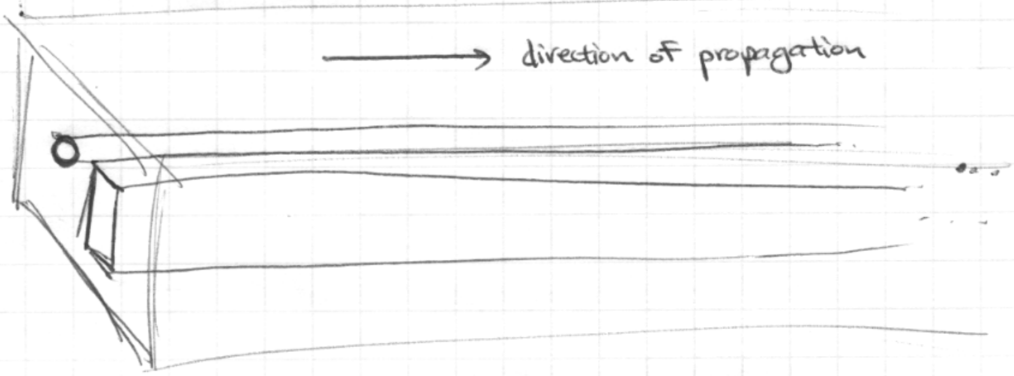
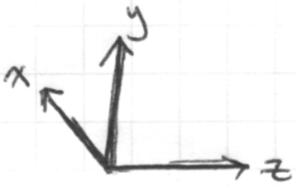




$\epsilon = \epsilon(x, y)$

$\mu = \mu_0$

Maxwell's equations:
(time-harmonic)

$$\nabla \times E = -i\omega\mu_0 H$$

$$\nabla \times H = i\omega\epsilon E$$

We seek solutions of the form $A_{x,y,z} = A_{x,y,z}(x,y)\, e^{i\omega t - i\beta z}$

and eventually we want an eigenvalue equation, where the eigenvalue is directly related to β.

The curl operator is:

$$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\partial_y A_z - \partial_z A_y\right) + \hat{y}\left(\partial_z A_x - \partial_x A_z\right) + \hat{z}\left(\partial_x A_y - \partial_y A_x\right)$$

and we now have a set of 6 coupled equations:

$$i\omega\epsilon_x E_x = \partial_y H_z + i\beta H_y \qquad -i\omega\mu_0 H_x = \partial_y E_z + i\beta E_y$$

$$i\omega\epsilon_y E_y = -i\beta H_x - \partial_x H_z \qquad -i\omega\mu_0 H_y = -i\beta E_x - \partial_x E_z$$

$$i\omega\epsilon_z E_z = \partial_x H_y - \partial_y H_x \qquad -i\omega\mu_0 H_z = \partial_x E_y - \partial_y E_x$$

We substitute for E_z and H_z.

$$i\omega\epsilon_x E_x = \frac{-1}{i\omega\mu_0}\partial_y\left(\partial_x E_y - \partial_y E_x\right) + i\beta H_y$$

$$i\omega\epsilon_y E_y = -i\beta H_x + \frac{1}{i\omega\mu_0}\partial_x\left(\partial_x E_y - \partial_y E_x\right)$$

$$-i\omega\mu_0 H_x = \partial_y \frac{1}{i\omega\epsilon_z}\left(\partial_x H_y - \partial_y H_x\right) + i\beta E_y$$

$$-i\omega\mu_0 H_y = -i\beta E_x - \partial_x \frac{1}{i\omega\epsilon_z}\left(\partial_x H_y - \partial_y H_x\right)$$

Then, we attempt to get E_x and E_y out of the picture, and be left with just H_x and H_y.

For H_x,

$$\beta H_x = \underbrace{-\omega\epsilon_y E_y}\ \underbrace{-\frac{1}{\omega\mu_0}\partial_x(\partial_x E_y - \partial_y E_x)}$$

$$-\frac{\omega\epsilon_y}{\beta}\left(-\omega\mu_0 H_x + \partial_y \frac{1}{\omega\epsilon_z}(\partial_x H_y - \partial_y H_x)\right)$$

$$\partial_x\partial_x E_y - \partial_x\partial_y E_x = -\frac{\omega\mu_0}{\beta}\partial_x\partial_x H_x + \cancel{\frac{1}{\omega\beta}\partial_x\partial_x\partial_y \frac{1}{\epsilon_z}(\partial_x H_y - \partial_y H_x)}$$

$$-\left(\frac{\omega\mu_0}{\beta}\partial_x\partial_y H_y + \cancel{\frac{1}{\omega\beta}\partial_x\partial_y\partial_x \frac{1}{\epsilon_z}(\partial_x H_y - \partial_y H_x)}\right)$$

$$= -\frac{\omega\mu_0}{\beta}(\partial_x\partial_x H_x + \partial_x\partial_y H_y)$$

Finally,

$$\beta^2 H_x = \omega^2\mu_0\epsilon_y H_x + \epsilon_y\partial_y\frac{1}{\epsilon_z}(\partial_y H_x - \partial_x H_y) + \partial_x\partial_x H_x + \partial_x\partial_y H_y$$

Likewise for H_y we get,

$$\beta^2 H_y = \omega^2\mu_0\epsilon_x H_y + \epsilon_x\partial_x\frac{1}{\epsilon_z}(\partial_x H_y - \partial_y H_x) + \partial_y\partial_y H_y + \partial_y\partial_x H_x$$

So now we have our eigenvalue equation

$$\begin{bmatrix} \omega^2\mu_0\epsilon_y + \epsilon_y\partial_y\frac{1}{\epsilon_z}\partial_y + \partial_x\partial_x & -\epsilon_y\partial_y\frac{1}{\epsilon_z}\partial_x + \partial_x\partial_y \\ -\epsilon_x\partial_x\frac{1}{\epsilon_z}\partial_y + \partial_y\partial_x & \omega^2\mu_0\epsilon_x + \epsilon_x\partial_x\frac{1}{\epsilon_z}\partial_x + \partial_y\partial_y \end{bmatrix} \begin{bmatrix} H_x \\ H_y \end{bmatrix} = \beta^2 \begin{bmatrix} H_x \\ H_y \end{bmatrix}$$

Now we can find the other field components via

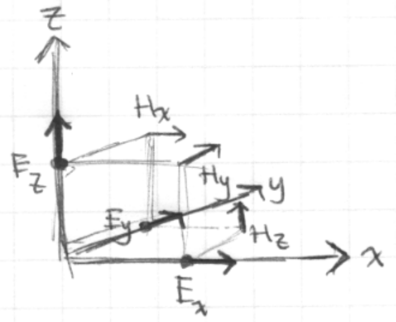
$$E_z = \frac{-i}{\omega\epsilon_z}\left(\partial_x H_y - \partial_y H_x\right)$$

$$E_x = \frac{\omega\mu_0}{\beta} H_y + \frac{i}{\beta}\partial_x E_z$$

$$E_y = \frac{-\omega\mu_0}{\beta} H_x + \frac{i}{\beta}\partial_y E_z$$

$$H_z = \frac{i}{\omega\mu_0}\left(\partial_x E_y - \partial_y E_x\right)$$

Numerically, we implement this on the Yee grid



which means we have to be careful which derivatives we use. ~~For~~

The following picture is helpful