Derivatives of

INVERSE TRIGONOMETRIC FUNCTIONS

Derivatives of Inverse Trigonometric Functions

1. $d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$

2. $d(\arccos u) = \frac{-1}{\sqrt{1-u^2}} \cdot du$

3. $d(\arctan u) = \frac{1}{1+u^2} \cdot du$

4. $d(\operatorname{arccot} u) = \frac{-1}{1+u^2} \cdot du$

5. $d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$

5. $d(\operatorname{arccsc} u) = \frac{-1}{u\sqrt{u^2-1}} \cdot du$

Example 1: Find the derivative of $y = \arcsin x^2$

Solution:

$$y = \arcsin x^2$$

$$y' = \frac{1}{\sqrt{1-(x^2)^2}} \cdot d(x^2)$$

$$y' = \frac{1}{\sqrt{1-x^4}} \cdot (2x)$$

$$\boldsymbol{y' = \frac{2x}{\sqrt{1-x^4}}}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 2: Find the derivative of $f(w) = \arctan(1 + 4w)$

Solution: $f(w) = \arctan(1 + 4w)$

$$f'(w) = \frac{1}{1+(1+4w)^2} \cdot d\ (1 + 4w)$$

$$f'(w) = \frac{1}{1+(1+4w)^2} \cdot (4)$$

$$f'(w) = \frac{4}{1 + 1 + 8w + 16w^2}$$

$$f'(w) = \frac{4}{2 + 8w + 16w^2}$$

$$\boldsymbol{f'(w) = \frac{2}{1 + 4w + 8w^2}}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 3: Find the derivative of $y = \operatorname{arcsec} 3x$

Solution: $y = \operatorname{arcsec} 3x$

$$y' = \frac{1}{|3x|\sqrt{(3x)^2 - 1}} \cdot d(3x)$$

$$y' = \frac{1}{|3x|\sqrt{9x^2 - 1}}(3)$$

$$\boldsymbol{y' = \frac{1}{|x|\sqrt{9x^2 - 1}}}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1 - u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1 + u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2 - 1}} \cdot du$$

Example 4: Find the derivative of $y = \arccos\left(-\frac{x}{3}\right)$

Solution:

$$y = \arccos\left(-\frac{x}{3}\right)$$

$$y' = \frac{-1}{\sqrt{1-\left(-\frac{x}{3}\right)^2}} \cdot d\left(-\frac{x}{3}\right)$$

$$y' = \frac{-1}{\sqrt{1-\frac{x^2}{9}}} \cdot \left(-\frac{1}{3}\right)$$

$$y' = \frac{-1}{\frac{\sqrt{9-x^2}}{3}} \cdot \left(-\frac{1}{3}\right)$$

$$y' = \frac{-3}{\sqrt{9-x^2}} \cdot \left(-\frac{1}{3}\right)$$

$$y' = \frac{-1}{\frac{\sqrt{9-x^2}}{9}} \cdot \left(-\frac{1}{3}\right)$$

$$\boldsymbol{y' = \frac{1}{\sqrt{9-x^2}}}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 5: Find the derivative of $y = \arctan^2 x^3$

Solution: $y = \arctan^2 x^3$

$$y' = 2arctan\, x^3 \cdot d\ (\arctan^2 x^3)$$

$$y' = 2arctan\, x^3 \cdot \frac{1}{1+(x^3)^2} \cdot d\ (x^3)$$

$$y' = 2arctan\, x^3 \cdot \frac{1}{1+(x^3)^2}\ (3x^2)$$

$$y' = 2arctan\, x^3 \cdot \frac{3x^2}{1+x^6}$$

$$y' = \frac{6x^2}{1+x^6} arctan\, x^3$$

$$\boldsymbol{y' = \frac{6x^2 arctan\, x^3}{1+x^6}}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 6: Find the derivative of $y = (1 + x^2) \arctan x - x$

