Summer

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Abstract

This is my summer project, a mixture of revision and research.

Contents

1 Real Analysis

1.1 Sequences

We denote set of positive integers as $\mathbb{N}$.

Proposition 1.1:
Let $a_i = (a_i^1, \ldots, a_i^n) \in \mathbb{R}^n$ be a sequence, $\forall n \in \mathbb{N}$. Then $a_i \to a := (a^1, \ldots, a^n) \in \mathbb{R}^n$, as $i \to \infty$ iff $a_i^k \to a^k$ as $i \to \infty$, $\forall k \in \{1, \ldots, n\}$.

Proof. We know that

$$\max_{k=1,\ldots,n} |a^k| \leq \|a\| \leq \sqrt{n} \max_{k=1,\ldots,n} |a^k|.$$

To show this, observe that:

$$\sqrt{\max_k ((a^k)^2)} \leq \sqrt{(a^1)^2 + \cdots + (a^n)^2} \leq \sqrt{n \max_k ((a^k)^2)}$$

$$\Rightarrow \sqrt{(\max_k |a^k|)^2} \leq \|x\| \leq \sqrt{n(\max_k |a^k|)^2}$$

$$\Rightarrow \max_k |a^k| \leq \|x\| \leq \sqrt{n} \max_k |a^k|$$

First assume that $a_i \to a$ as $i \to \infty$. Then $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that $\|a_i - a\| < \epsilon$ whenever $i \geq N$.

Then $\forall k = 1 \ldots n$, $|a_i^k - a^k| \leq \max_k |a_i^k - a^k| \leq \|a_i - a\| < \epsilon$ whenever $i \geq N$, so $a_i^k \to a^k$ as $i \to \infty$.

Conversely, assume that $a_i^k \to a^k$ as $i \to \infty$, $\forall k \in \{1, \ldots, n\}$. Then $\forall \epsilon > 0$, $\exists N^k \in \mathbb{N}$ such that $|a_i^k - a^k| \leq \frac{\epsilon}{\sqrt{n}}$ whenever $i \geq N^k$, $\forall k = 1, \ldots, n$. Let

$$N = \max_{k=1,\ldots,n} N^k.$$

Then $\max_k |a_i^k - a^k| < \frac{\epsilon}{\sqrt{n}}$ whenever $i \geq N$, so $\|a_i - a\| \leq \sqrt{n} \max_k |a_i^k - a^k| < \epsilon$. $\square$

Proposition 1.2:
The limit of a sequence in $\mathbb{R}^n$ is unique.

Proof. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence with $x_i \in \mathbb{R}^n$. Suppose for a contradiction that $x_i \to a$ and $x_i \to b$ as $i \to \infty$, with $a \neq b$.

Then $\forall \epsilon > 0$, $\exists N_a, N_b \in \mathbb{N}$ such that $\|x_i - a\| < \frac{\epsilon}{2}$ whenever $i \geq N_a$, and $\|x_i - b\| < \frac{\epsilon}{2}$ whenever $i \geq N_b$. Let

$$N = \max\{N_a, N_b\}.$$

Then whenever $i \geq N$, we have:

$$\begin{aligned}\|a - b\| &= \|a - x_i + x_i - b\| \\ &\leq \|a - x_i\| + \|x_i - b\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

And since ϵ can be made arbitrarily small, we find that $a = b$, which is a contradiction. $\square$

Theorem 1.3: [Monotone convergence]
An increasing and bounded above sequence is convergent. Likewise, a decreasing and bounded below sequence is convergent.

Proof. Suppose (a_i) is bounded above and increasing. Then the set $A = \{a_i : i \in \mathbb{N}\}$ has a supremum, say $a = \sup A$. I claim that $a_i \to a$ as $i \to \infty$.

Proof of claim. Fix $\forall \epsilon > 0$. Then $\exists a_N$, with $N \in \mathbb{N}$ such that

$$a - \epsilon < a_N \leq a < a + \epsilon.$$

Since (a_i) is increasing, $a_i \geq a_N$ when $i \geq N$, and since it is also bounded above by a, we have

$$a - \epsilon < a_N \leq a \leq a_i < a + \epsilon,$$

and so

$$|a - a_i| < \epsilon.$$

So $a_i \to a$ as $i \to \infty$. Simililary, we can show that if (a_i) is bounded below and decreasing, $a_i \to \inf\{a_i : i \in \mathbb{N}\}$ as $i \to \infty$. □

Lemma 1.4:
Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Then $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Proof. Since (a_i) is bounded, $\exists K \geq 0$ such that $a_i \in I_0 := [-K, K]\,\forall i \in \mathbb{N}$. Then one of $[-K, 0]$, $[0, K]$ contains infinitely many elements of (a_i). Choose the leftmost one, and call it I_1. Repeat this inductively, such that $I_j := [c_j, d_j]$ contains infinitely many elements of (a_i). We then inductively define

$$i_j = \min_{i > i_{j-1}} i : a_i \in I_j.$$

In other words, a_{i_j} is the first element of (a_i) which lies in I_j. Now, since

$$I_j \subset I_{j-1} \subset \cdots \subset I_1 \subset I_0,$$

The sequences (c_j) and (d_j) are monotically increasing and decreasing, respectively, and since they are bounded, they both converge, so say that

$$c_j \to c \in [-K, K] \text{ as } j \to \infty,$$

$$d_j \to d \in [-K, K] \text{ as } j \to \infty.$$

Since $d_j - c_j = \text{length}(I_j) = 2^{1-j}K \to 0$ as $j \to \infty$, we have that $c = d$, and since $c_j \leq a_{i_j} \leq d_j$, we also have that $a_{i_j} \to c = d$ as $j \to \infty$. □

Corollary:
Let $(a_i)_{i\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^n$. Then (a_i) has a convergent subsequence.

Proof. Since (a_i) is bounded, and the component sequences (a_i^k) are bounded $\forall k = 1, \ldots, n$.

Then by the previous lemma, $(a_i^1), \dots, (a_i^n)$ have subsequences $(a_{i_{j_1}}^1), \dots, (a_{i_{j_n}}^n)$ converging to $a^1, \dots, a^n$ respectively. Let

$$\begin{aligned} J_1 &= \{j_{1_1}, j_{1_2}, \dots\}, \\ J_2 &= \{j_{2_1}, j_{2_2}, \dots\}, \\ &\vdots \\ J_n &= \{j_{n_1}, j_{n_2}, \dots\}, \end{aligned}$$

and $J = J_1 \cap \dots \cap J_n$.

Then each component subsequence $(a_j^k)_{j \in J}$ converges to a^k, so $(a_j)_{j \in J}$ converges to a. □

Theorem 1.5: [Bolzano-Weierstrass]
Let $E \subset \mathbb{R}^n$ be compact and $(a_i)_{i \in \mathbb{N}}$ a sequence in E. Then (a_i) has a subsequence converging to a point in E.

Proof. Since (a_i) is bounded, it has a subsequence $(a_{i_j}) \to a$ as $j \to \infty$.

