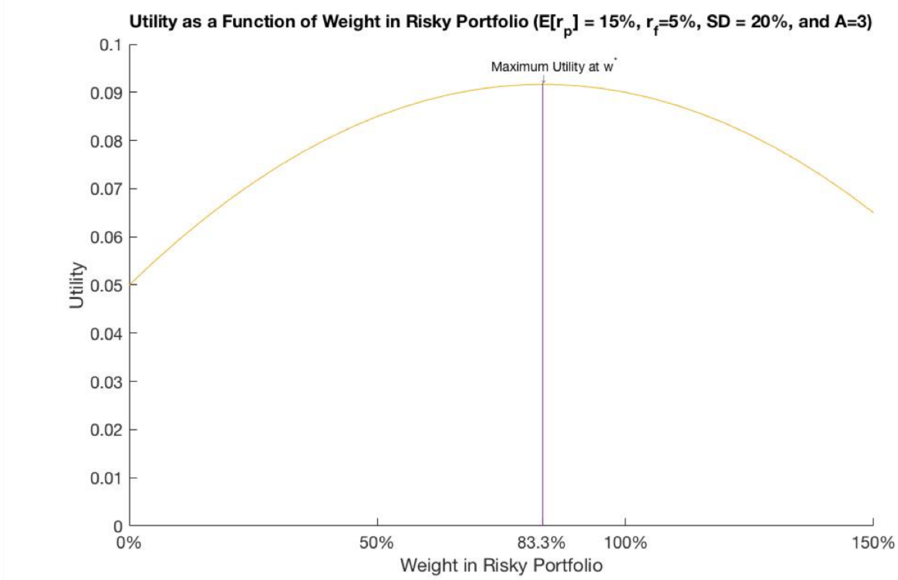
$$E[r_c] = r_f + w_p(E[r_p] - r_f)$$
$$\text{Since } \sigma_c^2 = w_p^2 \sigma_p^2 \Rightarrow \sigma_c = w_p \sigma_p \Rightarrow w_p = \frac{\sigma_c}{\sigma_p}$$
$$\Rightarrow E[r_c] = r_f + \frac{\sigma_p}{\sigma_c}(E[r_p] - r_f)$$
$$\text{Re-arranging... } E[r_c] = r_f + \frac{E[r_p] - r_f}{\sigma_p} \sigma_c$$



The optimal complete portfolio for an investor will choose a weight in the risky portfolio (w_p) to maximize their utility:

$$
\begin{aligned}
\max_{w_p} E[r_c] - \frac{\gamma}{2}\sigma_c^2 \\
\Rightarrow r_f + w_p(E[r_p] - r_f) - \frac{\gamma}{2}(w_p\sigma_p)^2 \\
F.O.C.: \quad (E[r_p] - r_f) - \gamma w_p \sigma_p^2 = 0 \\
w_p^* = \frac{E[r_p] - r_f}{\gamma\sigma_p^2}
\end{aligned}
$$

Example utility curve as function of weight in risky portfolio:

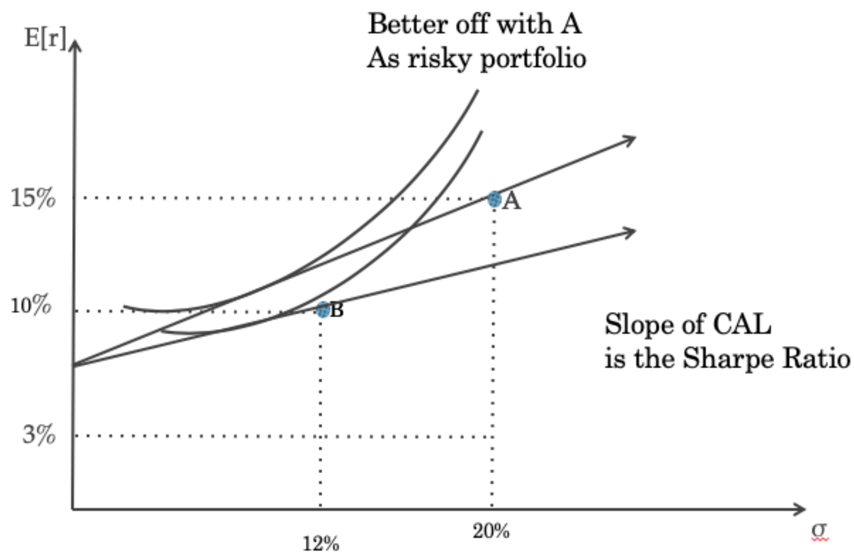


Maximum achievable utility is:

$$
\begin{aligned}
U^* = r_f + w_p^*(E[r_p] - r_f) - \frac{\gamma}{2}(w_p^*\sigma_p)^2 \\
= r_f + \frac{E[r_p] - r_f}{\gamma\sigma_p^2}(E[r_p] - r_f) - \frac{\gamma}{2}(\frac{E[r_p] - r_f}{\gamma\sigma_p^2}\sigma_p)^2 \\
= r_f + \frac{1}{\gamma}(\frac{E[r_p] - r_f}{\sigma_p})^2 - \frac{1}{2\gamma}(\frac{E[r_p] - r_f}{\sigma_p})^2 \\
= r_f + \frac{1}{2\gamma}Sharpe_p^2
\end{aligned}
$$

Our maximum achievable Sharpe ratio is increasing in the magnitude of the Sharpe ratio.
When selecting a risky portfolio to combine with the risk-free asset: (1) choose the highest Sharpe ratio risky portfolio as the optimal risky portfolio, and (2) combine the highest Sharpe ratio risky portfolio optimally with the risk-free asset (w_p^* weight in risky and $1 - w_p^*$ weight in risk-free) to maximize utility. Notice the first step is the same for everyone. No matter your risk aversion, the optimal risky portfolio is the highest Sharpe ratio portfolio. This result, that the selection of the optimal portfolio can decomposed into two steps, is called *two-fund separation.*

Example:



One additional note on the CAL... Typically, we cannot borrow at the risk-free rate. This will create a kink in the CAL when the weight in the risky portfolio exceeds 100% (weight in the risk-free falls below zero). The kink is because we are borrowing at a higher rate than the risk-free rate, so the slope of the line is: $\frac{E[r_p]-r_b}{\sigma_p}$, where r_b is the cost to borrow. Here is an illustrative example: