RULES FOR DIFFERENTIATION

Differentiation Rules

(i) Constant Rule: The derivative of a constant function is 0.

$$\frac{d}{dx}(c) = 0$$

Examples:

(1) $f(x) = 3$

$f'(x) = 0$

(2) $g(x) = \frac{-4}{5}$

$g'(x) = 0$

(3) $y = \pi$

$y' = 0$

(ii) Derivative of x: The derivative of x is 1

$$\frac{d}{dx}(x) = 1$$

Examples:

(1) $f(x) = 3\mathrm{x}$

$f'(x) = 3\frac{d}{dx}(x)$

$f'(x) = 3(1)$

$f'(x) = 3$

(2) $y = -\frac{2}{3}\mathrm{x}$

$y' = -\frac{2}{3}(1)$

$y' = -\frac{2}{3}$

(iii) **Power Rule:** The derivative of the function $f(x) = x^n$ is the exponent multiplied by x raised to $n - 1$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Examples:

(1) $y = x^5$

$y' = 5x^{5-1}$

$y' = 5x^4$

(2) $f(x) = \frac{1}{x^3} = x^{-3}$

$f'(x) = -3x^{-3-1}$

$f'(x) = -3x^{-4}$

$f'(x) = -\frac{3}{x^4}$

(3) $h(x) = \sqrt{x} = x^{\frac{1}{2}}$

$h'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$

$h'^{(x)} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

(iv) Constant times a Function: The derivative of a constant c times the function $f(x) = u$ is the product of the constant times the derivative of the function .

$$\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$$

Examples:

(1) $f(x) = 3x^4$

$$f'(x) = 3\frac{d}{dx}(x^4)$$

$$f'(x) = 3(4x^{4-1})$$

$$f'(x) = 12x^3$$

(2) $y = \frac{3}{x^7} = 3x^{-7}$

$$y' = 3\frac{d}{dx}(x^{-7})$$

$$y' = 3(-7x^{-7-1})$$

$$y' = -21x^{-8} = \frac{-21}{x^8}$$

Examples:

$$(3)\, g(w) = \sqrt[3]{w^2} - \frac{2\sqrt[3]{w}}{3}$$

$$g'(w) = \frac{d}{dw}\left(w^{2/3}\right) - \frac{2}{3}\frac{d}{dw}\left(w^{1/3}\right)$$

$$g'(w) = \frac{2}{3}w^{-1/3} - \frac{2}{3}\cdot\frac{1}{3}w^{-2/3}$$

$$g'(w) = \frac{2}{3}w^{-1/3} - \frac{2}{9}w^{-2/3}$$

$$g'(w) = \frac{2}{3w^{1/3}} - \frac{2}{9w^{2/3}}$$

$$g'(w) = \frac{2}{3\sqrt[3]{w}} - \frac{2}{9\sqrt[3]{w^2}}$$

$$(4)\; R(t) = \frac{3}{2\sqrt{t}} - \frac{2}{5\sqrt[3]{t}}$$

$$R'(t) = \frac{3}{2}\frac{d}{dt}\left(t^{-\frac{1}{2}}\right) - \frac{2}{5}\frac{d}{dt}\left(t^{-\frac{1}{3}}\right)$$

$$R'(t) = \frac{3}{2}\cdot -\frac{1}{2}t^{-\frac{3}{2}} - \frac{2}{5}\cdot -\frac{1}{3}t^{-\frac{4}{3}}$$

$$R'(t) = -\frac{3}{4}t^{-\frac{3}{2}} + \frac{2}{15}t^{-\frac{4}{3}}$$

$$R'(t) = -\frac{3}{4t^{\frac{3}{2}}} + \frac{2}{15t^{\frac{4}{3}}}$$

$$R'(t) = -\frac{3}{4\sqrt{t^3}} + \frac{2}{15\sqrt[3]{t^4}}$$

(v) Sum and Difference Rule: The derivative of the sum or difference of functions $f(x) = u,\ g(x) = v,\ h(x) = w, \ldots$ is the sum or difference of their derivatives.

$$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v) \pm \frac{d}{dx}(w) \pm \cdots$$

Examples:

(1) $y = 3x^2 - 4x + 2$

$$y' = 3\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) + \frac{d}{dx}(2)$$

$$y' = 3(2x) - 4(1) + 0$$

$$y' = 6x - 4$$

(2) $f(x) = -2x^3 - \dfrac{3}{x^4} + 4\sqrt{x}$

$$f'(x) = -2\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^{-4}) + 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$f'(x) = -2(3x^2) - 3(-4x^{-5}) + 4\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$f'(x) = -6x^2 + \frac{12}{x^5} + \frac{2}{\sqrt{x}}$$

(vi) Product Rule: The derivative of the product of the functions $f(x) = u$ and $g(x) = v$ is equal to the first function $f(x)$ times the derivative of the second function $g'(x)$ plus the second function $g(x)$ times the derivative of the first function $f'(x)$

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Examples:

