21241
Matrices and transformation

ZhenTing Liu

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Chapter 1

1.1 Eigentheory

Let's begin by talking about why we care about this topic

Consider the following System

Question 1

A particle jumps between A and B, suppose this particle starts at A, what are the probability that it ends up at A after

1. one step
2. n steps
3. ∞ steps

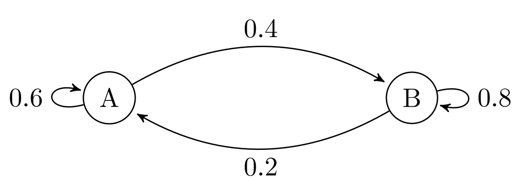
Solution: let $M = \begin{array}{cc} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} A \\ B \end{matrix} & \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \end{array}$ and $P_0 = \begin{array}{cc} \begin{matrix} A \\ B \end{matrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$ where M is called a stochastic Matrix, row 1 of P_0 is 1 because the particle is initially at A. So to answer (1), we simply multiply M and P_0, let $P_1 = MP_0$, we have $P_1 = \begin{array}{cc} \begin{matrix} A \\ B \end{matrix} & \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \end{array}$

For (2), notice that we just multiply all the "states" together

$$\begin{aligned} P_n &= MP_{n-1} && (1.1) \\ &= M(MP_{n-2}) && (1.2) \\ &= \underbrace{M \dots M}_{n \text{ times}} P_0 = M^n P_0 && (1.3) \end{aligned}$$

The answer to (3) is a bit tricky, which we will discuss after seeing some tools to simplify our computation

Claim 1.1.1 $\exists$ invertible matrix X and diagnol matrix Λ such that $M = X\Lambda X^{-1}$



$$
\begin{aligned}
M &= X\Lambda X^{-1} \\
M^2 &= M(M) \\
&= (X\Lambda X^{-1})(X\Lambda X^{-1}) \\
&= X\Lambda^2 X^{-1}
\end{aligned}
$$

hence we must have

$$M^n = X\Lambda^n X^{-1}$$

This result is very useful to us because we essentially avoided calculating very large power of M, instead, we take the diagonal matrix Λ to its nth power

Note:-

Let $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ Recall that exponent of a diagonal matrix is just its entries to the same power, so

$$\Lambda^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

Now, going back to (3), the computation is skipped here as we haven't talked must about it yet but we will discuss more later on. Λ for the problem happens to be $\begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}$ and X is $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

$$\lim_{n\to\infty} M^n P_0 = \lim_{n\to\infty} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & 0.4^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.67 \end{pmatrix}$$

So the probablity of the particle being at A after infinitely many steps is roughly 0.5

Definition 1.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 2

Is the set x−axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 1.1.2 Topology

Topology is cool

Example 1.1.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup_{x \in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

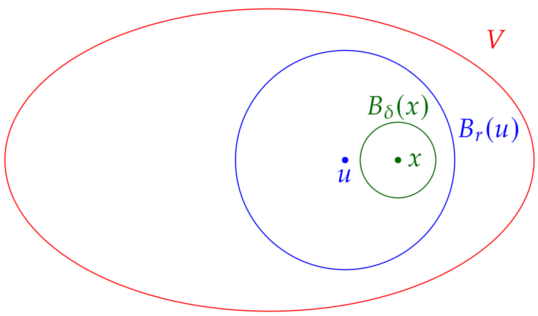
Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 1.1.1

If $x \in$ open set V then $\exists\ \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

😄

Corollary 1.1.1

By the result of the proof, we can then show...

Lenma 1.1.1

Suppose $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 1.1.1

$1 + 1 = 2$.

1.2 Random

Definition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \longrightarrow \mathbb{R}_{\geqslant 0}$ satisfying

① $\|x\| = 0 \iff x = 0\ \forall\ x \in V$

② $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\ x \in V$

③ $\|x + y\| \leqslant \|x\| + \|y\|\ \forall\ x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \longrightarrow \mathbb{R}_{\geqslant 0}$) where ② becomes $\|\lambda x\| = |\lambda|\|x\|$ $\forall\ \lambda \in \mathbb{C},\ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 1.2.1 ($p-$Norm)
$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 3

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property ③ for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2$$

$$\implies \sum_i (x_i^2 + 2x_i y_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_i y_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_i y_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$

- $\langle \cdot, \cdot \rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned} \langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle \end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geqslant 0$

1.3 Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output
/* This is a comment */

some code here;
$x \leftarrow 0$;
$y \leftarrow 0$;
if $x > 5$ **then**
    x is greater than 5 ; // This is also a comment
else
    x is less than or equal to 5;
end
foreach *y in 0..5* **do**
    $y \leftarrow y + 1$;
end
for y *in* 0..5 **do**
    $y \leftarrow y - 1$;
end
while $x > 5$ **do**
    $x \leftarrow x - 1$;
end
return Return something here;