LIMIT
OF A FUNCTION

The limit of a function f as x approaches p is the number L if the value of the function gets closer and closer to L as x gets closer and closer to p.

$$\lim_{x \to p} f(x) = L$$

"the limit of $f(x)$ as x approaches p is L"

Example: $\lim_{x \to 3}(x^2 - 1) = 8$

Inputs approaching 3 from the left

p	2.2	2.5	2.7	2.9	2.99	2.999	2.9999
$f(p)$	3.84	5.25	6.29	7.41	7.9401	7.9940	7.9994

Inputs approaching 3 from the right

p	3.0001	3.001	3.01	3.1	3.5	3.7	4
$f(p)$	8.0006	8.0060	8.0601	8.61	11.25	12.69	15

One-sided limits are those that approach to a single real value from one side of a function, either the left or the right.

left-sided or *left-hand limit:* $\lim_{x \to p^-} f(x)$ *ex.* $\lim_{x \to 3^-} (x^2 - 1)$

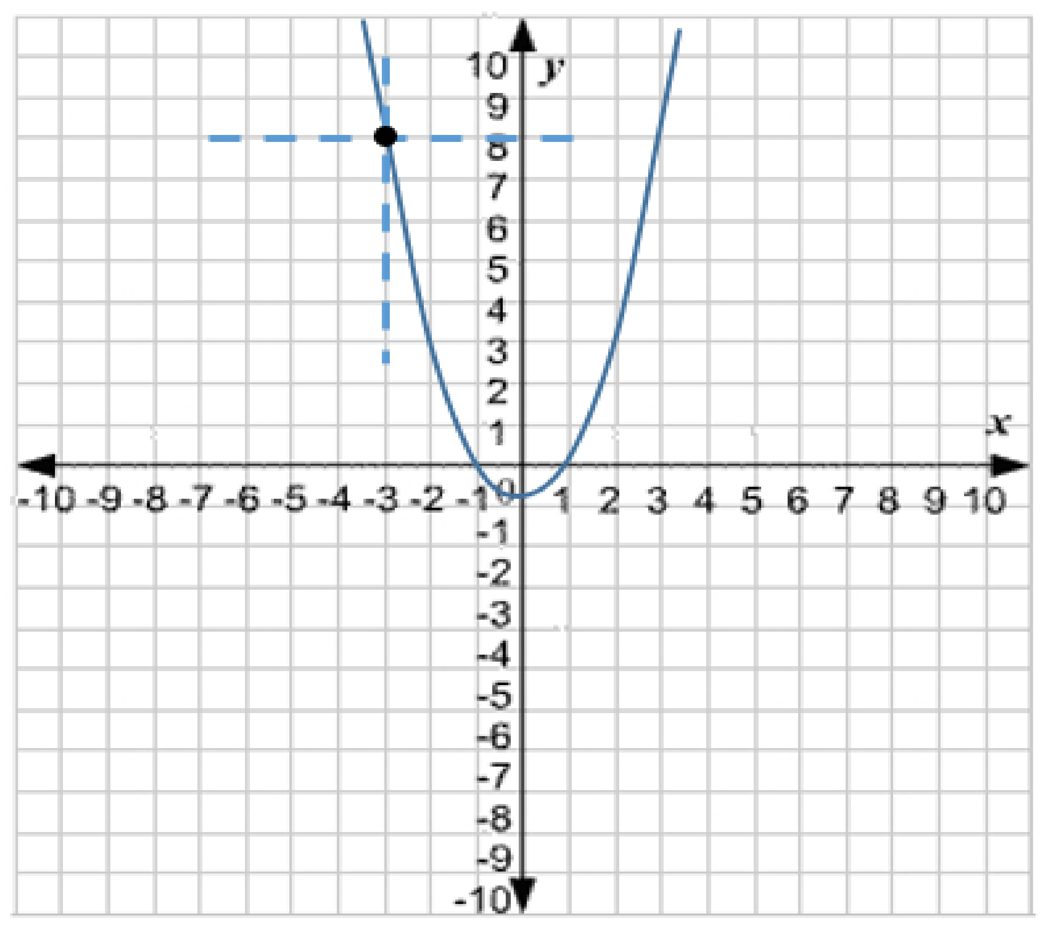
right-sided or right-hand limits: $\lim_{x \to p^+} f(x)$ *ex.* $\lim_{x \to 3^+} (x^2 - 1)$

Note: The limit of a function exists if the left-hand and right-hand limits exist and are the same.

