Exponential Functions

DERIVATIVES

Derivatives of Exponential Functions

i. $d(a^u) = a^u \ln a \, d(u)$

ii. $d(e^u) = e^u d(u)$

iii. $d(u^v) = u^v d(v \ln u)$

where a and e are constant
u and v are differentiable functions of x

Example: Find y' if $y = 4^{\sin 2x}$

Solution: Let $a = 4$

$u = \sin 2x$

$$y' = 4^{\sin 2x} \ln 4 \, d(\sin 2x)$$

$$y' = 4^{\sin 2x} \ln 4 \, (2 \cos 2x)$$

$$y' = (2^2)^{\sin 2x} \ln 2^2 \, (2 \cos 2x)$$

$$y' = 2^{2\sin 2x} 2 \ln 2 \, (2 \cos 2x)$$

$$y' = 2^{2+2\sin 2x} \ln 2 \, (\cos 2x)$$

$$\boldsymbol{y' = (\cos 2x) 2^{2+2\sin 2x} \ln 2}$$

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Example: Find y' if $y = x^3 e^{3x}$

Solution: $y' = x^3 d(e^{3x}) + e^{3x} d(x^3)$

$$y' = x^3(e^{3x} \cdot 3) + e^{3x}(3x^2)$$

$$y' = 3x^3 e^{3x} + 3x^2 e^{3x}$$

$$\boldsymbol{y' = 3x^2 e^{3x}(x+1)}$$

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Example: Find y' if $y = \sin x^{\cos x}$

Solution:

$$y' = \sin x^{\cos x} d(\cos x \ln \sin x)$$

$$y' = \sin x^{\cos x}\left[\cos x \cdot \frac{1}{\sin x} \cdot \cos x \cdot 1 + \ln \sin x \cdot -\sin x \cdot 1\right]$$

$$y' = \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\right]$$

$$\boldsymbol{y' = \sin x^{\cos x}[\cos x \cot x - \sin x \ln \sin x]}$$

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Example: Find y' if $y = (e^{-2x} + 2e^{x})^{3/2}$

Solution:

$$y' = \frac{3}{2}(e^{-2x} + 2e^{x})^{1/2}[e^{-2x}(-2) + 2e^{x}(1)]$$

$$y' = \frac{3}{2} \cdot 2(e^{-2x} + 2e^{x})^{1/2}[-e^{-2x} + e^{x}]$$

$$\boldsymbol{y' = 3\left(e^{-2x} + 2e^{x}\right)^{1/2}\left[-e^{-2x} + e^{x}\right]}$$

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Example: Find y' if $y = \frac{e^{3x}+1}{e^{3x}-1}$

Solution:

$$y' = \frac{(e^{3x}-1)(3e^{3x}) - (e^{3x}+1)(3e^{3x})}{(e^{3x}-1)^2}$$

$$y' = \frac{(3e^{3x})[e^{3x}-1-e^{3x}-1]}{(e^{3x}-1)^2}$$

$$y' = \frac{(3e^{3x})[-2]}{(e^{3x}-1)^2}$$

$$\boldsymbol{y' = \frac{-6e^{3x}}{(e^{3x}-1)^2}}$$

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Example: Find y' if $e^x + e^y = xe^y - ye^x$

Solution:

$$e^x \cdot 1 + e^y \cdot 1 \cdot y' = (xe^y \cdot 1 \cdot y' + e^y \cdot 1) - (ye^x \cdot 1 + e^x \cdot 1 \cdot y')$$

$$e^x + e^y \cdot y' = xe^y \cdot y' + e^y - ye^x - e^x \cdot y'$$

$$e^y \cdot y' - xe^y \cdot y' + e^x \cdot y' = e^y - ye^x - e^x$$

$$y'(e^y - xe^y + e^x) = e^y - ye^x - e^x$$

$$\boldsymbol{y' = \frac{e^y - ye^x - e^x}{e^y - xe^y + e^x}}$$

Example: Find y'', given $y = e^{-x} \ln x$

Solution:

$$y' = e^{-x} \cdot \frac{1}{x} \cdot 1 + \ln x \cdot e^{-x} \cdot -1$$

$$y' = \frac{e^{-x}}{x} - e^{-x} \ln x$$

$$y' = \frac{e^{-x}}{x} - y$$

$$y'' = \frac{xe^{-x} \cdot -1 - e^{-x} \cdot 1}{x^2} - y'$$

$$y'' = \frac{-xe^{-x} - e^{-x}}{x^2} - y'$$

$$y'' = \frac{-e^{-x}}{x} - \frac{e^{-x}}{x^2} - y'$$

$$y'' = \frac{-e^{-x}}{x} - \frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} - y$$

$$y'' = \frac{-e^{-x}}{x} - \frac{e^{-x}}{x^2} - \frac{e^{-x}}{x} - e^{-x} \ln x$$

$$y'' = \frac{-2e^{-x}}{x} - \frac{e^{-x}}{x^2} - e^{-x} \ln x$$

$$\boldsymbol{y'' = -e^{-x}\left[\frac{2}{x} + \frac{1}{x^2} + \ln x\right]}$$

Example: Find y'', given $y = e^{-2x} \sin 3x$

Solution:

$$y' = e^{-2x} \cdot 3\cos 3x + \sin 3x \cdot -2\, e^{-2x}$$

$$y' = 3e^{-2x} \cos 3x - 2e^{-2x} \sin 3x$$

$$y' = 3e^{-2x} \cos 3x - 2y$$

$$y'' = 3[e^{-2x} \cdot -3\sin 3x + \cos 3x \cdot -2e^{-2x}] - 2y'$$

$$y'' = -9e^{-2x} \sin 3x - 6e^{-2x} \cos 3x - 2y'$$

$$y'' = -9e^{-2x} \sin 3x - 6e^{-2x} \cos 3x - 2(3e^{-2x} \cos 3x - 2y)$$

$$y'' = -9e^{-2x} \sin 3x - 6e^{-2x} \cos 3x - 6e^{-2x} \cos 3x + 4y$$

$$y'' = -9e^{-2x} \sin 3x - 12e^{-2x} \cos 3x + 4y$$

$$y'' = -9e^{-2x} \sin 3x - 12e^{-2x} \cos 3x + 4(e^{-2x} \sin 3x)$$

$$y'' = -5e^{-2x} \sin 3x - 12e^{-2x} \cos 3x$$

$$\boldsymbol{y'' = -e^{-2x}[5\sin 3x + 12\cos 3x]}$$

Home Work #13:

Find dy/dx and simplify whenever possible:

1. $y = e^{\arctan x}$

2. $y = \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

3. $y = 3^{5x} \arctan 5x$

4. $y = (\ln x)^{\cos x}$

5. $y = x^{e^x}$

6. $y = x^{2x}$

7. $e^{\ln 4x} + e^{\ln 4y} = 1$

8. $3^{2x} + 3^{2y} = 6$

9. $5^{\ln x} y^2 + x^2 \ln y = x - y$

10. $y = \dfrac{9^{\tan(\sin x)}}{e^{\cos x}}$