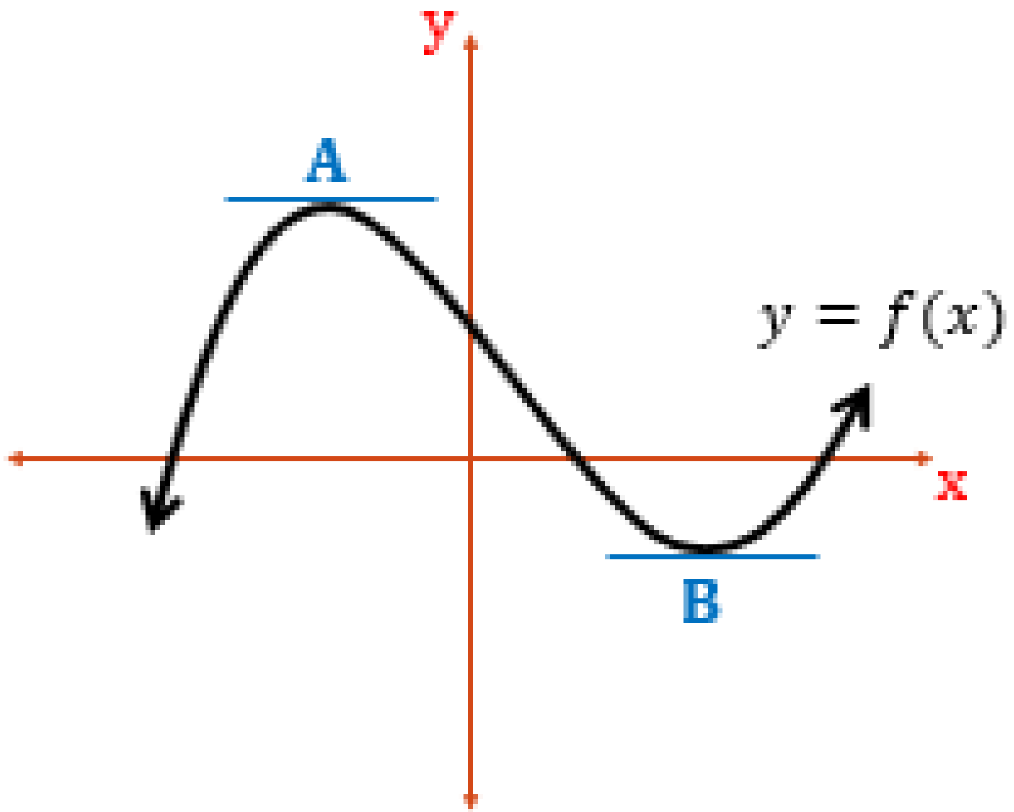
MAXIMA AND MINIMA

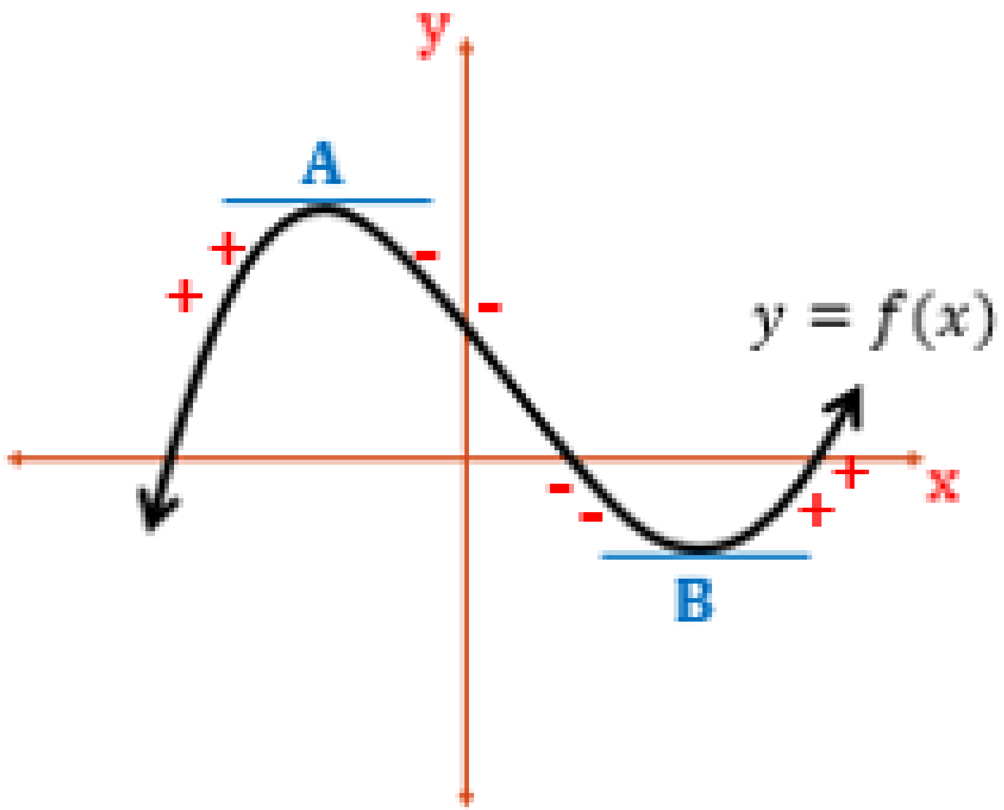
Curve Sketching

MAXIMA AND MINIMA



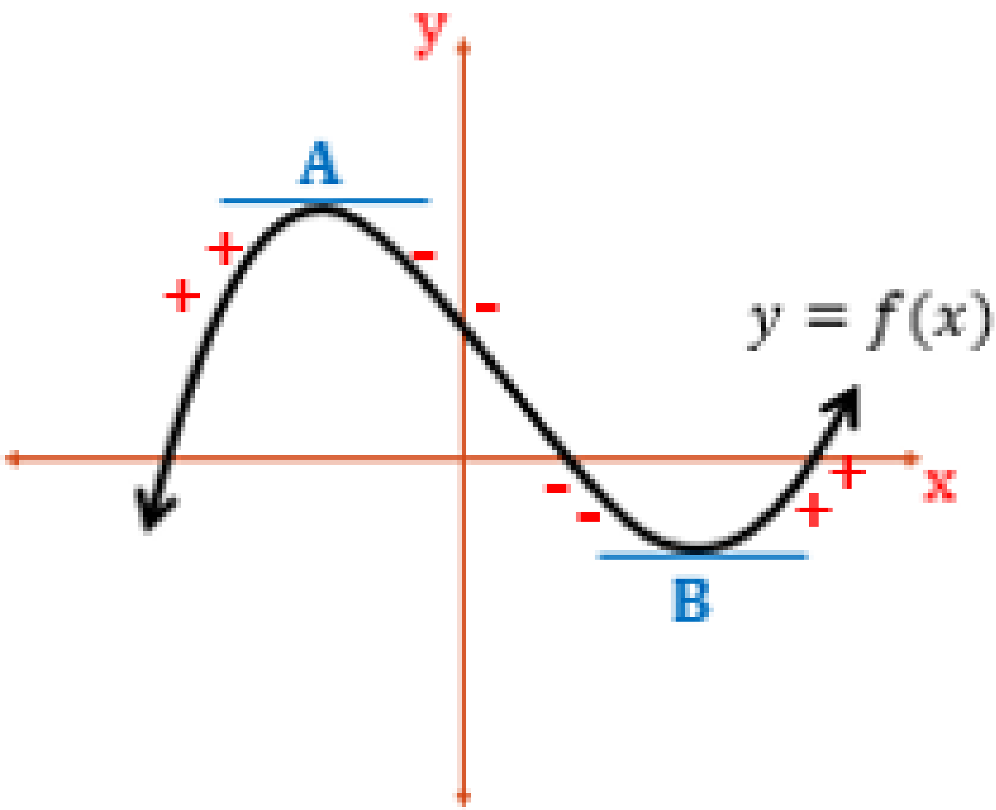
A function $f(x)$ has a relative maximum value (local maximum) at $x = a$, if $f(a)$ is *greater* than any value immediately preceding or following.

A function $f(x)$ has a relative minimum value (local minimum) at $x = b$, if $f(b)$ is *less* than any value immediately preceding or following.



Observe the gradients of tangents to the curve:

- To the left of A, the gradients are positive (+)
- Between A and B, the gradients are negative (-)
- To the right of B, the gradients are positive (+)



