Logarithmic Functions

DERIVATIVES

Laws of Exponents

i. $a^m a^n = a^{m+n}$

ii. $\frac{a^m}{a^n} = a^{m-n}$ $where\ a \neq 0$

iii. $(a^m)^n = a^{mn}$

iv. $(ab)^n = a^n b^n$

v. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $where\ b \neq 0$

Laws of Radicals

i. $\sqrt[n]{a^n} = a$

ii. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

iii. $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

iv. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

v. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$

Laws of Logarithms

i. $\log_b MN = \log_b M + \log_b N$

ii. $\log_b \frac{M}{N} = \log_b M - \log_b N$

iii. $\log_b N^p = p \log_b N$

iv. $\log_b b = 1$

v. $b^{\log_b N} = N$

Derivative of Logarithmic Functions

$$\text{i. } d(\log u) = \frac{M}{u}(d)u$$

$$\text{ii. } d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$\text{iii. } d(\ln u) = \frac{1}{u}(d)u$$

Where u is a differentiable function of x,

M is called the modulus of common logarithm

$M = 0.4343$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = \log_5(4x - 1)$

Solution: Let $u = 4x - 1$

$$y' = \frac{1}{4x - 1}\log_5 e \cdot d(4x - 1)$$

$$y' = \frac{1}{4x - 1}\log_5 e(4)$$

$$y' = \frac{4}{4x - 1}\log_5 e$$

$$\boldsymbol{y' = \frac{4\,log_5\,e}{4x - 1}}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = \ln\frac{x-2}{x+2}$

Solution: Let $u = \frac{x-2}{x+2}$

$$y' = \frac{1}{\frac{x-2}{x+2}}\,d\left(\frac{x-2}{x+2}\right)$$

$$y' = \frac{x+2}{x-2}\left[\frac{(x+2)(1)-(x-2)(1)}{(x+2)^2}\right]$$

$$y' = \frac{1}{x-2}\left[\frac{x+2-x+2}{x+2}\right]$$

$$y' = \frac{1}{x-2}\left[\frac{4}{x+2}\right]$$

$$\boldsymbol{y' = \frac{4}{(x-2)(x+2)} = \frac{4}{x^2-4}}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = \ln^2(\sin 3x)$

Solution: Let $u = \sin 3x$

$$y' = 2\ln(\sin 3x) \cdot \frac{1}{\sin 3x} d(\sin 3x)$$

$$y' = 2\ln(\sin 3x) \cdot \frac{1}{\sin 3x}(3\cos 3x)$$

$$y' = 2\ln(\sin 3x) \cdot \frac{3\cos 3x}{\sin 3x}$$

$$\boldsymbol{y' = 6\cot 3x \ln(\sin 3x)}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = ln\,(\sin 3x)^2$

Solution 1:

$$y' = \frac{1}{(\sin 3x)^2} \cdot 2(\sin 3x)(3\cos 3x)$$

$$y' = \frac{6\cos 3x}{\sin 3x}$$

$$\boldsymbol{y' = 6\cot 3x}$$

Solution 2:

$$y = \ln(\sin 3x)^2$$

$$y = 2\ln(\sin 3x)$$

$$y' = 2 \cdot \frac{1}{\sin 3x} \cdot 3\cos 3x$$

$$y' = 6 \cdot \frac{\cos 3x}{\sin 3x}$$

$$\boldsymbol{y' = 6\cot 3x}$$

Example:

Find y' if $y = x^2 \ln 3x$

Solution:

$$y' = x^2 \cdot \frac{1}{3x} \cdot 3 + \ln 3x \cdot 2x$$

$$y' = x + 2x \ln 3x$$

$$\boldsymbol{y' = x(1 + 2\ ln\, 3x)}$$

Example:

Find y' if $y = \ln(x^3 + 2)\,(x^2 + 3)$

$$y = \ln(x^3 + 2) + \ln(x^2 + 3)$$

Solution:

$$y' = \frac{1}{x^3 + 2} \cdot 3x^2 + \frac{1}{x^2 + 3} \cdot 2x$$

$$\boldsymbol{y' = \frac{3x^2}{x^3 + 2} + \frac{2x}{x^2 + 3}}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = \ln \frac{x^4}{(3x-4)^2}$

