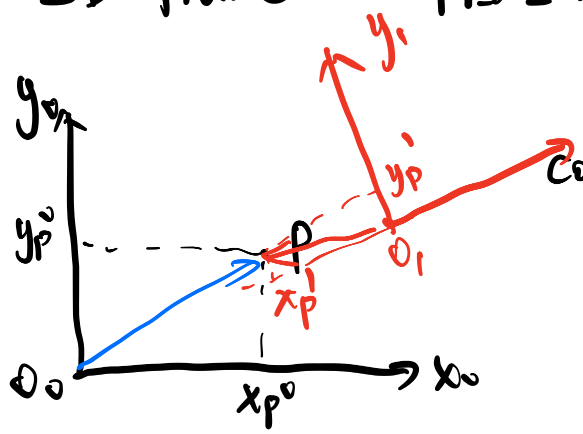
Kinematics :

① Frame , vector ② composition of Rotation

③ Rigid motion

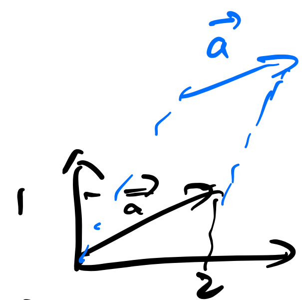
① Frame, vector, rotation

- 2D frame 平面坐标系 $\{O_0 : x_0, y_0\}$.



Coordinate $[x_p^0, y_p^0]$ in frame $\{O_0\}$.

连接 O_0P : vector : $\vec{v_0} = \overrightarrow{O_0P} = (x_p^0, y_p^0)$



vector ① direction ② magnitude 方向和大小

与始末位置无关

$|\vec{a}| = \sqrt{2^2+1^2} = \sqrt{5}$

新建坐标系 $\{1\}$

P (x_p^1, y_p^1) in $\{1\}$

连接 $\overrightarrow{O_1P}$: $\vec{v_1} = \overrightarrow{O_1P}$

- P point :

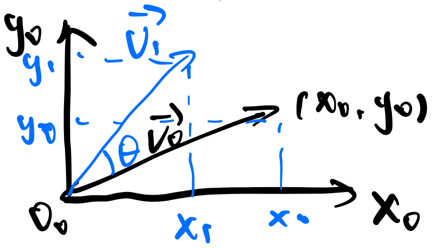
① a particular location

② different coordinates in different frames

i.e. $(x_p^0, y_p^0) \neq (x_p^1, y_p^1)$

- Rotation
  - Rotation of a vector
  - Rotation of a frame

① Rotation of a vector:



固定坐标系下，求旋转后 $\vec{V_1}$ 坐标.

find $\vec{V_1} = R(\theta)\,\vec{V_0}$ ← given

Rotation Matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

注: ① θ: 逆时针旋转. $\begin{cases} \text{逆}30^\circ: \theta = 30^\circ \\ \text{顺}30^\circ: \theta = -30^\circ. \end{cases}$

② $R(\theta)$ 的转置

③ $R(\theta)$: 正交矩阵 $A \cdot A^T = E \rightarrow$ 正交.

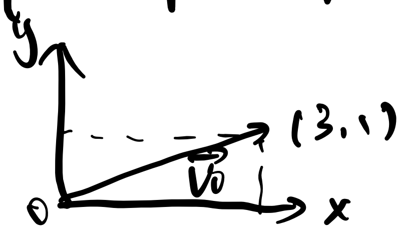
$R(\theta)\,R(\theta)\,(R(\theta))^T = \cancel{R(\theta)}E \quad \Rightarrow \quad R^{-1}(\theta) = R^T(\theta)$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad R^T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

if $|R(\theta)| = 1$, $\Rightarrow$ special orthogonal

$|R(\theta)| = \pm 1$

[例]:



1. $\vec{V_0}$ 逆时针旋转 $60^\circ \Rightarrow \vec{V_1} = (\ ?\ )$

2. $\vec{V_0}$ 顺时针旋转 $60^\circ \Rightarrow \vec{V_2} = (\ ?\ )$

1. $\theta_1 = 60^\circ$

$$V_1 = R(\theta_1) \cdot V_0 = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} + \frac{1}{2} \end{bmatrix}$$

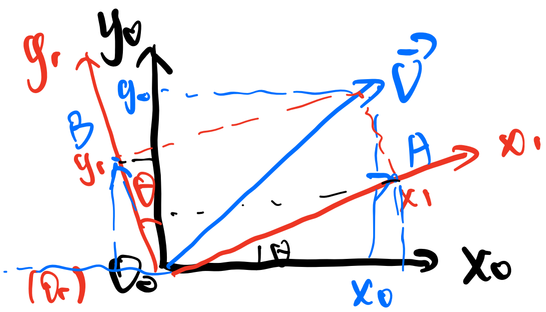
2. $\theta_2 = -60^\circ$

$$V_2 = R(\theta_2)\, V_1 = \begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} + \frac{\sqrt{3}}{2} \\ -\frac{3\sqrt{3}}{2} + \frac{1}{2} \end{bmatrix}$$

② Rotation of a frame



$\vec{v} = [x_0, y_0]$ in $\{0\}$

$\{0\} \xrightarrow{\text{逆}\theta} \{1\}$

$\vec{v} = [x_1, y_1]$ in $\{1\}$

求固定向量在新坐标系下的位置坐标.

