Practical Session

The Rijke tube

MOD 4.3: Combustion for propulsion

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ÉCOLE CENTRALE LYON

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by

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Contents

List of Figures

Nomenclature

Abbreviations

Abbreviation	Definition
FTF	Flame Transfer Function

Symbols

Symbol	Definition	Unit
L	Length of the tube	$[m]$
n	Normalized flame response	[.]
c	Speed of sound	$[m/s]$
ρ	Air density	$[kg/m^3]$
x	Position of the flame	[.]
U_0	Mean speed of the flow	$[m/s]$
τ	Convection time	$[s]$
f	Frequency	$[Hz]$
ω	Pulsation	$[rad/s]$
k	Wave number	$[rad/m]$

1 Introduction

Figure 1.1: Illustration of a flame organ

This second practical session was focused on the study of an experiment of the flame organ. The objectives of this study were to identify the expected acoustic frequencies and to plot the structure of the standing pressure eigenmodes inside the tube.
To do so, the model was simplified to consider the flame as an interface between two zones: one with the cold airflow and the other with the hot flow. This configuration will be further introduced in the following sections of this report.
Using the model and parameters provided for this simulation, we will build the stability map of the Rijke tube to identify the most unstable acoustic frequencies and their associated tones. We will then be able to plot the structure of the standing pressure unstable eigenmodes inside the tube and use the Rayleigh criterion to describe their unstable nature.

2

Model description

2.1. Configuration and parameters

First of all, we will be introducing the model configuration and parameters used in this simulation to achieve the objectives described in the previous section.

2.1.1. Configuration

To simplify the problem we will be focusing on the following configuration :

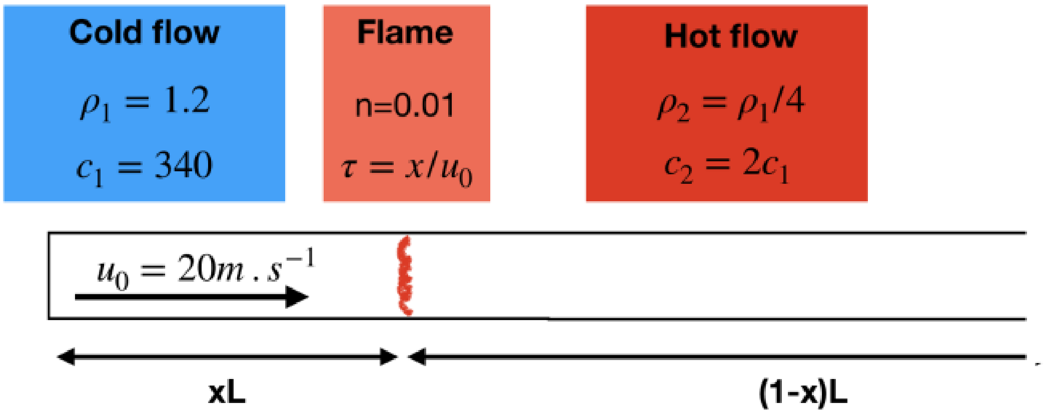


Figure 2.1: Model configuration considered for the simulation

In this case, we consider the flame as an interface between two flows. On the left, there is a cold air flow and on the right a hot one. The useful parameters of this simulation are the following :

- Convection time $\tau = \frac{xL}{U_0}$
- Length of the tube L = 1.783 m;
- Normalized flame response n = 0.01;
- Speed of sound in the first part of the tube c_1=340 m/s;
- Air density in the first part of the tube ρ_1= 1.2 kg/m^3
- Position of the flame x=0.2;
- Mean speed of the flow U_0=20 m/s;

