

36225
Probability

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Discrete Random Variables

Chapter 1

1.1 Probability and Mass functions

A **discrete random variable** X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined to be a mapping $X : \Omega \mapsto \mathbb{R}$ such that

the image $X(\Omega)$ is a countable subset of \mathbb{R} and
 $\{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F}$

A discrete random variable X takes value in \mathbb{R} , but we cannot predict the actual value of X with certainty since the underlying experiment involves chance. Instead, we would like to measure the probability that X takes a given value, x say.

The most interesting things about a discrete random variable are the values that it may take and the probability associated with these values. If X is a discrete random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then its **image** $\text{im}X$ is the image of Ω under X , that is, the set of values taken by X .

Henceforth, we abbreviate events of the form $\{\omega \in \Omega : X(\omega) = x\}$ to the more convenient form $\{X = x\}$.

Definition 1.1.1: The (probability) mass function

the pmf of the discrete random variable X is the function $p_X : \mathbb{R} \mapsto [0, 1]$ defined by

$$p_X(x) = \mathbb{P}(X = x)$$

Thus, $p_X(x)$ is the probability that the mapping X takes the value x . Note that $\text{im}X$ is countable for any discrete random variable X , and

$$p_X(x) = 0 \quad \text{if } x \notin \text{im } X, \quad (1.1)$$

$$\sum_{x \in \text{im } X} p_X(x) = \mathbb{P}\left(\bigcup_{x \in \text{im } X} \{\omega \in \Omega : X(\omega) = x\}\right) = \mathbb{P}(\Omega) = 1 \quad (1.2)$$

$$0 \leq p_X(x) \leq 1 \quad \forall x \quad (1.3)$$

1.2 Random

Definition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over \mathbb{R} (or \mathbb{C}). A norm on V is function $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- ① $\|x\| = 0 \iff x = 0 \quad \forall x \in V$
- ② $\|\lambda x\| = |\lambda| \|x\| \quad \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C}), x \in V$
- ③ $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.

• Same definition works with V a vector space over \mathbb{C} (again $\|\cdot\| \rightarrow \mathbb{R}_{\geq 0}$) where ② becomes $\|\lambda x\| = |\lambda| \|x\|$ $\forall \lambda \in \mathbb{C}, x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 1.2.1 (p -Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_m|^p \right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \dots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \rightarrow \infty$ (\mathbb{R}^m with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_m|\}$

For $m = 1$ these p -norms are nothing but $|x|$. Now exercise

Question 1

Prove that triangle inequality is true if $p \geq 1$ for p -norms. (What goes wrong for $p < 1$?)

Solution: For Property ③ for norm-2

When field is \mathbb{R} :

We have to show

$$\begin{aligned} \sum_i (x_i + y_i)^2 &\leq \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2 \\ \Rightarrow \sum_i (x_i^2 + 2x_i y_i + y_i^2) &\leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right]} + \sum_i y_i^2 \\ \Rightarrow \left[\sum_i x_i y_i \right]^2 &\leq \left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right] \end{aligned}$$

So in other words prove $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

Note:-

- $\|x\|^2 = \langle x, x \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle \cdot, \cdot \rangle$ is \mathbb{R} -linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned} \langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle \end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is \mathbb{C} :

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geq 0$

1.3 Algorithms

Algorithm 1: what

Input: This is some input

Output: This is some output

/ This is a comment */*

```
1 some code here;
2  $x \leftarrow 0$ ;
3  $y \leftarrow 0$ ;
4 if  $x > 5$  then
5   |  $x$  is greater than 5 ;                                // This is also a comment
6 else
7   |  $x$  is less than or equal to 5;
8 end
9 foreach  $y$  in 0..5 do
10  |  $y \leftarrow y + 1$ ;
11 end
12 for  $y$  in 0..5 do
13  |  $y \leftarrow y - 1$ ;
14 end
15 while  $x > 5$  do
16  |  $x \leftarrow x - 1$ ;
17 end
18 return Return something here;
```
