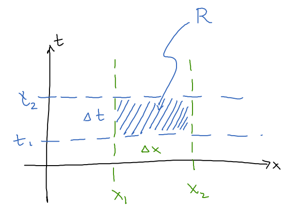
Given a conservation law

$$q_t + f(q)_x = 0$$

we consider the integral form defined over the rectangle $[x_1, x_2] \times [t_1, t_2]$



$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} q_t + f(q)_x \, dx \, dt = 0 \longrightarrow \int_{x_1}^{x_2} q \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} f(q)\Big|_{x_1}^{x_2} dt = 0$$

$$\int_{x_1}^{x_2} q(x, t+\Delta t) - q(x,t) \, dx \approx \int_{x_1}^{x_2} \cancel{q(x,t_1)} + \Delta t \, q_t(x,t_1) + \frac{\Delta t^2}{2} q_{tt}(x,t_1) + \dots - \cancel{q(x,t_1)} \, dx$$

$$\approx \Delta t \int_{x_1}^{x_2} q_t(x,t_1) + \frac{\Delta t}{2} q_{tt}(x,t_1) + \dots \, dx$$

$$\approx \Delta t \int_{x_1}^{x_2} f(q)_x \, dx + O(\Delta t)^2 = f(q(x_2,t_1)) - f(q(x_1,t_1))$$

$$\int_{t_1}^{t_2} f(q(x_1+\Delta x, t)) - f(q(x_1,t)) \, dt \approx \int_{t_1}^{t_2} f(q(x_1,t) + q_x(x_1,t)\Delta x + \dots) - f(q(x_1,t)) \, dt$$

$$\approx \int f(q(x_1,t)) + f'(q(x_1,t)) q_x(x_1,t)\Delta x - f(q(x_1,t)) \, dt$$

$$\approx \int_{t_1}^{t_2} f'(q(x_1,t)) q_x(x_1,t) \, \Delta x + O(\Delta x^2)$$

$$\approx \int_{t_1}^{t_2} \partial_x \big(f(q(x_1,t))\big) \Delta x \, dt + O(\Delta x^2) = \Delta x \, q \Big|_{t_1}^{t_2}$$

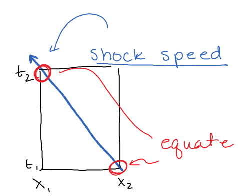
$$\approx \Delta x \; q(x_1,t_2) - q(x_1,t_1)$$

$\boxed{\text{I'm off by a sign somewhere}}$

So, we find $\Delta t \cdot f(q_r) - \Delta t \, f(q_l) + \Delta x \, q(x_1,t_2) - \Delta q(x_1,t_1) \approx 0$

$$\Delta t \left[f(q(x_2,t_1)) - f(q(x_1,t_1)) \right] + \Delta x \left[q(x_1,t_2) - q(x_1,t_1) \right] = 0$$

replace



$$\Delta t \left[f(q_r) - f(q_l) \right] + \Delta x \left[q_l - q_r \right] \approx 0$$

If the shock speed is "s" then $\Delta x = -s\Delta t$. We will replace Δx with the above and then simplify

$$\Delta t \left[f(q_r) - f(q_l) \right] - s\Delta t \left[q_l - q_r \right] \approx 0 \longrightarrow s(q_l - q_r) = \left[f(q_r) - f(q_l) \right] \quad (*)$$

↳ this is the Rankine - Hugoniot condition

So, we have $s \cdot [[q]] = -[[f(q)]]$. For a scalar conservation law, we have

$$s = \frac{f(q_r) - f(q_l)}{q_l - q_r}$$

Note: the correct condition

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}$$

according to the book.

In the limit as $q_l \to q_r$, $s \approx f'(q)$

Example

$$u_t + u u_x = 0 \longrightarrow (u)_t + (\tfrac{1}{2}u^2)_x = 0 \quad *$$

lagniappe

If we assume $u \to 0$ as $|x| \to \infty$ and integrate over all space, we have

$$\int_{-\infty}^{\infty} u_t + u u_x \, dx = 0 \Rightarrow \partial_t \int_{-\infty}^{\infty} u \, dx + \cancel{\tfrac{1}{2}u^2 \Big|_{-\infty}^{\infty}}^{\,0} = \partial_t \int_{-\infty}^{\infty} u \, dx = 0$$

So, $\partial_t \int_{-\infty}^{\infty} u \, dx = 0$ thus $\int_{-\infty}^{\infty} u \, dx$ is a conserved quantity. That is,

$$\int_{-\infty}^{\infty} u(x,0)\, dx = \int_{-\infty}^{\infty} u(x,t)\, dx \quad \forall\, t > 0.$$

looking at the integral form, we have

$$0 = \int_{x_1}^{x_2} \left(u_t + \tfrac{1}{2}(u^2)_x\right) dx = \partial_t \int_{x_1}^{x_2} u \, dx + \tfrac{1}{2} u^2 \Big|_{x_1}^{x_2} \longrightarrow \partial_t \int_{x_1}^{x_2} u \, dx + \tfrac{1}{2} u_r^2 - \tfrac{1}{2} u_\ell^2 = 0$$

$$\partial_t \int_{x_1}^{x_2} u \, dx = \tfrac{1}{2}\left(u_\ell^2 - u_r^2\right)$$

provided our solution has continuous derivatives this integral eqn, valid $\forall\, x_1, x_2$ is completely equiv to $*$

Assume that the location of the shock $x_s = s(t)$ is between x_1 and x_2. That is

$$x_1 < s(t) < x_2$$

So our integral becomes

$$\partial_t \left[\int_{x_1}^{s(t)} + \int_{s(t)}^{x_2}\right] u \, dx = \tfrac{1}{2}\left(u_\ell^2 - u_r^2\right)$$

Now, also assume that $\lim_{x \to s^\pm} u(x,t) = u^\pm$. Then the above becomes

$$\partial_t \left(u^- s - u^+ s\right) = \tfrac{1}{2}\left[(u^-)^2 - (u^+)^2\right] \rightarrow (u^- - u^+)\dot{s} = \tfrac{1}{2}(u^- - u^+)(u^- + u^+)$$

$$\dot{s} = \tfrac{1}{2}(u^- + u^+) \longrightarrow s = \tfrac{1}{2}(u^- + u^+)\, t$$

We can also find how the shock propagates