APPLICATIONS OF DERIVATIVE

Optimization Problems

Time-rates Problems

OPTIMIZATION PROBLEMS

MAXIMA AND MINIMA PROBLEMS

Optimization Problems: Maxima and Minima Problems

Steps in solving optimization problems:

1: Read and understand the problem. Identify the given and what is required

2: Draw the diagram and label the given and unknown quantities to help visualize.

3: Formulate the equations involved and express the quantity to be optimized as a function in terms of a single variable.

4: Differentiate the function and solve for the critical values. Do not forget to consider the domain.

5: Use the first derivative test to identify the optimum values.

6: Write your answer. Go back to the problem and review if your answer makes sense and always check the units.

Example 1: The sum of one number and twice another is 32. Find the two numbers so that their product is a maximum.

Solution: Let x and y be the numbers.
Let P be the product (to be maximized)

Required: the two numbers, x and y whose product is the maximum

Equation: $x + 2y = 32$

$$y = \frac{32 - x}{2}$$

Maximize: $P = xy$

$$P(x) = x\left(\frac{32-x}{2}\right)$$

$$P(x) = 16x - \frac{1}{2}x^2$$

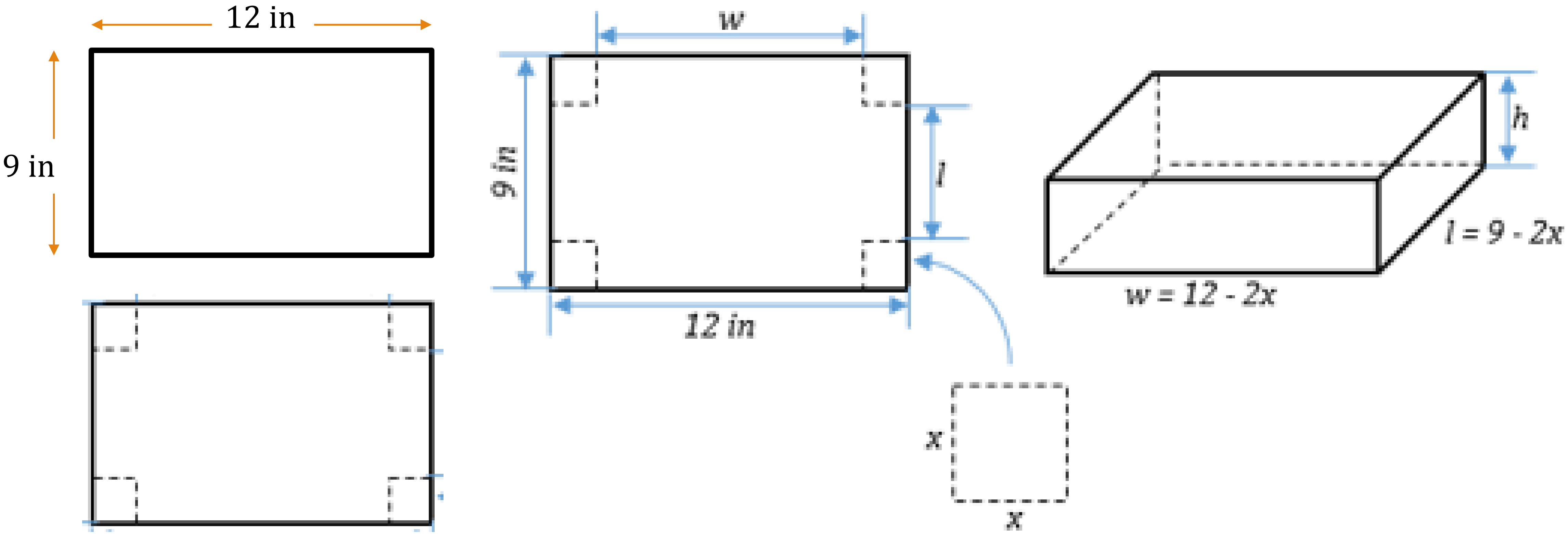
Differentiate: $P'(x) = 16 - x$

$$16 - x = 0$$

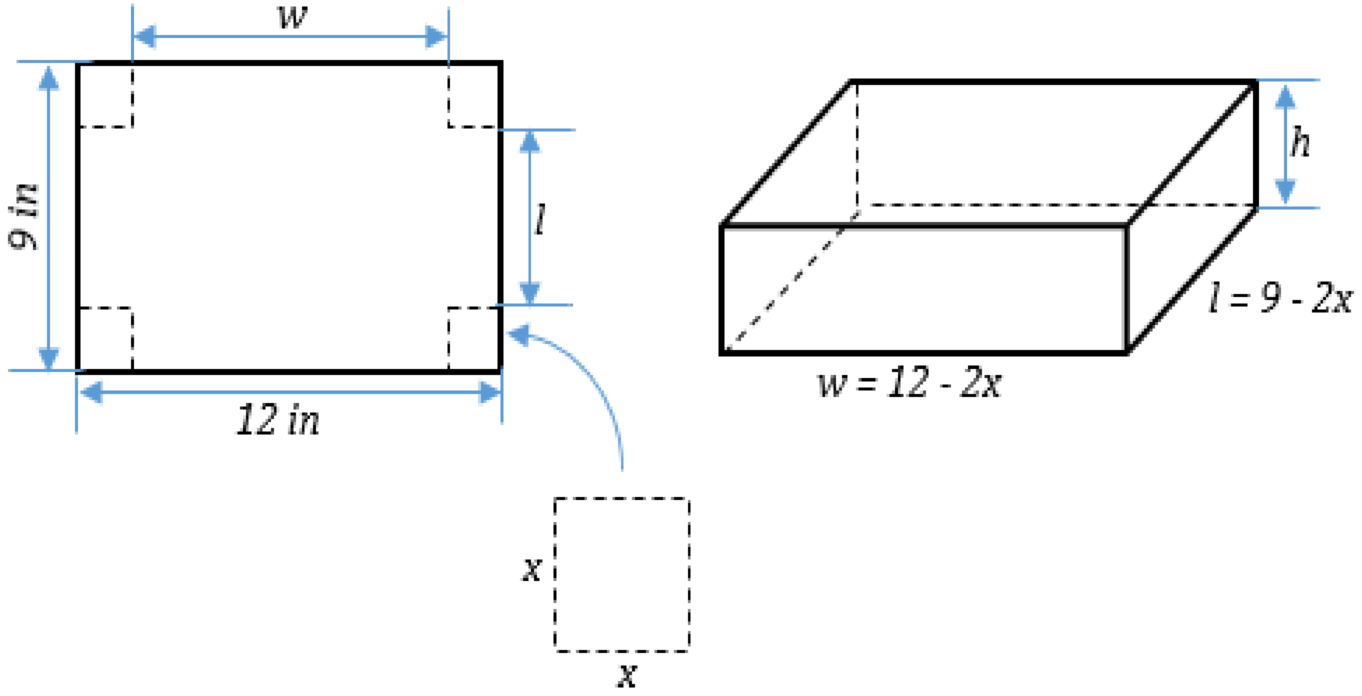
$$x = 16 \text{ and } y = \frac{32-16}{2} = \frac{16}{2} = 8$$