2.1.2. Equations and assumptions

In this section, we will be introducing the different equations that will allow us to solve this acoustic problem.
Using the conservation of mass, momentum and energy and assuming no heat flux, radiation or viscous effects, it is possible to derive the following fundamental wave equation in the presence of a flame :

$$\frac{\partial^2}{\partial t^2}ln(P) - \frac{\partial}{\partial x_i}({c_0}^2\frac{\partial}{\partial x_i}ln(P)) = \frac{\partial}{\partial t}(\frac{\dot{\omega_T}}{\rho C_v T}) \tag{2.1}$$

This equation can be used to describe waves of finite amplitude in the non-linear case. However, for our study and to simplify the calculations, we will be assuming a linear model (P_0 does not vary in space and $P_1 << P_0$) so that the previous equation can be simplified as follows :

$$\frac{\partial^2 P_1}{\partial t^2} - \frac{\partial}{\partial x_i}({c_0}^2\frac{\partial P_1}{\partial x_i}) = (\gamma - 1)\frac{\partial \dot{\omega_T}}{\partial t} \tag{2.2}$$

This linear model can be used because in our case, it is possible to assume that the frequency of the note does not depend on the amplitude of the fluctuations induced by the flame.
To represent the flame and its response to external acoustic perturbations, we will be using Crocco's model Transfer Function. This representation assumes that the response of the flame is the same regardless of the frequency of excitation (n) and that the time delay between the excitation and the flame response is constant (τ) :

$$FTF(\omega) = ne^{-i\omega\tau} \tag{2.3}$$

By introducing this transfer function in the linear wave equation previously described, we can get the following dispersion relation :

$$cos(k_2(1-x)L)cos(k_1 xL) - \frac{\rho_2 c_2 S_1}{\rho_1 c_1 S_2} sin(k_2(1-x)L)sin(k_1 xL)(1 + ne^{i\omega\tau}) = 0 \tag{2.4}$$

Where k_1=$\frac{\omega}{c_1}$, k_2=$\frac{\omega}{c_2}$ and x is the position of the flame.

Taking into account the configuration of figure 2.1, we can rewrite the dispersion relation :

$$cos(k_2(1-x)L)cos(k_1 xL) - \frac{1}{2} sin(k_2(1-x)L)sin(k_1 xL)(1 + ne^{i\omega\tau}) = 0 \tag{2.5}$$

We will now be trying to solve this equation to determine the most unstable acoustic frequencies of the system by building its stability map.

3

Stability map of the Rijke tube

3.1. Solving the dispersion relation

In this section, we will use the Newton-Raphson algorithm (Appendix A) in the complex plane to solve the dispersion relation. For this matter, we will try to find the zeros of the function f defined by:

$$f(\omega) = cos(k_2(1-x)L)cos(k_1 xL) - \frac{\rho_2 c_2}{\rho_1 c_1} sin(k_2(1-x)L)sin(k_1 xL)(1 + ne^{i\omega\tau}) \tag{3.1}$$

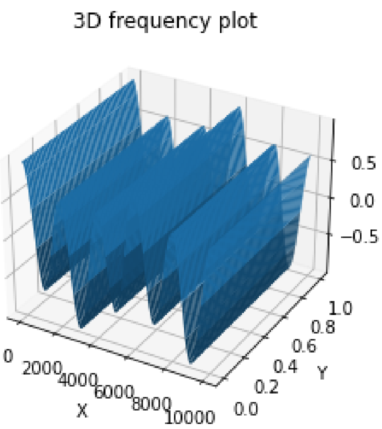
We also need its derivative which is calculated below:

$$\begin{aligned} f'(\omega) = & - \quad b_2 sin(b_2\omega)cos(b_1\omega) - b_1 cos(b_2\omega)sin(b_1 * \omega) - \frac{c_2}{4c_1} sin(b_2\omega)sin(b_1\omega)(1 + in\tau e^{i\omega\tau}) \\ & - \quad \frac{c_2}{4c_1} \left(b_2 cos(b_2\omega)sin(b_1\omega) + b_1 sin(b_2\omega)cos(b_1 * \omega)\right)(1 + ne^{i\omega\tau}) \end{aligned}$$

