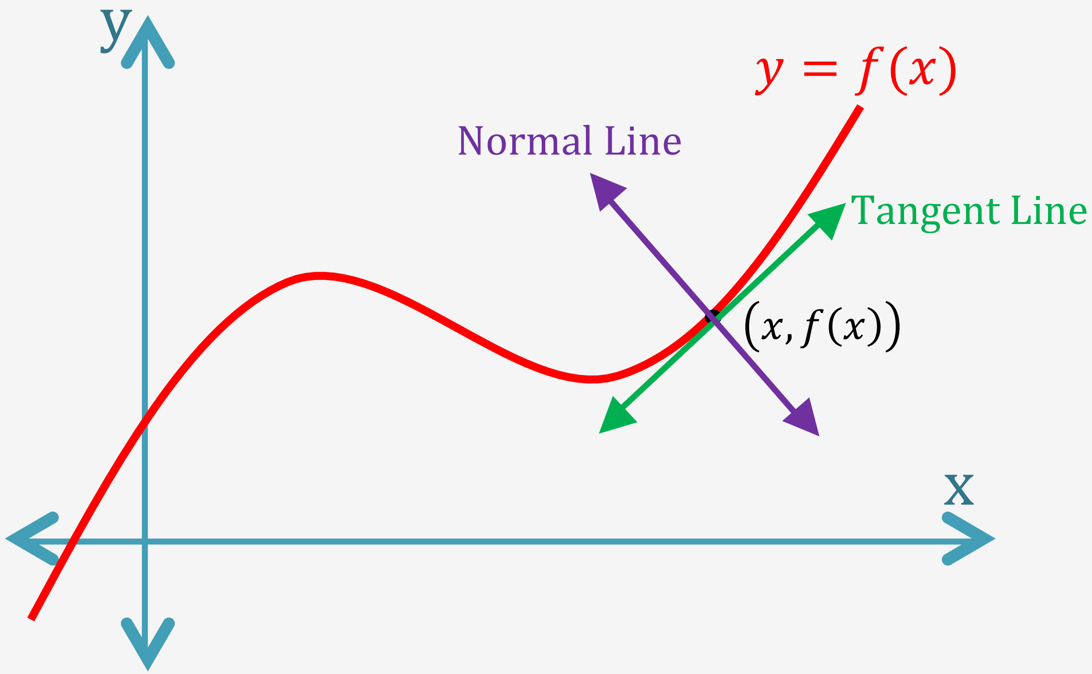
TANGENT AND NORMAL LINES TO A CURVE



Recall :

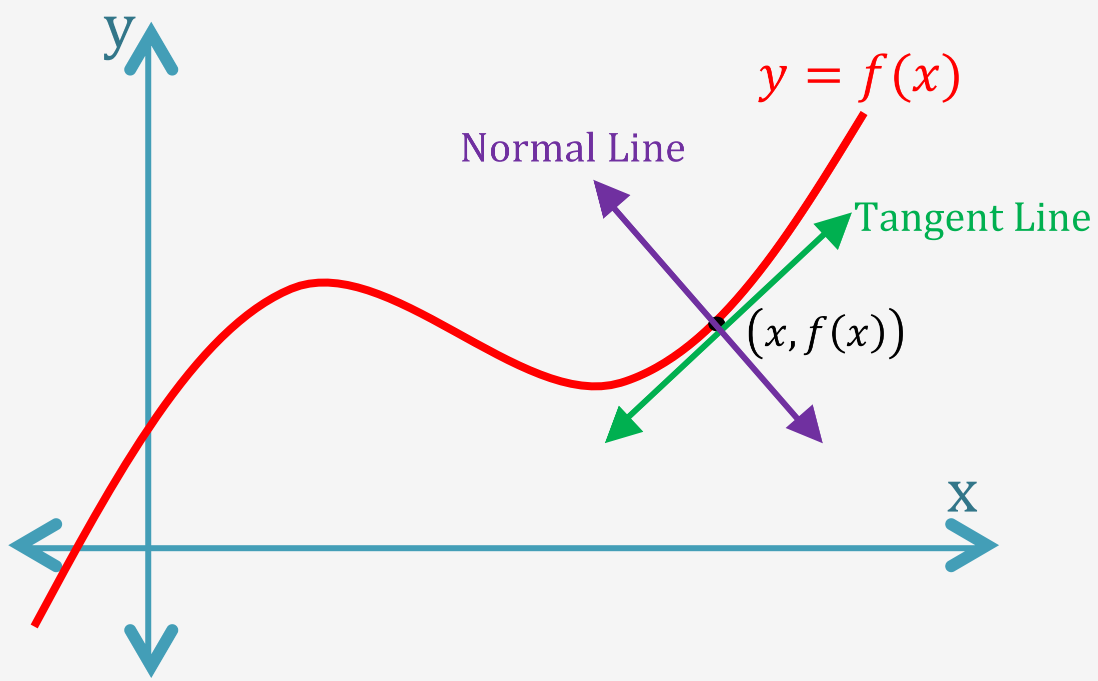
$$slope = m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = y'$$

