Studies in Computational Intelligence 863

Oscar Castillo
Dipak Kumar Jana
Debasis Giri
Arif Ahmed *Editors*

Recent Advances in Intelligent Information Systems and Applied Mathematics

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Recent Advances in Intelligent Information Systems and Applied Mathematics

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Preface

The 2nd International Conference on Information Technology and Applied Mathematics (ICITAM 2019) was held at the Haldia Institute of Technology, Haldia, from March 07 to March 09, 2019. Haldia is a city and a municipality in Purba Medinipur in the Indian state of West Bengal, and Haldia Institute of Technology is a premier institution training engineers and computer scientists for the past several years. It has gained its reputation through its institutional dedication to teaching and research.

In response to the call for papers for ICITAM 2019, 157 papers were submitted for presentation and inclusion in the proceedings of the conference. The papers were evaluated and ranked on the basis of their significance, novelty, and technical quality by at least two reviewers per paper. After a careful blind peer review, 70 papers were selected for inclusion in the conference proceedings. The papers cover current research in intelligent systems, soft computing, machine learning, natural language processing, image and video processing, computer network and security, cryptography, data hiding, rough set, fuzzy logic, operations research, optimization, uncertain theory, applications, etc. At ICITAM 2019, eminent personalities both from India and from all over the world (China, Mexico, USA, South Korea, Singapore, to just mention a few) participated, who deliver invited talks.

The speakers from India were recognized leaders in government, industry, and academic institutions like the Indian Statistical Institute (ISA) Kolkata, Indian Institute of Technology (IIT) Kharagpur, Indian Institute of Science Education and Research (IISER) Mohali, Indian Institute of Technology (IIT) Bombay, Vidyasagar University, etc. All of them are involved in research dealing with the current issues of interest related to the theme of the conference. The conference program included two keynote talks by Professor Oscar Castillo (Tijuana Institute of Technology, Mexico) and Professor Baoding Liu (Tsinghua University, Beijing, China) and ten invited talks.

A conference of this kind would not be possible to organize without the full support from different people across different committees. All logistics and general organizational aspects were looked after by the Organizing Committee members who spent their time and energy in making the conference a reality. We also thank all the Technical Program Committee members and external reviewers for thoroughly reviewing the papers submitted for the conference and sending their constructive suggestions within the deadlines. Our hearty thanks go to Springer for agreeing to publish the proceedings in its Advances in Intelligent System and Computing series.

We are indebted to Haldia Institute of Technology, Haldia, India, for sponsoring the event. Their support has significantly helped in raising the profile of the conference.

Last but not least, our sincere thanks go to all authors who submitted papers to ICITAM 2019 and to all speakers and participants. We sincerely hope that the readers will find the proceedings stimulating and inspiring.

Oscar Castillo
Dipak Kumar Jana
Debasis Giri
Arif Ahmed

Message from the General Chair

As we all are aware, mathematics has always been a discipline of interest not only to theoreticians but also to all practitioners irrespective of their specific profession. Be it science, technology, economics, commerce, or even sociology, new mathematical principles and models have been emerging and helping in new research and in drawing inferences from practical data as well as through logic. Past few decades have seen enormous growth in applications of mathematics in different areas which are multidisciplinary in nature. Artificial intelligence, fuzzy, and machine learning applications are such areas which have got more focus recently due to need of various emerging fields of applications. With emerging computing facilities and speeds, a phenomenal growth has happened in problem-solving area. Earlier, some observations were made and conjectures were drawn which remained conjectures till somebody either could prove it theoretically or found counter-examples. But today, we can write algorithms and use computers for long calculations, verifications, or generation of huge amount of data. With available computing capabilities, we can find factors of very large integers of the size of hundreds of digits; we can find inverses of very large-size matrices, solve a large set of linear equations, and so on. Thus, mathematics and computations have become more integrated areas of research these days, and it was thought to organize an event where thoughts may be shared by researchers and new challenging problems could be deliberated for solving these.

As conferences, seminars, and workshops are the mechanisms to share knowledge and new research results giving us a chance to get new innovative ideas for futuristic needs as threats and computational capabilities of adversaries are ever increasing, it was thought appropriate to organize the present conference focused on mathematics and computations covering theoretical as well as practical aspects of research.

Eminent personalities working in the mathematical sciences and related areas have been invited from abroad as well as from within the country, who would deliver invited talks and tutorials for participants. The talks by these speakers intend to cover a wide spectrum, viz. intelligent computing, uncertainty, fuzzy logic, etc. The conference is spread over three days (from March 07 to March 09, 2019). The main conference is planned with special talks by experts and paper presentations in each session.

I hope that the conference would meet the aspirations of the participants and meet its objective of ideas and current research being shared and new targets/problems identified; more so the young researchers and students would get new directions to pursue their future research.

Baoding Liu

Message from the Program Chairs

It is a great pleasure for us to organize the 2nd International Conference on Information Technology and Applied Mathematics (ICITAM 2019) held from March 07 to March 09, 2019, at the Haldia Institute of Technology, Purba Medinipur, West Bengal, India. Our main goal is to provide an opportunity to the participants to learn about contemporary research in applied mathematics and computing, and exchange ideas among themselves and with experts present in the conference as the plenary as well as invited speakers. With this aim in mind, we carefully selected the invited speakers. It is our sincere hope that the conference will help the participants in their research and training and open new avenues for work for those who are either starting their research or are looking for extending their area of research to a different area of current research in applied mathematics and computing.

The conference program included two keynote talks by Professor Oscar Castillo (Tijuana Institute of Technology, Mexico) and Professor Baoding Liu (Tsinghua University, Beijing, China) and ten invited talks that enriched the technical level of the international event.

After an initial call for papers, 157 papers were submitted for presentation at the conference. All submitted papers were sent to external referees, and after refereeing, 70 papers were recommended for publication for the conference proceedings that will be published by Springer in its Advances in Intelligent System and Computing series.

We are grateful to the speakers, participants, referees, organizers, sponsors, and funding agencies for their support and help without which it would have been impossible to organize the conference, the workshops, and the tutorials. We owe our gratitude to the volunteers who work behind the scene tirelessly in taking care of the details in making this conference a success.

Oscar Castillo
Debasis Giri
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Fractional Order Generalized EOQ Model with Demand-Dependent Unit Purchasing Cost Under Space Constraints

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Abstract. In the present article, we introduce a fractional order generalized EOQ model with limited storage capacity where demand is inversely related to unit production cost. Here fractional calculus has been utilized to develop our traditional classical EOQ model to a generalized EOQ model. Here the fractional derivative has been assumed in terms of Caputo-fractional derivative sense and the fractional differential equation has been solved using Laplace transform method. Geometric programming techniques have been applied to get the optimum result of fractional order EOQ model. A numerical example is presented to illustrate the model.

Keywords: Fractional differentiation · Fractional differential equation · Set up cost · Holding cost · Economic order quantity · Geometric programming

Mathematics Subject Classification 2010: 90B05 · 34A08 · 26A33

1 Introduction

Fractional calculus generalizes derivative and integration of a function to non-integer order. This generalization is a rather old problem, as demonstrated by a correspondence, which lasted several months in 1695, between Leibniz and L'Hopital. Many other famous scientists of the past studied and contributed to the development of fractional calculus in the field of pure mathematics [17–20]. The mathematics involving fractional order derivatives or integrals are appeared very different from that of integer order calculus. Initially there were almost no practical application of this field and due to this, it was considered that fractional calculus as an abstract area containing only rigorous mathematical manipulations. So for past three centuries this subject was with only mathematicians. But in last few decades, fractional calculus attracted a huge number of physicists and mathematicians [11, 21, 24]. Fractional differential and integral equations play an important role in the modeling of real problems in scientific fields and engineering.

In recent years the concept of fractional differential calculus has been applied to several fields of engineering, science and economics [8–10]. Some of the areas where Fractional Calculus has made an important role that are included viscoelasticity and

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rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, electromagnetic theory, mechanics, fluid mechanics, signal and image processing theory, particle physics, control theory [8] and many other field [10, 14].

Only recently, fractional calculus was applied to classical EOQ model to generalize this model in inventory model. In previous papers [4–7], Das and Roy developed some generalized inventory model based on generalized EOQ and generalized EPQ model using the concept of fractional calculus. Two major assumptions in the classical EOQ model are that the demand rate is constant and deterministic and that the unit production cost is constant and independent of demand [1, 23, 26, 28]. However, in the real market, the demand for any product cannot be constant for all situations [13]. Researchers have paid much attention to inventory modeling with time dependent demand. Silver and Meal [27] developed a heuristic approach to determine EOQ in the general case of a deterministic time-varying demand pattern. Donaldson [12] discussed the classical no-shortage inventory policy for the case of a linear, time dependent demand. It has also been seen that in a realistic situation, the demand rate and unit production cost may be related to each other. Such a relationship exist when demand is high, a company can justify the use of more efficient production process to produce item at lower production unit cost. Thus it is justified to assume that the demand rate and unit production cost are inversely related to each other. In [3] (1989) Cheng try to developed an EOQ model with demand-dependent unit production cost. In industry total expenditure for production and storage (space) area are normally limited and imprecise. Our objective in this paper is to develop a generalized EOQ model under a space constraint (with limited storage capacity) where demand is inversely related to the unit purchasing cost using the concept of fraction differential calculus.

Here the fractional differentiations have been assumed in terms of Caputo-fractional-derivative sense. Fractional derivatives and fractional integrals have some interesting mathematical properties that may be utilized to develop our motivation.

This paper is organized as follows: In Sect. 2, we represent a basic conception on Fractional Calculus and short history, description related to Fractional Differential Calculus. In Sect. 3, we represent the basic concept of Classical EOQ model. In Sect. 4, we introduce our main work which emphasizes on techniques and procedure for finding our optimum results. A numerical example and the associated table has been presented in Sect. 5. Finally, In Sect. 6, we present the conclusion of our work.

2 A Short Description on Fractional Differential Calculus

The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. S.F Lacroix was the first to mention in some two pages a derivative of arbitrary order in a 700 pages text book of 1819.

He developed the formula for the nth derivative of y $= x^m$, m is a positive integer,

$$D^n y = \frac{m!}{(m-n)!} x^{m-n}, \tag{1}$$

where n($\leq$m) is an integer.

Replacing the factorial symbol by the well-known Gamma function, he obtained the formula for the fractional derivative,

$$D^{\alpha}(x^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{2}$$

Where α, β are fractional numbers.

In particular he had,

$$D^{\frac{1}{2}}(x) = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{1/2} = 2\sqrt{\frac{x}{\pi}}. \tag{3}$$

Again the normal derivative of a function f is defined as,

$$D' f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \tag{4}$$

$$\begin{aligned} \text{And } D'' f(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+2h) - f(x+h) + f(x)}{h}. \end{aligned}$$

Iterating this operation yields an expression for the nth derivative of a function. As can be easily seen and proved by induction for any natural number n,

$$D^{n} f(x) = \lim_{h\to 0} h^{-n} \sum_{r=0}^{n} (-1)^{n} \binom{n}{r} \mathrm{f(x + (n - r)h)}. \tag{5}$$

Where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{6}$$

Or equivalently,

$$D^{n} f(x) = \lim_{h\to 0} h^{-n} \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} f(x - rh) \tag{7}$$

The case of n = 0 can be included as well.

The fact that for any natural number n, the calculation of *n*th derivative is given by an explicit formulas (5) or (7).

Now the generalization of the factorial symbol (!) by the gamma function allows

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \tag{8}$$

This is also valid for non-integer values of n.

Thus on using of the idea (8), fractional derivative leads as the limit of a sum given by

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^{r} \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} f(x-rh). \tag{9}$$

Provided the limit exists. Using the identity

$$(-1)^{r} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)} = \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)} \tag{10}$$

The result (9) becomes,

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{r=0}^{n} \frac{\Gamma(r-\alpha)}{\Gamma(r+1)} f(x-rh) \tag{11}$$

When α is an integer, the result (9) reduce to the derivative of integral order n as follows in (5). Again in 1927 Marchaud formulated the fractional derivative of arbitrary order α in the form given by,

$$D^{\alpha} f(x) = \frac{f(x)}{\Gamma(1-\alpha)x^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{f(x)-f(t)}{(x-t)^{\alpha+1}} dt, \text{ where } 0 < \alpha < 1 \tag{12}$$

In 1987, Samko et al. had shown that (12) and (9) are equivalent.

Replacing n by $(-m)$ in (7), it can be shown that

$$\begin{aligned} {}_{0}D_{x}^{-m} f(x) &= \lim_{h \to 0} h^{m} \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} f(x-rh) \\ &= \frac{1}{\Gamma(m)} \int_{0}^{x} (x-t)^{(m-1)} f(t) dt \end{aligned} \tag{13}$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{m(m+1)(m+2)\ldots\ldots..(m+r+1)}{r!} \tag{14}$$

This observation naturally leads to the idea of generalization of the notations of differentiation and integration by allowing m in (13) to be an arbitrary real or even complex number.

2.1 Fractional Derivatives and Integrals

The idea of fractional derivative or fractional integral can be described in another different ways.

First, we consider a linear non homogeneous nth order ordinary differential equation,

$$D^{n} y = f(x), \quad \mathrm{b} \leq \mathrm{x} \leq \mathrm{c} \tag{15}$$

Then $\{1,\ x,\ x^2,\ x^3,\ \ldots\ldots..,\ x^{n-1}\}$ is a fundamental set the corresponding homogeneous equation $D^n y = 0$, *f(x)* is any continuous function in $[b, c]$, then for any $a \in (b, c)$,

$$y(x) = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt \tag{16}$$

is the unique solution of the Eq. (15) with the initial data $y^{(k)}(a) = 0$,
for $0 \leq k \leq n-1$. Or equivalently,

$$y(x) = {}_0D_x^{-m} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt \tag{17}$$

Replacing n by a, where $\mathrm{Re}(\alpha) > 0$ in the above formula (17), we obtain the Riemann-Liouville definition of fractional integral that was reported by Liouville in 1832 and by Riemann in 1876 as

$$_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{18}$$

where $_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$ is the Riemann-Liouville integral operator. When $a = 0$, (18) is the Riemann definition of integral and if $a = -\infty$, (18) represents Liouville definition. Integral of this type were found to arise in theory of linear ordinary differential equations where they are known as Eulier transform of first kind.

If $a = 0$ and $x > 0$, then the Laplace transform solution the initial value problem

$$D^n y(x) = f(x) \qquad x > 0,\ y^{(k)}(0) = 0,\ 0 \leq k \leq n-1 \tag{19}$$

is

$$\bar{y}(s) = s^{-n} \bar{f}(s) \tag{20}$$

Where $\bar{y}(s)$ and $\bar{f}(s)$ are respectively the Laplace transform of the function y(x) and f(x).

The inverse Laplace transform gives the solution of the initial value problem (19) as

$$y(x) = {}_aD_x^{-n} f(x)$$

Again from (20) we have

$$\begin{aligned} y(x) &= L^{-1}\{\bar{y}(s)\} \\ &= L^{-1}\{s^{-n} \bar{f}(s)\} \end{aligned}$$

Thus we have

$$ {}_aD_x^{-n} f(x) = L^{-1}\{s^{-n} \bar{f}(s)\} \tag{21} $$

$$ \text{i.e } L^{-1}\{s^{-n} \bar{f}(s)\} = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)dt \tag{22} $$

$$ \therefore\ y(x) = {}_aD_x^{-n} f(x) = L^{-1}\{s^{-n} \bar{f}(s)\} = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)dt $$

This is the Riemann-Liouville integral formula for an integer *n*. Replacing *n* by real α gives the Riemann-Liouville fractional integral (17) with $a = 0$.

In complex analysis the Cauchy integral formula for the *n*th derivative of an analytic function *f(z)* is given by

$$ D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt \tag{23} $$

Where C is closed contour on which *f(z)* is analytic, and $t = z$ is any point inside C and $t = z$ is a pole.

If *n* is replaced by an arbitrary number α and $n!$ by $\Gamma(n+1)$, then a derivative of arbitrary order a can be defined by,

$$ D^\alpha f(z)\mathrm{f(z)} = \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(t)}{(t-z)^{\alpha+1}} dt \tag{24} $$

where $t = z$ is no longer a pole but a branch point.

In (24) C is no longer appropriate contour, and it is necessary to make a branch cut along the real axis from the point $z = x > 0$ to negative infinity.

Thus we can define a derivative of arbitrary α order by loop integral

$$ {}_aD_x^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_a^x (t-z)^{-\alpha-1} f(t) \tag{25} $$

Where $(t-z)^{-\alpha-1} = exp[-(\alpha+1)ln(t-z)]$ and *ln(t-z)* is real when $t\text{-}z > 0$. Using the classical method of contour integration along the branch cut contour *D*, it can be shown that

$$ \begin{aligned} {}_aD_x^\alpha f(z) &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_D (t-z)^{-\alpha-1} f(t) \\ &= \frac{\Gamma(\alpha+1)}{2\pi i} \left[1 - \exp\{-2\pi i(\alpha+1)\}\right] \int_0^z (t-z)^{-\alpha-1} d(t) = \frac{1}{\Gamma(-\alpha)} \int_0^z (t-z)^{-\alpha-1} f(t)dt \end{aligned} \tag{26} $$

which agrees with Riemann-Liuville definition (17) with $z = x,$ and $a = 0$, when a is replaced by $-\alpha$

2.2 Fractional Integration, Fractional Differential Equation Using Laplace Transformed Method

One of the very useful results is formula for Laplace transform of the derivative of an integer order n of a function *f(t)* is given by

$$\mathrm{L}\{f^{(n)}(t)\} = s^n \bar{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \tag{27}$$

$$= s^n \bar{f}(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)$$

$$= s^n \bar{f}(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0) \tag{28}$$

Where $f^{(n-k)}(0) = c_k$ represents the physically realistic given initial conditions and $\bar{f}(s)$ being the Laplace transform of the function *f(t).*

Like Laplace transform of integer order derivative, it is easy to shown that the Laplace transform of fractional order derivative is given by

$$\mathrm{L}\{{}_0D_t^{\alpha} f(t)\} = s^{\alpha} \bar{f}(s) - \sum_{k=0}^{n-1} s^k [{}_0D_t^{\alpha-k-1} f(t)]_{t=0} \tag{29}$$

$$= s^{\alpha} \bar{f}(s) - \sum_{k=1}^{n} s^{k-1} c_k, \tag{30}$$

where $\mathrm{n} - 1 \leq \alpha < n$ and

$$c_k = [{}_0D_t^{\alpha-k} f(t)]_{t=0} \tag{31}$$

represents the initial conditions which do not have obvious physical interpretation. Consequently, formula (30) has limited applicability for finding solutions of initial value problem in differential equations.

We now replace a by an integer-order integral J^n and $D^n f(t) \equiv f^{(n)}(t)$ is used to denote the integral order derivative of a function f(t). It turns out that

$$D^n J^n = \mathrm{I}, \quad J^n D^n \neq \mathrm{I}. \tag{32}$$

This simply means that D^n is the left inverse (not the right inverse) of J^n. It also follows that when $\mathrm{a} = n$, we have

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, \qquad \mathrm{t} > 0 \tag{33}$$

Similarly, D^{α} can also be defined as the left inverse of J^{α}. We define the fractional derivative of order $\alpha > 0$ with $n - 1 \leq \alpha < n$ by

$${}_0D_t^{\alpha} f(t) = D^n D^{-(n-\alpha)} f(t) \quad = D^n J^{n-\alpha} \mathrm{f(t)}$$

$$= D^n\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau\right] \quad (34)$$

Or,

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau$$

Where n is an integer and the identity operator 'I' is defined by
$D^0 f(t) = J^0 f(t) = If(t) = f(t)$, so that $D^\alpha J^\alpha = \mathrm{I}$, $\alpha \geq 0$.

Due to the lack of physical interpretation of initial data c_k in (30), Caputo and Mainardi adopted as an alternative new definition of fractional derivative to solve initial value problems. This new definition was originally introduced by Caputo in the form

$$\begin{aligned} {}_0^C D_t^\alpha f(t) &= J^{n-\alpha} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau \end{aligned} \quad (35)$$

Where $\mathrm{n}-1 \leq \alpha < n$ and n is an integer.
It follows from (34) and (35) that

$${}_0D_t^\alpha f(t) = D^n J^{n-\alpha} f(t) \neq J^{n-\alpha} D^n f(t) = {}_0^C D_t^\alpha f(t) \quad (36)$$

Unless *f(t)* and its first *(n − 1)* derivatives vanish at *t = 0*.
Furthermore, it follows from (35) and (36) that

$$J^\alpha {}_0^C D_t^\alpha f(t) = J^\alpha J^{n-\alpha} D^n f(t) = J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!} \quad (37)$$

This implies that

$$\begin{aligned} {}_0^C D_t^\alpha f(t) &= {}_0D_t^\alpha\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0)\right] \\ &= {}_0D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) \end{aligned} \quad (38)$$

This shows that Caputo's fractional derivative incorporates the initial values $f^{(k)}(0)$, for $k = 0, 1, 2, \ldots\ldots., n-1$.

The Laplace transform of Caputo's fractional derivative (38) gives an interesting formula

$$\mathrm{L}\left\{{}_0^C D_t^\alpha f(t)\right\} = s^\alpha \bar{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0)s^{\alpha-k-1} \quad (39)$$

This is a natural generalization of the corresponding well known formula for the Laplace transform of $f^{(n)}(t)$ when $\alpha = \mathrm{n}$ and can be used to solve the initial value problems in fractional differential equation with physically realistic initial conditions.

3 Basic Concept on Classical EOQ Model

The order quantity means the quantity produced or procured in one production cycle or order cycle (the time period between placement of two successive orders (or production) is referred to as an order cycle (or production cycle). This is also termed re-order quantity when the size of order increases, the order costs (cost of purchasing, inspection, etc.) will decrease whereas the inventory carrying costs will increase. Thus in the production or purchasing case, there are two opposite costs, one encourages the increase in the order size and the other discourages. Economic order quantity (EOQ) is that size of order which minimizes total annual costs of carrying inventory and cost of ordering.

Notations and Assumptions:

D	Demand rate
Q	Order quantity
C_1	Holding cost per unit time, which is time dependent
C_3	Set up cost
q(t)	Stock level at any time t $\geq$ 0
T	Cycle of length of given inventory
TAC(T)	Total average cost per unit time
W_0	Space area per unit quantity
W	Total storage space area of the inventory
w	Dual variable of T in geometric programming
U	Unit purchasing cost is inversely related to the demand and take the form: U $= \frac{a}{D^b}$, a $>$ 0 and b $>$ 1

In classical EOQ based inventory model, we already have

$$\begin{aligned}\frac{dq(t)}{dt} &= -D\,, \quad \text{for}\, 0 \leq t \leq T \\ &= 0, \quad \text{otherwise.}\end{aligned} \tag{40}$$

With the initial condition q(0) = Q and with the boundary condition q(T) = 0 (Fig. 1). By solving the Eq. (40), we have

$$q(t) = Q - Dt,\ \text{for}\, 0 \leq t \leq T \tag{41}$$

And on using the boundary condition

$$q(T) = 0,\ \text{we have}\ Q = DT. \tag{42}$$

Holding cost,

$$HC(T) = C_1 \int_{t=0}^{T} q(t)dt = C_1 \int_{t=0}^{T} (Q - Dt)dt = C_1[Qt - \frac{Dt^2}{2}]_{t=0}^{T} = C_1(QT - \frac{DT^2}{2}) = \frac{C_1 DT^2}{2} \tag{43}$$

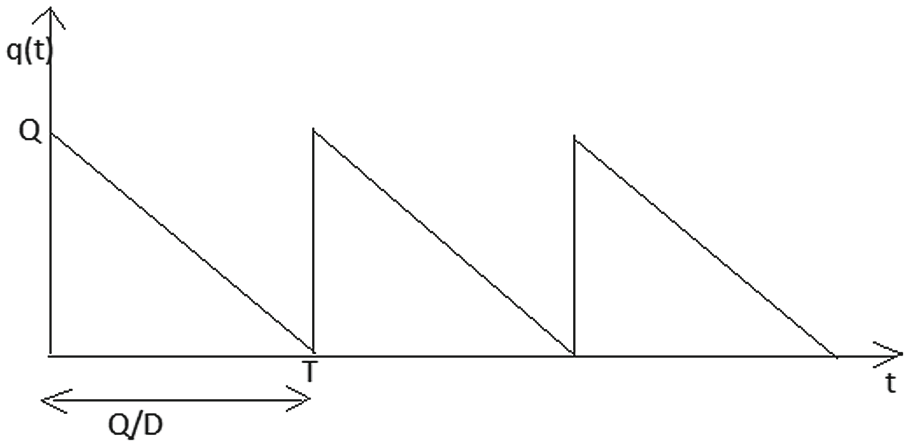

Fig. 1. Development of inventory level over time.

Total cost, $TC(T)$ = Purchasing cost(PC) + Holding cost(HC) + Set up cost(SC)

$$= \mathrm{UQ} + \frac{C_1 D T^2}{2} + C_3. \tag{44}$$

Total average cost over $[0, T]$ is given by

$$\begin{aligned} TAC(T) &= \frac{1}{T}[UQ + \frac{C_1 D T^2}{2} + C_3] \\ &= \frac{UQ}{T} + \frac{C_1 DT}{2} + \frac{C_3}{T} \end{aligned} \tag{45}$$

And the storage area is $W_0 Q$
Then the classical EOQ model can be written as

$$\mathrm{Min}\, TAC(T) = UD + \frac{C_1 DT}{2} + \frac{C_3}{T}$$

Subject to,

$$W_0 Q \leq W, \quad T > 0. \tag{47}$$

4 Generalized EOQ Model with Demand Dependent Unit Cost

We now generalize our discussion by accepting the Eq. (40) as a differential equation of fractional order instead of the linear order. i.e. we here consider the fractional order rate of instantaneous inventory level say a, where $0 < \alpha \leq 1$ instead of first order. Here instantaneous inventory level becomes

$$\frac{d^{\alpha}q(t)}{dt^{\alpha}} = -D \quad \text{for } 0 \le \mathrm{t} \le \mathrm{T}\ ,\ 0 < \alpha \le 1$$
$$= 0 \quad \text{otherwise.} \tag{48}$$

with the same initial and boundary condition as described in the previous problem in Eq. (40). i.e. $q(0) = Q$ and with $q(T) = 0$, where D is a constant.

Equation (48) can be rewritten as

$${}_0^C D_t^{\alpha} q(t) = -D \quad \text{for } 0 \le \mathrm{t} \le \mathrm{T}$$
$$= 0 \qquad \text{otherwise.} \tag{49}$$

where ${}_0^C D_t^{\alpha} \equiv J^{1-\alpha} D^1$ is the Caputo fractional derivative as described in (35) and $D^1 \equiv \frac{d}{dt}$.

To solve the initial value problem of fractional order differential Eq. (49) we apply the Laplace transform method. So taking Laplace transform of the Eq. (49), we have, $\mathrm{L}\{{}_0^C D_t^{\alpha}\mathrm{q(t)}\} = -\mathrm{DL}\{1\}$

$$\Rightarrow s^{\alpha}\bar{q}(s) - s^{\alpha-0-1}q(0) = -\frac{D}{s},$$

$\bar{q}(s)$ being Laplace transform of $q(t)$.

$$\Rightarrow s^{\alpha}\bar{q}(s) = \mathrm{Q}s^{\alpha-1} - \frac{D}{s}$$

$$\Rightarrow \bar{q}(s) = \frac{Q}{s} - \frac{D}{s^{\alpha+1}}$$

Taking Laplace inversion of above equation we have,

$$q(t) = L^{-1}\{\bar{q}(s)\} = Q - \frac{Dt^{\alpha}}{\Gamma(\alpha+1)}$$

So the inventory level at any time t based on α ordered decreasing rate of demand is

$$q_{\alpha}(t) = Q - \frac{DT^{\alpha}}{\Gamma(\alpha+1)} \quad \text{for } 0 \le \mathrm{t} \le \mathrm{T}. \tag{50}$$

on using the boundary condition $q(T) = 0$ implies that

$$\mathrm{Q} = \frac{DT^{\alpha}}{\Gamma(\alpha+1)} \tag{51}$$

[for a = 1 in (50) and (51) gives results as in (41) and (42)].

4.1 Generalized Holding Cost

Holding cost is

$$
\begin{aligned}
HC_\alpha(T) &= C_1 \int_0^T q_\alpha(t)dt \\
&= C_1 \int_0^T [Q - \frac{Dt^\alpha}{\Gamma(\alpha+1)}]dt \\
&= C_1[QT - \frac{DT^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}] \\
&= C_1[\frac{DT^{\alpha+1}}{\Gamma(\alpha+1)} - \frac{DT^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}] \\
&= \frac{C_1 \alpha DT^{\alpha+1}}{\Gamma(\alpha+2)}
\end{aligned} \tag{52}
$$

Purchasing cost $(PC) = UQ$, As D is inversely related to U, here U is assumed as, $U = \frac{a}{D^b}$, $a > 0$ and $b > 1$

Then

$$
PC = \frac{a}{D^b}\frac{DT^\alpha}{\Gamma(\alpha+1)} = \frac{aD^{1-b}}{\Gamma(\alpha+1)}T^\alpha \tag{53}
$$

Set up cost $(SC) = C_3$

$$
\begin{aligned}
\therefore \text{ total cost } (TC) &= PC + HC + SC \\
&= \frac{aD^{1-b}}{\Gamma(\alpha+1)}T^\alpha + \frac{C_1 \alpha DT^{\alpha+1}}{\Gamma(\alpha+2)} + C_3
\end{aligned} \tag{54}
$$

$$
\therefore \text{ TAC(D, T) } = \frac{aD^{1-b}}{\Gamma(\alpha+1)}T^{\alpha-1} + \frac{C_1 \alpha DT^{\alpha}}{\Gamma(\alpha+2)} + \frac{C_3}{T} \tag{55}
$$

Now our problem is to minimize $TAC(D,T)$ subject to the space constraint

$$
W_0 Q \leq W \text{ i.e } \frac{W_0}{W}\frac{DT^\alpha}{\Gamma(\alpha+1)} \leq 1 \tag{56}
$$

Then our problem becomes,

$$
TAC(D, T) = AD^{1-b}T^{\alpha-1} + BDT^\alpha + \frac{C_3}{T}, \tag{57}
$$

Subject to,

$$
LDT^\alpha \leq 1 \tag{58}
$$

Where $A = \frac{a}{\Gamma(\alpha+1)}$, $B = \frac{C_1\alpha}{\Gamma(\alpha+2)}$, $L = \frac{W_0}{W\Gamma(\alpha+1)}$

To optimize the problem (57) subject to (58), we apply geometric programming, where degree of difficulty (DD) = 4 − 2 − 1 = 1 (57) can be taken as a primal geometric programming problem with degree of difficulty (DD) = 1. Dual form of (57) subject to (58)

$$\text{Max}\, d(w) = \left(\frac{A}{w_{01}}\right)^{w_{01}} \left(\frac{B}{w_{02}}\right)^{w_{02}} \left(\frac{C_3}{w_{03}}\right)^{w_{03}} \left(\frac{L}{w_{11}}\right)^{w_{11}} w_{11}^{w_{11}}, \tag{59}$$

Subject to,

$$w_{01} + w_{02} + w_{03} = 1, \quad \text{(normalized condition)} \tag{60}$$

$$(\alpha - 1)w_{01} + \alpha\, w_{02} - w_{03} + \alpha w_{11} = 0, \quad \text{(orthogonal condition)} \tag{61}$$

$$(1 - b)w_{01} + w_{02} - w_{03} + w_{11} = 0 \tag{62}$$

Where $w_{01}, w_{02}, w_{03}, w_{11} \geq 0$

$w_{01}, w_{02}, w_{03}, w_{11}$ are the dual variables appearing in the geometric programming problem.

Solving for w_{01}, w_{02}, w_{03} in terms of w_{11} from above simultaneous linear Eqs. (60), (61) and (62) we get

$$w_{01} = \frac{1 + w_{11}}{b\alpha + b - 1}, \; w_{02} = \frac{b - 1 - b\alpha w_{11}}{b\alpha + b - 1}, \; w_{03} = \frac{(b\alpha - 1)(1 + w_{11})}{b\alpha + b - 1} \tag{63}$$

Again the primal-dual relations are

$$AD^{1-b}T^{\alpha-1} = w_{01}d(w), \; BDT^{\alpha} = w_{02}d(w), \; \frac{C_3}{T} = w_{03}d(w) \,\&\, LDT^{\alpha} = \frac{w_{11}}{w_{11}} \text{ i.e } LDT^{\alpha} = 1 \tag{64}$$

From second and third relations of (64) we get $\frac{w_{02}}{w_{03}} = \frac{BDT^{\alpha+1}}{C_3} = \frac{B}{C_3}\frac{T}{L}$

Again from first and third relation of (64) we get $\frac{w_{01}}{w_{03}} = \frac{AD^{1-b}T^{\alpha}}{C_3}$

$$\Rightarrow \frac{AD^{-b}}{C_3} DT^{\alpha} = \frac{1 + w_{11}}{(b\alpha - 1)(1 + w_{11})} \Rightarrow D^b = \frac{A}{LC_3}(b\alpha - 1)$$

Hence optimum value of D will be

$$D^* = \{\frac{A}{LC_3}(b\alpha - 1)\}^{\frac{1}{b}} \tag{65}$$

Hence optimum value of T will be

$$T^* = \left(\frac{1}{LD}\right)^{\frac{1}{\alpha}} \tag{66}$$

[using (64)].

Thus optimum value of D and T are given in (65), (66).

Then minimum value of *TAC(D, T)* will be given by

$TAC^*(D, T) = A(D^*)^{1-b}(T^*)^{\alpha-1} + BD^*(T^*)^{\alpha} + \frac{C_3}{T^*}$, Where T^*, D^* are given in (65), (66).

5 Numerical Example

For a particular EOQ problem $C_1 = \$6$, a = 100, b = 12, $C_3 = \$50$, $W_0 = 10$, W = 55.

For these values, the optimum values of demand rate D^*, optimum value of time period T^*, and the minimum total average cost $TAC^*(D, T)$ are obtained in the given Table 1.

Table 1. Optimum value of T*, D^* & $TAC^*_{\alpha,1}(T^*)$ for different values of a

α	$A = \frac{a}{\Gamma(\alpha+1)}$	$B = \frac{C_1\alpha}{\Gamma(\alpha+2)}$	$L = \frac{W_0}{W\Gamma(\alpha+1)}$	D^*	T^*	$TAC^*(D, T)$
0.1	101.114	0.573347	0.191116	1.06791	7974120	5.04879
0.2	108.912	1.08912	0.198023	1.25591	1051.06	5.58151
0.3	111.424	1.5428	0.20259	1.3224	80.6734	8.47355
0.4	112.706	1.9321	0.20492	1.36485	24.171	12.0415
0.5	112.838	2.25676	0.20516	1.39647	12.1829	15.9249
0.6	111.917	2.51814	0.203486	1.42172	7.90228	19.7229
0.7	110.055	2.719	0.2001	1.44284	5.8988	23.21
0.8	107.367	2.86312	0.195213	1.46102	4.79782	26.2999
0.9	103.975	2.95509	0.189046	1.47701	4.12677	28.984
1.0	100	3.0	0.1818	1.49131	3.6884	31.2901

Above table shows optimal results of total average cost, time period and demand for different values of α where $0 < \alpha \leq 1$. It has been seen that as α increases D^*increases [Fig. 2] and T^* decreases [Fig. 3] and *TAC*(T*)* increases [Fig. 4]

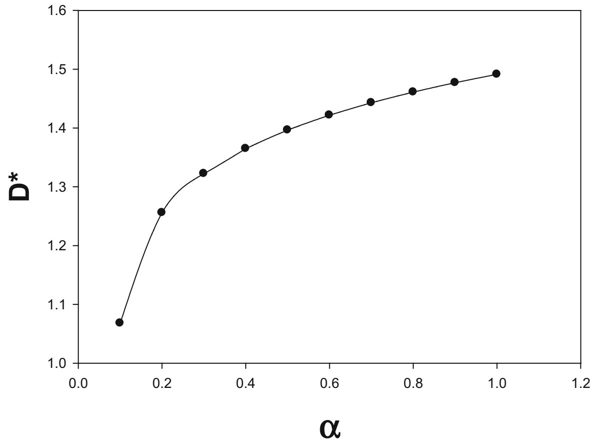

Fig. 2. Rough sketch of α versus D^* graph

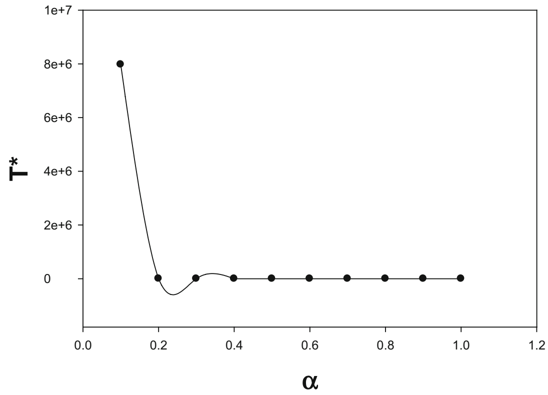

Fig. 3. Rough sketch of α versus T^* graph

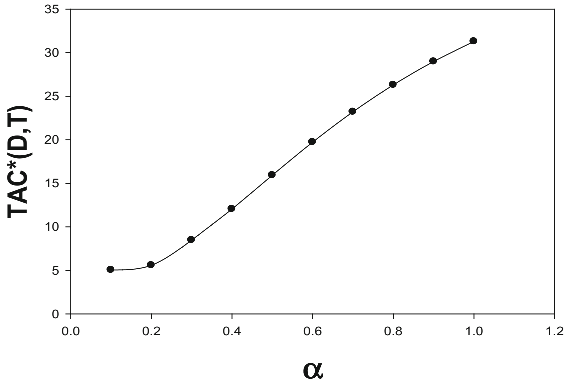

Fig. 4. Rough sketch of α versus $TAC^*(D, T)$ graph

6 Conclusion

In this paper, we present a real life inventory model in more generalized way using the concept of fractional differential calculus based on generalized EOQ model with a space constraint where demand is inversely proportional to the unit purchasing cost having no shortages. We formulate the generalized EOQ problem as geometric programming (GP) and apply the theories of GP to help derive the optimal solution. Finally a numerical example has been presented for determining the optimum value of demand rate, time period and total average cost. This method is quite general and can be extended to other similar inventory models including the ones with shortages. In future, the similar fractional order generalized inventory model can be developed with shortages.

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Interval Type-2 Fuzzy Logic and Its Application to Profit Maximization Solid Transportation Problem in Mustard Oil Industry

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Abstract. In this paper, a new design of interval type-2 fuzzy logic systems (IT2FLS) for a profit maximization solid transportation problem is presented. It has been proposed that the antecedent membership function parameters of the IT2FLS are produced randomly with the consequent part parameters being determined practically by the mustard oil producing factory in India. The application of mustered oil factory data sets has been used in order to exhibit the effectiveness of the proposed design of IT2FLS.The unit Transportation cost is determined by using IT2FLS, three inputs- supplies, demands and road conditions and one output parameter. A profit solid transportation problem is then formulated with volume and capacity constraints of oil packets and Generalized Reduced Gradient Technique (LINGO-12.0) is used to solve the proposed model. Lastly the optimal results are presented in tabular form and graphically.

Keywords: Fuzzy inference system · Interval type-2 fuzzy logic · Solid transportation problem · Generalized Reduced Gradient

1 Introduction

The type-2 fuzzy logic system (T2FLS) as an extension of the conventional type-1(T1) fuzzy logic system (T1FLS) is gradually more in use in modeling real world problems [8]. In contrast to T1FLS, a T2FLS assigns a fuzzy number to the membership grades [2]. Experimental results have been reported presenting improvements of the T2FLS over its T1 counterpart in terms of accuracy [3]. Since of the big computing resources is required for the computationally intensive T2FLSs. The IT2FLS is commenced as its simplest version of IT1FLS. Using IT2FLSs, the membership grade for every point is a crisp set in an interval $[0, 1]$ rather than fuzzy. Though improvements of IT2FLSs to its earlier version have been evidenced, yet it still lacks a systematic and coherent design procedure. In order to determine the

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parameters of an IT2FLS optimally, different learning algorithms are anticipated in literature that contain dynamical optimal training method [4].

Recently, many research papers have been developed in using type-1 fuzzy (T1F) and type-2 fuzzy(T2F) logic system. Castillo et al. [5] have developed a comparative study of type-1 fuzzy logic, interval T2F logic and generalized T2F logic systems in control problems. They studied different types of fuzzy logic systems for designing the fuzzy controllers of complex non-linear plants. Jana et al. [8] has presented a new approach to predict the quality of polypropylene in petrochemical plants. The proposed approach predicts the polypropylene quality. Sabahi et al. [6] have investigated a combination of type-2 fuzzy logic system (T2FLS) and a conventional feedback controller (CFC) for the load frequency control (LFC) of a nonlinear time-delay power system. Salles et al. [7] has shown how the use of fuzzy systems can improve the application of system dynamics (SD), with the construction of virtual worlds for organizational learning. Here, the main part is to suggest a fuzzy-SD integrated methodology that permits a natural language modeling of decision strategies. Hong et al. [9] has presented a paper to assess the effectiveness of protection layers and the residual risk of an incident scenario.

A transportation problem (TP) is often connected with additional costs which are also known as fixed costs besides usual transportation costs. The fixed charge transportation problem was first proposed by Hirsch and Dantzig [11] considering two types of costs mainly direct and fixed charge costs. Due to permit fees, toll charges, etc. this fixed charge costs are introduced. After the beginning of TPs by Hitchcock [13], there have been lots of developments in this area by a number of researchers. Chanas et al. [15] formulated and solved TPs with fuzzy supply and demand values. Fegad et al. [14] found optimal solutions to TPs using interval and ordinary triangular membership functions. Kaur and Kumar [10] provided a new method to solve TPs with transportation costs (TCs) as generalized trapezoidal fuzzy numbers. Ojha et al. [1] have developed a transportation policy for single and multi-objective transportation problems using fuzzy logic in type 1 fuzzy environment. Recently, many researchers have shown their interests in this field of TP optimization problems (cf. Yang et al. [16], Giri et al. [12], Chakraborty et al. [18]).