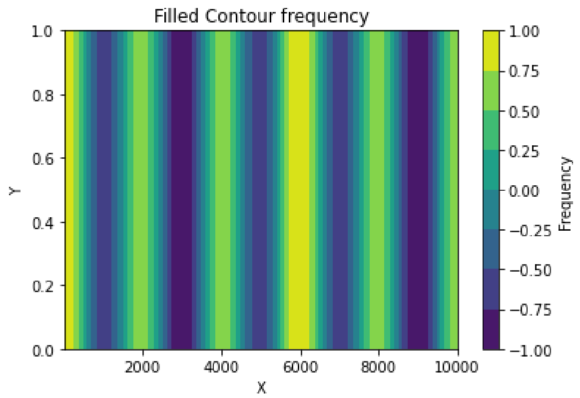
Where $b_1 = \frac{x_0 L}{c_1}$ and $b_2 = \frac{(1-x_0)L}{c_2}$

3.1.1. Geometric representation of the problem

The objective is to determine the zeros of the complex function f, represented in a 3D plot. This will provide insight into the potential zeros of the function.



(a) 3D plane equation



(b) Contours of the plane

Figure 3.1: Representation of the dispersion plane equation

3.1.2. Implementation of the algorithm and results

Our algorithm uses the Newton-Raphson method to find all zeros of the function. We only keep those with a positive imaginary part as they are the ones that are going to be amplified and threaten the stability of the flame.

Mode number	Frequency [Hz]	Growth rate (ω_i)	Associated tone (just click)
1	93.5	0.153	F♯2/G♭2
2	1046.9	0.153	C6 SOPRANO
3	2000.4	0.153	B6
4	2953.8	0.152	F♯7/G♭7
5	3907.3	0.153	B7
6	4860.7	0.153	D♯8/E♭8
7	5814.2	0.153	F♯8/G♭8
8	6767.6	0.152	G♯8/A♭8
9	7721.1	0.153	B8
10	8674.5	0.152	Above B8
11	9628.0	0.153	Above B8

Table 3.1: The most unstable acoustic frequencies of the Rijke tube

3.1.3. Stability map

Thanks to the Newton-Raphson algorithm, 11 unstable acoustic frequencies were identified between 200 and 10 000 Hz : these values were used to build the stability map of the tube which is pictured below.

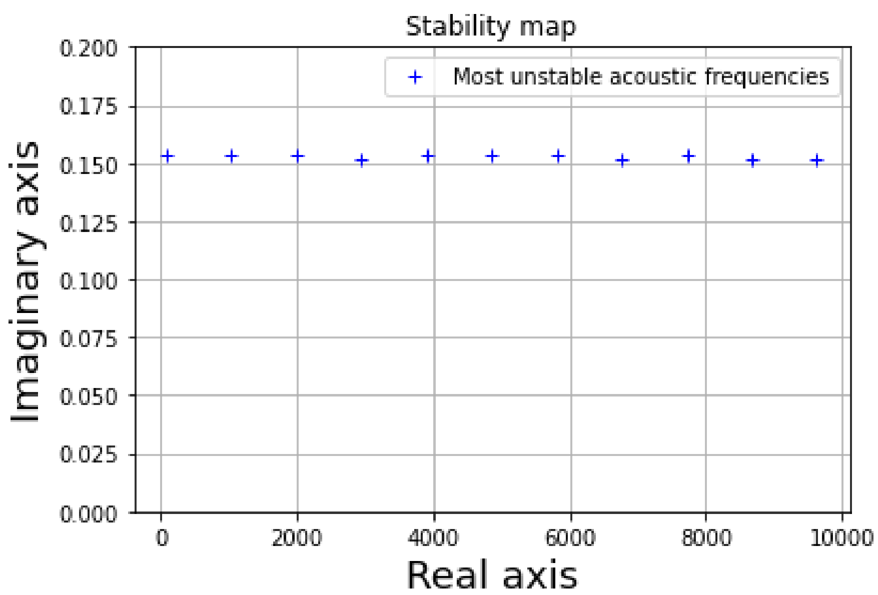


Figure 3.2: Stability map of the Rijke tube

We will examine the internal pressure of the tube for unstable modes previously identified. Through this, we aim to understand the underlying structures and patterns of these modes. This information will aid in developing control strategies to mitigate negative effects on system performance.

