DERIVATIVES OF

Trigonometric Functions

Derivatives of Trigonometric Functions

1. $d(\sin u) = \cos u \, du$
2. $d(\cos u) = -\sin u \, du$
3. $d(\tan u) = sec^2 \, u \, du$
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Some Identities of Trigonometric Functions:

Pythagorean Identities:

$$sin^2\theta + cos^2\theta = 1$$
$$1 + tan^2\theta = sec^2\theta$$
$$1 + cot^2\theta = csc^2\theta$$

RatioIdentities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities:

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\tan\theta = \frac{1}{\cot\theta}$$

Double Angle Identities:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - tan^2\theta}$$
$$\cos 2\theta = cos^2\theta - sin^2\theta$$
$$\cos 2\theta = 2cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2sin^2\theta$$

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Example 1: Find the first derivative of $y = \sin 4x$

Solution:

$$y' = \cos 4x\, d(4x)$$

$$y' = \cos 4x\ (4)$$

$$\mathbf{y' = 4 \cos 4x}$$

Example 2: Find $\frac{dx}{dt}$ of the function $x = \cos t^2$

Solution:

$$x' = -\sin t^2\, d(t^2)$$

$$x' = -\sin t^2\ (2t)$$

$$\mathbf{x' = -2t \sin t^2}$$

Example 3: Given $y = \cot(3x^2 - 2)$, find its 1st derivative

Solution:

$$y' = -csc^2 (3x^2 - 2)\, d(3x^2 - 2)$$
$$y' = -csc^2 (3x^2 - 2)\ (6x)$$
$$\boldsymbol{y' = -6x\, csc^2 (3x^2 - 2)}$$

Example 4: Find the first derivative of $f(x) = \sec\sqrt{x}$

Solution:

$$f'(x) = \sec\sqrt{x}\tan\sqrt{x}\, d(\sqrt{x})$$
$$f'(x) = \sec\sqrt{x}\tan\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$$
$$\boldsymbol{f'(x) = \frac{1}{2\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}}$$
$$\boldsymbol{f'(x) = \frac{sec\sqrt{x}tan\sqrt{x}}{2\sqrt{x}}}$$

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Example 5: Given $y = \tan \frac{2}{x^3}$, find y'

Solution:

$$y' = sec^2 \frac{2}{x^3} d\left(\frac{2}{x^3}\right)$$

$$y' = sec^2 \frac{2}{x^3} \left(\frac{-6}{x^4}\right)$$

$$\boldsymbol{y' = \frac{-6}{x^4} sec^2 \frac{2}{x^3}}$$

Example 6: Find the first derivative of $y = \sin{(2x-1)^3}$

Solution:

$$y' = \cos{(2x-1)^3} d\,(2x-1)^3$$
$$y' = \cos{(2x-1)^3}\,[3(2x-1)^2(2)]$$
$$\boldsymbol{y' = 6(2x-1)^2 \cos{(2x-1)^3}}$$

Example 7: Given $f(y) = \csc \frac{y^2}{3y+2}$, find $f'(y)$

Solution:

$$f'(y) = -\csc \frac{y^2}{3y+2} \cot \frac{y^2}{3y+2}\, d\left(\frac{y^2}{3y+2}\right)$$

$$f'(y) = -\csc \frac{y^2}{3y+2} \cot \frac{y^2}{3y+2} \left[\frac{(3y+2)(2y)-y^2(3)}{(3y+2)^2}\right]$$

$$f'(y) = -\csc \frac{y^2}{3y+2} \cot \frac{y^2}{3y+2} \left[\frac{6y^2+4y-3y^2}{(3y+2)^2}\right]$$

$$\boldsymbol{f'(y) = -\frac{3y^2+4y}{(3y+2)^2} \csc \frac{y^2}{3y+2} \cot \frac{y^2}{3y+2}}$$

$$\boldsymbol{f'(y) = -\frac{y(3y+4)}{(3y+2)^2} \csc \frac{y^2}{3y+2} \cot \frac{y^2}{3y+2}}$$