And since E is closed, and a a limit point of E, $a \in E$. □

1.2 Continuous Functions

Definition 1.6: Continuity
Let $A \subset \mathbb{R}^n$ be open. We say $f : A \to \mathbb{R}^m$ is *continuous* on A if, $\forall a \in A$, $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\|f(x) - f(a)\| < \epsilon$, whenever $\|x - a\| < \delta$.

Generally, we can think of continuous functions are those for which points that are ‘near’ are mapped to by points that are also ‘near’, and so showing that a function is continuous amounts to finding a positive relationship between the definition of ‘nearness’ in the domain and the definition of ‘nearness’ in the range (these are not generally the same). In this case, points $x, y \in \mathbb{R}^n$ are near if the distance between them, $\|x - y\|$, is small.

In a more general topological sense, we say that two points are near, in a certain capacity, if they are both elements of the same open set. Thus, very vaguely, the more open sets the points are both elements of, the nearer they are. In Euclidean space, open sets are defined as arbitary unions of open balls $B_\epsilon(x)$, for $x \in \mathbb{R}^n$ and $\epsilon > 0$. Thus, $x = y$ precisely when $y \in B_\epsilon(x)$ (and $x \in B_\epsilon(y)$) $\forall \epsilon > 0$.

This is the Euclidean case of a general property called non-seperatedness, which we use to describe points that both lie in every open set. Since they share every open set in common, their nearness is maximal, and so we can think of them as essentially being indistinguishable - i.e. the same point. We will revist this later, when we discuss seperatedness conditions.

Proposition 1.7:
For $f : A \to \mathbb{R}^m$, we write $f(x) = (f_1(x), \dots, f_m(x))$.

Then f is continuous if and only if each of the component functions $f_i : A \to \mathbb{R}$ are continuous.

Proof. First assume f is continuous at $a \in A$. Then $\forall \epsilon > 0\ \exists \delta > 0$ such that $\|f(x) - f(a)\| < \epsilon$ whenever $\|x - a\| < \delta$. Then

$$\begin{aligned} |f_i(x) - f_i(a)| &\leq \max_i |f_i(x) - f_i(a)| \\ &\leq \|f(x) - f(a)\| \\ &< \epsilon \quad \forall i = 1, \ldots, m. \end{aligned}$$

Hence each f_i is continuous at a. Now assume that each f_i is continuous at $a \in A$.

Then $\forall \epsilon > 0$, $\exists \delta_i > 0$ such that $|f_i(x) - f_i(a)| < \frac{\epsilon}{m\sqrt{m}}$ whenever $\|x - a\| < \delta_i$, $\forall i = 1, \ldots, m$. Now let $\delta = \min\{\delta_1, \ldots, \delta_m\}$

Then when $\|x - a\| < \delta$, $\|x - a\| < \delta_i$, so $|f_i(x) - f_i(a)| < \frac{\epsilon}{m\sqrt{m}}$ $\forall i = 1, \ldots, m$. Then

$$\begin{aligned} \|f_i(x) - f_i(a)\| &\leq \sqrt{m} \max_i |f_i(x) - f_i(a)| \\ &\leq \sqrt{m} \sum_{i=1}^{m} |f_i(x) - f_i(a)| \\ &< \epsilon \end{aligned}$$

□

Example:
Every linear function is continuous.

Proposition 1.8:
Let $A \subset \mathbb{R}^n$ be open. Then $f : A \to \mathbb{R}^m$ is continuous at $x \in A$ if and only if $f(x_i) \to f(x)$ as $i \to \infty$ for any sequence (x_i) converging to x.

Proof. Let (x_i) be a sequence in A converging to $x \in A$, and assume $f : A \to \mathbb{R}^m$ to be continuous at x. Then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ whenever $0 < \|x - y\| < \delta$.

Since $x_i \to x$, $\exists N \in \mathbb{N}$ such that $\|x_i - x\| < \delta$ whenever $i \geq N$, and hence $\|f(x_i) - f(x)\| < \epsilon$, so we have that $f(x_i) \to f(x)$ as $i \to \infty$.

Now assume the converse holds, and for a contradiction that f is not continuous. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$ $\|f(x) - f(x_i)\| \geq \epsilon$ whenever $0 < \|x - x_i\| < \delta$, and in particular whenever $0 < \|x - x_i\| < \frac{1}{i}$, $\forall i \in \mathbb{N}$.

Then the sequence (x_i) converges to x, so by assumption the image sequence $(f(x_i))$ converges to $f(x)$.

Thus $\exists N \in \mathbb{N}$ such that $\|f(x_i) - f(x)\| < \epsilon$ whenever $i \geq N$, which is a contradiction. Hence, f is continuous at x. □

Theorem 1.9: [Extreme Value Theorem]
Let $E \subset \mathbb{R}^n$ be compact and non-empty. Let $f : E \to \mathbb{R}$ be continuous. Then f achieves its maximum and minimum values on E.

Proof. We prove that f reaches its maximum on E. First suppose for a contradiction that f is unbounded above on E. Then $\forall M \geq 0$, and in particular $\forall i \in \mathbb{N}$, $\exists x_i \in E$ such that $f(x_i) > i$. In other words, $f(x_i) \to \infty$ as $i \to \infty$.

Then since E is compact, it is bounded, so the sequence $(x_i)_{i\in\mathbb{N}}$ is bounded. Then by the Bolzano-Weierstrass theorem it has a subsequence $(x_{i_j})_{j\in\mathbb{N}}$ convergent to $x \in E$.

Now since f is continuous on E, it is continuous at x, so $f(x_{i_j}) \to f(x)$ as $j \to \infty$. Since E is compact, it is closed, so $f(x) \in E$, and therefore $f(x)$ is finite. This is a contraction since f was assumed to diverge to infinity for any subsequence of (x_i), including (x_{i_j}).

So f is bounded above on E. Now by completeness of $\mathbb{R}$, $f(E)$ has a supremum, say

$$M = \sup_{x\in E} f(x).$$

We show that that f reaches its maximum on E, i.e. that $M \in f(E)$, or that $\exists p_+ \in E$ such that $f(p_+) = M$. Now $\forall i \in \mathbb{N}$, $\exists x_i \in E$ such that

$$M - \frac{1}{i} < f(x_i) \leq M.$$

Then clearly $f(x_i) \to M$ as $i \to \infty$, so by the Bolzano-Weierstrass theorem $(x_i)_{i\in\mathbb{N}}$ has a subsequence $(x_{i_j})_{j\in\mathbb{N}}$ convergent in to some $p_+ \in E$. Since f is continuous at p_+, $f(x_{i_j}) \to f(p_+)$ as $j \to \infty$. Hence $f(p_+) = M$, so f achieves its maximum on E.

Now in a similar fashion we prove that f reaches its minimum on E. Suppose that f is unbounded below on E. Then $\forall i \in \mathbb{N}$, $\exists x_i \in E$ such that $f(x_i) < -i$. So $f(x_i) \to -\infty$ as $i \to \infty$.