(1) $y = (4x - 1)(3x + 2)$

$$y' = (4x - 1)\left[\frac{d}{dx}(3x + 2)\right] + (3x + 2)\left[\frac{d}{dx}(4x - 1)\right]$$

$$y' = (4x - 1)[3] + (3x + 2)[4]$$

$$y' = 12x - 3 + 12x + 8$$

$$y' = 24x + 5$$

(vi) Product Rule:

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$$

Examples:

(2) $h(x) = (x^2 - 4x + 3)(3 - 2x)$

$$h'(x) = (x^2 - 4x + 3)\left[\frac{d}{dx}(3 - 2x)\right] + (3 - 2x)\left[\frac{d}{dx}(x^2 - 4x + 3)\right]$$

$$h'(x) = (x^2 - 4x + 3)[-2] + (3 - 2x)[2x - 4]$$

$$h'(x) = -2x^2 + 8x - 6 + 6x - 12 - 4x^2 + 8x$$

$$h'(x) = -6x^2 + 22x - 18$$

(vii) **Quotient Rule:** The derivative of a quotient of two functions $\frac{f(x)}{g(x)} = \frac{u}{v}$ where the denominator $g(x) = v \neq 0$, is equal to the denominator $g(x)$ times the derivative of the numerator $f'(x)$, minus the numerator $f(x)$ times the derivative of the denominator $g'(x)$, all over the square of the denominator.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Examples:

$$(1)\ y = \frac{x}{2x - x^2}$$

$$y' = \frac{(2x - x^2)\left[\frac{d}{dx}(x)\right] - (x)\left[\frac{d}{dx}(2x - x^2)\right]}{(2x - x^2)^2}$$

$$y' = \frac{(2x - x^2)[1] - (x)[2 - 2x]}{(2x - x^2)^2}$$

$$y' = \frac{2x - x^2 - 2x + 2x^2}{(2x - x^2)^2}$$

$$y' = \frac{x^2}{(2x - x^2)^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Examples:

$$(2)\ y = \frac{x^2 - 3x + 4}{3x + 1}$$

$$y' = \frac{(3x+1)\left[\frac{d}{dx}(x^2 - 3x + 4)\right] - (x^2 - 3x + 4)\left[\frac{d}{dx}(3x+1)\right]}{(3x+1)^2}$$

$$y' = \frac{(3x+1)[2x-3] - (x^2 - 3x + 4)[3]}{(3x+1)^2}$$

$$y' = \frac{6x^2 - 9x + 2x - 3 - (3x^2 - 9x + 12)}{(3x+1)^2}$$

$$y' = \frac{6x^2 - 9x + 2x - 3 - 3x^2 + 9x - 12}{(3x+1)^2}$$

$$y' = \frac{3x^2 + 2x - 15}{(3x+1)^2}$$

Practice Task #5: Basic Differentiation Formulas

Differentiate the following using the rules of differentiation.

(1) $y = 2x^5 + x^4 - 5x^3 - 2x^2 + 3$

(2) $f(x) = \dfrac{2}{x^4} - \dfrac{3}{x^2} + \dfrac{4}{x}$

(3) $g(x) = \sqrt{x} - 4\sqrt{x^3} + \sqrt[3]{x}$

(4) $y = \sqrt{3x^3} - \dfrac{1}{\sqrt{2x}}$

(5) $y = (x - 2)(x + 3)$

(6) $y = 6x(x^2 - 1)$

(7) $y = \dfrac{3 - 2x}{3 + 2x}$

(8) $y = \dfrac{2x + 1}{x^2 - 1}$

(9) $y = \dfrac{6x^4 - 18x^2 - 12x}{2x^3 + 2x - 1}$

(10) $y = (3x^2 - 2x + 5)(5x^2 + 3x - 2)$

(viii) Chain Rule for Differentiation (Extended Power Rule)

The derivative of a function $f(x) = u$ raised to n is equal to n the product of the exponent n times the function $f(x)$ raised to $n - 1$ multiplied by the derivative of the function $f'(x)$.

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}(u)$$

Examples:

(1) $y = (4x - 5)^3$

$$y' = 3(4x - 5)^{3-1}\frac{d}{dx}(4x - 5)$$

$$y' = 3(4x - 5)^2(4)$$

$$y' = 12(4x - 5)^2$$

(2) $f(x) = \dfrac{3}{(2x + 1)^2} = 3(2x + 1)^{-2}$

$$f'(x) = 3(-2)(2x + 1)^{-2-1}\frac{d}{dx}(2x + 1)$$

$$f'(x) = -6(2x + 1)^{-3}(2)$$

$$f'(x) = \frac{-12}{(2x + 1)^3}$$

Chain Rule for Differentiation

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}(u)$$

Examples:

