Linear Control

Dynamics $\Rightarrow$ EOM: $D(q)\ddot{q} + C\dot{q} + g = \tau$

$$\Rightarrow \ddot{q} = D^{-1}(\tau - C\dot{q} - g)$$

System state $X = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$ Control input $\tau = u$

$\Rightarrow$ system dynamics: $\dot{X} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} X(2) \\ D^{-1}(\tau - C\dot{q} - g) \end{bmatrix} = f(q, \dot{q}, \tau) = f(x, u)$

$\Rightarrow$ nonlinear system: $\dot{x} = f(x, u)$

nonlinear function

+ Linearization

→ approximation → linear system

Control goal: drive $\left.\begin{matrix} x \to x_d \\ u \to u_d \end{matrix}\right\} \Rightarrow \begin{matrix} \delta x = x - x_d \to 0 \\ \delta u = u - u_d \to 0 \end{matrix}$

$\delta x \to 0 \quad \delta u \to 0$

$$\dot{x} = f(x, u)$$

→ approximation around x_d, u_d

$$\dot{x} \approx \underbrace{f(x_d, u_d)}_{\dot{x}_d} + \underbrace{\frac{\partial f}{\partial x}(x_d, u_d)}_{A}\underbrace{(x - x_d)}_{\delta x} + \underbrace{\frac{\partial f}{\partial u}(x_d, u_d)}_{B}\underbrace{(u - u_d)}_{\delta u}$$

dynamics:

$\dot{x}_d = f(x_d, u_d)$

x_d, u_d?

+ manually

+ trajectory optimization

$\Rightarrow x_d(t) \quad u_d(t)$

$$\Rightarrow \boxed{\delta\dot{x} = A\,\delta x + B\,\delta u}$$

(Approximation)

$\Downarrow$

only accurate

if x is close to x_d

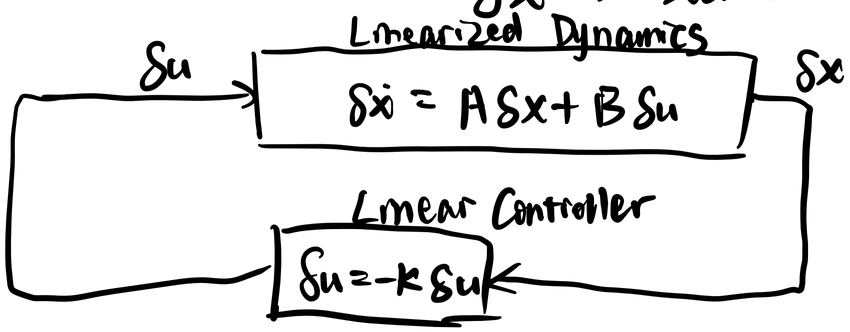
u is close to u_d

+ Linear Control

$$\boxed{\delta u = -K\,\delta x}$$

control goal : drive $x \to x_d$

$\Rightarrow \delta x = x - x_d \to 0$

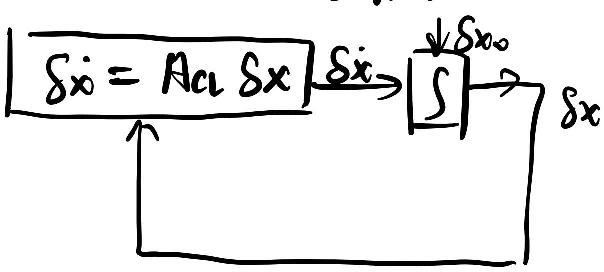


$\Rightarrow$ close-loop dynamics

$$\Rightarrow \delta\dot{x} = A\delta x + B\delta u$$
$$= A\delta x + B(-K\delta x)$$
$$= A_{CL}\,\delta x \qquad CL: \text{close-loop}$$