$$m_T = y' \qquad m_N = -\frac{1}{y'}$$

If L_1 and L_2 are parallel, then $m_1 = m_2$

If L_1 and L_2 are perpendicular, then $m_1 m_2 = -1$

If m is positive,	If $m = 0$,
If m is negative,	If m is undefined, $m = \frac{n}{0}$



Recall :

Equations of a Line :

a) General Form: $Ax + By + C = 0$

b) Two-Point Form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

c) Point-Slope Form: $y - y_1 = m(x - x_1)$

d) Slope-Intercept Form: $y = mx + b$

e) Intercept Form: $\frac{x}{a} + \frac{y}{b} = 1$

Example 1: Find the equation of the tangent and normal line to the curve

$$y = x^3 - 2x^2 + 4 \text{ at point } (2, 4).$$

Solution:

$$y = x^3 - 2x^2 + 4$$

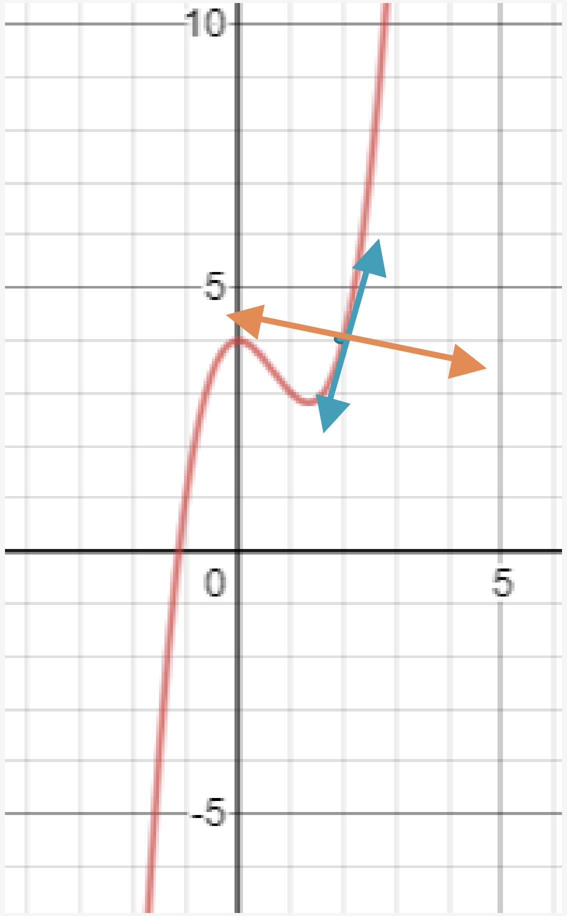
$$\boldsymbol{y' = 3x^2 - 4x}$$

at **point** $(\mathbf{2}, \mathbf{4})$ the slope of the TL is:

$$y' = f'(x) = m_T$$

$$m_T = f'(2)$$
$$= 3(2)^2 - 4(2)$$

$$\boldsymbol{m_T = 4} \qquad \boldsymbol{m_N = -\frac{1}{4}}$$



$$\mathbf{point}\ (\mathbf{2},\mathbf{4}), \boldsymbol{m_T} = \mathbf{4}, \boldsymbol{m_N} = -\frac{\mathbf{1}}{\mathbf{4}}$$

