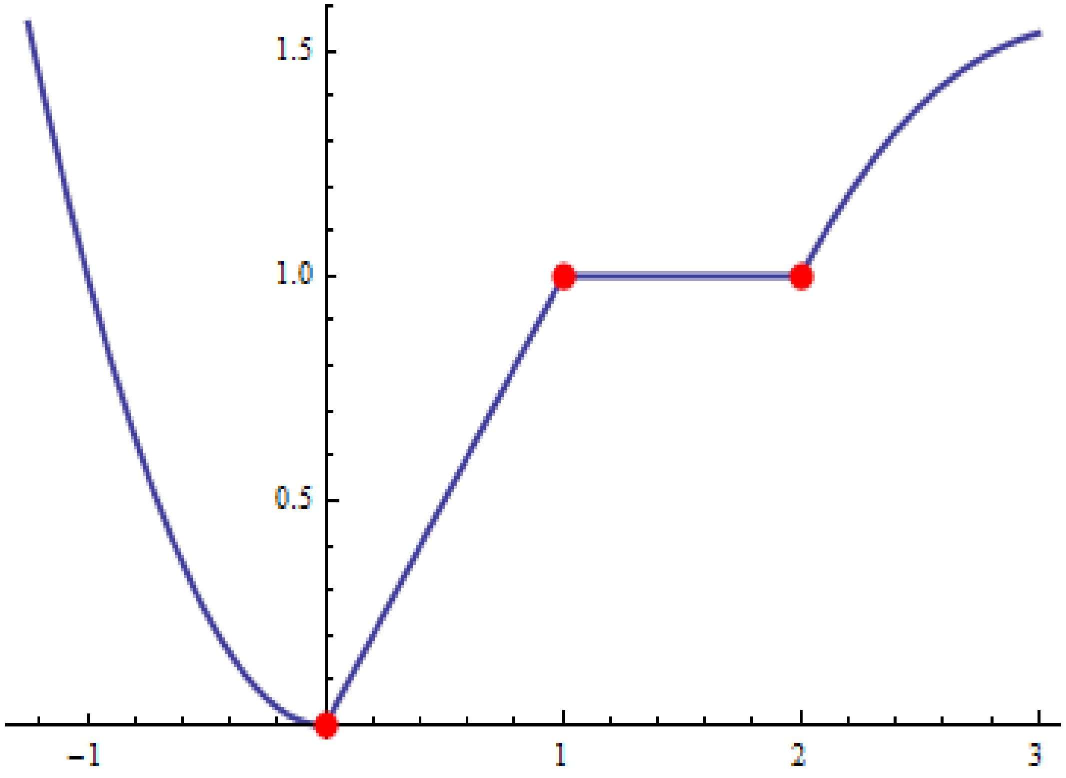
CONTINUOUS and DISCONTINUOUS FUNCTION

Continuous and Discontinuous Functions

A function can either be *continuous* or *discontinuous*.



an example of a graph of a continuous function

Continuity at a Point

A function is continuous at a point $x = p$ if all three conditions are satisfied:

(i) $f(p)$ exists

(ii) $\lim_{x \to p} f(x)$ exists, that is $\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x)$,

(iii) $\lim_{x \to p} f(x) = f(p)$

If at least one of these conditions is not satisfied, the function f is said to be discontinuous at p.

Continuity on an Interval

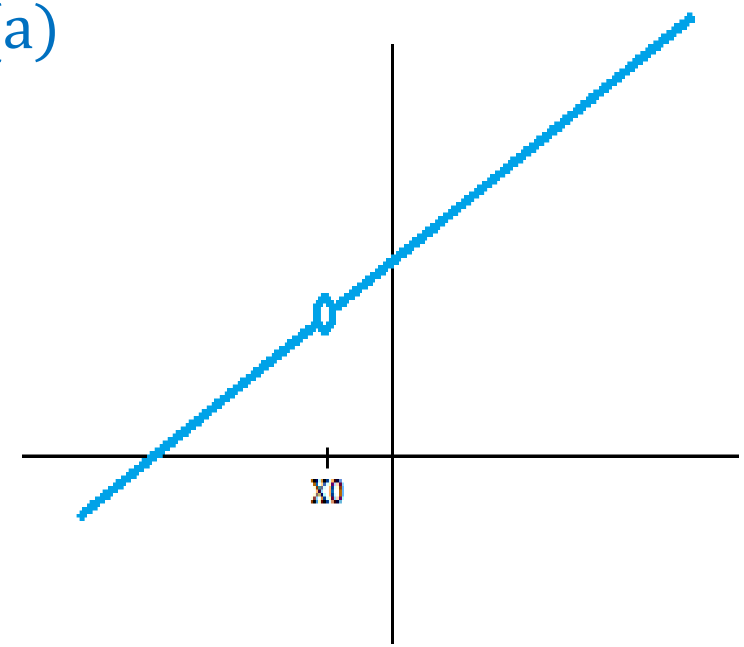
A function is continuous over an open interval (a, b) if it is continuous at every number on the interval (a, b). A function f is continuous over a closed interval $[a, b]$ if it is continuous on (a, b). In other words, $\lim_{x \to p} f(x) = f(p)$ for every p in the interval (a, b), $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to b^-} f(x) = f(b)$.