In other words, at a maximum,

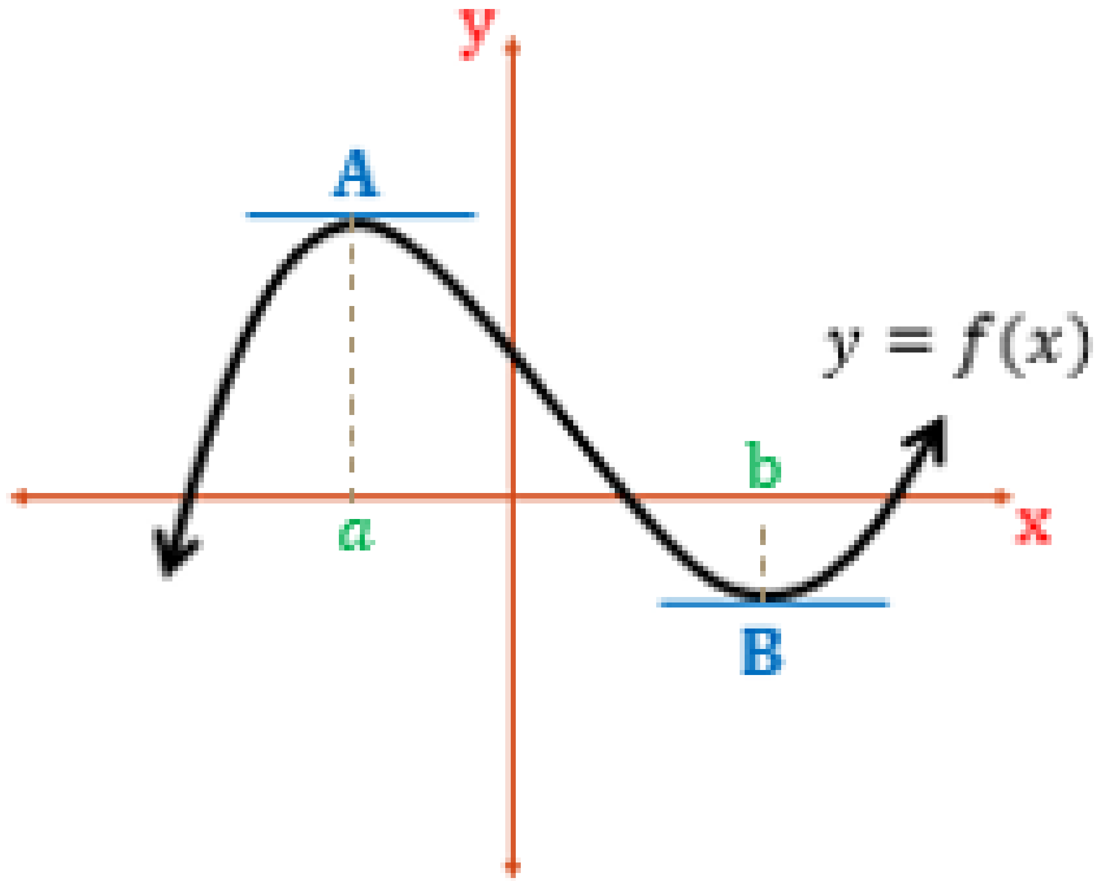
$f\,'(x)$ changes sign from + to − .

At a minimum,

$f\,'(x)$ changes sign from − to + .

We can also observe that at a maximum, at *A*, the graph is concave downward.

While at a minimum, at *B*, it is concave upward.



A value of x at which the function has either a maximum or a minimum is called a critical value.

The critical values determine turning points, at which the tangent is parallel to the x-axis. The critical values -- if any -- will be the *solutions* to $f\,'(x) = 0$.

The critical values are $x = a$ and $x = b$

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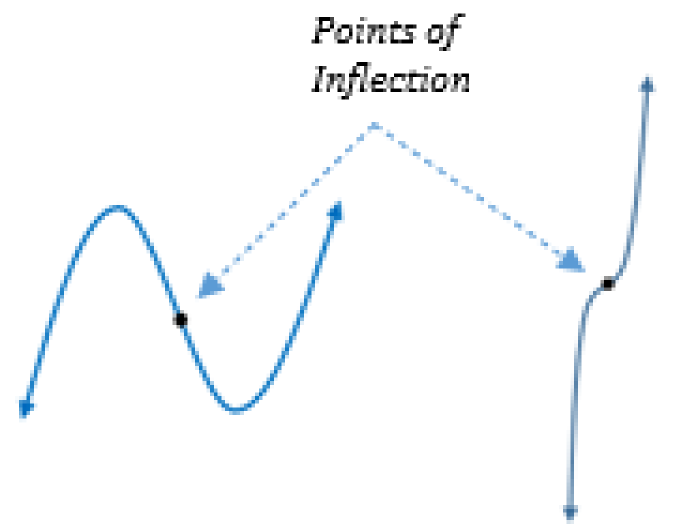
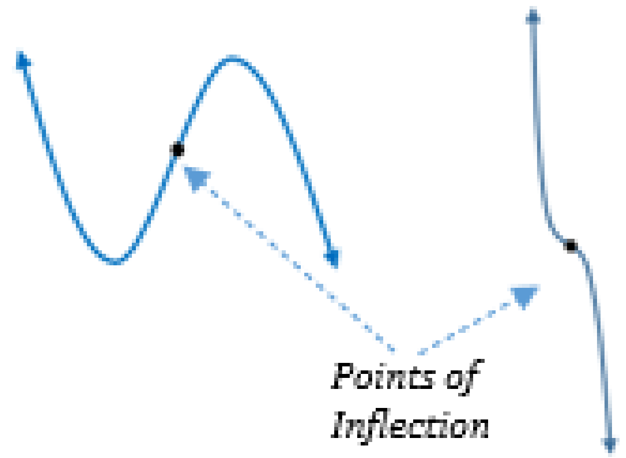
Solutions to $f''(x) = 0$ indicate a point of inflection at those solutions, not a maximum or minimum.

If the second derivative if positive, it is concave upward and if it is negative, the function is concave downward.



concave downward to concave upward



concave upward to concave downward

Example 1. Let $f(x) = x^2 - 6x + 5$.