Therefore, the two numbers are **16 and 8**.

Example 2: A rectangular piece of cardboard has dimensions 12 inches by 9 inches. An open box is formed by cutting out equal square pieces at the corners and bending upward the projecting portions which remain. Find the maximum volume that can be obtained.



A rectangular piece of cardboard has dimensions 12 inches by 19 inches. An open box is formed by cutting out equal square pieces at the corners and bending upward the projecting portions which remain. Find the maximum volume that can be obtained.



Equations: $l = 9 - 2x$

$w = 12 - 2x$

$h = x$

Required: The maximum volume, V

Maximize: $V = lwh$

Let l, w and h be the dimensions of the box

Let V be the volume of the box (to be maximized)

Equations: $l = 9 - 2x$

$$w = 12 - 2x$$

$$h = x$$

Required: The maximum volume, V

Maximize: $V = lwh$

$$V(x) = (9 - 2\text{x})(12 - 2\text{x})(\text{x})$$

$$V(x) = 108x - 42x^2 + 4x^3$$

Differentiate:

$$V'(x) = 108 - 84x + 12x^2$$

$$108 - 84x + 12x^2 = 0$$

$$12(9 - 7x + x^2) = 0$$

(using the quadratic formula):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$
$b = -7$
$c = 9$

$x = 5.30$ and $x = 1.69$

A rectangular piece of cardboard has dimensions 12 inches by 19 inches. An open box is formed by cutting out equal square pieces at the corners and bending upward the projecting portions which remain. Find the maximum volume that can be obtained.

