$$V' = \phi^{-1}(0) \cap (\mathcal{O} \times \mathbb{C}^s)$$

$$Z' = \{ (x,\theta) \in V' \ / \ \dim(\pi(T_{x\theta} V')) < s \}$$

Claim 1: $\overline{\pi(Z')} \subsetneq \mathbb{C}^s$.

Claim 2: For (x,θ) in V', $(x,\theta) \in Z' \iff \operatorname{rank}(M_{x,\theta}) < s+m$

$\iff \operatorname{rank}(J_{x,\theta}) < m$.

Claim 3: For $\theta \in \mathbb{C}^s - \overline{\pi(Z')}$, 0 is a reg. value of $\phi_\theta|_{\mathcal{O}}$.

Set $\mathcal{Z} := \{ (x,\theta,l) \in \mathbb{C}^{n+s+m} \ / \ \phi(x,\theta) = 0 \text{ and } l \cdot J_{x,\theta} = 0 \}$

and $\mathcal{Z}' := \overline{\mathcal{Z} \cap \mathcal{O} \times \mathbb{C}^s \times \mathbb{C}^m - \{x,\theta,0\cdots 0\}}$

1) take (x,θ) in Z'. Then $\begin{cases} \exists\, l \neq (0..0) \text{ st } l \cdot J_{x,\theta} = 0 \\ x \in \mathcal{O} \end{cases}$

so $(x,\theta) \in \pi_{x\theta}(\mathcal{Z}')$. $\Rightarrow$ $Z' \subset \pi_{x\theta}(\mathcal{Z}')$.

2) Let Y be an irred. comp. of $\mathcal{Z}'$. $\exists$ open dense $Y^\circ \subset Y$ st $\forall (x,\theta,l) \in Y^\circ$, $x \in \mathcal{O}$ and $l \neq (0..0)$.

$\rightarrow$ $\pi_{x,\theta}(Y^\circ) \subset Z'$ and so $\pi_{x\theta}(\bigcup_Y Y^\circ) \subset Z'$.

This gives $\overline{\pi_{x\theta}(\cup Y^\circ)} \subset \overline{Z'}$.

Since $\overline{\bigcup_Y Y^\circ} = \mathcal{Z}$, we get $\overline{\pi(\mathcal{Z})} = \overline{\pi(\overline{\bigcup_Y Y^\circ})} = \overline{\pi(\bigcup_Y Y^\circ)} \subset \overline{Z'}$.

So $\underset{(\text{by } 1)}{Z' \subset} \pi_{x\theta}(\mathcal{Z}') \subset \underset{(\text{by } 2)}{\overline{\pi_{x\theta}(\mathcal{Z}')} \subset \overline{Z'}}$. Call this W.

$$\Rightarrow \pi_\theta(Z') \subset \pi_\theta(W) \subset \pi_\theta(\overline{Z'}) \Rightarrow \overline{\pi_\theta(Z')} \subset \overline{\pi_\theta(W)} \subset \overline{\pi_\theta(\overline{Z'})} = \overline{\pi_\theta(Z')}$$

$$\Rightarrow \overline{\pi_\theta(Z')} = \overline{\pi_\theta(W)}.$$

Now, $\deg(Z_i') \le \deg(Z) \le$ (Bézout number)

$$\deg(W) = \deg \overline{\pi_{x,\theta}(Z_i')} \le \deg(Z_i')$$

$$\deg\left(\overline{\pi_\theta(Z')}\right) = \deg\left(\overline{\pi_\theta(W)}\right) \le \deg(W).$$