The closed interval $[a, b]$

The open interval (a, b)

Discontinuous Functions

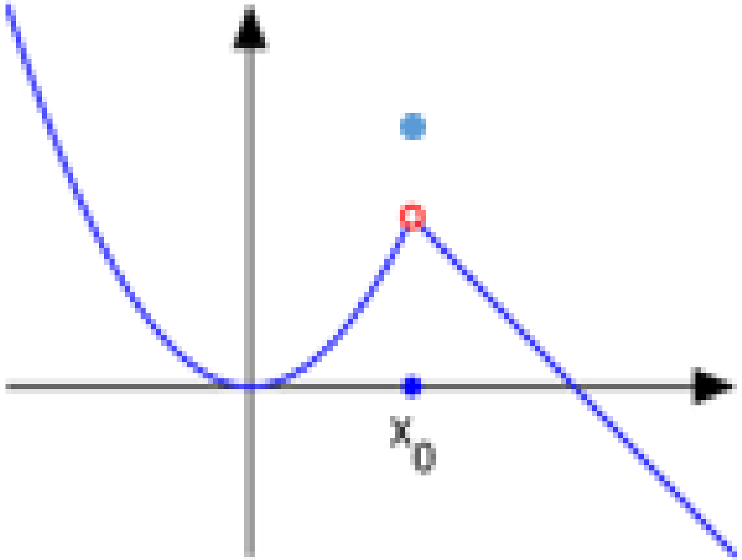
(a)



POINT DISCONTINUITY.

It happens if condition (i) is not satisfied, that is $f(x_0)$ does not exist.

(b)

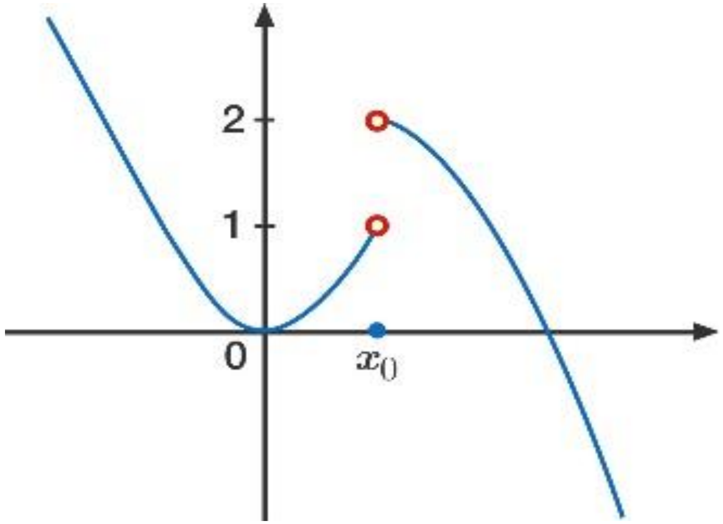


REMOVABLE DISCONTINUITY

It happens if condition (iii) is not satisfied. A continuity is called removable if it can be redefined at a different point to make is continuous.

Discontinuous Functions

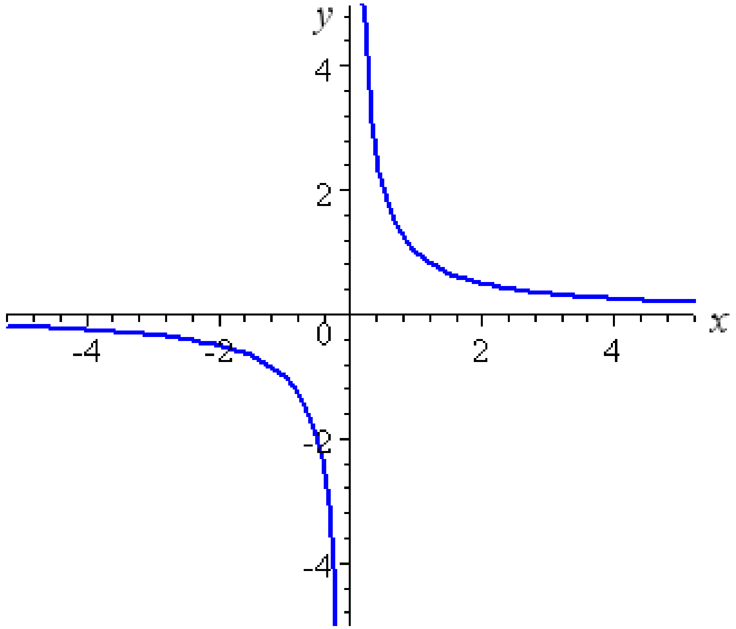
(c)



JUMP DISCONTINUITY

This type of discontinuity happens when (ii) fails to hold.

(d)



INFINITE OR NON-REMOVABLE DISCONTINUITY

It happens when at least one of the one-sided limits does not exist.