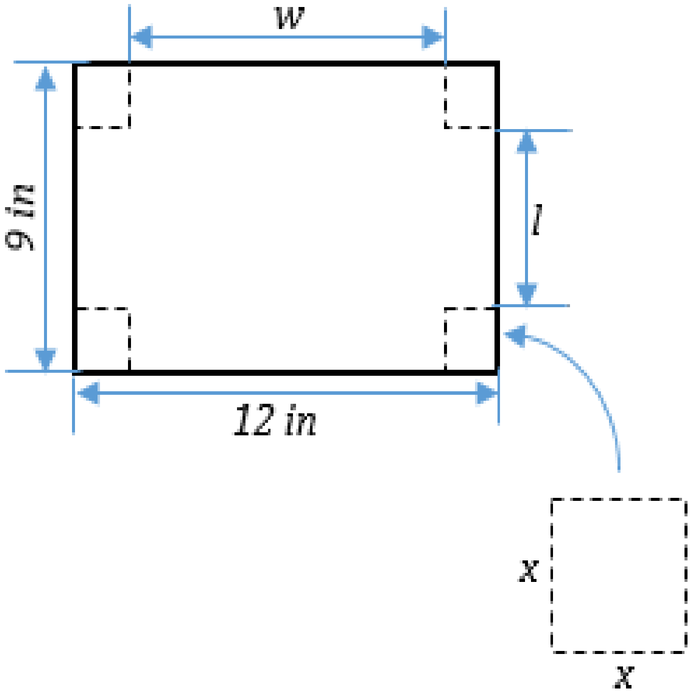
Equations:

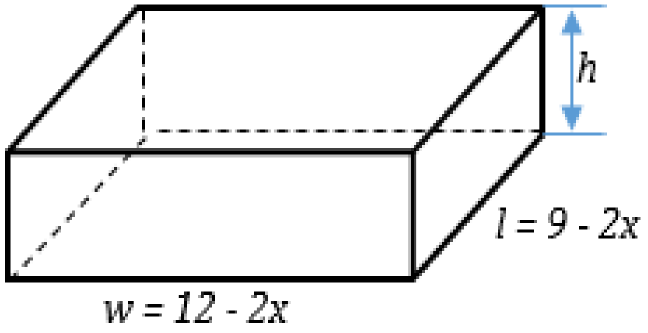
$$l = 9 - 2x$$

$$w = 12 - 2x$$

$$h = x$$

$x = 5.30$ and $x = 1.69$





$$l = 9 - 2(1.69) = 5.62\ in$$

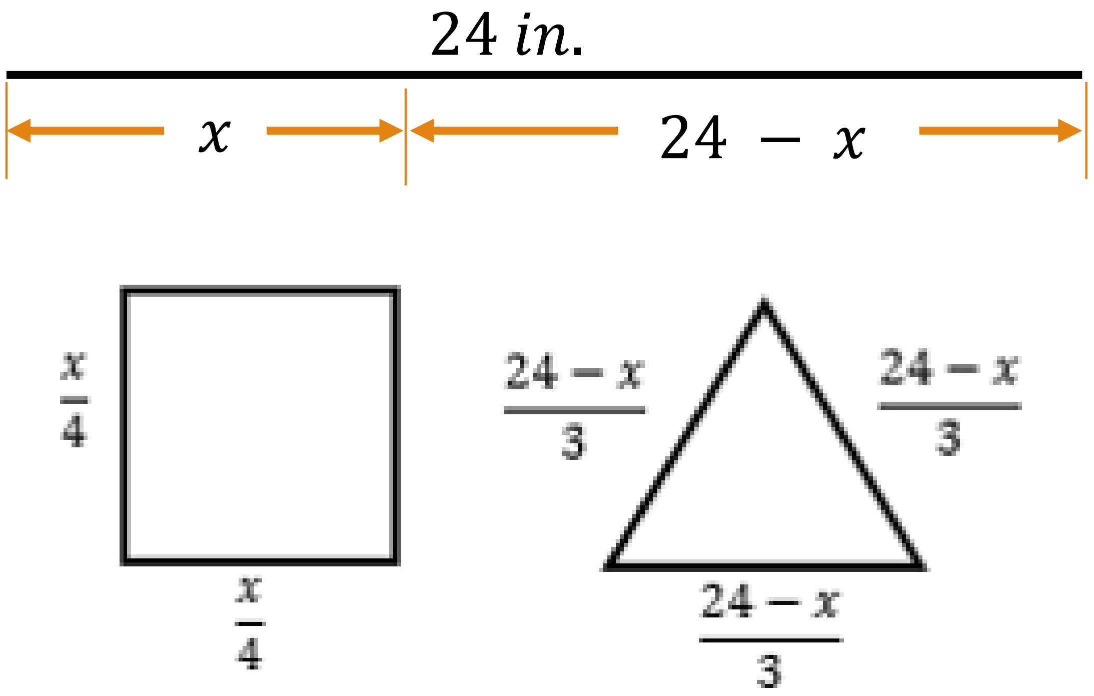
$$w = 12 - 2(1.69) = 8.62\ in$$

$$h = 1.69\ in$$

Therefore, $V = (5.62\ in)(8.62\ in)(1.69\ in)$

$$\boldsymbol{V = 81.871\ in^3}.$$

Example 3: A string 24 inches long is to be cut in two pieces. One part will be used to form a square and the other part to form an equilateral triangle. How should the string be cut so that the sum of the areas is a minimum?