$\lim_{x \to p} f(x)$ exists and is equal to L iff the

$\lim_{x \to p^-} f(x) = L$ and $\lim_{x \to p^+} f(x) = L$

Example: By inspection, determine the $\lim_{x \to -3} (x^2 - 1)$



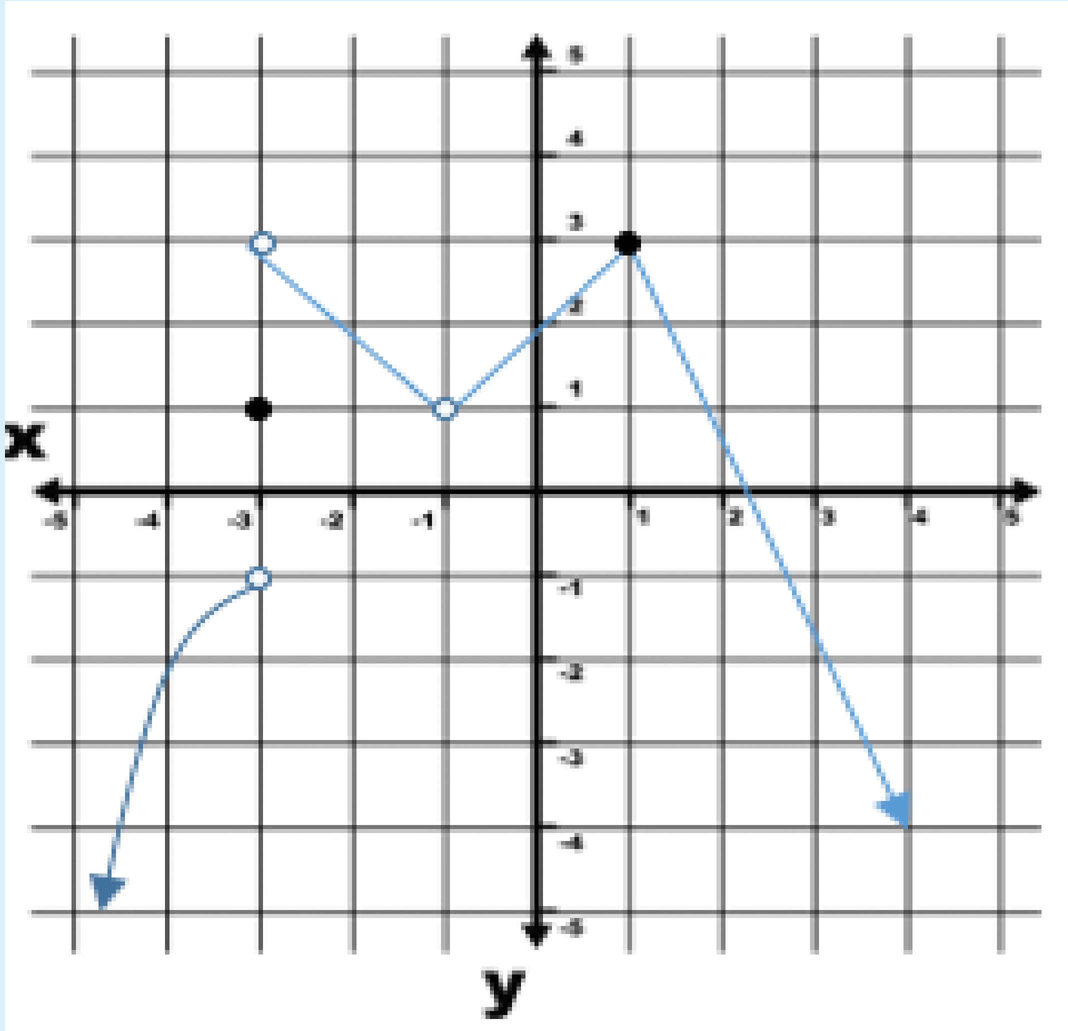
$$\lim_{x \to -3^-} (x^2 - 1) = 8$$

$$\lim_{x \to -3^+} (x^2 - 1) = 8$$

Therefore,

$$\boldsymbol{\lim_{x \to -3} \left(x^2 - 1\right) = 8}$$

Example: Let us consider the function whose graph is shown below.



a: $f(1) = 3$

$$\lim_{x\to 1} f(x) = 3$$

b: $f(-1) = DNE$

$$\lim_{x\to -1} f(x) = 1$$

c: $f(-3) = 1$

$$\lim_{x\to -3} f(x) = DNE$$

LIMIT LAWS:

Law 1: Limit of a Constant

If $f(x) = c$, where c is a constant, then $\lim\limits_{x \to p} f(x) = c$

Example: (1) $\lim\limits_{x \to 2} 7 = \mathbf{7}$ (2) $\lim\limits_{x \to -1} \frac{\pi}{2} = \boldsymbol{\frac{\pi}{2}}$

Law 2: Identity Law of Limits

If $f(x) = x$, then $\lim\limits_{x \to p} x = p$

Example: (1) $\lim\limits_{x \to 3} x = \mathbf{3}$ (2) $\lim\limits_{w \to 1/2} w = \frac{1}{2}$

For the Operational Laws for Limits, we let the $\lim_{x\to p} f(x) = L$ and $\lim_{x\to p} g(x) = M$,

Law 3: $\lim_{x\to p} k \cdot f(x) = k \cdot \lim_{x\to p} f(x)$

$$\boldsymbol{\lim_{x\to p} k \cdot f(x) = k \cdot L}$$

Example: (1) $\lim_{x\to -2} 2x = 2 \cdot \lim_{x\to -2} x = 2 \cdot -2 = \mathbf{-4}$

(2) $\lim_{x\to 2} \frac{3x}{4} = \frac{3}{4} \cdot \lim_{x\to 2} x = \frac{3}{4} \cdot 2 = \frac{6}{4} = \boldsymbol{\frac{3}{2}}$

For the Operational Laws for Limits, we let the $\lim_{x \to p} f(x) = L$ and $\lim_{x \to p} g(x) = M$,

Law 4: The limit of a sum or difference of two functions as x approaches p is equal to the sum or difference of their limits.

$$\lim_{x \to p} [f(x) \pm g(x)] = \lim_{x \to p} f(x) \pm \lim_{x \to p} g(x)$$

$$\boldsymbol{\lim_{x \to p} [f(x) \pm g(x)] = L \pm M}$$

Example: (1) $\lim_{x \to 2} (4x - 3) = \lim_{x \to 2} 4x - \lim_{x \to 2} 3$

$$= 4 \cdot \lim_{x \to 2} x - \lim_{x \to 2} 3$$

$$= 4 \cdot 2 - 3$$

$$\lim_{x \to 2} (4x - 3) = \mathbf{5}$$

For the Operational Laws for Limits, we let the $\lim\limits_{x \to p} f(x) = L$ and $\lim\limits_{x \to p} g(x) = M$,

Law 5: The limit of a product of two functions as x approaches p is equal to the product of their limits.

$$\lim_{x \to p} [f(x) \cdot g(x)] = \lim_{x \to p} f(x) \cdot \lim_{x \to p} g(x)$$

$$\boldsymbol{\lim_{x \to p} [f(x) \cdot g(x)] = L \cdot M}$$

Example: (1) $\lim\limits_{x \to -1} (x+3)(x-2)$

$$\begin{aligned} \lim_{x \to -1} (x+3)(x-2) &= \lim_{x \to -1} (x+3) \cdot \lim_{x \to -1} (x-2) \\ &= \left[\lim_{x \to -1} (x+3)\right]\left[\lim_{x \to -1} (x-2)\right] \\ &= \left[\lim_{x \to -1} x + \lim_{x \to -1} 3\right]\left[\lim_{x \to -1} x - \lim_{x \to -1} 2\right] \\ &= [-1+3][-1-2] \\ &= [2][-3] \end{aligned}$$

$$\lim_{x \to -1} (x+3)(x-2) = \boldsymbol{-6}$$

For the Operational Laws for Limits, we let the $\lim_{x \to p} f(x) = L$ and $\lim_{x \to p} g(x) = M$,

Law 6: The limit of a quotient of two functions as x approaches p is equal to the quotient of their limits.

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)} \text{ then } \boldsymbol{\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{L}{M}}, \text{ where } M \neq 0$$

Example: (1) $\lim_{x \to 2} \frac{2x - 3}{x + 1} = \frac{\lim_{x \to 2}(2x - 3)}{\lim_{x \to 2}(x + 1)} = \frac{\lim_{x \to 2} 2x - \lim_{x \to 2} 3}{\lim_{x \to 2} x + \lim_{x \to 2} 1}$

$$= \frac{2 \cdot \lim_{x \to 2} x - \lim_{x \to 2} 3}{\lim_{x \to 2} x + \lim_{x \to 2} 1} = \frac{2 \cdot 2 - 3}{2 + 1}$$

$$\lim_{x \to 2} \frac{2x - 3}{x + 1} = \boldsymbol{\frac{1}{3}}$$

For the Operational Laws for Limits, we let the $\lim_{x \to p} f(x) = L$ and $\lim_{x \to p} g(x) = M$,