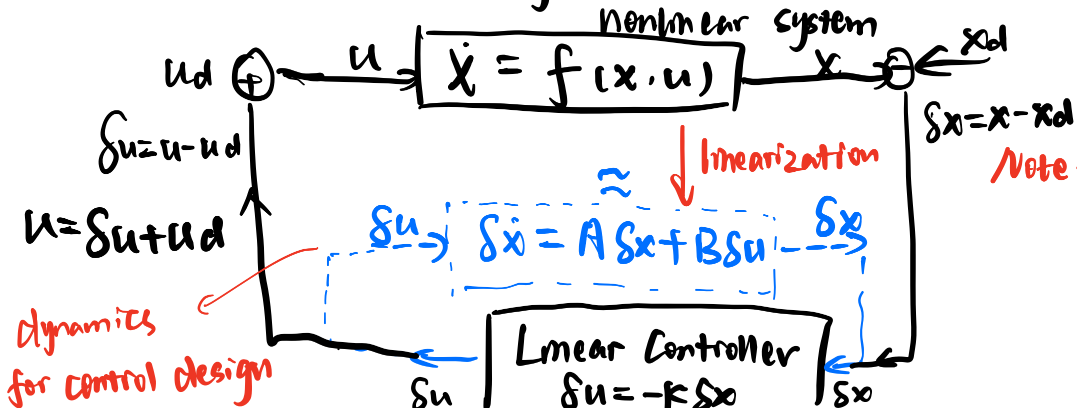
$\delta\dot{x} = A_{CL}\,\delta x$ close-loop dynamics

$\Rightarrow$ ode solver



$$\Rightarrow \delta x(t) \Rightarrow \underbrace{0 \quad (t \to \infty)}_{\text{goal}}$$

+ Linear Control Design



Note: apply linear controller

$\Rightarrow$ nonlinear system

$\Rightarrow$ ~~linearized~~ system

(approximation of the

nonlinear system)

linearization: help to design controller & verify stability ($\delta x \to 0$)

$\Rightarrow$ approximation around x_d, u_d

$\left.\begin{array}{l} \delta x = x - x_d \text{ is small enough} \\ \delta u = u - u_d \text{ is small enough} \end{array}\right\} \Rightarrow$ local stability

what happens if we apply the linear controller to the linear system.

(locally δx, δu are small : nonlinear system $\approx$ linear system)

Example:

$\left.\begin{array}{l} \text{close-loop dynamics} \\ \delta \dot{x} = A_{cl}\, \delta x \end{array}\right\} \Rightarrow \delta x(t) \to 0 \quad (t \to \infty)$

$\Rightarrow$ stability

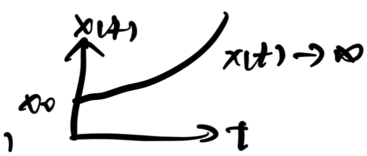
$\dot{x} = a x \quad (a \in \mathbb{R})$

$\Rightarrow \frac{dx}{dt} = ax \Rightarrow \int_0^t \frac{dx}{dt} = \int_0^t ax$

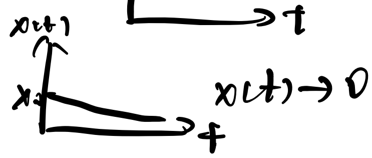
$\Rightarrow \ln \frac{x}{x_0} = at \Rightarrow \boxed{x(t) = x_0 e^{at}}$

+ $a = 0 \Rightarrow x(t) = x_0 \ (\forall t)$

(a>0) + $a = 1 \Rightarrow x(t) = x_0 e^{t}$



(a<0) + $a = -1 \Rightarrow x(t) = x_0 e^{-t}$



$\lim_{t \to \infty} x(t) = 0 \Rightarrow$ exponential stability

+ $\delta \dot{x} = A_{cl}\, \delta x$

A_{cl}: matrix; δx: vector $\in \mathbb{R}^n$

$\Rightarrow \delta x = \delta x_0 e^{-A_{cl} t}$

$\delta x \in \mathbb{R}^n \quad \delta \dot{x} \in \mathbb{R}^n \Rightarrow A_{cl} \in \mathbb{R}^{n \times n}$