In spite of the above mentioned developments, we have developed the following:

- Interval type-2 fuzzy logic control (T2FLC) approach for prediction of unit transportation cost has been studied.
- A new method is proposed to generate all optimal solutions of the Interval type 2 fuzzy version of the profit maximization solid transportation problem.
- In T2FLC, assists are developed to trace inputs and outputs in a well-organized manner for building the inferences train so that various types of transportation cost can be predicted.
- Finally, a case study has been studied for mustard oil factory and solved by GRG method using LINGO.

The present paper presents a complete interval type-2 fuzzy logic system (IT2FLS) which gives an essential indicator of parameters of supply, demand, road conditions and unit transportation cost in a mustard oil industry. There are so many possibilities to select supply, demand and operators as well as inference, implication, aggregation and defuzzification methods. So the search for the perfect mathematical model can be included among the most vital topics in development of rules based models. For this, we have made a primary investigation on the parameters controlling the unit cost assessment and choose the most appropriate input and output variables. A Mamdani interval type 2 fuzzy inference systems (MFT2IS) is then developed using these inputs and outputs. Depending on the membership functions (MF), a model has been established and sensitivity analyses are performed. After selection of unit transportation cost, we formulate a profit maximization solid transportation problem with different types of constraints and solved by a soft computing technique and Generalized Reduced Gradient (GRG) method.

1.1 Type-2 Fuzzy Sets

A type-2 fuzzy set (Jana et al. [8]) expresses the non-deterministic truth degree with imprecision and uncertainty for an element that belongs to a set. A type-2 fuzzy set (cf. Castillo and Melin [17]) denoted by $\tilde{\tilde{A}}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u)$ where $x \in X, \forall u \in J_x^u \subseteq [0,1]$ and $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$ defined by $\tilde{\tilde{A}} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$, that is it can be in Eq. (1) as

$$\tilde{\tilde{A}} = \{(x, u, \mu_{\tilde{A}}(x,u)) | x \in X, \forall u \in J_x^u \subseteq [0,1]\} \tag{1}$$

If $\tilde{\tilde{A}}$ is fuzzy type 2 (FT2) continuous variable, it is denoted as

$$\tilde{\tilde{A}} = \left\{ \int_{x \in X} \left[\int_{u \in J_x^u} f_x(u)/u \right] /x \right\} \tag{2}$$

where $\int\int$ denotes the union of x and u. If A is FT2 discrete, then it is denoted by Eq. (3)

$$\tilde{\tilde{A}} = \left\{ \sum_{x \in X} \mu_{\tilde{\tilde{A}}}(x)/x \right\} = \left\{ \sum_{i=1}^{N} \left[\sum_{k=1}^{M_i} f_{x_i}(u_k)/u_{ik} \right] /x_i \right\} \tag{3}$$

where $\sum\sum$ denotes the union of x and u. If $f_x(u) = 1, \forall u \in [\underline{J}_x^u, \overline{J}_x^u] \subseteq [0,1]$, the type-2 membership function $\mu_{\tilde{\tilde{A}}}(x,u)$ is expressed by one type-1 inferior membership function, $\underline{J}_x^u = \mu_A(x)$ and one type-1 superior, $\overline{J}_x^u = \mu_A(x)$, then it is called an interval type-2 fuzzy set denoted by Eq. (4).

$$\tilde{\tilde{A}} = \left\{ (x, u, 1) | \forall x \in X, \forall u \in [\underline{\mu}_A(x), \overline{\mu}_A(x)] \subseteq [0,1] \right\} \tag{4}$$

Definition 1: A type-1 fuzzy set X is comprised of a domain D_X of real numbers (also called the universe of discourse of X) together with a membership function (MF) $\mu_x : D_X \to [0,1]$, i.e.

$$X = \int_{D_x} \mu_x(x)/x \tag{5}$$

Here $\int$ denotes the collection of all points $x \in D_X$ with associated membership grade $\mu_x(x)$.

Definition 2 (Mendel [2]). An IT2 FS $\tilde{X}$ is characterized by its MF $\mu_x(x, u)$, i.e

$$= \int_{x \in D_x} \left[\int_{u \in J_x \subseteq [0,1]} 1/u \right] / x \tag{6}$$

where x, called the primary variable, has domain $D_{\tilde{X}} : u \in [0,1]$, called the secondary variable, has domain $J_x \subseteq [0,1]$ at each $x \in D_{\tilde{X}}$; J_x is also called the support of the secondary MF and the amplitude of $\mu_{\tilde{X}}(x, u)$, called a secondary grade of $\tilde{X}$, equals 1 for $\forall x \in D_{\tilde{X}}$ and $\forall u \in J_x \subseteq [0,1]$.

For general type-2 FSs $\mu_X(x, u)$ can be any number in $[0,1]$, and it varies as x and/or u vary. about $\tilde{X}$ is suggested by the union of all its primary memberships, which is said the footprint of uncertainty (FOU) of $\tilde{X}$, i.e.,

$$FOU(\tilde{X}) = \bigcup_{\forall x \in D_{\tilde{X}}} J_x = \left\{ (x, u) : u \in J_x \subseteq [0,1] \right\} \tag{7}$$

The size of an FOU is directly related to the uncertainty that is conveyed by an IT2 FS. So, an FOU with more area is more uncertain than one with less area. The upper membership function (UMF) and lower membership function (LMF) of $\tilde{X}$ are two T1 MFs X and X that bound the FOU (see Fig. 6).

$$J_x = [\mu_{\underline{X}}(x), \mu_{\overline{X}}(x)] \tag{8}$$

Using (8), FOU $(\tilde{X})$ can also be expressed as

$$FOU(\tilde{X}) = \bigcup_{x \in D_x} [\mu_{\underline{X}}(x), \mu_{\overline{X}}(x)] \tag{9}$$

2 Model Formulation for Fuzzy Logic Controller

The paper presents an interval type-2 fuzzy logic control (T2FLC) method of approach, for making prediction of various types of accidents in an environment with imprecise surroundings. Since no predictable rule is followed in such fields it is difficult in establishing a relation between investment and the type of accidents. Rather in other words it is associated in a non-linear manner. The type-2 fuzzy logics assist in a way such that the inputs and outputs are traced in a well-organized manner for building interference train for prediction of various types of accidents. The prediction of various types of accidents helps the managers in formulating policies of the organization and performs in a way such that

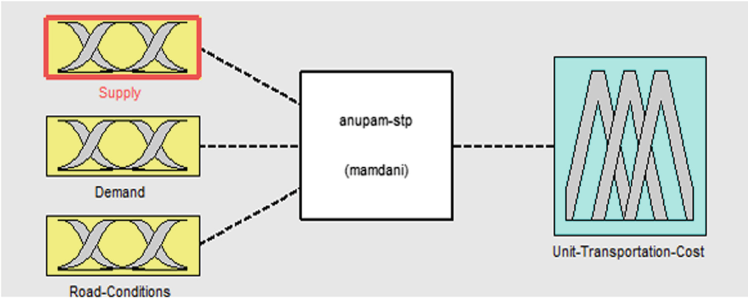

Fig. 1. The structure of Mamdani FIS

the safety performance increases minimizing rate of accidents. The next case study shows the command line editing procedure of the Mamdani interval type-2 fuzzy logic inference system structure implemented in the IT2FLS Toolbox. The proposed Mamdani type 2 fuzzy inference model with 3 input and 2 output variables is displayed in Fig. 1. Min and Max operators employed to evaluate the logical conjunction AND and OR. We have used Min and Max operators for implication and aggregation method respectively. Centroid method is applied for the defuzzification.

2.1 Supply

By using Table 1, the experts only determine the range of percentages associated with each category of supply of mustered oil packets (e.g. low, medium, etc), and provided definitions, as shown in the first column of Table 1. Fuzzy supplies set definitions are computed using these definitions, as shown in Fig. 2. The model uses the values given in Table 1 to define the linguistic expressions for supply.

Table 1. Linguistic variables (LVs) for input interval fuzzy 2 fuzzy values supply and demand

LVs	Supply	LVs	Demand
Low	[132.2 152.2 172.2 139.2 159.2 179.2]	Low	[146.5 161.5 176.5 151.5 166.5 181.5]
Medium	[146.7 166.7 186.7 153.3 173.3 193.3]	Medium	[157.5 172.5 187.5 162.5 177.5 192.5]
High	[158.7 178.7 198.7 165.3 185.3 205.3]	High	[168.9 183.9 198.9 173.9 188.9 203.9]

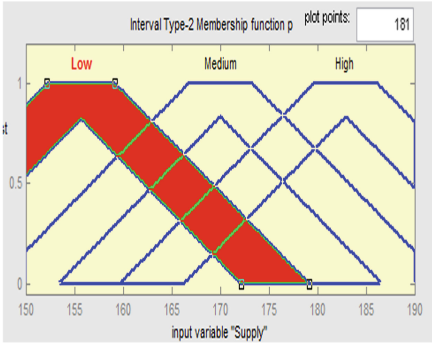

Fig. 2. MFs for supply

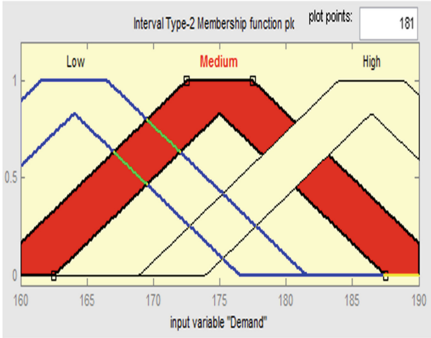

Fig. 3. MFs for demand

2.2 Demand

With the help of literature study and the rank arrangements it is possible to determine the demand of mustered oil packets. The set definitions of Interval type 2 fuzzy supply are calculated as per the definition given in the Table 1. Next to this the judgement of different group of experts were taken on subjective basis for determining the use of mustered oil packets. The designed model makes use of the average expert scores given in Fig. 3, so as for defining the interval type 2 fuzzy linguistic expressions. The input parameters cannot be changed by the user but on basis of personal view.

2.3 Road Conditions

During transportation of mustard oil products, there is a traffic problem. So, we introduced a new concept as linguistic variables for road conditions. Depending on the condition, manager will decide the cost of unit transportation and he put marks in $[0, 1]$. The nature of linguistic variables (LVs) for input IT2F variables road conditions are given in Table 2 and graphically shown in Fig. 5 (Table 3).

Table 2. Linguistic variables (LVs) for input IT2F variable road conditions

LVs	Road conditions
Worst	[−0.2917 −0.04167 0.2083 −0.2083 0.04167 0.2917]
Bad	[−0.04167 0.2083 0.4583 0.04167 0.2917 0.5417]
Average	[0.2083 0.4583 0.7083 0.2917 0.5417 0.7917]
Good	[0.4583 0.7083 0.9583 0.5417 0.7917 1.042]
Best	[0.7083 0.9583 1.208 0.7917 1.042 1.292]

Table 3. Linguistic variables (LVs) for out put IT2F variable unit transportation cost

LVs	Unit transportation cost
Very low	[−10.28 −0.6111 9.056 −7.056 2.611 12.28]
Low	[−0.6111 9.056 18.72 2.611 12.28 21.94]
Medium	[9.056 18.72 28.39 12.28 21.94 31.61]
High	[18.72 28.39 38.06 21.94 31.61 41.28]

2.4 Assumptions

In this transportation problem, the following assumptions are used.

(i) Given a set of DCs where a certain number of vehicles of different types are located, determine vehicle trips that maximize the total transportation profit, such that (a) each delivery point receives the required quantity of each product type, (b) all vehicle constraints are satisfied, (c) the distribution centers product availability is respected.

(ii) the total quantity available at DCs for each product type j is assumed to be sufficient to cover the demand of all delivery points for this product type i.e.

$$\sum_{i=1}^{n} \tilde{b}_{ij} \leq \sum_{l=1}^{u} a_{jl} \tag{10}$$

3 Formulation of Solid Transportation Problem

The profit maximization solid transportation problem (PSTP) can be mathematically defined as follows (Fig. 4):

Haldia mustard oil producing factory is situated in East Medinipur, India. The factory produces various type of mustard oil of different sizes of packets and sells into nearest markets. Let u be the number of distribution centers (DC) from which the oil products are supplied and n be the number of delivery points (DP), where people can purchase this products. Let p denotes the total number of products types which are needed for people. The quantity of product j available at DC l is p_{jl} and the quantity of product j required at delivery point i is d_{ij}. In addition, at each distribution center l, it is assumed that there are m_l vehicle types, and u_{hl} vehicles of each type h. There are two types of economic costs associated with this transportation problem, direct economic costs and indirect economic costs which is treated as fixed charge cost like toll tax and value added tax. The direct economic costs are the costs that relate to the restoration or reconstruction of the transportation system which is depends on unit transportation costs which are imprecise in nature. The proposed profit solid transportation can be mathematically stated as follows: The first objective

Fig. 4. The graphical representation of STP

function maximizes the total profit in this transportation problem of all trips is

$$\max f_2 = \sum_{i=1}^{n}\sum_{j=1}^{p}\sum_{l=1}^{u}\sum_{h=1}^{m_l}\sum_{k=1}^{u_{hl}}\sum_{v=1}^{r}\Big((SP_{ijlhkv} - C_{ijlhkv})Q_{ijlhkv} - x_{ijlhkv}F_{ijlhkv}\Big)$$

where

$$x_{ilhkv} = \begin{cases} 1, & \text{if } Q_{ijlhkv} > 0; \\ 0, & \text{if } Q_{ijlhkv} = 0 \end{cases} \tag{11}$$

The constraints (12) guarantee that the total quantity of a given product j delivered from a DC l does not exceed this DCs capacity.

$$\sum_{i=1}^{n}\sum_{h=1}^{m_l}\sum_{k=1}^{u_{hl}}\sum_{v=1}^{r} Q_{ijlhkv} \le a_{jl}, \begin{array}{l} j = 1, 2, \cdots, p, \\ l = 1, 2, \cdots, m \end{array} \tag{12}$$

The constraints (13) ensure that each delivery point i receives the requested quantity of each product j.

$$\sum_{l=1}^{u}\sum_{h=1}^{m_l}\sum_{k=1}^{u_{hl}}\sum_{v=1}^{r} Q_{ijlhkv} \ge b_{ij}, \begin{array}{l} i = 1, 2, \cdots, n, \\ j = 1, 2, \cdots, p \end{array} \tag{13}$$

Constraints (14) and (15) impose the vehicle capacity constraints for each trip, in terms of weight (Q_h) and volume (V_h)

$$\sum_{j=1}^{p} w_j Q_{ijlhkv} \le Q_h x_{ilhkv}, \begin{array}{l} i = 1, 2, \cdots, n, \\ l = 1, 2, \cdots, m \\ h = 1, 2, \cdots, m_l \end{array} \begin{array}{l} k = 1, 2, \cdots, u_{hl}, \\ v = 1, 2, \cdots, r \end{array} \tag{14}$$

$$\sum_{j=1}^{p} s_j Q_{ijlhkv} \le V_h x_{ilhkv}, \begin{array}{l} i = 1, 2, \cdots, n, \\ l = 1, 2, \cdots, m \\ h = 1, 2, \cdots, m_l \end{array} \begin{array}{l} k = 1, 2, \cdots, u_{hl}, \\ v = 1, 2, \cdots, r \end{array} \tag{15}$$

Finally, the constraints (16) is non-negativity constraints on the quantity and routing variables

$$Q_{ijlhkv} \in R^{+}, \begin{array}{ll} i = 1, 2, \cdots, n, & h = 1, 2, \cdots, m_l, \\ j = 1, 2, \cdots, p, & k = 1, 2, \cdots, u_{hl}, \\ l = 1, 2, \cdots, m & v = 1, 2, \cdots, r \end{array} \tag{16}$$

The above single objective profit maximization problem in Eq. (15) together with constraints from the Eqs. (11) to (16) has been solved using GRG technique.

4 Numerical Experiment

Haldia mustard oil producing factory produces different types of oil packets and transport into nearest markets. The factory wishes to maximize the profit. Hence the following parameters are assumed for two origins, two destinations, and two types of conveyances or vehicles. Here $n = 2, u = 2, p = 2, m_l = 2, u_{hl} = 2$. Let $F_{ijlhkv} = 240$ and $Q_1 = 300, Q_1 = 600, V_1 = 400, V_2 = 800$. The selling price of oil packets are given Table 4.

Table 4. Selling price for each items

20	11	12	18	10	18	17	21
18	19	12	11	15	20	17	29
22	19	11	23	10	19	21	29
29	18	12	18	19	25	17	29
22	18	10	14	17	15	19	31
22	19	17	18	21	19	25	23
23	24	16	28	19	25	17	19
21	19	16	18	21	19	17	29

5 Results and Discussion

On the basis of the input parameters, we have optimized the total profit function and the optimum results are given in Tables 5 and 6. From Table 6, it is cleared that the total profit decreases as fixed charge cost increases. It's a natural that ideas are shown in this proposed paper. The consequences of unit costs of transportation are obtained from IT2FLS. There revolutionizes are graphically shown in Figs. 5 and 6. The % change of equal selling prise of each oil packets is \$15, the corresponding changes of total profit are shown in Table 7. If selling prise increase, the total profit should increase. These events follow the present model.

Table 5. Total received amount during transportation when different FC is applied

FC	TP	Received amount	FC	TP	Received amount
5	7809.287	680	9	7709.345	673.01
6	7784.276	678.23	10	7684.623	672.85
7	7759.843	677.01	15	7559.432	671.43
8	7734.421	675.28	20	7434.987	670.89

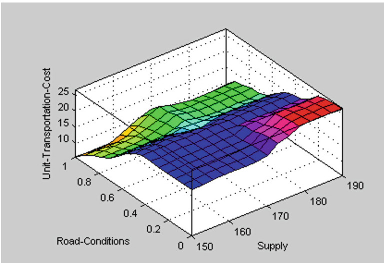

Fig. 5. Unit transportation with respect to supply and road condition

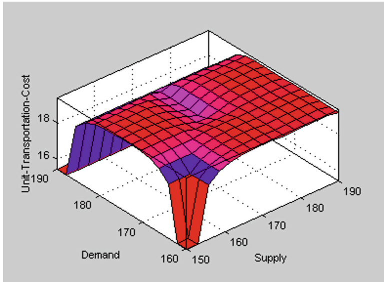

Fig. 6. Unit transportation with respect to demand and supply

Table 6. Total received amount during transportation when FC is 6$

Variables	Amount	Variables	Amount	Variables	Amount
Q(1, 1, 1, 1, 1, 1)	25	Q(2, 1, 1, 1, 1, 1)	0	Q(1, 2, 1, 2, 1, 2)	0
Q(1, 1, 1, 1, 1, 2)	25	Q(2, 1, 1, 1, 1, 2)	0	Q(1, 2, 1, 2, 2, 1)	23.07692
Q(1, 1, 1, 1, 2, 1)	0	Q(2, 1, 1, 1, 2, 1)	0	Q(1, 2, 1, 2, 2, 2)	34.29204
Q(1, 1, 1, 1, 2, 2)	0	Q(2, 1, 1, 1, 2, 2)	0	Q(1, 2, 2, 1, 1, 1)	23.07692
Q(1, 1, 1, 2, 1, 1)	0	Q(2, 1, 1, 2, 1, 1)	0	Q(1, 2, 2, 1, 1, 2)	0
Q(1, 1, 1, 2, 1, 2)	0	Q(2, 1, 1, 2, 1, 2)	0	Q(1, 2, 2, 1, 2, 1)	0
Q(1, 1, 1, 2, 2, 1)	25	Q(2, 1, 1, 2, 2, 1)	50	Q(1, 2, 2, 1, 2, 2)	16.57659
Q(1, 1, 1, 2, 2, 2)	0	Q(2, 1, 1, 2, 2, 2)	50	Q(1, 2, 2, 2, 1, 1)	0
Q(1, 1, 2, 1, 1, 1)	0	Q(2, 1, 2, 1, 1, 1)	0	Q(1, 2, 2, 2, 1, 2)	39.90061
Q(1, 1, 2, 1, 1, 2)	0	Q(2, 1, 2, 1, 1, 2)	0	Q(1, 2, 2, 2, 2, 1)	0
Q(1, 1, 2, 1, 2, 1)	5	Q(2, 1, 2, 1, 2, 1)	10	Q(1, 2, 2, 2, 2, 2)	0
Q(1, 1, 2, 1, 2, 2)	0	Q(2, 1, 2, 1, 2, 2)	0	Q(2, 2, 1, 2, 1, 2)	24.19048
Q(1, 1, 2, 2, 1, 1)	0	Q(2, 1, 2, 2, 1, 1)	0	Q(2, 2, 1, 2, 2, 1)	0
Q(1, 1, 2, 2, 1, 2)	0	Q(2, 1, 2, 2, 1, 2)	0	Q(2, 2, 1, 2, 2, 2)	0
Q(1, 1, 2, 2, 2, 1)	50	Q(2, 1, 2, 2, 2, 1)	50	Q(2, 2, 2, 1, 1, 1)	0
Q(1, 1, 2, 2, 2, 2)	50	Q(2, 1, 2, 2, 2, 2)	0	Q(2, 2, 2, 1, 1, 2)	0
Q(1, 2, 1, 1, 1, 1)	0	Q(2, 2, 1, 1, 1, 1)	23.07692	Q(2, 2, 2, 1, 2, 1)	7.8762

Table 6. (*continued*)

Variables	Amount	Variables	Amount	Variables	Amount
Q(1, 2, 1, 1, 1, 2)	0	Q(2, 2, 1, 1, 1, 2)	7.104896	Q(2, 2, 2, 1, 2, 2)	12.19
Q(1, 2, 1, 1, 2, 1)	0	Q(2, 2, 1, 1, 2, 1)	7.104896	Q(2, 2, 2, 2, 1, 1)	46.153
Q(1, 2, 1, 1, 2, 2)	23.076	Q(2, 2, 1, 1, 2, 2)	23.076	Q(2, 2, 2, 2, 1, 2)	0
Q(1, 2, 1, 2, 1, 1)	0	Q(2, 2, 1, 2, 1, 1)	0	Q(2, 2, 2, 2, 2, 1)	0
-	-	-	-	Q(2, 2, 2, 2, 2, 2)	29.22316

Table 7. The change of Total profit with respect to the % change of selling price

% change selling Price	5	10	15	20	25	30	35	40
Total Profit (TP)	4943.83	5461.74	5978	6496.28	7013	7531.59	8048.42	8566.64
% cange of TP	11.70	23.40	35.07	46.78	58.45	70.17	81.84	93.55

Table 8. Linguistic variables (LVs) for out put IT2F variable unit transportation cost

LVs	Unit transportation cost
Very low	[−10.28 −0.6111 9.056 −7.056 2.611 12.28]
Low	[−0.6111 9.056 18.72 2.611 12.28 21.94]
Medium	[9.056 18.72 28.39 12.28 21.94 31.61]
High	[18.72 28.39 38.06 21.94 31.61 41.28]

6 Conclusions and Future Research Work

The solid transportation problems have various applications in logistics and supply chains for reducing costs and maximizing total profit. In real-life situations, the parameters of transportation problems may not be known precisely because of uncontrollable factors. Herein, we propose a new method for solving profit solid transportation problems (PSTPs) in which the transportation costs and supply, demand and unit transportation costs are represented by non Interval type-2 fuzzy (IT2F) numbers. The IT2F parameters are controlled Interval type-2 fuzzy logic control (T2FLC) approach. Finally, the PSTP problem has been solved using GRG technique.A numerical example has been provided to illustrate our solution procedure. Besides that, our paper has some limitations in terms of including only one objective function and considering the IT2F parameters as IT2FVs forms. For future work this method can be extended to solve multi-objective form of STP for other types of fuzzy numbers (Table 8).

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Novel Derivations and Application of Complex LR Numbers on Fully Fuzzy Complex Linear System

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Abstract. In this paper, a fully fuzzy complex system of linear equations of the form $\tilde{A}\tilde{X} = \tilde{B}$ is presented, where $\tilde{A}$ is an LR fuzzy complex matrix, $\tilde{X}$ is an unknown LR fuzzy complex vector and $\tilde{B}$ is a known LR fuzzy complex vector. The definition of its fuzzy complex solution is proposed and discussion on a direct solution method of the fully fuzzy complex system of equation is discussed. Conditions on existence and uniqueness of fuzzy complex solution have been investigated. Numerical examples are presented to justify the applicability of the proposed method.

Keywords: LR fuzzy number · Fuzzy system of linear equations · Fuzzy complex number · Fuzzy complex system of linear equations · Fully fuzzy complex system of linear equations

1 Introduction

System of linear equations play an important role in solving problems in the field of engineering, economics, physics, chemistry, social sciences etc. Many real life situations demand the parameters to be uncertain or vague which can be achieved by use of fuzzy numbers. Solution procedure of $n \times n$ fuzzy linear system was first investigated by Friedman et al. [22]. Many researchers proposed analytic and iterative methods for solving real fuzzy linear systems [1–4,13]. Solution procedures of fully fuzzy linear systems can be found in the literature in [5,14–16]. Till date very limited work has been published on complex fuzzy system with almost no mention of fully fuzzy complex system of linear equations.

The concept of fuzzy complex numbers was introduced by Buckley [10] in 1989. Buckley defined a fuzzy complex number $\tilde{z}$ by its membership function $\mu_{\tilde{z}}$

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which is a mapping from the set of complex numbers C into $[0, 1]$ and discussed the arithmetic of fuzzy complex numbers through the idea of its α-cuts. Buckley and Qu [11] and later Buckley [12] carry forwarded the study on fuzzy complex analysis by developing the theory of differentiation and integration involving fuzzy complex numbers. Qiu et al. [27] further studied the convergence of the sequence and series of fuzzy complex numbers. Qiu et al. [28] mapped complex numbers into fuzzy complex numbers and fuzzy complex numbers into complex numbers and restudied the continuity and differentiability of such mapping functions.

In [17], Djanybekov considered interval Householder method for outer estimation of solution sets for system of linear equations with complex interval parameters. A few years later, Rahgooy et al. [29] investigated the application of system of linear equations $\tilde{A}X = \tilde{B}$ in the circuit analysis (CA), where the elements of the coefficient matrix $\tilde{A}$ and the constant column $\tilde{B}$ were considered to be fuzzy complex numbers. They used the embedding method employed by Friedman et al. [22] which involved replacing the original fuzzy complex linear system with a crisp linear system and then solving the crisp linear system. This study led to the beginning of research on fuzzy complex system of linear equations. Jahantigh [26] proposed a numerical method for solution of a fuzzy complex system of linear equations. Hlalik [25] investigated solution sets of systems of complex interval equations, where the form of the complex intervals is rectangular. Based on the concept of fuzzy center and width, Behera and Chakraberty [6] developed a procedure for solving real and complex fuzzy system of linear equations. Later, Behera and Chakraberty [7] defined fuzzy complex center and used the idea to present solution to fuzzy complex system. In [8], Behera and Chakraberty developed a new method for solving fuzzy complex system and applied it to an electric circuit and later in [9], they corrected a few definitions and theorems that were presented in [8]. Recently, Guo et al. [23] proposed a numerical procedure for calculating the complex fuzzy solution to an LR complex linear systems and presented a sufficient condition for the existence of strong fuzzy solution. In [30], Zang and Guo investigated QR-decomposition method for solving complex fuzzy linear equation and in [21], Farahani et al. developed an algorithm of finding all the solutions of complex fuzzy linear equation based on the eigenvalue method. Very recently, Guo et al. [24] presented a general method of solution to an LR complex fuzzy matrix equation $\tilde{Z}C = \tilde{W}$.

The lack of research on fully fuzzy complex system of linear equations motivated our work towards bridging this gap. A system of fully fuzzy complex linear equations is presented and its solution procedure is proposed. Condition for existence and uniqueness of the solution has also been developed.

2 Preliminaries

In this section, some basic concepts, notations and background of fuzzy set theory are recalled for better understanding of the paper [18–20].

2.1 Fuzzy Number

Definition 1. ***(Fuzzy set)*** *Let X be an universal set. Then a subset $\tilde{A}$ of X is called a fuzzy subset of X if $\tilde{A}$ is represented by its membership function $\mu_{\tilde{A}}$, where the function $\mu_{\tilde{A}} : X \to [0,1]$ assigns a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$, to each element $x \in X$. The real number $\mu_{\tilde{A}}(x)$ denotes the grade of membership of x in $\tilde{A}$.*

Definition 2. ***(Fuzzy number)*** *A fuzzy number $\tilde{A}$ is a fuzzy subset of the real number line R such that $\mu_{\tilde{A}} : R \to [0,1]$, where $\mu_{\tilde{A}}$ is the membership function of the subset $\tilde{A}$, satisfying the following properties:*

1. *$\mu_{\tilde{A}}$ is upper semi-continuous.*
2. *$\mu_{\tilde{A}}$ is a fuzzy convex set, i.e., $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ for all $x, y \in R$ and $\lambda \in [0,1]$.*
3. *$\mu_{\tilde{A}}$ is normal, i.e., there exists $r \in R$ such that $\mu_{\tilde{A}}(r) = 1$.*
4. *The support of $\mu_{\tilde{A}}$ is given by $supp(\mu_{\tilde{A}}) = \{x \in R | \mu_{\tilde{A}}(x) > 0\}$ and its closure $cl(supp(\mu_{\tilde{A}}))$ is a compact set.*

Let the set of all fuzzy numbers on R be denoted by E^1.

Definition 3. ***(LR fuzzy number)*** *A fuzzy number $\tilde{A}$ is called a Left-Right fuzzy number (LR fuzzy number) if its membership function $\mu_{\tilde{A}}$ is given by:*

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\alpha}),\ x \leq a, \alpha > 0 \\ R(\frac{x-a}{\beta}),\ x \geq a, \beta > 0 \end{cases}$$

where $a \in R$ is the mean value of $\tilde{A}$, $\alpha \in R$ and $\beta \in R$ are the left and right spreads respectively. The function $L(.)$ is called the left shape function satisfying the following:

1. *L(x) = L(-x).*
2. *L(0) = 1 and L(1) = 0.*
3. *L(x) is non-increasing on [0,1).*

The right shape function $R(.)$ is defined in similar manner as that of $L(.)$. An LR fuzzy number $\tilde{A}$ is symbolically written as $\tilde{A} = (a, \alpha, \beta)_{LR}$.

Definition 4. ***(Positive and negative fuzzy number)*** *A fuzzy number $\tilde{A}$ is called a positive (or negative) fuzzy number i.e $\tilde{A} > 0 (\tilde{A} < 0)$ if its membership function $\mu_{\tilde{A}}$ holds the condition $\mu_{\tilde{A}} = 0$ for $x < 0$ (or $\mu_{\tilde{A}} = 0$ for $x > 0$) to be true.*

A LR fuzzy number $\tilde{A} = (a, \alpha, \beta)_{LR}$ is positive if and only if $a - \alpha \geq 0$.

Definition 5. ***(Equal LR fuzzy numbers)*** *Two LR fuzzy numbers $\tilde{A} = (a, \alpha, \beta)_{LR}$ and $\tilde{B} = (b, \gamma, \delta)_{LR}$ are said to be equal if $a = b, \alpha = \gamma$ and $\beta = \delta$.*

On the basis of the extension principle, Dubois and Prade [20] developed exact formulas for addition, subtraction and scalar multiplication and approximate formula for multiplication of LR fuzzy numbers.

Definition 6. ***(Basic operations on LR fuzzy numbers)*** *For two LR fuzzy numbers* $\tilde{A} = (a, \alpha, \beta)_{LR}$ *and* $\tilde{B} = (b, \gamma, \delta)_{LR}$*, the arithmetic operations are as:*

- ***Addition:*** $\tilde{A} \oplus \tilde{B} = (a + b, \alpha + \gamma, \beta + \delta)_{LR}$
- ***Opposite:*** $-\tilde{A} = -(a, \alpha, \beta)_{LR} = (-a, \beta, \alpha)_{RL}$
- ***Subtraction:*** $\tilde{A} \ominus \tilde{B} = (a - b, \alpha + \delta, \beta + \gamma)_{LR}$
- ***Scalar Multiplication:*** *Let* $\lambda \in R$*. Then,*
  $\lambda \otimes \tilde{A} = \begin{cases} (\lambda a, \lambda\alpha, \lambda\beta)_{LR}, \lambda \geq 0, \\ (\lambda a, -\lambda\beta, -\lambda\alpha)_{RL}, \lambda \leq 0 \end{cases}$
- ***Approximate Multiplication:***
  If $\tilde{A} \geq 0$ *and* $\tilde{B} \geq 0$ *then,*
  $\tilde{A} \otimes \tilde{B} \approx (ab, a\gamma + b\alpha, a\delta + b\beta)$
  If $\tilde{A} \leq 0$ *and* $\tilde{B} \leq 0$ *then,*
  $\tilde{A} \otimes \tilde{B} \approx (ab, -b\beta - a\delta, -b\alpha - a\gamma)$
  If $\tilde{A} \geq 0$ *and* $\tilde{B} \leq 0$ *then,*
  $\tilde{A} \otimes \tilde{B} \approx (ab, b\alpha - a\delta, b\beta - a\gamma)$

Definition 7. ***(Fuzzy complex number)*** *An arbitrary LR fuzzy complex number* $\tilde{A}$ *can be represented as* $\tilde{A} = \tilde{p} + i\tilde{q}$*, where* $\tilde{p} = (p, p^l, p^r)$ *and* $\tilde{q} = (q, q^l, q^r)$ *and we write,* $\tilde{A} = (p, p^l, p^r) + i(q, q^l, q^r)$

Definition 8. ***(Fuzzy complex arithmetic)*** *Let* $\tilde{A} = \tilde{p}_1 + i\tilde{q}_1$ *and* $\tilde{B} = \tilde{p}_2 + i\tilde{q}_2$ *be two fuzzy complex numbers where* $\tilde{p}_1, \tilde{q}_1, \tilde{p}_2$ *and* $\tilde{q}_2$ *are all fuzzy numbers. The the fuzzy complex addition and multiplication is defined as:*

$$\tilde{A} \oplus \tilde{B} = (\tilde{p}_1 + i\tilde{q}_1) \oplus (\tilde{p}_2 + i\tilde{q}_2) = (\tilde{p}_1 \oplus \tilde{p}_2) + i(\tilde{q}_1 \oplus \tilde{q}_2)$$
$$\tilde{A} \otimes \tilde{B} = (\tilde{p}_1 + i\tilde{q}_1) \otimes (\tilde{p}_2 + i\tilde{q}_2) = (\tilde{p}_1 \otimes \tilde{p}_2 - \tilde{q}_1 \otimes \tilde{q}_2) + i(\tilde{p}_1 \otimes \tilde{q}_2 + (\tilde{q}_1 \otimes \tilde{p}_2)$$

2.2 Fuzzy Matrix System

Definition 9. ***(Fuzzy number matrix)*** *A matrix* $\tilde{A} = (\tilde{a}_{ij})_{n \times m}$ *is called a fuzzy matrix if* $\tilde{a}_{ij}$ *are fuzzy numbers for all* $1 \leq i \leq n$ *and* $1 \leq j \leq m$*.*

The matix $\tilde{A}$ *is said to be a positive (or negative) fuzzy matrix if all* $\tilde{a}_{ij}$ *are positive (or negative) fuzzy numbers and it is denoted as* $\tilde{A} > 0$*(or* $\tilde{A} < 0$*). Similar definitions can be stated for non-negative and non-positive fuzzy matrices.*

Definition 10. ***(Fuzzy matrix multiplication)*** *Let* $\tilde{A} = (\tilde{a}_{ij})_{n \times m}$ *and* $\tilde{B} = (\tilde{b}_{ij})_{m \times p}$ *be two LR fuzzy matrices. Then we define the matrix multiplication* $\tilde{A} \otimes \tilde{B} = \tilde{C}$ *where* $\tilde{C} = (\tilde{c}_{ij})_{n \times p}$ *and*

$$\tilde{c}_{ij} = \sum_{k=1,2,\cdots,n}^{\oplus} (\tilde{a}_{ik} \otimes \tilde{b}_{kj})$$

In our paper, we will work with positive LR fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ where each $\tilde{a}_{ij}$ is an LR fuzzy number given by $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$. The matrix $\tilde{A}$ can be represented as $\tilde{A} = (A, M, N)$ where the $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are all square matrices of of order n.

3 Fully Fuzzy Complex System of Linear Equations

Consider a system of linear equations written as:

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2 \\ \cdots \\ \cdots \\ \tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \cdots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n \end{cases} \tag{1}$$

where $\tilde{a}_{ij}, 1 \leq i, j \leq n$ are fuzzy complex numbers, the right hand side elements $\tilde{b}_i, 1 \leq i \leq n$ are fuzzy complex numbers and the unknown variables $\tilde{x}_i, 1 \leq i \leq n$ are also fuzzy complex numbers. Then such a system of linear equations is called a fully fuzzy complex system (FFCS) of linear equations.

The fuzzy complex elements $\tilde{a}_{ij}, 1 \leq i, j \leq n$; $\tilde{b}_i, 1 \leq i \leq n$ and $\tilde{x}_i, 1 \leq i \leq n$ can be written as:

$$\begin{cases} \tilde{a}_{ij} = \tilde{p}_{ij} + i\tilde{q}_{ij}, \ 1 \leq i, j \leq n \\ \tilde{b}_i = \tilde{m}_i + i\tilde{n}_i, \quad 1 \leq i \leq n \\ \tilde{x}_j = \tilde{y}_j + i\tilde{z}_j, \quad 1 \leq j \leq n \end{cases} \tag{2}$$

where $\tilde{p}_{ij}, \tilde{q}_{ij}, \tilde{m}_i, \tilde{n}_i, \tilde{y}_i$ and $\tilde{z}_i$ are all fuzzy numbers.

The matrix notation for the FFCS (1) is:

$$\tilde{A}\tilde{X} = \tilde{B} \tag{3}$$

If all the entries of $\tilde{A}$, $\tilde{X}$ and $\tilde{B}$ are ≥ 0 then the system (3) is called a positive FFCS of linear equations.

If the vector $\tilde{X} = (\tilde{x}_j)_{1\times n}$ satisfies (1), and all entries of $\tilde{X} = (\tilde{x}_j)_{1\times n}$ are positive fuzzy complex numbers then $\tilde{X}$ is called the positive fuzzy solution. Otherwise it is called a non-fuzzy solution.

4 Proposed Method for Solving Fully Fuzzy Complex System

In this section a novel method of solving a positive LR FFCS of linear equations have been proposed. For clear understanding of the method, a few related definitions and theorems are presented.

Definition 11. ***(LR fuzzy complex number)*** *An LR fuzzy complex number $\tilde{A} = \tilde{p} + i\tilde{q}$, where $\tilde{p} = (p, p^l, p^r)$ and $\tilde{q} = (q, q^l, q^r)$, is called a positive LR fuzzy complex number if $\tilde{p}$ and $\tilde{q}$ are both positive LR fuzzy numbers.*

Definition 12. ***(LR fuzzy complex matrix)*** *An LR fuzzy complex matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ with $\tilde{a}_{ij} = \tilde{p}_{ij} + i\tilde{q}_{ij}$ where $\tilde{p}_{ij}$ and $\tilde{q}_{ij}$ are LR fuzzy numbers, is called a positive LR fuzzy complex matrix if $\tilde{p}$ and $\tilde{q}$ are both positive LR fuzzy numbers.*

Definition 13. ***(LR Fully fuzzy complex linear system)*** *LR FFCS of linear equations* $\tilde{A}\tilde{X} = \tilde{B}$, *where* $\tilde{A} = \tilde{P} + i\tilde{Q} = (P, P^l, P^r) + i(Q, Q^l, Q^r)$, $\tilde{B} = \tilde{M} + i\tilde{N} = (M, M^l, N^r) + i(N, N^l, N^r)$ *and* $\tilde{X} = \tilde{Y} + i\tilde{Z} = (Y, Y^l, Y^r) + i(Z, Z^l, Z^r)$, *is called a positive LR FFCS of linear equations if* $\tilde{A}$, $\tilde{B}$ *and* $\tilde{X}$ *are all positive LR fuzzy matrices.*

In this paper we consider a positive LR FFCS (1), where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n} = (\tilde{p}_{ij} + i\tilde{q}_{ij})_{n\times n}$ with $\tilde{p}_{ij}$ and $\tilde{q}_{ij}$ as positive LR fuzzy numbers, the right hand side column matrix $\tilde{B} = (\tilde{b}_i)_{n\times 1} = (\tilde{m}_i + i\tilde{n}_i)_{n\times 1}$ consisting of $\tilde{m}_i$ and $\tilde{n}_i$ as positive LR fuzzy numbers and the unknown column matrix $\tilde{X} = (\tilde{x}_i)_{n\times 1} = (\tilde{y}_i + i\tilde{z}_i)_{n\times 1}$ with $\tilde{y}_i$ and $\tilde{z}_i$ as positive LR fuzzy numbers.

