HIGHER-ORDER DERIVATIVES

Higher-Order Derivative

Recall: Given a function $y = f(x)$, its derivative, $y' = f'(x)$.

The derivative of y', will then be referred to as the *second derivative* of y

$$y'' = f''(x)$$

Taking its derivative or the *third derivative* of y and it can go on and on.

Higher-Order Derivative

Function	y	$f(x)$	y	y
1st derivative	y'	$f'(x)$	$\dfrac{dy}{dx}$	$D_x(y)$
2nd derivative	y''	$f''(x)$	$\dfrac{d^2y}{dx^2}$	$D_x^2(y)$
3rd derivative	y'''	$f'''(x)$	$\dfrac{d^3y}{dx^3}$	$D_x^3(y)$
4th derivative	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$	$D_x^4(y)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth derivative	$y^{(n)}$	$f^{(4)}(x)$	$\dfrac{d^ny}{dx^n}$	$D_x^n(y)$

Example 1: For the function $y = 2x^5 - x^4 - 5x^2 + 2x + 3$, let us find the 1st, 2nd, 3rd and 4th derivatives.

Solution:

$$y' = 10x^4 - 4x^3 - 10x + 2$$

$$y'' = 40x^3 - 12x^2 - 10$$

$$y''' = 120x^2 - 24x$$

$$y^{(4)} = 240x - 24$$

Example 2: If $G(r) = \sqrt[3]{r}$, find $G'(8)$ and $G''(8)$.

Solution:

$$G(r) = \sqrt[3]{r} = r^{1/3}$$

$$G'(r) = \frac{1}{3} r^{-2/3} = \frac{1}{3r^{2/3}} = \boldsymbol{\frac{1}{3\sqrt[3]{r}^{2}}}$$

$$G'(8) = \frac{1}{3\sqrt[3]{8}^{2}} = \frac{1}{3\sqrt[3]{64}} = \frac{1}{3(4)} = \boldsymbol{\frac{1}{12}}$$

$$G''(r) = \frac{1}{3} \cdot -\frac{2}{3} r^{-5/3} = -\frac{2}{9r^{5/3}} = -\frac{2}{9\sqrt[3]{\boldsymbol{r}}^{\boldsymbol{5}}}$$

$$G''(8) = -\frac{2}{9\sqrt[3]{\mathbf{8}}^{\mathbf{5}}} = \boldsymbol{-\frac{1}{144}}$$

Example 3: Find $H'''(y)$, given the function $H(y) = \sqrt[3]{2y-7}$

Solution:

$$H(y) = \sqrt[3]{2y-7} = (2y-7)^{1/3}$$

$$H'(y) = \frac{1}{3}(2y-7)^{-2/3}[2]$$

$$= \frac{2}{3}(2y-7)^{-2/3}$$

$$\boldsymbol{H'(y) = \frac{2}{3(2y-7)^{2/3}}}$$

$$H''(y) = \frac{2}{3} \cdot -\frac{2}{3}(2y-7)^{-5/3}\,[2]$$

$$= -\frac{8}{9}(2y-7)^{-5/3}$$

$$\boldsymbol{H''(y) = -\frac{8}{9(2y-7)^{5/3}}}$$

$$H'''(y) = -\frac{8}{9} \cdot -\frac{5}{3}(2y-7)^{-8/3}[2]$$

$$= \frac{80}{27}(2y-7)^{-8/3}$$

$$\boldsymbol{H''(y) = \frac{80}{27(2y-7)^{8/3}}}$$

Example 4: Find y'', given the function $y = (2-3x)^3(2x-1)^2$

Solution:

$$y' = (2-3x)^3[2(2x-1)(2)] + (2x-1)^2[3(2-3x)^2(-3)]$$

$$= (4)(2-3x)^3(2x-1) + (-9)(2x-1)^2(2-3x)^2$$

$$= (2x-1)(2-3x)^2[(4)(2-3x) + (-9)(2x-1)]$$

$$= (2x-1)(2-3x)^2[8-12x-18x+9]$$

$$y' = \underbrace{(2x-1)}_{u}\underbrace{(2-3x)^2}_{v}\underbrace{[17-30x]}_{w}$$

General Form (Product Rule):

$$d(uvw) = uvdw + uwdv + vwdu$$

$$y' = \underbrace{(2x-1)}_{u}\underbrace{(2-3x)^2}_{v}\underbrace{[17-30x]}_{w}$$

General Form (Product Rule):

$$d(uvw) = uvdw + uwdv + vwdu$$

$$y'' = (2x-1)(2-3x)^2[-30] + (2x-1)(17-30x)[2(2-3x)(-3)] + (2-3x)^2(17-30x)[2]$$

...

$$y' = (2x-1)(2-3x)^2[17-30x]$$

$$y' = (2-3x)^2(-60x^2+64x-17)$$

$$y'' = (2-3x)^2(-120x+64) + (-60x^2+64x-17)[2(2-3x)(-3)]$$

$$= (-8)(2-3x)^2(15x+8) + (-6)(-60x^2+64x-17)(2-3x)$$

$$= (-2)(2-3x)[(4)(2-3x)(15x+8) + (3)(-60x^2+64x-17)]$$

$$= (-2)(2-3x)[-180x^2+24x+64-180x^2+192x-51]$$

$$\mathbf{y'' = (-2)(2-3x)\left[-360x^2+216x+13\right]}$$

IMPLICIT DIFFERENTIATION

EXPLICIT FUNCTION is a function where the dependent variable, y is given in terms of an independent variable, x.

$$y = x^3 - 3x^2 + 2x$$

$$f(x) = \frac{\sec 4x}{\tan 4x}$$

$$h'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$

$$y = \sin 3x + e^{-2x} + \ln 7x$$

IMPLICIT FUNCTION is given in terms of both the independent and dependent variables.

$$x^2 + y^2 = 9 \Rightarrow y = \pm\sqrt{9 - x^2}$$

$$3x - 4y - 4x^3y^3 = y^4 - 18$$

$$2x - 3y = 9 \Rightarrow y = \frac{2}{3}x - 3$$

$$x^2 \tan y + y^2 \sec x = 2x$$

Implicit Differentiation is the process of finding the derivative of a function that is implicitly defined.

Example 1: Differentiate $x^2 + y^2 = 9$ in terms of x

Solution 1 (express in explicit form):

$$y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$

we can differentiate either

$y = \sqrt{9 - x^2}$ or $y = -\sqrt{9 - x^2}$

using chain rule.

Solving for the derivative of

$y = \sqrt{9 - x^2} = (9 - x^2)^{1/2}$,

$$y' = \frac{1}{2}(9 - x^2)^{-1/2}[-2x]$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

Example 1: Differentiate $x^2 + y^2 = 9$ in terms of x

Solution 2 (by implicit differentiation):

$$2x + 2y \cdot y' = 0$$

$$2y \cdot y' = -2x$$

$$\frac{2y \cdot y'}{2y} = \frac{-2x}{2y}$$

$$\boldsymbol{y' = \frac{-x}{y}}$$

Answer from Solution 1:

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$y = \pm\sqrt{9 - x^2}$$

Example 2: Find the derivative of y in the given function

$$\mathbf{3x - 4y - 4x^3y^3 = y^4 - 18}$$

Solution:

$$3 - 4 \cdot y' - 4[x^3(3y^2 \cdot y') + y^3\,(3x^2)] = 4y^3 \cdot y'$$

$$3 - 4 \cdot y' - 12x^3y^2 \cdot y' - 12x^2y^3 = 4y^3 \cdot y'$$

$$-4 \cdot y' - 12x^3y^2 \cdot y' - 4y^3 \cdot y' = 12x^2y^3 - 3$$

$$y'(-4 - 12x^3y^2 - 4y^3) = 12x^2y^3 - 3$$

$$\frac{y'(-4 - 12x^3y^2 - 4y^3)}{-4 - 12x^3y^2 - 4y^3} = \frac{12x^2y^3 - 3}{-4 - 12x^3y^2 - 4y^3}$$

$$y' = \frac{12x^2y^3 - 3}{-4 - 12x^3y^2 - 4y^3}$$

Example 3: Find the derivative of y in the given function $\sqrt{\boldsymbol{x^2 - y^2}} = \dfrac{\boldsymbol{3x^2}}{\boldsymbol{2y^3}}$