Law 7: The **limit of an *n*th power** of a function as x approaches p is equal to the nth power of its limit.

$$\lim_{x \to p} [f(x)]^n = \left[\lim_{x \to p} f(x)\right]^n$$

$$\boldsymbol{\lim_{x \to p} [f(x)]^n = L^n}$$

Example:

(1) $\lim_{x \to 2} [x]^4 = \left[\lim_{x \to 2} x\right]^4$

$= [2]^4$

$\lim_{x \to 2} [x]^4 = \mathbf{16}$

(2) $\lim_{x \to 2} (x^3 - 2x^2) = \lim_{x \to 2} x^3 - \lim_{x \to 2} 2x^2$

$= \lim_{x \to 2} x^3 - 2 \cdot \lim_{x \to 2} x^2$

$= \left[\lim_{x \to 2} x\right]^3 - 2\left[\lim_{x \to 2} x\right]^2$

$= [2]^3 - 2[2]^2$

$\lim_{x \to 2} (x^3 - 2x^2) = \mathbf{0}$

For the Operational Laws for Limits, we let the $\lim_{x \to p} f(x) = L$ and $\lim_{x \to p} g(x) = M$,

Law 8: The **limit of an *n*th root** of a function as x approaches p is equal to the nth root of its limit.

$$\lim_{x \to p} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to p} f(x)}$$

$$\boldsymbol{\lim_{x \to p} \sqrt[n]{f(x)} = \sqrt[n]{L} \text{ or } L^{\frac{1}{n}}}$$

Example:

$$\lim_{x \to 2} \sqrt[3]{5x - 2} = \sqrt[3]{\lim_{x \to 2}(5x - 2)}$$

$$= \sqrt[3]{\lim_{x \to 2} 5x - \lim_{x \to 2} 2}$$

$$= \sqrt[3]{5 \cdot \lim_{x \to 2} x - \lim_{x \to 2} 2}$$

$$= \sqrt[3]{5 \cdot 2 - 2}$$

$$= \sqrt[3]{10 - 2}$$

$$= \sqrt[3]{8}$$

$$= \mathbf{2}$$

Note: The $\lim_{x \to p} f(x) = f(p)$.

Examples:

(1) $\lim_{x \to 3} 2x^3 = 2.[3]^3 = \mathbf{54}$

(2) $\lim_{x \to 2} (1 - 2x + 3x^2 - x^3)$

$= 1 - 2(2) + 3(2)^2 - (2)^3$

$= 1 - 2(2) + 3(4) - (8)$

$= 1 - 4 + 12 - 8$

$= \mathbf{1}$

(3) $\lim_{x \to -2} \dfrac{x^2 - 7x - 9}{4x - 5}$

$= \dfrac{(-2)^2 - 7(-2) - 9}{4(-2) - 5}$

$= \dfrac{4 + 14 - 9}{-8 - 5}$

$= \dfrac{\mathbf{9}}{\mathbf{-13}}$

However, evaluating the limit by direct substitution is not always possible.

Example: $\lim_{x \to 2} \frac{x^2 + 5x - 14}{x - 2} =$ 

Using direct substitution, $\lim_{x \to 2} \frac{x^2 + 5x - 14}{x - 2} = \frac{0}{0}$, the function indeterminate.

It means that 2 is not in the domain of $f(x) = \frac{x^2 + 5x - 14}{x - 2}$

x	1.899	1.999	1.99999	2	2.00001	2.001	2.25
$\frac{x^2 + 5x - 14}{x - 2}$	8.899	8.999	8.99999	DNE	9.00001	9.001	9.25

Example 1: $\lim_{x \to 2} \frac{x^2 + 5x - 14}{x - 2} = ?$

Solution: $\lim_{x \to 2} \frac{x^2 + 5x - 14}{x - 2} = \lim_{x \to 2} \frac{(x + 7)(x - 2)}{x - 2}$

$= \lim_{x \to 2} (x + 7)$

$= 2 + 7$

$\lim_{x \to 2} \frac{x^2 + 5x - 14}{x - 2} = \mathbf{9}$

Note:: This method of evaluating a limit is called the ***dividing out technique***.