Theorem 1. *The* $n \times n$ *positive LR FFCS of linear equations* $\tilde{A}\tilde{X} = \tilde{B}$ *represented explicitly in Eqs. (1) and (2), is equivalent to the system*

$$\tilde{A}'\tilde{X}' = \tilde{B}'$$

$$\textit{where}\, \tilde{A}' = \begin{pmatrix} \tilde{P} \; \tilde{Q} \end{pmatrix}, \tilde{X}' = \begin{pmatrix} \tilde{Y} & \tilde{Z} \\ -\tilde{Z} & \tilde{Y} \end{pmatrix}, \textit{and}\; \tilde{B}' = \begin{pmatrix} \tilde{M} \\ \tilde{N} \end{pmatrix} \tag{4}$$

Proof. The system $\tilde{A}\tilde{X} = \tilde{B}$ can be written as:

$$\begin{aligned} &(\tilde{P} + i\tilde{Q})(\tilde{Y} + i\tilde{Z}) = \tilde{M} + i\tilde{N} \\ \Rightarrow\; &(\tilde{P}\tilde{Y} - \tilde{Q}\tilde{Z}) + i(\tilde{P}\tilde{Z} + \tilde{Q}\tilde{Y}) = \tilde{M} + i\tilde{N} \end{aligned}$$

Comparing real and imaginary parts we have,

$$\begin{cases} \tilde{P}\tilde{Y} - \tilde{Q}\tilde{Z} = \tilde{M} \\ \tilde{P}\tilde{Z} + \tilde{Q}\tilde{Y} = \tilde{N} \end{cases}$$

$$\Rightarrow\; \begin{pmatrix} \tilde{P} \; \tilde{Q} \end{pmatrix} \begin{pmatrix} \tilde{Y} & \tilde{Z} \\ -\tilde{Z} & \tilde{Y} \end{pmatrix} = \begin{pmatrix} \tilde{M} \\ \tilde{N} \end{pmatrix}$$

Expressing the system in matrix form, we have

$$\tilde{A}'\tilde{X}' = \tilde{B}'$$

which is a $6n \times 6n$ system,

$$\text{where}\, \tilde{A}' = \begin{pmatrix} \tilde{P} \; \tilde{Q} \end{pmatrix}, \tilde{X}' = \begin{pmatrix} \tilde{Y} & \tilde{Z} \\ -\tilde{Z} & \tilde{Y} \end{pmatrix}, \tilde{B}' = \begin{pmatrix} \tilde{M} \\ \tilde{N} \end{pmatrix}$$

Theorem 2. *The $6n \times 6n$ positive LR FFCS $\tilde{A}'\tilde{X}' = \tilde{B}'$ can be extended to the system*

$$AX = B$$

$$where\, A = \begin{pmatrix} P & -Q & O & O & O & O \\ Q & P & O & O & O & O \\ P^l & Q^r & P & O & O & Q \\ Q^l & P^l & Q & P & O & O \\ P^r & Q^l & O & Q & P & O \\ Q^r & P^r & O & O & Q & P \end{pmatrix}, X = \begin{pmatrix} Y \\ Z \\ Y^l \\ Z^l \\ Y^r \\ Z^r \end{pmatrix}, B = \begin{pmatrix} M \\ N \\ M^l \\ N^l \\ M^r \\ N^r \end{pmatrix} \quad (5)$$

with $P = (p_{ij})_{n\times n}$, $Q = (q_{ij})_{n\times n}$, $P^l = (p^l_{ij})_{n\times n}$, $Q^l = (q^l_{ij})_{n\times n}$, $P^r = (p^r_{ij})_{n\times n}$, $Q^r = (q^r_{ij})_{n\times n}$, $Y = (y_i)_{n\times 1}$, $Z = (z_i)_{n\times 1}$, $Y^l = (y^l_i)_{n\times 1}$, $Z^l = (z^l_i)_{n\times 1}$, $Y^r = (y^l_i)_{n\times 1}$ and $Z^r = (z^r_{ij})_{n\times n}$.

Proof.

$$\begin{aligned} &\tilde{A}'\tilde{X}' = \tilde{B}' \\ \Rightarrow &\begin{cases} \tilde{P}\tilde{Y} - \tilde{Q}\tilde{Z} = \tilde{M} \\ \tilde{P}\tilde{Z} + \tilde{Q}\tilde{Y} = \tilde{N} \end{cases} \\ \Rightarrow &\begin{cases} (P, P^l, P^r)(Y, Y^l, Y^r) - (Q, Q^l, Q^r)(Z, Z^l, Z^r) = (M, M^l, M^r) \\ (P, P^l, P^r)(Z, Z^l, Z^r) + (Q, Q^l, Q^r)(Y, Y^l, Y^r) = (N, N^l, N^r) \end{cases} \end{aligned}$$

Applying basic operations on LR fuzzy numbers discussed in Definition (6) we get,

$$\Rightarrow \begin{cases} (PY, P^lY + PY^l, P^rY + PY^r) - (QZ, Q^lZ + QZ^l, Q^rZ + QZ^r) = \\ (M, M^l, M^r) \\ (PZ, P^lZ + PZ^l, P^rZ + PZ^r) + (QY, Q^lY + QY^l, Q^rY + QY^r) = \\ (N, N^l, N^r) \end{cases}$$

$$\Rightarrow \begin{cases} (PY - QZ, P^lY + PY^l + Q^rZ + QZ^r, P^rY + PY^r + Q^lZ + QZ^l) = \\ (M, M^l, M^r) \\ (PZ + QY, P^lZ + PZ^l + Q^lY + QY^l, P^rZ + PZ^r + Q^rY + QY^r) = \\ (N, N^l, N^r) \end{cases}$$

$$\Rightarrow \begin{cases} PY - QZ = M \\ PZ + QY = N \\ P^lY + PY^l + Q^rZ + QZ^r = M^l \\ P^lZ + PZ^l + Q^lY + QY^l = N^l \\ P^rY + PY^r + Q^lZ + QZ^l = M^r \\ P^rZ + PZ^r + Q^rY + QY^r = N^r \end{cases}$$

Rearranging the system taking O as an $n \times n$ zero matrix, we get

$$\Rightarrow \begin{cases} PY - QZ + OY^l + OZ^l + OY^r + OZ^r = M \\ QY + PZ + OY^l + OZ^l + OY^r + OZ^r = N \\ P^lY + Q^rZ + PY^l + OZ^l + OY^r + QZ^r = M^l \\ Q^lY + P^lZ + QY^l + PZ^l + OY^r + OZ^r = N^l \\ P^rY + Q^lZ + OY^l + QZ^l + PY^r + OZ^r = M^r \\ Q^rY + P^rZ + OY^l + OZ^l + QY^r + PZ^r = N^r \end{cases}$$

$$\Rightarrow \begin{pmatrix} P & -Q & O & O & O & O \\ Q & P & O & O & O & O \\ P^l & Q^r & P & O & O & Q \\ Q^l & P^l & Q & P & O & O \\ P^r & Q^l & O & Q & P & O \\ Q^r & P^r & O & O & Q & P \end{pmatrix} \begin{pmatrix} Y \\ Z \\ Y^l \\ Z^l \\ Y^r \\ Z^r \end{pmatrix} = \begin{pmatrix} M \\ N \\ M^l \\ N^l \\ M^r \\ N^r \end{pmatrix}$$

$$\Rightarrow AX = B$$

The linear system $AX = B$ is called the associated linear system of the FFCS $\tilde{A}\tilde{X} = \tilde{B}$.

If we can now show that solving the FFCS $\tilde{A}\tilde{X} = \tilde{B}$ is equivalent to solving the associated linear system $AX = B$, then the difficult task of dealing with fuzzy numbers can be simplified to working with only crisp values.

Theorem 3. *The solution to the fuzzy complex system of equations $\tilde{A}\tilde{X} = \tilde{B}$ given by (1) is equivalent to the solution of the associated linear system of equations $AX = B$, given by (5).*

Proof. The fuzzy complex matrix system $\tilde{A}\tilde{X} = \tilde{B}$ can be explicitly written as:

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \cdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \cdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \cdots \\ \tilde{b}_n \end{pmatrix}$$

$$\Rightarrow \begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2 \\ \cdots \\ \tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \cdots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n \end{cases}$$

$$\Rightarrow \begin{cases} (\tilde{p}_{11} + i\tilde{q}_{11}) \otimes (\tilde{y}_1 + i\tilde{z}_1) + (\tilde{p}_{12} + i\tilde{q}_{12}) \otimes (\tilde{y}_2 + i\tilde{z}_2) + \cdots + \\ (\tilde{p}_{1n} + i\tilde{q}_{1n}) \otimes (\tilde{y}_n + i\tilde{z}_n) = \tilde{m}_1 + i\tilde{n}_1 \\ (\tilde{p}_{21} + i\tilde{q}_{21}) \otimes (\tilde{y}_1 + i\tilde{z}_1) + (\tilde{p}_{21} + i\tilde{q}_{21}) \otimes (\tilde{y}_2 + i\tilde{z}_2) + \cdots + \\ (\tilde{p}_{2n} + i\tilde{q}_{2n}) \otimes (\tilde{y}_n + i\tilde{z}_n) = \tilde{m}_2 + i\tilde{n}_2 \\ \cdots \\ (\tilde{p}_{n1} + i\tilde{q}_{n1}) \otimes (\tilde{y}_1 + i\tilde{z}_1) + (\tilde{p}_{n2} + i\tilde{q}_{n2}) \otimes (\tilde{y}_2 + i\tilde{z}_2) + \cdots + \\ (\tilde{p}_{nn} + i\tilde{q}_{nn}) \otimes (\tilde{y}_n + i\tilde{z}_n) = \tilde{m}_n + i\tilde{n}_n \end{cases}$$

Applying arithmetic operations on fuzzy complex numbers and comparing real and imaginary parts we get,

$$\begin{cases} \sum\limits_{j=1}^{n} [(\tilde{p}_{ij} \otimes \tilde{y}_j) \ominus (\tilde{q}_{ij} \otimes \tilde{z}_j)] = \tilde{m}_i, \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} [(\tilde{p}_{ij} \otimes \tilde{z}_j) \oplus (\tilde{q}_{ij} \otimes \tilde{y}_j)] = \tilde{n}_i, \text{for } 1 \le i \le n \end{cases}$$

$$\Rightarrow \begin{cases} \sum\limits_{j=1}^{n} [((p_{ij}, p^l_{ij}, p^r_{ij}) \otimes (y_j, y^l_j, y^r_j)) \ominus ((q_{ij}, q^l_{ij}, q^r_{ij}) \otimes (z_j, z^l_j, z^r_j))] = \\ (m_i, m^l_i, m^r_i), \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} [((p_{ij}, p^l_{ij}, p^r_{ij}) \otimes (z_j, z^l_j, z^r_j)) \ominus ((q_{ij}, q^l_{ij}, q^r_{ij}) \otimes (y_j, y^l_j, y^r_j))] = \\ (n_i, n^l_i, n^r_i), \text{for } 1 \le i \le n \end{cases}$$

Applying arithmetic operations on LR fuzzy numbers and comparing we get,

$$\begin{cases} \sum\limits_{j=1}^{n} (p_{ij} y_j - q_{ij} z_j) = m_i, \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} (p_{ij} z_j + q_{ij} y_j) = n_i, \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} (p^l_{ij} y_j + p_{ij} y^l_j + q_{ij} z^r_j + q^r_{ij} z_j) = m^l_i, \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} (p_{ij} z^l_j + p^l_{ij} z_j + q_{ij} y^l_j + q^l_{ij} y_j) = n^l_i, \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} (p^r_{ij} y_j + p_{ij} y^r_j + q_{ij} z^l_j + q^l_{ij} z_i) = m^r_i, \text{for } 1 \le i \le n \\ \sum\limits_{j=1}^{n} (p_{ij} z^r_j + p^r_{ij} z_j + q_{ij} y^r_j + q^r_{ij} y_j) = n^r_i, \text{for } 1 \le i \le n \end{cases} \tag{6}$$

Solving the above system for $y_i, z_i, y^l_i, z^l_i, y^r_i$ and z^r_i we get the unique solution of $\tilde{A}\tilde{X} = \tilde{B}$.

Next we consider the associated linear system $AX = B$ given by (5). This system can be easily decomposed into the system given by (6), thus proving that solution of $\tilde{A}\tilde{X} = \tilde{B}$ is equivalent to solving $AX = B$.

We now discuss briefly the operations on block matrices. The process for manipulation of block matrices is similar to that used to work with ordinary matrices. As a result, the elementary row and column operations for general matrices can be generalized to block/partitioned matrices as the following operations:

1. By interchanging two rows (or columns) of blocks.
2. By multiplying a row (or column) block from the left or right by a non-singular matrix of appropriate size.
3. By adding a multiplication of a row (or column) block by a non-zero matrix from the left or right, to another row (or column).

The FFLS (1) with the associated linear system (5) can be uniquely solved for the solution X if and only if the matrix A is non-singular. We, now have to investigate whether A is non-singular and in what condition can A be non-singular.

Theorem 4. *The block matrix A in (5) is non-singular if and only if the matrices P and $(P + QP^{-1}Q)$ are both non-singular.*

Proof. In the block matrix A, the matrices P, Q, P^l, Q^l, P^r and Q^r are square matrices of common order n and so we can easily transform the block matrix A to an lower triangular block matrix by subtracting the sixth row multiplied by QP^{-1} from the third row, and adding the second row multiplied by QP^{-1} with the first row. We thus obtain the following:

$$A = \begin{pmatrix} P & -Q & O & O & O & O \\ Q & P & O & O & O & O \\ P^l & Q^r & P & O & O & Q \\ Q^l & P^l & Q & P & O & O \\ P^r & Q^l & O & Q & P & O \\ Q^r & P^r & O & O & Q & P \end{pmatrix} \rightarrow A_1 = \begin{pmatrix} P & -Q & O & O & O & O \\ Q & P & O & O & O & O \\ P^l - QP^{-1}Q^r & Q^r - QP^{-1}P^r & P & O & O & O \\ Q^l & P^l & Q & P & O & O \\ P^r & Q^l & O & Q & P & O \\ Q^r & P^r & O & O & Q & P \end{pmatrix}$$

$$\rightarrow \ A_2 = \begin{pmatrix} P + QP^{-1}Q & O & O & O & O & O \\ Q & P & O & O & O & O \\ P^l - QP^{-1}Q^r & Q^r - QP^{-1}P^r & P & O & O & O \\ Q^l & P^l & Q & P & O & O \\ P^r & Q^l & O & Q & P & O \\ Q^r & P^r & O & O & Q & P \end{pmatrix}$$

The determinant of a triangular block matrix is equal to the product of the determinant of the diagonal matrices. Thus,

$$|A| = |A_2| = |P + QP^{-1}Q||P|^5$$

Therefore, $|A| \neq 0$ if and only if $|P + QP^{-1}Q| \neq 0$ and $|P| \neq 0$.
This completes the proof.

We thus have the condition for existence and uniqueness of solution to the system (1).

Corollary 1. *If the crisp linear system $AX = B$ does not have a unique solution, then the associated fuzzy complex linear system $\tilde{A}\tilde{X} = \tilde{B}$ does not have one either.*

4.1 Example 1

Solve the FFCS for non-negative solutions:

$$\begin{cases} (\tilde{5} + \tilde{1}i)\tilde{x}_1 + (\tilde{6} + \tilde{1}i)\tilde{x}_2 = \tilde{50} + \tilde{50}i \\ (\tilde{7} + \tilde{1}i)\tilde{x}_1 + (\tilde{4} + \tilde{0}i)\tilde{x}_2 = \tilde{48} + \tilde{48}i \end{cases}$$

We mean,

$$\begin{cases} ((5,0.1,0.5)+i(1,0.3,0.2))\otimes(\tilde{y}_1+i\tilde{z}_1)+((6,0.2,0.2)+i(1,0.1,0.3))\otimes(\tilde{y}_2+i\tilde{z}_2)= \\ (50,30,30)+i(50,30,30) \\ ((7,0.3,0.1)+i(1,0.5,0.1))\otimes(\tilde{y}_1+i\tilde{z}_1)+((4,0.4,0.3)+i(0,0,0.2))\otimes(\tilde{y}_1+i\tilde{z}_1)= \\ (48,28,28)+i(48,28,28) \end{cases}$$

Using the proposed method, using Eq. (5) we obtain the following matrices:

$$\tilde{P}=\begin{pmatrix}5 & 6\\ 7 & 4\end{pmatrix}, \tilde{P}^l=\begin{pmatrix}0.1 & 0.2\\ 0.3 & 0.4\end{pmatrix}, \tilde{P}^r=\begin{pmatrix}0.5 & 0.2\\ 0.1 & 0.3\end{pmatrix},$$

$$\tilde{Q}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}, \tilde{Q}^l=\begin{pmatrix}0.3 & 0.1\\ 0.5 & 0.2\end{pmatrix}, \tilde{Q}^r=\begin{pmatrix}0.2 & 0.3\\ 0.1 & 0.2\end{pmatrix},$$

$$\tilde{M}=\begin{pmatrix}50\\ 48\end{pmatrix}, \tilde{M}^l=\begin{pmatrix}30\\ 28\end{pmatrix}, \tilde{M}^r=\begin{pmatrix}30\\ 28\end{pmatrix}, \tilde{N}=\begin{pmatrix}50\\ 48\end{pmatrix}, \tilde{N}^l=\begin{pmatrix}30\\ 28\end{pmatrix}, \tilde{N}^r=\begin{pmatrix}30\\ 28\end{pmatrix}$$

We further obtain the equivalent linear system as:

$$\begin{pmatrix}
5 & 6 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
7 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 7 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.1 & 0.2 & 0.2 & 0.3 & 5 & 6 & 0 & 0 & 0 & 0 & 1 & 1\\
0.3 & 0.4 & 0.1 & 0.2 & 7 & 4 & 0 & 0 & 0 & 0 & 1 & 0\\
0.3 & 0.1 & 0.1 & 0.2 & 1 & 1 & 5 & 6 & 0 & 0 & 0 & 0\\
0.5 & 0.2 & 0.3 & 0.4 & 1 & 0 & 7 & 4 & 0 & 0 & 0 & 0\\
0.5 & 0.2 & 0.3 & 0.1 & 0 & 0 & 1 & 1 & 5 & 6 & 0 & 0\\
0.1 & 0.4 & 0.5 & 0.2 & 0 & 0 & 1 & 0 & 7 & 4 & 0 & 0\\
0.2 & 0.3 & 0.5 & 0.2 & 0 & 0 & 0 & 0 & 1 & 1 & 5 & 6\\
0.1 & 0.2 & 0.1 & 0.3 & 0 & 0 & 0 & 0 & 1 & 0 & 7 & 4
\end{pmatrix}
\begin{pmatrix} y_1\\ y_2\\ z_1\\ z_2\\ y_1^l\\ y_2^l\\ z_1^l\\ z_2^l\\ y_1^r\\ y_2^r\\ z_1^r\\ z_2^r \end{pmatrix}
=
\begin{pmatrix} 50\\ 48\\ 50\\ 48\\ 30\\ 28\\ 30\\ 28\\ 30\\ 28\\ 30\\ 28 \end{pmatrix} \tag{7}$$

On solving the above crisp linear system we obtain the solution as:

$$y_1=3.9540, y_1^l=1.5133, y_1^r=1.8379, y_2=6.2759, y_2^l=2.5278, y_2^r=1.9634,$$
$$z_1=4.7816, z_1^l=2.5278, z_1^r=2.5918, z_2=2.6437, z_2^l=1.4556, z_2^r=1.2745$$

$$\text{or } \tilde{X}=\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}=\begin{pmatrix}(3.9540,1.5133,1.8379)+i(4.7816,2.5278,2.5918)\\ (6.2759,2.5278,1.9634)+i(2.6437,1.4556,1.2745)\end{pmatrix}$$

5 Conclusion

In this paper the solution of a fully fuzzy complex system of linear equations has been investigated, where the fuzzy number are represented by positive LR fuzzy numbers. Condition for existence and uniqueness of solution have been discussed. However, our future work would focus on finding the necessary and sufficient condition for existence of a positive fuzzy complex solution for any fully fuzzy complex system and investigating cases of negative and no solution to a fully fuzzy complex system of equations.

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A Heuristic Approach for Cluster TSP

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Abstract. This investigation took an attempt to solve the cluster traveling salesman problem (CTSP) by the heuristic approach. In this problem, nodes are clustered with given a set of vertices (nodes). Given the set of vertices is divided into a prespecified number of clusters. The size of each cluster is also pre-specified. The main aim is to find the least cost Hamiltonian tour based on the given vertices. Here vertices of each cluster visited contiguously, and the clusters are visited in a specific order. Standard GA is used to find a Hamiltonian path for each cluster. The performance of the algorithm has been examined against two existing algorithms for some symmetric TSPLIB instances of various sizes. The computational results show the proposed algorithm works well among the studied metaheuristics regarding the best result and computational time.

Keywords: Cluster TSP · GA · Heuristic

1 Introduction

Traveling salesman problem (TSP) has many different variations. The clustered traveling salesman problem (CTSP) is one of them. At first, CTSP was proposed by Chisman [4]. Different approaches are taken by various researcher during last decades to solve cluster traveling salesman problem (CTSP). Few of them are New Hybrid Heuristic approach by Mestria [11], using Neighborhood Random Local Search a heuristic approach by Mestria [10], another approach is based on with *d-relaxed priority rule* by Phuong et al. [12], a Metaheuristic approach by Zhang et al. [15], applying *the Lin-Kernighan-Helsgaun Algorithm* by Helsgaun [5], etc. CTSP is defined as follows: consider a complete undirected graph G. Where, G = (V, E). Here V = set of vertices and E = set of edges. If the number of node is N, then V = $\{v_1, v_2, v_3, \cdots, v_N\}$ and it is divided into K prespecified clusters. The prespecified clusters are $\{C_1, C_2, C_3, \cdots, C_k\}$. A cost matrix COST = $[c_{ij}]$ is present. This matrix represents the travel costs, distances, or travel times which is defined on the edge set E = $\{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$. Till now, different variants of CTSP is available based on different conditions.

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Suppose the number of clusters is two then it is treated as TSP with backhauls (TSPB) [8]. In the case of free CTSP, the effective number of cluster is determined dynamically, not determined by prespecified order. The routing between clusters is also an important part of this paper. In the case of free CTSP, it is determined simultaneously. If all variations of CTSP are colligation of classical TSP, they are all NP-hard. In real life, CTSP is important, and it also has a huge application like vehicle routing [3], warehouse routing [7], integrated circuit testing [6], production planning [6], etc. Chisman [4] first proposed that CTSP can be represented as a TSP by adding or subtracting a big impulsive constant I to or from the cost of every inter-cluster edge. So, at the end of conversion, a specific algorithm for the TSP also apply to solve the problem precisely. The use of the heuristic procedure is practical in CTSP when the number of nodes is large or very large. Most common heuristic algorithms are approximate algorithms, artificial neural network, tabu search, genetic algorithm (GA) and so on. To solve TSP and its variation, Genetic Algorithm (GA) is treated as best. Now our proposed algorithm *Heuristic Approach* is a variation of GA to find the optimal solution of given problem. The effectiveness of our proposed algorithm has been compared against lexisearch algorithm (LSA) [1] and hybrid GA(HGA) [2] for few symmetric TSPLIB [13] instances. At last, we have taken a set of solutions of large size TSPLIB [13] instances and compared with Hybrid GA (HGA).

The proposed algorithm have following key features:

- Cluster creation
- Genetic Algorithm (GA)
    - Probabilistic selection
    - Cyclic crossover
    - Random crossing point
    - Random mutation
- Routing between clusters
- Test on TSPLIB instances

The present paper is prepared as follows: Sect. 1, a short introduction is produced. In Sect. 2, required mathematical pre-requisite. In Sect. 3, the proposed algorithm is presented. In Sect. 4, a numerical tests are finished. Again in Sect. 5, a brief discussion is given. Finally, in Sect. 6, a conclusion with future scope is studied.

2 Classical Definition of CTSP

The CTSP is outlined on a loop-free undirected graph G. Where, $G = (V, E)$. Here $V =$ set of vertex and $E =$ set of edge. If the number of node is N, then, $V = \{v_1, v_2, v_3, \cdots, v_N\}$ and it is divided into K cluster. Here, K is pre-specified. The pre-specified clusters are $\{C_1, C_2, C_3, \cdots, C_k\}$. A cost matrix COST $= [c_{ij}]$ between i^{th} and j^{th} node is present. This matrix represents the travel costs, which is defined on the edge set E $= \{(v_i, v_j) : v_i, v_j \in V, i \neq j\}$. There is a decision variable x_{ij}, $x_{ij}=1$ iff a tour completed between v_i to v_j, otherwise, $x_{ij}=0$. The framing of CTSP can be represented as follows:

$$\left.\begin{array}{c}\text{Minimize} \quad Z=\sum_{i\neq j} c(i,j)x_{ij} \\ \text{subject to} \quad \sum_{i=1}^{N} x_{ij}=1 \ \text{ for } j=1,2,...,N \\ \sum_{j=1}^{N} x_{ij}=1 \ \text{ for } i=1,2,...,N \\ \sum_{i\in v_k}\sum_{j\in v_k} x_{ij}=|v_k|, \forall |v_k|\subset V, |v_k|\geq 1, k=1,2,3,\cdots,m \\ where\ x_{ij}\in\{0,1\}, i,j=1,2,\cdots,N\end{array}\right\} \quad (1)$$

Then the above CTSP reduces to

$$\left.\begin{array}{c}\text{determine a complete tour} \quad (x_1,x_2,...,x_N,x_1) \\ \text{to minimize} \quad Z=\sum_{i=1}^{N-1} c(x_i,x_{i+1})+c(x_N,x_1) \\ where\ x_i\neq x_j, i,j=1,2...,N.\end{array}\right\} \quad (2)$$

along with sub tour elimination criteria

$$\sum_{i\in S}^{N}\sum_{j\in S}^{N} x_{ij}\leq |S|-1, \forall S\subset Q \quad (3)$$

3 Proposed Heuristic Based Genetic Algorithm

A well-known heuristic based GA is used for solve the CTSP. Using GA or any other heuristic method we get a better solution for small size TSP or medium size TSP very easily. But when the size of TSP has increased then complexity increases in a parallel way. To overcome this problem, TSP transformed to CTSP, which is a variation of TSP. The proposed algorithm performs in three steps. First, all nodes are divided into the pre-specified number of cluster. The number of nodes in each cluster may be the same or not. Second, each cluster is optimized using GA. Third, reconstruct a Hamiltonian cycle using all optimized cluster. All optimized cluster contains a Hamiltonian path, not cycle.

3.1 Cluster Creation

The number of clusters is pre-specified. At first, we ensure the size of each cluster. Then, selected nodes are inserted into each specified cluster. It is clear that every cluster must contain a unique set of nodes. That is after the optimization of each cluster, generate a different and unique Hamiltonian path.

Algorithm:

1. Begin.
2. State the number of cluster.
3. Generate a random number(r) between (0 to N−1).
4. Calculate the cost(c) from node r to each node.
5. Select a node in a cluster depend on minimum cost(c).
6. Ignore the node selected in step 5.
7. Calculate the cost(c) from node r to each remain node.
8. Repeat steps 5 to 7 until all nodes are distributed based on previous cluster size of each cluster.
9. End.

3.2 Genetic Algorithm(GA)

Proposed algorithm have the concept of the generation of the cluster. Here given nodes are divided between pre-specified clusters based on subsection 3.1. Initially, each cluster contains a number of nodes. Based on these nodes initial population is created randomly. Each cluster strictly follows this step, and strictly GA is applied to each cluster to produce a Hamiltonian path. i.e GA is applied to optimize each cluster. So our proposed GA is as follows.

Genetic Algorithm is a well-known randomized search method. There is a natural rule that, survival of the fittest among the species based on their gene architecture of the chromosomes. Gene structure constructed based on random change on it and it is evolved from one iteration(generation) to next. Every iteration with the following three operations.

(a) Selection: It is a stochastic process which simulates the quotation **survival -of-fittest**. An objective function took a vital role and based on it few chromosomes are copied from a predefined population of the chromosome. All selected chromosomes are used for the next operation. Our proposed algorithm uses the Boltzman's probabilistic selection process [9].

(b) Crossover: It is known as a binary operator. It works with a pair of parent chromosome. Parents are selected with a significant probability, and as a result, two new offspring chromosomes are prepared. Its importance in GA is very much. The proposed algorithm uses Cyclic Crossover [14] as a crossover operator.

(c) Mutation: It is known as a unary operator. It is applied to every chromosome with a small probability. The mutation also important part to diversify the GA search space. The proposed algorithm uses random mutation as a mutation operator.

GA starts with a randomly generated initial population and repeat the above three operations until the stopping criterion is satisfied. Crossover creates a new opportunity over GA generating new offspring chromosomes. An example of a successful heuristic algorithm to solve a classical TSP and its variations is GA. It never gives the guarantee about the optimal solution, but it can find a near-optimal solution in a concise time.

3.3 Inter Cluster Re-linking

We aim to find a Hamiltonian cycle. Optimized each cluster have a Hamiltonian path. To produce a Hamiltonian cycle we have maintained the following steps.

1. Store the number of cluster
2. Store each Hamiltonian path of each cluster
3. Calculate possible combinations of given clusters
4. Arrange the cluster sequence based on combination sequence
5. Merge each combination and prepare a final path
6. Calculate the cost of each combination
7. The Least cost combination is treated as best result of our proposed algorithm

3.4 Proposed Algorithm

1. Start
2. Input the number of cluster.
3. Define the size of each cluster.
4. To determine the nodes for each cluster, do following steps:
   (A) Generate a random number(r) between (0 to N-1).
   (B) Calculate the cost(c) from node r to each node.
   (C) Select a node in a cluster depend on minimum cost(c).
   (D) Ignore the node selected in step (C).
   (E) Calculate the cost(c) from node r to each remain node.
   (F) Repeat steps (C) to (E) until all nodes are distributed based on previous cluster size of each cluster.
5. After creation of each cluster with its respective nodes, a randomly generated population is prepared on the basis of stored nodes of each cluster.
6. Proposed GA is applied to each cluster to generate a Hamiltonian path based on the specified nodes of each cluster.
7. Prepare possible combinations of given clusters.
8. Calculate objective function value of each combination(path).
9. Find minimum cost(objective function value) among all combinations, this will be the best solution of our proposed algorithm.
10. Stop

4 Numerical Tests

Proposed algorithm is guided by few parameters, namely, crossover probability (p_c), mutation probability (p_m) and population size (pv) and also termination condition. Proper functioning of GA depends on a proper selection of these parameters. Table 1 shows the comparison of performance between proposed *Heuristic based GA* (HbGA), LSA [1] and HGA [2] also.

Table 2 shows a comparative study between HGA and HbGA based on symmetric TSPLIB instances. Taken TSPLIB instances are larger than TSPLIB instances of Table 1.

Table 1. A comparative study between LSA, HGA and HbGA for few symmetric TSPLIB instances.

Instances	Clusters	Solution			Error (%)	
		LSA	HGA	HbGA	HGA vs HbGA	LSA vs HbGA
ulysses16	2	7303	7303	7373	0.96	0.96
gr17	2	2517	2517	2256	−10.37	−10.37
gr21	2	3465	3465	3499	0.98	0.98
gr24	2	1558	1558	1526	−2.05	−2.05
fri26	2	957	957	1276	33.33	33.33
bay29	2	2144	2144	1931	−9.93	−9.93
	3	2408	2408	2105	−12.58	−12.58
bayg29	2	2702	2702	2473	−8.48	−8.48
	3	2991	2991	2608	−12.81	−12.81
swiss42	2	1605	1605	1801	12.21	12.21
	3	1919	1919	1963	2.29	2.29
	4	1944	1944	1982	1.95	1.95
gr48	2	6656	6433	6518	1.32	−2.07
	3	7466	7466	7345	−1.62	−1.62
	4	8554	8554	7820	−8.58	−8.58
eil51	2	570	564	555	−1.60	−2.63
	3	689	681	647	−4.99	−6.10
	4	714	714	659	−7.70	−7.70

Table 2. A comparative study between HGA and HbGA based on symmetric TSPLIB instances

Instances	Clusters	Solution (HGA)	Solution (HbGA)	Error (%)
berlin52	2	10422	10257	−1.58
st70	2	916	895	−2.29
eil76	2	721	750	4.02
kroA100	4	45733	37650	−17.67
kroB100	4	45709	38855	−14.99
kroC100	4	46388	46558	0.37
kroD100	4	45681	42085	−7.87
kroE100	4	45431	43847	−3.49
rd100	4	15501	14628	−5.63
eil101	4	1080	1050	−2.78
pr107	4	51487	56136	9.03
bier127	4	174112	178416	2.47

(*continued*)

Table 2. (*continued*)

Instances	Clusters	Solution (HGA)	Solution (HbGA)	Error (%)
ch130	4	12000	10566	−11.95
kroA150	4	52824	45355	−14.14
kroB150	4	54008	46955	−13.06
ch150	4	13042	12093	−7.28
d198	4	17320	20956	20.99
kroA200	4	62514	56442	−9.71
kroB200	4	62842	55145	−12.25
gil262	4	4874	4173	−14.38
pr264	4	60161	68531	13.91
rd400	4	30821	30223	−1.94
fl417	4	20457	22428	9.63

Table 3. Parameter study for **kroA100 instance**

Cluster	p_c	p_m	pop_{size}	result	cpu-$time_{sec}$	Error (%)
4	0.34	0.01	50	37747	12.97	−17.46
		0.02	50	35119	13.04	−23.21
		0.04	50	33109	12.75	−27.60
		0.05	50	36775	13.74	−19.59
		0.001	50	49489	14.49	8.21
		0.003	50	44311	15.88	−3.11
		0.004	50	43029	13.92	−5.91
		0.005	50	42167	14.05	−7.80
		0.007	50	37634	16.31	−17.71
		0.009	50	36746	12.49	−19.65
4	0.10	0.43	50	40969	15.41	−10.42
	0.30		50	38267	15.33	−16.33
	0.35		50	33809	20	−26.07
	0.40		50	42305	25.68	−7.40
	0.45		50	36714	18	−19.72
	0.50		50	32936	17.72	−27.98
	0.55		50	38228	22.81	−16.41
	0.65		50	42604	20.84	−6.84
	0.70		50	37751	29.26	−17.45
	0.75		50	35228	20.55	−22.97
	0.80		50	34014	20.60	−25.62

(*continued*)

Table 3. (*continued*)

Cluster	p_c	p_m	pop_{size}	result	cpu-time$_{sec}$	Error (%)
4	0.34	0.43	50	34715	16.40	−24.09
			55	37499	18.20	−18.00
			60	50034	19.23	9.40
			65	31569	22.18	−30.97
			70	33665	24.57	−26.39
			75	46783	28.07	2.30
			80	40573	27.23	−11.28
			85	31703	31.86	−30.68
			90	44526	32.87	−2.64

Table 4. Comparative result based on different sizes cluster ($p_c = 0.34$, p_m=0.43, $pop_{size} = 50$)

Instance	Cluster	result	$cpu - time_{sec}$
kroA100	2	31186	17.37
	3	38372	16.80
	4	34715	16.40
	5	51670	20.01
	6	36372	19.73
	7	53503	22.98
	8	45696	22.52
	9	49470	22.98
	10	45106	27.63

5 Discussion

This article is a special attempt to find out a way to solve a large scale TSP in a convenient way. Here we have chosen the way as a cluster TSP (CTSP). Our proposed HbGA algorithm is implemented by considering some parametric values as probability of crossover (p_c), probability of mutation (p_m), maximum number of chromosome as a population (pv) and maximum generation. This proposed algorithm is written in C++. It is clear from Table 1 that our proposed HbGA algorithm is much efficient than LSA and HGA both. Results shown in Table 1 based on 10 benchmark TSP references in TSPLIB. These ten instances are between 16 and 51 cities. It is remarkable that our proposed HbGA is much efficient for bays29 for 29 cities problem and eil51 for 51 cities problem also. Compare to both LSA and HGA using our proposed HbGA, we got better results than existing, which are illustrated in Table 1. Table 2 is also prove the efficiency of HbGA based on a comparative study of instances in TSPLIB between 52 and

417 cities. So, all over performance of HbGA is better than HGA. Table 3 is a parametric study based on standard TSPLIB instance of 100 cities. Table 3 represents better results considering four(4) clusters and all different combination of parametric values by using our proposed HbGA. Also it is remarkably mention that, we got these better results within less CPU time than existing. From Table 4 we can observe that cluster size two(2) gives the better results than cluster size four(4). From above discussion, we may come to an end that our proposed HbGA is also applicable for solving real life optimization problems.

6 Conclusion

The present study, a heuristic based genetic algorithm modeled to solve cluster TSP. Here we developed an alternative methodology, i.e., heuristic to the creation and re-linking the inter-cluster and used GA for optimizing the path in intra-cluster also. Finally, an optimized path is generated. Again different numbers of the cluster are investigated because of such realistic happening found in the small scale tourism industry. In the tourism industry, it oftenly found that a different number of sight scenery are the demand by every group of tourist. Since tourism is travel for pleasure and business, so management prepares different kinds of travel plan in that case such proposed cluster model effectively works. Without cluster attempt to solve such TSP using a heuristic process like using GA, is a big headache regarding CPU time and complexity. The main motto of our prescribed article is to demonstrate the efficiency of our proposed cluster TSP algorithm than any other conventional Genetic Algorithms. We got a set of the heuristic solution by applying our proposed GA on CTSP. The effectiveness of clustering method has been examined with both lexisearch algorithm (LSA) and OCTSP [2] for few small TSPLIB instances. The experiment shows that CTSP is better than LSA and HGA also. Few TSPLIB instances also compared with HGA and the overall result is good enough. In the future, we can extend the algorithm using fuzzy distance for cluster creation and dynamic relinking of the inter-cluster also.

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Superconvergence of Iterated Galerkin Method for a Class of Nonlinear Fredholm Integral Equations

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Abstract. In this paper, we consider the Galerkin and iterated Galerkin methods for solving Fredholm-Hammestein integral equations with a Green's kernel, whose first derivative has singularity. We obtain error bounds and convergence rates for both the Galerkin and iterated Galerkin methods using graded mesh. In fact, by choosing the grading exponent appropriately, we obtain superconvergence results in iterated Galerkin method.

Keywords: Hammerstein integral equations · Galerkin method · Green's kernel · Superconvergence rates

1 Introduction

Consider the following nonlinear integral equation

$$x(t) = \alpha_1 + \frac{(\eta_1 - \beta_1\alpha_1)t^{1-\alpha}}{\beta_1 + \gamma_1(1-\alpha)} + \int_0^1 G(t,\xi)\psi(\xi, x(\xi))d\xi, \tag{1.1}$$

where $\alpha_1, \beta_1 > 0, \gamma_1$ and η_1 are any finite real constants, and $G(t,\xi)$ is given by

$$G(t,\xi) = \begin{cases} \frac{t^{1-\alpha}}{1-\alpha}\left[1 - \frac{\beta_1\xi^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right], & 0 \le t \le \xi, \\ \frac{\xi^{1-\alpha}}{1-\alpha}\left[1 - \frac{\beta_1 t^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right], & \xi \le t \le 1. \end{cases} \tag{1.2}$$

The above type of integral equations arise as reformulation of the following type of singular two-point boundary value problem

$$(t^\alpha x')' = \psi(t,x),\ 0 \le \alpha < 1,\ t \in (0,1], \tag{1.3}$$

subject to the boundary conditions

$$x(0) = \alpha_1,\ \beta_1 x(1) + \gamma_1 x'(1) = \eta_1. \tag{1.4}$$

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Several phenomena in chemical, mechanical engineering, physics, physiological studies and many other applied science problems are frequently reduced to solve the boundary value problem (1.3)–(1.4), for example in the study of distribution of heat sources in the human head [9] and steady-state oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics [11]. In particular, when $\alpha = 0$ and $\psi(t,x) = t^{-\frac{1}{2}}x^{\frac{3}{2}}$, (1.3) is known as Thomas-Fermi equation [12]. The numerical treatment of the singular boundary value problems has always been far from trivial, because of the singularity. In the last few years, considerable effort has been made in developing methods for numerically solving different types of boundary value problems and associated integral equations (see for example [3,6,8–15]). The main difficulty of the problem (1.3) is that the singularity behavior occurs at $t = 0$. In [13], numerical solvability of the singular two-point boundary value problem (1.3)–(1.4) was discussed after constructing the equivalent nonlinear Fredholm integral equation (1.1) with Green's kernel (1.2), whose first derivative has weakly singularity at $t = 0$. In [3,4], projection and iterated projection methods are applied to nonlinear Fredholm integral equations with Green's kernel. Superconvergence of the iterated Galerkin method for solving nonlinear Fredholm integral equations with weakly singular kernels was discussed in [10].

In this paper, we apply Galerkin and iterated Galerkin methods to solve the nonlinear Fredholm integral equation of the type (1.1) with Green's kernel (1.2), and investigate the convergence rates with respect to graded meshes. Choosing the grading exponent corresponding to the graded meshes suitably, we are able to obtain superconvergence results for the iterated Galerkin approximations.

We organize this paper as follows. In Sect. 2, we discuss spline based Galerkin and iterated Galerkin methods to solve the Eq. (1.1). In Sect. 3, we obtain superconvergence results for the iterated Galerkin approximations. Throughout this paper, we assume that c is a generic constant.

2 Galerkin and Iterated Galerkin Methods

Let $\mathbb{X} = L^\infty[0,1]$ and consider the following nonlinear integral equation

$$x(t) = \alpha_1 + \frac{(\eta_1 - \beta_1\alpha_1)t^{1-\alpha}}{\beta_1 + \gamma_1(1-\alpha)} + \int_0^1 G(t,\xi)\psi(\xi, x(\xi))d\xi, \tag{2.1}$$

where $G(t,\xi)$ is the Green's function given by (1.2). Note that $G(t,\xi) \in \mathcal{C}([0,1] \times [0,1])$ and the first derivative of $G(t,\xi)$ has weak singularity at $t = 0$ and the maximum value of the Green's function (1.2) can easily be obtained (see [13]) and it is given by

$$M = \max_{0 \le t,\xi \le 1} |G(t,\xi)| = \frac{\beta_1 + \gamma_1(1-\alpha)}{4\beta_1(1-\alpha)} < \infty. \tag{2.2}$$

We denote

$$y(t) = \alpha_1 + \frac{(\eta_1 - \beta_1\alpha_1)t^{1-\alpha}}{\beta_1 + \gamma_1(1-\alpha)}, \ t \in [0,1]. \tag{2.3}$$

Then (2.1) takes the form

$$x(t) = y(t) + \int_0^1 G(t,\xi)\psi(\xi, x(\xi))d\xi. \tag{2.4}$$

Let us define the operator $\mathcal{K} : \mathbb{X} \to \mathbb{X}$ as

$$(\mathcal{K}\psi)(x)(t) = \int_0^1 G(t,\xi)\psi(\xi, x(\xi))d\xi.$$

Then $\mathcal{K}$ is a compact operator on $\mathbb{X}$ and the Eq. (2.1) can be written as

$$x - \mathcal{K}\psi(x) = y. \tag{2.5}$$

Letting $\mathcal{T}(x) := y + \mathcal{K}\psi(x)$, $x \in \mathbb{X}$, Eq. (2.5) can be written as

$$x = \mathcal{T}(x). \tag{2.6}$$

Throughout the paper, we assume that the nonlinear function $\psi(s,x)$ and its first derivative w.r.t the second variable $\psi^{(0,1)}(s,x)$ are Lipschitz continuous in x, i.e., for any x_1, $x_2 \in \mathbb{R}$, $\exists\ c_1, c_2 > 0$, such that $\forall s \in [0,1]$,

$$\begin{aligned} |\psi(s,x_1) - \psi(s,x_2)| &\leq c_1|x_1 - x_2|, \\ |\psi^{(0,1)}(s,x_1) - \psi^{(0,1)}(s,x_2)| &\leq c_2|x_1 - x_2|. \end{aligned}$$

The following theorem gives the condition for the existence of unique solution of the Eq. (2.5) in $\mathbb{X}$.