Let x be one part of the string to be formed into a square and $24 - x$ be the part used to form the triangle

Let S be the sum of the areas

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Equations:

$$A_{Square} = (edge)^2 = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

$$A_{E\ Triangle} = \frac{\sqrt{3}}{4}(edge)^2$$

$$= \frac{\sqrt{3}}{4}\left(\frac{24-x}{3}\right)^2$$

$$= \frac{\sqrt{3}(24-x)^2}{36}$$

Required: How the string should be cut, x and $24 - x$

Minimize:

$$S = A_{Square} + A_{E\ Triangle}$$

$$S = \frac{x^2}{16} + \frac{\sqrt{3}(24-x)^2}{36}$$

Minimize: $S = A_{Square} + A_{E\ Triangle}$

$$S = \frac{x^2}{16} + \frac{\sqrt{3}(24-x)^2}{36}$$

$$S(x) = \frac{x^2}{16} + \frac{\sqrt{3}(576-48x+x^2)}{36}$$

$$S(x) = \frac{x^2}{16} + 16\sqrt{3} - \frac{4\sqrt{3}x}{3} + \frac{\sqrt{3}x^2}{36}$$

Required: How the string should be cut, x and $24 - x$

Therefore, the string should be cut where one part is $\mathbf{10.439}$ ***in*** and the other part $\mathbf{13.561}$ ***in*** so that the sum of the area is minimum.

$$S'(x) = \frac{2x}{16} - \frac{4\sqrt{3}}{3} + \frac{2\sqrt{3}x}{36}$$

$$S'(x) = \frac{x}{8} - \frac{4\sqrt{3}}{3} + \frac{\sqrt{3}x}{18}$$

$$S'(x) = \frac{4\sqrt{3}}{3} + \frac{9x+4\sqrt{3}x}{72}$$

$$S'(x) = \frac{4\sqrt{3}}{3} + \frac{(9+4\sqrt{3})x}{72}$$

$$\frac{4\sqrt{3}}{3} + \frac{(9+4\sqrt{3})x}{72} = 0$$

$x = 10.439\ in$ and

$24 - x = 13.561\ in$

RELATED RATES PROBLEMS

(TIME-RATES)

Steps in solving related-rates problems:

1: Read and understand the problem

2: Identify what is given and what is required.

3: Draw the figure and label with the given values and variables to represent unknown values.

4: Write the equations involved.

5: Solve the derivative of both sides of the equation with respect to time. It can be done by using implicit differentiation.

6: Solve for the required quantity.

Example 4: A spherical balloon is inflating at the rate of 12 $\frac{cm^3}{sec}$. How fast is its radius increasing when the radius is $5cm$?

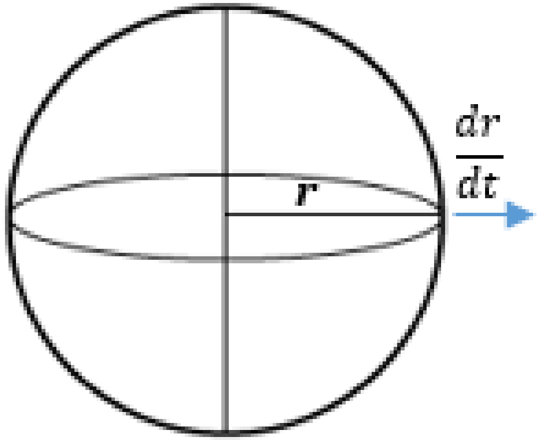
Solution: Let V be the volume of the balloon and r be the radius

Let $\frac{dV}{dt}$ be the rate of change in volume

and $\frac{dr}{dt}$ be the rate of change in radius.

