Today's topics : + Kinematics
+ Dynamics

- Rigid Motion : Translation + Rotation

$$\boxed{p^0 = R_1^0 p^1 + d^0}$$

New coordinate of P wrt {0}

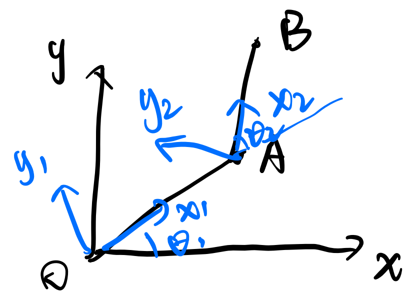
$\{0\} \xrightarrow{\{d, r\}} \{1\}$

fixed frame

P is attached to the moving frame {1}

⇓

p^1 = fixed (body / local coordinate)



$\{1\} \xrightarrow{\text{attached}}$ link 1 ⇒ P_A^0 ?

$\{2\} \xrightarrow{\text{attached}}$ link 2 ⇒ P_B^0 ?

$P_A^1 \rightarrow P_A^0$

local coordinate of A wrt {1}

local coordinate of B wrt {2}

$$\underbrace{P_B^2 \rightarrow P_B^1}_{\text{rigid motion between } \{1\} \rightarrow \{2\}} \underbrace{\rightarrow P_B^0}_{\text{rigid motion between } \{0\} \text{ to } \{1\}}$$

⇓

Forward Kinematics = Composition of multiple rigid motions

⇒ Systematic Method

⇒ Homogeneous Transformation (齐次变换)

$$\underset{\text{Square Matrix}}{H_{(4\times4)}} = \begin{array}{cc} \text{Rotation} & \text{Translation} \\ \left[\begin{matrix} R^{(3\times3)} \\ 0^{(1\times3)} \end{matrix}\right. & \left.\begin{matrix} d^{(3\times1)} \\ 1 \end{matrix}\right] \end{array}$$

to represent the rigid motion

$$P^0 = R_1^0 P^1 + d^0$$

⇒ extend the vector representation:

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix} \in R^{4\times 1}$$

⇒ $\underline{P^0 = H_1^0 P^1}$ Simpler Equation

$$= \begin{bmatrix} R_1^0 & d^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1^0 p^1 + d^0 \\ 1 \end{bmatrix}}_{p^0}$$

Composition of multiple rigid motion

(wrt the current frame)

$$\{0\} \xrightarrow{H_1^0} \{1\} \xrightarrow{H_2^1} \{2\}$$

↓ current frame

⇒ $\{0\} \to \{2\}$

⇓

$$H_2^0 = H_1^0 \cdot H_2^1$$

$H_1^0 = \begin{bmatrix} R_1^0 & d^0 \\ 0 & 1 \end{bmatrix}$ → $\overline{O_1 O_0}$

⇒ O_1 ?

$H_2^0 = \begin{bmatrix} R_2^0 & \overline{O_2 O_0} \\ 0 & 1 \end{bmatrix}$ ⇒ O_2 ? ⇒ A

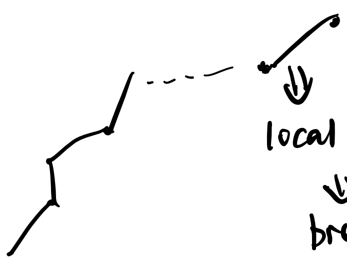
H_3^0 ⇒ O_3 ?

H_n^0 ⇒ O_n ?

⇒ $H_3^0 = H_2^0 H_3^2 = H_1^0 H_2^1 H_3^2$

$H_n^0 = H_{n-1}^0 \cdot H_n^{n-1}$ $(\{n-1\} \to \{n\})$

↓ local transformation



⇓ local transformation

⇓ break down the whole robot into kinematics of 2 consecutive links

Dynamics: q = joint angle $\quad \dot{q}$ = joint velocity $\quad \ddot{q}$ = joint acceleration

Kinematics: joint angles q $\Leftrightarrow$ end effector position / orientation

given q $\Rightarrow$ "draw" the robots
$= ?$

physical robot $\Rightarrow$ $\boxed{\text{sensors}}$ $\leftarrow$ Robot $\leftarrow$ $\boxed{\text{controller}}$

