

# Documentation for Level Set Method Package

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## 1 What is a level set?

A level set is an *implicit* definition of a boundary. For example, in two dimensions,

$$\phi(x, y) = x^2 + y^2 - 1 \tag{1}$$

can *implicitly* define a circle of radius 1 as the boundary

$$\phi(x, y) = 0, \tag{2}$$

which is the 0-level set of  $\phi$ .

We will use the convention of specifying boundaries using the 0-level set of  $\phi$ . Also, we consider the interior of a boundary as  $\phi < 0$  and its exterior as  $\phi > 0$ .

## 2 Basic mathematical properties

### 2.1 Derivative

We will use the notation  $\phi_x = \delta\phi/\delta x$  to signify the partial derivative.

On a discretized grid, the derivative can be calculated either as

$$\phi_x^+(i, j) = \frac{\phi(i+1, j) - \phi(i, j)}{\Delta x}, \quad (3)$$

$$\phi_x^0(i, j) = \frac{\phi(i+1, j) - \phi(i-1, j)}{2\Delta x}, \quad (4)$$

or

$$\phi_x^-(i, j) = \frac{\phi(i, j) - \phi(i-1, j)}{\Delta x}, \quad (5)$$

the appropriate choice usually given by stability and accuracy considerations.

Similarly, the second derivative can be calculated as

$$\phi_{xx}(i, j) = \frac{\phi(i-1, j) - 2\phi(i, j) + \phi(i+1, j))}{\Delta x^2}. \quad (6)$$

### 2.2 Gradient

The gradient of  $\phi$  is

$$\nabla\phi = (\phi_x, \phi_y, \phi_z), \quad (7)$$

where the appropriate partials are assumed.

Note that the outward (unit) normal of the boundary is then given by

$$\vec{N} = \frac{\nabla\phi}{|\nabla\phi|} \quad (8)$$

along the boundary.

### 2.3 Curvature

The curvature is defined as

$$\kappa = \nabla \cdot \vec{N} = \frac{\phi_x^2\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{xx}}{|\nabla\phi|^3}. \quad (9)$$

## 3 Signed distance function

A signed distance function  $\phi$  not only defines a boundary on its 0-level set but has the additional property,

$$|\nabla\phi| = 1, \quad (10)$$

which allows movement of the boundary to be “well-defined”.

To construct the signed distance function, find the stable solution of

$$\phi_t + S(\phi_0)(|\nabla\phi - 1|) = 0 \quad (11)$$

where

$$S(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 - \Delta x^2}} \quad (12)$$

and  $\phi_0$  is the initial description of the interface. See chapter 7 of ref. [1] for more details. Note that this requires moving an interface relative to its normal direction, covered below.

## 4 Interface motion using a Hamilton-Jacobi formulation

In order to add time dynamics (motion) to the interface, we cast  $\phi$  into the Hamilton-Jacobi equation

$$\phi_t + H(\nabla\phi) = 0. \quad (13)$$

### 4.1 Numerical discretization in time

We use the forward Euler time discretization, following Eq. 5.9 of ref. [1],

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} + H^n(\nabla\phi) = 0. \quad (14)$$

### 4.2 Numerical discretization in space

We employ Godunov's scheme as outline in sections 5.3.3 and 6.2–6.4 in ref. [1] to choose the most appropriate spatial derivative of  $\phi$ . The scheme is based on the following mechanism,

$$\hat{H} = \text{ext}_x \text{ext}_y H(\phi_x, \phi_y), \quad (15)$$

where

$$\text{ext}_x = \begin{cases} \min_{\phi_x \in I_x} H(\phi_x, \phi_y) & \text{if } \phi_x^- < \phi_x^+, \\ \max_{\phi_x \in I_x} H(\phi_x, \phi_y) & \text{if } \phi_x^- > \phi_x^+. \end{cases} \quad (16)$$

Here,  $I_x$  denotes the interval  $[\phi_x^-, \phi_x^+]$ .

## 5 Motion in an external velocity field

If  $H(\nabla\phi) = \vec{V} \cdot \nabla\phi = u\phi_x + v\phi_y$ , then Godunov's scheme chooses the derivatives in the following way,

$$\phi_x = \begin{cases} \phi_x^- & \text{if } u > 0, \\ \phi_x^+ & \text{if } u < 0. \end{cases} \quad (17)$$

Also, the time interval should be

$$\Delta t = \frac{\alpha}{\max \left\{ \frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} \right\}} \quad (18)$$

in order to obey the Courant-Friedrichs-Lewy (CFL) stability condition, with  $\alpha < 1$ . A good choice is apparently  $\alpha = 0.9$ .

## 6 Motion in the normal direction

If  $H(\nabla\phi) = a|\nabla\phi|$ , then Godunov's scheme is somewhat more complicated, but can be summarized as,

| Choice of $\phi_x$                                 | sign of $\phi_x^-$ | sign of $\phi_x^+$ | Type      |
|--|--------------------|--------------------|-----------|
| $\phi_x^-$   | positive           | positive           | upwinding |
| $\phi_x^+$   | negative           | negative           | upwinding |
| 0  | negative           | positive           | expansion |
| $\operatorname{argmax}_{\phi_x^\pm}  a\phi_x^\pm $ | positive           | negative           | shock     |

Alternatively, Eqs. 6.3 and 6.4 from ref. [1] can be used, which state

$$\phi_x^2 = \begin{cases} \max(\max(\phi_x^-, 0)^2, \min(\phi_x^+, 0)^2) & \text{if } a > 0, \\ \max(\min(\phi_x^-, 0)^2, \max(\phi_x^+, 0)^2) & \text{if } a < 0. \end{cases} \quad (19)$$

A stable time step for  $\alpha < 1$  is

$$\Delta t = \frac{\alpha}{\max \left\{ \frac{|H_x|}{\Delta x} + \frac{|H_y|}{\Delta y} \right\}}, \quad (20)$$

as expected from the CFL condition, where  $H_x = a\phi_x/|\nabla\phi|$ .

## 7 Motion involving mean curvature

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## References

- [1] Stanley Osher, Ronald Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces* (Springer 2003).