Theory for `lset-opt` package

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1 Definitions

grid A two-dimensional $m \times n$ regularly-spaced set of points. The grid spacing is 1 in both x and y directions.

cell A box with length and height equal to 1, centered at a grid point.

ϕ Level-set function defined on grid. The zero-crossing of ϕ defines the contour of the shapes on the grid.

p Fractional-filling function. Values of p range from -1 to 1 depending on the relative volume of each material found in its cell.

2 Initialization

Given a candidate level-set function, $\hat{\phi}$, a regularized ϕ is computed in the following way:

1. If any element of $\hat{\phi}$, $\hat{\phi}_i$, is exactly equal to 0, then let $\hat{\phi}_i = \epsilon$ where ϵ is the smallest positive number available.

2. For all $\hat{\phi}_i$ not adjacent to a boundary point, fix the corresponding ϕ_i as

$$\phi_i = \text{sign}(\hat{\phi}_i) = \begin{cases} -1 & \text{if } \hat{\phi}_i < 0, \\ +1 & \text{if } \hat{\phi}_i > 0. \end{cases} \tag{1}$$

3. To determine the remaining ϕ_i, solve the following:

$$\begin{aligned} \text{minimize} \quad & \|D\phi\|^2 && (2) \\ \text{subject to} \quad & A\phi = 0, && (3) \end{aligned}$$

where $D = [D_x \quad D_y]$, D_x and D_y being the difference matrices in the horizontal and vertical directions respectively. Also, A is a matrix which fixes the boundary points by forcing the ratio of the two adjacent values of ϕ_i to remain the same.

3 Conversion to fractional-filling

The fractional-filling, p, of each cell is calculated in the following way,

$$p_0 = \text{sign}(\phi_0)(\gamma_1 + \gamma_2)(\gamma_3 + \gamma_4), \tag{4}$$

where the subscript 0 refers to the value at the current grid point, and the subscripts 1, 2, 3, and 4 refer to values to the left, right, up, and down of the current grid point.

The values of γ are determined by

$$\gamma_i = \begin{cases} \frac{\phi_0}{\phi_0 - \phi_i} & \text{if } |\phi_0| < |\phi_i| \text{ and } \text{sign}(\phi_0) \neq \text{sign}(\phi_i), \\ 0.5 & \text{otherwise.} \end{cases} \tag{5}$$

for $i = 1, 2, 3, 4$.

4 Topology update

The topology defined by ϕ can be dynamically updated by specifying a desired change in p, $\Delta\hat{p}$. A realizable change in p, Δp, is then computed by first finding the matrix $\partial p/\partial\phi$,

$$\frac{\partial p}{\partial \phi} = \frac{\partial p}{\partial \gamma}\frac{\partial \gamma}{\partial \phi}, \tag{6}$$

where

$$\frac{\partial p_0}{\partial \gamma_i} = \begin{cases} \gamma_3 + \gamma_4 & \text{for } i = 1, 2, \\ \gamma_1 + \gamma_2 & \text{for } i = 3, 4. \end{cases} \tag{7}$$

and for $i = 1, 2$

$$\frac{\partial \gamma_i}{\partial \phi_0} = \begin{cases} \frac{-\phi_i}{(\phi_0 - \phi_i)^2} & \text{if } |\phi_0| < |\phi_i| \text{ and } \text{sign}(\phi_0) \neq \text{sign}(\phi_i), \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

$$\frac{\partial \gamma_i}{\partial \phi_i} = \begin{cases} \frac{-\phi_0}{(\phi_0 - \phi_i)^2} & \text{if } |\phi_0| < |\phi_i| \text{ and } \text{sign}(\phi_0) \neq \text{sign}(\phi_i), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

and for $i = 3, 4$

$$\frac{\partial \gamma_i}{\partial \phi_0} = \begin{cases} \frac{-\phi_i}{(\phi_0 - \phi_i)^2} & \text{if } |\phi_0| \leq |\phi_i| \text{ and } \mathrm{sign}(\phi_0) \neq \mathrm{sign}(\phi_i), \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

$$\frac{\partial \gamma_i}{\partial \phi_i} = \begin{cases} \frac{-\phi_0}{(\phi_0 - \phi_i)^2} & \text{if } |\phi_0| \leq |\phi_i| \text{ and } \mathrm{sign}(\phi_0) \neq \mathrm{sign}(\phi_i), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Once we have $\partial p / \partial \phi$ then we solve the following for $\Delta \phi$,

$$\text{minimize} \quad \|\Delta \phi\|^2 \tag{12}$$

$$\text{subject to} \quad \Delta \hat{p} = \frac{\partial p}{\partial \phi} \Delta \phi. \tag{13}$$

$p + \Delta p$ is then given by converting $\phi + \Delta \phi$ to fractional-filling. In general, there will be a large discrepancy between $p + \Delta p$ and $p + \Delta \hat{p}$ because only certain fractional-fillings correspond to valid topologies. However, for

$$\lim_{\|\Delta \hat{p}\| \to 0} \|\Delta p - \Delta \hat{p}\| = 0 \tag{14}$$

if $\Delta \hat{p} = 0$ wherever $p = -1, +1$.