(特征值)

δx is exponentially stable $\Longleftrightarrow$ eigenvalues of A_{CL} have strictly negative real parts

$\delta x \to 0$

- Eigenvalue of a matrix A

$$A \in \mathbb{R}^{n \times n} \quad v \in \mathbb{R}^n$$

$Av \Rightarrow$ change the direction of v

For certain vectors $Av /\!/ v \rightarrow Av = \underline{\underline{\lambda}} v^{/\!/ v}$ (λ is a scaler)

eigenvalue of A

$$Av = \lambda v$$

$$(A - \lambda I)v = 0$$

$$\mathbb{R}^{n \times n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$$

(行列式)

$\Rightarrow \det(A - \lambda I) = 0 \rightarrow$ get λ

Example:

$$A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \quad \lambda ?$$

Ans: $\det(A - \lambda I) = \left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$

$$= \begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-6-\lambda)(5-\lambda) - 3 \times 4 = 0$$

$$\lambda^2 + \lambda - 42 = 0$$

$$\Rightarrow \underbrace{\lambda_1 = 6}_{\text{positive}} \quad \underbrace{\lambda_2 = -7}_{\text{strictly negative}}$$

$\Rightarrow A$ is not stable.

f X is far away from X_d — linearization along a trajectory $(x_0 \to x_d)$
— nonlinear control

- $\begin{rcases} \delta\dot{x} = A\delta x + B\delta u \\ \delta u = -K\delta x \end{rcases} \Rightarrow \delta\dot{x} = (A-BK)\delta x = A_{cl}\delta x$

  $\Rightarrow$ close-loop dynamics: $\delta\dot{x} = A_{cl}\delta x$

design a controller $\delta u \Rightarrow \delta x \to 0$ (stable)

$A_{cl} \in \mathbb{R}^{n\times n} \Rightarrow$ eigenvalues: $\lambda_1, \lambda_2, \cdots \lambda_n$

$\Rightarrow$ real parts of λ_i, check if $Re(\lambda_i) < 0 \quad \forall i = 1 \cdots n$

$\Rightarrow$ all negative real parts

$\Rightarrow$ exponentially stable

$\Rightarrow \delta x \to 0$ (exponentially) $\quad \delta x = x - x_d$

$\Rightarrow x \to x_d$ (exponentially)

- Lyapunov Stability:

$$\begin{cases} \dot{x} = Ax + Bu \\ u = -Kx \end{cases} \Rightarrow \dot{x} = \overset{A_{cl}}{(A-BK)}x$$

Lyapunov function: $V(x) = x^T P x$

P is positive definite $\Rightarrow V(x) \geq 0$ (当且仅当 $x=0$ 时, $V(x)=0$)
($V(x) > 0$, $\forall x \neq 0$)

$V(x) \to 0 \Rightarrow x \to 0$

$x \in \mathbb{R}^n \Rightarrow V = \underset{\mathbb{R}^{1\times n}}{x^T}\ \underset{\mathbb{R}^{n\times n}}{P}\ \underset{\mathbb{R}^{n\times 1}}{x} \quad \in \mathbb{R}^1$ (scaler)

check the condition of a scaler $V \to 0$

⇒ check the condition of a vector $x \in \mathbb{R} \to 0$

$V(x) > 0 \quad (\forall x \neq 0)$

if design controller u: $\dot{V} < 0 \quad (\forall x \neq 0)$

$\Rightarrow V \to 0$

$V(x) = x^T P x$

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$

$\dot{x} = Ax + Bu = A_{cl} x$

$$\dot{V}(x) = x^T (\underbrace{A_{cl}^T P + P A_{cl}}_{-Q}) x$$

$\dot{V}(x) = -x^T Q x$

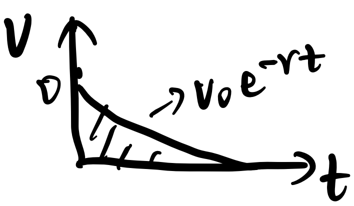
Q is positive definite $\Rightarrow \dot{V}(x) < 0 \quad (\forall x \neq 0)$