4

Standing pressure inside the tube

4.1. Expression of the standing pressure inside the tube

In this section, we will be attempting to compute the pressure fluctuations inside the Rijke tube. The general solutions to the Helmholtz equation for acoustic and pressure fluctuations can be written as follows :

$$p_1(x,t) = A_{1+}e^{ik_1x-i\omega t} + A_{1-}e^{-ik_1x-i\omega t} \tag{4.1}$$

$$u_1(x,t) = \frac{1}{\rho_1 c_1}(A_{1+}e^{ik_1x-i\omega t} - A_{1-}e^{-ik_1x-i\omega t}) \tag{4.2}$$

$$p_2(x,t) = A_{2+}e^{ik_2(x-L_1)-i\omega t} + A_{2-}e^{-ik_2(x-L_1)-i\omega t} \tag{4.3}$$

$$u_2(x,t) = \frac{1}{\rho_2 c_2}(A_{2+}e^{ik_2(x-L_1)-i\omega t} - A_{2-}e^{-ik_2(x-L_1)-i\omega t}) \tag{4.4}$$

Where $L_1 = xL$ and indices 1 refer to the first part of the tube and 2 to the second one.

As the tube is closed on the left side, $u_1(0,t) = 0$ which gives us $A_{1+} = A_{1-}$.
As the tube is opened on the right side, we have $p_2(L,t) = 0$ which gives us the condition $A_{2-} = -A_{2+}e^{2ik_2(L-L_1)}$.
We can also assume jump conditions through thin flames which state that $[p]_{L_1^-}^{L_1^+} = 0$ and that $[Su]_{L_1^-}^{L_1^+} = \frac{\gamma-1}{\gamma P_0}\dot{\omega_{T1}}$.

For the sake of this simulation and for simplification purposes, we will be assuming that $A_{1+} = 1$. Thanks to the conservation of the fluctuating pressure through thin flames, we get :

$$e^{ik_1L_1} + e^{ik_1L_1} = A_{2+} + A_{2-}$$

$$2cos(k_1L_1) = A_{2+}(1 - e^{2ik_2(L-L_1)})$$

$$cos(k_1L_1) = -iA_{2+}e^{ik_2(L-L_1)}sin(k_2(L-L_1)$$

Thanks to these different conditions, we can get the expressions of the several coefficients in the expressions of the acoustic and pressure fluctuations :

$$A_{2+} = ie^{-ik_2(L-L_1)}\frac{cos(k_1L_1)}{sin(k_2(L-L_1))} \tag{4.5}$$

$$A_{2-} = -ie^{i(2k_1-k_2)(L-L_1)}\frac{cos(k_1L_1)}{sin(k_2(L-L_1))} \tag{4.6}$$

$$A_{1+} = A_{1-} = 1 \tag{4.7}$$

This gives us the expression of the standing pressure $p_1(x)$ and $p_2(x)$ in the first and second part of the tube respectively :

$$p_1(x) = 2cos(k_1 x) \tag{4.8}$$

$$p_2(x) = |ie^{-ik_2(L-L_1)} \frac{cos(k_1 L_1)}{sin(k_2(L-L_1))} (e^{ik_2(x-L_1)} - e^{-ik_2(x-L_1)} e^{2ik_2(L-L_1)}| \tag{4.9}$$

Using the above-computed expressions, we will now be trying to plot the structure of the standing pressure unstable eigenmodes corresponding to the unstable frequencies previously determined for the Rijke tube.