Theorem 1. *Let $\mathbb{X} = L^\infty[0,1]$ and $M = \max_{0\leq t,s\leq 1}|G(t,s)|$. Assume that the function $\psi(s,x) \in \mathcal{C}([0,1]\times\mathbb{R})$ satisfies the Lipschitz condition in the second variable, i.e., for $x_1, x_2 \in \mathbb{X}$,*

$$|\psi(s,x_1) - \psi(s,x_2)| \leq c_1|x_1 - x_2|,\ \ \forall s \in [0,1],$$

with $Mc_1 < 1$. Then the operator equation $x = \mathcal{T}(x)$ has a unique solution $x_0 \in \mathbb{X}$, i.e., we have $x_0 = \mathcal{T}(x_0)$.

Proof of the above theorem can be easily done using Lemma 3.1 of [13].

For the rest of the paper we denote $G_t(\xi) = G(t,\xi)$. Note that the kernel $G_t \in \mathcal{C}[0,1],\ \forall t \in [0,1]$ and also G_t is sufficiently smooth in $(0,1]$. We assume for $0 < t \leq 1$, $G_t \in \mathcal{C}^k(0,t)$ and $G_t \in \mathcal{C}^k(t,1]$, with $k \geq 1$. Hence we have

$$G_t \in \mathcal{C}[0,1] \cap \mathcal{C}^k(0,t) \cap \mathcal{C}^k(t,1],\ \ k \geq 1, \tag{2.7}$$

uniformly in $t \in (0,1]$. We also observe from (2.3) that $y \in \mathcal{C}[0,1]$ and y is sufficiently smooth in $(0,1]$. We assume $y \in \mathcal{C}^k(0,1]$. Clearly with this, we can conclude that

$$x_0 \in \mathcal{C}[0,1] \cap \mathcal{C}^k(0,1]. \tag{2.8}$$

The Fréchet derivative $(\mathcal{K}\psi)'(x)$ is the linear integral operator on $\mathbb{X}$ given by

$$(\mathcal{K}\psi)'(x)h(t) = \int_0^1 G(t,\xi)\psi^{(0,1)}(\xi, x(\xi))h(\xi)\, d\xi.$$

Note that $(\mathcal{K}\psi)'(x)$ is a compact operator on $\mathbb{X}$.

Clearly using the estimate (2.2), we have

$$\begin{aligned}\|(\mathcal{K}\psi)'(x_0)y\|_\infty = \sup_{t\in[0,1]} |(\mathcal{K}\psi)'(x_0)y(t)| &= \sup_{t\in[0,1]} \left|\int_0^1 G(t,\xi)\psi^{(0,1)}(\xi, x_0(\xi))y(\xi)\, d\xi\right| \\ &\le \sup_{t,\xi\in[0,1]} |G(t,\xi)| \sup_{\xi\in[0,1]} |\psi^{(0,1)}(\xi, x_0(\xi))|\|y\|_\infty \\ &\le MB\|y\|_\infty, \end{aligned} \tag{2.9}$$

where we denote

$$B = \sup_{\xi\in[0,1]} |\psi^{(0,1)}(\xi, x_0(\xi))|. \tag{2.10}$$

Next we prove the following lemma, which is needed in our convergence analysis.

Lemma 1. *For any $x, y \in \mathbb{X}$, the following hold*

$$\begin{aligned}&\|[(\mathcal{K}\psi)'(x) - (\mathcal{K}\psi)'(y)]v\|_\infty \le Mc_2\|x-y\|_\infty\|v\|_\infty, \\ &\|(\mathcal{K}\psi)(x) - (\mathcal{K}\psi)(y)\|_\infty \le Mc_1\|x-y\|_\infty.\end{aligned}$$

Proof. Using Lipschitz continuity of $\psi^{(0,1)}(t,u)$ and estimate (2.2), for any $v \in \mathcal{C}[0,1]$, we have

$$\begin{aligned}\|[(\mathcal{K}\psi)'(x) - (\mathcal{K}\psi)'(y)]v\|_\infty &= \sup_{t\in[0,1]} |[(\mathcal{K}\psi)'(x) - (\mathcal{K}\psi)'(y)]v(t)| \\ &= \sup_{t\in[0,1]} \left|\int_0^1 G(t,\xi)[\psi^{(0,1)}(\xi, x(\xi)) - \psi^{(0,1)}(\xi, y(\xi))]v(\xi)d\xi\right| \\ &\le \sup_{t,\xi\in[0,1]} |G(t,\xi)| \int_0^1 \left|\psi^{(0,1)}(\xi, x(\xi)) - \psi^{(0,1)}(\xi, y(\xi))\right| |v(\xi)|d\xi \\ &\le c_2M\int_0^1 |x(\xi) - y(\xi)||v(\xi)|d\xi \\ &\le Mc_2\|x-y\|_\infty\|v\|_\infty. \end{aligned} \tag{2.11}$$

On the similar lines, using Lipschitz continuity of $\psi(t,u)$, we obtain

$$\|(\mathcal{K}\psi)(x) - (\mathcal{K}\psi)(y)\|_\infty \le Mc_1\|x-y\|_\infty. \tag{2.12}$$

This completes the proof. □

Next we will apply Galerkin method to solve the Eq. (2.5). To do this, we let for given $n \in \mathbb{N}$, $\Delta : 0 = t_0 < t_1 < ... < t_n = 1$ be a partition of the interval $[0, 1]$ with grid points

$$t_i = \left(\frac{i}{n}\right)^q, \ i = 0, 1, ..., n, \tag{2.13}$$

where the grading exponent $q \in \mathbb{R}$ will always be assumed to satisfy $q \geq 1$. For $q = 1$, the grid points $t_0 < t_1 < ... < t_n$ are distributed uniformly and for $q > 1$, the grid points are more densely clustered near the left endpoint of the interval $[0, 1]$. In addition, we define

$$h = \max\{h_i : 1 \leq i \leq n\}, \ h' = \min\{h_i : 1 \leq i \leq n\}, \tag{2.14}$$

where $h_i = t_i - t_{i-1}$. For any meshes of the form (2.13), we have $0 < h_0 = h' < h_1 < ... < h_{n-1} = h$, and also

$$h_i \leq h \leq qn^{-1}, \ 1 \leq i \leq n. \tag{2.15}$$

Thus the mesh diameters $h = \mathcal{O}(n^{-1})$ for a sequence of graded meshes of the form (2.13) on compact interval $[0, 1]$ (see [5]).

Next we denote $\Delta_i := [t_{i-1}, t_i]$, $i = 1, 2, ..., n$ and define $\mathcal{C}_\Delta := \prod_{i=1}^{n} \mathcal{C}(\Delta_i)$. Then $g = (g_1, g_2, ..., g_n) \in \mathcal{C}_\Delta$ consists of n components $g_i \in \mathcal{C}(\Delta_i)$ and is a piecewise-continuous function having (possibly) different left and right values at the partition points t_i. Note that $\mathcal{C}_\Delta$ is a Banach space with the norm $\|.\|_\Delta$ defined by $\|g\|_\Delta = \max_i \|g_i\|_\infty$, and since $\|g\|_\Delta = \|g\|_\infty$ for $g \in \mathcal{C}_\Delta$, we have $\mathcal{C}_\Delta \subset L^\infty(0, 1)$. More generally, we denote for a positive integer k, $\mathcal{C}^k_\Delta := \prod_{i=1}^{n} \mathcal{C}^k(\Delta_i)$. $g_i \in \mathcal{C}^k(\Delta_i)$ iff its k-th derivative $g_i^{(k)}$ is continuous on Δ_i. We define the norms

$$\|g_i\|_{2,\Delta_i} = \left(\int_{t_{i-1}}^{t_i} |g_i(t)|^2 dt\right)^{\frac{1}{2}} \text{ and } \|g\|_2 = \left(\sum_{i=1}^{n} \|g_i\|^2_{2,\Delta_i}\right)^{\frac{1}{2}}.$$

We will analyze the Galerkin method to solve the Eq. (2.5) in the piecewise polynomial space,

$$\mathbb{X}_n = \mathbb{P}_{r,\Delta} = \{u : u|_{(t_{i-1},t_i)} \in \mathbb{P}_r, \ 1 \leq i \leq n\}, \tag{2.16}$$

where $\mathbb{P}_r$ denotes, for given $r \geq 1$, the space of (real) polynomials of degree less than $r + 1$. For $g \in \mathbb{P}_{r,\Delta}$, if the value at t_i^- is defined by continuity, then $\mathbb{P}_{r,\Delta} \subset \mathcal{C}_\Delta$ and the projection $\mathcal{P}_n : \mathcal{C}_\Delta \to \mathbb{P}_{r,\Delta}$ is defined as $g = (g_1, g_2, ..., g_n) \to \mathcal{P}_n g = (\mathcal{P}g_1, \mathcal{P}g_2, ..., \mathcal{P}g_n)$, where $\mathcal{P}g_i$ is the orthogonal projection of $g_i \in \mathcal{C}(\Delta_i)$ on the polynomials of degree less than $r + 1$ on Δ_i.

We first quote the following Lemma from Chatelin [7].

Lemma 2. *Let $\mathcal{P}_n : \mathcal{C}_\Delta \to \mathbb{X}_n$ be the orthogonal projection then the following hold* *(i)* $\mathcal{P}_n$ *is uniformly bounded in infinity norm, i.e,* $\exists$ *a constant* p_1 *independent of* n *such that*

$$\|\mathcal{P}_n\|_\infty \le p_1 < \infty. \tag{2.17}$$

(ii) For $u \in \mathbb{X}$

$$\|\mathcal{P}_n u - u\|_\infty \to 0, \text{ as } n \to \infty. \tag{2.18}$$

(iii) If $u \in \mathcal{C}^k_\Delta$*, then*

$$\|(\mathcal{I} - \mathcal{P}_n)u\|_\infty \le ch^\beta \|u^{(\beta)}\|_\infty, \tag{2.19}$$

where $\beta = \min\{k, r+1\}$ *and* c *is a constant independent of* n*.*

The Galerkin method for the Eq. (2.5) is seeking an approximate solution $x_n \in \mathbb{X}_n$ such that

$$x_n - \mathcal{P}_n \mathcal{K}\psi(x_n) = \mathcal{P}_n y. \tag{2.20}$$

Let $\mathcal{T}_n$ be the operator defined by

$$\mathcal{T}_n(u) := \mathcal{P}_n y + \mathcal{P}_n \mathcal{K}\psi(u). \tag{2.21}$$

Then (2.20) can be written as

$$x_n = \mathcal{T}_n x_n. \tag{2.22}$$

In order to obtain more accurate approximation solution, we further consider the iterated Galerkin method for (2.5). To this end, we define the iterated solution as

$$\tilde{x}_n = y + \mathcal{K}\psi(x_n). \tag{2.23}$$

Applying $\mathcal{P}_n$ on both sides of the Eq. (2.23), we obtain

$$\mathcal{P}_n \tilde{x}_n = \mathcal{P}_n y + \mathcal{P}_n \mathcal{K}\psi(x_n). \tag{2.24}$$

From Eqs. (2.20) and (2.24), it follows that $\mathcal{P}_n \tilde{x}_n = x_n$. Using this, we see that the iterated solution $\tilde{x}_n$ satisfies the following equation

$$\tilde{x}_n - \mathcal{K}\psi(\mathcal{P}_n \tilde{x}_n) = y. \tag{2.25}$$

Letting $\widetilde{\mathcal{T}}_n(u) := y + \mathcal{K}\psi(\mathcal{P}_n u)$, $u \in \mathbb{X}$, the Eq. (2.25) can be written as $\tilde{x}_n = \widetilde{\mathcal{T}}_n \tilde{x}_n$.

3 Convergence Rates

In this section, we analyze the existence and convergence of the approximate and iterated approximate solutions in the Galerkin method. To do this, we let $B(\mathbb{X}, \mathbb{Y})$ denote the space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$. If $\mathbb{X} = \mathbb{Y}$, we denote the space by $B(\mathbb{X})$.

First we quote the following theorem from Vainikko [16], which gives us the conditions under which the solvability of one equation leads to the solvability of other equation, followed by the definition of collectively compact approximation from Anselone [2].

Theorem 2. *Let $\widehat{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ be continuous operators over an open set Ω in a Banach space $\mathbb{X}$. Let the equation $x = \widetilde{\mathcal{F}}x$ has an isolated solution $\tilde{x}_0 \in \Omega$ and let the following conditions be satisfied.*

(a) The operator $\widehat{\mathcal{F}}$ is Fréchet differentiable in some neighborhood of the point $\tilde{x}_0$, while the linear operator $\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{x}_0)$ is continuously invertible on $\mathbb{X}$, where $\widehat{\mathcal{F}}'(\tilde{x}_0)$ is the Fréchet derivative of $\widehat{\mathcal{F}}(x)$ at $\tilde{x}_0$.
(b) Suppose that for some $\delta > 0$ and $0 < q < 1$, the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|x - \tilde{x}_0\| \leq \delta$ is contained within Ω)

$$\sup_{\|x-\tilde{x}_0\|\leq\delta} \|(\mathcal{I} - \widehat{\mathcal{F}}'(x_0))^{-1}(\widehat{\mathcal{F}}'(x) - \widehat{\mathcal{F}}'(\tilde{x}_0))\| \leq q, \tag{3.1}$$

$$\gamma = \|(\mathcal{I} - \widehat{\mathcal{F}}'(x_0))^{-1}(\widehat{\mathcal{F}}(\tilde{x}_0) - \widetilde{\mathcal{F}}(\tilde{x}_0))\| \leq \delta(1 - q). \tag{3.2}$$

Then the equation $x = \widehat{\mathcal{F}}x$ has a unique solution $\hat{x}_0$ in the sphere $\|x - \tilde{x}_0\| \leq \delta$. Moreover, the inequality

$$\frac{\gamma}{1+q} \leq \|\hat{x}_0 - \tilde{x}_0\| \leq \frac{\gamma}{1-q} \tag{3.3}$$

is valid.

Definition 1. *A sequence $\{\mathcal{A}_n\}$ in $B(\mathbb{X}, \mathbb{Y})$ is said to be a collectively compact approximation of $\mathcal{A} \in B(\mathbb{X}, \mathbb{Y})$, i.e., $\mathcal{A}_n \xrightarrow{cc} \mathcal{A}$ if*

(i) $\mathcal{A}_n x \to \mathcal{A}x$, for each $x \in \mathbb{X}$,
(ii) for some positive integer n_0, the set $\bigcup_{n\geq n_0} \{(\mathcal{A}_n - \mathcal{A})x : \|x\| \leq 1, x \in \mathbb{X}\}$ is relatively compact.

In particular, if $\mathcal{A}$ is a compact operator, then the latter condition is equivalent to the condition that, for some positive integer n_0, the set $\bigcup_{n\geq n_0} \{\mathcal{A}_n x : \|x\| \leq 1, x \in \mathbb{X}\}$ is relatively compact.

Lemma 3 *(Ahues et al. [1]). Let $\mathbb{X}$ be a Banach space, $\mathcal{A} \in \mathbb{B}(\mathbb{X})$ and $\{\mathcal{A}_n\}$ be a bounded sequence in $\mathbb{B}(\mathbb{X})$. If $\|\mathcal{A}_n - \mathcal{A}\| \to 0$, as $n \to \infty$ or $\mathcal{A}_n \xrightarrow{cc} \mathcal{A}$, and $(\mathcal{I} - \mathcal{A})^{-1}$ exists, then $(\mathcal{I} - \mathcal{A}_n)^{-1}$ exists and uniformly bounded on $\mathbb{X}$, for sufficiently large n.*

Lemma 4. *Let $x_0 \in L^\infty[0,1]$ be an isolated solution of the Eq. (2.5). Assume that 1 is not an eigenvalue of $(\mathcal{K}\psi)'(x_0)$. Then for sufficiently large n, the operators $(\mathcal{I} - \mathcal{T}_n'(x_0))$ and $(\mathcal{I} - \widetilde{\mathcal{T}}_n'(x_0))$ are invertible on $L^\infty[0,1]$ and there exist constants $A_1, A_2 > 0$ independent of n such that $\|(\mathcal{I} - \mathcal{T}_n'(x_0))^{-1}\|_\infty \leq A_1$ and $\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(x_0))^{-1}\|_\infty \leq A_2$.*

Proof. Consider

$$\|\mathcal{T}_n'(x_0) - \mathcal{T}'(x_0)\|_\infty = \|(\mathcal{P}_n - \mathcal{I})(\mathcal{K}\psi)'(x_0)\|_\infty = \|(\mathcal{P}_n - \mathcal{I})\mathcal{T}'(x_0)\|_\infty. \tag{3.4}$$

Since $\mathcal{T}'(x_0) = (\mathcal{K}\psi)'(x_0)$ is compact operator, using estimates (2.18) and (3.4), we get

$$\|\mathcal{T}_n'(x_0) - \mathcal{T}'(x_0)\|_\infty \to 0, \ as\ n \to \infty. \tag{3.5}$$

This shows that $\{\mathcal{T}_n'(x_0)\}$ is norm convergent to the operator $\mathcal{T}'(x_0)$. Thus by direct application of Lemma 3, we can conclude that for sufficiently large n, $(\mathcal{I} - \mathcal{T}_n'(x_0))$ is invertible on $L^\infty[0,1]$, i.e., there exists a constant $A_1 > 0$, independent of n such that $\|(\mathcal{I} - \mathcal{T}_n'(x_0))^{-1}\|_\infty \leq A_1$.

Next using estimates (2.11), (2.17), (2.18) and (2.19), for any $x \in \mathbb{X}$, we have

$$\begin{aligned}
\|[\widetilde{\mathcal{T}}_n'(x_0) - \mathcal{T}'(x_0)]x\|_\infty &= \|(\mathcal{K}\psi)'(\mathcal{P}_n x_0)\mathcal{P}_n x - (\mathcal{K}\psi)'(x_0)x\|_\infty \\
&\leq \|[(\mathcal{K}\psi)'(\mathcal{P}_n x_0) - (\mathcal{K}\psi)'(x_0)]\mathcal{P}_n x\|_\infty + \|(\mathcal{K}\psi)'(x_0)(\mathcal{P}_n - \mathcal{I})x\|_\infty \\
&\leq Mc_2\|\mathcal{P}_n x_0 - x_0\|_\infty \|_\infty \|\mathcal{P}_n x\|_\infty + MB\|(\mathcal{P}_n - \mathcal{I})x\|_\infty \\
&\leq Mc_2 p_1\|\mathcal{P}_n x_0 - x_0\|_\infty \|x\|_\infty + MB\|(\mathcal{P}_n - \mathcal{I})x\|_\infty \\
&\to 0, \ as\ n \to \infty.
\end{aligned} \tag{3.6}$$

This shows that $\{\widetilde{\mathcal{T}}_n'(x_0)\}$ converges pointwise to the operator $\mathcal{T}'(x_0)$ in infinity norm.

Again using estimates (2.9), (2.11), (2.17) and (2.18) we have

$$\begin{aligned}
\|\widetilde{\mathcal{T}}_n'(x_0)\|_\infty &= \|(\mathcal{K}\psi)'(\mathcal{P}_n x_0)\mathcal{P}_n\|_\infty \\
&\leq \|[(\mathcal{K}\psi)'(\mathcal{P}_n x_0) - (\mathcal{K}\psi)'(x_0)]\mathcal{P}_n\|_\infty + \|(\mathcal{K}\psi)'(x_0)\mathcal{P}_n\|_\infty \\
&\leq Mc_2 p_1\|\mathcal{P}_n x_0 - x_0\|_\infty + p_1\|(\mathcal{K}\psi)'(x_0)\|_\infty \\
&\leq Mp_1\left[c_2\|\mathcal{P}_n x_0 - x_0\|_\infty + B\right] < \infty.
\end{aligned} \tag{3.7}$$

Hence $\{\widetilde{\mathcal{T}}_n'(x_0)\}$ is uniformly bounded in infinity norm.

Since $G(t,\xi) \in \mathcal{C}([0,1] \times [0,1])$, $G(t,\xi)$ is uniformly continuous in first variable t. Hence for any $\epsilon > 0$, however small, there exists some number $\delta > 0$ such that

$$|G(t,\xi) - G(t',\xi)| < \epsilon, \ \text{whenever}\ |t - t'| < \delta. \tag{3.8}$$

Then using Lipschitz continuity of $\psi^{(0,1)}(.,.)$ and estimates (2.10), (2.17) and (3.8), we obtain

$$
\begin{aligned}
|\widetilde{\mathcal{T}}_n'(x_0)x(t) - \widetilde{\mathcal{T}}_n'(x_0)x(t')| &= |(\mathcal{K}\psi)'(\mathcal{P}_n x_0)\mathcal{P}_n x(t) - (\mathcal{K}\psi)'(\mathcal{P}_n x_0)\mathcal{P}_n x(t')| \\
&= \left|\int_0^1 \left[G(t,\xi) - G(t',\xi)\right] \psi^{(0,1)}(\xi, \mathcal{P}_n x_0(\xi))\mathcal{P}_n x(\xi) d\xi\right| \\
&\le \left|\int_0^1 \left[G(t,\xi) - G(t',\xi)\right] [\psi^{(0,1)}(\xi, \mathcal{P}_n x_0(\xi)) - \psi^{(0,1)}(\xi, x_0(\xi))]\mathcal{P}_n x(\xi) d\xi\right| \\
&+ \left|\int_0^1 \left[G(t,\xi) - G(t',\xi)\right] \psi^{(0,1)}(\xi, x_0(\xi))\mathcal{P}_n x(\xi) d\xi\right| \\
&\le \sup_{0\le\xi\le1} |G(t,\xi) - G(t',\xi)| \int_0^1 \left|\psi^{(0,1)}(\xi, \mathcal{P}_n x_0(\xi)) - \psi^{(0,1)}(\xi, x_0(\xi))\right| |\mathcal{P}_n x(\xi)| d\xi \\
&+ \sup_{0\le\xi\le1} |G(t,\xi) - G(t',\xi)| \int_0^1 \left|\psi^{(0,1)}(\xi, x_0(\xi))\right| |\mathcal{P}_n x(\xi)| d\xi \\
&\le \epsilon \left[c_2 \int_0^1 |\mathcal{P}_n x_0(\xi) - x_0(\xi)| \, |\mathcal{P}_n x(\xi)| d\xi + B\|\mathcal{P}_n x\|_\infty\right] \\
&\le \epsilon p_1 \left[c_2\|\mathcal{P}_n x_0 - x_0\|_\infty + B\right] \|x\|_\infty \to 0, \ \ as \ t \to t'.
\end{aligned}
\tag{3.9}
$$

Estimates (3.7) and (3.9) imply that, $\exists$ a positive integer n_0 such that $\bigcup_{n\ge n_0} \{\widetilde{\mathcal{T}}_n'(x_0)x : \|x\| \le 1, x \in \mathbb{X}\}$ is relatively compact. Also from the estimate (3.6), we have that $\widetilde{\mathcal{T}}_n'(x_0)$ converge pointwise to the operator $\mathcal{T}'(x_0)$. Hence it is clear that $\widetilde{\mathcal{T}}_n'(x_0) \xrightarrow{cc} \mathcal{T}'(x_0)$. Since $\mathcal{T}'(x_0)$ is compact and $(\mathcal{I}-\mathcal{T}'(x_0))$ is invertible on $\mathbb{X}$, then using Lemma 3, we can conclude that for sufficiently large n, $(\mathcal{I}-\widetilde{\mathcal{T}}_n'(x_0))$ is invertible on $L^\infty[0,1]$, i.e., there exists a constant $A_2 > 0$, independent of n such that $\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(x_0))^{-1}\|_\infty \le A_2$. This completes the proof. $\square$

Theorem 3. *Let $x_0 \in \mathcal{C}[0,1]$ be an isolated solution of the Eq. (2.5). Assume that* 1 *is not an eigenvalue of the linear operator $\mathcal{T}'(x_0)$, where $\mathcal{T}'(x_0)$ denotes the Fréchet derivative of $\mathcal{T}(x)$ at x_0. Then the Eq. (2.20) has a unique solution $x_n \in B(x_0, \delta) = \{x : \|x - x_0\|_\infty < \delta\}$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$
\frac{\gamma_n}{1+q} \le \|x_n - x_0\|_\infty \le \frac{\gamma_n}{1-q},
$$

where $\gamma_n = \|(\mathcal{I} - \mathcal{T}_n'(x_0))^{-1}(\mathcal{T}_n(x_0) - \mathcal{T}(x_0))\|_\infty$.

Proof. From Lemma 4, we have $(\mathcal{I} - \mathcal{T}_n'(x_0))^{-1}$ exists and uniformly bounded on $\mathbb{X}$, for some sufficiently large n, i.e., there exists some $A_1 > 0$ such that $\|(\mathcal{I} - \mathcal{T}_n'(x_0))^{-1}\|_\infty \le A_1 < \infty$.

Now using estimates (2.11) and (2.17), we have for any $x \in B(x_0, \delta)$,

$$
\begin{aligned}
\|\mathcal{T}_n'(x_0) - \mathcal{T}_n'(x)\|_\infty &= \|\mathcal{P}_n(\mathcal{K}\psi)'(x_0) - \mathcal{P}_n(\mathcal{K}\psi)'(x)\|_\infty \\
&\le p_1\|(\mathcal{K}\psi)'(x_0) - (\mathcal{K}\psi)'(x)\|_\infty \\
&\le p_1 M c_2\|x_0 - x\|_\infty \le p_1 M c_2 \delta.
\end{aligned}
$$

Thus we have

$$\sup_{\|x-x_0\|_\infty \le \delta} \|(\mathcal{I}-\mathcal{T}_n'(x_0))^{-1}(\mathcal{T}_n'(x_0)-\mathcal{T}_n'(x))\|_\infty \le A_1 p_1 M c_2 \delta \le q \ (say).$$

Here we choose δ in such a way that, $0 < q < 1$. This proves the estimate (3.1) of Theorem 2.

Taking use of (2.18), we have

$$\begin{aligned}\gamma_n = \|(\mathcal{I}-\mathcal{T}_n'(x_0))^{-1}(\mathcal{T}_n(x_0)-\mathcal{T}(x_0))\|_\infty &\le A_1\|\mathcal{T}_n(x_0)-\mathcal{T}(x_0)\|_\infty \\ &= A_1\|\mathcal{P}_n\mathcal{K}\psi(x_0)-\mathcal{K}\psi(x_0)+\mathcal{P}_n y - y\|_\infty \\ &= A_1\|(\mathcal{P}_n-\mathcal{I})(\mathcal{K}\psi(x_0)+y)\|_\infty \\ &= A_1\|(\mathcal{P}_n-\mathcal{I})x_0\|_\infty \qquad (3.10) \\ &\to 0, \ as \ n\to\infty.\end{aligned}$$

By choosing n large enough such that $\alpha_n \le \delta(1-q)$, the estimate (3.2) of Theorem 2 is satisfied. Hence by applying Theorem 2, we obtain

$$\frac{\gamma_n}{1+q} \le \|x_n - x_0\|_\infty \le \frac{\gamma_n}{1-q}.$$

This completes the proof. $\square$

Theorem 4. *Let $x_0 \in \mathcal{C}[0,1]\cap\mathcal{C}^k(0,1]$ be an isolated solution of the equation (2.5) and x_n be the Galerkin approximation of x_0. If the grading exponent corresponding to the graded meshes (2.13) is given by*

$$q = \frac{\beta}{1-\alpha}.$$

Then there hold

$$\|x_0 - x_n\|_\infty = \mathcal{O}(n^{-\beta}),$$

where $\beta = \min\{k, r+1\}$.

Proof. From Theorem 3 and estimate (3.10), it follows that

$$\begin{aligned}\|x_n - x_0\|_\infty \le \frac{\gamma_n}{1-q} &\le c\|(\mathcal{I}-\mathcal{T}_n'(x_0))^{-1}(\mathcal{T}_n(x_0)-\mathcal{T}(x_0))\|_\infty \\ &\le cA_1\|(\mathcal{P}_n-\mathcal{I})x_0\|_\infty. \qquad (3.11)\end{aligned}$$

Since $x_0 \in \mathcal{C}^k[t_1,1]$, using (2.15) we have for $j = 2,3,...,n$,

$$\|(\mathcal{P}-\mathcal{I})(x_0)_j\|_{\infty,\Delta_j} \le c h_j^\beta \|(x_0)_j^{(\beta)}\|_\infty = \mathcal{O}(n^{-\beta}), \qquad (3.12)$$

where $\beta = \min\{k, r+1\}$.
Next since $x_0 \in \mathcal{C}[0,t_1]$, using (2.17) on Δ_1 we have

$$\begin{aligned}\|(\mathcal{P}-\mathcal{I})(x_0)_1\|_{\infty,\Delta_1} &= \|(\mathcal{P}-\mathcal{I})(\mathcal{K}\psi((x_0)_1)+y)\|_{\infty,\Delta_1} \\ &\le (1+p_1)\|\mathcal{K}\psi((x_0)_1)\|_{\infty,\Delta_1} + \|(\mathcal{P}-\mathcal{I})y\|_{\infty,\Delta_1}. \qquad (3.13)\end{aligned}$$

We have

$$\|(\mathcal{P}-\mathcal{I})y\|_{\infty,\Delta_1} \leq \|(\mathcal{P}-\mathcal{I})\alpha_1\|_{\infty,\Delta_1} + \left\|(\mathcal{P}-\mathcal{I})\left(\frac{(\eta_1-\beta_1\alpha_1)t^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right)\right\|_{\infty,\Delta_1}.$$

Since $\mathcal{P}$ is a orthogonal projection on the polynomials of degree less than $r+1$, we have $\|(\mathcal{P}-\mathcal{I})\alpha_1\|_{\infty,\Delta_1} = 0$. Hence using estimate (2.17), we get

$$\begin{aligned}\|(\mathcal{P}-\mathcal{I})y\|_{\infty,\Delta_1} &\leq (1+p_1)\sup_{t\in[0,t_1]}\left|\frac{(\eta_1-\beta_1\alpha_1)t^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right| \\ &\leq (1+p_1)\left|\frac{(\eta_1-\beta_1\alpha_1)t_1^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right| \\ &= (1+p_1)\left|\frac{(\eta_1-\beta_1\alpha_1)n^{-q(1-\alpha)}}{\beta_1+\gamma_1(1-\alpha)}\right| \\ &= \mathcal{O}(n^{-q(1-\alpha)}) = \mathcal{O}(n^{-\beta}). \end{aligned} \tag{3.14}$$

Again using (2.10), we obtain

$$\begin{aligned}\|\mathcal{K}\psi((x_0)_1)\|_{\infty,\Delta_1} = \sup_{t\in[0,t_1]}|\mathcal{K}\psi((x_0)_1)(t)| &= \sup_{t\in[0,t_1]}\left|\int_0^1 G(t,\xi)\psi(\xi,(x_0)_1(\xi))d\xi\right| \\ &\leq B\sup_{\substack{0\leq t\leq t_1\\0\leq\xi\leq 1}}|G(t,\xi)|. \end{aligned} \tag{3.15}$$

To estimate $\sup_{\substack{0\leq t\leq t_1\\0\leq\xi\leq 1}}|G(t,\xi)|$, we consider the following three cases:

Case 1. Let $\xi\in[0,t_1]$ and $t\in[0,\xi]$, we have

$$\max_{\substack{0\leq\xi\leq t_1\\0\leq t\leq\xi}}|G_t(\xi)| = \max_{\substack{0\leq\xi\leq t_1\\0\leq t\leq\xi}}\left|\frac{t^{1-\alpha}}{1-\alpha}\left[1-\frac{\beta_1\xi^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right]\right| = \frac{t_1^{1-\alpha}}{1-\alpha} = \frac{n^{-q(1-\alpha)}}{1-\alpha} = \mathcal{O}(n^{-\beta}). \tag{3.16}$$

Case 2. Let $\xi\in[0,t_1]$ and $t\in[\xi,t_1]$, we have

$$\max_{\substack{0\leq\xi\leq t_1\\\xi\leq t\leq t_1}}|G_t(\xi)| = \max_{\substack{0\leq\xi\leq t_1\\\xi\leq t\leq t_1}}\left|\frac{\xi^{1-\alpha}}{1-\alpha}\left[1-\frac{\beta_1 t^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right]\right| = \frac{t_1^{1-\alpha}}{1-\alpha} = \mathcal{O}(n^{-\beta}). \tag{3.17}$$

Case 3. Let $\xi\in[t_1,1]$ and $t\in[0,t_1]$, we have

$$\max_{\substack{t_1\leq\xi\leq 1\\0\leq t\leq t_1}}|G_t(\xi)| = \max_{\substack{t_1\leq\xi\leq 1\\0\leq t\leq t_1}}\left|\frac{t^{1-\alpha}}{1-\alpha}\left[1-\frac{\beta_1\xi^{1-\alpha}}{\beta_1+\gamma_1(1-\alpha)}\right]\right| \leq \frac{t_1^{1-\alpha}}{1-\alpha} = \mathcal{O}(n^{-\beta}). \tag{3.18}$$

Hence considering all the possible cases, we get

$$\sup_{\substack{0\leq t\leq t_1\\0\leq\xi\leq 1}}|G(t,\xi)| = \mathcal{O}(n^{-\beta}), \tag{3.19}$$

which implies

$$\|\mathcal{K}\psi((x_0)_1)\|_{\infty,\Delta_1} \leq B \sup_{\substack{0\leq t\leq t_1\\0\leq\xi\leq 1}} |G(t,\xi)| = \mathcal{O}(n^{-\beta}). \tag{3.20}$$

From estimates (3.13), (3.14) and (3.20), we get

$$\|(\mathcal{P}-\mathcal{I})(x_0)_1\|_{\infty,\Delta_1} = \mathcal{O}(n^{-\beta}). \tag{3.21}$$

Hence estimates (3.12) and (3.21) imply that

$$\|(\mathcal{P}_n-\mathcal{I})x_0\|_{\infty} = \max_i \|(\mathcal{P}-\mathcal{I})(x_0)_i\|_{\infty,\Delta_i} = \mathcal{O}(n^{-\beta}). \tag{3.22}$$

Combining estimates (3.11) and (3.22), we finally obtain

$$\|x_0 - x_n\|_\infty = \mathcal{O}(n^{-\beta}),$$

where $\beta = \min\{k, r+1\}$. This completes the proof. □

Next we discuss the existence and convergence of the iterated approximate solution $\tilde{x}_n$ to x_0.

Theorem 5. *Let $x_0 \in \mathcal{C}[0,1]$ be an isolated solution of the Eq. (2.5). Assume that 1 is not an eigenvalue of $\mathcal{T}'(x_0)$, then for sufficiently large n, the iterated solution $\tilde{x}_n$ defined by (2.25) is the unique solution in the sphere $B(x_0,\delta) = \{x : \|x-x_0\|_\infty < \delta\}$. Moreover, there exists a constant $0<q<1$, independent of n such that*

$$\frac{\zeta_n}{1+q} \leq \|\tilde{x}_n - x_0\|_\infty \leq \frac{\zeta_n}{1-q},$$

where

$$\zeta_n = \|(\mathcal{I}-\widetilde{\mathcal{T}}_n'(x_0))^{-1}(\widetilde{\mathcal{T}}_n(x_0)-\mathcal{T}(x_0))\|_\infty.$$

Proof. From Lemma 4, we have that $\|(\mathcal{I}-\widetilde{\mathcal{T}}_n'(x_0))^{-1}\|_\infty \leq A_2 < \infty$, for sufficiently large value of n.

Using estimates (2.11), (2.17), for any $x \in B(x_0,\delta)$, we have

$$\begin{aligned}\|[\widetilde{\mathcal{T}}_n'(x_0) - \widetilde{\mathcal{T}}_n'(x)]v\|_\infty &= \|[(\mathcal{K}\psi)'(\mathcal{P}_n x_0) - (\mathcal{K}\psi)'(\mathcal{P}_n x)]\mathcal{P}_n v\|_\infty \\ &\leq Mc_2\|\mathcal{P}_n(x_0-x)\|_\infty\|\mathcal{P}_n v\|_\infty \\ &\leq Mc_2 p_1^2\|x-x_0\|_\infty\|v\|_\infty \leq c_2Mp_1^2\delta\|v\|_\infty. \end{aligned}\tag{3.23}$$

This implies

$$\sup_{\|x-x_0\|_\infty\leq\delta} \|(\mathcal{I}-\widetilde{\mathcal{T}}_n'(x_0))^{-1}(\widetilde{\mathcal{T}}_n'(x)-\widetilde{\mathcal{T}}_n'(x_0))\|_\infty \leq A_2Mc_2p_1^2\delta \leq q \ (say).$$

We choose δ in such a way that $0<q<1$. This proves the estimate (3.1) of Theorem 2.

Next using estimates (2.12) and (2.18), we have

$$\begin{aligned}\zeta_n = \|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(x_0))^{-1}(\widetilde{\mathcal{T}}_n(x_0) - \mathcal{T}(x_0))\|_\infty &\le A_2\|\widetilde{\mathcal{T}}_n(x_0) - \mathcal{T}(x_0)\|_\infty \\ &= A_2\|\mathcal{K}\psi(\mathcal{P}_n x_0) - \mathcal{K}\psi(x_0)\|_\infty \\ &\le A_2 M c_1\|(\mathcal{I} - \mathcal{P}_n)x_0\|_\infty \to 0, \ as\ n \to \infty.\end{aligned}$$

Choose n large enough such that $\zeta_n \le \delta(1-q)$. Then the estimate (3.2) of Theorem 2 is satisfied. Thus by applying Theorem 2, we obtain

$$\frac{\zeta_n}{1+q} \le \|\tilde{x}_n - x_0\|_\infty \le \frac{\zeta_n}{1-q},$$

where

$$\zeta_n = \|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(x_0))^{-1}(\widetilde{\mathcal{T}}_n(x_0) - \mathcal{T}(x_0))\|_\infty.$$

This completes the proof. □

To prove the following convergence results, we denote $g_{1t} = g_t|_{[0,\xi]}$ and $g_{2t} = g_t|_{[\xi,1]}$.

Lemma 5. *Let x_0 be an isolated solution of the Eq. (2.5). Then there hold*

$$\|\mathcal{K}\psi(\mathcal{P}_n x_0) - \mathcal{K}\psi(x_0)\|_\infty \le M c_2\|(\mathcal{I} - \mathcal{P}_n)x_0\|_\infty^2 + \sup_{t\in[0,1]}|\langle g_t(.), (\mathcal{I} - \mathcal{P}_n)x_0(.)\rangle|,$$

where $g_t(.) = G(t,.)\psi^{(0,1)}(., x_0(.))$.

Proof. Using mean value theorem, we have

$$\begin{aligned}|[\mathcal{K}\psi(\mathcal{P}_n x_0) - \mathcal{K}\psi(x_0)](t)| &= \left|\int_0^1 G(t,\xi)[\psi(\xi, \mathcal{P}_n x_0(\xi)) - \psi(\xi, x_0(\xi))]d\xi\right| \\ &= \left|\int_0^1 G(t,\xi)[\psi^{(0,1)}(\xi, x_0(\xi) + \theta(\mathcal{P}_n x_0(\xi) - x_0(\xi)))](x_0 - \mathcal{P}_n x_0)(\xi)d\xi\right| \\ &= \left|\int_0^1 [g(t,\xi,x_0,\mathcal{P}_n x_0,\theta) - g_t(\xi) + g_t(\xi)](x_0 - \mathcal{P}_n x_0)(\xi)d\xi\right| \\ &\le \left|\int_0^1 [g(t,\xi,x_0,\mathcal{P}_n x_0,\theta) - g_t(\xi)](x_0 - \mathcal{P}_n x_0)(\xi)d\xi\right| \\ &+ \left|\int_0^1 g_t(\xi)(x_0 - \mathcal{P}_n x_0)(\xi)d\xi\right| \\ &= I_1 + I_2,\end{aligned}$$

where $g(t,\xi,x_0,\mathcal{P}_n x_0,\theta) = G(t,\xi)\psi^{(0,1)}(\xi, x_0(\xi) + \theta(\mathcal{P}_n x_0(\xi) - x_0(\xi)))$ and $g_t(\xi) = G(t,\xi)\psi^{(0,1)}(\xi, x_0(\xi))$.

Now for the estimate I_1, using the Lipschtiz's continuity of $\psi^{(0,1)}(., x_0(.))$ and using (2.2), we have

$$\begin{aligned}I_1 &= \left|\int_0^1 [g(t,\xi,x_0,\mathcal{P}_n x_0,\theta) - g_t(\xi)](x_0 - \mathcal{P}_n x_0)(\xi)d\xi\right| \\ &= \left|\int_0^1 G(t,\xi)[\psi^{(0,1)}(\xi, x_0(\xi) + \theta(\mathcal{P}_n x_0(\xi) - x_0(\xi))) - \psi^{(0,1)}(\xi, x_0(\xi))](x_0 - \mathcal{P}_n x_0)(\xi)d\xi\right| \\ &\le \sup_{t,\xi\in[0,1]}|G(t,\xi)|\int_0^1 c_2|(x_0 - \mathcal{P}_n x_0)(\xi)||(x_0 - \mathcal{P}_n x_0)(\xi)|d\xi \le M c_2\|x_0 - \mathcal{P}_n x_0\|_\infty^2.\end{aligned}$$

Now for the estimate I_2, we have

$$I_2 = \left| \int_0^1 g_t(\xi)(x_0 - \mathcal{P}_n x_0)(\xi) d\xi \right| = \left| \langle g_t(.), (\mathcal{I} - \mathcal{P}_n) x_0(.) \rangle \right|.$$

Hence we obtain

$$\|\mathcal{K}\psi(\mathcal{P}_n x_0) - \mathcal{K}\psi(x_0)\|_\infty \leq M c_2 \|(\mathcal{I} - \mathcal{P}_n)x_0\|_\infty^2 + \sup_{t\in[0,1]} \left| \langle g_t(.), (\mathcal{I} - \mathcal{P}_n) x_0(.) \rangle \right|. \tag{3.24}$$

This completes the proof. □

Lemma 6. *If the grading exponent corresponding to the graded meshes (2.13) is given by $q = \frac{\beta}{1-\alpha}$, then the following hold*

$$\|(\mathcal{I} - \mathcal{P})(g_t)_1\|_{2,\Delta_1} = \mathcal{O}(n^{-\left(\beta + \frac{1}{2}\right)}),$$

where $\beta = \min\{k, r+1\}$.

