Notes on the Bi-Conjugate Gradient Method

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1 Without a preconditioner

Found in section 3.6.1 of [1]. Initialize as follows:

$$r = b - Ax, \hat{r} = r, \rho_0 = 1, \hat{p} = p = 0, k = 0. \tag{1}$$

Repeat the following until a termination condition such as $\|r\|_2 < \epsilon\|b\|_2$ is satisfied,

$$\begin{align}
k &= k + 1 \tag{2a}\\
\rho_k &= \hat{r}^T r, \beta = \rho_k / \rho_{k-1} \tag{2b}\\
p &= r + \beta p, \hat{p} = \hat{r} + \beta \hat{p} \tag{2c}\\
v &= Ap \tag{2d}\\
\alpha &= \rho_k / (\hat{p}^T v) \tag{2e}\\
x &= x + \alpha p \tag{2f}\\
r &= r - \alpha v, \hat{r} = \hat{r} - \alpha A^T \hat{p}. \tag{2g}
\end{align}$$

2 From Wikipedia

First, choose initial vectors x_0, x_0^*, and b^*. Initialize using

$$\begin{align}
r_0 &= b - Ax_0 \tag{3a}\\
r_0^* &= b^* - x_0^* A \tag{3b}\\
p_0 &= r_0 \tag{3c}\\
p_0^* &= r_0^*. \tag{3d}
\end{align}$$

Then loop over the following for $k = 0, 1, \ldots$

$$\alpha_k = r_k^* r_k / p_k^* A p_k \tag{4a}$$
$$x_{k+1} = x_k + \alpha_k p_k \tag{4b}$$
$$x_{k+1}^* = x_k^* + \overline{\alpha_k} p_k^* \tag{4c}$$
$$r_{k+1} = r_k - \alpha_k A p_k \tag{4d}$$
$$r_{k+1}^* = r_k^* - \overline{\alpha_k} p_k^* A \tag{4e}$$
$$\beta_k = r_{k+1}^* r_{k+1} / r_k^* r_k \tag{4f}$$
$$p_{k+1} = r_{k+1} + \beta_k p_k \tag{4g}$$
$$p_{k+1}^* = r_{k+1}^* + \overline{\beta_k} p_k^*. \tag{4h}$$

The residuals r_k and r_k^* satisfy

$$r_k = b - A x_k \tag{5a}$$
$$r_k^* = b^* - x_k^* A, \tag{5b}$$

where x_k and x_k^* are the approximate solutions to

$$Ax = b \tag{6a}$$
$$x^* A = b^*. \tag{6b}$$

References

[1] C. T. Kelley, "Iterative Methods for Linear and Nonlinear Equations", SIAM (1995).

[2] `http://en.wikipedia.org/wiki/Biconjugate_gradient_method`