Now, the sequence $(x_i)_{i\in\mathbb{N}}$ is bounded. Then by the Bolzano-Weierstrass theorem it has a subsequence $(x_{i_j})_{j\in\mathbb{N}}$ convergent to $x \in E$. Since f is continuous at x, $f(x_{i_j}) \to f(x)$ as $j \to \infty$, and so $f(x) \in E$ (so $f(x)$ is finite). This is a contraction since f was assumed to diverge to negative infinity for any subsequence of (x_i), including (x_{i_j}).

So f is bounded below on E. Now by completeness of $\mathbb{R}$, $f(E)$ has an infimum, say

$$m = \inf_{x\in E} f(x).$$

We show that that f reaches its minimum on E, i.e. that $m \in f(E)$, or that $\exists p_- \in E$ such that $f(p_-) = m$. Now $\forall i \in \mathbb{N}$, $\exists x_i \in E$ such that

$$m \leq f(x_i) < m + \frac{1}{i}.$$

Then clearly $f(x_i) \to m$ as $i \to \infty$, so by the Bolzano-Weierstrass theorem $(x_i)_{i\in\mathbb{N}}$ has a subsequence $(x_{i_j})_{j\in\mathbb{N}}$ convergent in to some $p_- \in E$. Since f is continuous at p_-, $f(x_{i_j}) \to f(p_-)$ as $j \to \infty$. Hence $f(p_-) = m$, so f achieves its minimum on E. $\square$

1.3 Uniform Continuity and Convergence

Definition 1.10: Uniformly Continuous
Let $A \subset \mathbb{R}^n$ be open. We say $f : A \to \mathbb{R}^m$ is *uniformly continuous* on A if, $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\forall x, y \in A$, $\|f(x) - f(y)\| < \epsilon$ whenever $\|x - y\| < \delta$.

Remark: Clearly uniform continuity implies continuity.

Proposition 1.11:
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Then f is uniformly continuous.

Proof. Fix $\forall \epsilon > 0$. Then $\forall x, y \in \mathbb{R}$ with $x < y$, by the Mean Value Theorem $\exists c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$. By assumption $\exists M \geq 0$ such that $|f'(c)| = \frac{|f(y)-f(x)|}{|y-x|} \leq M$.

Set $\delta = \frac{\epsilon}{M+1}$. Then

$$\begin{aligned}|f(y) - f(x)| &\leq M|y - x| \\ &< \frac{M\epsilon}{M+1} \\ &< \frac{M\epsilon}{M} \\ &= \epsilon\end{aligned}$$

□

Proposition 1.12:
A continuous function on a compact subset $E \subset \mathbb{R}^n$ is uniformly continuous.

Proof. Let $E \subset \mathbb{R}^n$ be compact, and $f : E \to \mathbb{R}^m$ continuous. We assume for a contradiction that f is not uniformly continuous. Then $\exists \epsilon > 0$ such that $\forall i \in \mathbb{N}$ $\exists x_i, y_i \in E$ with $\|x_i - y_i\| < \frac{1}{i}$ such that $\|f(x_i) - f(y_i)\| \geq \epsilon$.

Then by the Bolzano-Weierstrass theorem, the sequence $(x_i)_{i\in\mathbb{N}}$ has a subsequences $(x_{i_j})_{j\in\mathbb{N}}$ converging to some $x \in E$. Furthermore, the sequence $(y_i)_{i\in\mathbb{N}}$ has a convergent subsequence $(y_{i_j})_{j\in\mathbb{N}}$. We now show that $y_{i_j} \to x$ as $j \to \infty$.

Choose $\epsilon' > 0$. Then $\exists N \in \mathbb{N}$ such that $\|x_{i_j} - x\| < \frac{\epsilon'}{2}$ whenever $i_j \geq N$.

$$\begin{aligned}\|x - y_{i_j}\| &= \|x - x_{i_j} + x_{i_j} - y_{i_j}\| \\ &\leq \|x - x_{i_j}\| + \|x_{i_j} - y_{i_j}\| \\ &< \frac{\epsilon'}{2} + \frac{1}{i_j} \\ &\leq \frac{\epsilon'}{2} + \frac{1}{j} \\ &< \frac{\epsilon'}{2} + \frac{\epsilon'}{2} \\ &= \epsilon'\end{aligned}$$

□

Theorem 1.13: [Heine-Borel]
Let $A \subset \mathbb{R}^n$ be closed and bounded. Then A is compact.

Definition 1.14: Uniform convergence
Let $A \subset \mathbb{R}^n$ and $(f_i)_{i\in\mathbb{N}}$ be a sequence of functions $f_i : A \to \mathbb{R}^m$.

We say $f_i \to f$ *uniformly* to $f : A \to \mathbb{R}^m$ if and only if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|f_i(x) - f(x)\| < \epsilon$ whenever $i \geq N$ and $x \in A$.

Theorem 1.15:
Let $A \subset \mathbb{R}^n$ and $(f_i)_{i \in \mathbb{N}}$ be a sequence of functions $f_i : A \to \mathbb{R}^m$ which converges uniformly to $f : A \to \mathbb{R}^m$ uniformly.

Then if f_i is uniformly continuous $\forall i \in \mathbb{N}$, f is uniformly continuous.

Proof. Fix $\epsilon > 0$. Then since $f_i \to f$ uniformly, $\exists N \in \mathbb{N}$ such that $\|f_i(x) - f(x)\| < \epsilon/3$ whenever $i \geq N$ and $x \in A$.

Also since each f_i is uniformly continuous, $\exists \delta > 0$ such that $\|f_i(x) - f_i(y)\|$ $\forall x, y \in A$. Then using the triangle inequality:

$$\begin{aligned}
\|f(x) - f(y)\| &\leq \|f(x) - f_i(x)\| + \|f_i(x) - f(y)\| \\
&< \epsilon/3 + \|f_i(x) - f_i(y)\| + \|f_i(y) - f(y)\| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon
\end{aligned}$$

$\forall x, y \in A$ such that $\|x - y\| < \delta$. Hence, f is uniformly continuous. □

Corollary:
Now assume that each f_i is just continuous instead. Then f is continuous.

Proof. Fix $a \in A$, and $\epsilon > 0$. Then since $f_i \to f$ uniformly, $\exists N \in \mathbb{N}$ such that $\|f_i(x) - f(x)\| < \epsilon/3$ whenever $i \geq N$ and $x \in A$.

Also since each f_i is continuous, $\exists \delta > 0$ such that $\|f_i(x) - f_i(a)\| < \epsilon/3$ $\forall x \in A$. Then using the triangle inequality:

$$\begin{aligned}
\|f(x) - f(a)\| &\leq \|f(x) - f_i(x)\| + \|f_i(x) - f(a)\| \\
&< \epsilon/3 + \|f_i(x) - f_i(a)\| + \|f_i(a) - f(a)\| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon
\end{aligned}$$

$\forall x \in A$ such that $\|x - a\| < \delta$. Hence, f is continuous. □

1.4 Integration

Definition 1.16: Partitions
Let $[a, b] \subset \mathbb{R}$. We define a *partition*, $\mathcal{P}$, of $[a, b]$ by an ordered tuple

$$\mathcal{P} = (x_0, x_1, \ldots, x_{k-1}, x_k),$$

such that $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$.