Dividing Out Technique

Example 2: $\lim_{x\to -4} \frac{x^2+x-12}{x^2+6x+8}$

$$= \lim_{x\to -4} \frac{(x+4)(x-3)}{(x+4)(x+2)}$$

$$= \lim_{x\to -4} \frac{(x-3)}{(x+2)}$$

$$= \frac{-4-3}{-4+2}$$

$$= \frac{-7}{-2}$$

$$= \mathbf{\frac{7}{2}}$$

Example 3: $\lim_{x\to 3} \frac{x^3-27}{x^2-9}$

$$= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x+3)(x-3)}$$

$$= \lim_{x\to 3} \frac{(x^2+3x+9)}{(x+3)}$$

$$= \frac{(3)^2+3(3)+9}{3+3}$$

$$= \frac{9+9+9}{3+3} = \frac{27}{6}$$

$$= \mathbf{\frac{9}{2}}$$

The next examples can be done by rationalizing the numerator or the denominator to get rid of the radical signs and make the dividing out technique possible.

Example 4:

$$\lim_{x\to 2}\frac{x-2}{\sqrt{x^2-4}} = \lim_{x\to 2}\frac{x-2}{\sqrt{x^2-4}}\cdot\frac{\sqrt{x^2-4}}{\sqrt{x^2-4}}$$

$$= \lim_{x\to 2}\frac{(x-2)\sqrt{x^2-4}}{\left(\sqrt{x^2-4}\right)^2}$$

$$= \lim_{x\to 2}\frac{(x-2)\sqrt{x^2-4}}{x^2-4}$$

$$= \lim_{x\to 2}\frac{(x-2)\sqrt{x^2-4}}{(x+2)(x-2)}$$

$$= \lim_{x\to 2}\frac{\sqrt{x^2-4}}{x+2}$$

$$= \frac{\sqrt{(2)^2-4}}{2+2}$$

$$= \frac{\sqrt{4-4}}{4}$$

$$= \frac{0}{4}$$

$$= \mathbf{0}$$

Example 5:

$$\lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2} = \lim_{x\to1}\frac{x-1}{\sqrt{x^2+3}-2}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}$$

$$= \lim_{x\to1}\frac{(x-1)\left(\sqrt{x^2+3}+2\right)}{\left(\sqrt{x^2+3}\right)^2-(2)^2}$$

$$= \lim_{x\to1}\frac{(x-1)\left(\sqrt{x^2+3}+2\right)}{(x^2+3)-4}$$

$$= \lim_{x\to1}\frac{(x-1)\left(\sqrt{x^2+3}+2\right)}{x^2-1}$$

$$= \lim_{x\to1}\frac{(x-1)\left(\sqrt{x^2+3}+2\right)}{(x+1)(x-1)}$$

$$= \lim_{x\to1}\frac{\left(\sqrt{x^2+3}+2\right)}{x+1}$$

$$= \frac{\left(\sqrt{(1)^2+3}+2\right)}{1+1}$$

$$= \frac{\left(\sqrt{4}+2\right)}{2}$$

$$= \frac{2+2}{2}$$

$$= \frac{4}{2} = \mathbf{2}$$

Limits at Infinity

To describe the behavior of a function as x increases or decreases without bound defines the limit of a function at infinity.

$$\lim_{x\to+\infty} f(x) \text{ or } \lim_{x\to-\infty} f(x)$$

Some properties of the *limits at infinity*:

$$(1)\ \lim_{x\to\infty} \frac{1}{x} = 0 \qquad (3)\ \lim_{x\to\infty} \frac{1}{\sqrt[n]{x}} = 0$$

$$(2)\ \lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad (4)\ \lim_{x\to\infty} \frac{1}{c^x} = 0$$

where $c > 0$, c is any real number

Some properties of *infinite limits*:

If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$ then for every c and L that are real,

(5) $\lim_{x \to c}[f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = \infty$

(6) If $L > 0$ then $\lim_{x \to c}[f(x)g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = \infty$

(7) If $L < 0$ then $\lim_{x \to c}[f(x)g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = -\infty$

(8) $\lim_{x \to c} \frac{g(x)}{f(x)} = \frac{\lim_{x \to c} g(x)}{\lim_{x \to c} f(x)} = 0$