Proof. Using estimates (2.15) and (2.17), we have

$$\|(\mathcal{I} - \mathcal{P})(g_t)_1\|_{2,\Delta_1} \leq c h_1^{\frac{1}{2}} \|(\mathcal{I} - \mathcal{P})(g_t)_1\|_{\infty,\Delta_1} \leq c h_1^{\frac{1}{2}} (1 + p_1)\eta_0 \leq c n^{-\frac{1}{2}} (1 + p_1)\eta_0, \tag{3.25}$$

where $\eta_0 = \inf_{v\in\mathbb{P}_{r,\Delta}} \max_{0\leq\xi\leq t_1} |g_t(\xi) - v(\xi)|$.

To estimate η_0, it is sufficient to take $v(t)$ as a constant or linear function. We choose $v(t) = g_t(0) = 0$. Hence it is enough to estimate $\max_{0\leq\xi\leq t_1} |g_t(\xi)|$.

From estimates (3.16) and (3.17), we already have that

$$\max_{\substack{0\leq\xi\leq t_1\\0\leq t\leq t_1}} |g_t(\xi)| = \max_{\substack{0\leq\xi\leq t_1\\0\leq t\leq t_1}} |G(t,\xi)\psi^{(0,1)}(\xi, x_0(\xi))| \leq B \max_{\substack{0\leq\xi\leq t_1\\0\leq t\leq t_1}} |G_t(\xi)| = \mathcal{O}(n^{-\beta}). \tag{3.26}$$

Also if $t \in [t_1, t_n]$, we have

$$\begin{aligned}
\max_{\substack{0\leq\xi\leq t_1\\t_1\leq t\leq t_n}} |G_t(\xi)| &= \max_{\substack{0\leq\xi\leq t_1\\t_1\leq t\leq t_n}} \left| \frac{\xi^{1-\alpha}}{1-\alpha} \left[1 - \frac{\beta_1 t^{1-\alpha}}{\beta_1 + \gamma_1(1-\alpha)} \right] \right| \\
&= \left| \frac{t_1^{1-\alpha}}{1-\alpha} \left[1 - \frac{\beta_1 t_1^{1-\alpha}}{\beta_1 + \gamma_1(1-\alpha)} \right] \right| \\
&\leq \frac{t_1^{1-\alpha}}{1-\alpha} = \frac{n^{-q(1-\alpha)}}{1-\alpha} = \mathcal{O}(n^{-\beta}).
\end{aligned} \tag{3.27}$$

This implies

$$\max_{\substack{0\leq\xi\leq t_1\\t_1\leq t\leq t_n}} |g_t(\xi)| = \max_{\substack{0\leq\xi\leq t_1\\t_1\leq t\leq t_n}} |G(t,\xi)\psi^{(0,1)}(\xi, x_0(\xi))| \leq B \max_{\substack{0\leq\xi\leq t_1\\t_1\leq t\leq t_n}} |G_t(\xi)| = \mathcal{O}(n^{-\beta}). \tag{3.28}$$

Hence combining estimates (3.25), (3.26) and (3.28), we obtain

$$\|(\mathcal{I} - \mathcal{P})(g_t)_1\|_{2,\Delta_1} = \mathcal{O}(n^{-\left(\beta + \frac{1}{2}\right)}). \tag{3.29}$$

This completes the proof. □

Following the result of Lemma 7.8 (pp-330) from [7] and the above lemma, we obtain the following theorem.

Theorem 6. *Let $x_0 \in \mathcal{C}[0,1] \cap \mathcal{C}^k(0,1]$ be an isolated solution of the equation (2.5). Then for $t \notin \Delta$,*

$$|\langle g_t(.), (\mathcal{I} - \mathcal{P}_n)x_0 \rangle| = \begin{cases} \mathcal{O}(n^{-(\beta+\beta^*)}), \, t \in (t_{i-1}, t_i), i = 2, 3, ...n, \\ \mathcal{O}(n^{-2\beta}), \qquad t \in (t_0, t_1), \end{cases}$$

and for $t \in \Delta$

$$|\langle g_t(.), (\mathcal{I} - \mathcal{P}_n)x_0 \rangle| = \mathcal{O}(n^{-2\beta}),$$

where $g_t(.) = G(t,.)\psi^{(0,1)}(., x_0(.))$, $\beta = \min\{k, r+1\}$ and $\beta^* = \min\{\beta, 2\}$.

Proof. Using orthogonality of $\mathcal{P}_n$, we have

$$\begin{aligned} |\langle g_t(.), (\mathcal{I} - \mathcal{P}_n)x_0(.) \rangle| &= \sum_{i=1}^{n} |\langle (g_t)_i, (\mathcal{I} - \mathcal{P})(x_0)_i \rangle| \\ &= \sum_{i=1}^{n} |\langle (\mathcal{I} - \mathcal{P})(g_t)_i, (\mathcal{I} - \mathcal{P})(x_0)_i \rangle| \\ &\leq \sum_{i=1}^{n} \left[\|(\mathcal{I} - \mathcal{P})(g_t)_i\|_{2,\Delta_i} \|(\mathcal{I} - \mathcal{P})(x_0)_i\|_{2,\Delta_i} \right]. \end{aligned} \tag{3.30}$$

From estimates (2.15), (3.12) and (3.21), for $l = 1, 2, ..., n$, we have

$$\|(\mathcal{I} - \mathcal{P})(x_0)_l\|_{2,\Delta_l} \leq c h_l^{\frac{1}{2}} \|(\mathcal{I} - \mathcal{P})(x_0)_l\|_{\infty,\Delta_l} = \mathcal{O}(n^{-\left(\beta + \frac{1}{2}\right)}), \tag{3.31}$$

where $\beta = \min\{k, r+1\}$.

Next we consider the following three cases:

Case 1. Let $t \notin \Delta$ and $t \in (t_{i-1}, t_i)$, for $i = 2, 3, ..., n$. Then $(g_t)_j \in \mathcal{C}^k(\Delta_j)$, for $j \neq i$, $(g_t)_i \in \mathcal{C}(\Delta_i)$ and also $(g_t)_1 \in \mathcal{C}(\Delta_1)$. Then from Lemma 7.8 of [7](pp. 330–331), we have for $j \neq i$ and $j = 2, 3, ..., n$

$$\begin{aligned} \|(\mathcal{I} - \mathcal{P})(g_t)_j\|_{2,\Delta_j} &\leq c h_j^{\beta} \max(\|(g_{1t})_j^{(\beta)}\|_{2,\Delta_j}, \|(g_{2t})_j^{(\beta)}\|_{2,\Delta_j}) \\ &\leq c h_j^{\beta + \frac{1}{2}} \max(\|g_{1t}^{(\beta)}\|_{\infty}, \|g_{2t}^{(\beta)}\|_{\infty}) = \mathcal{O}(n^{-\left(\beta + \frac{1}{2}\right)}), \end{aligned} \tag{3.32}$$

and on Δ_i,

$$\begin{aligned} \|(\mathcal{I} - \mathcal{P})(g_t)_i\|_{2,\Delta_i} &\leq c h_i^{\beta_1} \left[\left(\|g_{1t}^{(\beta_1)}\|_{2,[t_{i-1},t]} \right)^2 + \left(\|g_{2t}^{(\beta_1)}\|_{2,[t,t_i]} \right)^2 \right]^{\frac{1}{2}} \\ &\leq c h_i^{\beta_1 + \frac{1}{2}} \left[\left(\|g_{1t}^{(\beta_1)}\|_{\infty} \right)^2 + \left(\|g_{2t}^{(\beta_1)}\|_{\infty} \right)^2 \right]^{\frac{1}{2}} = \mathcal{O}(n^{-\left(\beta_1 + \frac{1}{2}\right)}), \end{aligned} \tag{3.33}$$

where $\beta = \min\{k, r+1\}$ and $\beta_1 = \min\{\beta, 1\}$.
Hence from estimates (3.29), (3.30), (3.31), (3.32) and (3.33) it follows that

$$\begin{aligned}
|\langle g_t(.), (\mathcal{I}-\mathcal{P}_n)x_0\rangle| &\leq \sum_{i=1}^{n} [\|(\mathcal{I}-\mathcal{P})(g_t)_i\|_{2,\Delta_i}\|(\mathcal{I}-\mathcal{P})(x_0)_i\|_{2,\Delta_i}] \\
&\leq \|(\mathcal{I}-\mathcal{P})(g_t)_1\|_{2,\Delta_1}\|(\mathcal{I}-\mathcal{P})(x_0)_1\|_{2,\Delta_1} \qquad (3.34)\\
&+ \sum_{\substack{j=2\\ j\neq i}}^{n} \left[\|(\mathcal{I}-\mathcal{P})(g_t)_j\|_{2,\Delta_j}\|(\mathcal{I}-\mathcal{P})(x_0)_j\|_{2,\Delta_j}\right] \\
&+ \|(\mathcal{I}-\mathcal{P})(g_t)_i\|_{2,\Delta_i}\|(\mathcal{I}-\mathcal{P})(x_0)_i\|_{2,\Delta_i} \\
&= \mathcal{O}(n^{-\min\{2\beta+1, 2\beta, \beta+\beta_1+1\}}) = \mathcal{O}(n^{-(\beta+\beta^*)}), \quad (3.35)
\end{aligned}$$

where $\beta^* = \min\{\beta, \beta_1+1\} = \min\{\beta, 2\}$.
Case 2. Let $t \notin \Delta$ and $t \in (t_0, t_1)$. Then we have $(g_t)_i \in \mathcal{C}^k(\Delta_i)$, for $i = 2, 3, ..., n$ and $(g_t)_1 \in \mathcal{C}(\Delta_1)$. Thus from (3.29), (3.31), (3.32), it follows that

$$\begin{aligned}
|\langle g_t(.), (\mathcal{I}-\mathcal{P}_n)x_0\rangle| &\leq \sum_{i=1}^{n} [\|(\mathcal{I}-\mathcal{P})(g_t)_i\|_{2,\Delta_i}\|(\mathcal{I}-\mathcal{P})(x_0)_i\|_{2,\Delta_i}] \\
&\leq \|(\mathcal{I}-\mathcal{P})(g_t)_1\|_{2,\Delta_1}\|(\mathcal{I}-\mathcal{P})(x_0)_1\|_{2,\Delta_1} \\
&+ \sum_{j=2}^{n} \left[\|(\mathcal{I}-\mathcal{P})(g_t)_j\|_{2,\Delta_j}\|(\mathcal{I}-\mathcal{P})(x_0)_j\|_{2,\Delta_j}\right] \\
&= \mathcal{O}(n^{-\min\{2\beta+1, 2\beta\}}) = \mathcal{O}(n^{-2\beta}). \qquad (3.36)
\end{aligned}$$

Case 3. Let $t \in \Delta$ and $t \neq t_0 = 0$, i.e., $t = t_i,\ i = 1, 2, ..., n$. In such case $g_t \in \mathcal{C}^k_\Delta$, then using Cauchy-Schwartz inequality and Corollary 7.7 of [7](pp 329), we have

$$\begin{aligned}
|\langle g_t(.), (\mathcal{I}-\mathcal{P}_n)x_0\rangle| &\leq \|(\mathcal{I}-\mathcal{P}_n)g_t\|_2\|(\mathcal{I}-\mathcal{P}_n)x_0\|_2 \\
&\leq ch^{2\beta}\|x_0^{\beta}\|_\infty \left[\left(\|g_{1t}^{(\beta)}\|_{\infty,[t_1,t]}\right)^2 + \left(\|g_{2t}^{(\beta)}\|_{\infty,[t,1]}\right)^2\right]^{\frac{1}{2}} \\
&= \mathcal{O}(n^{-2\beta}). \qquad (3.37)
\end{aligned}$$

In case of $t = t_0 = 0$, we have $g(0,\xi) = G(0,\xi)\psi^{(0,1)}(\xi, x_0(\xi)) = 0$. Hence we ignore such possibility. This completes the proof. □

Theorem 7. *Let $x_0 \in \mathcal{C}[0,1] \cap \mathcal{C}^k(0,1]$ be an isolated solution of the equation (2.5) and $\tilde{x}_n$ be the iterated Galerkin approximation of x_0. Then there hold*

$$\|x_0 - \tilde{x}_n\|_\infty = \mathcal{O}(n^{-\min\{2\beta, \beta+\beta^*\}}),$$

where $\beta = \min\{k, r+1\}$ and $\beta^* = \min\{\beta, 2\}$.

Proof. From Theorem 5, 6, Lemma 5 and estimate (3.22), we have

$$\begin{aligned}\|\tilde{x}_n - x_0\|_\infty \leq \frac{\zeta_n}{1-q} &\leq c\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(x_0))^{-1}(\widetilde{\mathcal{T}}_n(x_0) - \mathcal{T}(x_0))\|_\infty \\ &\leq cA_2\|\widetilde{\mathcal{T}}_n(x_0) - \mathcal{T}(x_0)\|_\infty \\ &= cA_2\|\mathcal{K}\psi(\mathcal{P}_n x_0) - \mathcal{K}\psi(x_0)\|_\infty \\ &\leq cA_2\left[Mc_2\|(\mathcal{I} - \mathcal{P}_n)x_0\|_\infty^2 + \sup_{t\in[0,1]}|\langle g_t(.), (\mathcal{I} - \mathcal{P}_n)x_0(.)\rangle|\right] \\ &= \mathcal{O}(n^{-\min\{2\beta, \beta+\beta^*\}}),\end{aligned}$$

where $\beta = \min\{k, r+1\}$ and $\beta^* = \min\{\beta, 2\}$. This completes the proof. □

Remark: From Theorems 4 and 7, we observe that the Galerkin and iterated Galerkin solutions of the Eq. (2.1) converges with the orders, $\mathcal{O}(n^{-\beta})$ and $\mathcal{O}(n^{-\min\{2\beta,\beta+\beta^*\}})$, respectively in infinity norm, where $\beta = \min\{k, r+1\}$ and $\beta^* = \min\{\beta, 2\}$. This shows that the iterated Galerkin approximation improves over the Galerkin approximation and exhibits superconvergence rates.

4 Numerical Example

In this section we present the numerical results. To apply Galerkin and iterated Galerkin methods, we choose the approximating subspaces $\mathbb{X}_n$ to be piecewise polynomial subspaces. Choosing the approximating subspaces to be the space of piecewise constants ($r = 0$) and the space of piecewise linear ($r = 1$) functions, we give the errors in infinity norm in the following Tables and Figures. For computations we use the Newton-Kantorovich method to solve the nonlinear systems. The numerical algorithms are compiled by using Matlab.

In Tables 1 and 3, we present the errors in Galerkin and iterated Galerkin methods with approximating subspace as the space of piecewise constant functions, and in Tables 2 and 4, we give the errors for Galerkin and iterated Galerkin methods with approximating subspace as the space of piecewise linear functions. We denote the Galerkin and iterated Galerkin solutions by x_n and $\tilde{x}_n$, respectively in the following tables. We let $e_n = x_0 - x_n$ and $\tilde{e}_n = x_0 - \tilde{x}_n$. For the following examples, numerical errors $\log_{10}||e_n||_\infty$ and $\log_{10}||\tilde{e}_n||_\infty$ with several values of n are displayed in Figs. 1 and 2. In Tables 1, 2, 3 and 4 and Figs. 1 and 2, n denote the the dimension of the approximating subspace.

Note that, for $r = 0$, the expected orders of convergence of Galerkin and iterated Galerkin approximations are 1 and 2, whereas for $r = 1$, the expected orders of convergence of Galerkin and iterated Galerkin approximations are 2 and 4.

Example 1. We consider the following problem

$$\begin{aligned}&(t^{\frac{1}{2}}x')' = \tfrac{3}{16}(-4 + 5t^{\frac{3}{2}})x^5,\ 0 < t < 1,\\ &x(0) = 1,\ x(1) = \sqrt{\tfrac{1}{2}},\end{aligned}$$

which is equivalent to the following integral equation

$$x(t) - \int_0^1 G(t,\xi)\psi(\xi, x(\xi))d\xi = 1 + \left(\frac{1}{\sqrt{2}} - 1\right)\sqrt{t},$$

where

$$\psi(\xi, x(\xi)) = \frac{3}{16}(-4 + 5\xi^{\frac{3}{2}})x^5,$$

and

$$G(t,\xi) = \left\{ \begin{array}{l} 2\sqrt{t}\left(1-\sqrt{\xi}\right),\ 0 \le t \le \xi, \\ 2\sqrt{\xi}(1-\sqrt{t}),\ \xi \le t \le 1. \end{array} \right\},$$

with exact solution $x(t) = \frac{1}{\sqrt{(1+t^{\frac{3}{2}})}}$.

Table 1. Galerkin method ($r = 0$) for Example 1

n	$\|x - x_n\|_\infty$	*order*	$\|x - \tilde{x}_n\|_\infty$	*order*
2	0.132059198442640	–	0.017792379629884	–
4	0.069651349143622	0.922961591239108	0.004646727136088	1.936972640907684
8	0.034285399785894	1.012556953148701	0.001201782889585	1.951038643047089
16	0.016913931913771	1.029382268635534	0.000303139033661	1.987124749073901

Table 2. Galerkin method ($r = 1$) for Example 1

n	$\|x - x_n\|_\infty$	*order*	$\|x - \tilde{x}_n\|_\infty$	*order*
2	0.008053349211828	–	0.741088614954255e−03	–
4	0.003058166309931	1.396922027522465	0.050676027724683e−03	3.870270714330239
8	0.000814100077235	1.909388804965587	0.002590881847508e−03	4.289788311640125
16	0.000208729394784	1.963572370176226	0.000108841167523e−03	4.573146983040751

Example 2. We consider the following problem

$$\begin{array}{l} (t^\alpha x')' = \beta t^{\alpha+\beta-2}(\beta t^\beta + \alpha + \beta - 1)x,\ 0 < t \le 1, \\ x(0) = 1,\ x(1) = e, \end{array}$$

which is equivalent to the following problem

$$x(t) - \int_0^1 G(t,\xi)\psi(\xi, x(\xi))d\xi = 1 + (e-1)\sqrt{t},$$

where $\psi(\xi, x(\xi)) = \beta\xi^{\alpha+\beta-2}(\beta\xi^\beta + \alpha + \beta - 1)x$ and the exact solution $x(t) = e^{t^\beta}$. We consider the case for $\alpha = 0.5$ and $\beta = 1$. Hence we take

$$G(t,\xi) = \left\{ \begin{array}{l} 2\sqrt{t}\left(1-\sqrt{\xi}\right),\ 0 \le t \le \xi, \\ 2\sqrt{\xi}(1-\sqrt{t}),\ \xi \le t \le 1. \end{array} \right.$$

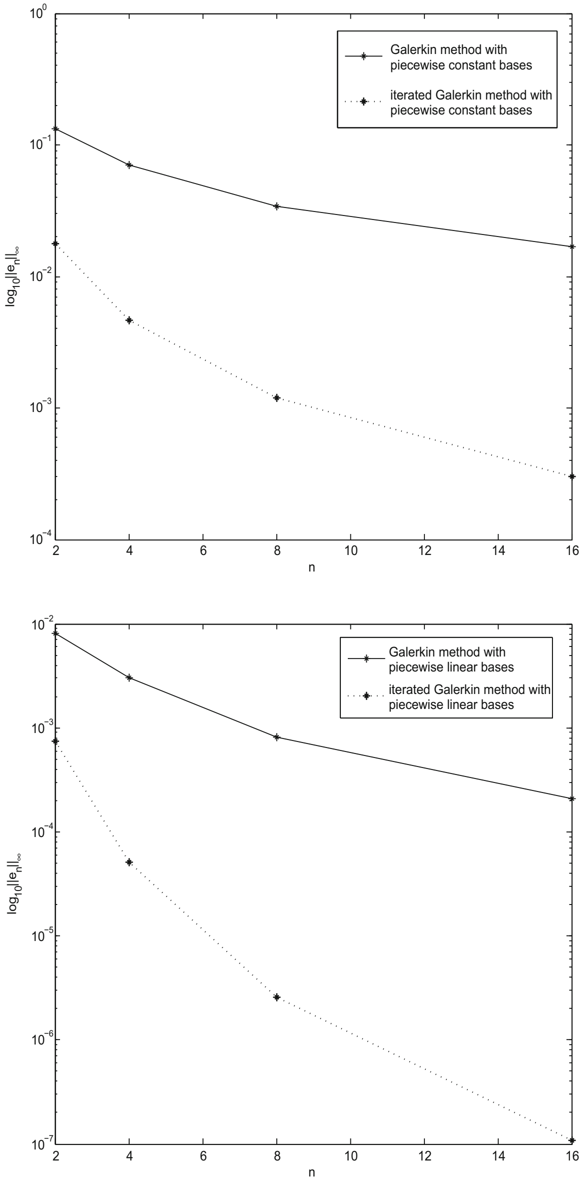

Fig. 1. Numerical error comparison between piecewise polynomial based Galerkin and iterated Galerkin method for Example 1

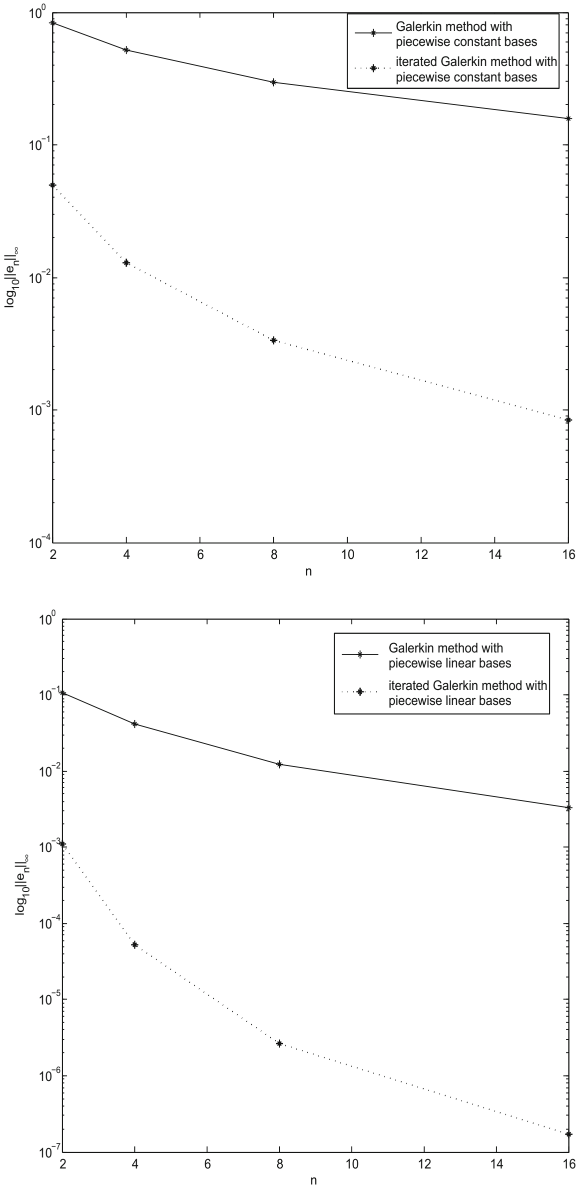

Fig. 2. Numerical error comparison between piecewise polynomial based Galerkin and iterated Galerkin method for Example 2

Table 3. Galerkin method ($r = 0$) for Example 2

n	$\|x - x_n\|_\infty$	*order*	$\|x - \tilde{x}_n\|_\infty$	*order*
2	0.829362419065241	–	0.049589588961522	–
4	0.522389237950712	0.666877500445172	0.012990218437040	1.932611576551135
8	0.295616770786710	0.821397057299279	0.003374372591959	1.944734503388001
16	0.157713940270086	0.906417933094777	0.000843420076282	2.000296013915911

Table 4. Galerkin method ($r = 1$) for Example 2

n	$\|x - x_n\|_\infty$	*order*	$\|x - \tilde{x}_n\|_\infty$	*order*
2	0.106866377186242	–	0.109295200681e−02	–
4	0.041034517053983	1.380898137740918	0.052498410171e−03	3.879812507463321
8	0.012207364667423	1.749086189798126	0.002636623206e−03	4.315510415335539
16	0.003293085749748	1.890239799793947	0.000170827800e−03	4.548076733232739

5 Conclusion

In this paper, after applying Galerkin and iterated Galerkin methods to solve the nonlinear Fredholm integral equation of the type (1.1) with Green's kernel (1.2), we are able to obtain superconvergence results for the iterated Galerkin approximations, using suitable graded mesh. It is also clear from the numerical errors given in from of tables and figures that the numerical results validate the theoretical results.

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A Learning Effected Imperfect Production Inventory Model for Several Markets with Fuzzy Trade Credit Period and Inflation

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Abstract. This paper considers the joint relationship among supplier, manufacturer-cum-retailer and multiple markets in which manufacturer-cum-retailer gets a facility of fuzzy credit period for purchasing of raw materials from supplier. The manufacturer produces the finished goods along with a constant defective rate. Here the finished product is transported to different markets in different seasons, with a transportation cost that depends on the amount of transportation and learning ratio. Also, the demand of the item is different in different markets. Further, the optimal operation policy that maximizes total profit of the integrated system is derived under a constant rate of inflation. But due to impreciseness in trade credit period, profit function becomes fuzzy in nature thereby determining the optimal values of decision variables, equivalent crisp profit function is procured by applying fuzzy expectation method. The necessity optimal conditions of the objective and its concavity properties have been derived to obtain maximum profit. Finally, the models are illustrated with certain numerical and graphical solutions provided with sensitivity analysis of model's parameters.

Keywords: Imperfect production · Learning effect · Fuzzy trade credit · Inflation · Several markets

1 Introduction

The main objective of inventory management deals with maximization of the total inventory profit for which it is required to determine the optimal inventory policy to meet the future demand. Generally a manufacturer produces an item and it is sold at different markets. But at-times, it has been observed that the markets have different selling seasons. Hence, manufacturer/supplier

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has to adopt the appropriate management policies/strategies in the business with the different markets. In a production inventory model with deteriorating items, He et al. [6] also considered multiple-market demands. Then Pal et al. [14] researched on multi-echelon supply chain model in multiple markets with supply disruption. However, in today's contemporary business era, it can be observed that suppliers often offer a period to settle the account to accumulate the future demand. The concept of trade credit was first introduced by Haley and Higgins [5]. Goyal [4] was the first who established an EOQ model with a constant demand rate under the condition of permissible delay in payments. Later, Aggarwal and Jaggi [1] generalized the EOQ model from non-deteriorating items to deteriorating items. Mahata and Goswami [9] introduced an inventory model of deteriorating items under trade credit financing in the fuzzy sense. Seifert et al. [18] organized a review of trade credit literature and provided a detailed agenda for future research. Taleizadeh et al. [19] investigated supply chain problem with stochastic demand, fuzzy lead-time using particle swarm optimization and fuzzy simulation. Das et al. [3] studied an integrated production inventory model under interactive fuzzy credit period for deteriorating item with several markets. Ouyang et al. [13] proposed an integrated inventory model with capacity constraint and a order-size dependent permissible delay payment period.

During last few decades, due to high inflation [20] and consequent sharp decline in the purchasing power of money in the developing countries like India, Bangladesh etc., the financial situation has been changed and it is not possible to ignore the effect of inflation and time value of money any further. Recently, Mousavi et al. [10] presented a seasonal multi-product multi-period inventory control model with inventory costs obtained under inflation and all-unit discount policy.

Salameh and Jaber [15] were the first to come up with the concept of imperfect quality. Later, Maddah and Jaber [8] amended Salameh and Jaber's model. Meanwhile, Sana [16] presented an EPL model with random imperfect production process and defective units were repaired immediately when they were produced. Chen [2] investigated a problem of production with preventive maintenance, inspection and inventory for an imperfect production process. Ouyang and Chang [12] then built up an EPQ model with imperfect production process and complete backlogging. Sarkar et al. [17] addressed an integrated inventory model for defective products with variable lead time and permissible delay in payments. In the context of earlier investigations, we have considered the following new deliberations:

- Integrated inventory model with multiple markets is considered.
- The finished products' transportation cost is reduced in each order due to the learning factors of the manufacturer-cum-retailer.
- An imprecise trade credit period is offered by the supplier to the manufacturer.
- Inflation and time value money are incorporated.

2 Mathematical Prerequisites for Defuzzification

Definition 1. Fuzzy Extension Principle [11]**:** *If* $\tilde{a}$, $\tilde{b} \subseteq \Re$ *and* $\tilde{c} = f(\tilde{a}, \tilde{b})$, *where* $f : \Re \times \Re \to \Re$ *is a binary operation, membership function* $\mu_{\tilde{c}}$ *of* $\tilde{c}$ *is defined as For each* $z \in \Re$, $\mu_{\tilde{c}}(z) = sup\{min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)),\ x, y \in \Re$ *and* $z = f(x, y)\}$

Lemma 1. *[7] The expected value of triangular fuzzy variable* $\tilde{A} = (a_1, a_2, a_3)$ *is given as* $E[\tilde{A}] = \frac{1}{2}[(1-\rho)a_1 + a_2 + \rho a_3]$ *where* ρ $(0 \leq \rho \leq 1)$ *is the degree of optimism/pessimism for decision maker.*

Lemma 2. *[7] If* $\tilde{A} = (a_1, a_2, a_3, a_4)$ *is a TrFN, then expected value of* $\tilde{A}$, $E[\tilde{A}]$, *is given by* $E[\tilde{A}] = \frac{1}{2}[(1-\rho)(a_1 + a_2) + \rho(a_3 + a_4)]$ *where* ρ $(0 \leq \rho \leq 1)$ *is the degree of optimism/pessimism for decision maker.*

3 Development of an Imperfect Production Inventory Model for a Manufacturer-Cum-Retailer and Several Seasonal Markets

A manufacturer produces finished goods along with a constant imperfect rate and delivered it to different seasonal markets where the demand rate and time duration of each market are different. The manufacturer gives an opportunity of initial part payment to those markets who receive the goods during the production run time and the remaining amount paid at the end of the business period. To produce the finished goods, manufacturer received the raw-material instantly rate from the supplier who offers an imprecise credit period to the manufacturer. The proposed model is formulated in terms of integrated total profit under the following notations:

$I(t)$	Inventory level at time t.
P	Manufacturer's production rate per year.
t_1	Length of time from the beginning and end of production. It has been taken as decision variable.
M	Manufacturer's trade credit period offered by raw materials supplier in years.
f	Unit usage of raw materials per finished product.
λ	Percentage of defective rate.
Cs_p	Manufacturer's set up cost.
Cs_m	Market's set up cost.
C_p	Unit production cost.
H_r	Unit stock holding cost of raw materials.
H_p	Unit stock holding cost per perfect finished product of manufacturer.
H_m	Unit stock holding cost per perfect finished product at markets.
r	The discount rate.
i	The inflation rate, which is varied by the social economical situations (e.g., consumer price index(CPI)and producer price index(PPI)), and the company operation status(e.g.,operation cost index,and productivity index).
R	$= r - i$, is the difference between discount rate and inflation rate.
d_i	Customers' demand rate per year for i^{th} market.
T_i	Starting time of the business of i^{th} market.
T_{ei}	Time at which the selling season ends for i^{th} market.
n	Number of markets where the products are transported from the manufacturer.
Q_i	Quantity received by i^{th} market from the manufacturer.
WTP	Total manufacturer's profit.
MTP	Total markets' profit.
ITP	Total profit for the integrated system.

3.1 Raw Material's Inventory for Manufacturer

The manufacturer receives all the required quantity of raw materials instantaneously from the raw material supplier to produce the finished good when he/she is going to start his/her production. Then the inventory of raw materials depletes gradually with time due to production and completely depleted at time t_1. So, the raw material's inventory $I_r(t)$ at time t satisfies $\frac{dI_r(t)}{dt} = -fP$ with the boundary condition $I_r(t_1) = 0$. Hence, the inventory level $I_r(t)$ at time t is $I_r(t) = fP(t_1 - t)$ and the quantity of raw materials received by the manufacturer is $Q_r = I_r(0) = fPt_1$. Present value of raw material's holding cost is

$$HC_r = H_r \int_0^{t_1} e^{-Rt} I_r(t)dt = H_r \int_0^{t_1} e^{-Rt} fP(t_1 - t)dt = H_r fP\left\{\frac{t_1}{R} - \frac{1 - e^{-Rt_1}}{R^2}\right\}$$

The manufacturer needs to make the full payments of the raw-material at the end of the credit period M, otherwise manufacturer will have to pay an interest to the supplier. So, present value of interest payable by the manufacturer is

$$IP_r = C_r i_c \int_M^{t_1} e^{-Rt} I_r(t)dt = C_r i_c fP\left\{\frac{(t_1 - M)e^{-RM}}{R} - \frac{e^{-RM} - e^{-Rt_1}}{R^2}\right\}$$

Present value of manufacturer's purchase cost for the raw materials will be

$$PC_r = C_r \int_0^{t_1} fPe^{-Rt}dt = \frac{C_r fP(1 - e^{-Rt_1})}{R}$$

3.2 Manufacturer's Finished Products' Inventory

The manufacturer starts production of the finished product along with a constant defective rate λ from $t = 0$ and then the inventory level increases over time. After time T_1, the first market receives its required quantity Q_1 instantaneously, then at time T_2, the second market receives its required quantity Q_2 instantaneously and so on. Gradually, as the production ceases at time t_1, the inventory level decreases due to the remaining markets' instantaneous replenishment and the inventory completely depleted after the receipt of the quantity Q_n by the last market. Let the manufacturer's inventory level in the interval $[T_{k-1}, T_k]$ $(k = 1, 2, \cdots, n)$ be $I_k(t)$ and $I_{m-}(t)$, $I_{m+}(t)$ be the inventory levels in time intervals $[T_{m-1}, t_1]$, $[t_1, T_m]$ respectively. Then the differential equations of the finished products' inventory levels at time t in $[0, T_n]$ are as follows (cf. Figure 1): $\frac{dI_1(t)}{dt} = (1 - \lambda)P,\ 0 \le t \le T_1$; $\frac{dI_k(t)}{dt} = (1 - \lambda)P,\ T_{k-1} \le t \le T_k,\ k = 2, 3, \cdots, m - 1$; $\frac{dI_{m-}(t)}{dt} = (1 - \lambda)P,\ T_{m-1} \le t \le t_1$; $\frac{dI_{m+}(t)}{dt} = 0,\ t_1 \le t \le T_m$; $\frac{dI_k(t)}{dt} = 0,\ T_{k-1} \le t \le T_k, k = m + 1, m + 2, \cdots, n$. with boundary conditions $I_1(0) = 0; I_k(T_{k-1}) = I_{k-1}(T_{k-1}) - Q_{k-1},\ k = 2, 3, \cdots, m - 1; I_{m-}(T_{m-1}) = I_{m-1}(T_{m-1}) - Q_{m-1}; I_{m+}(T_m) = I_{m+1}(T_m) + Q_m; I_k(T_{k-1}) = I_{k-1}(T_{k-1}) - Q_{k-1},\ k = m + 1, m + 2, \cdots, n$ and $I_n(T_n) = Q_n$.

Solving these equations, we get the level of inventory at different time t as $I_k(t) = (1 - \lambda)Pt - \sum_{i=1}^{k-1} Q_i,\ T_{k-1} \le t \le T_k\ k = 1, 2, \cdots, m - 1; I_{m-}(t) = (1 - \lambda)Pt - \sum_{i=1}^{m-1} Q_i,\ T_{m-1} \le t \le t_1; I_{m+}(t) = \sum_{i=m}^{n} Q_i, t_1 \le t \le T_m$ and $I_k(t) = \sum_{i=k}^{n} Q_i,\ T_{k-1} \le t \le T_k\ k = m + 1, m + 2, \cdots, n$. Using the continuity condition at $t = t_1$, we have $P = \frac{1}{(1-\lambda)t_1} \sum_{i=1}^{n} Q_i$ which gives the relation between the two variables P and t_1.

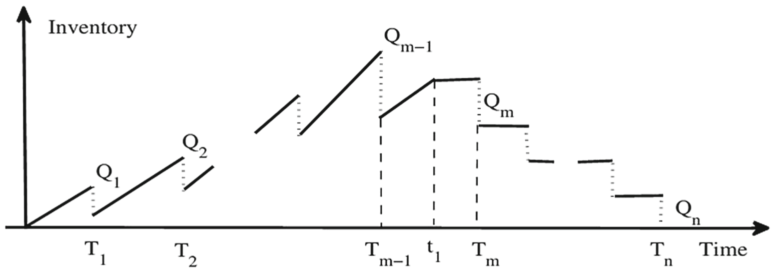

Fig. 1. Manufacturer's finished product inventory vs. time

The present value of holding cost for the manufacturer's finished product is $HC_p = H_p\Big[\sum_{k=1}^{m-1}\int_{T_{k-1}}^{T_k}\Big\{(1-\lambda)Pt - \sum_{i=1}^{k-1} Q_i\Big\}e^{-Rt}dt + \int_{T_{m-1}}^{t_1}\Big\{(1-\lambda)Pt - \sum_{i=1}^{m-1} Q_i\Big\}e^{-Rt}dt + \int_{t_1}^{T_m}\sum_{i=m}^{n} Q_i e^{-Rt}dt + \sum_{k=m+1}^{n}\int_{T_{k-1}}^{T_k}\sum_{i=k}^{n} Q_i e^{-Rt}dt\Big]$
$= H_p\Big[(1-\lambda)P\Big\{\frac{1-e^{-Rt_1}}{R^2} - \frac{t_1 e^{-Rt_1}}{R}\Big\} - \sum_{k=1}^{m-1}\frac{e^{-RT_{k-1}}-e^{-RT_k}}{R}\sum_{i=1}^{k-1} Q_i - \frac{e^{-RT_{m-1}}-e^{-Rt_1}}{R}\sum_{i=1}^{m-1} Q_i + \frac{e^{-Rt_1}-e^{-RT_m}}{R}\sum_{i=m}^{n} Q_i + \sum_{k=m+1}^{n}\frac{e^{-RT_{k-1}}-e^{-RT_k}}{R}\sum_{i=k}^{n} Q_i\Big]$

The present value of manufacturer's production cost is $PC_p = C_p P e^{-Rt}dt = \frac{C_p P(1-e^{-Rt_1})}{R}$. The present value for transportation cost of perfect products ordered by the market which is reduced at a learning rate γ is $TC_p = \sum_{i=1}^{n}\{T_{r0}+T_{r1}e^{-\gamma(i-1)}Q_i\}e^{-RT_i}$. The present value of manufacturer's sales revenue is $SR_p = S_p\sum_{i=1}^{n} Q_i e^{-RT_i}$. The present value of manufacturer's interest earned is $IE_p = \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_e S_p \rho Q_i \int_{T_i}^{T_n} e^{-Rt}dt + \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_e S_p(1-\rho)Q_i\int_{T_{ei}}^{T_n} e^{-Rt}dt + \sum_{\substack{i=m\\T_i>t_1}}^{n} i_e S_p Q_i \int_{T_i}^{T_n} e^{-Rt}dt$
$= \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_e S_p \rho Q_i \frac{e^{-RT_i}-e^{-RT_n}}{R} + \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_e S_p(1-\rho)Q_i\frac{e^{-RT_{ei}}-e^{-RT_n}}{R} + \sum_{\substack{i=m\\T_i>t_1}}^{n} i_e S_p Q_i \frac{e^{-RT_i}-e^{-RT_n}}{R}$.

The present value for manufacturer's total profit is given by $WTP(t_1) =$ *sales revenue + interest earned - set-up cost - raw material's purchase cost - raw material's holding cost - production cost - finished products holding cost - transportation cost - interest payable.* i.e.

$$WTP(t_1) = SR_p + IE_p - Cs_p - PC_r - HC_r - PC_p - HC_p - TC_p - IP_r$$

3.3 The Markets' Inventory

The i^{th} market receives its total required quantity Q_i of finished good from the manufacturer at the beginning of its selling season to fulfil the customers' demand rate d_i. These markets start their business on or before the production run time t_1, by paying ρ portion of the price amount initially and the remaining $(1-\rho)$ portion at the end of his business period. But these markets take the delivery after the production run time t_1, by paying the total amount at their business starting time. They pay the initial amount by getting loan from a bank at the rate of interest of i_c per year. Every market earns interest at the rate of i_d by depositing sales revenue continuously. The inventory level $J_i(t)$ for the i^{th} market is governed by the following differential equation: $\frac{dJ_i(t)}{dt} = -d_i, i = 1, 2, 3, \cdots, n.$ with the boundary condition $J_i(T_{ei}) = 0$. Using these boundary conditions, the solutions are $J_i(t) = d_i(T_{ei} - t)$. The quantity of products received by the each market is $Q_i = J_i(T_i) = d_i(T_{ei} - T_i)$. The present value of holding cost for all markets is $HC_m = \sum_{i=1}^{n} H_m \int_{T_i}^{T_{ei}} J_i(t)e^{-Rt}dt = \sum_{i=1}^{n} H_m d_i \left\{ \frac{(T_{ei}-T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i}-e^{-RT_{ei}}}{R^2} \right\}$. The present value of all markets' total sales revenue is $SR_m = \sum_{i=1}^{n} S_m d_i \int_{T_i}^{T_{ei}} e^{-Rt}dt = \sum_{i=1}^{n} S_m d_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R}$. The present value of all markets' purchase cost PC_m is equal to present value of sales revenue SR_p of the manufacturer. The present value of all markets' total interest payable is $IP_m = \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \int_{T_i}^{T_{ei}} e^{-Rt}dt + \sum_{\substack{i=m\\T_i > t_1}}^{n} i_c S_p Q_i \int_{T_i}^{T_{ei}} e^{-Rt}dt = \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_c S_p \rho Q_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R} + \sum_{\substack{i=m\\T_i > t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R}$. The present value of all markets' total interest earned is $IE_m = \sum_{i=1}^{n} i_e S_m d_i \int_{T_i}^{T_{ei}} (T_{ei} - t)e^{-Rt}dt = \sum_{i=1}^{n} i_e S_m d_i \left\{ \frac{(T_{ei}-T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i}-e^{-RT_{ei}}}{R^2} \right\}$. Therefore, the total profit of all markets is given by

$$MTP(t_1) = SR_m + IE_m - PC_m - HC_m - IP_m - nCs_m$$

3.4 Model-1: Imperfect Production Inventory Model Considering Together the Defective Production and Inflation on Sales Revenues and Inventory Cost

The total profit of the described model for the integrated system is sum of the manufacturer's total profit and that of from all markets. Substituting the value of P in the profit functions, finally integrated profit of the system is

$ITP(t_1) = \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_e S_p \rho Q_i \frac{e^{-RT_i}-e^{-RT_n}}{R} + \sum_{\substack{i=1\\T_i \le t_1}}^{m-1} i_e S_p (1-\rho) Q_i \frac{e^{-RT_{ei}}-e^{-RT_n}}{R} + \sum_{\substack{i=m\\T_i > t_1}}^{n} i_e S_p Q_i \frac{e^{-RT_i}-e^{-RT_n}}{R} - Cs_p - \frac{C_r f(1-e^{-Rt_1})}{(1-\lambda)Rt_1} \sum_{i=1}^{n} Q_i - \frac{H_r f}{(1-\lambda)} \left\{ \frac{1}{R} - \frac{(1-e^{-Rt_1})}{R^2 t_1} \right\} \sum_{i=1}^{n} Q_i - \frac{C_p(1-e^{-Rt_1})}{(1-\lambda)Rt_1} \sum_{i=1}^{n} Q_i - H_p \left[\left\{ \frac{1-e^{-Rt_1}}{R^2 t_1} - \frac{e^{-Rt_1}}{R} \right\} \sum_{i=1}^{n} Q_i - \sum_{k=1}^{m-1} \frac{e^{-RT_{k-1}}-e^{-RT_k}}{R} \sum_{i=1}^{k-1} Q_i - \frac{e^{-RT_{m-1}}-e^{-Rt_1}}{R} \sum_{i=1}^{m-1} Q_i + \frac{e^{-Rt_1}-e^{-RT_m}}{R} \sum_{i=m}^{n} Q_i + \sum_{k=m+1}^{n} \frac{e^{-RT_{k-1}}-e^{-RT_k}}{R} \sum_{i=k}^{n} Q_i \right] - \sum_{i=1}^{n} \{T_{r0} +$

$T_{r1}e^{-\gamma(i-1)}Q_i\}e^{-RT_i} - \frac{C_r i_c f}{(1-\lambda)}\left\{\frac{(t_1-M)e^{-RM}}{Rt_1} - \frac{e^{-RM}-e^{-Rt_1}}{R^2 t_1}\right\}\sum_{i=1}^{n} Q_i +$ $\sum_{i=1}^{n} S_m d_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R} + \sum_{i=1}^{n} i_e S_m d_i\left\{\frac{(T_{ei}-T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i}-e^{-RT_{ei}}}{R^2}\right\} -$ $\sum_{i=1}^{n} H_m d_i\left\{\frac{(T_{ei}-T_i)e^{-RT_i}}{R} - \frac{e^{-RT_i}-e^{-RT_{ei}}}{R^2}\right\} - \sum_{\substack{i=1\\T_i\le t_1}}^{m-1} i_c S_p \rho Q_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R} +$ $\sum_{\substack{i=m\\T_i>t_1}}^{n} i_c S_p Q_i \frac{e^{-RT_i}-e^{-RT_{ei}}}{R} - nCs_m$

3.5 Model-2: Imperfect Production Inventory Model Considering Without Inflation

This is the model same as the Model 1 without considering the inflation on the sales revenue and related inventory costs. Thus, taking $R \to 0$ in the integrated profit function, we have obtained the integrated profit of the Model 2.