Example: Determine if the functions are continuous at the given values of x.

(1) $f(x) = 2x^2 - 6x + 1 \; at \, x = 1$

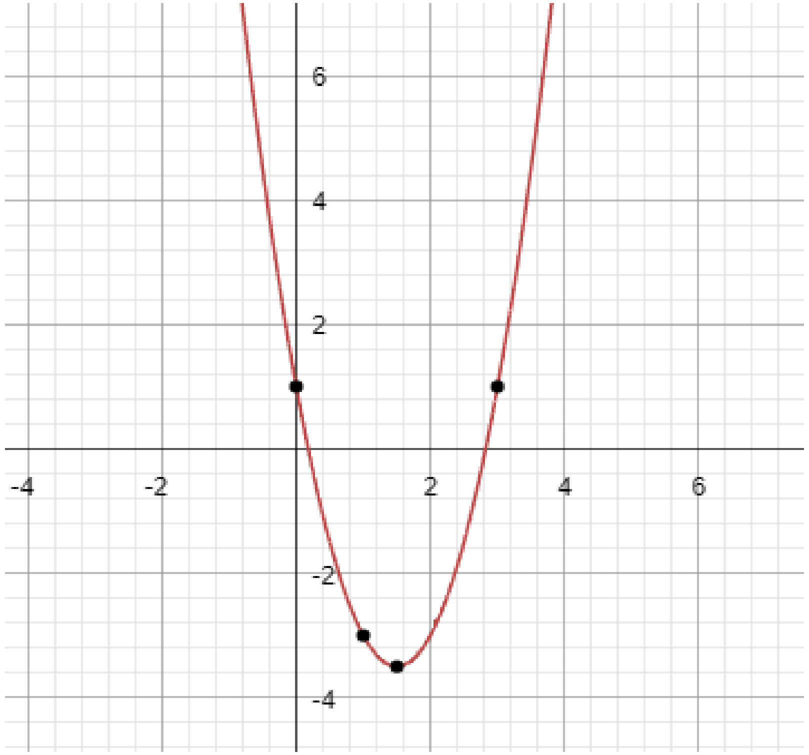
(i) $f\,(1) = 2(1)^2 - 6(1) + 1 = \; 2 - 6 + 1 = -3$

(ii) $\lim_{x \to 1^-} f(2x^2 - 6x + 1) = \; -3$ and

$\lim_{x \to 1^+} f(2x^2 - 6x + 1) = \; -3$

$\lim_{x \to 1} f(2x^2 - 6x + 1) = -3$

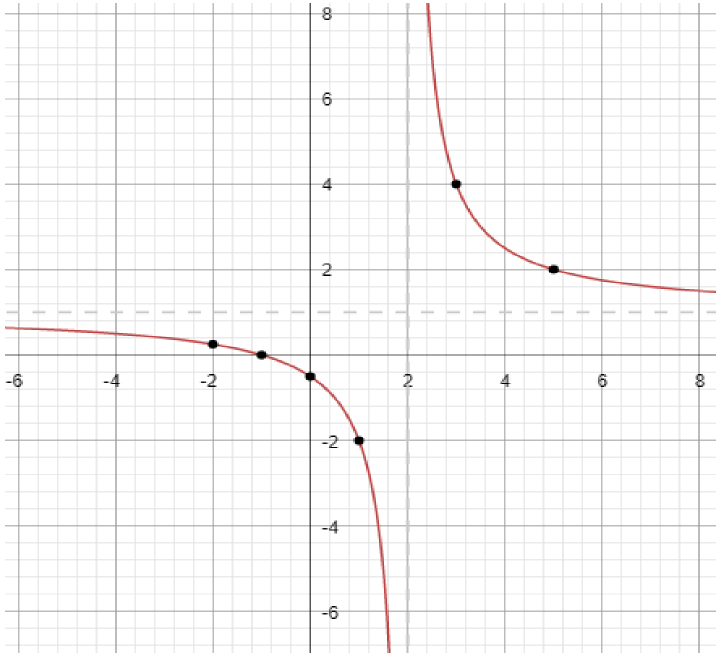
(iii) $\lim_{x \to 1} f(2x^2 - 6x + 1) = f(1) = -3$



$\therefore$ Therefore $f(x) = 2x^2 - 6x + 1$ is **continuous** at $x = 1$

Example: Determine if the functions are continuous at the given values of x.

