VECTOR SUBSPACES

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Definition : Let V(F) be a vector space and $W \subseteq V$. Then W is said to be a subspace of V if W itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.

Note : 1. Suppose W (F) is a subspace of V (F) then W is a sub-group of V.

2. Let V (F) be a vector space. Then clearly the zero vector space $\{\bar{o}\} \subseteq V$ and $V \subseteq V$. Hence $\{\bar{o}\}$ and V are trivial subspaces of V.

Theorem 1: *Let V (F) be a vector space and let $W \subseteq V$. Then the necessary and sufficient condition for W to be a subspace of V are (i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$*

(ii) $a \in F , \alpha \in W \Rightarrow a\alpha \in W$.

Proof : Given , V (F) is a vector space and $W \subseteq V$.

Part - I : *The conditions are necessary.*

Let W be a vector subspace of V.

To prove that conditions (i) and (ii) are true.

(i) Since W is a vector subspace of V, W itself is a vector space.

$\Rightarrow$ W is a subgroup of (V, +).

$\Rightarrow$ (W, +) is a group.

$\Rightarrow$ if $\alpha , \beta \in W$ then $\alpha - \beta \in W$.

(ii) Since W is a subspace of V, W itself is a vector space.

Hence it is closed under scalar multiplication (by the definition)

$\Rightarrow a \in F , \alpha \in W \Rightarrow a\alpha \in W$.

Hence conditions (i) & (ii) are satisfied.

Part – II : *The conditions are sufficient.*

Let W be a nonempty subset of V such that conditions (i) & (ii) are satisfied.

To prove that W is a subspace of V.

For this we need to prove that W itself is a vector space.

Let $\alpha , \alpha \in$ W then $\alpha - \alpha \in$ W (By (i))

$\Rightarrow \bar{0} \in$ W

$\therefore$ The zero of vector of V is also the zero vector of W.

Now $\bar{0} \in$ W, $\alpha \in$ W $\Rightarrow \bar{0} - \alpha \in$ W (By (i))

$\Rightarrow -\alpha \in$ W

$\Rightarrow$ additive inverse of each element of W is also in W.

Again $\alpha , \beta \in$ W, $\Rightarrow \alpha, (-\beta) \in$ W.

$\Rightarrow \alpha - (-\beta) \in$ W (By (i))

$\Rightarrow \alpha + \beta \in$ W.

i.e. W is closed under vector addition.

As W $\subseteq$ V, all the elements of W are also the elements of V. Hence vector addition in W will be associative and commutative.

$\therefore$ (W, +) is an abelian group.

Also by condition (ii), W is closed under scalar multiplication and hence all the postulates of vector space V hold in W as W $\subseteq$ V.

$\therefore$ W itself is a vector space under the operations of V.

$\Rightarrow$ W (F) is a vector subspace of V (F).

Theorem 2 : *Let V(F) be a vector space. A non-empty set W $\subseteq$ V. The necessary and sufficient condition for W to be a subspace of V is*

$$a , b \in F \text{ and } \alpha , \beta \in W \Rightarrow a\alpha + b\beta \in W \ldots\ldots (\text{I})$$

Proof : Given V(F) is a vector space and a non-empty set W $\subseteq$ V.

Part - I : *The condition is necessary.*

Let W (F) be a vector subspace of V (F).

To prove that condition (I) is true.

Since W is a vector subspace of V, W itself is a vector space.

$\therefore$ a $\in$ F , $\alpha \in$ W $\Rightarrow$ a$\alpha \in$ W and $b \in$ F , $\beta \in$ W $\Rightarrow$ $b\beta \in$ W.

Now a$\alpha \in$ W , $b\beta \in$ W. $\Rightarrow$ a$\alpha + b\beta \in$ W.

Hence the condition is true.

Part – II : *The condition is sufficient.*

Let W be a nonempty subset of V satisfying the given condition

a , b $\in$ F and $\alpha , \beta \in$ W $\Rightarrow$ aα + b$\beta \in$ W. … (I)

To prove that W is a subspace of V.

For this we need to prove that W itself is a vector space.

In (I), put a = 1, b = -1 and $\alpha , \beta \in$ W $\Rightarrow (1)\,\alpha + (-1)\,\beta \in$ W.