3.6 Model-3: Production Inventory Model Without both Defective and Inflation

This model is formulated assuming no number of produced quantity is defective of the Model 2. Therefore, taking $\lambda = 0$ in the Model 2, we have obtained profit expression of Model 3. Our objective is to find out optimal values of t_1 such that the integrated total profit functions $ITP(t_1)$s are maximum for the Models 1, 2 and 3.

Lemma 3. *Manufacturer's production run time* (t_1) *must satisfy the condition* $1 \le \frac{t_1}{T_k} \le \sum_{i=1}^{n} Q_i / \sum_{i=1}^{k} Q_i$ *for all* $k = 1, 2, \cdots, m-1$.

Proof. According to our assumption, for $k = 1, 2, \cdots, m-1$ we have $T_k \le t_1$ and $I_k(t) = (1-\lambda)Pt - \sum_{i=1}^{k-1} Q_i$. As shortages are not allowed for the system, for each k=1, 2, $\cdots$, m-1 we have $I_k(T_k) \ge Q_k$ or, $(1-\lambda)PT_k - \sum_{i=1}^{k-1} Q_i \ge Q_k$ or, $\frac{\sum_{i=1}^{n} Q_i}{t_1} T_k \ge \sum_{i=1}^{k-1} Q_i + Q_k$ [Using the value of P] or, $T_k \le t_1 \le T_k \frac{\sum_{i=1}^{n} Q_i}{\sum_{i=1}^{k} Q_i}$ [Since $T_k \le t_1$] or, $1 \le \frac{t_1}{T_k} \le \frac{\sum_{i=1}^{n} Q_i}{\sum_{i=1}^{k} Q_i}$

Lemma 4. *The integrated profit function* $ITP(t_1)$ *for Model 1 is maximum when* $\{C_r f R - H_r f + C_p R + (1-\lambda)H_p\}(1 - e^{-Rt_1} - Rt_1 e^{-Rt_1}) - C_r f i_c(e^{-RM} + RMe^{-RM} - e^{-Rt_1} - Rt_1 e^{-Rt_1}) = 0$ *and* $[\{C_r f R - H_r f + C_p R + (1-\lambda)H_p\}\{1-(1+Rt_1 + R^2 t_1^2)e^{-Rt_1}\} + C_r f i_c(R^2 t_1^2 e^{-Rt_1} + Rt_1 e^{-Rt_1} - RMe^{-RM} + e^{-RM} - e^{-Rt_1})] > 0$.

Proof. Taking the first and second order derivatives of $ITP(t_1)$ with respect to t_1 and setting first derivative equal to 0 and for the maximum value of $ITP(t_1)$, considering $\frac{d^2 ITP(t_1)}{dt_1^2} < 0$ the proposed lemma is proved easily.

Lemma 5. *The integrated profit function* $ITP(t_1)$ *for Model 2 is maximum at* $t_1^* = M\sqrt{\frac{C_r f i_c}{H_r f + C_r f i_c - (1-\lambda)H_p}}$ *and* t_1^* *is feasible if* $H_r f < (1-\lambda)H_p < H_r f + C_r f i_c$.

Proof. Taking the first and second order derivatives of $ITP(t_1)$ with respect to t_1 and setting first derivative equal to 0 gives: $\frac{dITP(t_1)}{dt_1} = -\frac{H_r f}{2(1-\lambda)}\sum_{i=1}^{n} Q_i - \frac{C_r f i_c}{2(1-\lambda)}(1-\frac{M^2}{t_1^2})\sum_{i=1}^{n} Q_i - H_p(\frac{1}{2}\sum_{i=1}^{n} Q_i - \sum_{i=1}^{m-1} Q_i - \sum_{i=m}^{n} Q_i) = 0$ or, $t_1 = M\sqrt{\frac{C_r f i_c}{H_r f + C_r f i_c - (1-\lambda)H_p}}$

$\frac{d^2 ITP(t_1)}{dt_1^2} = -\frac{C_r f i_c}{(1-\lambda)t_1^3}\sum_{i=1}^{n} Q_i < 0$, for all $t_1 > 0$. Therefore, $ITP(t_1)$ attends maximum at $t_1^* = M\sqrt{\frac{C_r f i_c}{H_r f + C_r f i_c - (1-\lambda)H_p}}$. Now, according to our assumption, the value t_1 obtained is feasible

if $t_1 < M$ and $H_r f + C_r f i_c - (1-\lambda)H_p > 0$

or, if $M\sqrt{\frac{C_r f i_c}{H_r f + C_r f i_c - (1-\lambda)H_p}} < M$ and $(1-\lambda)H_p < H_r f + C_r f i_c$

or, if $H_r f < (1-\lambda)H_p$ and $(1-\lambda)H_p < H_r f + C_r f i_c$

or, if $H_r f < (1-\lambda)H_p < H_r f + C_r f i_c$.

Lemma 6. *The integrated profit function $ITP(t_1)$ for Model 3 is maximum at $t_1^* = M\sqrt{\frac{C_r f i_c}{H_r f + C_r f i_c - H_p}}$ and t_1^* is feasible if $H_r f < H_p < H_r f + C_r f i_c$.*

Proof. The proof is similar as Lemma 5 with taking $\lambda = 0$.

3.7 Formulation of Models with Fuzzy Credit Period

Let us consider that the raw material supplier gives an opportunity to the manufacturer-cum-retailer by offering a fuzzy credit period $(\widetilde{M})$. Here, the credit period $\widetilde{M}$ is represented as triangular fuzzy number and Trapezoidal Fuzzy Number. So due to fuzzy credit period $(\widetilde{M})$, the optimum values of integrated profit function $ITP(t_1)$ will be different for various values of M with some degree of belongingness. Therefore in such situation, the profit function will be fuzzy in nature and it is denoted by $\widetilde{ITP}(t_1)$. The expected value of the decision variable t_1 is obtained from the following equation

$\{C_r f R - H_r f + C_p R + (1-\lambda)H_p\}(1 - e^{-Rt_1} - Rt_1 e^{-Rt_1}) - C_r f i_c(e^{-RE[M]} + RE[M]e^{-RE[M]} - e^{-Rt_1} - Rt_1 e^{-Rt_1}) = 0$.

and the corresponding concavity condition is reduced to

$[\{C_r f R - H_r f + C_p R + (1-\lambda)H_p\}\{1-(1+Rt_1+R^2 t_1^2)e^{-Rt_1}\} + C_r f i_c(R^2 t_1^2 e^{-Rt_1} + Rt_1 e^{-Rt_1} - RE[M]e^{-RE[M]} + e^{-RE[M]} - e^{-Rt_1})] > 0$.

Similarly, $\widetilde{ITP}(t_1)$ for the model without inflation (Model-2) and the model without both inflation and defective (Model-3) are obtained by substituting M by $\widetilde{M}$.

3.8 Defuzzification Algorithm to Get the Optimum Value of t_1

To get the optimum value of the production run time (t_1) in the proposed integrated models with fuzzy credit period, the following steps are used.

Step 1: At first to get the expression of fuzzy integrated profit $\widetilde{ITP}(t_1)$, M is replaced by $\widetilde{M}$.
Step 2: Calculate the expected value of $\widetilde{ITP}(t_1)$ i,e, $E[\widetilde{ITP}(t_1)]$ for the fuzzy credit period $\widetilde{M}$ by using Lemma 1 and 2 and the fuzzy extension principle.
Step 3: To get the optimal value of t_1, solve modified expected equations obtained from first order derivative of the objective.
Step 4: Putting the value of t_1, check the Lemma 3 and concavity condition of objective function.
Step 5: If both Lemma 3 and concavity condition are satisfied by t_1, Putting the value of t_1 in the expected profit function to obtain the maximum profit.

Similar procedure is applied to get the optimal solution for model without inflation (Model-2) and model without both inflation and defective rate (Model-3).

4 Numerical Problems and Results

To illustrate the proposed models, three different types of trade credit periods are considered. In **Problem 1**, deterministic model and in other two problems **Problem 2** and **Problem 3**, fuzzy models with as triangular and trapezoidal fuzzy trade credit period are considered respectively.

Problem 1: Let a manufacturer-cum-retailer sells the finished perfect product to three different markets in different selling seasons. The required raw-material is procured by the supplier. Here, the supplier offers a crisp credit period to the manufacturer to settle the account. We consider such a supply chain situation with the following data:
$M = 0.25$ year, $f = 1.75$ units, $H_r = \$2.00$/unit, $C_r = \$5.00$/unit, $Cs_p = \$5000.00$/cycle, $C_p = \$1.50$/unit, $H_p = \$4.0$/unit/year, $i_c = 0.15/\$$/unit, $i_e = 0.10/\$$/unit, $R = 0.10$, $\lambda = 0.10$, $\rho = 0.50$, $S_p = \$30$/unit, $T_{r0} = \$100.00$/transport, $T_{r1} = \$2.50$/unit, $\gamma = 0.25$, $T_1 = 0.20$ year, $T_2 = 0.45$ year, $T_3 = 1.00$ year, $T_{e1} = 0.60$ year, $T_{e2} = 0.95$ year, $T_{e3} = 1.40$ year, $Cs_m = \$2000$/order, $S_m = \$40$/unit, $H_m = \$6.0$/unit/year, $d_1 = 2000$ unit/year, $d_2 = 4000$ unit/year, $d_3 = 4000$ unit/year.

Table 1. Optimum results for Models 1, 2 and 3 under Problem 1

Model	t_1^*	ITP^*	P^*	IP_r^*	IE_p^*	PC_r^*	HC_r^*	PC_p^*	HC_p^*	TR_p^*
1	0.6707	79437	7288	814	3018	41374	5612	7093	4471	8162
2	0.2601	90371	18796	1	4740	42778	2225	7333	8351	8620
3	0.3177	95628	13847	42	4740	38500	2446	6600	7843	8620

Now, the manufacturer-cum-retailer is interested to find out the optimal profits jointly with the markets along with optimal production run time. The optimal solution are presented in Table 1 and the concavity property of the models are graphically shown in Figs. 2, 3 and 4.

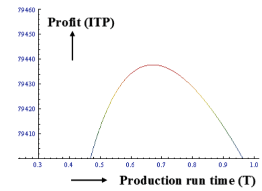

Fig. 2. Integrated profit against production run time for Model-1

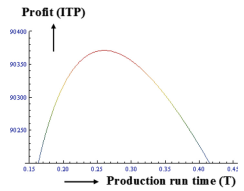

Fig. 3. Integrated profit against production run time for Model-2

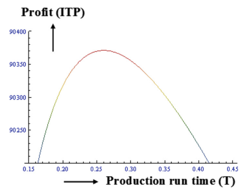

Fig. 4. Integrated profit against production run time for Model-3

Problem 2: Let supplier's offered trade credit period be a triangular fuzzy number & is defined as $\widetilde{M} = (M_0 - \Delta_1, M_0, M_0 + \Delta_2)$, where $M_0 = 0.25$ years, $0 < \Delta_1 < M_0$ and $0 < \Delta_2$. For different values of Δ_1 and Δ_2, Model-1, 2 and 3 are optimized for the same input data as in Problem 1 and optimum results are presented in Table 2.

Table 2. Optimum results for Models 1, 2 and 3 under Problem 2

Δ_1	Δ_2	Model-1			Model-2			Model-3		
		t_1^*	ITP^*	P^*	t_1^*	ITP^*	P^*	t_1^*	ITP^*	P^*
0.01	0.05	0.6980	79477	7004	0.2705	90374	18073	0.3304	95640	13315
0.02	0.04	0.6843	79457	7144	0.2653	90372	18427	0.3241	95634	13576
0.03	0.03	0.6707	79437	7288	0.2601	90371	18796	0.3177	95627	13847
0.04	0.02	0.6571	79418	7439	0.2549	90370	19179	0.3114	95621	14130
0.05	0.01	0.6435	79398	7597	0.2497	90368	19579	0.3050	95615	14424

Problem 3: This problem input are similar as Problem 1 with considering supplier's offered trade credit period be a trapezoidal fuzzy number which is represented as $\widetilde{M} = (M_0 - \Delta_1, M_0 - \Delta_2, M_0 + \Delta_3, M_0 + \Delta_4)$, where $M_0 = 0.25$ years, $0 < \Delta_2 < \Delta_1 < M_0$ and $0 < \Delta_3 < \Delta_4$. For different values of Δ_1, Δ_2, Δ_3 and Δ_4, Model-1, 2 and 3 are optimized and optimum results are presented in Table 3.

Table 3. Optimum results for Models 1, 2 and 3 under Problem 3

Δ_1	Δ_2	Δ_3	Δ_4	Model-1			Model-2			Model-3		
				t_1^*	ITP^*	P^*	t_1^*	ITP^*	P^*	t_1^*	ITP^*	P^*
0.01	0.006	0.010	0.05	0.7006	79481	6977	0.2715	90374	18004	0.3317	95641	13264
0.02	0.007	0.009	0.04	0.6857	79459	7129	0.2658	90372	18391	0.3247	95634	13549
0.03	0.008	0.008	0.03	0.6707	79437	7288	0.2601	90371	19796	0.3177	95628	13847
0.04	0.009	0.007	0.02	0.6558	79416	7455	0.2543	90370	19219	0.3107	95620	14159
0.05	0.010	0.006	0.01	0.6408	79394	7629	0.2486	90368	19661	0.3037	95614	14485

4.1 Discussion and Managerial Insights

(i) Table 1 reveals that Model-3 gives the highest profit i.e. model without both defective and inflation is more profitable then the other. We have observed that the raw material's purchase cost, production cost and holding cost of finished products decreases for the model without inflation meanwhile interest paid and holding cost of raw material increases. Also, interest earned and transportation cost are same whilst the total decreasing cost value dominates all other increasing costs. For the obvious reason, model with defective gives the worse profit than the model without defective.

(ii) By taking impreciseness in trade credit, decision-makers absorb all the turbulence in the cost due to market fluctuation. From Tables 2 and 3, it is also concluded that, the trade credit period is proportional to the production run time rate (t_1) and inversely proportional to production rate P. Therefore, results of the numerical Problems 2 and 3 confirm that the fuzzy trade credit, in general, has a positive role on improving the integrated supply chain's performance.

5 Conclusion

The results in this paper not only provides a valuable references for the decision-makers in planning and controlling the inventory but also furnish a useful model for many organisations such as domestic items and retail business industries. It can be used for domestic goods, fashion cloths, electronics components, medicines and other products. By rigorous mathematical derivation, we have shown the concavity of the objective function of the model and obtained the closed-form optimal solution of the model. For future research, this model can be extended for shortages and variable ordering cost. Another possible extension would be with the time dependent or stock dependent holding cost and others.

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Some Fixed Point Theorems in G-fuzzy Normed Linear Spaces

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Abstract. The purpose of this paper is to introduce some fixed point and coincidence point theorems for generalized contraction mappings in G-fuzzy normed linear spaces under H-type t-norm.

Keywords: Generalized Banach contraction mapping · G-norm · G-fuzzy metric · G-fuzzy norm · H-type t-norm

1 Introduction

The theory of fixed point has a wide range of applications in different areas such as, differential equations, equilibrium problems, variational inequality, optimization problems etc (for the reference please see [5]). In metric fixed point theory, the contractive conditions on underlying functions have an important role where as in probabilistic and fuzzy metric space, the underlying t-norm of the spaces also plays a pivotal role for ensuring the existence and uniqueness of the fixed point of a function. In 1972, Sehgal and Bharucha-Reid [1] introduced the notion of a contraction mapping in probabilistic metric spaces under the strongest triangular t-norm 'minimum'. Later, Radu [12] proved that the only continuous t-norms which could replace t-norm 'minimum' in the theorem of Sehgal and Bharucha-Reid are Hadžić-type (H-type) [7] t-norm. After that, many fixed point results have been established using different types of t-norm in fuzzy metric spaces as well as in fuzzy normed linear spaces (for the reference please see [3,4,6]).

The main objective of this paper is to establish some fixed point theorems in G-fuzzy normed linear spaces. Ćirić ([4]) and Choudhury ([3]) generalized the Banach contraction mapping in fuzzy metric spaces and following the idea of both the authors, we introduce a generalized contraction mapping in G-fuzzy normed linear spaces and establish some fixed point results under H-type t-norm. On the other hand, we give another notion of generalized contraction in G-fuzzy setting and show that Mustafa and Obiedat type ([11]) contraction mapping (in G-metric space) is a particular case of this generalized contraction. Finally, we prove some fixed point theorems using this type of contraction.

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The rest of the article is arranged in the following manner:

Section 2 deals with the preliminary results which are used in the subsequent sections. Section 3 deals with some basic results of G-fuzzy normed linear space. In Sect. 4, some common fixed point theorems and coincidence point theorems are established in G-fuzzy setting.

2 Preliminaries

Definition 1 [9]. *A t-norm* $* : [0,1] \times [0,1] \to [0,1]$ *is a binary operation which satisfies the following conditions:*
$(T1)$ $*$ *is associative and commutative;*
$(T2)$ $x * 1 = x \ \forall x \in [0,1]$;
$(T3)$ *For each* $x, y, z, w \in [0,1]$, $x * y \le z * w$ *whenever* $x \le z$ *and* $y \le w$.

Definition 2 [7]. *A t-norm* $*$ *is said to be Hadžić-type (H-type) t-norm if the family* $\{*^n\}_{n \in \mathbb{N}}$ *for each* $s \in (0,1)$ *defined by* $*^0(s) = s$, $*^{n+1}(s) = *^n(s) * s$, $\forall n \ge 0$ *is equi-continuous at* $s = 1$, *that is, for any* $1 > \epsilon > 0$ *there exists* $\eta(\epsilon) \in (0,1)$ *such that* $\forall n \in \mathbb{N}$, $*^{(n)}(s) > 1 - \epsilon$ *whenever* $1 \ge s \ge \eta(\epsilon)$.

Definition 3 [13]. *A 3-tuple* $(X, Q, *)$ *is called a Q-fuzzy metric space (Sun and Yang type) if* X *is an arbitrary nonempty set,* $*$ *is continuous t-norm and* Q *is a fuzzy set on* $X^3 \times (0, \infty)$ *satisfying the following conditions for each* $x, y, z, a \in X$ *and* $t, s > 0$*:*

(Q_N1) $Q(x,x,y,t) > 0 \ \forall x, y \in X$ *with* $x \neq y$, *and* $Q(x,x,y,t) \ge Q(x,y,z,t), \forall\ x,y,z \in X$ *with* $z \neq y$,
(Q_N2) $Q(x,y,z,t) = 1$ *if and only if* $x = y = z$,
(Q_N3) $Q(x,y,z,t) = Q(p(x,y,z),t)$, *(symmetry) where* p *is a permutation function.*
(Q_N4) $Q(x,y,z,t+s) \ge Q(x,a,a,t) * Q(a,y,z,s) \ \forall\ x,y,z,a \in X$,
(Q_N5) $Q(x,y,z,.) : (0,\infty) \to [0,1]$ *is continuous.*

Note 1. Through Out this paper Q-fuzzy metric space is denoted by G-fuzzy metric space.

Lemma 1 [13]. *Let* $(X, Q, *)$ *be a G-fuzzy metric space. Then* Q *is a continuous function on* $X^3 \times (0,\infty)$. *i.e.* $\lim_{n\to\infty} Q(x_n, y_n, z_n, t_n) = Q(x,y,z,t)$ *whenever* $\lim_{n\to\infty} Q(x_n, x_n, x, t) = 1$, $\lim_{n\to\infty} Q(y_n, y_n, y, t) = 1$, $\lim_{n\to\infty} Q(z_n, z_n, z, t) = 1$, $\lim_{n\to\infty} Q(x,y,z,t_n) = Q(x,y,z,t)$.

Definition 4 [2]. *A G-fuzzy normed linear space is a 3-tuple* $(X, G_N, *)$ *if* X *is a linear space,* $'*'$ *is a general t-norm and* G_N *is a fuzzy subset of* $X^3 \times \mathbb{R}$, *satisfying the following conditions:*

for each $x, y, z \in X, c \in F$

(G_N1) $\forall t \in \mathbb{R},\ t \le 0,\ G_N(x,y,z,t) = 0$,
(G_N2) $(\forall t \in \mathbb{R}, t > 0,\ G_N(x,y,z,t) = 1)\ iff\ x = y = z = \theta$,

(G_N3) $G_N(x, y, z, t) = G_N(p(x, y, z), t)$, *(symmetry) where p is a permutation function of x, y, z,*
(G_N4) $\forall t \in \mathbb{R}, t > 0,\ G_N(cx, cy, cz, t) = G_N(x, y, z, \frac{t}{|c|}), if\, c \neq 0,$
(G_N5) $\forall\ s, t \in \mathbb{R},\ x, y, z, x', y', z' \in X;\ G_N(x + x', y + y', z + z', t + s) \geq G_N(x, y, z, t) * G_N(x', y', z', s),$
(G_N6) $\lim_{t\to\infty} G_N(x, y, z, t) = 1,$
(G_N7) $G_N(x + y, \theta, z, t) \geq G_N(x, y, z, t),\ \forall x, y, z \in X.$

Proposition 1 [2]**.** *Let $(X, G_N, *)$ be a G-fuzzy normed linear space and $*$ is a continuous t-norm. Then the function $Q : X^3 \times (0, \infty) \to [0, 1]$ defined by $Q(x, y, z, t) = G_N(x - y, y - z, z - x, t)$ is a symmetric G-fuzzy metric space which is similar to Sun and Yang type.*

Definition 5 [8]**.** *If two self mappings f, g of a nonempty set X commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$, for each $x \in X$, then f and g are called weakly compatible pair.*

The following definition is inspired by the idea of Ψ-function defined in [3].

Definition 6. *(Ψ-function) A Ψ-function is a mapping $\psi : [0, 1]\times[0, 1]\times[0, 1] \to [0, 1]$ satisfying*

(1) ψ is monotone increasing and continuous.
(2) $\psi(t, t, t) \geq t,\ \forall\ 0 \leq t \leq 1.$

3 Some Basic Results in *G*-fuzzy Normed Linear Spaces

In this section some basic definitions and results of G-fuzzy normed linear space are established.

Definition 7. *Let $\{x_n\}$ be a sequence in a G-fuzzy normed space $(X, G_N, *)$. It is said to be convergent if $\exists\, x \in X$ such that $\lim_{n,m,l\to\infty} G_N(x_n - x, x_m - x, x_l - x, t) = 1,\ \forall t > 0.$*
Equivalently, $\{x_n\}$ is said to be convergent to $x \in X$, if for each $\alpha \in (0, 1),\ t > 0,\ \exists\ N_0(\alpha, t) \in \mathbb{N}$ such that $G_N(x_n - x, x_m - x, x_l - x, t) > 1 - \alpha\ \forall n, m, l \geq N_0$.

Then x is called the limit of the sequence $\{x_n\}$ and we denote it by $lim x_n$.

Note 2. In a G-fuzzy normed linear space limit of a sequence $\{x_n\}$ is unique if $'*'$ is continuous at $(1, 1)$.

Remark 1. From the definition of convergence it follows that, every subsequence of a convergent sequence in a G-fuzzy normed space converges to the same point.

Definition 8. *In a G-fuzzy normed space* $(X, G_N, *)$ *a sequence* $\{x_n\}$ *is said to be a Cauchy sequence if* $\lim\limits_{n,m,l\to\infty} G_N(x_n - x_m, x_m - x_l, x_l - x_n, s) = 1,\ \forall s > 0.$

Equivalently, $\{x_n\}$ *is said to be Cauchy sequence if for every* $\epsilon \in (0,1),\ s > 0,\ \exists\ N_0(\epsilon, s) \in \mathbb{N}$ *such that* $G_N(x_n - x_m, x_m - x_l, x_l - x_n, s) > 1-\epsilon\ \forall n, m, l \geq N_0$.

Definition 9. *A G-fuzzy normed space* $(X, G_N, *)$ *is said to be complete if Cauchyness implies convergence in* X.

Proposition 2. *In a G-fuzzy normed linear space* $(X, G_N, *)$ *with* $*$ *is continuous at* $(1, 1)$, *every convergent sequence is a Cauchy sequence.*

Proposition 3. *Consider a G-fuzzy normed space* $(X, G_N, *)$ *whose corresponding t-norm is continuous at* $(1, 1)$. *Then for a sequence* $\{x_n\} \subseteq X$ *and* $x \in X$ *the following are equivalent.*

(i) $\lim\limits_{n\to\infty} G_N(x_n - x, x_n - x, x_n - x, t) = 1, \forall\ t > 0.$
(ii) $\lim\limits_{n\to\infty} G_N(x_n - x, \theta, \theta, t) = 1, \forall\ t > 0.$
(iii) $\lim\limits_{n\to\infty} G_N(x_n - x, x - x_n, \theta, t) = 1, \forall\ t > 0.$

Now by the Proposition 3, the Definition 7 can also be written as $\lim\limits_{n\to\infty} G_N(x_n - x, x_n - x, x_n - x, t) = 1,\ \forall t > 0$ if the corresponding t-norm is continuous at $(1, 1)$.

Definition 10. $F \subseteq X$ *is said to be closed in a G-normed space* $(X, G_N, *)$ *if* $\lim\limits_{n\to\infty} G_N(x_n - x, x_n - x, x_n - x, t) = 1,\ \forall\ t > 0$ *implies that* $x \in F$.

Note 3. In a complete G-fuzzy normed linear space every closed subspace is complete if corresponding t-norm is continuous at $(1, 1)$.

Lemma 2. *If* $G_N(x - y, y - z, z - x, kt) \geq G_N(x - y, y - z, z - x, t),\ \forall\ t > 0,$ *for some* $0 < k < 1$, *in a G-fuzzy nls* $(X, G_N, *)$ *then* $x = y = z$.

Proof. By using (G_N6) and the property of non-decreasing of (G_N) w.r.t t, the result follows.

Definition 11. *Let* $(X, G_N, *)$ *be a G-fuzzy normed linear space. Then* G_N *is continuous function on* $X^3 \times \mathbb{R}$ *if it is continuous w.r.t to it's induced G-fuzzy metric.*

Proposition 4. *Let* $(X, G_N, *)$ *be a G-fuzzy normed linear space where* $'*'$ *is a continuous t-norm. Then* G_N *is continuous function on* $X^3 \times \mathbb{R}$ *if* G_N *is continuous w.r.t* t.

Proof. The proof follows from the Proposition 1 and Lemma 1.

4 Some Fixed Point Theorems

In this section, some common fixed point and coincidence point theorems are introduced in G-fuzzy normed linear space under H-type t-norm.

Lemma 3. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space, where $'*'$ is a H-type t-norm. If the sequence $\{x_n\}$ in X is such that*
$\forall n \in \mathbb{N}$,

$$\begin{aligned} &G_N(x_{n+2} - x_{n+1}, x_{n+1} - x_n, x_n - x_{n+2}, t) \\ &\geq G_N(x_{n+1} - x_n, x_n - x_{n-1}, x_{n-1} - x_{n+1}, \frac{t}{k}), \\ &\qquad \textit{where } 0 < k < 1,\ t > 0 \end{aligned} \tag{1}$$

then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. It follows from the condition (1) that, $\forall\ t > 0,\ n > 0$ and each $i \geq 1$,

$$\begin{aligned} &G_N(x_{n+i+2} - x_{n+i+1}, x_{n+i+1} - x_{n+i}, x_{n+i} - x_{n+i+2}, t) \\ &\geq G_N(x_{n+2} - x_{n+1}, x_{n+1} - x_n, x_n - x_{n+2}, \frac{t}{k^i}) \\ &\geq G_N(x_2 - x_1, x_1 - x_0, x_0 - x_2, \frac{t}{k^{n+i}}) \end{aligned} \tag{2}$$

Note that if $(X, G_N, *)$ is a G-fuzzy normed linear space then the function $Q_N(x, y, z, t) = G_N(x - y, y - z, z - x, t)$ forms a symmetric G-fuzzy metric space which is similar to Sun and Yang type. Now $\forall\ n < m < l$ we have,

$$\begin{aligned} &G_N(x_n - x_m, x_m - x_l, x_l - x_n, t) = Q_N(x_n, x_m, x_l, t) \\ &\geq Q_N(x_n, x_{n+1}, x_{n+1}, \frac{t}{2}) * Q_N(x_{n+1}, x_m, x_l, \frac{t}{2}) (\textit{by } Q_N 4) \\ &\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2}) * Q_N(x_{n+1}, x_m, x_l, \frac{t}{2})\ (\textit{by } Q_N 1) \\ &\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2}) * Q_N(x_{n+1}, x_m, x_m, \frac{t}{2^2}) * Q_N(x_m, x_m, x_l, \frac{t}{2^2}) (\textit{by } Q_N 4) \\ &\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2}) * Q_N(x_{n+1}, x_{n+2}, x_m, \frac{t}{2^2}) * Q_N(x_m, x_{m+1}, x_l, \frac{t}{2^2})\ (\textit{by } Q_N 1) \end{aligned}$$

Now $\frac{t}{2^2} = \frac{t}{2^2}(1-k)/(1-k) > \frac{t}{2^2}(1-k)(k + k^2 + \cdots + k^{m-n-2})$
Similarly $\frac{t}{2^2} > \frac{t}{2^2}(1-k)(1 + k + k^2 + \cdots + k^{l-m-2})$
Thus

$$\begin{aligned} &G_N(x_n - x_m, x_m - x_l, x_l - x_n, t) \\ &\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2}) * Q_N(x_{n+1}, x_{n+2}, x_m, \frac{t}{2^2}(1-k)(k + k^2 + \cdots + k^{m-n-2})) * \\ &Q_N(x_m, x_{m+1}, x_l, \frac{t}{2^2}(1-k)(1 + k + k^2 + \cdots + k^{l-m-2})) \\ &\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2}) * Q_N(x_{n+1}, x_{n+2}, x_{n+3}, \frac{t}{2^2} k(1-k)) * \end{aligned}$$

$$
\begin{aligned}
&Q_N(x_{n+3}, x_{n+3}, x_m, \frac{t}{2^2}(1-k)(k^2+k^3+\cdots+k^{m-n-2}))*\\
&Q_N(x_m, x_{m+1}, x_{m+2}, \frac{t}{2^2}(1-k))*\\
&Q_N(x_{m+2}, x_{m+2}, x_l, \frac{t}{2^2}(1-k)(k+k^2+\cdots+k^{l-m-2}))\\
&\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2})*Q_N(x_{n+1}, x_{n+2}, x_{n+3}, \frac{t}{2^2}k(1-k))*\\
&Q_N(x_{n+2}, x_{n+3}, x_m, \frac{t}{2^2}(1-k)(k^2+k^3+\cdots+k^{m-n-2}))*\\
&Q_N(x_m, x_{m+1}, x_{m+2}, \frac{t}{2^2}(1-k))*\\
&Q_N(x_{m+1}, x_{m+2}, x_l, \frac{t}{2^2}(1-k)(k+k^2+\cdots+k^{l-m-2}))
\end{aligned}
$$

Continuing this way we get,

$$
\begin{aligned}
&G_N(x_n - x_m, x_m - x_l, x_l - x_n, t)\\
&\geq Q_N(x_n, x_{n+1}, x_{n+2}, \frac{t}{2})*Q_N(x_{n+1}, x_{n+2}, x_{n+3}, \frac{t}{2^2}k(1-k))*\\
&Q_N(x_{n+2}, x_{n+3}, x_{n+4}, \frac{t}{2^2}k^2(1-k))*\cdots*Q_N(x_{m-2}, x_{m-1}, x_m, \frac{t}{2^2}(1-k)k^{m-n-2})*\\
&Q_N(x_m, x_{m+1}, x_{m+2}, \frac{t}{2^2}(1-k))*Q_N(x_{m+1}, x_{m+2}, x_{m+3}, \frac{t}{2^2}k(1-k))*\cdots*\\
&Q_N(x_{l-2}, x_{l-1}, x_l, \frac{t}{2^2}(1-k)k^{l-m-2})\\
&\geq Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2})*Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*\\
&Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*\cdots*Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*\\
&Q_N(x_m, x_{m+1}, x_{m+2}, \frac{t}{2^2}(1-k))*Q_N(x_m, x_{m+1}, x_{m+2}, \frac{t}{2^2}(1-k))*\cdots*\\
&Q_N(x_m, x_{m+1}, x_{m+2}, \frac{t}{2^2}(1-k))(by(2))\\
&\geq Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*\\
&Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*\cdots*Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))*
\end{aligned}
$$

$$
\begin{aligned}
&Q_N(x_{m+2}, x_{m+1}, x_m, \frac{t}{2^2}(1-k))*Q_N(x_{m+2}, x_{m+1}, x_m, \frac{t}{2^2}(1-k))*\cdots*\\
&Q_N(x_{m+2}, x_{m+1}, x_m, \frac{t}{2^2}(1-k))\ \text{(by symmetric property of } Q_N)\\
&\geq *^{m-n}Q_N(x_{n+2}, x_{n+1}, x_n, \frac{t}{2^2}(1-k))**^{l-m}Q_N(x_{m+2}, x_{m+1}, x_m, \frac{t}{2^2}(1-k))\\
&\geq *^{m-n}Q_N(x_2, x_1, x_0, \frac{t}{2^2}\frac{(1-k)}{k^n})**^{l-m}Q_N(x_2, x_1, x_0, \frac{t}{2^2}\frac{(1-k)}{k^m})(by(2))\\
&\geq *^{l-n}Q_N(x_2, x_1, x_0, \frac{t}{2^2}\frac{(1-k)}{k^n})
\end{aligned}
$$

Note that

$$\lim_{n\to\infty} Q_N(x_2, x_1, x_0, \frac{t}{2^2}\frac{(1-k)}{k^n}) = 1 \tag{3}$$

By hypothesis, the $t-$norm $'*'$ is of $H-$type, $\forall \epsilon \in (0,1),\ \exists\ \eta > 0$ such that

$$*^p\ (s) > 1 - \epsilon\ \forall\ s \in (1 - \eta(\epsilon), 1]\ \forall\ p \in \mathbb{N} \tag{4}$$

From (3),
$\exists N_0(t,\epsilon) \in \mathbb{N}$ such that,
$Q_N(x_2, x_1, x_0, \frac{t}{2^2}\frac{(1-k)}{k^n}) > \eta(\epsilon),\ \forall n \geq N_0(t,\epsilon)$.
Again by (4),

$$*^{l-n}\ [Q_N(x_2, x_1, x_0, \frac{t}{2^2}\frac{(1-k)}{k^n})] > 1 - \epsilon,\ \forall n \geq N_0(t,\epsilon) \tag{5}$$

Now from (5), $\forall\ l > m > n,\ \forall\ t > 0,\ \forall\ \epsilon \in (0,1)\ \exists\ N_0 \in \mathbb{N}$ such that,
$G_N(x_n - x_m, x_m - x_l, x_l - x_n, t) > 1 - \epsilon,\ \forall n \geq N_0$
$\Rightarrow \lim_{n,m,l\to\infty} G_N(x_n - x_m, x_m - x_l, x_l - x_n, t) = 1,\ \forall t > 0$
Thus $\{x_n\}$ is a Cauchy sequence.

Theorem 1. *Let $(X, G_N, *)$ be a complete G-fuzzy normed linear space, where $'*'$ is a H-type t-norm and A, B, C, g be four self mappings on X such that:*

(1) $g\,X$ *is closed*
(2) $AX,\ BX,\ CX \subseteq gX$
(3) $G_N(Ax - By, By - Cz, Cz - Ax, kt) + q(1 - max\{G_N(gx - By, By - gx, \theta, kt),$
$G_N(gy - Ax, Ax - gy, \theta, kt), G_N(gy - Cz, Cz - gy, \theta, kt),$
$G_N(gz - By, By - gz, \theta, kt), G_N(gz - Ax, Ax - gz, \theta, kt),$
$G_N(gx - Cz, Cz - gx, \theta, kt)\}) \geq G_N(gx - gy, gy - gz, gz - gx, t)$
for each $x, y, z \in X,\ \forall t > 0,\ 0 < k < 1,\ q(x,y,z,t) \geq 0$. *Then the mappings* $A, B, C,\ g$ *have a coincidence point.*

Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\}$ as follows:
$gx_1 = Ax_0, gx_2 = Bx_1, gx_3 = Cx_2, gx_4 = Ax_3$, and in general
$\forall n \in \mathbb{N}, gx_{3n-2} = Ax_{3n-3}, gx_{3n-1} = Bx_{3n-2}, gx_{3n} = Cx_{3n-1}$
Similarly $gx_{3n+1} = Ax_{3n}, gx_{3n+2} = Bx_{3n+1}, gx_{3n+3} = Cx_{3n+2}$
By condition (2), this construction is valid.

Now $G_N(gx_{3n+1} - gx_{3n+2}, gx_{3n+2} - gx_{3n+3}, gx_{3n+3} - gx_{3n+1}, t)$
$= G_N(Ax_{3n} - Bx_{3n+1}, Bx_{3n+1} - Cx_{3n+2}, Cx_{3n+2} - Ax_{3n}, t)$
So from condition (3), with $x = x_{3n}, y = x_{3n+1}, z = x_{3n+2}$ we have
$G_N(Ax_{3n} - Bx_{3n+1}, Bx_{3n+1} - Cx_{3n+2}, Cx_{3n+2} - Ax_{3n}, kt) + q(1 - max$
$\{G_N(gx_{3n} - Bx_{3n+1}, Bx_{3n+1} - gx_{3n}, \theta, kt), G_N(gx_{3n+1} - Ax_{3n}, Ax_{3n} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+1} - Cx_{3n+2}, Cx_{3n+2} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+2} - Bx_{3n+1}, Bx_{3n+1} - gx_{3n+2}, \theta, kt), G_N(gx_{3n+2} - Ax_{3n}, Ax_{3n} - gx_{3n+2}, \theta, kt),$
$G_N(gx_{3n} - Cx_{3n+2}, Cx_{3n+2} - gx_{3n}, \theta, kt)\})$

$\geq G_N(gx_{3n} - gx_{3n+1}, gx_{3n+1} - gx_{3n+2}, gx_{3n+2} - gx_{3n}, kt)$
Now, $max\{G_N(gx_{3n} - Bx_{3n+1}, Bx_{3n+1} - gx_{3n}, \theta, kt),$
$G_N(gx_{3n+1} - Ax_{3n}, Ax_{3n} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+1} - Cx_{3n+2}, Cx_{3n+2} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+2} - Bx_{3n+1}, Bx_{3n+1} - gx_{3n+2}, \theta, kt),$
$G_N(gx_{3n+2} - Ax_{3n}, Ax_{3n} - gx_{3n+2}, \theta, kt), G_N(gx_{3n} - Cx_{3n+2}, Cx_{3n+2} - gx_{3n}, \theta, kt)\} =$
$max\{G_N(gx_{3n} - gx_{3n+2}, gx_{3n+2} - gx_{3n}, \theta, kt),$
$G_N(gx_{3n+1} - gx_{3n+1}, gx_{3n+1} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+1} - gx_{3n+3}, gx_{3n+3} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+2} - gx_{3n+2}, gx_{3n+2} - gx_{3n+2}, \theta, kt),$
$G_N(gx_{3n+2} - gx_{3n+1}, gx_{3n+1} - gx_{3n+2}, \theta, kt),$
$G_N(gx_{3n} - gx_{3n+3}, gx_{3n+3} - gx_{3n}, \theta, kt)\} = 1$
$i.e. G_N(gx_{3n+1} - gx_{3n+2}, gx_{3n+2} - gx_{3n+3}, gx_{3n+3} - gx_{3n+1}, kt) + q(1-1) \geq$
$G_N(gx_{3n} - gx_{3n+1}, gx_{3n+1} - gx_{3n+2}, gx_{3n+2} - gx_{3n}, t)$

Hence $\forall\ t > 0,\ n \geq 0$, we have,

$$\begin{aligned} &G_N(gx_{3n+3} - gx_{3n+2}, gx_{3n+2} - gx_{3n+1}, gx_{3n+1} - gx_{3n+3}, kt) \\ &\quad \geq G_N(gx_{3n+2} - gx_{3n+1}, gx_{3n+1} - gx_{3n}, gx_{3n} - gx_{3n+2}, t) \end{aligned} \tag{6}$$

Again by similar argument putting $x = x_{3n}, y = x_{3n+1}, z = x_{3n-1},\ \forall t > 0,\ n \geq 0$, in condition (3) we have,

$$\begin{aligned} &G_N(gx_{3n+2} - gx_{3n+1}, gx_{3n+1} - gx_{3n}, gx_{3n} - gx_{3n+2}, kt) \\ &\geq G_N(gx_{3n+1} - gx_{3n}, gx_{3n} - gx_{3n-1}, gx_{3n-1} - gx_{3n+1}, t) \end{aligned} \tag{7}$$

In a similar manner putting $x = x_{3n+3}, y = x_{3n+1}, z = x_{3n+2}, \forall t > 0$, $n \geq 0$, in condition (3) we have,

$$\begin{aligned} &G_N(gx_{3n+4} - gx_{3n+3}, gx_{3n+3} - gx_{3n+2}, gx_{3n+2} - gx_{3n+4}, kt) \\ &\geq G_N(gx_{3n+3} - gx_{3n+2}, gx_{3n+2} - gx_{3n+1}, gx_{3n+1} - gx_{3n+3}, t) \end{aligned} \tag{8}$$

Now from (6), (7), (8) we have, $\forall n \geq 0$, $\forall t > 0$,
$G_N(gx_{n+2} - gx_{n+1}, gx_{n+1} - gx_n, gx_n - gx_{n+2}, kt)$
$\geq G_N(gx_{n+1} - gx_n, gx_n - gx_{n-1}, gx_{n-1} - gx_{n+1}, t)$
Now by Lemma 3, $\{g(x_n)\}$ is a Cauchy sequence.