(2) $f(x) = \frac{x+1}{x-2}$ at $x = 2$



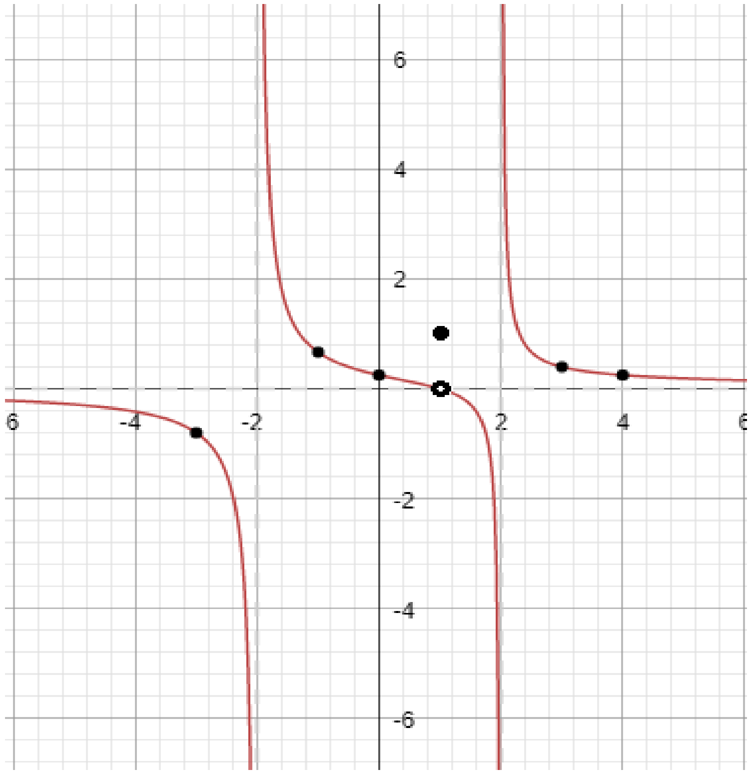
(i) $f(2) = \frac{2+1}{2-2} = \frac{3}{0}$ is undefined

$f(2)$ does not exists, condition (i) fails.

Therefore $f(x) = \frac{x+1}{x-2}$ is **not continuous** at $x = 2$ and this discontinuity is *asymptotic* at $x = 2$.

Example: Determine if the functions are continuous at the given values of x.

(3) $f(x) = \begin{cases} \dfrac{x-1}{x^2-4}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ $at\ x = 1$



(i) $f(1) = 1$, condition (i) is satisfied.

(ii) $\lim\limits_{x \to 1^-} \left(\dfrac{x-1}{x^2-4}\right) = 0$ and $\lim\limits_{x \to 1^+} \left(\dfrac{x-1}{x^2-4}\right) = 0$

$\lim\limits_{x \to 1} \left(\dfrac{x-1}{x^2-4}\right) = 0$, condition (ii) is satisfied.