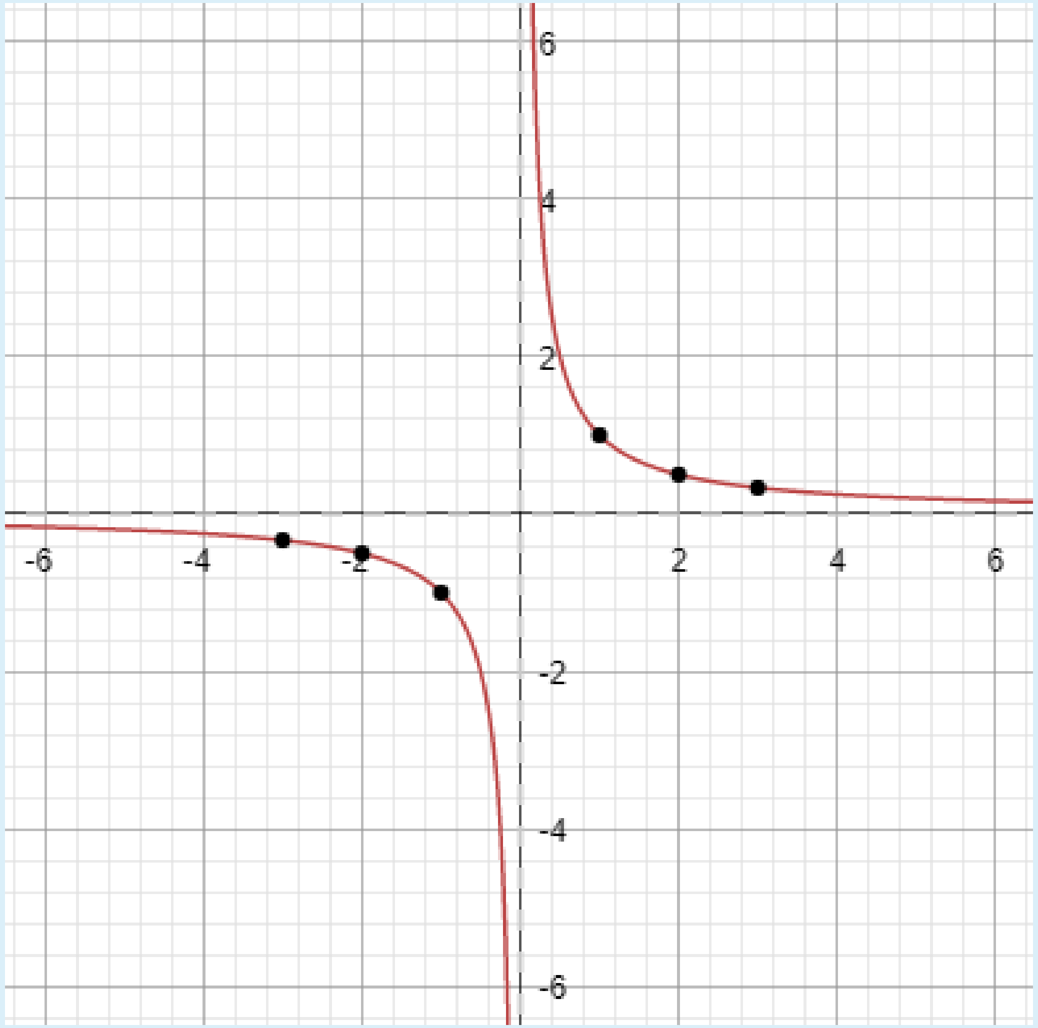
Example 1: Let us observe the graph of $f(x) = \frac{1}{x}$ and find the $\lim_{x\to\infty} \frac{1}{x}$

$as\ x \to \infty^-$

x	$f(x)$
-3	-0.33333
-2	-0.5
-1	-1
-0.5	-2
-0.25	-4
-0.10	-10

$as\ x \to \infty^+$

x	$f(x)$
3	0.33333
2	0.5
1	1
0.5	2
0.25	4
0.10	10



Example 2:

$$\lim_{x\to\infty}(4x) = 4\lim_{x\to\infty} x$$

$$= 4 \cdot \infty$$

$$\lim_{x\to\infty}(4x) = \infty$$

Example 3:

$$\lim_{x\to\infty}\left(3 + \frac{1}{3^x}\right) = \lim_{x\to\infty} 3 + \lim_{x\to\infty}\frac{1}{3^x}$$

$$= 3 + 0$$

$$\lim_{x\to\infty}\left(3 + \frac{1}{3^x}\right) = \mathbf{3}$$

Example 4:

$$\lim_{x\to\infty}(3x^3 - 2x) = \lim_{x\to\infty} 3x^3 - \lim_{x\to\infty} 2x \; \cancel{= \infty - \infty}$$

but we cannot subtract infinities!!!

Because infinity is not a number.