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Example 8: Find the 1st derivative of $f(x) = 3\underbrace{\sec 2x}_{u}\,\underbrace{\tan 2x}_{v}$

Solution:

$$f'(x) = 3[\sec 2x\, d(\tan 2x) + \tan 2x\, d(\sec 2x)]$$

$$f'(x) = 3[\sec 2x\,(sec^2\, 2x\, d(2x)) + \tan 2x\,(\sec 2x \tan 2x\, d(2x))]$$

$$f'(x) = 3[\sec 2x\,(sec^2\, 2x\,(2)) + \tan 2x\,(\sec 2x \tan 2x\,(2))]$$

$$f'(x) = 3[2(sec^3\, 2x) + 2(\sec 2x \tan^2 2x)]$$

$$f'(x) = 6 \sec 2x\,[sec^2\, 2x + \tan^2 2x]$$

$$f'(x) = 6 \sec 2x\,[1 + tan^2\, 2x + \tan^2 2x]$$

$$\boldsymbol{f'(x) = 6 \sec 2x\left[1 + 2tan^2\, 2x\right]}$$

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Example 9: Find $\frac{dz}{dw}$ of the given function $z = \frac{2\csc w - 1}{\csc w + 2}$ $\Big\}\ \frac{u}{v}$

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4. $d(\cot u) = -csc^2\, u\, du$
5. $d(\sec u) = \sec u \tan u\, du$
6. $d(\csc u) = -\csc u \cot u\, du$

Solution:

$$z' = \frac{(\csc w + 2)d(2\csc w - 1) - (2\csc w - 1)d(\csc w + 2)}{(\csc w + 2)^2}$$

$$z' = \frac{(\csc w + 2)(-2\csc w \cot w\,(1) - 0) - (2\csc w - 1)(-\csc w \cot w\,(1) - 0)}{(\csc w + 2)^2}$$

$$z' = \frac{(\csc w + 2)(-2\csc w \cot w) - (2\csc w - 1)(-\csc w \cot w)}{(\csc w + 2)^2}$$

$$z' = \frac{-2\csc^2 w \cot w - 4\csc w \cot w + 2\csc^2 w \cot w - \csc w \cot w}{(\csc w + 2)^2}$$

$$z' = \frac{-5\csc w \cot w}{(\csc w + 2)^2}$$

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Example 10: Find y' of the function $y = \tan(x \sin x)$

Solution:

$$y' = sec^2(x \sin x) d(x \sin x)$$

$$y' = sec^2(x \sin x)[x\, d(\sin x) + \sin x\, d(x)]$$

$$y' = sec^2(x \sin x)[x(\cos x)(1) + \sin x\,(1)]$$

$$\boldsymbol{y' = [x \cos x + \sin x] sec^2(x \sin x)}$$

Example 11: Find y' of $y = \sqrt{\frac{1-\cos x^3}{1+\cos x^3}} = \left(\frac{1-\cos x^3}{1+\cos x^3}\right)^{1/2}$

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4. $d(\cot u) = -csc^2\, u\, du$
5. $d(\sec u) = \sec u \tan u\, du$
6. $d(\csc u) = -\csc u \cot u\, du$

Solution:

$$y' = \frac{1}{2}\left(\frac{1-\cos x^3}{1+\cos x^3}\right)^{-1/2} \left[\frac{(1+\cos x^3)d(1-\cos x^3)-(1-\cos x^3)d(1+\cos x^3)}{(1+\cos x^3)^2}\right]$$

$$y' = \frac{1}{2}\left(\frac{1+\cos x^3}{1-\cos x^3}\right)^{1/2} \left[\frac{(1+\cos x^3)\left(0-(-\sin x^3)(3x^2)\right)-(1-\cos x^3)\left(0-sin\, x^3(3x^2)\right)}{(1+\cos x^3)^2}\right]$$