Using the Point-Slope Form, $y - y_1 = m(x - x_1)$

Equation of Tangent Line:

$$y - 4 = 4(x - 2)$$

$$y - 4 = 4x - 8$$

$$4x - y - 4 = 0$$

$$y = 4x - 4$$

Equation of Normal Line:

$$y - 4 = -\frac{1}{4}(x - 2)$$

$$4(y - 4) = -1(x - 2)$$

$$4y - 16 = -x + 2$$

$$x + 4y - 18 = 0$$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

Example 2: Find the equation of the tangent and normal line to the curve $x^2 + 3xy + y^2 = 5$ at $(1,1)$.

Solution:

$$x^2 + 3xy + y^2 = 5$$

$$2x + 3[x \cdot 1 \cdot y' + y(1)] + 2y \cdot y' = 0$$

$$2x + 3x \cdot y' + 3y + 2y \cdot y' = 0$$

$$3x \cdot y' + 2y \cdot y' = -2x - 3y$$

$$y'(3x + 2y) = -2x - 3y$$

$$\boldsymbol{y' = \frac{-2x - 3y}{3x + 2y}}$$

at $(\mathbf{1}, \mathbf{1})$, the slope of the tangent line is.

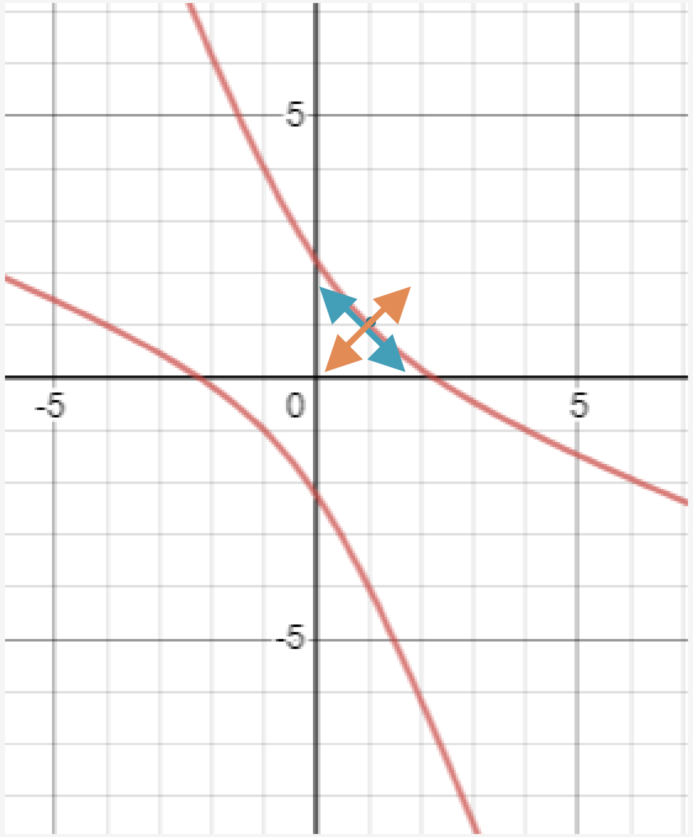
$$m_T = \frac{-2(1) - 3(1)}{3(1) + 2(1)}$$

$$m_T = \frac{-5}{5}$$

$$\boldsymbol{m_T = -1} \qquad \boldsymbol{m_N = 1}$$

$$\text{point } (\mathbf{1},\mathbf{1}), \boldsymbol{m_T} = -\mathbf{1}, \boldsymbol{m_N} = \mathbf{1}$$

