Documentation for Level Set Method Package

Jesse Lu, jesselu@stanford.edu

May 9, 2011

Contents

1 What is a level set?

A level set is an *implicit* definition of a boundary. For example, in two dimensions,

$$\phi(x, y) = x^2 + y^2 - 1 \tag{1}$$

can *implicitly* define a circle of radius 1 as the boundary

$$\phi(x, y) = 0, \tag{2}$$

which is the 0-level set of ϕ.

We will use the convention of specifying boundaries using the 0-level set of ϕ. Also, we consider the interior of a boundary as $\phi < 0$ and its exterior as $\phi > 0$.

2 Basic mathematical properties

2.1 Derivative

We will use the notation $\phi_x = \delta\phi/\delta x$ to signify the partial derivative.

On a discretized grid, the derivative can be calculated either as

$$\phi_x^+(i, j) = \frac{\phi(i+1, j) - \phi(i, j)}{\Delta x}, \tag{3}$$

$$\phi_x^0(i,j) = \frac{\phi(i+1,j) - \phi(i-1,j)}{2\Delta x}, \tag{4}$$

or

$$\phi_x^-(i,j) = \frac{\phi(i,j) - \phi(i-1,j)}{\Delta x}, \tag{5}$$

the appropriate choice usually given by stability and accuracy considerations.

Similarily, the second derivative can be calculated as

$$\phi_{xx}(i,j) = \frac{\phi(i-1,j) - 2\phi(i,j) + \phi(i-1,j)}{\Delta x^2}. \tag{6}$$

2.2 Gradient

The gradient of ϕ is

$$\nabla\phi = (\phi_x, \phi_y, \phi_z), \tag{7}$$

where the appropriate partials are assumed.

Note that the outward (unit) normal of the boundary is then given by

$$\vec{N} = \frac{\nabla\phi}{|\nabla\phi|} \tag{8}$$

along the boundary.

2.3 Curvature

The curvature is defined as

$$\kappa = \nabla \cdot \vec{N} = \frac{\phi_x^2\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{xx}}{|\nabla\phi|^3}. \tag{9}$$

3 Signed distance function

A signed distance function ϕ not only defines a boundary on its 0-level set but has the additional property,

$$|\nabla\phi| = 1, \tag{10}$$

which allows movement of the boundary to be "well-defined".

To construct the signed distance function, find the stable solution of

$$\phi_t + S(\phi_0)(|\nabla\phi - 1) = 0 \tag{11}$$

where

$$S(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 - \Delta x^2}} \tag{12}$$

and ϕ_0 is the initial description of the interface. See chapter 7 of ref. [1] for more details. Note that this requires moving an interface relative to its normal direction, covered below.

References

[1] Stanley Osher, Ronald Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces* (Springer 2003).