Solution: $f'(x) = 2x - 6$

$$2x - 6 = 0$$

$$2x = 6$$

critical value $\quad x = 3$

$$f(3) = (3)^2 - 6(3) + 5$$

$$f(3) = -4$$

$$y = -4$$

$$critical\ point: P(3, -4)$$

Is *P(3, -4)* a maximum or minimum?

Use the First Derivative Test (FDT)

$$f'(x) = 2x - 6$$

$$f'(2.9) = 2(2.9) - 6 = -0.20$$

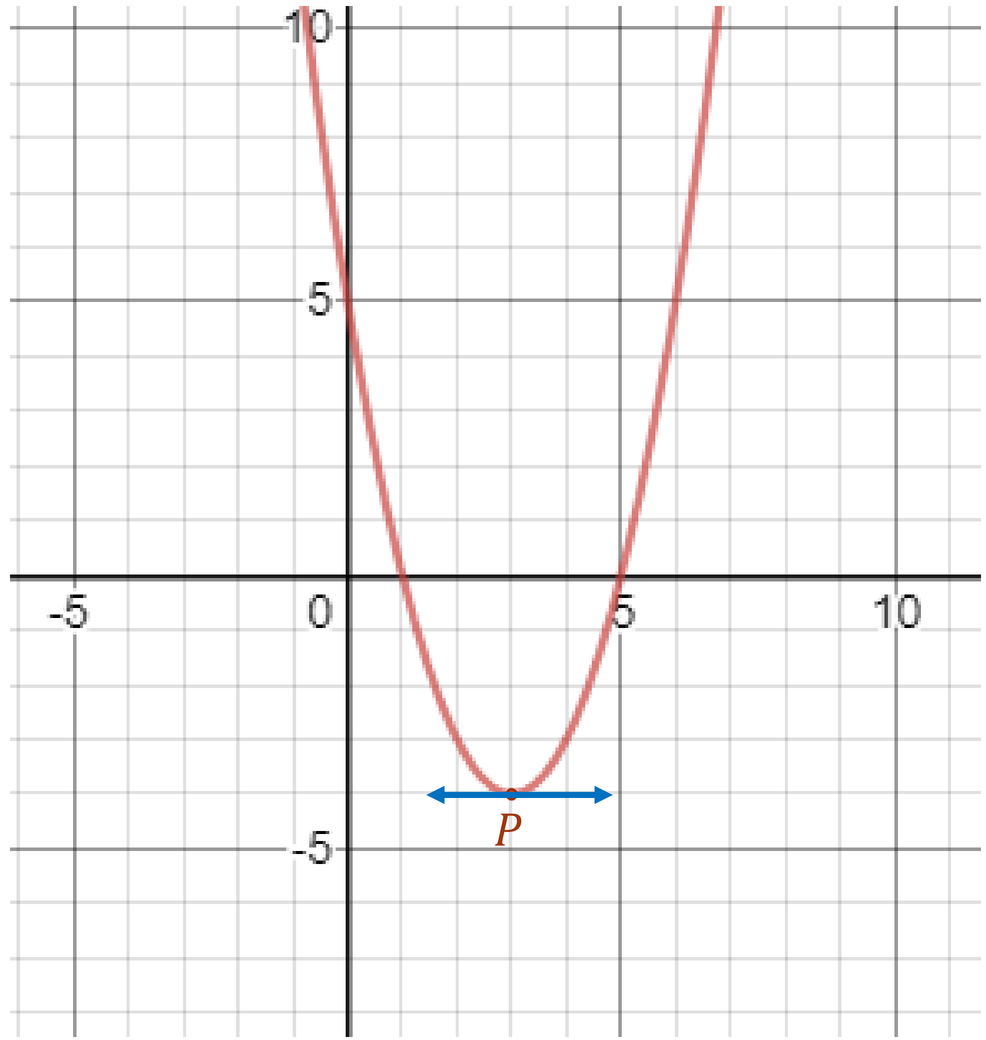
$$f'(3.1) = 2(3.1) - 6 = +0.20$$

Since the slope decreases as it approaches 3 then increases immediately after 3, then P is a relative minimum point.

$f(x) = x^2 - 6x + 5.$

$critical\ point: P(3, -4)$

P is a relative minimum point.



Example 2. Let $f(x) = 2x^3 - 9x^2 + 12x - 3$.