$$x^2 + 3xy + y^2 = 5$$



Using the Point-Slope Form, $y - y_1 = m(x - x_1)$

Equation of Tangent Line:

$$y - 1 = -1(x - 1)$$

$$y - 1 = -x + 1$$

$$x + y - 2 = 0$$

$$y = -x + 2$$

Equation of Normal Line:

$$y - 1 = 1(x - 1)$$

$$y - 1 = x - 1$$

$$x - y = 0$$

$$x = y$$

Example 3: Find the equation of the normal line(s) of slope $\frac{1}{3}$ to the curve $y^2 = 2x^3$

Solution:

$$y^2 = 2x^3$$

$$2y \cdot y' = 6x^2$$

$$y' = \frac{6x^2}{2y}$$

$$y' = \frac{3x^2}{y} = m_T$$

$$m_N = -\frac{y}{3x^2}$$

$$\frac{1}{3} = -\frac{y}{3x^2}$$

$$3x^2 = -3y$$

$$\boldsymbol{y = -x^2}$$

To solve for point (x, y),

$$(-x^2)^2 = 2x^3$$

$$x^4 = 2x^3$$

$$\boldsymbol{x = 2}$$

If $x = 2$, find y:

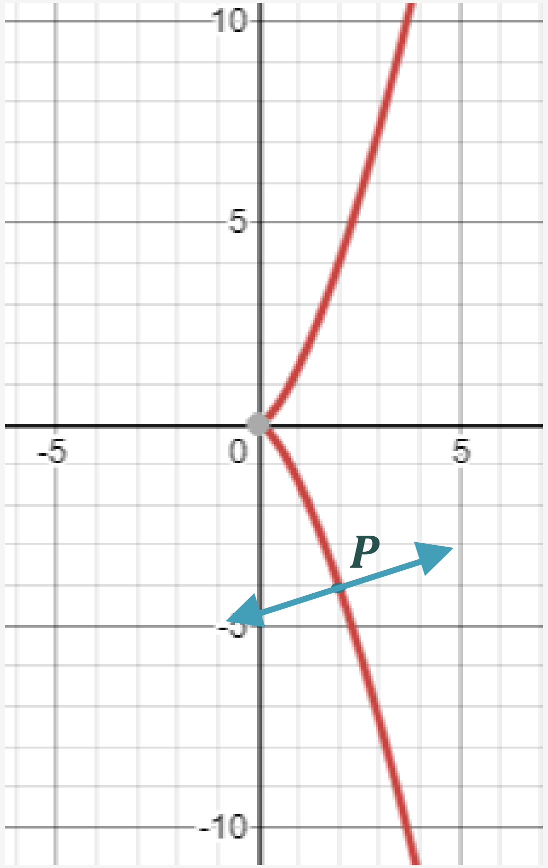
$$\boldsymbol{y = -x^2}$$

$$y = -(2)^2$$

$$\boldsymbol{y = -4}$$

$$\boldsymbol{P(2, -4)}$$

$\boldsymbol{P(2, -4)}, \ \boldsymbol{m_N = \frac{1}{3}}$ curve $y^2 = 2x^3$



Equation of the normal line at P,

$$y - (-4) = \frac{1}{3}(x - 2)$$

$$y + 4 = \frac{1}{3}(x - 2)$$

$$3(y + 4) = 1(x - 2)$$

$$3y + 12 = x - 2$$

$$x - 3y - 14 = 0$$

$$3y = x - 14$$

$$y = \frac{1}{3}x - \frac{14}{3}$$

Example 4: Find the equation of the tangent line with slope $m = -\frac{2}{9}$ to the ellipse $4x^2 + 9y^2 = 40$