Solution:

$$\begin{aligned}
\lim_{x\to\infty}(3x^3 - 2x) &= \lim_{x\to\infty} x^3(3 - 2x^{-2}) \\
&= \lim_{x\to\infty} x^3\left(3 - \frac{2}{x^2}\right) \\
&= \lim_{x\to\infty} x^3 . \lim_{x\to\infty}\left(3 - \frac{2}{x^2}\right) \\
&= \lim_{x\to\infty} x^3\left[\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{2}{x^2}\right] \\
&= \infty[3 - 0]
\end{aligned}$$

$$\lim_{x\to\infty}(3x^3 - 2x) = \infty$$

Example 5: $\lim_{x\to\infty} \dfrac{x^2 - 5x - 3}{2x^4 + 3x^3} = \lim_{x\to\infty} \dfrac{x^4(x^{-2} - 5x^{-3} - 3x^{-4})}{x^4(2 + 3x^{-1})}$

$$= \lim_{x\to\infty} \frac{(x^{-2} - 5x^{-3} - 3x^{-4})}{(2 + 3x^{-1})}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{5}{x^3} - \frac{3}{x^4}}{2 + \frac{3}{x}}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{x^2} - \lim_{x\to\infty} \frac{5}{x^3} - \lim_{x\to\infty} \frac{3}{x^4}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{3}{x}}$$

$$= \frac{0 - 0 - 0}{2 + 0} = \frac{0}{2}$$

$$\lim_{x\to\infty} \frac{x^2 - 5x - 3}{2x^4 + 3x^3} = \mathbf{0}$$

Limits involving Trigonometric Functions

<u>4 Basic Limits Properties:</u>

(1) $\lim_{x \to p} \sin x = \sin p$

(2) $\lim_{x \to p} \cos x = \cos p$

(3) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(4) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 1:

$$\lim_{x \to 0} \frac{\sin x}{\cos x - 3} = \frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} (\cos x - 3)}$$

$$= \frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} \cos x - \lim_{x \to 0} 3}$$

$$= \frac{\sin 0}{\cos 0 - 3}$$

$$= \frac{0}{1 - 3}$$

$$\lim_{x \to 0} \frac{\sin x}{\cos x - 3} = \mathbf{0}$$

(1) $\lim_{x \to p} \sin x = \sin p$

(2) $\lim_{x \to p} \cos x = \cos p$

(3) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(4) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 2:

$$\lim_{x \to 0} \frac{2 \sin 3x}{3x} = 2 \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} = 2 \cdot 1 = \mathbf{2}$$

Example 3:

$$\lim_{x \to 0} \frac{\sec x - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{\cos x} - 1}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1 - \cos x}{\cos x}}{x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\cos x} \cdot \frac{1}{x}$$

$$= \mathbf{\lim_{x \to 0} \frac{1 - \cos x}{x}} \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 0 \cdot \frac{1}{\cos 0}$$

$$= 0 \cdot 1$$

$$= \mathbf{0}$$

Example 4:

$$\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} \frac{\sin 5x}{x} \cdot \frac{5}{5}$$

$$= \lim_{x \to 0} \frac{5\sin 5x}{5x}$$

$$= 5 \cdot \lim_{x \to 0} \frac{\sin 5x}{5x}$$

$$= 5 \cdot 1$$

$$\lim_{x \to 0} \frac{\sin 5x}{x} = \mathbf{5}$$

(1) $\lim_{x \to p} \sin x = \sin p$

(2) $\lim_{x \to p} \cos x = \cos p$

(3) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(4) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 5:

$$\lim_{x\to -3} \frac{\sin(x+3)}{x^2+7x+12}$$ but for $x = -3$, the denominator is 0

$$= \lim_{x\to -3} \frac{\sin(x+3)}{(x+3)(x+4)}$$

$$= \lim_{x\to -3} \frac{\sin(x+3)}{(x+3)} \cdot \frac{1}{(x+4)}$$

$$= \lim_{x\to -3} \frac{\sin(x+3)}{(x+3)} \cdot \lim_{x\to -3} \frac{1}{(x+4)}$$

$$= 1 \cdot \frac{1}{-3+4} = 1 \cdot 1$$

$$\lim_{x\to -3} \frac{\sin(x+3)}{x^2+7x+12} = \mathbf{1}$$

(1) $\lim_{x\to p} \sin x = \sin p$

(2) $\lim_{x\to p} \cos x = \cos p$

(3) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(4) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Practice Task #3: Limits of Algebraic Functions