Solution 1:

$$y' = \frac{1}{\frac{x^4}{(3x-4)^2}} d\left[\frac{x^4}{(3x-4)^2}\right]$$

$$y' = \frac{1}{\frac{x^4}{(3x-4)^2}} \left[\frac{(3x-4)^2(4x^3) - x^4(2)(3x-4)(3)}{(3x-4)^4}\right]$$

$$y' = \frac{(3x-4)^2(2x^3)(3x-4)}{x^4} \left[\frac{(3x-4)(2) - 3x}{(3x-4)^4}\right]$$

$$y' = \frac{(3x-4)^3(2x^3)}{x^4} \left[\frac{6x-8-3x}{(3x-4)^4}\right]$$

$$y' = \frac{(3x-4)^3(2x^3)}{x^4} \left[\frac{3x-8}{(3x-4)^4}\right]$$

$$\boldsymbol{y' = \frac{2(3x-8)}{x(3x-4)}}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = \ln\frac{x^4}{(3x-4)^2}$

$$y = \ln x^4 - \ln(3x-4)^2$$

$$y = 4\ln x - 2\ln(3x-4)$$

Solution 2:

$$y' = 4\cdot\frac{1}{x}\cdot 1 - 2\cdot\frac{1}{3x-4}\cdot 3$$

$$y' = \frac{4}{x} - \frac{6}{3x-4}$$

$$y' = \frac{4(3x-4)-6x}{x(3x-4)}$$

$$y' = \frac{12x-16-6x}{x(3x-4)}$$

$$\boldsymbol{y' = \frac{6x-16}{x(3x-4)}}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y = \ln\left(x + \sqrt{1 + x^2}\right)$

Solution:

$$y' = \frac{1}{x + \sqrt{1 + x^2}}\,d\left(x + \sqrt{1 + x^2}\right)$$

$$y' = \frac{1}{x + \sqrt{1 + x^2}}\left[1 + \frac{1}{2}(1 + x^2)^{-\frac{1}{2}}(2x)\right]$$

$$y' = \frac{1}{x + \sqrt{1 + x^2}}\left[1 + \frac{2x}{2\sqrt{1 + x^2}}\right]$$

$$y' = \frac{1}{x + \sqrt{1 + x^2}}\left[1 + \frac{x}{\sqrt{1 + x^2}}\right]$$

$$y' = \frac{1}{x + \sqrt{1 + x^2}}\left[\frac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2}}\right]$$

$$\boldsymbol{y' = \frac{1}{\sqrt{1 + x^2}}}$$

Example: Find y' if $\log xy = 1 + \log(x+y)$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Solution:

$$\frac{M}{xy}(xy' + y \cdot 1) = \frac{M}{(x+y)}(1+y')$$

$$\frac{M}{y}y' + \frac{M}{x} = \frac{M}{(x+y)} + \frac{M}{(x+y)} \cdot y'$$

$$\frac{M}{y}y' - \frac{M}{(x+y)} \cdot y' = \frac{M}{(x+y)} - \frac{M}{x}$$

$$y'\left(\frac{M}{y} - \frac{M}{x+y}\right) = \frac{M}{(x+y)} - \frac{M}{x}$$

$$y'\left(\frac{M(x+y) - My}{y(x+y)}\right) = \frac{Mx - M(x+y)}{x(x+y)}$$

$$y' = \frac{y[Mx - M(x+y)]}{x[M(x+y) - My]}$$

$$y' = \frac{y[Mx - Mx - My]}{x[Mx + My - My]}$$

$$y' = \frac{y[-My]}{x[Mx]}$$

$$y' = \frac{-My^2}{Mx^2} = \frac{-y^2}{x^2}$$

$$d(\log u) = \frac{M}{u}(d)u$$

$$d(\log_b u) = \frac{1}{u}\log_b e\,(d)u$$

$$d(\ln u) = \frac{1}{u}(d)u$$

Example: Find y' if $y \ln x = 1 + \ln(\ln x) + \ln x$

Solution:

$$y \cdot \frac{1}{x} \cdot 1 + \ln x \cdot y' = \frac{1}{\ln x} \cdot \frac{1}{x} . 1 + \frac{1}{x} . 1$$