We call any interval of the form $[x_j, x_{j+1}]$ a *subinterval* of the partition $\mathcal{P}$. The length of this subinterval is $\Delta x_j = x_{j+1} - x_j$, and we define

$$\text{mesh}(\mathcal{P}) := \max_{j=0,\ldots,k-1} \Delta x_j.$$

Finally, for any two partitions $\mathcal{P}$ and $\mathcal{Q}$, we say $\mathcal{Q}$ *refines* $\mathcal{P}$ iff every point in $\mathcal{P}$ lies in $\mathcal{Q}$.

Definition 1.17: Darboux sums
Let $f : [a, b] \to \mathbb{R}$ be bounded, and fix a partition $\mathcal{P}$ of $[a, b]$. Then we define the *upper* and *lower Darboux sums* by

$$U(f, \mathcal{P}) = \sum_{j=0}^{k-1} M_j \Delta x_j,$$

where $M_j = \sup_{x \in [x_j, x_{j+1}]} f(x)$, and

$$L(f, \mathcal{P}) = \sum_{j=0}^{k-1} m_j \Delta x_j,$$

where $m_j = \inf_{x \in [x_j, x_{j+1}]} f(x)$.

Lemma 1.18:
Let $A \subset \mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ be bounded. Then

1. $\sup_{x \in A} -f(x) = -\inf_{x \in A} f(x)$
2. $\inf_{x \in A} -f(x) = -\sup_{x \in A} f(x)$

Lemma 1.19:
Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a, b]$, with $\mathcal{Q}$ a refinement of $\mathcal{P}$. Then

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P}).$$

Proof. Let $\mathcal{Q} = (x_0 = a, x_1, \ldots, x_k = b)$. Let I_j be the subinterval of $\mathcal{P}$ containing $[x_j, x_{j+1}]$, for $j = 0, \ldots, k-1$. Then

$$\inf_{x \in I_j} f(x) \leq \inf_{x \in [x_j, x_{j+1}]} f(x) \leq \sup_{x \in [x_j, x_{j+1}]} f(x) \leq \sup_{x \in I_j} f(x).$$

Then by multiplying by Δx_j and summing over j, we get

$$\sum_{j=0}^{k-1} \Delta x_j \inf_{x \in I_j} f(x) \leq \sum_{j=0}^{k-1} \Delta x_j m_j \leq \sum_{j=0}^{k-1} \Delta x_j M_j \leq \sum_{j=0}^{k-1} \Delta x_j \sup_{x \in I_j} f(x).$$

We can rewrite the two outer sums as

$$\sum_{j=0}^{k-1} \Delta x_j \inf_{x \in I_j} f(x) = \sum_{\text{subints. of } \mathcal{P},\ I} \left[\inf_{x \in I} f(x) \sum_{j \text{ s.t. } I = I_j} \Delta x_j \right] = L(f, \mathcal{P}),$$

$$\sum_{j=0}^{k-1} \Delta x_j \sup_{x \in I_j} f(x) = \sum_{\text{subints. of } \mathcal{P},\ I} \left[\sup_{x \in I} f(x) \sum_{j \text{ s.t. } I = I_j} \Delta x_j \right] = U(f, \mathcal{P}).$$

Hence,

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P}).$$

□

Corollary:

$$L(f,\mathcal{P}) \le U(f,\mathcal{P}')$$

for any two partitions $\mathcal{P}$ and $\mathcal{P}'$.

Proof. Let $\mathcal{Q}$ be a common refinement of $\mathcal{P}$ and $\mathcal{P}'$. Then

$$L(f,\mathcal{P}) \le L(f,\mathcal{Q}) \le U(f,\mathcal{Q}) \le U(f,\mathcal{P}').$$

$\square$

Definition 1.20: Darboux integrals
We define the following sets:

$$\begin{aligned} U_f &= \{U(f,\mathcal{P}') : \mathcal{P}' \text{ is a partition of } [a,b]\}, \\ \text{and } L_f &= \{L(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}. \end{aligned}$$

Let $\mathcal{Q}$ be any parition of $[a,b]$. Then U_f is bounded below by $L(f,\mathcal{Q})$ and L_f is bounded above by $U(f,\mathcal{Q})$. Thus we can define *upper* and *lower Darboux integrals* by:

$$\overline{\int_a^b} f(x)\mathrm{d}x = \inf U_f,$$

$$\underline{\int_a^b} f(x)\mathrm{d}x = \sup L_f.$$

If these two are equal, then we say f is *integrable*, and we let

$$\int_a^b f(x)\mathrm{d}x = \overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x.$$

Proposition 1.21:
A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if $\forall \epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

Proof. We assume that $\forall \epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$. Then

$$L(f,\mathcal{P}) \le \underline{\int_a^b} f(x)\mathrm{d}x \le \overline{\int_a^b} f(x)\mathrm{d}x \le U(f,\mathcal{P}),$$

so

$$\overline{\int_a^b} f(x)\mathrm{d}x - \underline{\int_a^b} f(x)\mathrm{d}x \le U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon,$$

and so

$$\overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x$$

so f is integrable.

Conversely, assume that f is integrable. Fix $\forall \epsilon > 0$. Then since $\underline{\int_a^b} f(x)\mathrm{d}x = \sup L_f$ and $\overline{\int_a^b} f(x)\mathrm{d}x = \inf U_f$, $\exists \mathcal{P}, \mathcal{P}'$ as paritItions of $[a, b]$, such that

$$\underline{\int_a^b} f(x)\mathrm{d}x \geq L(f, \mathcal{P}) > \underline{\int_a^b} f(x)\mathrm{d}x - \frac{\epsilon}{2},$$

$$\overline{\int_a^b} f(x)\mathrm{d}x \leq U(f, \mathcal{P}') < \overline{\int_a^b} f(x)\mathrm{d}x + \frac{\epsilon}{2}.$$

Since f is integrable, this simplifies to

$$L(f, \mathcal{P}) + \frac{\epsilon}{2} > 0,$$

$$U(f, \mathcal{P}') - \frac{\epsilon}{2} < 0.$$

We let $\mathcal{Q}$ be the common refinement of $\mathcal{P}$ and $\mathcal{P}'$. Then

$$U(f, \mathcal{Q}) - L(f, \mathcal{Q}) \leq U(f, \mathcal{P}') - L(f, \mathcal{P}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Proposition 1.22:
Every continuous function is integrable.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Since $[a, b]$ is compact, f is uniformly continuous, so $\forall x, y \in [a, b]$, $\forall \epsilon > 0 \exists \delta > 0$ such that $|f(x) = f(y)| < \epsilon / b - a$ when $|x - y| < \delta$. Now fix $\forall \epsilon > 0$, let $calP$ be a partition of $[a, b]$ with $k + 1$ uniformly spaced points, and such that $\mathrm{mesh}(calP) < \delta$.

