Notes on the Bi-Conjugate Gradient Method

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Contents

1 The algorithm

The algorithm attempts to solve $Ax = b$, where the matrix A is indefinite. The algorithm accepts as inputs

1. a method to multiply a vector by A,
2. a method to multiply a vector by A^T (note that this is the transpose of A, *not* its conjugate-transpose), and
3. the vector b.

The algorithm begins by initializing the following variables,

$$r_0 = b - Ax_0 \tag{1a}$$
$$\hat{r}_0 = b - A^T\hat{x}_0 \tag{1b}$$
$$p_0 = r_0 \tag{1c}$$
$$\hat{p}_0 = \hat{r}_0. \tag{1d}$$

$$\tag{2a}$$

Then loop over the following for $k = 0, 1, \ldots$

$$\alpha_k = \hat{r}_k^T r_k / \hat{p}_k^T A p_k \tag{3a}$$
$$x_{k+1} = x_k + \alpha_k p_k \tag{3b}$$
$$\hat{x}_{k+1} = \hat{x}_k + \alpha_k \hat{p}_k \tag{3c}$$
$$r_{k+1} = r_k - \alpha_k A p_k \tag{3d}$$
$$\hat{r}_{k+1} = \hat{r}_k - \alpha_k A^T \hat{p}_k \tag{3e}$$
$$\beta_k = \hat{r}_{k+1}^T r_{k+1} / \hat{r}_k^T r_k \tag{3f}$$
$$p_{k+1} = r_{k+1} + \beta_k p_k \tag{3g}$$
$$\hat{p}_{k+1} = \hat{r}_{k+1} + \beta_k \hat{p}_k. \tag{3h}$$

A Reference algorithm: From C. T. Kelley

Found in section 3.6.1 of [1]. Initialize as follows:

$$r = b - Ax, \hat{r} = r, \rho_0 = 1, \hat{p} = p = 0, k = 0. \tag{4}$$

Repeat the following until a termination condition such as $\|r\|_2 < \epsilon \|b\|_2$ is satisfied,

$$k = k + 1 \tag{5a}$$
$$\rho_k = \hat{r}^T r, \beta = \rho_k / \rho_{k-1} \tag{5b}$$
$$p = r + \beta p, \hat{p} = \hat{r} + \beta \hat{p} \tag{5c}$$
$$v = Ap \tag{5d}$$
$$\alpha = \rho_k / (\hat{p}^T v) \tag{5e}$$
$$x = x + \alpha p \tag{5f}$$
$$r = r - \alpha v, \hat{r} = \hat{r} - \alpha A^T \hat{p}. \tag{5g}$$

B Reference algorithm: From Wikipedia

First, choose initial vectors x_0, x_0^*, and b^*. Initialize using

$$r_0 = b - Ax_0 \tag{6a}$$
$$r_0^* = b^* - x_0^* A \tag{6b}$$
$$p_0 = r_0 \tag{6c}$$
$$p_0^* = r_0^*. \tag{6d}$$

Then loop over the following for $k = 0, 1, \ldots$

$$\alpha_k = r_k^* r_k / p_k^* A p_k \tag{7a}$$

$$x_{k+1} = x_k + \alpha_k p_k \tag{7b}$$

$$x_{k+1}^* = x_k^* + \overline{\alpha_k} p_k^* \tag{7c}$$

$$r_{k+1} = r_k - \alpha_k A p_k \tag{7d}$$

$$r_{k+1}^* = r_k^* - \overline{\alpha_k} p_k^* A \tag{7e}$$

$$\beta_k = r_{k+1}^* r_{k+1} / r_k^* r_k \tag{7f}$$

$$p_{k+1} = r_{k+1} + \beta_k p_k \tag{7g}$$

$$p_{k+1}^* = r_{k+1}^* + \overline{\beta_k} p_k^*. \tag{7h}$$

The residuals r_k and r_k^* satisfy

$$r_k = b - A x_k \tag{8a}$$

$$r_k^* = b^* - x_k^* A, \tag{8b}$$

where x_k and x_k^* are the approximate solutions to

$$Ax = b \tag{9a}$$

$$x^* A = b^*. \tag{9b}$$

References

[1] C. T. Kelley, "Iterative Methods for Linear and Nonlinear Equations", SIAM (1995).

[2] `http://en.wikipedia.org/wiki/Biconjugate_gradient_method`