Lyapunov equation: $\boxed{A_{cl}^T P + P A_{cl} = -Q}$

MATLAB: P = lyap(Acl, Q)

$$\left.\begin{array}{l} V(x) = x^T P x \\ \dot{V}(x) = -x^T Q x \end{array}\right\} \Rightarrow \dot{V}(x) \leq -r V(x), \quad r = \frac{\lambda_{min} Q}{\lambda_{max} P} > 0$$

$\Rightarrow \dot{V}(x) \leq -r V(x) \Rightarrow V(x) \leq V_0 e^{-rt} \quad (V_0 \geq 0)$



$\Rightarrow V \to 0 \quad (t \to \infty)$

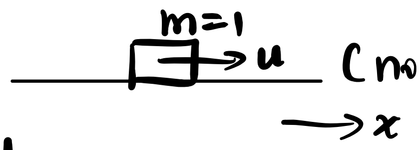
$\Rightarrow V = x^T P x > 0 \quad (\forall x \neq 0)$

$V \to 0 \Rightarrow x \to 0$

stability

Lyapunov function ⇒ design a controller

PD control



(no friction)

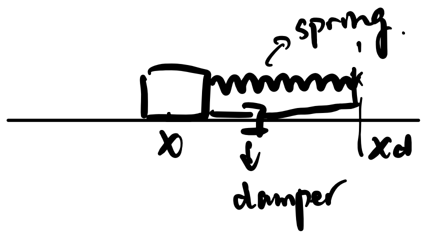
$m\ddot{x} = u$

$m = 1 \Rightarrow u = \ddot{x}$

$$\boxed{u = \underbrace{k_p}_{\text{P gain}}(x_d - x) + \underbrace{k_d}_{\text{D gain}}(\dot{x}_d - \dot{x})} \xrightarrow{?} x \to x_d$$

P gain (spring force)

D gain (Damper)



Spring : $x \to x_d$

Damper : $\dot{x} \to \dot{x}_d$

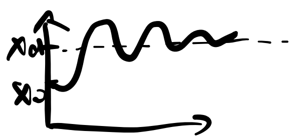
$\dot{x}_d = 0$: spring $\Rightarrow u \Rightarrow$ push $x \to x_d$

damper $\Rightarrow u \Rightarrow$ slow down when $x \to x_d$

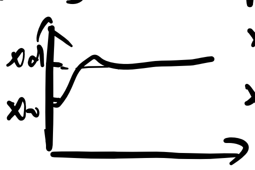
$x = x_d$

$\dot{x} = \dot{x}_d = 0$

Spring only :



