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1.

Consider the linear function $f(x) = ax + b$. Prove whether $f(x)$ is convex using the definition of convexity.

proof

Definition 1.2: Convex Function

$f : S \to \mathbb{R}$ is a convex function if $dom f = S$ is a convex set, and
$\forall u, v \in S, \forall \alpha \in [0, 1] : f(\alpha v + (1-\alpha)u) \le \alpha f(v) + (1-\alpha) f(u)$.
i.e. the line connecting any 2 points in the domain of the function is above the function's values.

$S = \mathbb{R} \Rightarrow$ is a convex set: closed under multiplication & addition

RHS evaluate $f(\alpha \cdot v + (1-\alpha)u)$: $= a\cdot(\alpha \cdot v + (1-\alpha)u) + b$

$= a \cdot \alpha v + u - \alpha u + b$

LHS: evaluate: $\alpha f(v) + (1-\alpha) f(u) = \alpha \cdot (a \cdot v + b) + (1-\alpha)\cdot(a u + b)$

$= \alpha \cdot a \cdot v + \cancel{\alpha b} + a \cdot u + b - \alpha \cdot a \cdot u - \cancel{\alpha b}$

$= \alpha \cdot a \cdot v - a \cdot u + b$

hence $RHS = LHS \Rightarrow f(\alpha \cdot v + (1-\alpha)u) \le \alpha \cdot f(v) + (1-\alpha) f(u)$

2.

Consider a quadratic function $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Investigate the convexity of $f(x)$ and determine under what conditions it is convex.

the proof for the domain is as above.

we can use:

Theorem: Second order characterization ($\mathbb{R} \to \mathbb{R}$)

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then f is convex iff I is a convex set and

$$\forall\, x \in I \quad f''(x) \ge 0$$

since f is a second degree polynomial
it is twice differentiable:

$f'(x) = 2ax +$

$f''(x) = 2a$

$2a \ge 0 \iff a \ge 0 \iff f$ is convex ■

3.

Consider the exponential function $f(x) = e^x$. Prove whether $f(x)$ is convex

f: is $\mathbb{R} \to \mathbb{R}$, hence its domain is positive.

by def $f(x) = f'(x) = f''(x) = e^x$

$\forall x \in \mathbb{R}$ $e^x \ge 0$

$\Rightarrow$ hence it is convex ■

4.

Let $f(x) = \max(x, c)$, where $c \in \mathbb{R}$ is a constant. Prove or disprove the convexity of $f(x)$

$F(X)$ is convex.

$F(x)$ is the pointwise supremum of:

$g(x) = x$, $h(x) = c$

$g(x) = x$ is convex as a case of g_1 with $b = 0$, $a = 1$.

$h(X)$. is similarly convex with $a = 0$ $b = c$.

hence $F(X)$ is convex.

5.

Consider the cosine function $f(x) = \cos(x)$. Prove or disprove the convexity of $f(x)$.

It is not convex:

$f(X)$ is twice differentiable.

$f''(X) = -\cos(X)$.

let $x = 0 \Rightarrow f''(0) = -1$.

Since

Theorem: Second order characterization ($\mathbb{R} \to \mathbb{R}$)

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then f is convex iff I is a convex set and

$$\forall\, x \in I \quad f''(x) \geq 0$$

$f(X)$ is not convex.

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