Given: $\frac{dV}{dt} = 12\frac{cm^3}{sec}$

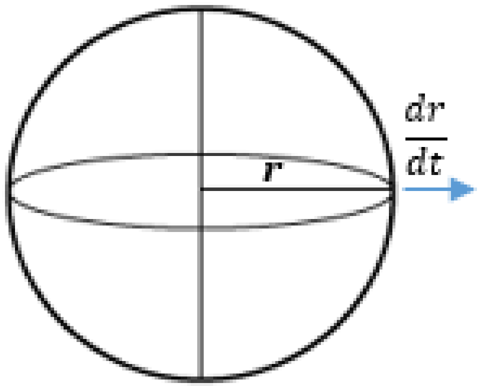
Required: $\frac{dr}{dt}$ when $r = 5\ cm$



Equation: $V_{sphere} = \frac{4}{3}\pi r^3$

Given: $\frac{dV}{dt} = 12\frac{cm^3}{sec}$

Required: $\frac{dr}{dt}$ when $r = 5\ cm$



Equation: $V_{sphere} = \frac{4}{3}\pi r^3$

Derivative: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}$$

Substitute: $\frac{dr}{dt} = \frac{1}{4\pi(5\ cm)^2} \cdot 12\frac{cm^3}{sec}$

$$\frac{dr}{dt} = \frac{12\ cm^3}{(100\pi cm^2)\ sec}$$

$$\frac{dr}{dt} = \frac{3}{25\pi}\frac{cm}{sec} \text{ or } 0.038\frac{cm}{sec}$$

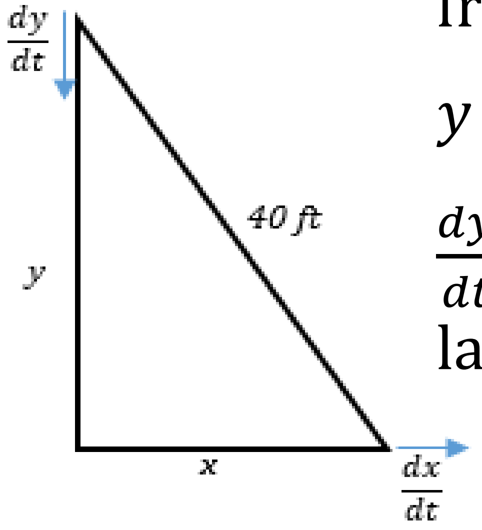
Example 5: A 40 ft ladder is leaning against a tall wall. The base of the ladder slips away from the wall at the rate of 3 ft/min. How fast is the top of the ladder moving down the wall when the base of the ladder is15 ft from the wall?

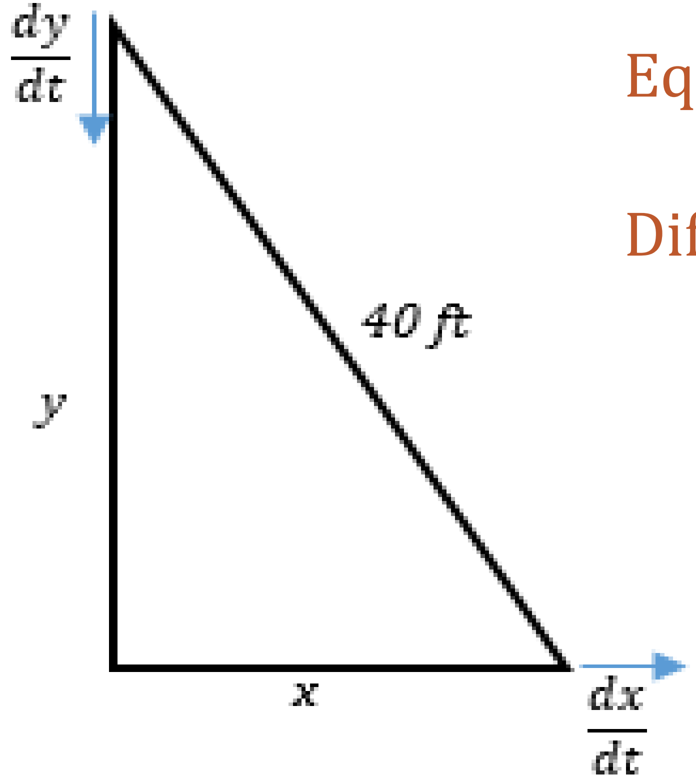
Solution: Let x be the distance of the foot of the ladder from the wall

$\frac{dx}{dt}$ be the rate of change of the distance of the foot of the ladder from the wall

y be distance of the top of the ladder from the foot of the wall

$\frac{dy}{dt}$ be the rate of change of the distance of the top of the ladder from the foot of the wall