Since gX is closed, there exists $x \in X$ such that $\lim_{n\to\infty} gx_n = gx$.

Without loss of generality, we assume that $x_n \neq x$, $\forall n \in \mathbb{N}$, otherwise there exists a subsequence with this property.

Putting $x = x_{3n}, y = x_{3n+1}, z = x$, in condition (3)
$\forall t > 0$ and $n \in \mathbb{N}$ we have,

$G_N(Ax_{3n} - Bx_{3n+1}, Bx_{3n+1} - Cx, Cx - Ax_{3n}, kt) + q(1 - max$
$\{G_N(gx_{3n} - Bx_{3n+1}, Bx_{3n+1} - gx_{3n}, \theta, kt), G_N(gx_{3n+1} - Ax_{3n}, Ax_{3n} - gx_{3n+1}, \theta, kt),$
$G_N(gx_{3n+1} - Cx, Cx - gx_{3n+1}, \theta, kt), G_N(gx - Bx_{3n+1}, Bx_{3n+1} - gx, \theta, kt),$
$G_N(gx - Ax_{3n}, Ax_{3n} - gx, \theta, kt), G_N(gx_{3n} - Cx, Cx - gx_{3n}, \theta, kt)\})$

$$\geq G_N(gx_{3n} - gx_{3n+1}, gx_{3n+1} - gx, gx - gx_{3n}, t)$$
$$\Rightarrow G_N(Ax_{3n} - Bx_{3n+1}, Bx_{3n+1} - Cx, Cx - Ax_{3n}, kt) + q(1 - max$$
$$\{G_N(gx_{3n} - gx_{3n+2}, gx_{3n+2} - gx_{3n}, \theta, kt),$$
$$G_N(gx_{3n+1} - gx_{3n+1}, gx_{3n+1} - gx_{3n+1}, \theta, kt), G_N(gx_{3n+1} - Cx, Cx - gx_{3n+1}, \theta, kt),$$
$$G_N(gx - gx_{3n+2}, gx_{3n+2} - gx, \theta, kt), G_N(gx - gx_{3n+1}, gx_{3n+1} - gx, \theta, kt),$$
$$G_N(gx_{3n} - Cx, Cx - gx_{3n}, \theta, kt)\}) \geq G_N(gx_{3n} - gx, gx - gx_{3n}, \theta, \frac{t}{2}) *$$
$$G_N(gx - gx_{3n+1}, gx_{3n+1} - gx, \theta, \frac{t}{2})$$

Taking $n \to \infty$ both side and using the Proposition 3 we get,

$$\lim_{n\to\infty} G_N(Ax_{3n} - Bx_{3n+1}, Bx_{3n+1} - Cx, Cx - Ax_{3n}, kt) \geq 1 * 1 = 1$$
$$\Rightarrow \lim_{n\to\infty} G_N(Ax_{3n} - Cx, Cx - Ax_{3n}, \theta, t) = 1, \forall t > 0 \ (by\ G_N7)$$

Now again by the Proposition 3,

$$\lim_{n\to\infty} G_N(Ax_{3n} - Cx, Ax_{3n} - Cx, Ax_{3n} - Cx, t) = 1, \forall t > 0$$
$$\Rightarrow \lim_{n\to\infty} G_N(gx_{3n+1} - Cx, gx_{3n+1} - Cx, gx_{3n+1} - Cx, t) = 1, \forall t > 0$$

Now by Remark 1, $gx = Cx$

In condition (3) Putting $x = x, y = x_{3n+1}, z = x_{3n+2}$ and again $x = x_{3n}, y = x, z = x_{3n+2}$ we get by a similar argument, $gx = Ax$ and $gx = Bx$ respectively.

Thus, $gx = Ax = Bx = Cx$

$\therefore$ x is a coincidence point of g, A, B and C.

Definition 12. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space and T be a mapping of X into itself. T is said to be a generalized Banach type contraction mapping if there exist $k \in (0, 1)$ and $q = q(x, y, z, t) \geq 0$ such that*
$G_N(Tx - Ty, Ty - Tz, Tz - Tx, kt) + q(1 - max\{G_N(x - Ty, Ty - x, \theta, kt), G_N(y - Tx, Tx - y, \theta, kt), G_N(y - Tz, Tz - y, \theta, kt), G_N(z - Ty, Ty - z, \theta, kt), G_N(z - Tx, Tx - z, \theta, kt), G_N(x - Tz, Tz - x, \theta, kt)\}) \geq G_N(x - y, y - z, z - x, t)$ *for each $x, y, z \in X$ and for all $t > 0$.*

If $q(x, y, z, t) = 0$, T satisfies the condition
$G_N(T(x) - T(y), T(y) - T(z), T(z) - T(x), kt) \geq G_N(x - y, y - z, z - x, t)$, $\forall t > 0$, $\forall x, y, z \in X$ where $0 < k < 1$, which is similar to the Banach type contraction mapping defined in [10]. So the contraction mapping defined as in Definition 12 is called generalized Banach type contraction.

Theorem 2 *(Ćirić [4] type common fixed point theorem for generalized Banach contraction mappings in G-fuzzy setting)***.** *Let A, B, C be three self mappings of a complete G-fuzzy normed space $(X, G_N, *)$ such that there exists $k \in (0, 1)$ and $q(x, y, z, t) \geq 0$ such that $G_N(Ax - By, By - Cz, Cz - Ax, kt) + q(1 - max\{G_N(x - By, By - x, \theta, kt), G_N(y - Ax, Ax - y, \theta, kt), G_N(y - Cz, Cz - y, \theta, kt), G_N(z - By, By - z, \theta, kt), G_N(z - Ax, Ax - z, \theta, kt), G_N(x - Cz, Cz - x, \theta, kt)\}) \geq G_N(x - y, y - z, z - x, t)$ for each $x, y, z \in X$, $\forall\ t > 0$. If the family of t-norms is H-type, then A, B, C have a common fixed point in X.*

Proof. Proof follows from Theorem 1 by putting $gx = x$, $\forall x \in X$.

Theorem 3. *Let $(X, G_N, *)$ be a complete G-fuzzy normed linear space. Let A be a self mapping of X such that A satisfies generalized Banach type contraction defined in Definition 12. If the family of t-norms is H-type, then A has a fixed point in X.*

Proof. Proof follows by putting $A = B = C$ in Theorem 2.

Theorem 4. *Let $(X, G_N, *)$ be a complete G-fuzzy normed linear space such that $G_N(x, y, z, t)$ is a strictly increasing and continuous with respect to t for all $x, y, z \in X$, where $*$ is a H-type t-norm. Let A, g be two self mappings on X such that the following conditions are satisfied:*

(1) gX *is closed*
(2) $AX \subseteq gX$
(3) $G_N(Ax-Ay, Ay-Az, Az-Ax, kt)+q(1-max\{G_N(gx-Ay, Ay-gx, \theta, kt), G_N(gy-Ax, Ax-gy, \theta, kt), G_N(gy-Az, Az-gy, \theta, kt), G_N(gz-Ay, Ay-gz, \theta, kt), G_N(gz-Ax, Ax-gz, \theta, kt), G_N(gx-Az, Az-gx, \theta, kt)\}) \geq \psi(G_N(gx-Ax, Ax-gx, \theta, t), G_N(gy-Ay, Ay-gy, \theta, t), G_N(gz-Az, Az-gz, \theta, t))$
for each $x, y, z \in X$, $\forall\ t > 0$, $0 < k < 1$, $q(x, y, z, t) \geq 0$ *and* ψ *is a* Ψ*-function.*

Then A and g have coincidence point. Further if (A, g) is a weakly compatible pair, then A and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\}$ as follows: $y_1 = gx_1 = Ax_0,\ y_2 = gx_2 = Ax_1,\ y_3 = gx_3 = Ax_2$ and in general $y_n = gx_n = Ax_{n-1},\ \forall n \in \mathbb{N}$.

This is possible by condition (2).

Further we assume that, $y_n \neq y_{n+1} \neq y_{n+2} (\neq y_n),\ \forall n \in \mathbb{N}$, otherwise g and A have a coincidence point. Since G_N is strictly increasing w.r.t t, then

$$0 < G_N(y_{n+2} - y_{n+1}, y_{n+1} - y_n, y_n - y_{n+2}, t) < 1 \tag{9}$$

Again G_N is continuous w.r.t t then by Proposition 4, G_N is continuous function on $X^3 \times \mathbb{R}$. In condition (3) putting $x = x_{n+1},\ y = x_n,\ z = x_{n-1}, \forall t > 0$, we have,

$$\begin{aligned}
&G_N(Ax_{n+1} - Ax_n, Ax_n - Ax_{n-1}, Ax_{n-1} - Ax_{n+1}, kt) + q(1 - max\\
&\{G_N(gx_{n+1} - Ax_n, Ax_n - gx_{n+1}, \theta, kt), G_N(gx_n - Ax_{n+1}, Ax_{n+1} - gx_n, \theta, kt),\\
&G_N(gx_n - Ax_{n-1}, Ax_{n-1} - gx_n, \theta, kt), G_N(gx_{n-1} - Ax_n, Ax_n - gx_{n-1}, \theta, kt),\\
&G_N(gx_{n-1} - Ax_{n+1}, Ax_{n+1} - gx_{n-1}, \theta, kt), G_N(gx_{n+1} - Ax_{n-1}, Ax_{n-1} - gx_{n+1}, \theta, kt)\})\\
&\geq \psi(G_N(gx_{n+1} - Ax_{n+1}, Ax_{n+1} - gx_{n+1}, \theta, t), G_N(gx_n - Ax_n, Ax_n - gx_n, \theta, t),\\
&G_N(gx_{n-1} - Ax_{n-1}, Ax_{n-1} - gx_{n-1}, \theta, t))\\
&\Rightarrow G_N(gx_{n+2} - gx_{n+1}, gx_{n+1} - gx_n, gx_n - gx_{n+2}, kt) + q(1 - max\\
&\{G_N(gx_{n+1} - gx_{n+1}, gx_{n+1} - gx_{n+1}, \theta, kt), G_N(gx_n - gx_{n+2}, gx_{n+2} - gx_n, \theta, kt),\\
&G_N(gx_n - gx_n, gx_n - gx_n, \theta, kt), G_N(gx_{n-1} - gx_{n+1}, gx_{n+1} - gx_{n-1}, \theta, kt),
\end{aligned}$$

$$G_N(gx_{n-1} - gx_{n+2}, gx_{n+2} - gx_{n-1}, \theta, kt), G_N(gx_{n+1} - gx_n, gx_n - gx_{n+1}, \theta, kt)\})$$
$$\geq \psi(G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, t), G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, t),$$
$$G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, t))$$
$$\Rightarrow G_N(gx_{n+2} - gx_{n+1}, gx_{n+1} - gx_n, gx_n - gx_{n+2}, kt) + q(1-1) \geq$$
$$\psi(G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, t), G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, t),$$
$$G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, t))$$

Thus $G_N(gx_{n+2} - gx_{n+1}, gx_{n+1} - gx_n, gx_n - gx_{n+2}, kt)$
$\geq \psi(G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, t), G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, t),$
$G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, t))$

If $min\{G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, s), G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, s),$
$G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s)\} = G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s)$ for some $s > 0$, from the above inequality, using the property of ψ and (9) we get, $G_N(gx_{n+2} - gx_{n+1}, gx_{n+1} - gx_n, gx_n - gx_{n+2}, ks) \geq$

$$\psi(G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s), G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s),$$
$$G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s))$$
$$\geq G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s)$$

Now by $(G_N7), G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, ks)$
$\geq G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s)$
This is a contraction.

Again if $min\{G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, s),$
$G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, s), G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, s)\} =$
$G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, s), for some s > 0$, again we arise at a contraction by similar argument.

Thus $min\{G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, t), G_N(gx_n - gx_{n+1}, gx_{n+1} - gx_n, \theta, t),$
$G_N(gx_{n+1} - gx_{n+2}, gx_{n+2} - gx_{n+1}, \theta, t)\} = G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, t)$
for all $t > 0$.
Thus for all $n \in \mathbb{N}$ and $t > 0$ we get

$$\begin{aligned} &G_N(gx_{n+2} - gx_{n+1}, gx_{n+1} - gx_n, gx_n - gx_{n+2}, kt) \\ &\geq G_N(gx_{n-1} - gx_n, gx_n - gx_{n-1}, \theta, t) \\ &\geq G_N(gx_{n+1} - gx_n, gx_n - gx_{n-1}, gx_{n-1} - gx_{n+1}, t)(by(G_N7)) \end{aligned}$$

Then, from Lemma 3, we conclude that $\{y_n\}$ is a Cauchy sequence. By completeness of X, there exists $l \in X$, such that $\lim_{n\to\infty} y_n = l$.

Since gX is closed $\exists\ u \in X$ such that $gu = l$.

Putting $x = u,\ y = x_{n-1},\ z = x_{n-2}$, in condition (3) $\forall\ t > 0$, we have,

$$G_N(Au - Ax_{n-1}, Ax_{n-1} - Ax_{n-2}, Ax_{n-2} - Au, kt) + q(1 - max$$
$$\{G_N(gu - Ax_{n-1}, Ax_{n-1} - gu, \theta, kt), G_N(gx_{n-1} - Au, Au - gx_{n-1}, \theta, kt),$$
$$G_N(gx_{n-1} - Ax_{n-2}, Ax_{n-2} - gx_{n-1}, \theta, kt), G_N(gx_{n-2} - Ax_{n-1}, Ax_{n-1} - gx_{n-2}, \theta, kt),$$
$$G_N(gx_{n-2} - Au, Au - gx_{n-2}, \theta, kt), G_N(gu - Ax_{n-2}, Ax_{n-2} - gu, \theta, kt)\}) \geq$$

$$\begin{aligned}
&\psi(G_N(gu-Au,Au-gu,\theta,t),G_N(gx_{n-1}-Ax_{n-1},Ax_{n-1}-gx_{n-1},\theta,t),\\
&G_N(gx_{n-2}-Ax_{n-2},Ax_{n-2}-gx_{n-2},\theta,t))\\
&\Rightarrow G_N(Au-gx_n,gx_n-gx_{n-1},gx_{n-1}-Au,kt)+q(1-max\\
&\{G_N(gu-gx_n,gx_n-gu,\theta,kt),G_N(gx_{n-1}-Au,Au-gx_{n-1},\theta,kt),\\
&G_N(gx_{n-1}-gx_{n-1},gx_{n-1}-gx_{n-1},\theta,kt),G_N(gx_{n-2}-gx_n,gx_n-gx_{n-2},\theta,kt),\\
&G_N(gx_{n-2}-Au,Au-gx_{n-2},\theta,kt),G_N(gu-gx_{n-1},gx_{n-1}-gu,\theta,kt)\})\\
&\geq\psi(G_N(gu-Au,Au-gu,\theta,t),G_N(gx_{n-1}-gx_n,gx_n-gx_{n-1},\theta,t),\\
&G_N(gx_{n-2}-gx_{n-1},gx_{n-1}-gx_{n-2},\theta,t))\\
&\Rightarrow G_N(Au-gx_n,gx_n-gx_{n-1},gx_{n-1}-Au,kt)+q(1-1)\\
&\geq\psi(G_N(gu-Au,Au-gu,\theta,t),G_N(gx_{n-1}-gx_n,gx_n-gx_{n-1},\theta,t),\\
&G_N(gx_{n-2}-gx_{n-1},gx_{n-1}-gx_{n-2},\theta,t))
\end{aligned}$$

Letting $n\to\infty$ both side of the above inequality, $\forall\ t>0$, we have,

$$\begin{aligned}
&G_N(Au-l,\theta,l-Au,kt)\geq\psi(G_N(gu-Au,Au-gu,\theta,t),G_N(\theta,\theta,\theta,t),\\
&G_N(\theta,\theta,\theta,t))\geq\psi(G_N(gu-Au,Au-gu,\theta,t),G_N(gu-Au,Au-gu,\theta,t),\\
&G_N(gu-Au,Au-gu,\theta,t))\geq G_N(gu-Au,Au-gu,\theta,t)\,\text{(using the properties of }\psi)
\end{aligned}$$

The above inequality implies that $Au=l$. (by Lemma 2)
$\Rightarrow Au=l=gu$.
$\therefore u$ is a coincidence point of A and g.
Further let (A,g) be a weakly compatible pair of mappings.
Then we have $gl=gAu=Agu=Al$.
Now putting $x=l,\ y=x_{n-1},\ z=x_{n-2}$ in condition (2), $\forall\ t>0$ we have,

$$\begin{aligned}
&G_N(Al-Ax_{n-1},Ax_{n-1}-Ax_{n-2},Ax_{n-2}-Al,kt)+q(1-max\\
&\{G_N(gl-Ax_{n-1},Ax_{n-1}-gl,\theta,kt),G_N(gx_{n-1}-Al,Al-gx_{n-1},\theta,kt),\\
&G_N(gx_{n-1}-Ax_{n-2},Ax_{n-2}-gx_{n-1},\theta,kt),G_N(gx_{n-2}-Ax_{n-1},Ax_{n-1}-gx_{n-2},\theta,kt),\\
&G_N(gx_{n-2}-Al,Al-gx_{n-2},\theta,kt),G_N(gl-Ax_{n-2},Ax_{n-2}-gl,\theta,kt)\})\\
&\geq\psi(G_N(gl-Al,Al-gl,\theta,t),G_N(gx_{n-1}-Ax_{n-1},Ax_{n-1}-gx_{n-1},\theta,t),\\
&G_N(gx_{n-2}-Ax_{n-2},Ax_{n-2}-gx_{n-2},\theta,t))\\
&\Rightarrow G_N(Al-gx_n,gx_n-gx_{n-1},gx_{n-1}-Al,kt)+q(1-max\\
&\{G_N(gl-gx_n,gx_n-gl,\theta,kt),G_N(gx_{n-1}-Al,Al-gx_{n-1},\theta,kt),\\
&G_N(gx_{n-1}-gx_{n-1},gx_{n-1}-gx_{n-1},\theta,kt),G_N(gx_{n-2}-gx_n,gx_n-gx_{n-2},\theta,kt),\\
&G_N(gx_{n-2}-Al,Al-gx_{n-2},\theta,kt),G_N(gl-gx_{n-1},gx_{n-1}-gl,\theta,kt)\})\\
&\geq\psi(G_N(gl-Al,Al-gl,\theta,t),G_N(gx_{n-1}-gx_n,gx_n-gx_{n-1},\theta,t),\\
&G_N(gx_{n-2}-gx_{n-1},gx_{n-1}-gx_{n-2},\theta,t))\\
&\Rightarrow G_N(Al-gx_n,gx_n-gx_{n-1},gx_{n-1}-Al,kt)+q(1-1)\\
&\geq\psi(G_N(gl-Al,Al-gl,\theta,t),G_N(gx_{n-1}-gx_n,gx_n-gx_{n-1},\theta,t),\\
&G_N(gx_{n-2}-gx_{n-1},gx_{n-1}-gx_{n-2},\theta,t))
\end{aligned}$$

Letting $n\to\infty$ both side of the above inequality by the Proposition 4, $\forall\ t>0$, we have, $G_N(Al-l,\theta,l-Al,kt)\geq 1$
$Al=l=gl$.
So, l is a fixed point of A and g.

To prove the uniqueness, let z_1, z_2 be two distinct points such that $Az_1 = z_1 = gz_1, Az_2 = z_2 = gz_2$.
Putting $x = z = z_1$ and $y = z_2$ in condition (3),

$$
\begin{aligned}
&G_N(Az_1 - Az_2, Az_2 - Az_1, Az_1 - Az_1, kt) + \\
&q(1 - max\{G_N(gz_1 - Az_2, Az_2 - gz_1, \theta, kt), G_N(gz_2 - Az_1, Az_1 - gz_2, \theta, kt), \\
&G_N(gz_2 - Az_1, Az_1 - gz_2, \theta, kt), G_N(gz_1 - Az_2, Az_2 - gz_1, \theta, kt), \\
&G_N(gz_1 - Az_1, Az_1 - gz_1, \theta, kt), G_N(gz_1 - Az_1, Az_1 - gz_1, \theta, kt)\}) \\
&\geq \psi(G_N(gz_1 - Az_1, Az_1 - gz_1, \theta, t), G_N(gz_2 - Az_2, Az_2 - gz_2, \theta, t), \\
&G_N(gz_1 - Az_1, Az_1 - gz_1, \theta, t)) \\
&\Rightarrow G_N(Az_1 - Az_2, Az_2 - Az_1, \theta, kt) + q(1 - 1) \geq \psi(1, 1, 1) \geq 1, \ \forall t > 0. \\
&\Rightarrow G_N(Az_1 - Az_2, Az_2 - Az_1, \theta, kt) = 1, \ \forall t > 0. \\
&\Rightarrow z_1 = z_2 \ (by\ G_N 2)
\end{aligned}
$$

This completes the proof.

Example 1. We consider the complete G-normed linear space $X = \mathbb{R}$ with G-norm defined by a function $\|\cdot,\cdot,\cdot\| : X^3 \to \mathbb{R}$ by $\|x, y, z\| = |x-y|+|y-z|+|z-x|$ $(x, y, z \in \mathbb{R})$. Define $a * b = min\{a, b\}$ and consider the G-fuzzy norm $(X, G_N, *)$ by $G_N(x, y, z, t) = \begin{cases} \frac{t}{t+\|x,y,z\|} & \forall x, y, z \in \mathbb{R}, and\ t \in (0, \infty) \\ 0, & if\ t \leq 0 \end{cases}$

Let $A, g : X \to X$ be defined by $Ax = 2, \ \forall x \in X$ and $gx = \begin{cases} 2, & x \in \mathbb{Q} \\ 0, & otherwise \end{cases}$
and $\psi(x, y, z) = min\{x, y, z\}$. Then all the conditions of Theorem 4 is satisfied with $q = 0$ and 2 is the unique common fixed point.

Corollary 1. *Let $G_N(x, y, z, t)$ be strictly increasing and continuous w.r.t t, for each $x, y, z \in X$, in a complete G-fuzzy normed space $(X, G_N, *)$ with H-type t-norm $'*'$. Let A be a self mapping on X which satisfies the following conditions for all $x, y, z \in X$:*

$$
\begin{aligned}
&G_N(Ax - Ay, Ay - Az, Az - Ax, kt) \geq \\
&\psi(G_N(x - Ax, Ax - x, \theta, t), G_N(y - Ay, Ay - y, \theta, t), G_N(z - Az, Az - z, \theta, t))
\end{aligned} \tag{10}
$$

for each $x, y, z \in X, \ \forall t > 0, \ 0 < k < 1$ and ψ is a Ψ-function. Then A has a unique fixed point.

Proof. Proof follows from Theorem 4 by choosing $gx = x$ and $q = 0$.

Corollary 2. *Let $(X, \|\cdot,\cdot,\cdot\|)$ be a complete G-normed linear space and A be a self map which satisfies the following inequality:*

$$
\|Ax-Ay, Ay-Az, Az-Ax\| \leq \frac{k}{3}\{\|x-Ax, Ax-x, \theta\|+\|y-Ay, Ay-y, \theta\|+\|z-Az, Az-z, \theta\|\} \tag{11}
$$

where $0 < k < 1$, $x, y, z \in X$. Then A has a unique fixed point.

Proof. Consider a G-fuzzy normed linear space $(X, G_N, *)$ where
$$G_N(x,y,z,t) = \begin{cases} [exp(\frac{\|x,y,z\|}{t})]^{-1}, & \forall x,y,z \in \mathbb{R}, and\ t \in (0,\infty) \\ 0, & if\ t \leq 0 \end{cases}$$
and $a * b = ab$.
We prove that the inequality (11) implies the inequality (10) with $\psi(x,y,z) = min\{x,y,z\}$. If otherwise, then from (10) for some t,

$$\{exp(\frac{\|Ax-Ay,Ay-Az,Az-Ax\|}{kt})\}^{-1} < min\{\{exp(\frac{\|x-Ax,Ax-x,\theta\|}{t})\}^{-1}, \{exp(\frac{\|y-Ay,Ay-y,\theta\|}{t})\}^{-1}, \{exp(\frac{\|z-Az,Az-z,\theta\|}{t})\}^{-1}\}$$

$$\Rightarrow \{exp(\frac{\|Ax-Ay,Ay-Az,Az-Ax\|}{kt})\}^{-1} < \{exp(\frac{\|x-Ax,Ax-x,\theta\|}{t})\}^{-1} \tag{12}$$

$$\{exp(\frac{\|Ax-Ay,Ay-Az,Az-Ax\|}{kt})\}^{-1} < \{exp(\frac{\|y-Ay,Ay-y,\theta\|}{t})\}^{-1} \tag{13}$$

and

$$\{\exp(\frac{\|Ax-Ay,Ay-Az,Az-Ax\|}{kt})\}^{-1} < \{exp(\frac{\|z-Az,Az-z,\theta\|}{t})\}^{-1} \tag{14}$$

Now from (12),

$$\{exp(\frac{\|Ax-Ay,Ay-Az,Az-Ax\|}{kt})\} > \{exp(\frac{\|x-Ax,Ax-x,\theta\|}{t})\}$$
$$\Rightarrow \frac{\|Ax-Ay,Ay-Az,Az-Ax\|}{kt} > \frac{\|x-Ax,Ax-x,\theta\|}{t}$$
$$\Rightarrow \|Ax-Ay,Ay-Az,Az-Ax\| > k\|x-Ax,Ax-x,\theta\|$$

Similarly from (13) and (14)

$\|Ax-Ay,Ay-Az,Az-Ax\| > k\|y-Ay,Ay-y,\theta\|\ and$
$\|Ax-Ay,Ay-Az,Az-Ax\| > k\|z-Az,Az-z,\theta\|$
$i.e.\ \|Ax-Ay,Ay-Az,Az-Ax\| > \frac{k}{3}\{\|x-Ax,Ax-x,\theta\| + \|y-Ay,Ay-y,\theta\| + \|z-Az,Az-z,\theta\|\}$, which is a contradiction.

This completes the proof.

5 Conclusion

Study on fixed point theory in G-fuzzy normed space is a recent development. In this paper, fixed point theorems for generalized contraction mappings in G-fuzzy normed linear space under H-type t-norm have been established. There is a possibility to establish these results using different type of t-norm other than H-type. So, this might be a scope for the researcher to develop fixed point theorems in this field.

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Vision Based Automatic Landing of Unmanned Aerial Vehicle

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Abstract. Aerial robotics is a growing field with tremendous civil and military applications. Potential applications include surveying and maintenance tasks, aerial transportation and manipulation, search and rescue, and surveillance. The challenges associated with tackling robotics tasks in complex, three dimensional, indoor and outdoor environments bring into focus some of the limitations of accepted solutions to classical robotics problems in sensing, planning, localization, and mapping. A quadcopter which is capable of autonomous landing on a stationary platform using only onboard parameters such as sensing, recognition and computation is presented. We present state-of-the-art computer vision, deep learning neural net inception model, algorithms, detection and state estimation of the target for our project. We have deployed and tested in indoor environment due to limitations of resources and controlled environmental features. We rely on Faster-RCNN-Inception-V2-COCO model but other robust training model could be used for devices with limited computation power like Raspberry pi with same procedures and improved results. The Tensorflow model is rapidly growing with current version 1.12.0 with extended Keras support makes our project more dynamic and facile. To the best of our knowledge, this is the first demonstration of a fully autonomous quadrotor system capable of landing on a stationary target, using only onboard sensing and computing, without relying on external infrastructure which uses deep learning for target recognition, state estimation and target tracking.

Keywords: Autonomous · Multirotor · UAV · Quad copter · Deep learning · Image analysis · Position estimation · State estimation · Advanced Kalman filter · Template matching

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1 Introduction

Computer vision is a current open and inexhaustive research area, Visual service algorithms have been extensively developed in the field of robotics over the last decade, with a passive and informative source helping the sensor suite for control and guidance of Unmanned Aerial Vehicles (UAV). A vision system on board a UAV typically augments a sensor suite that might include Global Positioning System (GPS), Inertial Navigation Sensors(INS), laser range finders [1]. The design of any real-time vision system is a challenging task. It involves a systematic integration of hardware, low level image processing (like segmentation and corner and edge detection); multiple view geometry and synthesis of real-time micro controllers. Because of the structured nature of UAV, the task of autonomous landing is well-suited for vision-based state estimation and control and has recently been an active topic of research [2]. In [3] a technique is presented for estimating the Position-based visual control to a known object given a scaled orthographic project ion model of a camera. In [4], the use of vanishing points of parallel lines on a landmark is proposed for the purpose of estimating the location and landing of a UAV upon a landing pad. Since their technique relies on vanishing points of parallel lines, their algorithms most sensitive to noise and does not give the particular position estimates when it matters the most: when the UAV is directly landing upon the pad. An important task for UAVs is the capability to perform automatic landing with safety. Towards fully automation, the landing-pad detection via digital images is a critical mission. Methodologies can be categorized according to the characteristics of the landing site. Roughly, it is possible to distinguish methods that detect a known target from methods that identify suitable sites in unknown environment. If we have to really treat quadcopter as fully autonomous then all the computation, estimation and recognition have to be done using on board computer. We have used raspberry pi with camera and Clean Flight simulator. Pose estimates are computed at standard frequency [10] using advanced kalman filters and state estimation is done by monocular visual-inertial odometry [11] to estimate the state of the quadrotor. The target recognition is done by image classifier trained on images of h and x but it can be generated for any sign. Models like SSD-MobileNet model allows faster detection but have less accuracy whereas models like Faster-RCNN model allows slower detection but with improved accuracy. We have preferred accuracy by considering rescue mission in hazards as main target. We have assumed that rescue targets are mainly stationary in certain conditions and we have restricted the movement of our target.

2 Research Methodology

Supervised training methodologies seem robust for detection of known features. Due to the flexibility within the landing environment, which for every mission scenario could be subjected to changes, inspire the present research to limit and focus on the landing-pads with artificial markers that resemble characters "H" an "X" inscribed into a circle (Fig. 1).

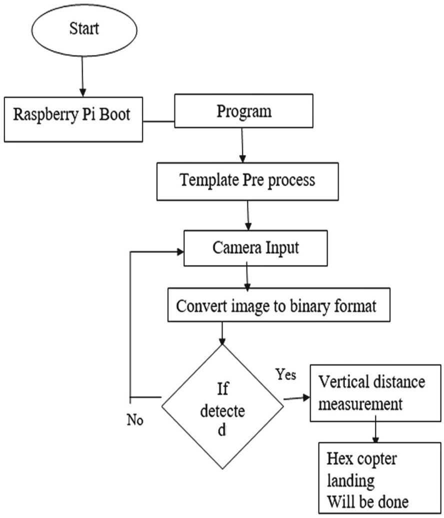

Fig. 1. Program flow chart for template matching

2.1 Quadrotor State Estimation

Pose estimates are computed at standard frequency [10] using advanced kalman filters and state estimation is done by monocular visual-inertial odometry [11] to estimate the state of the quadrotor. The pipeline provides accurate estimate of quadrotor position, linear velocity and orientation with respect to world frame. The complete pipeline uses onboard computation (e.g. Fig. 2).

2.2 Detection of the Landing Platform

In order the running algorithm detect the landing platform first of all we need to specify how the landing platform.

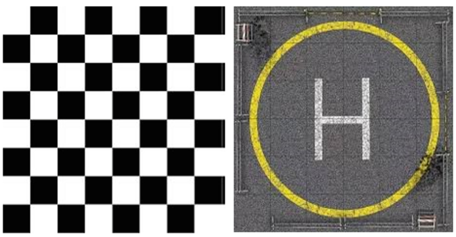

Fig. 2. Landing platform.

The recognition process consists of the following subtasks:

1. Detection of landmark in the video stream.
2. Recognition of the type of this landmark.

For landmark recognition problem we considered the approach of classifier mentioned in further section.

2.3 Trajectory Planning

The procedure followed for trajectory planning is highly determined by the environmental factors because indoor environment like hall have less colliding objects whereas outdoor fuzzy environment like busy street have more colliding factors. So taking minimum collision under consideration we have used [13] to plan optimal, feasible trajectories that prevent the vehicle from colliding with obstacles. In that paper it is proposed that a fast polynomial trajectory generation method minimizes the third derivative of the position (namely, the jerk). A recursive computation is performed for target state estimation and future position of target with respect to quadrotor in its frame with respect to drone frame by using advanced kalman filters [10] (Fig. 3).

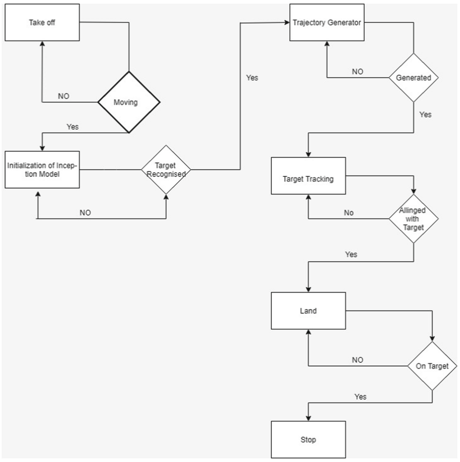

Fig. 3. Flowchart of state machine

2.4 Quadrotor Control

The state-of-the-art, nonlinear control is used to drive the quadrotor along the desired trajectory. As suggested [9] the high-level controller is used for position and attitude corrections, and a low-level controller for reaching the required body rates. The high level controller has to compute difference between estimated position and real position continuously and it feeds these data to low level controller. Depending on its feed torque given to rotors are controlled by low level controller. The whole procedure is followed from [14,15].

2.5 Boosting

Boosting is a set of methods, which improves the accuracy of analytic model. In machine learning boosting means a procedure, which sequentially construct a composition of machine learning algorithms, where each of following algorithms tries to improve and compensate weaknesses of all previous algorithms compositions. Boosting is a greedy algorithm of algorithms composition construction. The main purpose of boosting approach is finding a local optimal solution on each stage. Boosting on decision trees is one of the most effective method for classification issue.

2.6 Image Classifier

The open source machine learning framework tensorflow is used to train image classifier. The actual training process is very time consuming and inefficient strategy so we have used the retraining process of machine learning which is known as transfer learning. We have used inception model which utilizes a pre-trained neural network. The **pipeline** of the training model is viewed as:

Algorithm 1. Pipeline

procedure INITIALIZATION
    $num \leftarrow$ number of $class$
procedure IMAGE RESIZER
    $dimension <$ max dimension
procedure FEATURE EXTRACTOR
    type and stride setting
procedure FIRST STAGE ANCHOR GENERATOR
    height and width stride setting
    scales setting
    aspect ratio setting
procedure FIRST STAGE BOX PREDICTOR
    weighted regularizer
    truncated initializer
procedure SECOND STAGE BOX PREDICTOR
    regularizer after threshold and loss weight
procedure SECOND STAGE POST PROCESSING
    Softmax score convertor
procedure OPTIMIZER
procedure GRADIENT CLIPPING BY NORM

2.7 Training Data Set and Results

The training was carried out by google tensorflow developers for inception model. The transfer learning was carried out on Windows 10 PC, 4 GB RAM, 2 GB Nvdia graphics for 7 h over training images of h and x. The training images were taken from pi camera mounted over drone and some standard data-set images from various distances and various angles.The images were taken in indoor environment with human and material noise added deliberately to create robust classifier. The classifier generated was giving around 89% accuracy at testing images, although this accuracy may vary from system to system but it will be roughly around this value.

2.8 Retraining

The retraining involves several steps but first of all the bottlenecks are calculated. It took 50 min but the speed is machine dependent and can vary from one to another. As initially the inception v3 is trained on different set of images it analyzes all the image on the disc or cloud for image classification. We have kept image locally. The bottleneck is the penultimate layer which is trained on certain images of landing pattern with label **'H'**. Since it uses bottleneck for all images we have saved its cache in /temp/himage/bottleneck folder so that it can be used again and again by python script automatically. Next step begins actual training of classifier after computing the bottlenecks. By default it would run 4000 steps but we have changed it to 10000 steps for better feature extraction. The entropy, gain function and validation accuracy is seen at each step with the help of **tensorboard api by issuing the command tensorboard** $--$ $logdir/tmp/retrain_logs$. Once it oppens go to browser and connect to local host with default port 6006. After everything works smoothly it is ready to use on local datasets. Create directory filled with images for training and testing with ratio 80:20 with appropriate labels. Run the script retrain.py and if satisfied with accuracy halt otherwise increase $--how-many-training-steps$. The rate of accuracy curve becomes flat after sometime but since we have used high quality images filled with noise we have gained stability around 8500 steps but we have trained upto 10000 steps just to be sure. The penultimate layer of the **CNN** can be viewed mathematically as the evidences which can be calculated based on a sum of weights detected by the intensity of the pixels with added bias,

$$evidence_i = \sum_j W_{i,j} x_j + b_i \tag{1}$$

i: i^{th} class
W_i: Weight of the class
b_i: bias added

We have usedsoftmax activation in place of classical logistical sigmoid function. Then the probabilities are calculated by passing the evidence through the **softmax function.**

$$y = softmax(evidence) \tag{2}$$

The visualization from Fig. 4 can be shown using probability matrix as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = softmax \begin{bmatrix} W_{1,1}x_1 + W_{1,2}x_2 + W_{1,3}x_3 + b1 \\ W_{2,1}x_1 + W_{2,2}x_2 + W_{2,3}x_3 + b2 \\ W_{3,1}x_1 + W_{3,2}x_2 + W_{3,3}x_3 + b3 \end{bmatrix} \tag{3}$$

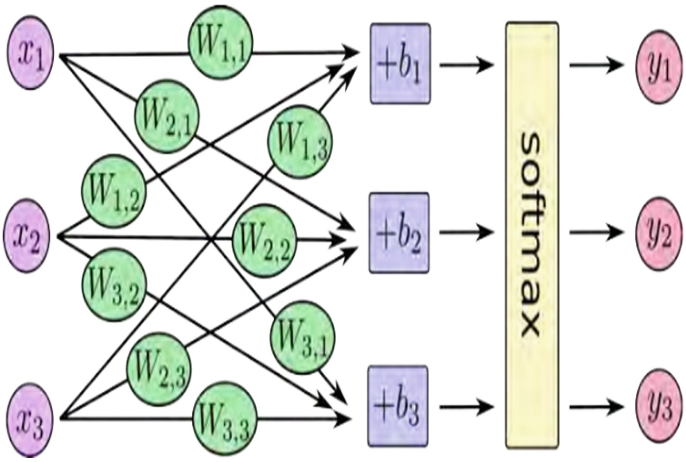

Fig. 4. Visual representation of softmax function

$$softmax = \begin{bmatrix} 34 & 5 \\ 12 & 43 \end{bmatrix} \tag{4}$$

a_{11}: **True Positive**-for correctly predicted event values
a_{12}: **False Positive**-for incorrectly predicted event values
a_{21}: **True Negative**-for correctly predicted no-event values
a_{12}: **False Negative**-for incorrectly predicted event values

2.9 Experimental Results and Comparison

In [9] and other similar works authors have used direct computation at each stage where a shape is recognized through human developed algorithms which may work in certain environments and may fail in others. We have left the algorithm deciding part to computer which recursively learns from its bottlenecks and create perfect algorithms for each environment (Fig. 5).

