Notes on symmetrizing the FDFD matrix with stretched-coordinate PML.

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We want to find a diagonal matrix

$$D = \mathrm{diag}(d) \tag{1}$$

such that

$$(AD)^T = AD, \tag{2}$$

where $A = \nabla \times \nabla \times$ is the system matrix for the finite-difference frequency-domain problem for Maxwell's equations.

In fact, if we use an equally-spaced Cartesian grid (constant Δx, Δy, and Δz), A is already symmetric. However, when applying a stretched-coordinate perfectly-matched layer (SC-PML) this no longer holds true, and one requires a diagonal preconditioning matrix D in order to restore the symmetry of A. Note that we assume that both A and D are complex-valued, and that our goal is to make A symmetric, not Hermitian.

First we note that,

$$(AD)^T = DA^T \tag{3}$$

$$(AD)_{ij} = A_{ij} d_j \tag{4}$$

$$(DA^T)_{ij} = d_i (A^T)_{ij} = A_{ji} d_i, \tag{5}$$

and so state our problem as finding d such that

$$A_{ij} d_j = A_{ji} d_i, \quad \text{for all } i, j. \tag{6}$$

To do so, we observe that from figure 1 we can deduce that the $\nabla \times \nabla \times$ operation will result in

$$A_{ba} = -\frac{1}{\Delta x_b \Delta y_b} \tag{7}$$

$$A_{ab} = -\frac{1}{\Delta x_a \Delta y_a}, \tag{8}$$

from which we conclude that

$$d_i \propto \frac{1}{\Delta x_i \Delta y_i}. \tag{9}$$

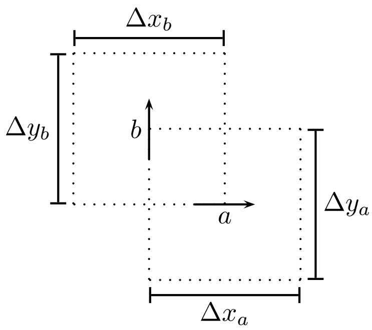

Figure 1: Schematic used to find the values of A_{ab} and A_{ba}, where a is a vector component in the x-direction and b is a vector component in the y-direction. A similar schematic can be used to determine the values for the A_{ac} and A_{ca}, where c is a vector component in the z-direction, by replacing $b \to c$ and $\Delta y_{a,b} \to \Delta z_{a,c}$.

Similarly, if we replace $b \to c$ and $\Delta y_{a,b} \to \Delta z_{a,c}$ in the figure, then we have

$$A_{ca} = -\frac{1}{\Delta x_b \Delta z_b} \tag{10}$$

$$A_{ac} = -\frac{1}{\Delta x_a \Delta z_a}, \tag{11}$$

which leads to

$$d_i \propto \frac{1}{\Delta x_i \Delta z_i}. \tag{12}$$

We then conclude that the choice of

$$d_i = \frac{1}{\Delta x_i \Delta y_i \Delta z_i} \tag{13}$$

will lead to symmetric AD. Note that none of the $A_{aa'}$ terms need to be included, where a and a' point in the same direction, since $A_{aa'} = A_{a'a}$ will hold both with or without the preconditioner.