Solution: $f'(x) = 6x^2 - 18x + 12$

$$6x^2 - 18x + 12 = 0$$

$$6(x^2 - 3x + 2) = 0$$

$$6(x-1)(x-2) = 0$$

critical values $\Rightarrow$ $x = 1$ *and* $x = 2$

$$f(1) = 2(1)^3 - 9(1)^2 + 12(1) - 3$$

$f(1) = 2$ $\quad P_1(1,2)$

$$f(2) = 2(2)^3 - 9(2)^2 + 12(2) - 3$$

$f(2) = 1$ $\quad P_2(2,1)$

Example 2. Let $f(x) = 2x^3 - 9x^2 + 12x - 3.$

$critical\ points$: $P_1(1,2)$ & $P_2(2,1)$

$f'(x) = 6x^2 - 18x + 12$

First Derivative Test (FDT):

$P_1(1,2)$

$f'(0.9) = 6(0.9)^2 - 18(0.9) + 12$

$f'(0.9) = +0.66$

$f'(1.1) = 6(1.1)^2 - 18(1.1) + 12$

$f'(1.1) = -0.54$

$P_1(1,2)$ is relative maximum point

$P_2(2,1)$

$f'(1.9) = 6(1.9)^2 - 18(1.9) + 12$

$f'(1.9) = -0.54$

$f'(2.1) = 6(2.1)^2 - 18(2.1) + 12$

$f'(2.1) = +0.66$

$P_2(2,1)$ is relative minimum point

Given: $y = 2x^3 - 9x^2 + 12x - 3.$

$$f'(x) = 6x^2 - 18x + 12$$

$P_1(1,2)$ is a rel. max pt.

$P_2(2,1)$ is a rel. min pt.

$$f''(x) = 12x - 18$$

$$12x - 18 = 0$$

$$6(2x - 3) = 0$$

$$x = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 12\left(\frac{3}{2}\right) - 3$$

$$f\left(\frac{3}{2}\right) = \frac{3}{2}$$

$P_3\left(\frac{3}{2}, \frac{3}{2}\right) \Leftarrow$ *point of inflection*

Second Derivative Test (SDT):