$\Rightarrow \alpha - \beta \in$ W (Since $\alpha \in W$ we have $\alpha \in V$ *and* $1\alpha = \alpha$ in V)

We know from group theory that H $\subseteq$ G and a , $b \in$ H $\Rightarrow$ a $o\, b^{-1} \in$ H then (H, o) is a subgroup of (G, o).

$\therefore$ (W, +) is a subgroup of the abelian group (V, +).

$\Rightarrow$ (W, +) is an abelian group.

Now put a =a and b = 0 in condition (I).

Then a , 0 $\in$ F *and* $\alpha , \beta \in W$ $\Rightarrow$ a$\alpha + 0\beta \in$ W.

$\Rightarrow$ a$\alpha \in$ W

i.e. a $\in$ F *and* $\alpha \in$ W $\Rightarrow$ a$\alpha \in$ W.

$\therefore$ W is closed under scalar multiplication.

Since W $\subseteq$ V, the remaining postulates of a vector space hold in W.

$\therefore$ W itself is a vector space and hence it is a subspace of V (F).

Theorem 3 : *A non-empty set W is a subset of a vector space V (F). Then W s a subspace of V if and only if* $a \in F$ *and* $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$. …… (I)

Proof : Given V(F) is a vector space and a non-empty set $W \subseteq V$.

Part - I : *The condition is necessary.*

Let W (F) be a vector subspace of V (F).

To prove that condition (I) is true.

Since W is a vector subspace of V, W itself is a vector space.

$\therefore a \in F,\ \alpha \in W \Rightarrow a\alpha \in W$

Now again $a\alpha \in W,\ \beta \in W. \Rightarrow a\alpha + \beta \in W.$

Hence the condition is true.

Part – II **:** *The condition is sufficient.*

Let W be a nonempty subset of V satisfying the given condition

$a \in F$ *and* $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W.$ … (I)

To prove that W is a subspace of V.

For this we need to prove that W itself is a vector space.

In (I), put a = -1 then for $\alpha,\ \alpha \in W$ *we have* $\Rightarrow (-1)\,\alpha + \alpha \in W.$

$\Rightarrow \overline{0} \in W$

$\therefore$ Zero vector i.e. additive identity exist in W

Now $a \in F$ *and* $\alpha, \overline{0} \in W \Rightarrow a\alpha + \overline{0} \in W$ (Since by (I))

$\Rightarrow a\alpha \in W$

$\therefore$ W is closed under scalar multiplication.

Again $-1 \in F$ *and* $\alpha, \overline{0} \in W \Rightarrow (-1)\alpha + \overline{0} \in W$ (Since by (I))

$\Rightarrow -\alpha \in W$

$\therefore$ Additive inverse exists in W.

Since the elements of W are the elements of V, the remaining postulates of vector space hold good in W also.

Hence W is a subspace of V (F).

Example : *Let p, q, r be the fixed elements of a field F. Show that the set W of all triads (x, y, z) of elements of F, such that px + qy + rz = 0 is a vector subspace of* V_3 *(F).*

Solution : Given, p, q, r are the fixed elements of a field F.

To show that the set W of all triads (x, y, z) of elements of F, such that $px + qy + rz = 0$ is a vector subspace of V_3 (F).

By the definition $W \neq \phi$.

Let α, $\beta \in W$, where $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$

for some $x_1, y_1, z_1, x_2, y_2, z_2 \in F$

By the definition of W, $px_1 + qy_1 + rz_1 = 0$ .... (1)

$px_2 + qy_2 + rz_2 = 0$ .... (2)

For a, b $\in$ F, consider $a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (a x_1 + ay_1 + a z_1) + (b x_2 + by_2 + b z_2)$

$= (a x_1 + b x_2,\ ay_1 + b y_2,\ a z_1 + b z_2)$

We now see whether the element $a\alpha + b\beta$ satisfies the condition $px + qy + rz = 0$

Consider $p(a x_1 + b x_2) + q(ay_1 + by_2) + r(az_1 + bz_2)$

$= a(p x_1 + qy_1 + r z_1) + b(p x_2 + q y_2 + r z_2)$

$= a(0) + b(0)$ (By (1) & (2)).

$= 0.$

$a\alpha + b\beta = (a x_1 + b x_2,\ ay_1 + b y_2,\ a z_1 + b z_2) \in W.$

Hence W is a subspace of V_3 (F).

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