4.2. Structure of the standing pressure inside the tube

Using the previous expressions, we were able to get the following structure for the standing pressure inside the Rijke tube :

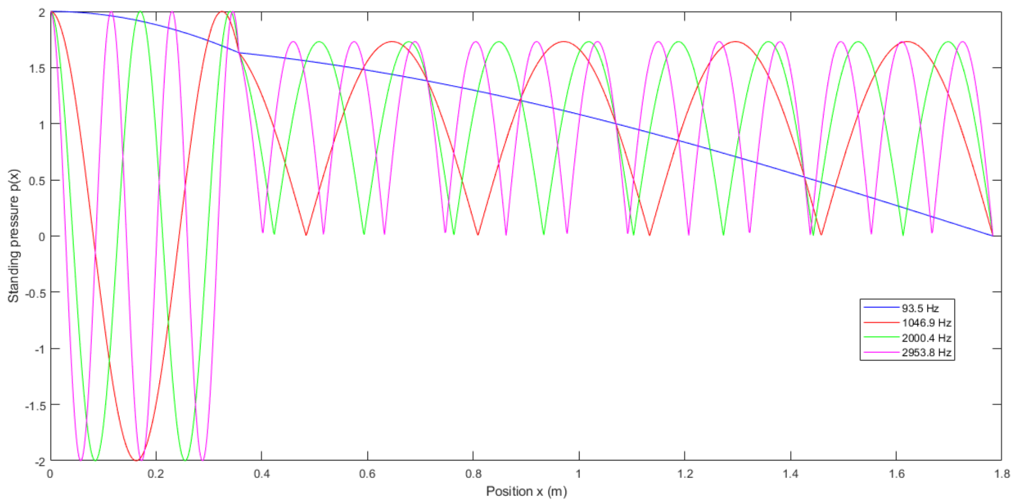


Figure 4.1: Standing pressure inside the Rijke tube for frequencies f_1 to f_4

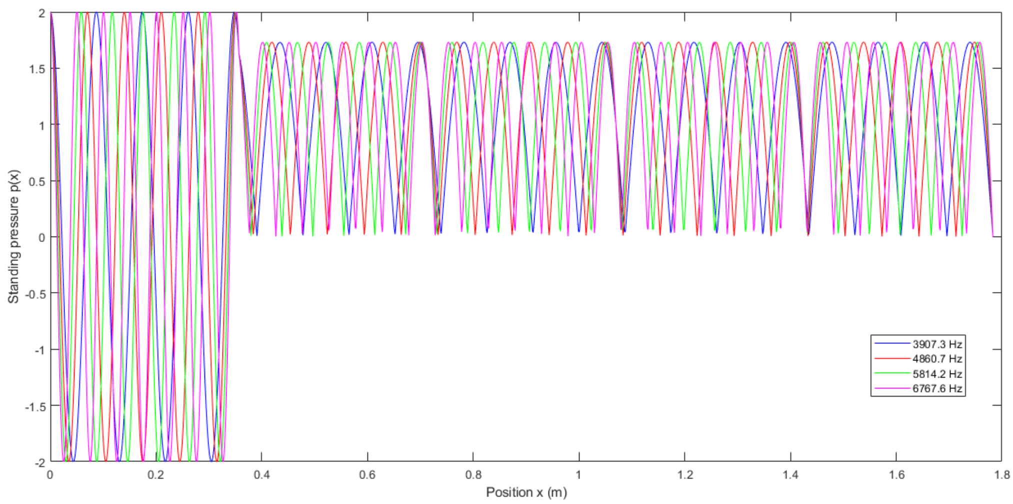


Figure 4.2: Standing pressure inside the Rijke tube for frequencies f_5 to f_8

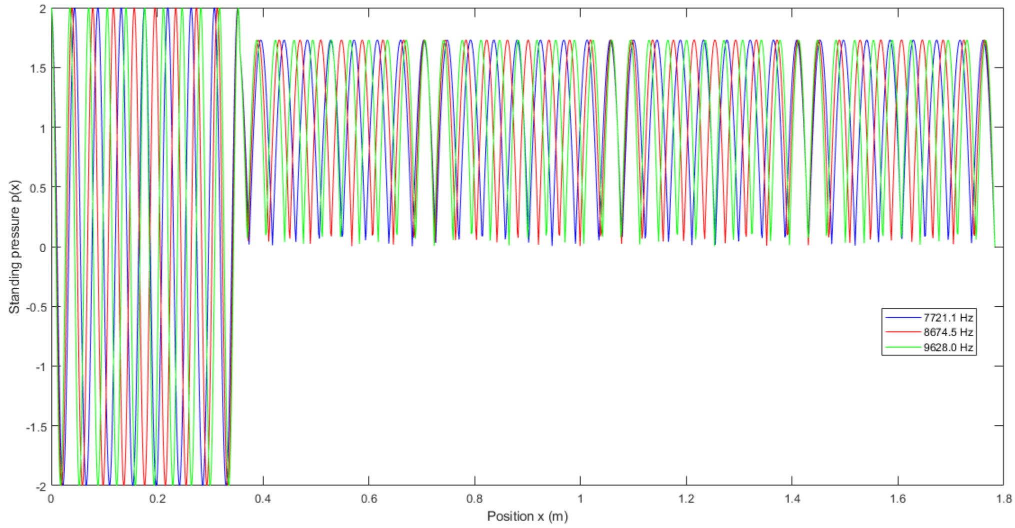


Figure 4.3: Standing pressure inside the Rijke tube for frequencies f_9 to f_{11}

4.3. Rayleigh criterion