Then $\forall x, y \in [x_j, x_{j+1}]$, we have $|f(x) - f(y)| < \epsilon / b - a$, for $j = 0, 1, \ldots, k - 1$. In particular $M_j - m_j < \epsilon / b - a$, so $\Delta x_j (M_j - m_j) < \Delta x_j \epsilon / b - a$. We then sum over j, noting that $\Delta x_j = b - a/k$

$$\sum_{j=0}^{k-1} \Delta x_j (M_j - m_j) < \sum_{j=0}^{k-1} \frac{\epsilon(b - a)}{k(b - a)}$$

$$\sum_{j=0}^{k-1} \Delta x_j M_j - \sum_{j=0}^{k-1} \Delta x_j m_j < \frac{k\epsilon}{k}$$

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

And so f is integrable by the previous proposition. □

Proposition 1.23: [Linearity of the integral]
Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be integrable functions, and $\lambda \in \mathbb{R}$. Then $f + \lambda g$ is inetgrable, and

$$\int_a^b f(x) + \lambda g(x)\, dx = \int_a^b f(x)\, dx + \lambda \int_a^b g(x)\, dx.$$

Proof. □

Proposition 1.24: [Additivity of the integral]

2 Algebra and Topology

2.1 Quotients

Definition 2.1: Binary relation
Let X be a non-empty set. A *binary relation*, $\sim$, on X is a map

$$\cdot \sim \cdot : X \times X \to \{\texttt{True, False}\}.$$

Definition 2.2: Equivalence relation
Let $\sim$ be a binary operation on a non-empty set X. We say $\sim$ is an *equivalence relation* if it is

1. Reflexive: $x \sim x \quad \forall x \in X$
2. Symmetric: $x \sim y \Leftrightarrow y \sim x \quad \forall x, y \in X$
3. Transitive: $x \sim y,\, y \sim z \Rightarrow x \sim z \quad \forall x, y, z \in X$

Proposition 2.3:
Let $\sim$ be an equivalence operation on a non-empty set X. Given an $x \in X$, we define the equivalence class of x, $[x]$, to be the set of elements in X equivalent to x under $\sim$. In other words:

$$[x] = \{y \in X : y \sim x\}.$$

Then the equivalence classes partition X.

Proof. Since every element $x_i \in X$, $i \in I$, is equivalent to itself, $x_i \in [x_i]$, so

$$X = \bigcup_{i \in I} [x_i].$$

So equivalence classes cover X.

Then if $x \nsim y$, $y \nsim x$, so $y \notin [x]$. And if $z \in [y]$, $z \sim y$ so $z \nsim x$, and so $z \notin [x]$. Hence

$$x \nsim y \Leftrightarrow [x] \cap [y] = \varnothing.$$

So equivalence classes partition X. □

Definition 2.4: Quotient set
Let X be a non-empty set, and $\sim$ an equivalence relation on X. We define the *quotient set*, or *quotient space*, $X \,/\!\sim$ to be the set of equivalence classes of X. I.e.

$$X \,/\!\sim\, = \{[x] : x \in X\}.$$

Example: Quotient group
Let $(G, +)$ be a group, and N a normal subgroup of G, where we write xy for $x + y$ $\forall x, y \in G$. We define an equivalence relation $\sim$ on G to be such that

$$x \sim y \text{ if and only if } Nx = Ny.$$

Then equivalence classes $[x]$ are the cosets of N in G, $Nx = \{nx : n \in N\}$, so the quotient set

$$G \,/\!\sim\, = \{Nx : x \in G\}.$$

Equivalently, two elements of G are equivalent if they lie in the same coset. Then we define the *quotient group*

$$\frac{G}{N} = (G/\sim\,, +)$$

where the binary operation $\cdot + \cdot : \frac{G}{N} \times \frac{G}{N} \to \frac{G}{N}$ is defined by

$$(Nx)(Ny) = N(xy).$$

Proposition 2.5:
$\frac{G}{N}$ is indeed a group under $+$.

Proof. We first check that $+$ is well-defined on $\frac{G}{N}$.

Suppose that $Nx = Nx'$, and $Ny = Ny'$. So $xx'^{-1} \in N$ and $yy'^{-1} \in N$. Then $xx'^{-1}yy'^{-1} \in N$ this implies that $xy(x'y')^{-1} \in N$, since N is a normal subgroup.

Then $Nxy = Nx'y'$, which implies that $(Nx)(Ny) = (Nx')(Ny')$.

So $+$ is well-defined. We now check the group axioms:

1. *Closure*: Since G is closed under $+$, $\frac{G}{N}$ is clearly closed under $+$.
2. *Associativity*:

$$\begin{aligned}\forall x, y, z \in G, (Nx)((Ny)(Nz)) &= (Nx)(N(yz)) \\ &= Nx(yz) \\ &= N(xy)z \\ &= (Nxy)(Nz) \\ &= ((Nx)(Ny))(Nz)\end{aligned}$$

3. *Identity*: N is the identity. To see this, observe that $N(Nx) = Nx = (Nx)N$.
4. *Inverses*: Given Nx, $(Nx)^{-1} = Nx^{-1}$. To see this, observe that $(Nx)(Nx^{-1}) = N = (Nx^{-1})(Nx)$.

□

Example: Quotient ring
Let $(R, +, \times)$ be a ring, and I an ideal of R. We can define an equivalence relation, $\sim$, on R by

$$r \sim s \iff I + r = I + s.$$

Then $(R/\sim, +, \times)$ forms a ring under $+$ and $\times$, called the *quotient ring*, R/I.

We can construct the field of complex numbers, $\mathbb{C}$, by taking the following quotient:

Consider $I := (x^2 + 1)\mathbb{R}[x]$ as an ideal of $\mathbb{R}[x]$. We define an equivalence relation $\sim$ on $\mathbb{R}[x]$ by

$$f(x) \sim g(x) \iff I + f(x) = I + g(x) \quad , \forall f(x), g(x) \in \mathbb{R}[x].$$

I claim that the quotient ring $\mathbb{R}[x]/\sim$,

$$\frac{\mathbb{R}[x]}{I} = \{a(I + x) + (I + b) : a, b \in \mathbb{R}\}$$

.

Proof. Fix a coset $I+p(x)$, $p(x) \in \mathbb{R}[x]$. Since $\mathbb{R}$ is a field, $\mathbb{R}[x]$ is a Euclidean domain, so $\exists q(x), r(x) \in \mathbb{R}[x]$, with $deg(r(x)) < deg(x^2+1) = 2$, or else $r(x) = 0$, such that

$$p(x) = q(x)(x^2+1) + r(x).$$

Then we write the quotient ring as

$$\begin{aligned}\frac{\mathbb{R}[x]}{I} &= \{I + p(x) : p(x) \in \mathbb{R}[x]\} \\ &= \{I + q(x)(x^2+1) + r(x) : deg(r) < 2\} \\ &= \{I + r(x) : deg(r) < 2\} \\ &= \{I + (ax+b) : a, b \in \mathbb{R}\} \\ &= \{a(I+x) + (I+b) : a, b \in \mathbb{R}\}\end{aligned}$$

□

Note that $(I+x)^2 = I + x^2 = (I+x^2) + (I+1) - (I+1) = -(I+1)$. This is clearly isomorphic to $\mathbb{C}$, since $i^2 = -1$ when we identify i with $I + x$.

