

Another Proof For the Existence of the Free Convolution Semigroup for Compactly Supported Measures

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Abstract

The original proof for the existence of the free convolution semigroup for compactly supported measures due to Nica and Speicher is a construction based proof. This paper presents an alternate non-constructive proof following as a direct corollary of Voiculescu's characterization of R-transforms of compactly supported measures via Nevanlinna Pick theory.

Proof

In Lemma 3.3 of [1] Voiculescu showed a sort of Pick characterization of R-transforms of compactly supported measures:

Lemma 1: For a power series $R(z) = \sum_{n \in \mathbb{N}} \kappa_n z^{n-1}$ the following are equivalent:

- (i) There exists a compactly supported measure μ whose R-transform R_μ coincides with R .
- (ii) R is convergent in some ball of radius C about 0, $\kappa_n \in \mathbb{R}$, and there exists a $C' > C^{-1}$ such that $\forall z_1, z_2 \in \overline{\mathbb{C}^+} \setminus B_{C'}(0)$ the following Hermitian matrix is positive semi-definite:

$$\begin{bmatrix} \frac{\Im(z_1)}{\Im(z_1) + \Im(R(\frac{1}{z_1}))} & \frac{z_1 - \bar{z}_2}{z_1 - \bar{z}_2 + R(\frac{1}{z_1}) - R(\frac{1}{\bar{z}_2})} \\ \frac{z_2 - \bar{z}_1}{z_2 - \bar{z}_1 + R(\frac{1}{z_2}) - R(\frac{1}{\bar{z}_1})} & \frac{\Im(z_2)}{\Im(z_2) + \Im(R(\frac{1}{z_2}))} \end{bmatrix}$$

Remarks:

- (i) We now know due to Speicher ([3]) that the coefficients of the R-transform are the free cumulants of the corresponding measure. This is perhaps worth keeping in mind when reading Voiculescu's paper, as it was published before the theory of free cumulants had been developed (before [3]).
- (ii) Note that in Voiculescu's proof, $K(z)$ is defined as $z + \varphi(1/z)$ (φ corresponding to R here) and $H(z)$ as $K(z)$'s inverse (i.e. the inverse of the reciprocal of the Cauchy transform $1/G(z)$).

This lemma along with the Hadamard product's conservation of positive semi-definiteness can be used to show Lemma 4.2 in [1]:

Lemma 2: If R is holomorphic on some neighborhood of $\{0\} \cup \mathbb{C} \setminus \mathbb{R}$ such that $R(\bar{z}) = \overline{R(z)}$ and $\Im R(z) \geq 0$ if $\Im z > 0$, then R is the R-transform of some compactly supported measure.

From this second lemma, the existence of the free convolution semigroup (corollary 1.14 in [4] or theorem 4.16 from [2]) easily follows:

Theorem 3: For a compactly supported probability measure μ on \mathbb{R} there exists a (unique) semigroup $(\{\mu_t\}_{t \geq 1}, \boxplus)$ (the semigroup operation is the free additive convolution denoted by \boxplus) of compactly supported probability measures on \mathbb{R} such that:

- (i) $\mu_1 := \mu$

- (ii) $\forall s, t \geq 1 \mu_{s+t} = \mu_s \boxplus \mu_t$
- (iii) $\forall n \in \mathbb{N}$, the nth moment and cumulant maps $t \mapsto \kappa_n^{\mu_t}, t \mapsto m_n^{\mu_t}$ are continuous in t
- (iv) $\forall t \geq 1$, for sufficiently small $|z|$, we have the R-transform's "additivity property" $R_{\mu_t}(z) = tR_{\mu_1}(z) = tR_\mu(z)$

Proof: Multiplication by a constant t does not change the realness of μ 's cumulants (μ is a measure on \mathbb{R}) nor the convergence of the series $\sum_{n \in \mathbb{N}} t\kappa_n^\mu z^{n-1} = tR_\mu(z)$. Thus, $R_{\mu_t} := tR_\mu$ is holomorphic/analytic on the same neighborhood of 0 as R_μ and obviously $\Im z > 0 \implies \Im R_\mu(z) \geq 0$ yields $\Im z > 0 \implies \Im R_{\mu_t}(z) = t\Im R_\mu(z) \geq 0$. By the previous, lemma, we have that there exists some μ_t with R-transform R_{μ_t} . The continuity of the cumulant maps is obvious since, by the definition of R_{μ_t} , $\kappa_n^{\mu_t} = t\kappa_n^\mu$. We have by the moment-cumulant formula (see section 4 of [3]), $m_n^{\mu_t} = \sum_{\pi \in NC(n)} \kappa_\pi^{\mu_t} = \sum_{\pi \in NC(n)} t\kappa_\pi^\mu = t \sum_{\pi \in NC(n)} \kappa_\pi^\mu = tm_n^\mu$ and hence also the continuity of $m_n^{\mu_t}$. Here $NC(n)$ are the set of non-crossing partitions over n elements and $\kappa_\pi^{\mu_t} = \prod_{E \in \pi} \kappa_E^{\mu_t}$. \square

References

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- [4] A. Nica, R. Speicher, On the multiplication of free n -tuples of non-commutative random variables.