$\Downarrow$ get q

$\Downarrow$

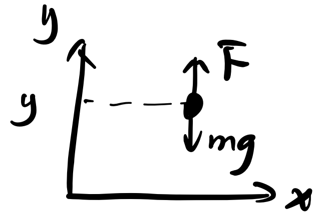
physical simulation: simulate the robot.

given input $\tau(t) \rightarrow q(t)$

$\ddot{q}$ = joint acceleration
$\Downarrow$
motor torque τ
$\Downarrow$
solve for $q(t)$
$\Downarrow$
kinematics
$\Downarrow$
draw the robot with $q(t)$
$\Downarrow$
motion (simulation)

Example:

- a particle m $\Rightarrow$ moves vertically
- apply an external force F (F_y)
- Newton's Second Law



$$ma = \sum F$$

(m, g constant parameters)

$$\Rightarrow m\ddot{y} = F - mg \quad \Rightarrow \boxed{\ddot{y} = \frac{F}{m} - g} \quad \text{dynamics}$$

$$\left.\begin{array}{l} q = y \quad \Rightarrow \dot{q} = \dot{y}\ ;\ \ddot{q} = \ddot{y} \\ u = F \end{array}\right\} \Rightarrow \ddot{q} = \frac{u}{m} - g$$

$$= f(q, \dot{q}, u)$$

(这里仅与u有关)

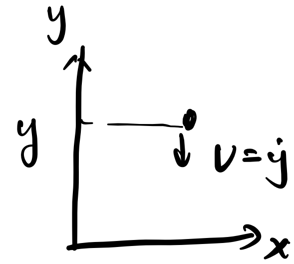
Control input $u(t) \Rightarrow \ddot{q}(t) = f(q, \dot{q}, u(t))$

$\Rightarrow$ ode solver

$$\ddot{q}(t) \Rightarrow \int \rightarrow \dot{q}(t) \Rightarrow \int \Rightarrow q(t)$$

initial condition: $\dot{q}(0)$ $\quad q(0)$

$\Rightarrow$ more systematic method $\Rightarrow$ dynamics



$\Rightarrow$ Euler - Lagrange Dynamics :

Lagrangian : $L = K - P$

K → kinetic Energy

P → potential Energy

Kinetic Energy : $K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$

Potential Energy : $P = mgy$

$\Rightarrow$ Lagrangian : $L = K - P = \frac{1}{2}m\dot{y}^2 - mgy$

Euler - Lagrange Equation :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = F$$

fix base

$$\frac{\partial L}{\partial \dot{y}} = \frac{\partial K}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = m\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y}$$

$$\frac{\partial L}{\partial y} = -\frac{\partial P}{\partial y} = -\frac{\partial}{\partial y}(mgy) = -mg$$

$\Rightarrow m\ddot{y} - (-mg) = F$

$\Rightarrow \boxed{m\ddot{y} = F - mg}$

∘ Euler - Lagrange equation (for a n-link robot arm)

+ Lagrangian : $L = K - P$
+ Dynamics : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \tau_k \quad (k = 1, 2 \cdots n)$

2 different types of joints :

(1) revolute joint

$\Rightarrow$ rotation

joint angle (rad)

$\Downarrow$

joint torque τ_k (N-m)

(2) prismatic joint

q_k = displacement (m)

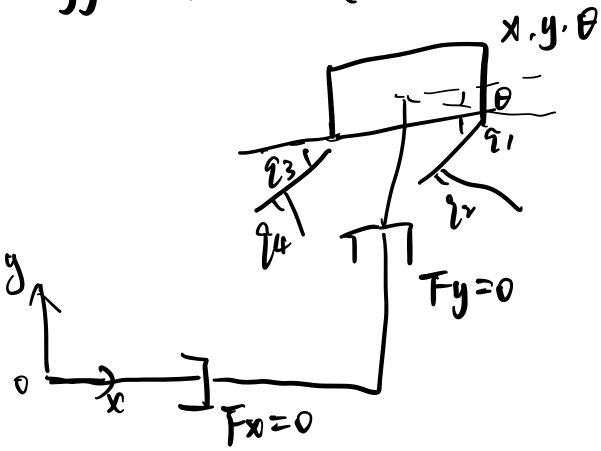
$\Downarrow$

joint Force τ_k (N)

different unit

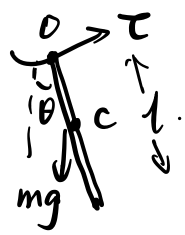
+ Mobile robots (floating based)