$\vec{v} = \vec{OA} + \vec{OB}$

$$\begin{cases} x_0 = OA \cdot \cos\theta - OB \sin\theta \\ y_0 = OA \cdot \sin\theta + OB \cos\theta \end{cases}$$

$OA = x_1$, $OB = y_1$ $\Rightarrow$ $\begin{cases} x_0 = x_1 \cos\theta - y_1 \sin\theta \\ y_0 = x_1 \sin\theta + y_1 \cos\theta \end{cases}$.

$$\underline{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \underline{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}$$

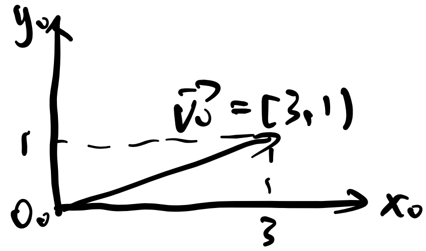
Given → $v^0 = R_1^0 \, v^1$ ← find

$v^1 = ?$　　$v^1 = (R_1^0)^T v^0 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} v^0$

注: ① θ 逆时针

② R_1^0: $\{0\} \to \{1\}$.

[3,1]



1. {0} 逆时针旋转 60° → {1}. $\vec{v}$ in {1}.

2. {0} 顺时针旋转 60° → {2}　$\vec{v}$ in {2}.

1. $\theta = 60^\circ$

$$v^1 = (R_1^0)^T \cdot v^0$$
$$= \begin{bmatrix} \cos 60^\circ & \sin 60^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{2} + \frac{\sqrt{3}}{2} \\ -\frac{3\sqrt{3}}{2} + \frac{1}{2} \end{bmatrix}.$$

2. $\theta = -60^\circ$

$$v^2 = (R_2^0)^T \cdot v^0$$
$$= \begin{bmatrix} \cos(-60^\circ) & \sin(-60^\circ) \\ -\sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

R(θ)

$$= \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} + \frac{1}{2} \end{bmatrix}$$

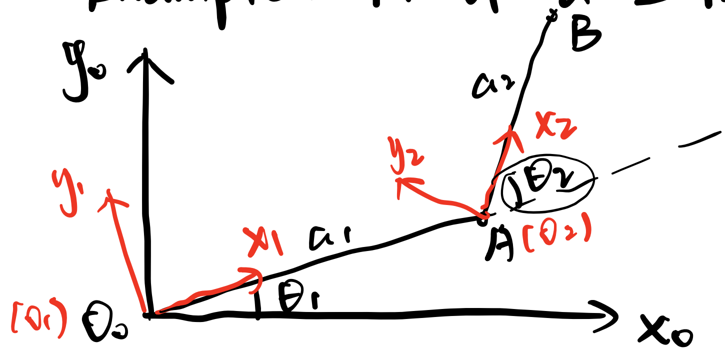
发现: ① 逆时针旋转向量 <=> 顺时针旋转坐标系

② $(R_1^0)^T$ $\quad R(-\theta)$

Rigid motion

刚体：平移+旋转运动

Example: FK of a 2-link robot arm



$$\begin{cases} x_E = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ y_E = a_1\sin\theta_1 + a_2\cdot\sin(\theta_1+\theta_2) \end{cases}$$

in $\{1\}$: $A(a_1, 0)$ $\qquad$ in $\{2\}$: $B(a_2, 0)$

$P_A^1 = (a_1, 0)$ $\qquad$ $P_B^2 = (a_2, 0)$

local coordinates $\qquad$ $\{1\}\{2\}$: moving frame $\quad$ body frame

$\{0\}$: fixed frame

求 A, B在$\{0\}$的坐标 P_A^0, P_B^0

坐标系旋转角度:

$\{0\} \xrightarrow{\text{逆}\theta} \{1\}$: O_1与O_0重合

$$P_A^0 = R_1^0 P_A^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} a_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \end{bmatrix}$$

$\{1\} \to \{2\}$ 平移 $\vec{O_1O_2}$, 旋转θ $\qquad$ $P_B^2 = \begin{bmatrix} a_2 \\ 0 \end{bmatrix}$

$P_B^1 = P_A^1 + R_2^1 P_B^2$ $\qquad$ $P_A^1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ $\quad$ $R_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

