1 Part A: Calculus

1.1 Calculus-Q1

CQ1.1 Use the chain rule to calculate the gradient of

$$h(\boldsymbol{\sigma}) = \frac{1}{2}\|f(\boldsymbol{\sigma}) - y\|^3$$

where $\boldsymbol{\sigma} \in \mathbb{R}^m$ and f is some arbitrary function from $\mathbb{R}^m$ to $\mathbb{R}^n$.
CQ1.2 Compute the expression in the case where:

$$f(\boldsymbol{\sigma}) = \begin{bmatrix} \sigma_1 \cdot \sigma_2 \\ \sigma_1^2 + \sigma_2^2 \\ \sigma_1 \end{bmatrix}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$h(\sigma) = \frac{1}{2} \cdot \|r(\sigma)\|^3$

let:

$r(\sigma) = f(\sigma) - y, \quad g(u) = \frac{1}{2}\|u\|^3$

hence:

$h(\sigma) = g(r(\sigma))$

let us find gradiants:

$\|u\| = (u \cdot u^T)^{\frac{1}{2}}$

$g(u) = \frac{1}{2}\|u\|^3 = \frac{1}{2}(u \cdot u^T)^{\frac{3}{2}}$

$g'(u) = \frac{1}{2} \cdot \frac{3}{2} \cdot (u \cdot u^T)^{\frac{1}{2}} \cdot (u \cdot u^T)' = \frac{1}{2} \cdot \frac{3}{2} \cdot (u \cdot u^T)^{\frac{1}{2}} \cdot (u^2)'$

$= \frac{1}{2} \cdot \frac{3}{\cancel{2}} \cdot \|u\| \cdot \cancel{2}u = \frac{3}{2}\|u\| \cdot u =$

let $u = r(\sigma)$:

$g'(u) = \frac{3}{2}\|r(\sigma)\| r(\sigma)$

we evaluate $f'(\alpha)$ as the jacobian:

$$J_f(\sigma) = \begin{bmatrix} \frac{\partial f_1}{\partial \sigma_1} & \cdots & \frac{\partial f_1}{\partial \sigma_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial \sigma_1} & \cdots & \frac{\partial f_n}{\partial \sigma_m} \end{bmatrix} \in \mathbb{R}^{n \times m}.$$

$f'(\sigma) = \begin{pmatrix} 2 & 1 \\ 2 & 4 \\ 1 & 0 \end{pmatrix}$