(iii) $\lim\limits_{x \to 1} \left(\dfrac{x-1}{x^2-4}\right) = 0$ and $f(1) = 1$

$\lim\limits_{x \to 1} \left(\dfrac{x-1}{x^2-4}\right) \neq f(1)$

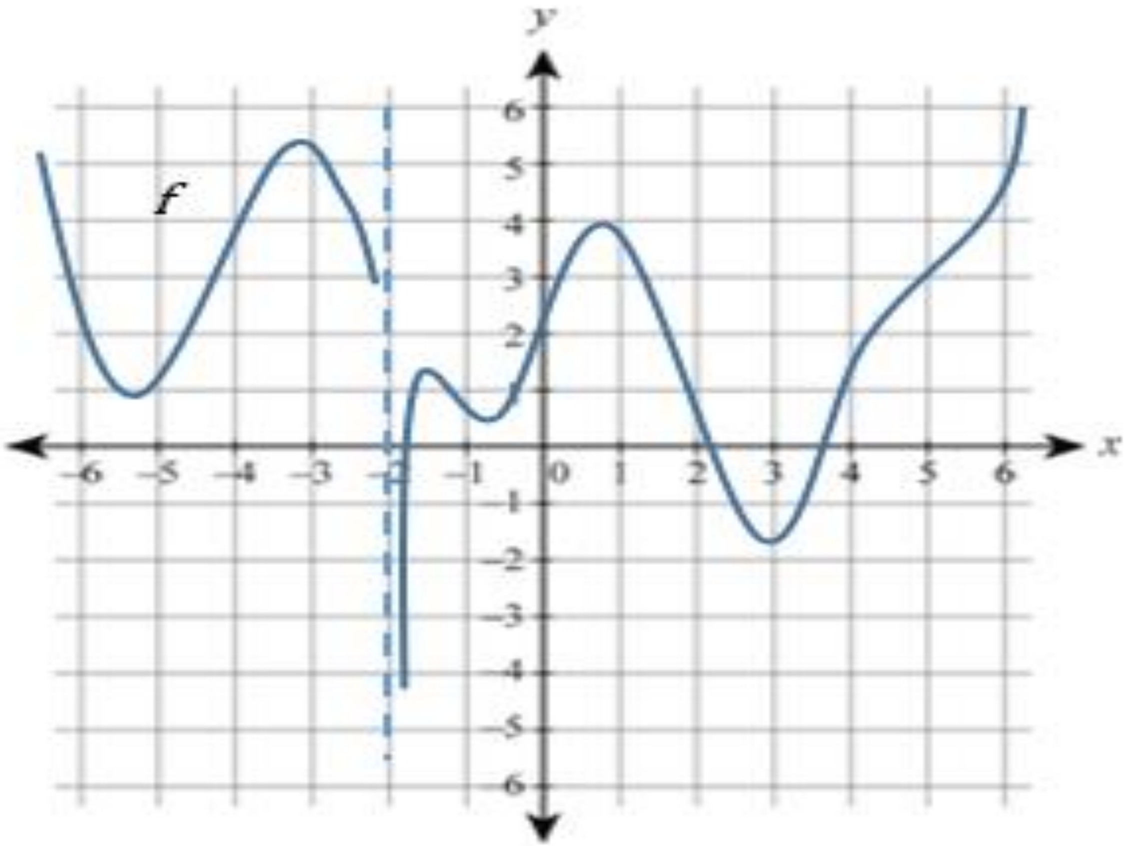
$0 \neq 1$

Therefore $f(x) = \begin{cases} \dfrac{x-1}{x^2-4}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ is **not continuous** at $x = 1$.

It is a *removable* discontinuity at $x = 1$.

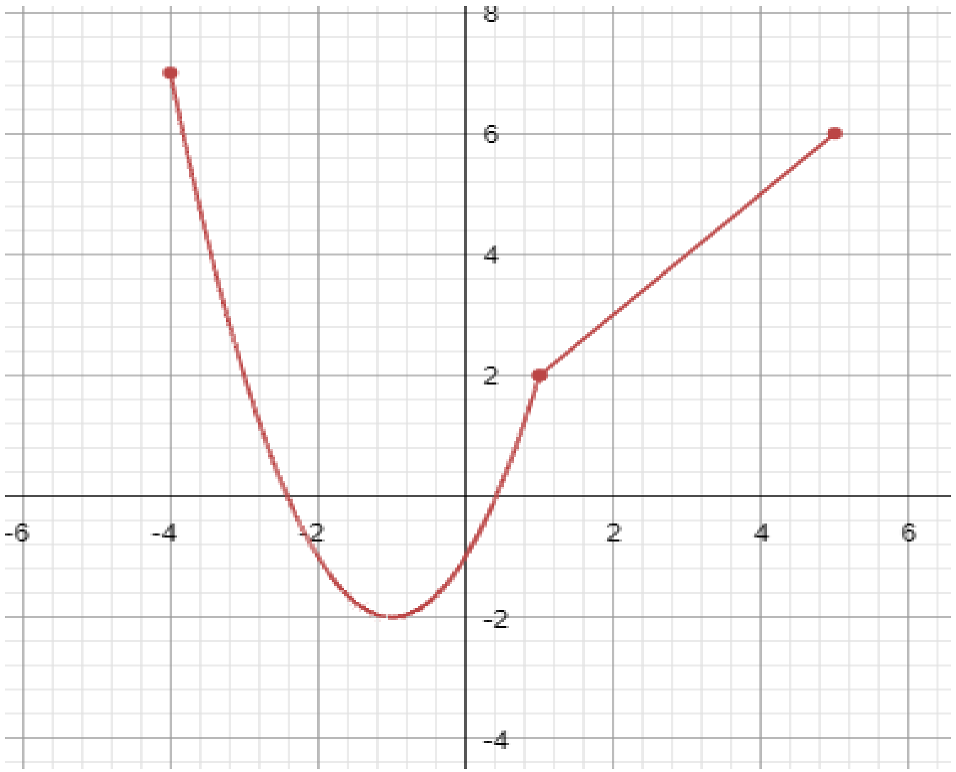
Refer to the graph for the next examples.

(4) The figure shows the that the function f is continuous over the interval $[-1, 4]$



Graph also show continuity on the interval $(-\infty, -2)$ and the interval $(-2, \infty)$.

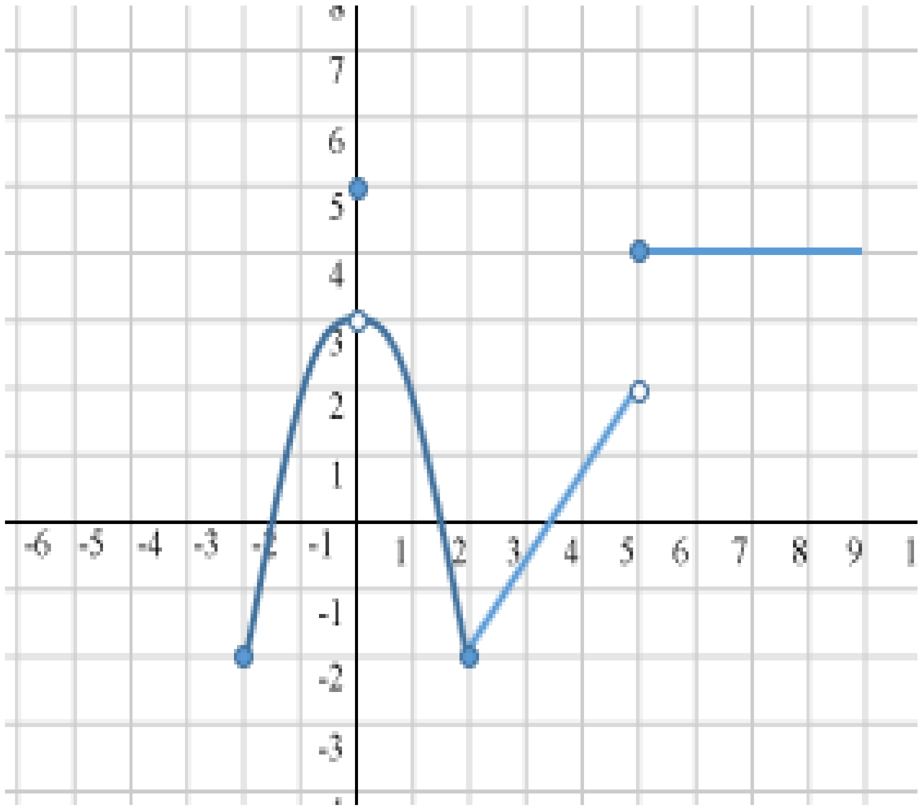
(5) The piecewise function $f(x) = \begin{cases} x^2 + 2x - 1, & -4 \leq x \leq 1 \\ x + 1, & 1 \leq x \leq 5 \end{cases}$ shows continuity.