The Rayleigh criterion states that for a localized heat source if ϕ represents the phase between the fluctuating pressure p_1 and $\dot{\omega_{T1}}$, the system will be unstable if $0 < \phi < \pi$ and will be stable if $\pi < \phi < 2\pi$. In order to check the relevance of the Rayleigh criterion in our study of the Rijke tube, we derived the phase of the standing pressure inside the system as pictured in the figures below :

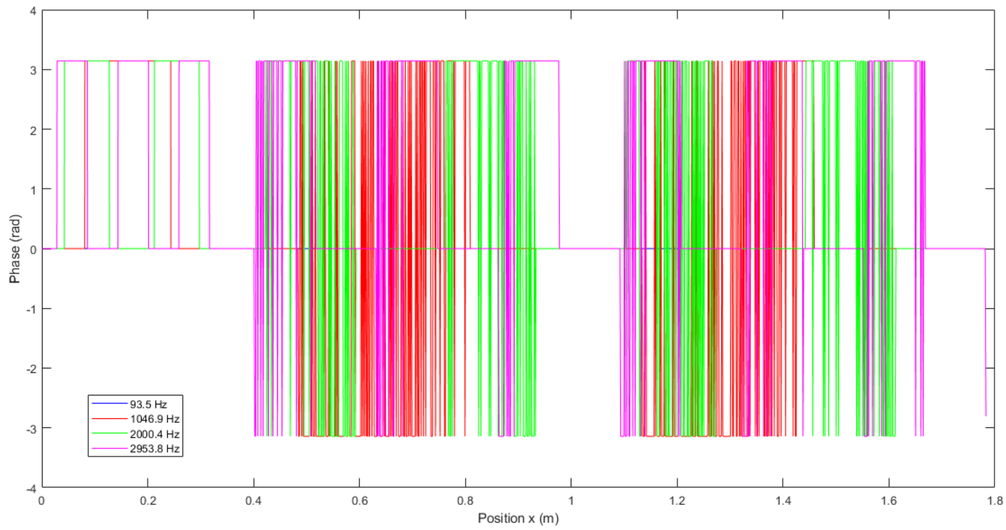


Figure 4.4: Phase of the standing pressure inside the Rijke tube for frequencies f_1 to f_4