Example: Quotient topology
Let (X, τ_x) be a topological space, Y a set, and $f : X \to Y$ a surjective function. Then a subset $U \subset Y$ is open in Y if and only if $f^{-1}(U)$ is open in X. In other words, this topology, called the *quotient topology*, is the finest topology we can define on Y such that f is continuous, called the *final topology* on Y with respect to f.

The key example here is given by defining an equivalence relation, $\sim$ on X, and then taking $Y = X\,/\sim$. We then define the *natural map*, $p : X \to X\,/\sim$, by $p(x) = [x]\ \forall x \in X$.

Then $(X\,/\sim, \tau)$ is a topological space, where the quotient topology, τ, is defined by

$$\tau = \{U \in X\,/\sim\, : p^{-1}(U) \in \tau_X\}.$$

Note that τ is the final topology on $X\,/\sim$ with respect to p.

Definition 2.6: Separateness conditions

1. Hausdorff: A topological space X is called *Hausdorff* iff $\forall x, y \in X$, $\exists U_x, U_y$ such that $U_x \cap U_y = \varnothing$, where U_x, U_y are open neighbourhoods of x, y respectively.

   This asserts that every two points in X can be 'separated' by open sets - open neighbourhoods that are disjoint. Conversely, X is non-Hausdorff if there exists a pair of points $x, y \in X$ such for every open neighbourhood U_x of x and U_y of y, $U_x \cap U_y \neq \varnothing$.

   For example, if $X = \{x, y, z\}$ and $\tau = \{\varnothing, \{x\}, \{x, y\}, \{x, z\}, X\}$, then X is non-Hausdorff since the open neighbourhoods of x are $\{x\}, \{x, y\}, \{x, z\}$, and the only open neighbourhoods of y, z $\{x, y\}, \{x, z\}$ respectively, which are all clearly non-disjoint.
2. Regular: A topological space X is called *regular* iff for any $x \in X$ and any closed set $V \subset X$, there exists an open neighbourhood U_x of x and an open superset U_V of V, such that $U_x \cap U_V = \varnothing$.
3. Normal: A topological space X is called *normal* iff for any pair of closed sets $V_1, V_2 \subset X$ there exist open supersets U_1, U_2 of V_1, V_2 such that $U_1 \cap U_2 = \varnothing$.

Example:
Let X be a non-Hausdroff topological space. Then we can turn X into a Hausdorff space by considering the equivalence relation $\sim$ on X defined by

$$x \sim y \iff U_x \cap U_y \neq \varnothing$$

for all open neighbourhoods U_x, U_y of x, y respectively.

Then the quotient space $X / \sim$ is Hausdorff when given the quotient topology.

Proof. So $U \subset X / \sim$ is open if and only if $p^{-1}(U)$ is open, where $p : X \to X / \sim$ maps $x \in X$ to the set of points which cannot be separated from x by open sets, denoted $[x]$.

So we fix distinct $[x], [y] \in X / \sim$. Now, $p^{-1}(\{[x]\})$ is open in X, so $\{[x]\}$ is open in $X / \sim$. Similarly, $\{[y]\}$ is open in $X / \sim$. Finally, note that $\{[x]\} \cap \{[y]\} = \varnothing$, and so $X / \sim$ is Hausdorff. □

Definition 2.7: Quotient map
We say $q : X \to Y$ between topological spaces X and Y is a *quotient map* if it is surjective, and $U \subset Y$ is open in Y if and only if $q^{-1}(U) \subset X$ is open in X.

Example:
The natural map $p : X \to X / \sim$ is a quotient map.

Proof. p is surjective, since $\forall [x] \in X / \sim$, $p(x) = [x]$. The second property follows from the definition of p. □

Proposition 2.8:
Let $q : X \to Y$ be a quotient map. Then $f : Y \to Z$ is continuous if and only if $f \circ q : X \to Z$ is continuous.

Proof. If f is continuous, then $f \circ q$ is continuous by composition. Now assume $f \circ q$ to be continuous, and let $U \subset Z$ be open in Z. Then $(f \circ q)^{-1}(U) = q^{-1}(f^{-1}(U))$ is open in X, so $f^{-1}(U)$ is open in Y. □

Proposition 2.9: [Universal property of quotients]
If $f : X \to Y$ is continuous and constant on equivalence classes of X. I.e. $f|_{[x]}$ is constant $\forall [x] \in X / \sim$.

Then there exists a unique continuous map $g : X / \sim \to Y$ such that

$$f = g \circ p.$$

We say that f descends to the quotient.

Proof. So given $x_i \in X$, $f(x) = c_i \ \forall x \in [x_i]$. Then we define $g : X / \sim \to Y$ by $g([x_i]) = c_i$. Clearly this choice of g is unique. We first check this is well-defined:

Suppose $[x_i] = [y_i]$. Then if $y \in [y_i]$, $y \in [x_i]$, so $g([y_i]) = c_i$. Now, $g \circ p(x_i) = g(p(x_i)) = g([x_i]) = c_i = f(x_i)$. Hence $g \circ p = f$.

We now check g is continuous. Since p is a quotient map and $g \circ p$ is continuous, g is continuous by the previous proposition. □

2.2 Group Actions

Definition 2.10: Automorphism group
Given a mathematical structure X, an *automorphism* is an isomorphism, ϕ, from X to itself. Explicitly,

$$\phi : X \to X.$$

Coloquially, automorphisms are invertible mappings that preserve structure. In fact, they form a group under composition, called the *automorphism group* of X, which we denote $\mathrm{Aut}(X)$.

Example:

1. If X is a set with n elements, then
$$\mathrm{Aut}(X) = S_n,$$
the group of permutations of the elements of X.

2. If V is an n-dimensional vector space over a field F, then
$$\mathrm{Aut}(V) = \mathrm{GL}_n(F),$$
the group of invertible $n \times n$ matrices over F, or equivalently the group of invertible linear transformations from V to itself.

3. If (X, τ) is a topological space, then
$$\mathrm{Aut}(X) = \mathrm{Homeo}(X),$$
the group of homeomorphisms of X.

Given a set X, we can think of $\mathrm{Aut}(X)$ as *acting* on X in an invertible way. Explicitly, we describe the *group action* of $\mathrm{Aut}(X)$ on X by the function:

$$\mathrm{Aut}(X) \times X \to X : (\sigma, x) \mapsto \sigma \cdot x, \quad \forall \sigma \in \mathrm{Aut}(x),$$

such that the following hold $\forall x \in X$, where $\sigma \cdot x$ refers to outcome of acting on $x \in X$ by an automorphism $\sigma \in \mathrm{Aut}(X)$, and the identity in $\mathrm{Aut}(X)$ is denoted e:

1. *Identity*: $e \cdot x = x$
2. *Compatibility*: $(\sigma\rho) \cdot x = \sigma \cdot (\rho \cdot x), \quad \forall \sigma, \rho \in \mathrm{Aut}(X)$

In fact, this is a special case of an arbitrary group G acting on X. We specify the action by describing a homomorphism
$$\phi : G \to \mathrm{Aut}(X).$$

Which amounts to assigning an automorphism of X to each element of G in such a way that

1. The identity automorphism in $\mathrm{Aut}(X)$ is assigned to the identity in G,
2. The composition of the two automorphisms assigned to two elements $g, h \in G$ is assigned to the product gh.