$$f''(1.4) = 12(1.4) - 18 = -1.2$$

$$f''(1.6) = 12(1.6) - 18 = +1.2$$

Given: $y = 2x^3 - 9x^2 + 12x - 3.$

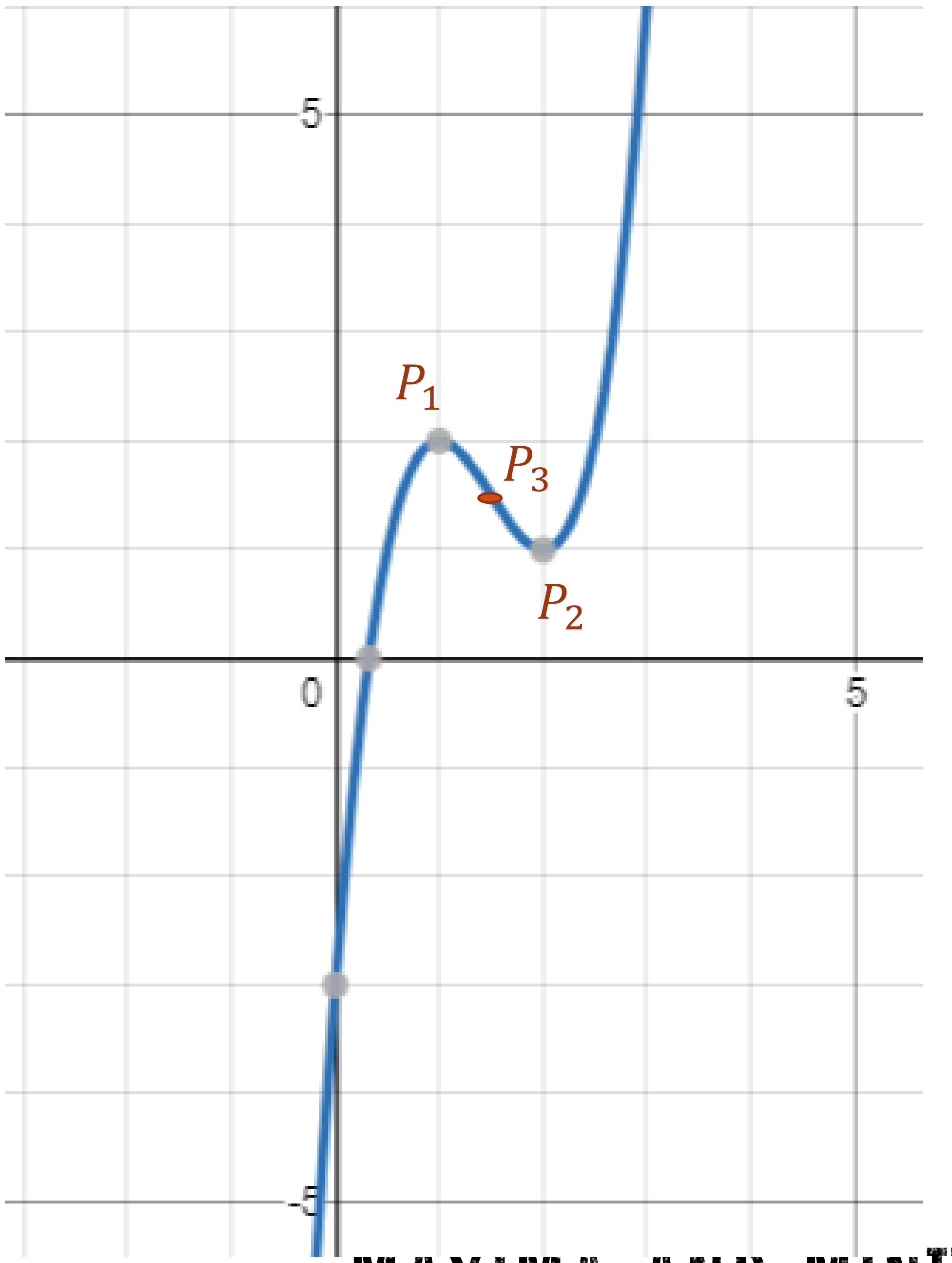
$$f'(x) = 6x^2 - 18x + 12$$

$P_1(1,2)$ is a rel. max pt.

$P_2(2,1)$ is a rel. min pt.

$$f''(x) = 12x - 18$$

$P_3\left(\frac{3}{2}, \frac{3}{2}\right)$ point of inflection



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Example 3. Graph $\boldsymbol{y = x^3 - 3x^2 - 1}$

Solution: $y' = 3x^2 - 6x$

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

critical values $\Rightarrow$ $x = 0 \text{ and } x = 2$

$$f(0) = (0)^3 - 3(0)^2 - 1 = -1$$

$$P_1(0, -1)$$

$$f(2) = (2)^3 - 3(2)^2 - 1 = -5$$

$$P_2(2, -5)$$

First Derivative Test (FDT):

$P_1(0, -1)$

$$f'(-0.1) = 3(-0.1)^2 - 6(-0.1) = 0.63$$

$$f'(0.1) = 3(0.1)^2 - 6(0.1) = -0.57$$

$P_1(0, -1)$ is a maximum point

$P_2(2, -5)$

$$f'(1.9) = 3(1.9)^2 - 6(1.9) = -0.57$$

$$f'(2.1) = 3(2.1)^2 - 6(2.1) = 0.63$$

$P_1(2, -5)$ is a minimum point

Given: $\boldsymbol{y = x^3 - 3x^2 - 1}$

$y' = 3x^2 - 6x$

$P_1(0, -1)$ is a max point

$P_1(2, -5)$ is a min point

$y'' = 6x - 6$

$6x - 6 = 0$

$6(x - 1) = 0$

$x = 1$

$f(1) = (1)^3 - 3(1)^2 - 1 = -3$

$Is\ \ P_3(1, -3)$ P.O.I?