Solution: $y = \underbrace{(1 + x^2)}_{u} \underbrace{\arctan x}_{v} - x$

$$y' = \underbrace{(1 + x^2)}_{u} \cdot \underbrace{d(\arctan x)}_{dv} + \underbrace{\arctan x}_{v}\, \underbrace{d(1 + x^2)}_{du} - 1$$

$$y' = (1 + x^2) \cdot \frac{1}{1+x^2}(1) + \arctan x\,(2x) - 1$$

$$y' = 1 + \arctan x\,(2x) - 1$$

$$\boldsymbol{y' = 2x\,arctan\,x}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 7: Find the derivative of $y = x \arcsin x + \sqrt{1 - x^2}$

Solution:

$$y = x \arcsin x + \sqrt{1 - x^2}$$

$$y' = [x\, \mathrm{d}(\arcsin \mathrm{x}) + \arcsin x\, d(x)] + \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)$$

$$y' = \left[x\left(\frac{1}{\sqrt{1 - x^2}} \cdot 1\right) + \arcsin x\,(1)\right] + \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)$$

$$y' = \frac{x}{\sqrt{1 - x^2}} + \arcsin x - \frac{x}{\sqrt{1 - x^2}}$$

$$\boldsymbol{y' = arcsin\, x}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1 - u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1 + u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2 - 1}} \cdot du$$

Example 8: Find the derivative of $y = \cos(\arcsin x)$

Solution:

$$y = \cos(\arcsin x)$$

$$y' = -\sin(\arcsin x)d\ (\arcsin x)$$

$$y' = -\sin(\arcsin x)\frac{1}{\sqrt{1-x^2}} \cdot 1$$

$$y' = -\mathrm{x} \cdot \frac{1}{\sqrt{1-x^2}} \cdot 1$$

$$\boldsymbol{y' = \frac{-x}{\sqrt{1-x^2}}}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 9: Find the derivative of $y = \operatorname{arccot}\frac{1}{x} - \arctan x$

Solution:

$$y = \operatorname{arccot}\frac{1}{x} - \arctan x$$

$$y' = \frac{-1}{1+\left(\frac{1}{x}\right)^2} d\left(\frac{1}{x}\right) - \frac{1}{1+(x)^2} d(x)$$

$$y' = \frac{-1}{1+\frac{1}{x^2}} \left(\frac{-1}{x^2}\right) - \frac{1}{1+(x)^2} (1)$$

$$y' = \frac{-1}{\frac{x^2+1}{x^2}} \left(\frac{-1}{x^2}\right) - \frac{1}{1+(x)^2}$$

$$y' = \frac{-x^2}{x^2+1} \left(\frac{-1}{x^2}\right) - \frac{1}{1+(x)^2}$$

$$y' = \frac{1}{x^2+1} - \frac{1}{1+x^2}$$

$$\boldsymbol{y' = 0}$$

$$d(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1+u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2-1}} \cdot du$$

Example 10: Find the y' of $\operatorname{arcsec}\frac{x}{y} - \operatorname{arccsc}\frac{y}{x} = x - y$

Solution:

$$\frac{1}{\frac{x}{y}\sqrt{\left(\frac{x}{y}\right)^2 - 1}} d\left(\frac{x}{y}\right) - \frac{-1}{\frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}} d\left(\frac{y}{x}\right) = d(x) - d(y)$$

$$\left[\frac{1}{\frac{x}{y}\sqrt{\left(\frac{x}{y}\right)^2 - 1}}\right]\left(\frac{y(1) - x(1)y'}{y^2}\right) - \left[\frac{-1}{\frac{y}{x}\sqrt{\left(\frac{y}{x}\right)^2 - 1}}\right]\left(\frac{x(1)y' - y(1)}{x^2}\right) = 1 - (1)(y')$$

$$\left[\frac{1}{\frac{x}{y}\sqrt{\frac{x^2}{y^2} - 1}}\right]\left(\frac{y - xy'}{y^2}\right) - \left[\frac{-1}{\frac{y}{x}\sqrt{\frac{y^2}{x^2} - 1}}\right]\left(\frac{xy' - y}{x^2}\right) = 1 - y'$$