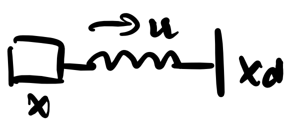
Spring + damper



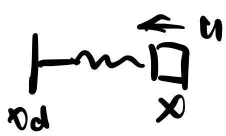
$x \to x_d$

$\dot{x} \to 0$

$k_p(x_d - x)$

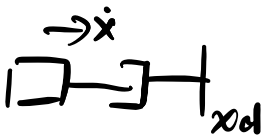


$x_d > x \Rightarrow k_p(x_d - x) > 0 \Rightarrow x \to x_d$



$x > x_d$: $\Rightarrow k_p(x_d - x) < 0 \Rightarrow x \to x_d$

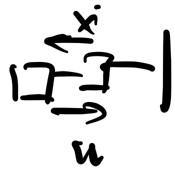
$k_D(\dot{x}_d - \dot{x})$

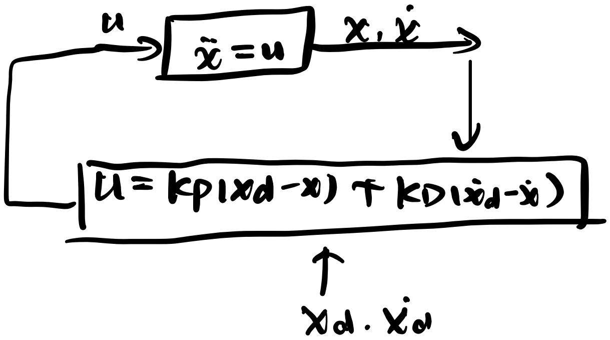


$\dot{x}_d = 0 \Rightarrow -k_D\dot{x}$

$\dot{x} > 0 \Rightarrow -k_D\dot{x} < 0$

$\dot{x} < 0 \Rightarrow -k_D\dot{x} > 0$





$\Rightarrow$ close-loop system : $x(t)$?

$$\ddot{x} = u = k_P(\underbrace{x_d - x}_{-e}) + k_D(\underbrace{\dot{x}_d - \dot{x}}_{-\dot{e}})$$

errors : $e = x - x_d$ $\quad \dot{e} = \dot{x} - \dot{x}_d$ $\quad \ddot{e} = \ddot{x} - \underbrace{\ddot{x}_d}_{0} = \ddot{x}$

$\Rightarrow$ error dynamics : $\ddot{e} = \ddot{x} = -k_P e - k_D\dot{e}$

$\Rightarrow$ $\boxed{\ddot{e} + k_D\dot{e} + k_P e = 0}$ second-order error dynamics

$\Rightarrow$ system dynamics : $\ddot{x} = u \Rightarrow \ddot{e} = u$

$$X = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \Rightarrow \dot{X} = \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} x^{(2)} \\ u \end{bmatrix}$$

$$\dot{X} = AX + Bu$$

$$\dot{X} = \begin{bmatrix} x^{(2)} \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} X + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

PD controller : $u = -k_P e - k_D\dot{e} = -\underbrace{\begin{bmatrix} k_P & k_D \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} e \\ \dot{e} \end{bmatrix}}_{X}$

$u = -KX$ $\quad (K = [k_P \ k_D])$

$\Rightarrow$ close-loop system : $\dot{X} = AX + B(-KX) = \underbrace{(A - BK)}_{A_{cl}} X$

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_P & k_D \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -k_P & -k_D \end{bmatrix}$$

$$\Rightarrow \dot{X} = A_{cl} X \quad \left(A_{cl} = \begin{bmatrix} 0 & 1 \\ -k_P & -k_D \end{bmatrix} \right)$$

$\downarrow$

stable $\to$ A_{cl}? stable $\to$ eigenvalues of A_{cl}

$\to$ $k_P > 0$ & $k_D > 0$

Midterm:

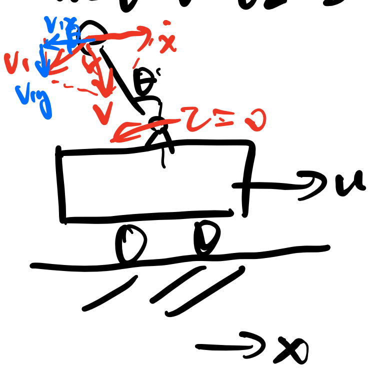
1) Kinematics : Rotation, Rigid motion, composition of rigid motion FK

2) Dynamics : Euler-Lagrange equation, system dynamics, Simulate using MATLAB ode solver

3) Linear Control : Linearization $\begin{pmatrix} x_d \\ u_d \end{pmatrix}$, $\underline{\text{Linear control, PD control}}$ stability

MATLAB implementation

$W = \dot{\theta}$ $\quad V = \dot{\theta} L$ $\quad \frac{1}{2} m V^2$



① q

② $L = E_k - E_p$

③ $\frac{\partial L}{\partial q}$ $\quad \frac{\partial L}{\partial \dot{q}}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$