Equation: $x^2 + y^2 = (40\ ft)^2$

Differentiate: $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$

$$2y \cdot \frac{dy}{dt} = -2x \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{2x}{2y} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

when $x = 15\ ft,$

$$(15\ ft)^2 + y^2 = (40\ ft)^2$$

$$y = 37.081\ ft$$

Given: $40\ ft$ – length of the ladder

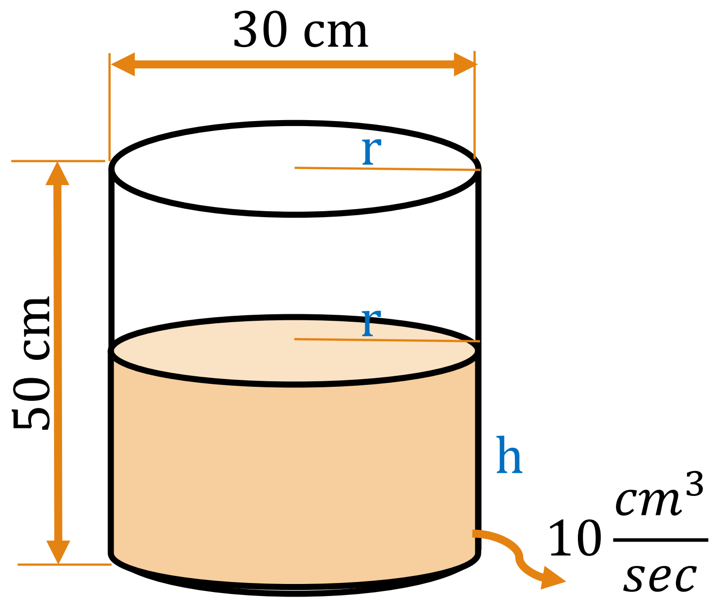
Required: $\frac{dy}{dt}$ when $x = 15\ ft$

Substitute: $\frac{dy}{dt} = -\frac{15\ ft}{37.081\ ft} \cdot 3\,\frac{ft}{min}$

$$\boldsymbol{\frac{dy}{dt} = -1.214\,\frac{ft}{min}}$$

Example 5: A cylindrical container 50 cm deep and 30 cm in diameter. Water leaks at the bottom of the container at $10\,\frac{cm^3}{sec}$. How fast is the water level dropping when the height of the water is 20 cm?

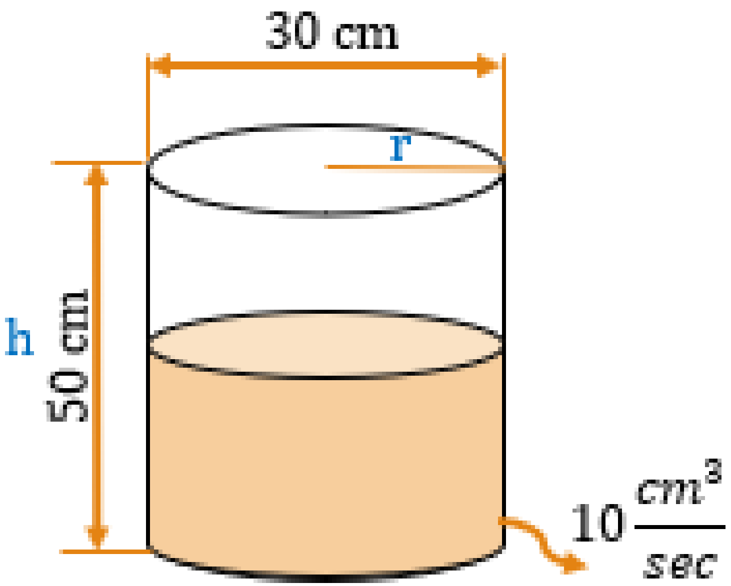
Solution:



Let h be the height and r be the radius of the water in the container.

Let $\frac{dh}{dt}$ be the rate of change in height of the water in the container

Let $\frac{dV}{dt}$ be the rate of change in the volume



Required: $\frac{dh}{dt}$ when $h = 20\ cm$

Equation: $V_{cylinder} = \pi r^2 h$

Differentiate:

$$\frac{dV}{dt} = \pi\left[r^2 \cdot \frac{dh}{dt} + h \cdot 2r \cdot \frac{dr}{dt}\right]$$