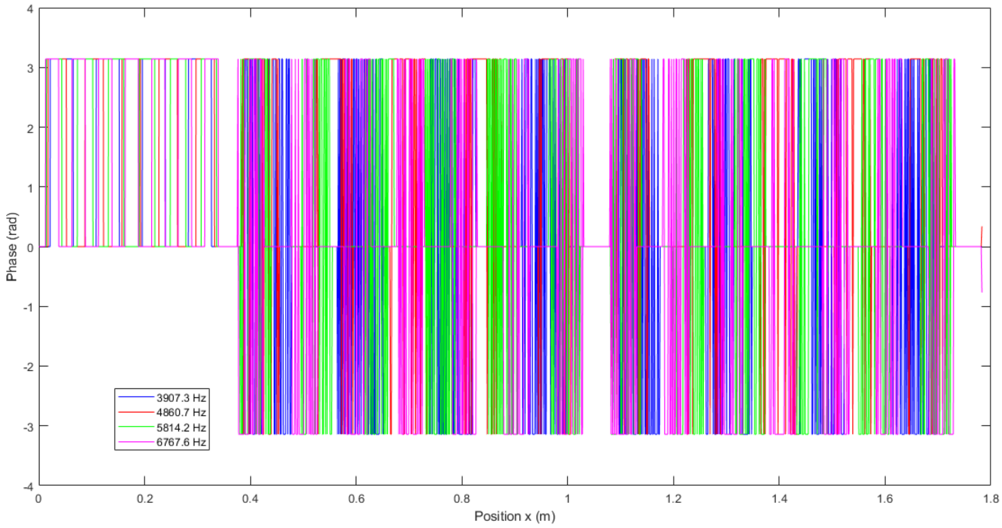


Figure 4.5: Phase of the standing pressure inside the Rijke tube for frequencies f_5 to f_8

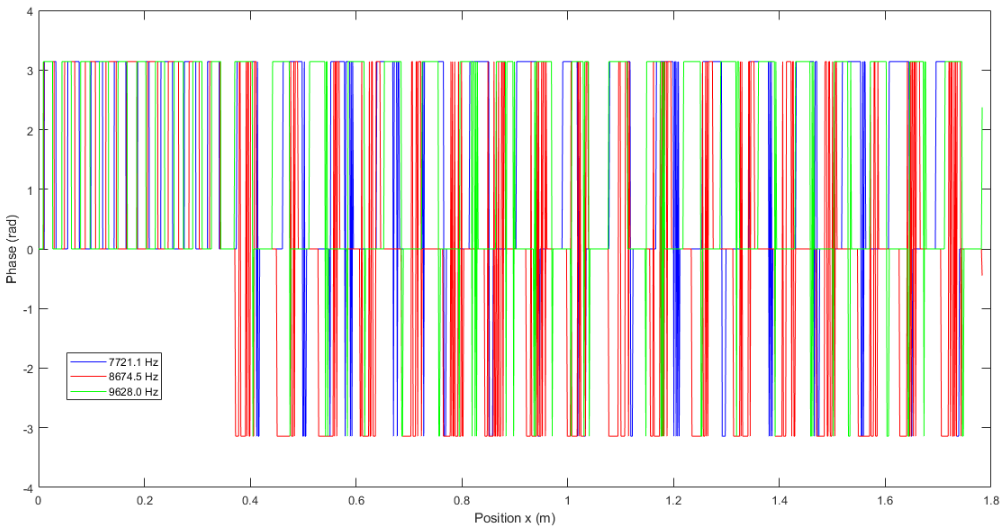


Figure 4.6: Phase of the standing pressure inside the Rijke tube for frequencies f_9 to f_{11}

Thanks to these considerations, we can see that the phase angle of the standing pressure inside the tube is always between $-\pi$ and π. According to the Rayleigh criterion, this means that these modes are indeed unstable. As a result, the Rayleigh criterion contributes to validating the unstable nature of the modes we identified in our study of the Rijke tube.