Since $\phi(G)$ is a subgroup of $\mathrm{Aut}(X)$, we define the action of G on X by the group action of the homomorphic image of G, in $\mathrm{Aut}(X)$, on X. Specifically:

Definition 2.11: Group action
Let G be a group and X a set. The *(left) group action*, is a function

$$G \times X \to X : (g, x) \mapsto g \cdot x,$$

such that the following hold $\forall x \in X$, where we e is the identity in G:

1. *Identity*: $e \cdot x = x$
2. *Compatibility*: $(gh) \cdot x = g \cdot (h \cdot x) \quad \forall g, h \in G$

Here G is called a *transformation group*, X is called a *G-set*.

Definition 2.12: Orbit
Consider a group G acting on a set X. We define the *orbit*, $G \cdot x$, of $x \in X$ to be the set of elements in X to which x can be mapped to by the elements of G. Explicitly:

$$G \cdot x = \{g \cdot x : g \in G\}.$$

We can define an equivalence relation $\sim$ on X by

$$x \sim y \text{ if and only if } G \cdot x = G \cdot y, \quad x, y \in X.$$

Equivalently, $x \sim y$ if and only if $\exists g \in G$ such that $g \cdot x = y$ (and so $g^{-1} \cdot y = x$). Then the orbits of X, under the action of G, partition X. The quotient set induced by $\sim$ is

$$X / \sim = \{G \cdot x : x \in X\}.$$

This is the set of all orbits of X under the action of G, called the *quotient* of the action and denoted $\frac{X}{G}$.

Definition 2.13: Stabiliser subgroup
Given $g \in G$ and $x \in X$ such that $g \cdot x = x$, we say that g fixes x, in other words g acts as the identity on x. So for all $x \in X$, we define the *stabiliser subgroup*, $G_x \subset G$, of G with respect to x as the set of elements in G that fix x. Explicitly, given x,

$$G_x = \{g \in G : g \cdot x = x\}.$$

Proposition 2.14:
G_x is indeed a subgroup of G.

G_x is generally not a normal subgroup of G. However, given an $x \in X$, we can define an equivalent relation, $\sim$, on G by

$$g \sim h \iff G_x g = G_x h, \quad g, h \in G.$$

We can then form the quotient set $G / \sim$ to be the set of cosets of G_x in G, denoted G/G_x. Explicitly, given $x \in X$,

$$\frac{G}{G_x} = \{G_x g : g \in G\}.$$

Lemma 2.15:
Given $x, y \in X$, and $g \in G$ such that $y = g \cdot x$, the two stabiliser groups G_x, G_y are related by

$$G_y = g^{-1} G_x g.$$

Proof. Fix $x, y \in X$, and let $g \in G$ be such that $y = g \cdot x$, and fix $h \in G_y$. Then

$$\begin{aligned} h \cdot y = y &\iff h \cdot (g \cdot x) = g \cdot x \\ &\iff (hg) \cdot x = g \cdot x \\ &\iff g^{-1} \cdot ((hg) \cdot x) = g^{-1} \cdot (g \cdot x) \\ &\iff (g^{-1}hg) \cdot x = (g^{-1}g) \cdot x = x \\ &\iff g^{-1}hg \in G_x \\ &\iff h \in g^{-1}G_x g \end{aligned}$$

□

Theorem 2.16: [Orbit-stabiliser theorem]
For a fixed $x \in X$, there exists a bijection $\psi : G/G_x \to G \cdot x$.

Proof. We define a map $\psi : G/G_x \to G \cdot x$ by $\psi(G_x g) = g \cdot x$, $\forall g \in G$. First, we show this an injective, well-defined function: Fix $x \in X$, and let $G_x g = G_x h$. Then

$$\begin{aligned} G_x g^{-1} h = G_x &\iff g^{-1}h \in G_x \\ &\iff (g^{-1}h) \cdot x = g^{-1} \cdot (h \cdot x) = x \\ &\iff h \cdot x = g \cdot x \\ &\iff \psi(G_x g) = \psi(G_x h) \end{aligned}$$

Now we show that ψ is surjective: Fix $x \in X$, and $g \cdot x \in G \cdot x$.

$$G_x g = \{hg : h \cdot x = x, \forall h \in G\}$$

□

Corollary: [Burnside's lemma]
The number of orbits of G in X, $|X/G|$, is given by

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where $X^g \subset X$ is the set of elements in X fixed by g.

In fact, if G is finite, then by Lagrange's theorem,

$$|G| = |G \cdot x||G_x|.$$

Definition 2.17: Topological group
A *topological group* is a non-empty set G which is both a group and topological space, in which the group operations of product:

$$G \times G \to G : (x, y) \mapsto xy, \quad x, y \in G$$

and taking inverses:

$$G \to G : x \mapsto x^{-1}, \quad x \in G$$

are continuous, where $G \times G$ is given the product topology.

For example, we can turn every finite group into a topological group by giving it the discrete topology.

Definition 2.18: Homomorphism of topological groups
Let G, H be topological groups. Then a *homomorphism of topological groups* is a structure-preserving map (i.e. a group homomorphism), $\phi : G \to H$, which is continuous. Explicitly,

1. $\phi(xy) = \phi(x)\phi(y), \quad x, y \in G$
2. $\phi^{-1}(U)$ is open in G whenever U is an open subset of H.

An *isomorphism* is a bijective group homomorphism which is also a homeomorphism of the underlying topological spaces.

Definition 2.19: Continuous group action
Let X be a topological space, and G a topological group. Then we say the action of G on X,

$$\phi : G \times X \to X : (g, x) \mapsto g \cdot x,$$

is *continuous* if $\phi^{-1}(U)$ is open in $G \times X$, with respect to the product topology, whenever U is open in X.

We then call X a *G-space*.

2.3 Initial and Final Topologies

Given a set X, the indiscrete topology is the coursest topology we can give to X. However, for a family of topological spaces Y_i indexed by $i \in I$, the functions

$$f_i : X \to Y_i$$

will almost never be continuous. The initial topology is the coursest topology such that they are.

Definition 2.20: Initial topology
Given a family of topological spaces Y_i indexed by $i \in I$, the *initial topology* with respect to the functions

$$f_i : X \to Y_i$$

is that in which open sets $U \subset X$ are those for which $f_i(U)$ is open in Y_i $\forall i \in I$. If X already has a topology, then these functions are continuous if and only if the existing topology is finer than the initial topology. Thus, the initial topology is the coursest topology that can be defined on X such that these functions are continuous.