$$\left[\frac{1}{\frac{x}{y}\sqrt{\frac{x^2 - y^2}{y^2}}}\right]\left(\frac{y - xy'}{y^2}\right) - \left[\frac{-1}{\frac{y}{x}\sqrt{\frac{y^2 - x^2}{x^2}}}\right]\left(\frac{xy' - y}{x^2}\right) = 1 - y'$$

$$\left[\frac{y^2}{x\sqrt{x^2 - y^2}}\right]\left(\frac{y - xy'}{y^2}\right) - \left[\frac{-x^2}{y\sqrt{y^2 - x^2}}\right]\left(\frac{xy' - y}{x^2}\right) = 1 - y'$$

$$d(\arcsin u) = \frac{1}{\sqrt{1 - u^2}} \cdot du$$

$$d(\arctan u) = \frac{1}{1 + u^2} \cdot du$$

$$d(\operatorname{arcsec} u) = \frac{1}{u\sqrt{u^2 - 1}} \cdot du$$

Example 10: Find the y' of $\operatorname{arcsec}\frac{x}{y} - \operatorname{arccsc}\frac{y}{x} = x - y$

Solution:

$$\left[\frac{y^2}{x\sqrt{x^2-y^2}}\right]\left(\frac{y-xy'}{y^2}\right) - \left[\frac{-x^2}{y\sqrt{y^2-x^2}}\right]\left(\frac{xy'-y}{x^2}\right) = 1 - y'$$

$$\frac{y}{x\sqrt{x^2-y^2}} - \frac{y'}{\sqrt{x^2-y^2}} + \frac{xy'}{y\sqrt{y^2-x^2}} - \frac{1}{\sqrt{y^2-x^2}} = 1 - y'$$

$$y' - \frac{y'}{\sqrt{x^2-y^2}} + \frac{xy'}{y\sqrt{y^2-x^2}} = 1 - \frac{y}{x\sqrt{x^2-y^2}} + \frac{1}{\sqrt{y^2-x^2}}$$

$$y'\left(1 - \frac{1}{\sqrt{x^2-y^2}} + \frac{x}{y\sqrt{y^2-x^2}}\right) = 1 - \frac{y}{x\sqrt{x^2-y^2}} + \frac{1}{\sqrt{y^2-x^2}}$$

$$y'\left(\frac{y\sqrt{y^2-x^2}\sqrt{x^2-y^2} - y\sqrt{y^2-x^2} + x\sqrt{x^2-y^2}}{y\sqrt{y^2-x^2}\sqrt{x^2-y^2}}\right) = \frac{x\sqrt{y^2-x^2}\sqrt{x^2-y^2} - y\sqrt{y^2-x^2} + x\sqrt{x^2-y^2}}{x\sqrt{y^2-x^2}\sqrt{x^2-y^2}}$$

$$\boldsymbol{y' = \frac{y\left(x\sqrt{y^2-x^2}\sqrt{x^2-y^2} - y\sqrt{y^2-x^2} + x\sqrt{x^2-y^2}\right)}{x\left(y\sqrt{y^2-x^2}\sqrt{x^2-y^2} - y\sqrt{y^2-x^2} + x\sqrt{x^2-y^2}\right)}}$$

Home Work #11

Find the first derivative of the following functions and simplify the result whenever possible.

1. $y = x \arcsin\frac{x}{2} - \frac{\sqrt{4-x^2}}{x}$

2. $y = \frac{1}{4}\arctan\frac{4\sin x}{3+5\cos x}$

3. $y = \sqrt{\arcsin(1-x^2)}$

4. $y = arccot\, x + \arctan\frac{2+x}{1-2x}$

5. $y = x\, arcsin^2 x - 2x + 2\sqrt{1-x^2}\arcsin x$

6. $\arctan\frac{x}{y} = x - y$

7. $\arcsin\frac{x}{y} + \arccos\frac{y}{x} = 1$

8. $y = \frac{(x+3)^2}{\text{arccsc}(x+3)}$