Theory for Constrained Gradient Optimization

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Suppose we have an *objective function*, $f : \mathbb{C}^n \to \mathbb{R}$, whose value we would like to decrease. The most naive and straightforward approach would be to first form a linear approximation to f at x_0,

$$f(x) \approx f(x_0) + \mathrm{Re}\{g(x_0)^*(x - x_0)\} \tag{1}$$

where $g : \mathbb{C}^{n\times 1}$ is the *gradient*,

$$g(x) = \nabla_x f(x). \tag{2}$$

The optimal point x_{opt} of the linearized function is

$$x = x_0 - ag(x_0) \tag{3}$$

where a is a positive, scalar step-size.

Of course, when we want to optimize the actual objective function $f(x)$, we will have to limit a since our linear approximation will break down as we leave the vicinity of x_0. Optimizing $f(x)$ by taking steps in the opposite direction of its gradient is commonly referred to as a *gradient-descent* algorithm.

2 Constrained gradient optimization

For many practical problems, we would like to put a constraint on the permissible values of x. A simple constraint may have the form

$$Ax = b. \tag{4}$$

In this case, we must modify the descent direction so that the constraint is satisfied. In all cases, we assume that x_0 already satisfies the constraint, and so we only need to ensure that our step $\Delta x = x - x_0$ satisfies it as well.

This is accomplished by ensuring that

$$A\Delta x = 0, \tag{5}$$

which is satisfied if we simply project $-g(x_0)$ onto the null-space of A. We do so in the following manner

$$\Delta x = -Pg(x_0), \tag{6}$$

where $P = I - A(A^*A)^{-1}A^*$ is the projector onto the null-space of A.

For constraints involving non-linear functions or inequalities there may not be a unique way to project Δx back into allowable range of values. A heuristic will then be required. Additionally, it may not be obvious *which* allowed value it should be projected to, although the one with the minimum distance, $\|\Delta x - \Delta x_{\text{allowed}}\|$ is an intuitive choice.