$$\frac{y}{x} + \ln x \cdot y' = \frac{1}{x \ln x} + \frac{1}{x}$$

$$\ln x \cdot y' = \frac{1}{x \ln x} + \frac{1}{x} - \frac{y}{x}$$

$$\ln x \cdot y' = \frac{1 + \ln x - y \ln x}{x \ln x}$$

$$\boldsymbol{y' = \frac{1 + \ln x - y \ln x}{x \ln^2 x}}$$

Example: Find y' of $y = (x^2+2)^3(1-x^3)^4$

Solution: Solve by **Logarithmic Differentiation**

$$\ln y = \ln(x^2+2)^3(1-x^3)^4$$

$$\ln y = \ln(x^2+2)^3 + \ln(1-x^3)^4$$

$$\ln y = 3\ln(x^2+2) + 4\ln(1-x^3)$$

$$\frac{1}{y}\cdot y' = 3\cdot\frac{1}{x^2+2}\cdot 2x + 4\cdot\frac{1}{1-x^3}\cdot -3x^2$$

$$\frac{1}{y}\cdot y' = \frac{6x}{x^2+2} - \frac{12x^2}{1-x^3}$$

$$y' = y\left[\frac{6x}{x^2+2} - \frac{12x^2}{1-x^3}\right]$$

$$\boldsymbol{y' = (x^2+2)^3(1-x^3)^4\left[\frac{6x}{x^2+2} - \frac{12x^2}{1-x^3}\right]}$$

Example: Find y' of $y = \frac{x\left(1-x^2\right)^2}{(1+x^2)^{1/2}}$

Solution: Solve by **Logarithmic Differentiation**

$$\ln y = \ln \frac{x(1-x^2)^2}{(1+x^2)^{1/2}}$$

$$\ln y = \ln x(1-x^2)^2 - \ln(1+x^2)^{1/2}$$

$$\ln y = \ln x + \ln(1-x^2)^2 - \ln(1+x^2)^{1/2}$$

$$\ln y = \ln x + 2\ln(1-x^2) - \frac{1}{2}\ln(1+x^2)$$

$$\frac{1}{y}\cdot y' = \frac{1}{x}\cdot 1 + 2\cdot\frac{1}{1-x^2}\cdot -2x - \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x$$

$$\frac{1}{y}\cdot y' = \frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}$$

$$y' = y\left[\frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}\right]$$

$$\boldsymbol{y' = \frac{x\left(1-x^2\right)^2}{(1+x^2)^{1/2}}\left[\frac{1}{x} - \frac{4x}{1-x^2} - \frac{x}{1+x^2}\right]}$$

Example: Find y' of $y = x^{\ln x}$

Solution: Solve by **Logarithmic Differentiation**

$$\ln y = \ln x^{\ln x}$$

$$\ln y = (\ln x) \ln x$$

$$\ln y = \ln^2 x$$

$$\frac{1}{y} \cdot y' = 2 \ln x \cdot \frac{1}{x} \cdot 1$$

$$\frac{1}{y} \cdot y' = \frac{2}{x} \ln x$$

$$y' = y \left[\frac{2 \ln x}{x}\right]$$

$$\boldsymbol{y' = x^{\ln x} \left[\frac{2 \ln x}{x}\right]}$$

Example: Find y' of $y = (\cos x)^{2x}$

Solution: Solve by **Logarithmic Differentiation**

$$\ln y = \ln(\cos x)^{2x}$$

$$\ln y = 2x \ \ln \cos x$$

$$\frac{1}{y} \cdot y' = 2x \cdot \frac{1}{\cos x} \cdot - \sin x + \ln \cos x \cdot 2$$

$$\frac{1}{y} \cdot y' = \frac{-2x \sin x}{\cos x} + 2 \ln \cos x$$

$$\frac{1}{y} \cdot y' = -2x \tan x + 2 \ln \cos x$$

$$y' = y[-2x \tan x + 2 \ln \cos x]$$

$$\boldsymbol{y' = \cos x^{2x} [-2x \tan x + 2 \ln \cos x]}$$

Home Work #12:

Find $\frac{dy}{dx}$ and simplify whenever possible.

1. $y = \log\sqrt{2x-8}$

2. $y = \ln\sqrt{\frac{x-1}{x+1}}$

3. $y = \log(4 + 3\sin 2x)$

4. $y = ln^4(x+3)$

5. $y = \sqrt[3]{\frac{x^2(x+1)}{(x-4)^3}}$

6. $y = \log\sqrt[3]{\sqrt{12x}}$

7. $y = x^{x^x}$

8. $y = \frac{1}{2}(x^2+4)\ln(x^2+4) - x^2$

9. $\sin y = \ln(x+y)$

10. $x\ln y + y\ln x = 1$