Solution:

$$4x^2 + 9y^2 = 40$$

$$8x + 18y \cdot y' = 0$$

$$18y \cdot y' = -8x$$

$$y' = -\frac{8x}{18y}$$

$$y' = -\frac{4x}{9y} = m_T$$

$$-\frac{2}{9} = -\frac{4x}{9y}$$

$$-18y = -36x$$

$$2y = 4x$$

$$\boldsymbol{y = 2x}$$

$$4x^2 + 9(2x)^2 = 40$$

$$4x^2 + 36x^2 = 40$$

$$40x^2 = 40$$

$$x^2 = 1$$

$$x = \pm\sqrt{1}$$

$$\boldsymbol{x = \pm 1}$$

ellipse $4x^2 + 9y^2 = 40$

$x = \pm 1,\ \boldsymbol{y = 2x}$

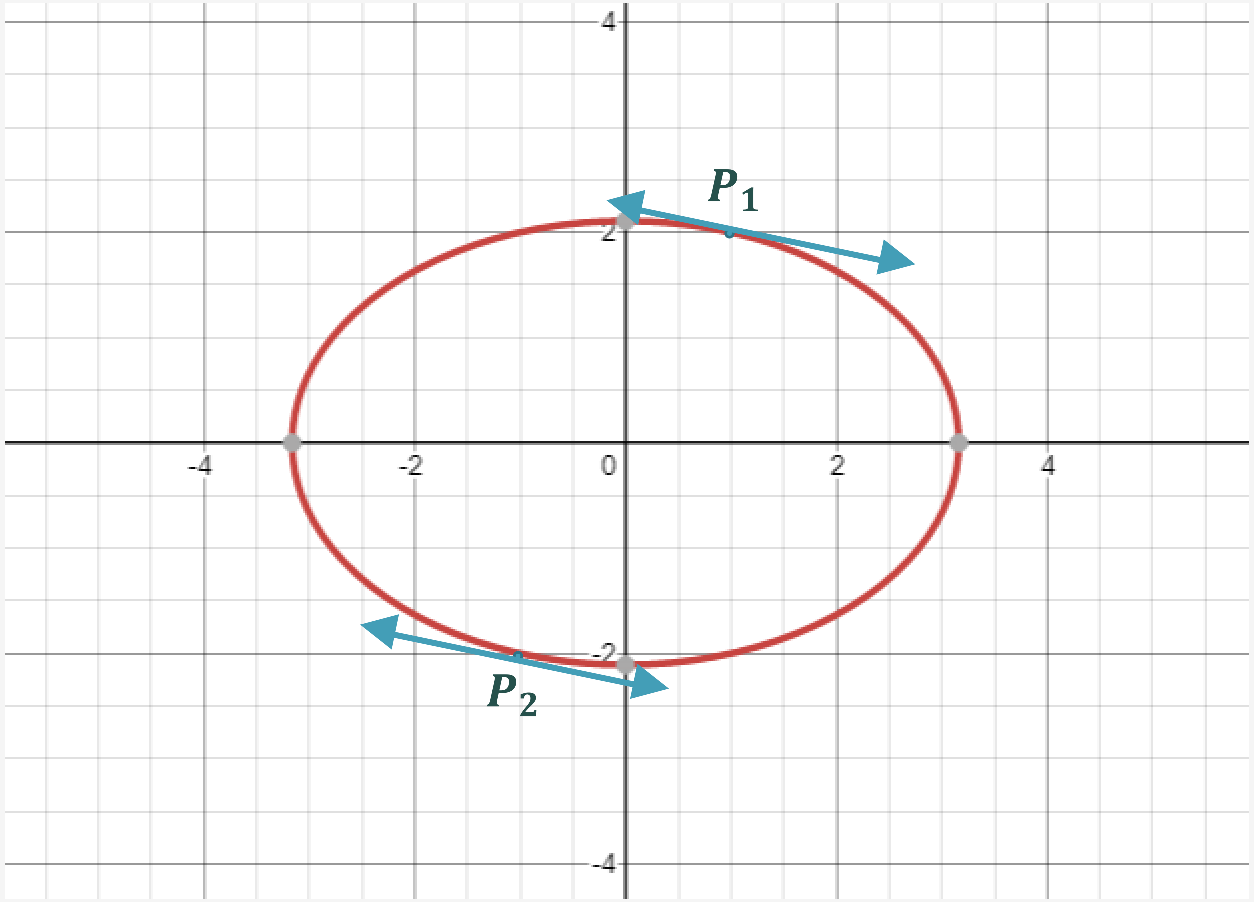
if $x = 1,\ \ y = 2(1)$

$y = 2$

if $x = -1,\ \ y = 2(-1)$

$y = -2$

$\boldsymbol{P_1(1, 2)}$**and** $\boldsymbol{P_2(-1, -2)}$



$$\boldsymbol{m_T = -\frac{2}{9}} \quad \boldsymbol{P_1(1,2)} \textbf{and } \boldsymbol{P_2(-1,-2)}$$

Equation of the tangent line at P_1,

$$y - 2 = -\frac{2}{9}(x - 1)$$

$$9(y - 2) = -2(x - 1)$$

$$9y - 18 = -2x + 2$$

$$2x + 9y - 20 = 0$$

$$9y = -2x + 20$$

$$y = -\frac{2}{9}x + \frac{20}{9}$$

Equation of the tangent line at P_2,

$$y - (-2) = -\frac{2}{9}[x - (-1)]$$

$$y + 2 = -\frac{2}{9}(x + 1)$$

$$9(y + 2) = -2(x + 1)$$

$$9y + 18 = -2x - 2$$

$$2x + 9y + 20 = 0$$

$$9y = -2x - 20$$

$$y = -\frac{2}{9}x - \frac{20}{9}$$

Example 5: Find the tangent to $x^2 + y^2 = 5$ and parallel to $2x - y = 4$.

Solution:

