Topological Spaces and Continuous Functions

Topological Space

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Definition

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$; if $\mathcal{T}'$ *properly* contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$. We also say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$, or *strictly coarser*, in these two respective situations. We say that $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Basis for a Topology

If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If X belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Lemma 13.1

Let X be a set; let $\mathcal{B}$ be a basis for topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the set of all unions of elements of $\mathcal{B}$.

Lemma 13.2

Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Definition

If $\mathcal{B}$ the collection of all open intervals in the real line,

$$(a, b) = \{x | a < x < b\}$$

the topology generated by $\mathcal{B}$ is called the *standard topology* on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise. If $\mathcal{B}'$ is the collection of all half-open intervals of the form

$$[a, b) = \{x | a \leq x < b\}$$

, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$ denoted $\mathbb{R}_l$. Let K denote all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ will be called the **K-topology** on $\mathbb{R}$ denoted by $\mathbb{R}_K$.