5

Conclusion and recommendations

To conclude, thanks to this practical session, we put into equation an experiment on a flame organ to identify the most unstable acoustic frequencies of a single tube. To do so, we implemented the Newton-Raphson method as a root-finding algorithm adapted to complex functions to solve the dispersion equation. Due to the flame inside the tube, we noticed a larger variation in standing pressure near the flame region by plotting the structure of the standing pressure which was determined thanks to the expressions of the pressure we computed analytically for the two regions separated by the flame that we considered as a thin flame. By deriving the phase of the standing pressure inside the tube, the Rayleigh criterion highlighted that the modes were unstable, thus validating the results obtained thanks to the solving of the dispersion relation.

To go further in this practical session, we could have used another root-finding algorithm, such as the Method of Delves and Lyness, in order to compare the unstable frequencies obtained with these different methods. We could also try to model the Rijke tube on commercial software to see how our hypothesis (linear model, constant position of the flame, flame response independent of frequency of excitation, constant time delay...etc) affects the results of this simulation.

A

Newton-Raphson algorithm

A.1. Reminders on the Newton–Raphson algorithm

In numerical analysis, the Newton–Raphson method is a root-finding algorithm which makes successively better approximation zeroes of a real-valued function. Its application can be extended to complex functions, we recall the principle of the algorithm below:

Let f be a function, f' its derivative and x_0 our initial guess for a root of f. If the function is close to the original guess and enough assumptions are met, then:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{A.1}$$

The process can be iterated as follows until a sufficiently precise value is reached:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{A.2}$$

A.2. Python code

```python

def newtonRaphson(x_guess,e,N,c1,c2,rho1,x0,L,Tau,n):
    """
    Implementation of the algorithm
    Parameters
    ----------
    x_guess : guess value of function zeros.
    e : error
    N : max number iteration

    Returns
    -------
    x1 : the angular frequency

    """
    #print('\n\n*** NEWTON RAPHSON METHOD IMPLEMENTATION ***')
    step = 1
    flag = 1
    condition = True
    while condition:
        if g(x_guess,c1,c2,rho1,x0,L,Tau,n) == 0.0:
            print('Divide by zero error!')
            break

        x1 = x_guess - f(x_guess,c1,c2,rho1,x0,L,Tau,n)/g(x_guess,c1,c2,rho1,x0,L,Tau,n)
        freq = x1/(2*np.pi)
        #print('Iteration-%d, x1 = %0.3f and f(x1) = %0.3f' % (step, x1, f(x1,c1,c2,rho1,x0,L
            ,Tau,n)))
        x_guess = x1
        step = step + 1

        if step > N: #we limit the maximum number of iterations if it diverges
            flag = 0
            break

        condition = abs(f(x1,c1,c2,rho1,x0,L,Tau,n)) > e #test the distance

    if flag==1:
        freq = x1/(2*np.pi)
        #print('\nRequired root is: %0.8f' % freq)
        #print('({0.real:.2f} + {0.imag:.2f}i)'.format(freq))
    else:
        print('\nNot Convergent.')

    return (freq)
```

B

Task Division

Table B.1: Distribution of the workload

	Task	Student Name(s)
Chapter 1	Introduction	Nezha Bourakkadi
Chapter 2	Model description	Nezha Bourakkadi
Chapter 3	Stability map of the Rijke tube	Jérémy Archier
Chapter 4	Standing pressure inside the tube	Nezha Bourakkadi
Chapter 5	Conclusion	Jérémy Archier & Nezha Bourakkadi
	Editors	Jérémy Archier & Nezha Bourakkadi
	Newton-Raphson algorithm	Jérémy Archier
	Document Design and Layout	Jérémy Archier