Note that the tangent line is parallel to $2x - y = 4$

$$m_T = m_L$$

$$m_L = ?$$

$$\boldsymbol{y = mx - b}$$

$$y = 2x - 4$$

$$m_L = 2 = m_T = y'$$

$$2x + 2y \cdot y' = 0$$

$$2y \cdot y' = -2x$$

$$y' = \frac{-2x}{2y}$$

$$y' = -\frac{x}{y}$$

$$2 = -\frac{x}{y}$$

$$\boldsymbol{x = -2y}$$

$$(-2y)^2 + y^2 = 5$$

$$4y^2 + y^2 = 5$$

$$5y^2 = 5$$

$$y^2 = 1$$

$$\boldsymbol{y = \pm 1}$$

$$if\ \boldsymbol{y = 1}, x = -2(1) = \boldsymbol{-2}$$

$$if\ \boldsymbol{y = -1}, x = -2(-1) = \boldsymbol{2}$$

$$\boldsymbol{P_1(-2, 1) \text{and } P_2(2, -1)}$$

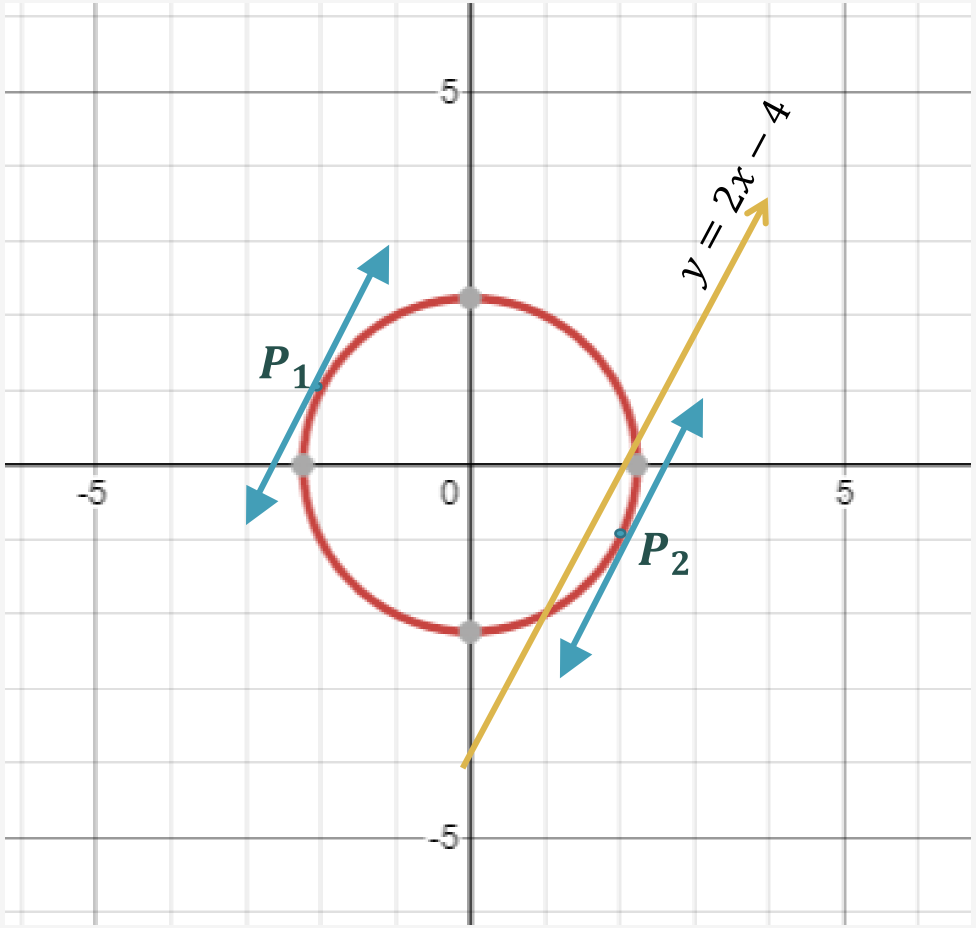
$x^2 + y^2 = 5$

tangent line is parallel to $2x - y = 4$

$$y = 2x - 4$$

$\boldsymbol{P_1(-2,1)}$ **and** $\boldsymbol{P_2(2,-1)}$

$m_L = 2 = m_T = y'$



$$\boldsymbol{P_1(-2,1)\text{and } P_2(2,-1), m_T = 2}$$

Equation of the Tangent Line at $P_1(-2,1)$:

$$y-(1)=2(x-(-2))$$

$$y-1=2(x+2)$$

$$y-1=2x+4$$

$$2x-y+5=0$$

$$y=2x+5$$

Equation of the Tangent Line at $P_1(2,-1)$:

$$y-(-1)=2[x-2]$$

$$y+1=2[x-2]$$