<u>Second Derivative Test (sDT):</u>

$f''(0.9) = 6(0.9) - 6 = -5.46$

$f''(1.1) = 6(1.1) - 6 = +0.60$

$P_3(1, -3)$ is a point of inflection

Given: $\boldsymbol{y = x^3 - 3x^2 - 1}$

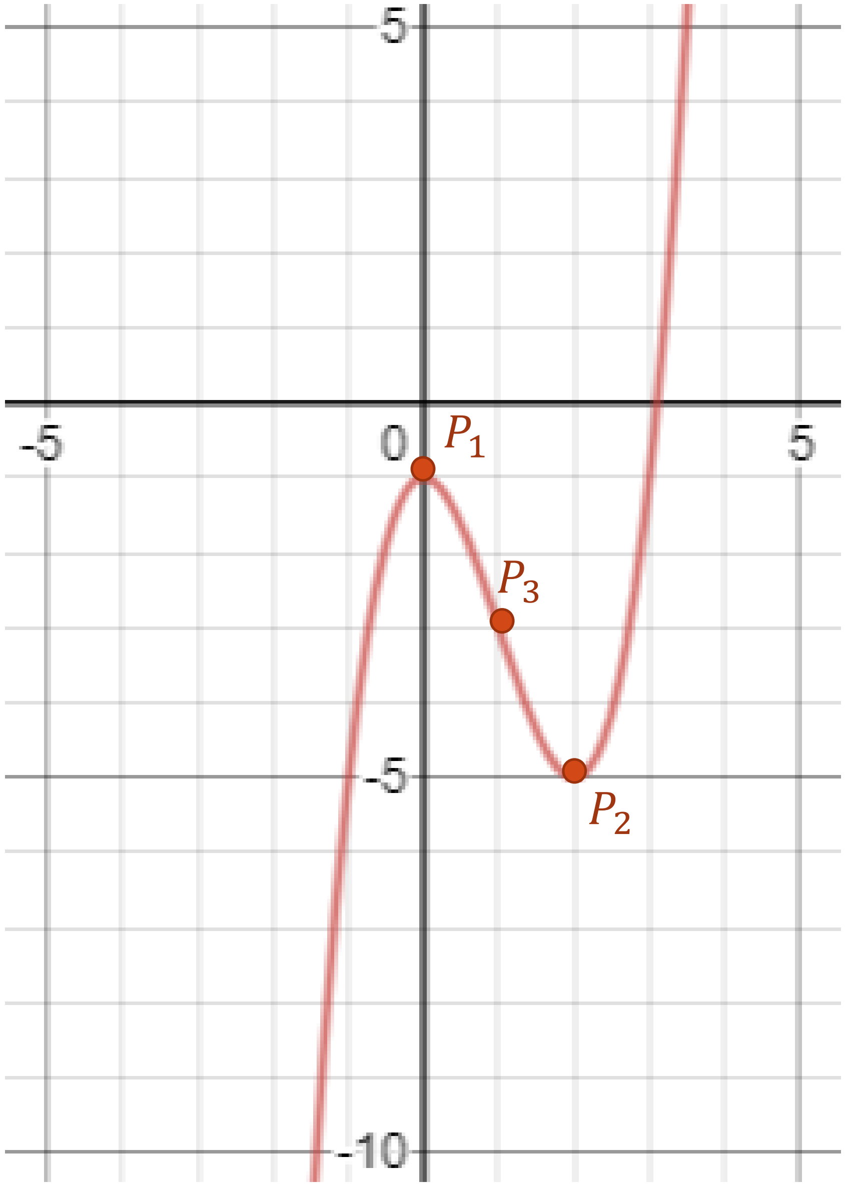
$y' = 3x^2 - 6x$

$P_1(0, -1)$ is a max point

$P_2(2, -5)$ is a min point

$y'' = 6x - 6$

$P_3(1, -3)$ is a point of inflection



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Example 4. Graph $\boldsymbol{y = x^4 - 6x^2}$

Solution: $y' = 4x^3 - 12x$

$4x(x^2 - 3) = 0$

critical values ⇒

$x = 0$

$x = \sqrt{3} = 1.73$

$x = -\sqrt{3} = -1.73$

$f(0) = (0)^4 - 6(0)^2 = 0$

$P_1(0, 0)$

$f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2 = -9$

$P_2(\sqrt{3}, -9)$

$f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 = -9$

$P_3(-\sqrt{3}, -9)$

Given: $\boldsymbol{y = x^4 - 6x^2}$

$y' = 4x^3 - 12x$

First Derivative Test (FDT):

$P_1(0,0), P_2(\sqrt{3},-9), P_3(-\sqrt{3},-9)$

$P_1(0,0)$

$f'(-0.1) = 4(-0.1)^3 - 12(-0.1)$

$f'(-0.1) = 1.196$

$f'(0.1) = 4(0.1)^3 - 12(0.1)$

$f'(0.1) = -1.196$

$P_1(0,0)$ is a relative max point

$P_2(\sqrt{3},-9)$

$f'(1.6) = 4(1.6)^3 - 12(1.6) = -2.816$

$f'(1.8) = 4(1.8)^3 - 12(1.8) = 1.728$

$P_2(\sqrt{3},-9)$ is a relative min point

$P_3(-\sqrt{3},-9)$

$f'(-1.8) = 4(-1.8)^3 - 12(-1.8)$

$f'(-1.8) = -1.728$

$f'(-1.6) = 4(-1.6)^3 - 12(-1.6)$

$f'(-1.6) = 2.816$

$P_3(-\sqrt{3},-9)$ is a relative min point

Given: $\boldsymbol{y = x^4 - 6x^2}$

$y' = 4x^3 - 12x$

$P_1(0,0)$ is a rel. max pt.

$P_2\left(\sqrt{3},-9\right)$ is a rel. min pt.

$P_3\left(-\sqrt{3},-9\right)$ is a rel. min pt.