$$y' = \frac{1}{2}\left(\frac{1+\cos x^3}{1-\cos x^3}\right)^{1/2} \left[\frac{3x^2 \sin x^3+3x^2 \sin x^3 \cos x^3+3x^2 \sin x^3-3x^2 \sin x^3 \cos x^3}{(1+\cos x^3)^2}\right]$$

$$y' = \frac{1}{2}\left(\frac{1+\cos x^3}{1-\cos x^3}\right)^{1/2} \left[\frac{6x^2 \sin x^3}{(1+\cos x^3)^2}\right]$$

$$\boldsymbol{y' = \frac{3x^2\, sin\, x^3}{(1-cos\, x^3)^{1/2}(1+cos\, x^3)^{3/2}}}$$

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Example 12: Let $y = tan^2(3x - 2)$, find y''

Solution:

$$y' = 2\tan(3x-2)\, d\tan(3x-2)$$

$$y' = 2\tan(3x-2)\sec^2(3x-2)d(3x-2)$$

$$y' = 2\tan(3x-2)\sec^2(3x-2)(3)$$

$$\boldsymbol{y' = 6\tan(3x-2)\sec^2(3x-2)}$$

$$y'' = 6\left[\tan(3x-2)d\left(\sec^2(3x-2)\right) + \sec^2(3x-2)d(\tan(3x-2))\right]$$

$$y'' = 6\begin{bmatrix}\tan(3x-2)(2)\sec(3x-2)\sec(3x-2)\tan(3x-2)(3) \\ +\sec^2(3x-2)sec^2(3x-2)(3)\end{bmatrix}$$

$$y'' = 6[6\tan^2(3x-2)\sec^2(3x-2) + 3\sec^4(3x-2)]$$

$$\boldsymbol{y'' = 18sec^2(3x-2)\left[2tan^2(3x-2) + sec^2(3x-2)\right]}$$

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Example 13: Find y' of the function $y^2 = \sin(x+y)$

Solution:

$$2y \cdot y' = \cos(x+y)\,[1 + 1.y']$$

$$2y \cdot y' = \cos(x+y) + \cos(x+y)y'$$

$$2y \cdot y' - \cos(x+y)y' = \cos(x+y)$$

$$y'[2y - \cos(x+y)] = \cos(x+y)$$

$$\boldsymbol{y' = \frac{\cos(x+y)}{2y - \cos(x+y)}}$$

Example 14: Let $\sin y + \cos x = 1$, find y''

Solution:

$$\cos y \cdot 1 \cdot y' + (-\sin x \cdot 1) = 0$$

$$\cos y \cdot y' - \sin x = 0$$

$$\cos y \cdot y' = \sin x$$

$$y' = \frac{\sin x}{\cos y}$$

$$y'' = \frac{(\cos y)d(\sin x) - (\sin x)d(\cos y)}{(\cos y)^2}$$

$$y'' = \frac{(\cos y)(\cos x)(1) - (\sin x)(-\sin y)(y')}{(\cos y)^2}$$

$$y'' = \frac{\cos x \cos y + \sin x \sin y \left(\frac{\sin x}{\cos y}\right)}{(\cos y)^2}$$

$$\boldsymbol{y'' = \frac{\cos x \cos y + \sin^2 x \tan y}{\cos^2 y}}$$

$$\boldsymbol{y'' = \frac{\cos x \cos^2 y + \sin^2 x \sin y}{\cos^3 y}}$$

HOME WORK #10:

Find the first derivative of the following functions and simplify the result whenever possible.

1. $y = 3x\cos\frac{x}{3} - 9sin\,\frac{x}{3}$

2. $y = 2\csc(1 - 3x)$

3. $y = \dfrac{\tan 2x}{1 - \cot 2x}$

4. $y = \dfrac{1 - \cos 4x}{\sin 4x}$

5. $f(x) = (tan^2x - x^2)^2$

6. $sec^2 2x + csc^2 2y = 4$

7. $\cos(xy) = x + y$

8. $x\cos x = \sin(x + y)$