The function $f(x)$ is continuous over the interval $[-4, 5]$.

Practice Task #4: Continuity of a Function

1. Consider the graph of the function $f(x)$.



(a) Find all the points where $f(x)$ is discontinuous.

(b) What kind of discontinuity is each point?

2. Determine whether the function is continuous on the given point. Sketch the graph.

(a) $f(x) = |x| - x$

(b) $f(x) = \begin{cases} 5, & x = 1 \\ 2x + 3, & x \neq 1 \end{cases}$

3. For what value(s) of x is the function $f(x) = \frac{x^3-27}{x^2-9}$ discontinuous? What kind of discontinuity is each point?

4. Determine if the function is discontinuous. What type of discontinuity is it?

(a) $f(x) = \begin{cases} \frac{x^2-2x-3}{x-3} & if\ x \neq 3 \\ 5 & if\ x = 3 \end{cases}$

(b) $f(x) = \begin{cases} 3 - x\ if\ x < 2 \\ 2\ if\ x = 2 \\ \frac{x}{2}\ if\ x > 2 \end{cases}$

DERIVATIVES

DERIVATIVES

$$slope = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

The derivative is simply the slope or rate of change of the function with respect to its independent variable. The process of calculating the derivative is known as differentiation.

Notations for Differentiation:

Given the function: $y = f(x)$

The derivative of y may be written in the ff ways:

$$\frac{dy}{dx} = \frac{df(x)}{dx} = \frac{d}{dx}f(x)$$

Leibniz's Notation

$$y' = f'(x)$$

Lagrange's Notation

$$D_x y = (Df)(x)$$

Euler's Notation

$$\dot{y} = \frac{\dot{y}}{\dot{x}}$$

Newton's Notation

Derivative Of A Function As The *Instantaneous Rate Of Change*

Given a function $y = f(x)$,

the derivation of the function is $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 1: Find the derivative of $f(x) = x^2$

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + \Delta x^2) - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x + 0$$

$$= \mathbf{2x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example 2: Find the derivative of $f(x) = 2x^2 - 3x + 5$

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 - 3(x + \Delta x) + 5] - (2x^2 - 3x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2[x^2 + 2x(\Delta x) + (\Delta x)^2] - 3(x + \Delta x) + 5 - (2x^2 - 3x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x^2 + 4x(\Delta x) + 2(\Delta x)^2 - 3x - 3\Delta x + 5 - 2x^2 + 3x - 5}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x(\Delta x) + 2(\Delta x)^2 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[4x + 2\Delta x - 3]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [4x + 2\Delta x - 3]$$

$$= 4x + 2(0) - 3$$

$$\boldsymbol{f'(x) = 4x - 3}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 3: Find the derivative of the function $g(x) = \frac{x}{2x+1}$

Solution:

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x}{2(x+\Delta x)+1} - \frac{x}{2x+1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{x+\Delta x}{2(x+\Delta x)+1} - \frac{x}{2x+1}\right]\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)(2x+1) - x(2(x+\Delta x)+1)}{[2(x+\Delta x)+1][2x+1]}\right]\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{2x^2 + x + 2x(\Delta x) + \Delta x - 2x^2 - 2x(\Delta x) - x}{[2(x+\Delta x)+1][2x+1]}\right]\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{\Delta x}{[2(x+\Delta x)+1][2x+1]}\right]\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{1}{[2(x+\Delta x)+1][2x+1]}\right]$$

$$= \left[\frac{1}{[2(x+0)+1][2x+1]}\right]$$

$$= \left[\frac{1}{[2x+1][2x+1]}\right]$$

$$\boldsymbol{g'(x) = \frac{1}{[2x+1]^2}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 4: Find the derivative of the function

$$h(y) = \sqrt{3y-5}$$

Solution:

$$h'(y) = \lim_{\Delta y \to 0} \frac{h(y+\Delta y) - h(y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\sqrt{3(y+\Delta y)-5} - \sqrt{3y-5}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\sqrt{3(y+\Delta y)-5} - \sqrt{3y-5}}{\Delta y} \cdot \frac{\sqrt{3(y+\Delta y)-5} + \sqrt{3y-5}}{\sqrt{3(y+\Delta y)-5} + \sqrt{3y-5}}$$

$$= \lim_{\Delta y \to 0} \frac{(3(y+\Delta y)-5) - (3y-5)}{\Delta y\left(\sqrt{3(y+\Delta y)-5} + \sqrt{3y-5}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{3y + 3\Delta y - 5 - 3y + 5}{\Delta y\left(\sqrt{3(y+\Delta y)-5} + \sqrt{3y-5}\right)}$$

Example 4: Find the derivative of the function

$$h(y) = \sqrt{3y - 5}$$

Solution:

$$h'(y) = \lim_{\Delta y \to 0} \frac{3\Delta y}{\Delta y\left(\sqrt{3(y + \Delta y) - 5} + \sqrt{3y - 5}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{3}{\sqrt{3(y + \Delta y) - 5} + \sqrt{3y - 5}}$$

$$= \frac{3}{\sqrt{3(y + 0) - 5} + \sqrt{3y - 5}}$$

$$= \frac{3}{\sqrt{3y - 5} + \sqrt{3y - 5}}$$

$$\boldsymbol{h'(y) = \frac{3}{2\sqrt{3y - 5}}}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example 5: Find the derivative of the function

$$J(w) = \frac{3}{\sqrt{2w+1}}$$

Solution:

$$J'(w) = \lim_{\Delta w \to 0} \frac{J(w+\Delta w) - J(w)}{\Delta w} = \lim_{\Delta w \to 0} \frac{\dfrac{3}{\sqrt{2(w+\Delta w)+1}} - \dfrac{3}{\sqrt{2w+1}}}{\Delta w}$$

$$= \lim_{\Delta w \to 0} \left[\frac{3}{\sqrt{2(w+\Delta w)+1}} - \frac{3}{\sqrt{2w+1}}\right]\frac{1}{\Delta w}$$

$$= \lim_{\Delta w \to 0} \left[\frac{3\sqrt{2w+1} - 3\sqrt{2(w+\Delta w)+1}}{\sqrt{2(w+\Delta w)+1} \cdot \sqrt{2w+1}}\right]\frac{1}{\Delta w}$$

$$= \lim_{\Delta w \to 0} \frac{1}{\Delta w} \cdot \frac{3\sqrt{2w+1} - 3\sqrt{2(w+\Delta w)+1}}{\sqrt{(2w+2\Delta w+1)(2w+1)}} \cdot \frac{3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}}{3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}}$$

Example 5: Find the derivative of the function $J(w) = \frac{3}{\sqrt{2w+1}}$

$$J'(w) = \lim_{\Delta w \to 0} \frac{1}{\Delta w} \cdot \frac{3\sqrt{2w+1} - 3\sqrt{2(w+\Delta w)+1}}{\sqrt{(2w+2\Delta w+1)(2w+1)}} \cdot \frac{3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}}{3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}}$$

$$= \lim_{\Delta w \to 0} \frac{1}{\Delta w} \cdot \frac{9(2w+1) - 9[2(w+\Delta w)+1]}{\sqrt{(2w+2\Delta w+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}\right]}$$

$$= \lim_{\Delta w \to 0} \frac{1}{\Delta w} \cdot \frac{18w + 9 - 18w - 18\Delta w - 9}{\sqrt{(2w+2\Delta w+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}\right]}$$

$$= \lim_{\Delta w \to 0} \frac{1}{\Delta w} \cdot \frac{-18\Delta w}{\sqrt{(2w+2\Delta w+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}\right]}$$

$$= \lim_{\Delta w \to 0} \frac{-18}{\sqrt{(2w+2\Delta w+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}\right]}$$

Example 5: Find the derivative of the function $J(w) = \frac{3}{\sqrt{2w+1}}$

$$J'(w) = \lim_{\Delta w \to 0} \frac{-18}{\sqrt{(2w+2\Delta w+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2(w+\Delta w)+1}\right]}$$

$$= \frac{-18}{\sqrt{(2w+2(0)+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2(w+(\mathbf{0}))+1}\right]}$$

$$= \frac{-18}{\sqrt{(2w+1)(2w+1)}\left[3\sqrt{2w+1} + 3\sqrt{2w+1}\right]}$$

$$= \frac{-18}{(2w+1)\left[6\sqrt{2w+1}\right]}$$

$$= \frac{-3}{(2w+1)\sqrt{2w+1}}$$

$$\boldsymbol{J'(w) = \frac{-3}{(2w+1)^{3/2}}}$$

Note: If f(x) is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$

Example 6: Given $y = 2 - x^2 + x^3$, find $\frac{dy}{dx}$.