$y'' = 12x^2 - 12$

$12x^2 - 12 = 0$

$12(x^2 - 1) = 0$

$x = 1 \; and \; x = -1$

$f(1) = (1)^4 - 6(1)^2 = -5$

$P_4(1,-5)$

$f(-1) = (-1)^4 - 6(-1)^2 = -5$

$P_5(-1,-5)$

P_4: $f''(0.9) = 12(0.9)^2 - 12 = -2.28$

$f''(1.1) = 12(1.1)^2 - 12 = +2.25$

P_5: $f''(-1.1) = 12(-1.1)^2 - 12 = +2.25$

$f''(-0.9) = 12(-0.9)^2 - 12 = -2.28$

$P_4(1,-5)$ and $P_5(-1,-5)$ are points of inflection

Given: $\boldsymbol{y = x^4 - 6x^2}$

$y' = 4x^3 - 12x$

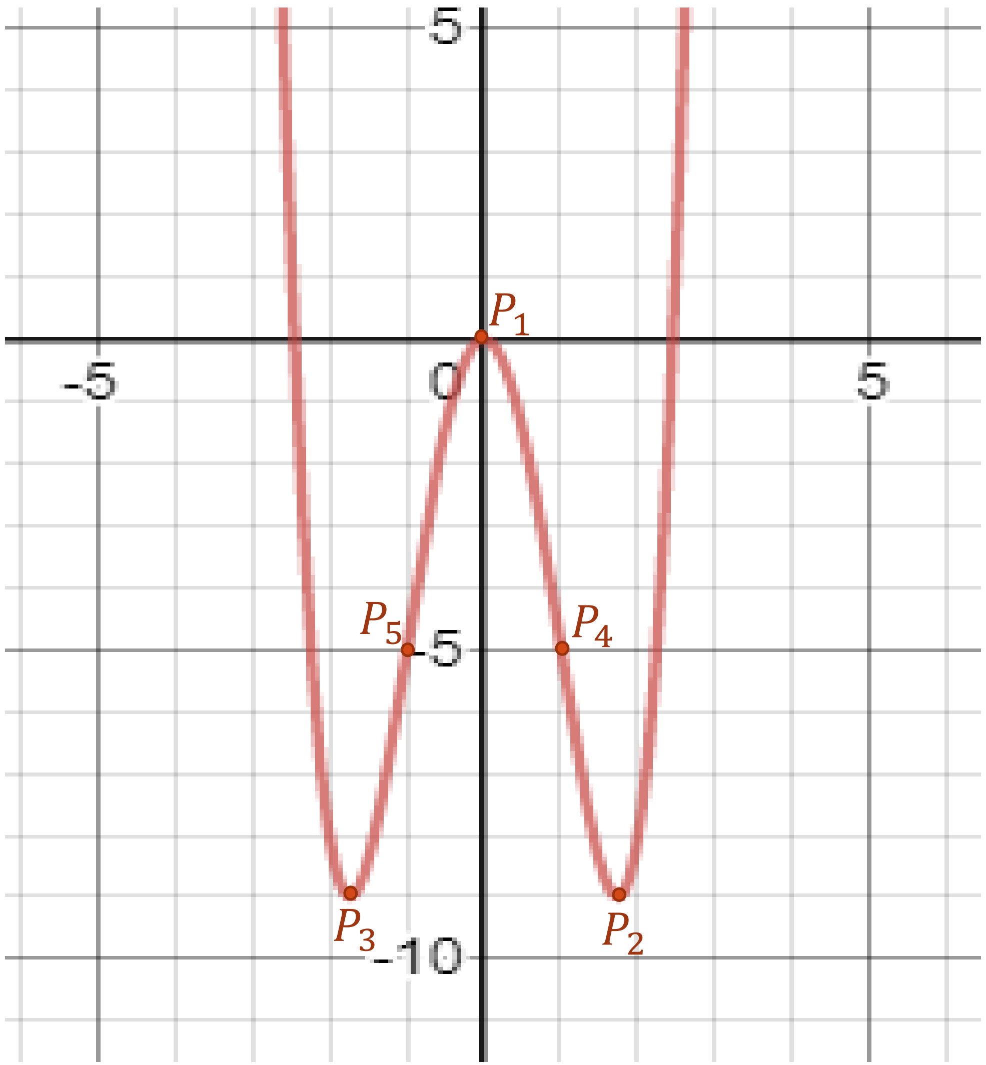
$P_1(0,0)$ is a rel. max pt.

$P_2(\sqrt{3},-9)$ is a rel. min pt.

$P_3(-\sqrt{3},-9)$ is a rel. min pt.

$y'' = 12x^2 - 12$

$P_4(1,-5)$ and $P_5(-1,-5)$ are points of inflection



Example 5. Find the point of inflection of the function $\boldsymbol{y = 3x^5 + 5x^4 - 20x^3}$

Solution:

$$y' = 15x^4 + 20x^3 - 60x^2$$

$$y'' = 60x^3 + 60x^2 - 120x$$

$$60x^3 + 60x^2 - 120x = 0$$

$$60x(x^2 + x - 2) = 0$$

$$60x(x + 2)(x - 1) = 0$$

$$x = 0;\ x = -2\,;\ x = 1$$

$$f(0) = 3(0)^5 + 5(0)^4 - 20(0)^3$$

$$f(0) = 0$$

$$f(-2) = 3(-2)^5 + 5(-2)^4 - 20(-2)^3$$

$$f(-2) = 144$$

$$f(1) = 3(1)^5 + 5(1)^4 - 20(1)^3$$

$$f(1) = -12$$

$\boldsymbol{The\ points\ of\ inflection\ are}$

$\boldsymbol{(0,0), (-2,144)\, and\ (1,-12)}$

HOME WORK #8

Find the relative maximum point(s), relative minimum point(s) and point(s) of inflection of the following functions. Sketch the graph.

1. $y = x^2 - 4x + 3$
2. $y = x^2(x-2)^2$
3. $y = x^3 - 3x^2 - 4x + 5$

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