Documentation for Level Set Method Package

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1 What is a level set?

A level set is an *implicit* definition of a boundary. For example, in two dimensions,

$$\phi(x, y) = x^2 + y^2 - 1 \tag{1}$$

can *implicitly* define a circle of radius 1 as the boundary

$$\phi(x, y) = 0, \tag{2}$$

which is the 0-level set of ϕ.

We will use the convention of specifying boundaries using the 0-level set of ϕ. Also, we consider the interior of a boundary as $\phi < 0$ and its exterior as $\phi > 0$.

2 Basic mathematical properties

2.1 Derivative

We will use the notation $\phi_x = \delta\phi/\delta x$ to signify the partial derivative.

On a discretized grid, the derivative can be calculated either as

$$\phi_x^+(i,j) = \frac{\phi(i+1,j) - \phi(i,j)}{\Delta x}, \tag{3}$$

$$\phi_x^0(i,j) = \frac{\phi(i+1,j) - \phi(i-1,j)}{2\Delta x}, \tag{4}$$

or

$$\phi_x^-(i,j) = \frac{\phi(i,j) - \phi(i-1,j)}{\Delta x}, \tag{5}$$

the appropriate choice usually given by stability and accuracy considerations.

Similarily, the second derivative can be calculated as

$$\phi_{xx}(i,j) = \frac{\phi(i-1,j) - 2\phi(i,j) + \phi(i-1,j)}{\Delta x^2}. \tag{6}$$

2.2 Gradient

The gradient of ϕ is

$$\nabla\phi = (\phi_x, \phi_y, \phi_z), \tag{7}$$

where the appropriate partials are assumed.

Note that the outward (unit) normal of the boundary is then given by

$$\vec{N} = \frac{\nabla\phi}{|\nabla\phi|} \tag{8}$$

along the boundary.

2.3 Curvature

The curvature is defined as

$$\kappa = \nabla \cdot \vec{N} = \frac{\phi_x^2\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{xx}}{|\nabla\phi|^3}. \tag{9}$$

3 Signed distance function

A signed distance function ϕ not only defines a boundary on its 0-level set but has the additional property,

$$|\nabla\phi| = 1, \tag{10}$$

which allows movement of the boundary to be "well-defined".

To construct the signed distance function, find the stable solution of

$$\phi_t + S(\phi_0)(|\nabla\phi - 1) = 0 \tag{11}$$

where

$$S(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 - \Delta x^2}} \tag{12}$$

and ϕ_0 is the initial description of the interface. See chapter 7 of ref. [1] for more details. Note that this requires moving an interface relative to its normal direction, covered below.

4 Interface motion using a Hamilton-Jacobi formulation

In order to add time dynamics (motion) to the interface, we cast ϕ into the Hamilton-Jacobi equation

$$\phi_t + H(\nabla\phi) = 0. \tag{13}$$

4.1 Numerical discretization in time

We use the forward Euler time discretization, following Eq. 5.9 of ref. [1],

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + H^n(\nabla\phi) = 0. \tag{14}$$

4.2 Numerical discretization in space

We employ Godunov's scheme as outline in sections 5.3.3 and 6.2–6.4 in ref. [1] to choose the most appropriate spatial derivative of ϕ. The scheme is based on the following mechanism,

$$\hat{H} = \text{ext}_x \text{ext}_y H(\phi_x, \phi_y), \tag{15}$$

where

$$\text{ext}_x = \begin{cases} \min_{\phi_x \in I_x} H(\phi_x, \phi_y) & \text{if } \phi_x^- < \phi_x^+, \\ \max_{\phi_x \in I_x} H(\phi_x, \phi_y) & \text{if } \phi_x^- > \phi_x^+. \end{cases} \tag{16}$$

Here, I_x denotes the interval $[\phi_x^-, \phi_x^+]$.

5 Motion in an external velocity field

If $H(\nabla\phi) = \vec{V} \cdot \nabla\phi = u\phi_x + v\phi_y$, then Godunov's scheme chooses the derivatives in the following way,

$$\phi_x = \begin{cases} \phi_x^- & \text{if } u > 0, \\ \phi_x^+ & \text{if } u < 0. \end{cases} \tag{17}$$

Also, the time interval should be

$$\Delta t = \frac{\alpha}{\max\left\{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y}\right\}} \tag{18}$$

in order to obey the Courant-Friedreichs-Lewy (CFL) stability condition, with $\alpha < 1$. A good choice is apparently $\alpha = 0.9$.

6 Motion in the normal direction

If $H(\nabla\phi) = a|\nabla\phi|$, then Godunov's scheme is somewhat more complicated, but can be summarized as,

Choice of ϕ_x	sign of ϕ_x^-	sign of ϕ_x^+	Type
ϕ_x^-	positive	positive	upwinding
ϕ_x^+	negative	negative	upwinding
0	negative	positive	expansion
$\mathrm{argmax}_{\phi_x^\pm}\|a\phi_x^\pm\|$	positive	negative	shock

Alternatively, Eqs. 6.3 and 6.4 from ref. [1] can be used, which state

$$\phi_x^2 = \begin{cases} \max(\max(\phi_x^-, 0)^2, \min(\phi_x^+, 0)^2) & \text{if } a > 0, \\ \max(\min(\phi_x^-, 0)^2, \max(\phi_x^+, 0)^2) & \text{if } a < 0. \end{cases} \tag{19}$$

A stable time step for $\alpha < 1$ is

$$\Delta t = \frac{\alpha}{\max\left\{\frac{|H_x|}{\Delta x} + \frac{|H_y|}{\Delta y}\right\}}, \tag{20}$$

as expected from the CFL condition, where $H_x = a\phi_x^2/|\nabla\phi|$.

7 Motion involving mean curvature

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References

[1] Stanley Osher, Ronald Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces* (Springer 2003).