④ 代入 Equation

$$q = \begin{pmatrix} x \\ \theta \end{pmatrix}$$

$$E_k = E_{k1} + E_{k2}$$

$$E_{k1} = \frac{1}{2} m \dot{x}^2 \qquad \left(E_k = \frac{1}{2} m v^2 = \frac{1}{2} I \cdot w^2 \right)$$

$$V_{ball}^2 = V_{ballx}^2 + V_{bally}^2$$

$$V_{ball}^2 = (\dot{x} - \dot{\theta} L \cos\theta)^2 + (\dot{\theta} L \sin\theta)^2$$

$$E_{k2} = \frac{1}{2} m V_{ball}^2 = \frac{1}{2} m \left[(\dot{x} - \dot{\theta} L \cos\theta)^2 + (\dot{\theta} L \sin\theta)^2 \right]$$

$$E_k = E_{k1} + E_{k2} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2} m \dot{\theta}^2 L^2 - m \dot{\theta} L \dot{x} \cos\theta$$

$$E_p = mgL\cos\theta$$

$$L = E_k - E_p = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\dot{\theta}L^2 - m\dot{\theta}L\dot{x}\cos\theta - mgL\cos\theta$$

对 x: $\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} - m\dot{\theta}L\cos\theta$ $\quad \frac{d}{dt}(\dot{\theta}\cos\theta)$

$$\frac{\partial L}{\partial x} = 0 \qquad (xy)' = x'y + xy'$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} - mL\left[\ddot{\theta}\cos\theta + \dot{\theta}(-\sin\theta)\cdot\dot{\theta}\right]$$

代入: $(M+m)\ddot{x} - mL\ddot{\theta}\cos\theta + mL\sin\theta\cdot\dot{\theta}^2 = u$

对 θ: $\frac{\partial L}{\partial \dot{\theta}} = mL^2\dot{\theta} - mL\dot{x}\cos\theta$

$$\frac{\partial L}{\partial \theta} = m\dot{\theta}L\dot{x}\sin\theta + mgL\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta} - mL(\ddot{x}\cos\theta + \dot{x}\cdot(-\sin\theta)\cdot\dot{\theta})$$

$$\Rightarrow mL^2\ddot{\theta} - mL\ddot{x}\cos\theta - mgL\sin\theta = 0$$

$$☆\begin{cases}(M+m)\ddot{x} - mL\ddot{\theta}\cos\theta + mL\sin\theta\cdot\dot{\theta}^2 = u \\ mL^2\ddot{\theta} - mL\ddot{x}\cos\theta - mgL\sin\theta = 0\end{cases}$$

Equation of motion (EoM)

$$D(q)\cdot\ddot{q} + \underbrace{C(q,\dot{q})\dot{q} + g(q)}_{N(q,\dot{q})} = u.$$

$$\Rightarrow D(q)\cdot\ddot{q} + N(q,\dot{q}) = u.$$

$$q = \begin{pmatrix}x\\\theta\end{pmatrix} \qquad \ddot{q} = \begin{pmatrix}\ddot{x}\\\ddot{\theta}\end{pmatrix}$$

$$D(q) = \begin{pmatrix}M+m & -mL\cos\theta \\ -mL\cos\theta & mL^2\end{pmatrix} \qquad D(q)\ddot{q} = \begin{pmatrix}\_ & \_\end{pmatrix}\begin{pmatrix}\ddot{x}\\\ddot{\theta}\end{pmatrix}$$

$$N(q,\dot{q}) = \begin{pmatrix}mL\sin\theta\cdot\dot{\theta}^2 \\ -mgL\sin\theta\end{pmatrix} \qquad u = \begin{pmatrix}u\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}M+m & -mL\cos\theta \\ -mL\cos\theta & mL^2\end{pmatrix}\begin{pmatrix}\ddot{x}\\\ddot{\theta}\end{pmatrix} + \begin{pmatrix}mL\sin\theta\cdot\dot{\theta}^2 \\ -mgL\sin\theta\end{pmatrix} = \begin{pmatrix}u\\0\end{pmatrix}$$