

# Notes on symmetrizing the FDFD matrix with stretched-coordinate PML.

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We want to find a diagonal matrix

$$D = \text{diag}(d) \tag{1}$$

such that

$$(DA)^T = DA, \tag{2}$$

where  $A = \nabla \times \nabla \times$  is the system matrix for the finite-difference frequency-domain problem for Maxwell's equations.

In fact, if we use an equally-spaced Cartesian grid (constant  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$ ),  $A$  is already symmetric. However, when applying a stretched-coordinate perfectly-matched layer (SC-PML) this no longer holds true, and one requires a diagonal preconditioning matrix  $D$  in order to restore the symmetry of  $A$ . Note that we assume that both  $A$  and  $D$  are complex-valued, and that our goal is to make  $A$  symmetric, not Hermitian.

First we note that,

$$(DA)^T = A^T D \tag{3}$$

$$(DA)_{ij} = A_{ij} d_i \tag{4}$$

$$(A^T D)_{ij} = (A^T)_{ij} d_j = A_{ji} d_j, \tag{5}$$

and so state our problem as finding  $d$  such that

$$A_{ij} d_i = A_{ji} d_j, \quad \text{for all } i, j. \tag{6}$$

To do so, we observe that from figure 1 we can deduce that the  $\nabla \times \nabla \times$  operation will result in

$$A_{ba} = -\frac{1}{\Delta x_b \Delta y_b} \tag{7}$$

$$A_{ab} = -\frac{1}{\Delta x_a \Delta y_a}, \tag{8}$$

from which we conclude that

$$d_i \propto \Delta x_i \Delta y_i. \tag{9}$$

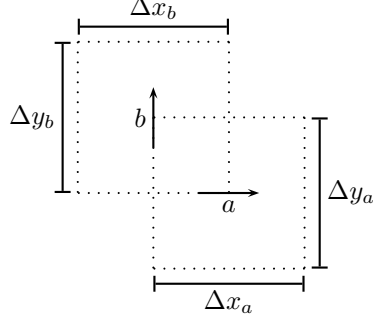


Figure 1: Schematic used to find the values of  $A_{ab}$  and  $A_{ba}$ , where  $a$  is a vector component in the  $x$ -direction and  $b$  is a vector component in the  $y$ -direction. A similar schematic can be used to determine the values for the  $A_{ac}$  and  $A_{ca}$ , where  $c$  is a vector component in the  $z$ -direction, by replacing  $b \rightarrow c$  and  $\Delta y_{a,b} \rightarrow \Delta z_{a,c}$ .

Similarly, if we replace  $b \rightarrow c$  and  $\Delta y_{a,b} \rightarrow \Delta z_{a,c}$  in the figure, then we have

$$A_{ca} = -\frac{1}{\Delta x_b \Delta z_b} \quad (10)$$

$$A_{ac} = -\frac{1}{\Delta x_a \Delta z_a}, \quad (11)$$

which leads to

$$d_i \propto \Delta x_i \Delta z_i. \quad (12)$$

We then conclude that the choice of

$$d_i = \Delta x_i \Delta y_i \Delta z_i \quad (13)$$

will lead to symmetric  $DA$ . Note that none of the  $A_{aa'}$  terms need to be included, where  $a$  and  $a'$  point in the same direction, since  $A_{aa'} = A_{a'a}$  will hold both with or without the preconditioner.