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} + \pi 2rh \cdot \frac{dr}{dt}$$

$$\pi r^2 \cdot \frac{dh}{dt} = \frac{dV}{dt} - \pi 2rh \cdot \frac{dr}{dt}$$

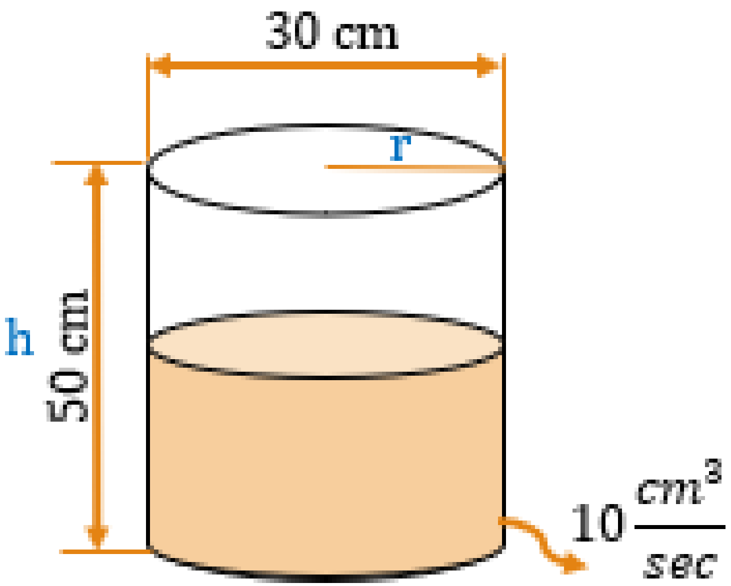
$$\frac{dh}{dt} = \frac{1}{\pi r^2} \cdot \frac{dV}{dt} - \frac{\pi 2rh}{\pi r^2} \cdot \frac{dr}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi r^2} \cdot \frac{dV}{dt} - \frac{2h}{r} \cdot \frac{dr}{dt}$$

Given: $h = 50\ cm, r = 15\ cm$

$$\frac{dV}{dt} = -10\frac{cm^3}{sec}$$

rate is negative because
the it is decreasing



Given: $h = 50\ cm, r = 15\ cm$

$$\frac{dV}{dt} = -10\,\frac{cm^3}{sec}$$

Required: $\frac{dh}{dt}$ when $h = 20\ cm$

$$\frac{dh}{dt} = \frac{1}{\pi r^2}\cdot\frac{dV}{dt} - \frac{2h}{r}\cdot\frac{dr}{dt}$$

Substitute:

$$\frac{dh}{dt} = \frac{1}{\pi(15\ cm)^2}\left(-10\,\frac{cm^3}{sec}\right) - \frac{2(20\ cm)}{15\ cm}(0)$$

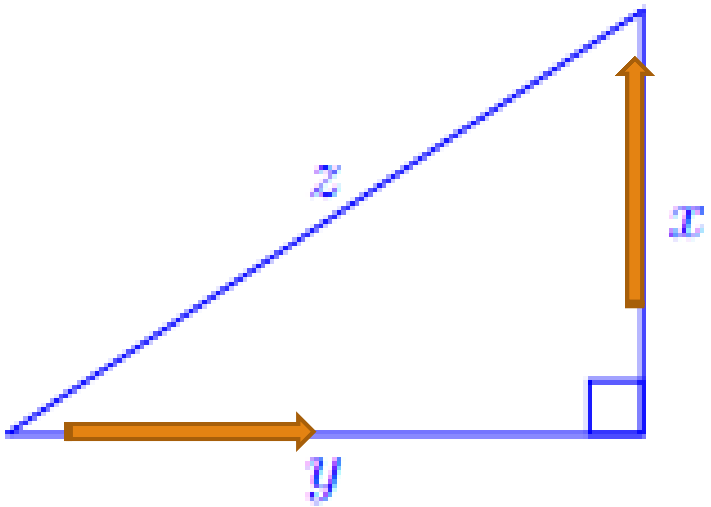
$\frac{dr}{dt} = 0$ because r is constant

$$\frac{dh}{dt} = \frac{-10}{225\pi cm^2}\cdot\frac{cm^3}{sec}$$

$$\frac{dh}{dt} = -\frac{2}{45\pi}\,\frac{cm}{sec} = -0.014\,\frac{cm}{sec}$$

Example 6: One car leaves a given point and travels north at 30 mph. Another car leaves 1 hour later, and travels east at 40 mph. At what rate is the distance between the cars changing at the instant the second car has been traveling for 1 hour?