Legged robots (2D)



$\Rightarrow$ 3 virtual links + $\begin{bmatrix} F_x = 0 \\ F_y = 0 \\ \tau_\theta = 0 \end{bmatrix}$

Example: 1-link robot arm



+ m : mass

+ I_c : moment of inertia about the center of mass C

1) Newton Law

Dynamics: $I\ddot{\theta} = \Sigma M \Rightarrow \Sigma M = \tau - mg \cdot \overset{\frac{l}{2}}{OC} \sin\theta$

$\Downarrow$

moment of inertia wrt the rotation axis O $(OC = \frac{l}{2})$

$\Rightarrow I = I_c + md^2 \quad (d = OC = \frac{l}{2})$

$= I_c + \frac{1}{4}ml^2$

$\Rightarrow (I_c + \frac{1}{4}ml^2)\ddot{\theta} = \tau - mg\frac{l}{2}\sin\theta$

2) Euler-Lagrange Equation:

$$L = K - P$$

$$K = \frac{1}{2} I w^2 = \frac{1}{2} I_c w^2 + \frac{1}{2} m V_c^2$$

$$w = \dot{\theta} \qquad V_c = w \cdot \overline{OC} = w \cdot \frac{l}{2}$$

$$\Rightarrow K = \frac{1}{2} (I_c + m \cdot \frac{l^2}{4}) \cdot w^2$$

$$P = -mgh = -mg\frac{l}{2}\cos\theta \quad (P = 0,\ at\ O)$$

$$L = K - P = \frac{1}{2} I \dot{\theta}^2 + mg\frac{l}{2}\cos\theta$$

$$q = \theta \Rightarrow \dot{q} = \dot{\theta} = w; \quad \ddot{q} = \ddot{\theta}$$

$$\Rightarrow L = \frac{1}{2} I \dot{q}^2 + mg\frac{l}{2}\cos q$$

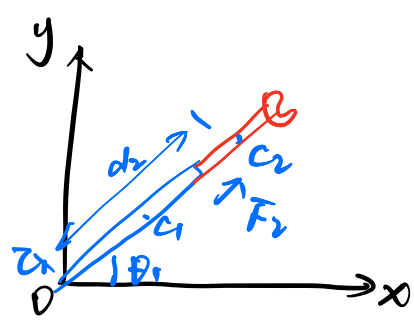
$$\begin{cases} \dfrac{\partial L}{\partial \dot{q}} = I\dot{q} \quad \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = I\ddot{q} \\[2ex] \dfrac{\partial L}{\partial q} = -mg\dfrac{l}{2}\sin q \end{cases}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

$$\Rightarrow I\ddot{q} + mg\frac{l}{2}\sin q = \tau$$

Example:

2-link robot arm



+ link 1: revolute joint $\Rightarrow \theta_1, \tau_1$
+ link 2: prismatic joint $\Rightarrow d_2, F_2$

$q_1 = \theta_1$; $u_1 = \tau_1$

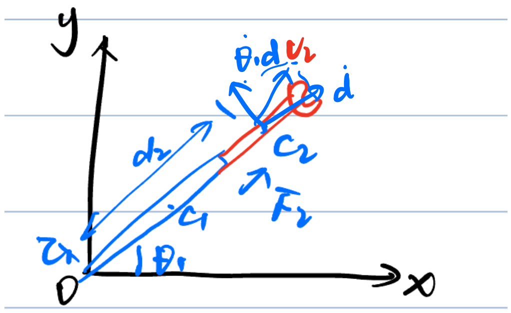
$q_2 = d_2$; $u_2 = F_2$

$$\Rightarrow q = \begin{bmatrix} \theta_1 \\ d_2 \end{bmatrix}; \quad u = \begin{bmatrix} \tau_1 \\ F_2 \end{bmatrix}$$