(3) $g(x) = 12\sqrt[3]{x^2 - 1}$

$$= 12(x^2 - 1)^{\frac{1}{3}}$$

$$g'(x) = 12\left(\frac{1}{3}\right)(x^2 - 1)^{\frac{1}{3}-1}\frac{d}{dx}(x^2 - 1)$$

$$g'(x) = 4(x^2 - 1)^{-\frac{2}{3}}(2x)$$

$$g'(x) = \frac{8x}{(x^2 - 1)^{\frac{2}{3}}}$$

$$g'(x) = \frac{8x}{\sqrt[3]{(x^2 - 1)^2}}$$

(4) Find the derivative of

$$y = (x - 4)^3(2x + 3)^5$$

(5) Find the derivative of

$$y = \frac{(2-5x)^4}{(1+x^3)^5}$$

(6) Find f'(x), given that

$$f(x) = \sqrt{\frac{7x+2}{(8x-3)^3}}$$

Chain Rule for Differentiation

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}(u)$$

Examples:

(4) Find the derivative of $y = (x-4)^3(2x+3)^5$

Let $u = (x-4)^3$ $\quad du = 3(x-4)^2(1) = 3(x-4)^2$

$v = (2x+3)^5$ $\quad dv = 5(2x+3)^4(2) = 10(2x+3)^4$

Using the product formula $u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$.

$$y' = (x-4)^3[10(2x+3)^4] + (2x+3)^5[3(x-4)^2]$$

$$y' = (x-4)^2(2x+3)^4\{(x-4)[10] + (2x+3)[3]\}$$

$$y' = (x-4)^2(2x+3)^4\{10x - 40 + 6x + 9\}$$

$$y' = (x-4)^2(2x+3)^4(16x-31)$$

Examples:

Chain Rule for Differentiation

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}(u)$$

(5) Find the derivative of y $= \frac{(2-5x)^4}{(1+x^3)^5}$

Let $u = (2-5x)^4$ $\qquad du = 4(2-5x)^3(-5) = -20(2-5x)^3$

$v = (1+x^3)^5$ $\qquad dv = 5(1+x^3)^4(3x^2) = 15x^2(1+x^3)^4$

Using the quotient formula $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

$$y' = \frac{(1+x^3)^5[-20(2-5x)^3] - (2-5x)^4[15x^2(1+x^3)^4]}{[(1+x^3)^5]^2}$$

$$y' = \frac{5(2-5x)^3(1+x^3)^4\{(1+x^3)[-4] - (2-5x)[3x^2]\}}{(1+x^3)^{10}}$$

$$y' = \frac{5(2-5x)^3(1+x^3)^4\{-4-4x^3-6x^2+15x^3\}}{(1+x^3)^{10}} = \frac{5(2-5x)^3(-4-6x^2+11x^3)}{(1+x^3)^6}$$

Examples:

Chain Rule for Differentiation

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}(u)$$

(6) Find f'(x), given that $f(x) = \sqrt{\frac{7x+2}{(8x-3)^3}}$

Let $u = (7x+2)^{\frac{1}{2}}$ $\quad du = \frac{1}{2}(7x+2)^{-\frac{1}{2}}(7) = \frac{7}{2}(7x+2)^{-\frac{1}{2}}$

$v = (8x-3)^{\frac{3}{2}}$ $\quad dv = \frac{3}{2}(8x-3)^{\frac{1}{2}}(8) = 12(8x-3)^{\frac{1}{2}}$

$$y' = \frac{(8x-3)^{\frac{3}{2}}\left[\frac{7}{2}(7x+2)^{-\frac{1}{2}}\right] - (7x+2)^{\frac{1}{2}}\left[12(8x-3)^{\frac{1}{2}}\right]}{\left[(8x-3)^{\frac{3}{2}}\right]^2}$$

$$y' = \frac{(8x-3)^{\frac{1}{2}}(7x+2)^{-\frac{1}{2}}\left\{(8x-3)\left[\frac{7}{2}\right] - (7x+2)[12]\right\}}{(8x-3)^3}$$

$$y' = \frac{(8x-3)^{\frac{1}{2}}(7x+2)^{-\frac{1}{2}}\left\{28x - \frac{21}{2} - 96x - 24\right\}}{(8x-3)^3}$$

$$y' = \frac{(8x-3)^{\frac{1}{2}}(7x+2)^{-\frac{1}{2}}\left\{-68x - \frac{69}{2}\right\}}{(8x-3)^3}$$

$$y' = \frac{\left\{\frac{-136x-69}{2}\right\}}{(8x-3)^{\frac{5}{2}}(7x+2)^{\frac{1}{2}}}$$

$$y' = \frac{-136x-69}{2(8x-3)^{\frac{5}{2}}(7x+2)^{\frac{1}{2}}}$$

Home Work #5: Chain Rule

Find the derivative of the following using the rules of differentiation.

(1) $f(x) = (x^2 - 3)^4$

(2) $H(x) = \dfrac{3}{(4 - x^2)^2}$

(3) $G(x) = \sqrt{3 + 5x - x^2}$

(4) $h(x) = \dfrac{12}{\sqrt[3]{2x^2 - 3x + 1}}$

(5) $f(x) = x^2(x + 1)^3$

(6) $H(x) = \dfrac{(x^2 - 1)^2}{(x^2 + 1)^3}$

(7) $Y(x) = \dfrac{1}{\left(1 + \sqrt{1 - x}\right)^2}$

(8) $Y(x) = \sqrt{\dfrac{(2x + 1)^5}{2x - 1}}$