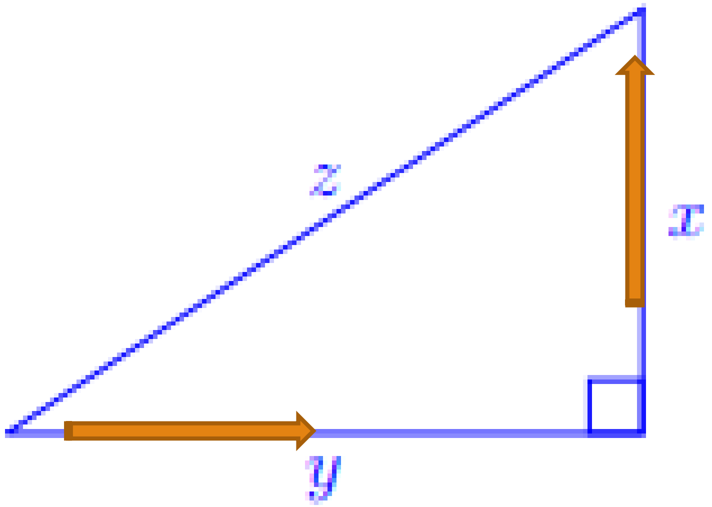
Solution:



Let $\frac{dx}{dt}$ be the rate of change in x (1^{st} car)

Let $\frac{dy}{dt}$ be the rate of change in y (2^{nd} car)

Let $\frac{dz}{dt}$ be the rate of change in the distance between car1 and car 2



Given:

$$\frac{dx}{dt} = 30\,mph \text{ and } x(t) = 30t$$

$$\frac{dy}{dt} = 40\,mph \text{ and } y(t) = 40(t-1)$$

Required: $\frac{dz}{dt}$ when $t = 2hrs$

Equation: $x^2 + y^2 = z^2$

Differentiate:

$$2x.\frac{dx}{dt} + 2y.\frac{dy}{dt} = 2z.\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{2x.\ \frac{dx}{dt} + 2y.\frac{dy}{dt}}{2z}$$

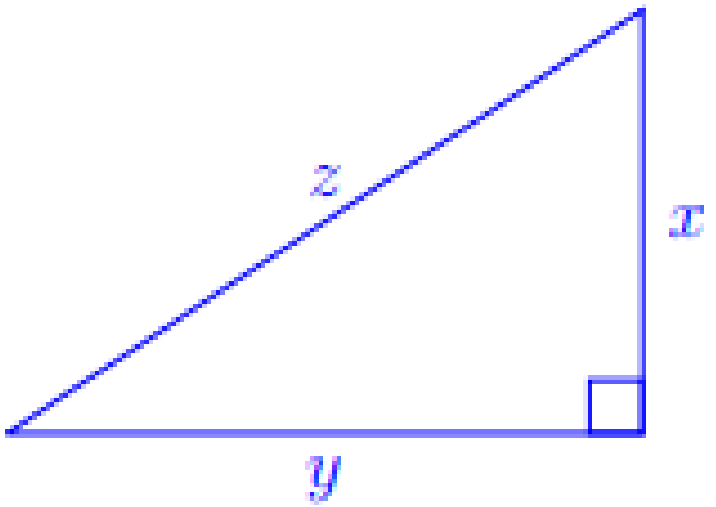
$$\frac{dz}{dt} = \frac{x.\ \frac{dx}{dt} + y.\frac{dy}{dt}}{z}$$

when $t = 2hrs, x = 60\ and\ y = 40$

$$z = \sqrt{x^2 + y^2}$$

$$= \sqrt{(60)^2 + (40)^2}$$

$$z = 72.111$$



Given:

$$\frac{dx}{dt} = 30\ mph \text{ and } x(t) = 30t$$

$$\frac{dy}{dt} = 40\ mph \text{ and } y(t) = 40(t-1)$$

Required: $\frac{dz}{dt}$ when $t = 2hrs$

when $t = 2hrs, x = 60\ and\ y = 40$

$$z = 72.111$$

Substitute: $\frac{dz}{dt} = \frac{x.\ \frac{dx}{dt} + y.\frac{dy}{dt}}{z}$

$$\frac{dz}{dt} = \frac{60 \cdot\ 30 + 40 \cdot 40}{72.111}$$

$$\boldsymbol{\frac{dz}{dt} = 47.150\ mph}$$

Home Work #9

Solve the following problems.

1. Find two numbers whose difference is 80 and whose product is a minimum.
2. A farmer wants to fence off a rectangular field and use a well as one side. He has 500 meters of fencing. What are the dimensions of the field that has the largest area?
3. Find the volume of the maximum right cylinder that can be inscribed in a sphere of radius 15 cm.
4. A balloon is at the height of 60 meters, and is rising at the constant rate of 4 m/sec. A tricycle passes beneath it, traveling in a straight line at the constant speed of 20 m/sec. How fast is the distance between the tricycle and the ballo0n increasing 5 seconds later?
5. Two cars, one going due West at the rate of 100 km/hr and the other going North at the rate of 70 km/hr, are traveling toward the intersection of the two roads. At what rate are the cars approaching each other at the instant when the first car is0.3 km and the second car is 0.2 km from the intersection?