Robot parameters:

link 1 : m_1, I_1, l_1 $\rightarrow$ moment of inertia of link 1 wrt C_1

link 2 : m_2, I_2 $\rightarrow$ moment --- C_2



- Lagrangian : $L = K - P$

$$K = K_1 + K_2$$

$$K_1 = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_1 V_1^2$$

$$V_1 = \dot{\theta}_1 \frac{l}{2}$$

$$\Rightarrow K_1 = \frac{1}{2}\left(I_1 + \frac{1}{4} m_1 l_1^2\right)\dot{\theta}_1^2$$

$$\cancel{+\ K_2 = \frac{1}{2} I_2 \dot{\theta}_1^2 + \frac{1}{2} m_2 d^2 \dot{\theta}_1^2}$$

$$K_2 = \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} m_2 V_c^2$$

$$\omega_2 = \omega_1 = \dot{\theta}_1$$

$$V_2^2 = (\dot{\theta}_1 d_2)^2 + (\dot{d}_2)^2$$

$$\Rightarrow K_2 = \frac{1}{2} I_2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (d_2^2 \dot{\theta}_1^2 + \dot{d}_1^2)$$

$$= \frac{1}{2}(I_2 + m_2 d_2^2)\dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{d}_2^2$$

+ Potential Energy $\qquad P = P_1 + P_2$

$$P_1 = m_1 g\, y_{c_1} = m_1 g\, l_1 \sin\theta_1$$ $\rightarrow$ const

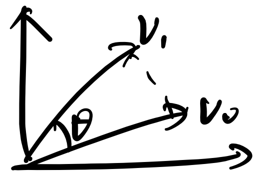
$$P_2 = m_2 g\, y_{c_2} = m_2 g\, d_2 \sin\theta_1$$ $\rightarrow$ joint variable

$$\Rightarrow L = K - P$$

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_1}\right) - \dfrac{\partial L}{\partial \theta_1} = \tau_1 & \text{(motor torque)} \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{d}_2}\right) - \dfrac{\partial L}{\partial d_2} = F_2 & \text{(motor Force)} \end{cases}$$

HW2:

Q7:



$$V_1 = R(\theta) V_0 \qquad (2,1)^T$$

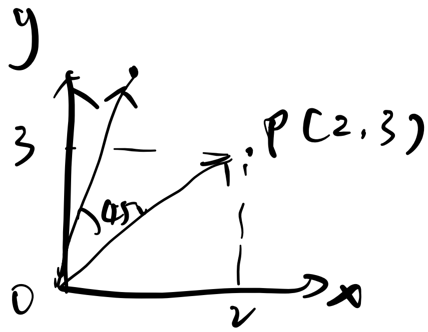
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$V_1 = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} - \frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}$$

Q8: $P(2,3)$



$$v' = R(\theta) \cdot v^0 \qquad \theta = 45^\circ$$

① $$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

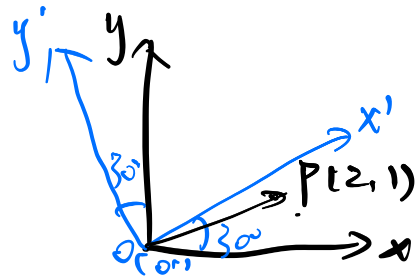
$$= \begin{bmatrix} \sqrt{2} - \frac{3\sqrt{2}}{2} \\ \sqrt{2} + \frac{3\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix} \begin{matrix} +3 \\ -2. \end{matrix}$$

② $$\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} + 3 \\ \frac{5\sqrt{2}}{2} - 2 \end{bmatrix}$$

$$H_{3\times3} = \begin{bmatrix} \cos45^\circ & -\sin45^\circ & 3 \\ \sin45^\circ & \cos45^\circ & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad H_{4\times4} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 3 \\ \sin\theta & \cos\theta & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q9: $\overset{(2,1)}{V^0} = R_1^0 V^1$

$$R_1^0 = \begin{pmatrix} \cos30^\circ & -\sin30^\circ \\ \sin30^\circ & \cos30^\circ \end{pmatrix}$$



$$V^1 = (R_1^0)^T V^0 = \begin{pmatrix} \cos30^\circ & \sin30^\circ \\ -\sin30^\circ & \cos30^\circ \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3} + \frac{1}{2} \\ -1 + \frac{\sqrt{3}}{2} \end{pmatrix}$$