Solution:

$$(x^2 - y^2)^{1/2} = \frac{3x^2}{2y^3}$$

$$\frac{1}{2}(x^2 - y^2)^{-1/2}[2x - 2y \cdot y'] = \frac{(2y^3)(6x) - (3x^2)(6y^2 \cdot y')}{(2y^3)^2}$$

$$\frac{1}{2}(x^2 - y^2)^{-1/2}[2x] - \frac{1}{2}(x^2 - y^2)^{-1/2}[2y]y' = \frac{12xy^3 - 18x^2y^2 \cdot y'}{4y^6}$$

$$\frac{x}{(x^2 - y^2)^{1/2}} - \frac{y}{(x^2 - y^2)^{1/2}} \cdot y' = \frac{3x}{y^3} - \frac{9x^2}{2y^4} \cdot y'$$

$$\frac{9x^2}{2y^4} \cdot y' - \frac{y}{(x^2 - y^2)^{1/2}} \cdot y' = \frac{3x}{y^3} - \frac{x}{(x^2 - y^2)^{1/2}}$$

Example 3: $\sqrt{x^2 - y^2} = \dfrac{3x^2}{2y^3}$

Continuation: $\dfrac{9x^2}{2y^4} \cdot y' - \dfrac{y}{(x^2 - y^2)^{1/2}} \cdot y' = \dfrac{3x}{y^3} - \dfrac{x}{(x^2 - y^2)^{1/2}}$

$$y'\left[\frac{9x^2}{2y^4} - \frac{y}{(x^2 - y^2)^{1/2}}\right] = \frac{3x}{y^3} - \frac{x}{(x^2 - y^2)^{1/2}}$$

$$y'\left[\frac{(9x^2)(x^2 - y^2)^{1/2} - (y)(2y^4)}{2y^4(x^2 - y^2)^{1/2}}\right] = \frac{(3x)(x^2 - y^2)^{1/2} - xy^3}{y^3(x^2 - y^2)^{1/2}}$$

$$y' = \left[\frac{2y^4(x^2 - y^2)^{1/2}}{(9x^2)(x^2 - y^2)^{1/2} - (y)(2y^4)}\right]\left[\frac{(3x)(x^2 - y^2)^{1/2} - xy^3}{y^3(x^2 - y^2)^{1/2}}\right]$$

$$y' = \frac{2xy\left[3(x^2 - y^2)^{1/2} - y^3\right]}{\left[9x^2(x^2 - y^2)^{1/2} - 2y^5\right]} = \frac{2xy\left[3(x^2 - y^2)^{1/2} - y^3\right]}{9x^2(x^2 - y^2)^{1/2} - 2y^5}$$

Home Work #6: Higher-Order Derivatives and Implicit Differentiation

A. Solve for the indicated Higher-Order Derivative for each:

(1)Find $f'''(x)$, if $f(x) = 3x^4 + 2x^3 - 5x + 8$

(2)Find $f''(0)$, if $f(x) = 2(2x - 3)^4$

(3)Find y''' of the function $y = -x^2 + 2\sqrt[3]{x^7}$

(4)A function g is defined by $g(w) = \frac{5-2w}{w}$, find $g''(w)$

(5)Find the 3rd derivative of the function $J(p) = \frac{3p}{(p-4)^2}$

Home Work #6: Higher-Order Derivatives and Implicit Differentiation

B. For the given functions: (a) express as explicit function and find y' and

(b) find y' by implicit differentiation.

(1) $x^4 + y^2 = 5$

(2) $\frac{x^2}{y^3} = 2$

(3) $xy + 2x - y = 0$

C. For the following functions, use implicit differentiation to find y'.

(4) $4\,x^2 = 2y^3 - 4y$

(5) $3x^2y^2 = \sqrt{4x^2 - 2\mathrm{y}}$