A. Complete the table of values ang give the limit of the given functions.

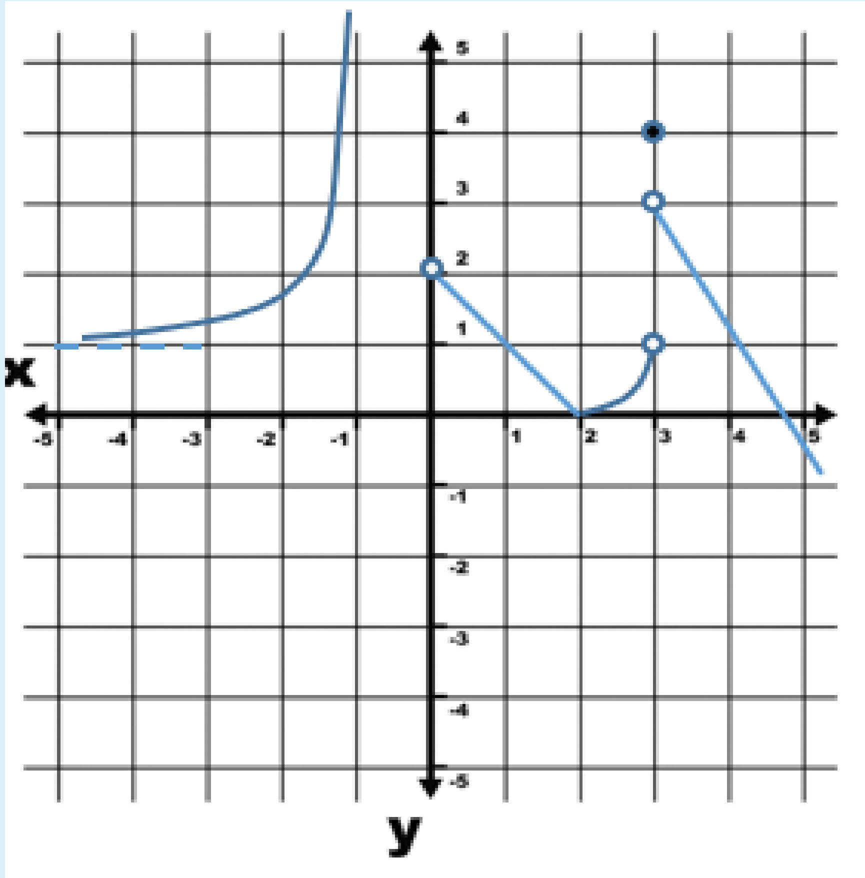
$$\lim_{x \to -2^-} (x^2 - 3x + 2) = \_\_\_\_\_\_$$

x	$f(x)$
-3	
-2.75	
-2.50	
-2.25	
-2.10	
-2.001	
-2.0001	

$$\lim_{x \to 1^+} \frac{x^2 + 3x + 2}{x - 1} = \_\_\_\_\_\_$$

x	$f(x)$
2	
1.75	
1.50	
1.25	
1.001	
1.0001	
1.000001	

B. Given the graph of the function $f(x)$, fill in the table with the missing values.



a.	$f(0)$	=
b.	$f(2)$	=
c.	$f(3)$	=
d.	$\lim_{x\to 0^-} f(x)$	=
e.	$\lim_{x\to 0^+} f(x)$	=
f.	$\lim_{x\to 3^-} f(x)$	=
g.	$\lim_{x\to 3^+} f(x)$	=

Home Work #3:

Evaluate the limit of the given functions.

$$(a)\ \lim_{x\to 2}(3x^2-4x+1)$$

$$(b)\ \lim_{x\to 0}\frac{x^2-25}{x^2-4x-5}$$

$$(c)\ \lim_{x\to 1}\frac{7x^2-4x-3}{3x^2-4x+1}$$

$$(d)\ \lim_{x\to 3}\frac{\sqrt{x^2+7}-3}{x+3}$$

$$(e)\ \lim_{x\to 5}\frac{\frac{1}{x}-\frac{1}{5}}{x-5}$$

$$(f)\ \lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$$

$$(g)\ \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$$

$$(h)\ \lim_{x\to \infty}\frac{x+1}{2x+1}$$

Home Work #3:

$(i)\ \lim_{x\to\infty}(3^{-x}+2)$

$(j)\ \lim_{x\to\infty}\frac{9x^2}{x+2}$

$(k)\ \lim_{x\to-\infty}\frac{2x^2-5}{x^3-2x^2-1}$

$(l)\ \lim_{x\to\infty}(x^3-x^2+2x)$

$(m)\ \lim_{x\to 0}\frac{\cos x}{\sin x-3}$

$(n)\ \lim_{x\to 0}\frac{\sin 3x}{2x}$

$(o)\ \lim_{x\to 0}\frac{1-\cos x}{\sin x}$

$(p)\ \lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$

$Hint\!: \ \sin\alpha - sin\beta = 2\sin\frac{\alpha-\beta}{2}cos\frac{\alpha+\beta}{2}$