$$y+1=2x-4$$

$$2x-y-5=0$$

$$y=2x-5$$

Practice Task #7:

Find the equations fo the tangent and normal lines at the indicated point.

(1) $y = 3x^2 - 2x + 1 \; at \; (1,2)$

(2) $y = 2 + 4x - x^3 \; at \; x = -1$

(3) $y = 1 + 3\sqrt{x}$ at $(4,7)$

(4) $y = \dfrac{2}{x}$ at $(1,2)$

(5) $2xy + 5x - 3y = 0 \; at \; x = 4$

Home Work#7:

Solve the following problems:

(1) Find the equation(s) of the line(s) tangent to the curve $y = x^3 - 6x + 2$ and parallel to the line $y = 6x - 2$.

(2) Find the equation(s) of the line(s) normal to the curve $xy + 2x - y = 0$ and parallel to the line $2x + y = 0$.

(3) Find the equation(s)of the tangent line(s)to the curve $x^2 + 4y^2 = 8$ and parallel to the line $x + 2y = 6$.

(4) The line is tangent to the parabola $y^2 = 6x - 3$, and is perpendicular to the line $x + 3y = 7$. Determine the equation of the tangent line.

(5) Find the equation of the normal line if the tangent line to the ellipse $x^2 - xy + 2y^2 - 4x + 2y + 2 = 0$ is parallel to the line $x - 4y = 2$.