Explicitly, it is the topology generated by taking all finite intersections and arbitrary unions of sets of the form $f_i^{-1}(U_i)$, where U_i is open in Y_i. Consider the following examples:

1. Let $A \subset X$. The subspace topology $\tau_A = \{A \cap U : U \text{ is open in } X\}$ is the initial topology on A with respect to the canonical inclusion map

$$i : A \hookrightarrow X.$$

2. Given a family of topological spaces X_i indexed by I, let $X = \prod_{i \in I} X_i$. the product topology on X is the initial topology on X with respect to the canonical projections

$$p_i : X \to X_i \quad \forall i \in I.$$

If f is injective, the initial topology can be identified with the *subspace* topology of X, when X is viewed as a subset of Y_i.

Theorem 2.21: [Characteristic property of initial topologies]
Let X, Y, Z_i be topological spaces, with Z_i indexed by $i \in I$ and Y given the initial topology with respect to the functions

$$g_i : Y \to Z_i.$$

Let $f : X \to Y$. Then f is continuous if and only if $g_i \circ f$ is continuous $\forall i \in I$.

In other words, the following diagram is commutative for all $i \in I$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g_i \circ f"'] & Y \arrow[d, "g_i"] \\ & Z_i \end{tikzcd}$$

Proof. Assume first that f is continuous. Then since g_i is continuous $\forall i \in I$, $g_i \circ f$ is continuous.

Now assume $g_i \circ f$ is continuous $\forall i \in I$. Let $U \subset Y$ be open in Y. Then U is a finite intersection and arbitrary union of open sets of the form $g_i^{-1}(V_i)$, where each V_i is open in Z_i.

Then $f^{-1}(U)$ is a finite intersection and arbitrary union of open sets of the form $f^{-1}(g_i^{-1}(V_i)) = (g_i \circ f)^{-1}(V_i)$. This is equal to $(g_i \circ f)^{-1}(W_i)$, where W_i is a finite intersection and arbitrary union of the open sets V_i.

Then since each W_i is open, and each $g_i \circ f$ is continuous, $(g_i \circ f)^{-1}(W_i) = f^{-1}(g_i^{-1}(W_i))$ is open in X. $g_i^{-1}(W_i)$ is a finite intersection and arbitrary union of $g_i^{-1}(V_i)$, which is open in Y. Hence $f^{-1}(U)$ is open in X. □

Definition 2.22: Final topology
Given a family of topological spaces X_i indexed by $i \in I$, the *final topology* with respect to the functions

$$f_i : X_i \to Y$$

is that in which open sets $A \subset Y$ are those for which $f_i^{-1}(U)$ is open in X_i $\forall i \in I$. If Y already has a topology, the f_i are continuous precisely when the existing topology is *courser* than the final topology on Y. Thus, the final topology is the finest topology that can be given to Y such that these functions are continuous.

If each f_i is surjective, we call this topology the *quotient topology* under the equivalence relation $\sim$ defined by

$$x \sim y \iff f_i^{-1}(\{x\}) = f_i^{-1}(\{y\}) \quad \forall x, y \in X_i,\, i \in I.$$

Theorem 2.23: [Universal property of final topologies]
Let X_i, Y, Z be topological spaces, with X_i indexed by $i \in I$ and Y given the final topology with respect to the functions

$$f_i : X_i \to Y$$

Then $g : Y \to Z$ is continuous if and only if $g \circ f_i$ is continuous $\forall i \in I$.

In other words, the following diagram is commutative $\forall i \in I$:

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[dr, "g\circ f_i"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

2.4 Review of point-set topology

Proposition 2.24:
Any compact subset of a Hausdroff space is closed.

Proposition 2.25:
Any closed subset of a compact space is compact.

Proposition 2.26:
Let $f : X \to Y$ be injective and continuous. Then if Y is Hausdorff, X is Hausdorff.

Proposition 2.27:
Let X be a compact topological space, and Y a Hausdorff space. Let $f : X \to Y$ be surjective and continuous. Then f is a quotient map.

Proof. Let $U \subset Y$ be non-empty. Since f is surjective, $f^{-1}(U)$ is non-empty. Suppose $f^{-1}(U)$ is open in X. Then since $X\backslash f^{-1}(U)$ is closed and X is compact, $X\backslash f^{-1}(U)$ is compact. Then since f is continuous, $f(X\backslash f^{-1}(U))$ is compact.

Then since Y is Hausdorff, $f(X\backslash f^{-1}(U)) = Y\backslash f(f^{-1}(U)) = Y\backslash U$ is closed, and hence U is open, so f is a quotient map. $\square$

2.5 Construction of topological spaces

Definition 2.28: Disjoint union
Consider a family of non-empty sets X_i indexed by I. Let $X_i^* = \{(x, i) : x \in X_i\}$. Then we define the *disjoint union* of the X_i as

$$\coprod_{i\in I} X_i = \bigcup_{i\in I} X_i^*.$$

Definition 2.29: Disjoint union topology
Let $X = \coprod_{i\in I} X_i$. For each $i \in I$, we define

$$\phi_i : X_i \to X : x \mapsto (x, i)$$

to be the *canonical injection*. Then we say U is open in X if and only if $\phi_i^{-1}(U)$ is open in X_i, $\forall i \in I$.

Equivalently, U is open in X if and only if $U \cap X_i$ is open in X_i, $\forall i \in I$.

Note that this topology is the finest topology on X such that all of the canonical injections ϕ_i are continuous. In other words, it is the final topology on X with respect to the ϕ_i.

We use this to construct some topological spaces.

1. Let A be a subset of a topological space X, and $q : A \to Y$ be a quotient map. We define an equivalence relation $\sim$ on A by

$$x \sim y \Leftrightarrow q(x) = q(y), \quad \forall x, y \in A.$$

Then we define $X \cup_q Y = X \,/\sim$.

If Y is a singlet set, then we call this *collapsing*, and denote it X/A.

2. Instead let $f : A \to Y$ be continuous, and define $\sim$ on Y by $x \sim f(x) \quad \forall x \in A$. Then we define

$$X \cup_q Y = X \sqcup Y \,/\sim .$$

2.6 Manifolds

Definition 2.30: Chart
A *chart*, (U, ϕ), for a topological space X is an ordered pair of an open subset $U \subset X$ to an open subset of R^n.

Definition 2.31: Atlas
An *atlas* of X is a collection of charts $\{(U_i, \phi_i) : i \in I\}$ such that

$$\bigcup_{i \in I} U_i = X.$$

Definition 2.32: Manifold
If $\phi_i(U_i) = \mathbb{R}^n \; \forall i \in I$, then X is called an n-dimensional *manifold*.

Definition 2.33: Transition map

Definition 2.34: Lie group
A *Lie group* is a topological group in which the group operations are smooth (inifinitely differentiable), and which is also a smooth manifold.

3 Additional Topics

3.1 Lebesgue Integration

3.2 Differential Forms

3.3 The Banach-Tarski Paradox