求 P_B^1

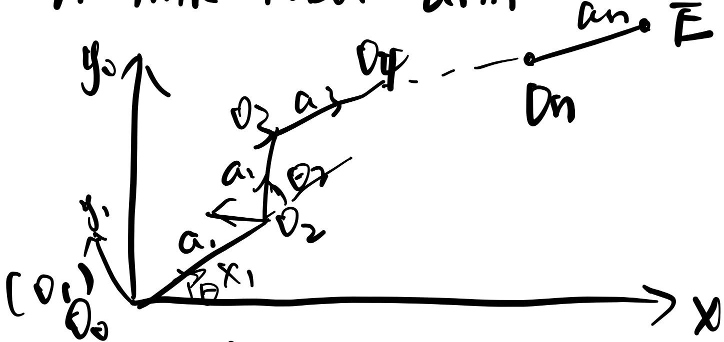
$$P_B^1 = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} a_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_2\cos\theta_2 \\ a_2 \sin\theta_2 \end{bmatrix}$$

$$P_B^0 = R_1^0 \, P_B^1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} a_1 + a_2\cos\theta_2 \\ a_2\sin\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\cos\theta_1 + \underbrace{a_2\cos\theta_1\cos\theta_2 - a_2\sin\theta_1\sin\theta_2}_{a_2\cos(\theta_1+\theta_2)} \\ a_1\sin\theta_1 + \underbrace{a_2\sin\theta_1\cos\theta_2 + a_2\sin\theta_2\cos\theta_1}_{a_2\sin(\theta_1+\theta_2)} \end{bmatrix}$$

n-link robot arm



已知 E in $\{O_n\}$: $(a_n, 0)$ $\Rightarrow$ $P_E^n = (a_n, 0)$

求 P_E^0

$\{0\} \to \{1\} \to \{2\} \to \{3\} \to \{4\} \to \cdots \{n-1\} \; \{n\}$

$P_E^0 \, ? \Leftarrow P_E^1 \leftarrow \cdots \leftarrow P_E^{n-2} \leftarrow P_E^{n-1} \leftarrow P_E^n$

$\{1\} \to \{2\}$: translation + rotation

$p^1 = R_1^0 P^2 + d^1$ $\quad$ $p^1 \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1} \to 3\times 1$, $\quad R_1^0 : 2\times 2 \quad d: 2\times 1$

$\Rightarrow P_B^1 = T_2^1 \, P_B^2$

Transformation Matrix

增加维数：

$$T = \begin{bmatrix} R_{2\times2} & \overset{\text{Translation}}{\begin{pmatrix} \; \end{pmatrix}_{2\times1}} \\ 0 \; 0 & 1 \end{bmatrix} \quad 3\times 3$$

$P\begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix}$

$(a_1, 0)$

$\{1\} \rightarrow \{2\}$: 逆θ + 平移 d.

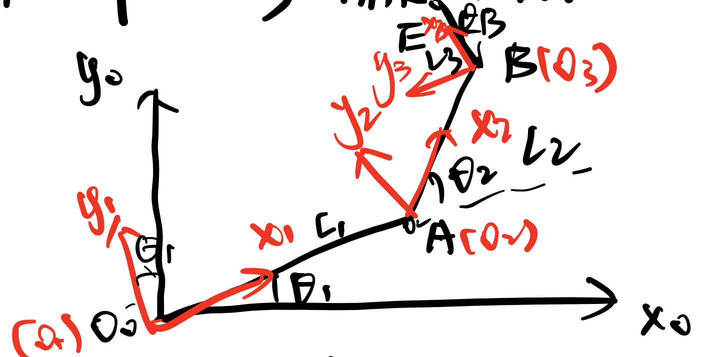
$$T_2^1 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & a_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_B^2 = \begin{pmatrix} a_2 \\ 0 \\ 1 \end{pmatrix}$$

$$P_B^1 = T_2^1 P_B^2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & a_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2\cos\theta_2 + a_1 \\ a_2\sin\theta_2 \\ 1 \end{pmatrix}$$

Example: 3-link robot arm:



已知 $P_E^3 = (L_3, 0)$ in $\{3\}$

求 P_E^0

$$P_E^3 \rightarrow P_E^2 \rightarrow P_E^1 \rightarrow P_E^0$$

$P_E^2 = T_3^2 P_E^3 \qquad P_E^1 = T_2^1 \cdot P_E^2 \qquad P_E^0 = T_1^0 P_E^1$

$$\Rightarrow P_E^0 = T_1^0 T_2^1 T_3^2 P_E^3$$

$$T_1^0 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_2^1 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_3^2 = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P_E^3 = \begin{pmatrix} L_3 \\ 0 \\ 1 \end{pmatrix}$$