Also, find $\frac{dy}{dx}$ when (a) $x = 2$, (b) $x = 0$ and (c) $x = -1$

Solution:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[2 - (x + \Delta x)^2 + (x + \Delta x)^3] - (2 - x^2 + x^3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[2 - (x^2 + 2x(\Delta x) + (\Delta x)^2) + (x^3 + 3x^2\Delta \mathrm{x} + 3\mathrm{x}(\Delta x)^2 + (\Delta x)^3)] - (2 - x^2 + x^3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2 - x^2 - 2x(\Delta x) - (\Delta x)^2 + x^3 + 3x^2\Delta \mathrm{x} + 3\mathrm{x}(\Delta x)^2 + (\Delta x)^3 - 2 + x^2 - x^3}{\Delta x}$$

Example 6: Given $y = 2 - x^2 + x^3$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{2 - x^2 - 2x(\Delta x) - (\Delta x)^2 + x^3 + 3x^2\Delta\text{x} + 3\text{x}(\Delta x)^2 + (\Delta x)^3 - 2 + x^2 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x(\Delta x) - (\Delta x)^2 + 3x^2\Delta\text{x} + 3\text{x}(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\,[-2x - \Delta x + 3x^2 + 3\text{x}(\Delta x) + (\Delta x)^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [-2x - \Delta x + 3x^2 + 3\text{x}(\Delta x) + (\Delta x)^2]$$

$$= -2x - 0 + 3x^2 + 3\text{x}(0) + (0)^2$$

$$\boldsymbol{\frac{dy}{dx} = -2x + 3x^2}$$

Example 6: Given $y = 2 - x^2 + x^3$, find $\frac{dy}{dx}$.

Also, find $\frac{dy}{dx}$ when (a) $x = 2$, (b) $x = 0$ and (c) $x = -1$

$$\frac{dy}{dx} = -2x + 3x^2$$

find $\frac{dy}{dx}$ when (a) $x = 2$

$$\frac{dy}{dx} = -2(2) + 3(2)^2 = -4 + 3(4) = -4 + 12 = \mathbf{8}$$

find $\frac{dy}{dx}$ when (b) $x = 0$

$$\frac{dy}{dx} = -2(0) + 3(0)^2 = 0 + 3(0) = 0 + 0 = \mathbf{0}$$

find $\frac{dy}{dx}$ when (c) $x = -1$

$$\frac{dy}{dx} = -2(-1) + 3(-1)^2 = 2 + 3(1) = 2 + 3 = \mathbf{5}$$

Example 7: Find the derivative of $f(x) = \sin x$

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - \sin x + \sin x \cos \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - \sin x\,(1 - \cos \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\cos x \sin \Delta x}{\Delta x} - \frac{\sin x\,(1 - \cos \Delta x)}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x}{\Delta x} - \lim_{\Delta x \to 0} \frac{\sin x\,(1 - \cos \Delta x)}{\Delta x}$$

$$= \cos x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} - \sin x \lim_{\Delta x \to 0} \frac{(1 - \cos \Delta x)}{\Delta x}$$

$$= \cos x\,(1) - \sin x\,(0)$$

$$= \boldsymbol{\cos x}$$

$$\therefore \boldsymbol{d(\sin x) = \cos x}$$

Example 8: Find the slope of the curve $y = \frac{4}{x+1}$ at the point $x = 1$

Solution:

$$m = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$m = \lim_{\Delta x \to 0} \frac{\frac{4}{(x+\Delta x)+1} - \frac{4}{x+1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{4}{(x+\Delta x)+1} - \frac{4}{x+1}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{4(x+1) - 4\big((x+\Delta x)+1\big)}{\big((x+\Delta x)+1\big)(x+1)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{-4\Delta x}{\big((x+\Delta x)+1\big)(x+1)}\right)$$

$$= \lim_{\Delta x \to 0} \frac{-4}{\big((x+\Delta x)+1\big)(x+1)}$$

$$= \frac{-4}{\big((x+0)+1\big)(x+1)}$$

$$\mathbf{m} = \frac{\mathbf{-4}}{(\boldsymbol{x}+\mathbf{1})^{\mathbf{2}}}$$

at the point $x = 1$,

$$m = \frac{-4}{(x+1)^2} = \frac{-4}{(1+1)^2}$$

$$= \frac{-4}{(2)^2} = \frac{-4}{4} = \mathbf{-1}$$

Home Work #4: Derivative

A. Find the derivative of the following functions. Write your complete solution for each.

(1) $y = x^2 + 4x - 3$

(2) $F(x) = \dfrac{2x+3}{4x-5}$

(3) $h(x) = \sqrt{3-4x}$

B. Find the slope of the following curves at the given point.

(1) $y = x^2 - 4x + 11 \; at \; x = 2$

(2) $y = 2 - 3x^2 \; at \; x = -1$

(3) $f(x) = -x^2 + 3x - 5$ at $\mathrm{x} = 3$

(4) $g(x) = \dfrac{2}{x} \; at \; x = 9$