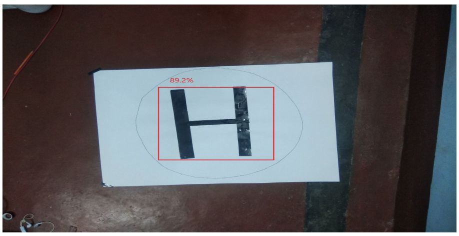

Fig. 5. Testing a positive sample

In comparison with classifiers like haar cascade and multiscale local binary patterns our classifier is slow but this latency is only in initialization steps. Whereas in terms of accuracy our convolution neural network designed is almost similar to them but in terms of versatility and robustness our classifier exceeds them. It can detect images at distorted angles during rotation which these classifiers cannot. It can detect images with noises (certain permissible limit) which is tedious for normal classifier but our classifier will handle this. So overall rescue operations where noises and distorted angles are to be expected our classifier will give better performance. The **sensitivity** is calculated around 0.7391 for few batches by:

$$TPR = TP/(TP + FN) \tag{5}$$

The specificity is the another part of our classifier statistical analysis giving promising results of 0.8958 and calculated by:

$$SPC = TN/(FP + TN) \tag{6}$$

Precision is numerically equals to 0.8718 calculated by:

$$PPV = TP/(TP + FP) \tag{7}$$

Negative Predictive Value equals to 0.7818 and is calculated by:

$$NPV = TN/(TN + FN) \tag{8}$$

False Positive Rate equals to 0.1042 and is calculated by:

$$FPR = FP/(FP + TN) \tag{9}$$

False Discovery Rate equals to 0.1282 and is calculated by:

$$FDR = FP/(FP + TP) \tag{10}$$

False Negative Rate equals to 0.2609 and is calculated by:

$$FNR = FN/(FN + TP) \tag{11}$$

The accuracy of classifier is misleading in conventional manner as they vary according to the batches. So confusion matrix gives overall accuracy independently. It is numerically equals to 0.8191 and is calculated by:

$$ACC = (TP + TN)/(P + N) \tag{12}$$

F1 score or test accuracy is equals to 0.80 and is calculated by:

$$F1 = 2TP/(2TP + FP + FN) \tag{13}$$

Matthews Correlation Coefficient is equals to 0.6442 as it suggests the balanced measure of our classifier for various test sets.

2.10 Mathematical Supervisions

The height measurement above ground is a distance measurement of the (vertical) distance from the ground. There are some tried and tested sensors for this purpose. To complete the process, all images were converted to grayscale and had their histograms equalized. The process of creating a positive sample is summarized in Fig. 6.

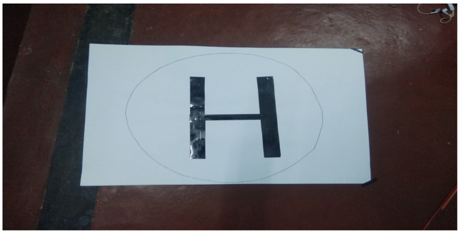

Fig. 6. Generating a positive sample

For larger drone aircraft the camera can be positioned at the ground. The measurement object would then be for example the wingspan of the aircraft. For greater heights (depending on the drone from about 50 m), this measurement method is too imprecise. It is better to carry the camera like a **Raspberry pi Camera Module V2** in the drone. The measurement object at the bottom can be easily enlarged to the expected flight altitudes as vice versa. The static position calculation method is not very efficient. We have used second order discrete linear adaptive kalman filter: First Riccati equation

$$x_k^- = \delta x_{k-1}^+ + C_k U_k + W_k \tag{14}$$

x_k^-: estimation of the prior state of quadrotor.
δ_k: discrete fundamental matrix (state transition of quadrotor).
x_{k-1}^+: previous known posterior state.
U_k: input control matrix.
W_k: process noise(proportional to environmental and state dependent factors).
C_k: conversion matrix to go from control space to the state space (coupled with air drag effect) This could be simplified as

$$\begin{bmatrix} x_k \\ y_k \\ v_{xk} \\ v_{yk} \\ a_{xk} \\ a_{yk} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta t & \frac{1}{2}\Delta t^2 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2}\Delta t^2 \\ 0 & 0 & 1 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k & 1 \\ y_k & 1 \\ v_{xk} & 1 \\ v_{yk} & 1 \\ a_{xk} & 1 \\ a_{yk} & 1 \end{bmatrix} + \begin{bmatrix} 3.3 \\ 3.3 \\ 11.3 \\ 11.2 \\ 39.5 \end{bmatrix} - \begin{bmatrix} \psi_a \\ \psi_b \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

ψ_a: deducing factor due to air drag in x direction.
ψ_b: deducing factor due to air drag in y direction.

These equations are developed and tested on Hu moments of landing target given by h_0, h_1, h_3, h_5 and ignoring higher moments for sake of simplicity

1. Correct: Helipad detected (normal case)
   h_0: 0.509797
   h_1: 0.003082
   h_3: 3.812e-5
   h_5: -2.069e-6

2. incorrect: Helipad detected (extreme case) h_0: 0.500000
   h_1: 0.060000
   h_3: 9.86259e-05
   h_5: 1.69799e-05

The calculation are done using following standard equations of moments from newton euler equation of dynamics.

$$h1 = \mu_2 0 + \mu_0; h2 = \mu_2 0 - \mu_0 2^2 + 4\mu^2 \tag{16}$$

Similarly h3 and h5 can also be calculated, whereas ψ can be calculated using newton euler equation with drag effect according to environment dynamically. We first recognize the landing target through our trained classifier and align centre of mass of quadrotor with c_z given by classifier and apply newton euler equation with respect to fixed body frame of quadrotor with synchronization with first order Ricatti equation (Fig. 7).

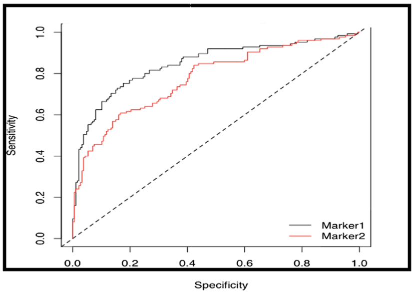

Fig. 7. Roc curve analysis as per softmax function

3 Conclusion and Future Works

We have presented the development of a real-time vision based system for detecting a landing target. The vision is simple, fast and computationally inexpensive. This work proposed vertical take-off and landing unmanned aerial vehicle (VTOL UAV) applications with vision based landing system. Our vision system is designed using Raspberry Pi as a host. It uses free software including Tensorflow framework, open CV library and python language. The datasets could be varied according to use and training model could also be adjusted. We have applied deep learning in target recognition process and have followed target tracking and state estimation using standard algorithms. The biggest hindrance of vision based approach are blind spots where camera cant see the target and state estimation and target tracking can only be done in linear or some predefined path. We are planning to use regression and unsupervised learning in target tracking and state estimation to overcome this hindrance. In this work solution of navigation problem was not considered which will be a task of the next step of this work. We are also looking to implement some encryption algorithm such that target recognition and tracking will be free from any cyber hindrance [17].

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A Group Evaluation Method for Supplier Selection Based on GSCM Practices in an Indian Manufacturing Company

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Abstract. Due to ever increasing consciousness and considerable environmental pressures from various parties, companies have begun to understand the significance of including green practices into their daily activities. As the green issues are novel, evolving continuously and hence it requires a regular study and constant monitoring in this area to understand the problems properly. In this paper we propose a framework using interval type-2 (IT2) fuzzy Vlsekriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method for evaluating suppliers in an uncertain environment. The proposed model is developed considering various green supply chain management (GSCM) factors. The trapezoidal IT2 fuzzy numbers are used in this study to handle the imprecision involved in the decision process. The proposed model is then demonstrated with a case study in an Indian manufacturing organization for selection of suitable green suppliers. Sensitivity analysis of the results is carried out to observe the influence of the preferences given by the decision makers.

Keywords: Group decision making · Interval Type-2 fuzzy sets · Interval Type-2 fuzzy VIKOR · Green supply chain management · Supplier selection

1 Introduction

In recent years, due to growing worldwide consciousness of environmental protection, ever-increasing government regulations, and stronger public awareness in environmental protection have forces the organizations to focus on the environmental issues for their survival in the global market [11]. In a view to decrease the

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hazardous environmental effects, organizations have been compelled to improve the environmental issues in their manufacturing processes, hazardous effects of their products, logistic activities and so forth. It is a critical issue to assess suppliers based on their performance along with the green criteria. So, in view of environmental safety, organizations are obliged to consider the environmental practices in their daily activities to strengthen the green image of their own company [13]. In recent times, the companies are enforcing more standards and regulations on their daily activities such as raw material extraction methods and overflowing of waste sites to prevent pollution and environmental deterioration [21].

GSCM is an approach to the management philosophy, which integrates all the components of the supply chain and seen as a progressive practice for the organizations [24]. This concentrates on how organization utilizes the supplier's technologies and processes, as well as the ability to integrate environmental issues and improve the firm's competitive advantage. However, to advance the knowledge related to GSCM needs recognition of appropriate actions. The development of reliable and valid factors can benefit the organization and also practitioners can use these factors as benchmarks to achieve continuous improvement. Incomplete information and uncertainties are always present in the implementation process. Thus, the selection of green supplier considering GSCM practices is always a Multi Criteria Decision Making (MCDM) problem. Lin et al. [12] pointed out that if properly implemented GSCM can bring economic benefits to the manufacturers. Several researchers found that still there exist some areas in industrial countries where much attention is needed to implement GSCM practices.

At present, a large number of articles on GSCM can be found in the literature [22]. GSCM is an integration of environmental factors and supply chain management activities and it includes material selection, product design, manufacturing activities, delivery of final product and end-of-life product management [22]. However, by using GSCM, the firms can select the suitable suppliers and eliminate hazardous environmental effects. The objectives of the proposed research are given below:

(1) This study proposes a novel IT2 fuzzy VIKOR group decision making approach for effective green supplier evaluation and selection.
(2) IT2 fuzzy sets are used in this model to consider the uncertainty and unknown information involved in the supplier selection process.
(3) IT2 fuzzy VIKOR method is used to find out the ranking of prospective suppliers for better accuracy.
(4) A case study in an Indian manufacturing industry from eastern India is provided to validate the proposed model.

The remaining part of the paper is structured as follows. Section 2 describes a brief review of past literature related to green supply chain management, green supplier selection methods and criteria. Basic concepts of IT2 fuzzy sets, their operations and proposed ranking method are described in Sect. 3. Section 4 is devoted to the details of the proposed IT2 fuzzy VIKOR model. In Sect. 5, a numerical example from an Indian manufacturing company is developed for finding the applicability and usefulness of the proposed model. Conclusions, limitations and future research directions are presented in Sect. 6.

2 Related Works

Now-a-days, GSCM factors have been applied by the organizations to control the use of materials, develop the green strategies and the flow of information. The studies in this area shows that GSCM is extended from green purchasing to supply chain integration and even to reverse logistics [22], also it includes supplier, manufacturer and customers. Srivastava [22] defined GSCM as the inclusion of environmental thinking into supply chain management (SCM). He proposed that GSCM can be included in material sourcing, product design, manufacturing, and transportation as well as in the end-of-life management activity of the product. Several researchers have used these criteria to assess green suppliers and perspective customers to improve the competitiveness [23].

2.1 Green Supply Chain Management

GSCM activities are mainly associated with product, production process and selection of materials. The requirements of buyer are often combined in the concept of GSCM. Rao et al. [18] found that the association between firms and suppliers can occur simultaneously and GSCM advocates the efficiency and willingness among partners in a view to improve environmental performance, reduce waste of materials and cost saving. In a study Zhu and Sarkis [20] pointed out that Chinese firms have greater environmental awareness by successfully implementing GSCM practices and establishing a long term associations with other firms and exporters. Vachon and Klassen [23] proposed that the aim of environmental collaboration is to discover the inter-organizational interactions i.e. goal setting, environmental planning and reduction of pollution effects. Thus, in a view to increase the competitiveness, and to reduce the harmful environmental effects this study provides the selection of suitable green suppliers for the organization considering various GSCM factors.

2.2 Green Supplier Evaluation Methods

Numerous methods are used in the past to solve green supplier evaluation problem. Various strategic purchasing plans developed to promote long term relations and cooperations among buyers and suppliers to achieve responsiveness of their suppliers. Chen et al. [1] developed many models considering qualitative and quantitative criteria such as quality, flexibility and price using TOPSIS method. Humphreys et al. [6] developed a hierarchical fuzzy model to help the supplier evaluation process by considering environmental criteria. Further improvement of the AHP method using a fuzzy rough set and multi-objective formulation is developed by Xia and Wu [25] to determine the number of suppliers required and allocation of appropriate orders to them.

2.3 GSCM Criteria for Supplier Evaluation

This study establishes a supplier evaluation method considering various GSCM practices adopted by the organizations in an uncertain environment. The decision making committee identified six most important GSCM criteria for the evaluation and selection of the suitable green suppliers. The selection criteria are defined in the following way.

Green Design (C_1): Eco-design or green design also known as design for the environment which includes all the activities that are required to be considered to minimize environmental impacts of products during their useful life [4]. This process starts with acquiring resources, processing of materials, and reuse of the materials that ends its life without any consideration of performance and cost [7]. This is an important consideration because the most of the hazardous practices arise directly in the design stage of the products. Eco-design programs differ from firm to firm and from one product to another product.

Green Purchasing (C_2): Green purchasing (GP) should be considered to all the activities that aim to ensure that the purchased items should possess the required environmental aspects i.e. recyclability, reusability and absence of hazardous materials. Several researchers have studied extensively on this issue to reduce harmful environmental effects.

Green Production (C_3): The objective of Green production is the continuous improvements of industrial activities to prevent or reduce water, air and land pollution effects thus reducing the risks to human being and species [7]. It is seen that green production can reduce cost of raw material, increase production efficiency, and reduce environmental and occupational safety expenses. Srivastava [22] pointed out that atmospheric pollution can occur at all the stages of production and thus companies must consider green initiatives to reduce the environmental impacts.

Green Warehousing (C_4): Now-a-days, more firms have understood the usefulness of green warehousing to save cost and energy. But the initial investment and time required to convert or construct a green warehouse is very high. Thus, to minimize the cost and increase social responsibility, many warehouses are implementing environmentally friendly practices to minimize carbon footprint and environmental pollution.

Green Transportation (C_5): The dangerous gases emitted from transports have major impact on the environment. According to the Salimifard et al. [19] about 15% of greenhouse gas emissions and 23% of CO_2 emissions are the direct result of the transportation sector. From 1990–2007 a growth of 45% is recorded in CO_2 emissions, and it is predicted that from 2007–2030 another 40% growth in CO_2 emissions will endanger the global health.

Green Recycling (C_6): The purpose of green recycling (GR) is to get back the used product from customers rather than moving product to customers [16]. It includes all the activities that required to return back the materials or products for remanufacture, reuse, recycle, repair or refurbishing.

3 Basic Concepts

3.1 IT2 Fuzzy Sets

Fuzzy set theory developed by Zadeh [26] is appropriate for handling uncertainty and imprecision involved in the information related to various parameters. Interval type-2 fuzzy (IT2F) sets, is an extension of type-1 fuzzy sets was also proposed by Zadeh [27]. IT2F sets allow the rating of alternatives using linguistic variables in a logical manner. The grade of type-1 fuzzy sets is a crisp numbers whereas the grades of membership function of IT2F sets are themselves fuzzy. Due to computational difficulty of using general type-2 fuzzy sets, IT2F sets are generally used in practice. IT2F sets are very helpful in situations where it is not possible to obtain correct membership function. Hence, they are helpful for considering linguistic uncertainties [10]. IT2F sets allow the measurement of dispersion to capture inherent uncertainties, thus, very helpful in problems where it is not possible to determine the exact membership function [15]. In this sub-section, some definitions of general type-2 fuzzy sets are presented [2,14]:

Definition 1: An IT2 fuzzy number $\tilde{\tilde{A}}$ in the universe of discourse X can be represented by a type-2 membership function $\mu_{\tilde{\tilde{A}}}$, shown as follows [14]:

$$\tilde{\tilde{A}} = \{((x,u), \mu_{\tilde{\tilde{A}}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1], 0 \le \mu_{\tilde{\tilde{A}}}(x,u) \le 1\} \quad (1)$$

Where J_x denotes an interval in $[0,1]$. Moreover, the type-2 fuzzy number $\tilde{\tilde{A}}$ also can be represented as follows:

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{\tilde{A}}}(x,u)/(x,u) \quad (2)$$

Definition 2: The upper and lower membership function of an IT2F number are type-1 membership function and is represented by [2].

$$\tilde{\tilde{A}}_i = (\tilde{A}_i^U, \tilde{A}_i^L) = ((a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1(\tilde{A}_i^U), H_2(\tilde{A}_i^U)), ((a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1(\tilde{A}_i^L), H_2(\tilde{A}_i^L))$$

where $\tilde{A}_i^U$ and $\tilde{A}_i^L$ are type - 1 fuzzy sets, $a_{i1}^U; a_{i2}^U, a_{i3}^U, a_{i4}^U, a_{i1}^L, a_{i2}^L, a_{i3}^L$ and a_{i4}^L are the reference points of the IT2F number $\tilde{\tilde{A}}_i; H_j(\tilde{A}_i^U)$ denotes the membership value of the element $a_{i(j+1)}^U$ in the upper trapezoidal membership function $\tilde{A}_i^U; 1 \le j \le 2, H_j(\tilde{A}_i^L)$ denotes the membership value of the element $a_{i(j+1)}^L$ in the lower trapezoidal membership function $\tilde{A}_i^L; 1 \le j \le 2, H_j(\tilde{A}_i^L), H_1(\tilde{A}_i^U) \in [0,1], H_2(\tilde{A}_i^U) \in [0,1], H_1(\tilde{A}_i^L) \in [0,1], H_2(\tilde{A}_i^L) \in [0,1],$ and $1 \le i \le n$.

3.2 The Arithmetic Operations of Trapezoidal IT2F Numbers

Definition 3: The addition between $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is defined as [2]:

$$\tilde{\tilde{A}}_1 \oplus \tilde{\tilde{A}}_2 = ((a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; min((H_1(\tilde{A}_1^U), H_1(\tilde{A}_2^U), H_2(\tilde{A}_1^U), H_2(\tilde{A}_2^U)))), (a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; min((H_1(\tilde{A}_1^L), H_1(\tilde{A}_2^L), H_2(\tilde{A}_1^L), H_2(\tilde{A}_2^L))) \quad (3)$$

Definition 4: The subtraction between $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is defined as [2]:

$$\tilde{\tilde{A}}_1 \ominus \tilde{\tilde{A}}_2 = ((a_{11}^U - a_{24}^U, a_{12}^U - a_{23}^U, a_{13}^U - a_{22}^U, a_{14}^U - a_{21}^U; min((H_1(\tilde{A}_1^U), H_1(\tilde{A}_2^U), H_2(\tilde{A}_1^U), H_2(\tilde{A}_2^U)))), (a_{11}^L - a_{24}^L, a_{12}^L - a_{23}^L, a_{13}^L - a_{22}^L, a_{14}^L - a_{21}^L; min((H_1(\tilde{A}_1^L), H_1(\tilde{A}_2^L), H_2(\tilde{A}_1^L), H_2(\tilde{A}_2^L))) \tag{4}$$

Definition 5: The multiplication between $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is defined as [3]:

$$\tilde{\tilde{A}}_1 \otimes \tilde{\tilde{A}}_2 = ((X_1^U, X_2^U, X_3^U, X_4^U; min(H_1(\tilde{A}_1^U), H_1(\tilde{A}_2^U), min(H_2(\tilde{A}_1^U), H_2(\tilde{A}_2^U))), ((X_1^L, X_2^L, X_3^L, X_4^L; min(H_1(\tilde{A}_1^L), H_1(\tilde{A}_2^L), min(H_2(\tilde{A}_1^L), H_2(\tilde{A}_2^L))) \tag{5}$$

where

$$X_i^T = \begin{cases} min(a_{1i}^T a_{2i}^T, a_{1i}^T a_{2(5-i)}^T, a_{1(5-i)}^T a_{2i}^T, a_{1(5-i)}^T a_{2(5-i)}^T) \text{ if } i = 1, 2 \\ max(a_{1i}^T a_{2i}^T, a_{1i}^T a_{2(5-i)}^T, a_{1(5-i)}^T a_{2i}^T, a_{1(5-i)}^T a_{2(5-i)}^T) \text{ if } i = 3, 4 \end{cases}$$

and $T \in \{U, L\}$.

Definition 6: The division between $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is defined as [3]:

$$\tilde{\tilde{A}} \oslash \tilde{\tilde{B}} = ((Y_1^U, Y_2^U, Y_3^U, Y_4^U; min(H_1(\tilde{A}_1^U), H_1(\tilde{A}_2^U), min(H_2(\tilde{A}_1^U), H_2(\tilde{A}_2^U))), ((Y_1^L, Y_2^L, Y_3^L, Y_4^L; min(H_1(\tilde{A}_1^L), H_1(\tilde{A}_2^L), min(H_2(\tilde{A}_1^L), H_2(\tilde{A}_2^L))) \tag{6}$$

where

$$Y_i^T = \begin{cases} min(a_{1i}^T / a_{2i}^T, a_{1i}^T / a_{2(5-i)}^T, a_{1(5-i)}^T / a_{2i}^T, a_{1(5-i)}^T / a_{2(5-i)}^T) \text{ if } i = 1, 2 \\ max(a_{1i}^T / a_{2i}^T, a_{1i}^T / a_{2(5-i)}^T, a_{1(5-i)}^T / a_{2i}^T, a_{1(5-i)}^T / a_{2(5-i)}^T) \text{ if } i = 3, 4 \end{cases}$$

and $T \in \{U, L\}$.

Definition 7: Some arithmetic operations of trapezoidal IT2F numbers [8]:

$$\begin{aligned} k\tilde{\tilde{A}}_1 = (k\tilde{A}_1^U, k\tilde{A}_1^L) = ((k \times a_{11}^U, k \times a_{12}^U, k \times a_{13}^U, k \times a_{14}^U; H_1(\tilde{A}_1^U), H_2(\tilde{A}_1^U)), \\ ((k \times a_{11}^L, k \times a_{12}^L, k \times a_{13}^L, k \times a_{14}^L; H_1(\tilde{A}_1^L), H_2(\tilde{A}_1^L)) \end{aligned}$$

$$and, \tag{7}$$

$$\begin{aligned} \frac{\tilde{\tilde{A}}_1}{k} = ((\frac{1}{k} \times a_{11}^U, \frac{1}{k} \times a_{12}^U, \frac{1}{k} \times a_{13}^U, \frac{1}{k} \times a_{14}^U; H_1(\tilde{A}_1^U), H_2(\tilde{A}_1^U)), \\ ((\frac{1}{k} \times a_{11}^L, \frac{1}{k} \times a_{12}^L, \frac{1}{k} \times a_{13}^L, \frac{1}{k} \times a_{14}^L; H_1(\tilde{A}_1^L), H_2(\tilde{A}_1^L)) \end{aligned} \tag{8}$$

Where k is a constant.

Definition 8: The defuzzified (crisp) value of a trapezoidal IT2F number proposed by Kahraman et al. [9] is defined as follows:

$$Def(\tilde{\tilde{A}}) = \frac{1}{2}(\sum_{T \in \{U,T\}} \frac{a_1^T + (1 + H_1(\tilde{A}^T))a_2^T + (1 + H_2(\tilde{A}^T))a_3^T + a_4^T}{4 + H_1(\tilde{A}^T) + H_2(\tilde{A}^T)}) \tag{9}$$

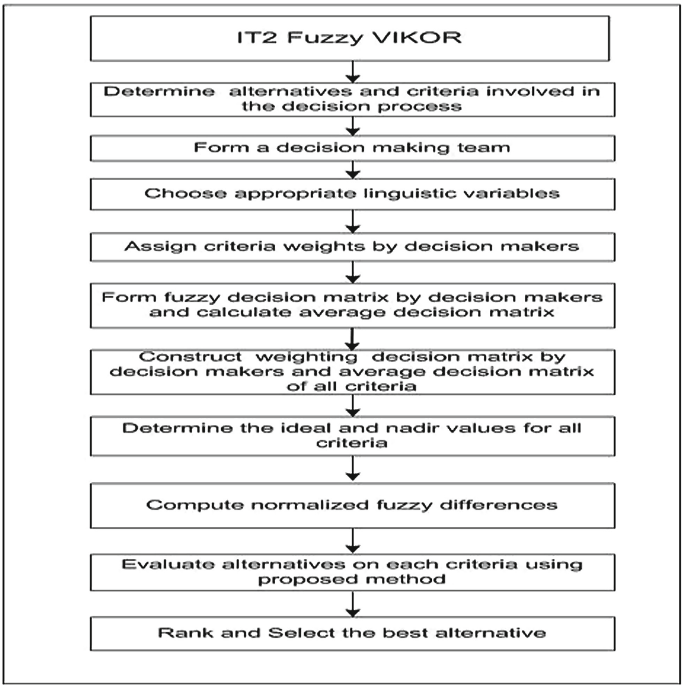

Fig. 1. The flowchart of the suggested IT2 fuzzy VIKOR methodology

4 IT2 Fuzzy VIKOR Method

VIKOR method was proposed as a MCDM tool to solve the discrete multi criteria problems with non-commensurable and conflicting factors [17] and gives a solution near to the ideal one. It is an effective technique developed to solve the complex problem by sorting, ranking and selecting suitable alternatives from a group of conflicting alternatives. This method reflects the decision maker's preferences by considering group utility maximization and individual regret minimization [17]. Because of its capability and advantages, the application of this method has been increased now-a-days. Recently, Gul et al. [5] presented a review paper on VIKOR method and discussed various fuzzy extensions of the method. The framework of the proposed model is shown in Fig. 1. Now, we will present the step by step procedure of IT2F VIKOR method for handling multi-criteria group decision-making problem. In the evaluation process, it is assumed that there are m alternatives $(A = A_1, A_2, \cdots, A_m)$, a set of n criteria $(C = C_1, C_2, \cdots, C_n)$ and k decision makers $(DM_1, DM_2, \cdots, \text{and } DM_k)$. The proposed method is described by using following steps:

Step - 1: Construct the decision matrix DM_p of the p^{th} decision maker, shown as follows:

$$DM_p = [\tilde{\tilde{x}}_{ij}^p]_{m \times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \tilde{\tilde{x}}_{11}^p & \tilde{\tilde{x}}_{12}^p & \cdots & \tilde{\tilde{x}}_{1n}^p \\ \tilde{\tilde{x}}_{21}^p & \tilde{\tilde{x}}_{22}^p & \cdots & \tilde{\tilde{x}}_{2n}^p \\ \dots & \dots & \dots & \dots \\ \tilde{\tilde{x}}_{m1}^p & \tilde{\tilde{x}}_{m2}^p & \dots & \tilde{\tilde{x}}_{mn}^p \end{pmatrix} \end{array} \quad (10)$$

where $\tilde{\tilde{x}}_{ij}^P$ denotes the performance value of alternative A_i on criterion C_j assigned by the p^{th} decision maker, $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq p \leq k$.

Step - 2: Construct the average decision matrix $\bar{Y}$, shown as follows:

$$\tilde{\tilde{x}}_{ij} = \frac{\tilde{\tilde{x}}_{ij}^1 \oplus \tilde{\tilde{x}}_{ij}^2 \oplus \cdots \oplus \tilde{\tilde{x}}_{ij}^k}{k} \quad \text{and} \quad \bar{Y} = [\tilde{\tilde{x}}_{ij}]_{m \times n} \quad (11)$$

where $\tilde{\tilde{X}}_{ij}$ denotes the average performance value of alternative A_i on criterion $C_j, 1 \leq i \leq m, 1 \leq j \leq n$.

Step - 3: Construct the weighting matrix W_p of the criteria of the p^{th} decision maker, shown as follows:

$$W_p = [\tilde{\tilde{w}}_j^p]_{n \times 1} = [\tilde{\tilde{w}}_1^p, \tilde{\tilde{w}}_2^p, \cdots, \tilde{\tilde{w}}_n^p]^T \quad (12)$$

where $\tilde{\tilde{w}}_j^p$ denotes the weight of criterion C_j assigned by the p^{th} decision maker, $1 \leq j \leq n, 1 \leq p \leq k$.

Step - 4: Construct the average weighting matrix $\bar{W}$, shown as follows:

$$\tilde{\tilde{w}}_j = \frac{\tilde{\tilde{w}}_j^1 \oplus \tilde{\tilde{w}}_j^2 \oplus \cdots \oplus \tilde{\tilde{w}}_j^k}{k}, \bar{W} = [\tilde{\tilde{w}}_j]_{n \times 1} \quad (13)$$

Step - 5: Determine the ideal($\tilde{\tilde{X}}_j^*$) and the nadir($\tilde{\tilde{X}}_j^0$) values for all criteria according to the beneficial and non-beneficial criteria.

$$\tilde{\tilde{x}}_j^* = max_i \tilde{\tilde{X}}_{ij}, \text{ and } \tilde{\tilde{X}}_j^0 = min_i \tilde{\tilde{X}}_{ij}, \text{ for beneficial criteria} \quad (14)$$

$$\tilde{\tilde{x}}_j^* = min_i \tilde{\tilde{X}}_{ij}, \text{ and } \tilde{\tilde{X}}_j^0 = max_i \tilde{\tilde{X}}_{ij}, \text{ for non-beneficial criteria} \quad (15)$$

where $1 \leq i \leq m, 1 \leq j \leq n$ and, $\tilde{\tilde{X}}_j^*$ and $\tilde{\tilde{X}}_j^0$ are trapezoidal interval type-2 fuzzy sets.

Step - 6: Compute normalized fuzzy differences:

$$\tilde{\tilde{d}}_{ij} = \left(\tilde{\tilde{X}}_j^* \ominus \tilde{\tilde{X}}_{ij} \right) / D(\tilde{\tilde{X}}_j^*, \tilde{\tilde{X}}_j^0) \text{ for benefit criteria} \quad (16)$$

$$\tilde{\tilde{d}}_{ij} = \left(\tilde{\tilde{X}}_j \ominus \tilde{\tilde{X}}^*_{ij}\right) / D(\tilde{\tilde{X}}^0_j, \tilde{\tilde{X}}^*_j) \text{ for non-benefit criteria} \tag{17}$$

Step - 7: Determine $\tilde{\tilde{S}}_i$. and $\tilde{\tilde{R}}_i$ for all alternatives, shown as follows:

$$\tilde{\tilde{S}}_i = \left(\tilde{\tilde{W}}_1 \otimes \tilde{\tilde{d}}_{i1}\right) \oplus \left(\tilde{\tilde{W}}_2 \otimes \tilde{\tilde{d}}_{i2}\right) \oplus \cdots \oplus \left(\tilde{\tilde{W}}_n \otimes \tilde{\tilde{d}}_{in}\right) \tag{18}$$

$$\tilde{\tilde{R}}_i = max_j \left(\tilde{\tilde{W}}_j \otimes \tilde{\tilde{d}}_{ij}\right) \tag{19}$$

where $\tilde{\tilde{S}}_i$ and $\tilde{\tilde{R}}_i$ are trapezoidal interval type-2 fuzzy sets.

Step - 8: Compute the best values of $\tilde{\tilde{S}}_i$ and $\tilde{\tilde{R}}_i$, shown as follows:

$$\tilde{\tilde{S}}^* = min_i \tilde{\tilde{S}}_i \quad \text{and} \quad \tilde{\tilde{R}}^* = min_i \tilde{\tilde{R}}_i \tag{20}$$

Step - 9: Compute $\tilde{\tilde{Q}}_i$ for all alternatives, shown as follows:

$$\tilde{\tilde{Q}}_i = \nu \frac{\tilde{\tilde{S}}_i \ominus \tilde{\tilde{S}}^*}{S^0 - S^{*U}_{i1}} \oplus (1 - \nu) \frac{\tilde{\tilde{R}}_i \ominus \tilde{\tilde{R}}^*}{r^0 - r^{*U}_{i1}} \tag{21}$$

where $s^0 = max\ s^U_{i4}, r^0 = max\ r^U_{i4}$, ν is the weight of the strategy of maximum group utility and $1-\nu$ is the weight of individual regret. The $\tilde{\tilde{Q}}_i$ of all alternatives are defuzzified according to Eq. (10) and is denoted as Q_i.

Step - 10: Ranking the order of alternatives are obtained according to the increasing order of Q_i.

5 Numerical Example

In this section, the proposed algorithm is applied to solve green supplier selection problem in a manufacturing industry located in India and to observe its suitability from the points of view of viability and feasibility. A reputed manufacturing company working in Eastern part of India desires to select suitable green suppliers for supplying appropriate material required for its new product line. To meet this specific goal, the company has set up a decision making team consist of three Experts. All the decision makers are chosen because of their vast experience (more than 10 years at least) in the field of supplier selection and innovations. It is assumed that the committee is homogeneous in spite of difference in Decision Maker's age, experience, and sex. After preliminary screening, the committee has short listed five feasible suppliers (A_1, A_2, A_3, A_4 and A_5) for further assessment. The committee has prepared a set of questionnaires for selection suitable green criteria after consulting with researchers in this domain.

Table 1. Linguistic variables and their corresponding IT2TrFNs.

Linguistic terms	Corresponding IT2TrFNs
Very low (VL)	((0,0,0,0.1;1,1), (0,0,0, .05; 0.9, 0.9))
Low (L)	((0, 0.1, 0.15, 0.3; 1,1),(0.05, 0.1, 0.15, 0.2; 0.9, 0.9))
Medium low (ML)	((0.1, 0.3, 0.35, 0.5; 1,1), (0.2, 0.3, 0.35, 0.4; 0.9, 0.9))
Medium (M)	((0.3, 0.5, 0.55, 0.7; 1,1), (0.4, 0.5, 0.55, 0.6; 0.9,0.9))
Medium high (MH)	((0.5, 0.7, 0.75, 0.9; 1,1), (0.6, 0.7, 0.75,0.8; 0.9, 0.9))
High (H)	((0.7, 0.85, 0.9, 1; 1,1), (0.8, 0.85, 0.9, 0.95; 0.9, 0.9))
Very high (VH)	((0.9, 1,1,1; 1,1), (0.95,1,1,1; 0.9,0.9))

Table 2. The aggregated decision matrix ($\bar{Y}$) of suppliers with respect to criteria.

	$\tilde{X}_{ij}^U$						$\tilde{X}_{ij}^L$					
	x_{1ij}^U	x_{2ij}^U	x_{3ij}^U	x_{xij}^U	$H_1(\tilde{X}_{ij}^U)$	$H_2(\tilde{X}_{ij}^U)$	x_{1ij}^L	x_{2ij}^U	x_{3ij}^L	x_{4ij}^L	$H_1(\tilde{X}_{ij}^L)$	$H_2(\tilde{X}_{ij}^L)$
X_{11}	0.07	0.23	0.28	0.43	1.00	1.00	0.15	0.23	0.28	0.33	0.90	0.90
X_{21}	0.77	0.90	0.92	0.97	1.00	1.00	0.83	0.90	0.92	0.93	0.90	0.90
X_{31}	0.63	0.80	0.85	0.97	1.00	1.00	0.73	0.80	0.85	0.90	0.90	0.90
X_{41}	0.17	0.37	0.42	0.57	1.00	1.00	0.27	0.37	0.42	0.47	0.90	0.90
X_{51}	0.57	0.75	0.80	0.93	1.00	1.00	0.67	0.75	0.80	0.85	0.90	0.90
	...	...	...	...	...	...	...	...	...	...	...	...
X_{16}	0.07	0.20	0.23	0.37	1.00	1.00	0.13	0.20	0.23	0.28	0.90	0.90
X_{26}	0.63	0.80	0.85	0.97	1.00	1.00	0.73	0.80	0.85	0.90	0.90	0.90
X_{36}	0.17	0.37	0.42	0.57	1.00	1.00	0.27	0.37	0.42	0.47	0.90	0.90
X_{46}	0.43	0.63	0.68	0.83	1.00	1.00	0.53	0.63	0.68	0.73	0.90	0.90
X_{56}	0.23	0.42	0.48	0.63	1.00	1.00	0.33	0.43	0.48	0.53	0.90	0.90

The committee identified six main criteria i.e. Green design (C_1), Green purchasing (C_2), Green production (C_3), Green transportation (C_4), Green recycling (C_5) and Green warehousing (C_6) for the evaluation and selection of the suitable green suppliers. The selection criteria are defined in Subsect. 2.3.

For determining the performance ratings, the DMs have consulted with competent authorities of the prospective suppliers. The information collected is verified by cross checking and then used for this decision making process. The weights and performance ratings of suppliers related to various GSCM criteria are expressed by suitable linguistic terms as the collected data are complex, vague and fuzzy in nature. Then the linguistic terms are converted into corresponding IT2 Trapezoidal fuzzy numbers as shown in Table 1. The aggregated rating matrix of suppliers for each factor is obtained by using Eq. (11) and the results are shown in Table 2. The aggregated weighting matrix of factors $\bar{W}_j$ is obtained by using Eq. (13). The result is displayed in Table 3.

Table 3. The aggregated weighting matrix $\bar{W}$ of criteria.

	$\tilde{W}_j^U$						$\tilde{W}_j^L$					
	w_{1j}^U	w_{2j}^U	w_{3j}^U	w_{4j}^U	$H_1(\tilde{w}_j^U)$	$H_2(\tilde{w}_j^U)$	w_{1j}^L	w_{2j}^L	w_{3j}^L	w_{4j}^L	$H_1(\tilde{w}_j^L)$	$H_2(\tilde{w}_j^L)$
$\tilde{\tilde{w}}_1$	0.83	0.95	0.97	1.00	1.00	1.00	0.90	0.95	0.97	0.98	0.90	0.90
$\tilde{\tilde{w}}_2$	0.37	0.57	0.62	0.77	1.00	1.00	0.47	0.57	0.62	0.67	0.90	0.90
$\tilde{\tilde{w}}_3$	0.63	0.80	0.85	0.97	1.00	1.00	0.73	0.80	0.85	0.90	0.90	0.90
$\tilde{\tilde{w}}_4$	0.77	0.90	0.92	0.97	1.00	1.00	0.83	0.90	0.92	0.93	0.90	0.90
$\tilde{\tilde{w}}_5$	0.57	0.75	0.80	0.93	1.00	1.00	0.67	0.75	0.80	0.85	0.90	0.90
$\tilde{\tilde{w}}_6$	0.63	0.80	0.85	0.97	1.00	1.00	0.73	0.80	0.85	0.90	0.90	0.90

Thereafter, the ideal and nadir values of all the criteria are calculated by using Eqs. (14) and (15). Tables 4 and 5 shows the result of calculation. The values of $\tilde{\tilde{S}}_i$, $\tilde{\tilde{R}}_i$ and the corresponding ranking values are calculated by using Eqs. (18), (19) and (9) respectively.

Table 4. The ideal values for all criteria.

	$\tilde{X}_j^{*U}$						$\tilde{X}_j^{*L}$					
	x_{1j}^{*U}	x_{2j}^{*U}	x_{3j}^{*U}	x_{4j}^{*U}	$H_1(\tilde{X}_j^{*U})$	$H_2(\tilde{X}_j^{*U})$	x_{1j}^{*L}	x_{2j}^{*L}	x_{3j}^{*L}	x_{4j}^{*L}	$H_1(\tilde{X}_j^{*L})$	$H_2(\tilde{X}_j^{*L})$
$\tilde{\tilde{X}}_1^*$	0.07	0.23	0.28	0.43	1.00	1.00	0.15	0.23	0.28	0.33	0.90	0.90
$\tilde{\tilde{X}}_2^*$	0.83	0.95	0.97	1.00	1.00	1.00	0.90	0.95	0.97	0.98	0.90	0.90
$\tilde{\tilde{X}}_3^*$	0.30	0.50	0.55	0.70	1.00	1.00	0.40	0.50	0.55	0.60	0.90	0.90
$\tilde{\tilde{X}}_4^*$	0.83	0.95	0.97	1.00	1.00	1.00	0.90	0.95	0.97	0.98	0.90	0.90
$\tilde{\tilde{X}}_5^*$	0.63	0.80	0.85	0.97	1.00	1.00	0.73	0.80	0.85	0.90	0.90	0.90
$\tilde{\tilde{X}}_6^*$	0.63	0.80	0.85	0.97	1.00	1.00	0.73	0.80	0.85	0.90	0.90	0.90

Table 5. The nadir values for all criteria.

	$\tilde{X}_j^{0U}$						$\tilde{X}_j^{0L}$					
	x_{1j}^{0U}	x_{2j}^{0U}	x_{3j}^{*0U}	x_{4j}^{0U}	$H_1(\tilde{X}_j^{0U})$	$H_2(\tilde{X}_j^{0U})$	x_{1j}^{0L}	x_{2j}^{0L}	x_{3j}^{0L}	x_{4j}^{0L}	$H_1(\tilde{X}_j^{0L})$	$H_2(\tilde{X}_j^{0L})$
$\tilde{\tilde{X}}_1^0$	0.77	0.90	0.92	0.97	1.00	1.00	0.83	0.90	0.92	0.93	0.90	0.90
$\tilde{\tilde{X}}_2^0$	0.00	0.07	0.10	0.23	1.00	1.00	0.03	0.07	0.10	0.15	0.90	0.90
$\tilde{\tilde{X}}_3^0$	0.03	0.13	0.17	0.30	1.00	1.00	0.08	0.13	0.17	0.22	0.90	0.90
$\tilde{\tilde{X}}_4^0$	0.03	0.17	0.22	0.37	1.00	1.00	0.10	0.17	0.22	0.27	0.90	0.90
$\tilde{\tilde{X}}_5^0$	0.03	0.17	0.22	0.37	1.00	1.00	0.10	0.17	0.22	0.27	0.90	0.90
$\tilde{\tilde{X}}_6^0$	0.07	0.20	0.23	0.37	1.00	1.00	0.13	0.20	0.23	0.28	0.90	0.90

Table 6. The calculated values for $\tilde{\tilde{S}}_i$.

	$\tilde{S}_i^U$						$\tilde{S}_i^L$					
	s_{i1}^U	s_{i2}^U	s_{i3}^U	s_{i4}^U	$H_1(\tilde{S}_i^U)$	$H_2(\tilde{S}_i^U)$	s_{i1}^L	s_{i2}^L	s_{i3}^L	s_{i4}^L	$H_1(\tilde{S}_i^L)$	$H_2(\tilde{S}_i^L)$
$\tilde{\tilde{S}}_1$	−0.48	0.50	0.80	1.86	1	1	0.09	0.50	0.76	1.23	0.9	0.9
$\tilde{\tilde{S}}_2$	−0.24	0.70	0.99	1.93	1	1	0.29	0.70	0.94	1.35	0.9	0.9
$\tilde{\tilde{S}}_3$	0.05	1.04	1.38	2.53	1	1	0.62	1.04	1.32	1.86	0.9	0.9
$\tilde{\tilde{S}}_4$	−0.32	0.66	0.98	2.04	1	1	0.24	0.66	0.93	1.42	0.9	0.9
$\tilde{\tilde{S}}_5$	0.07	1.17	1.52	2.63	1	1	0.70	1.17	1.45	1.97	0.9	0.9

Table 7. The calculated values for $\tilde{\tilde{R}}_i$.

	$\tilde{R}_i^U$						$\tilde{R}_i^L$					
	r_{i1}^U	r_{i2}^U	r_{i3}^U	r_{i4}^U	$H_1(\tilde{R}_i^U)$	$H_2(\tilde{R}_i^U)$	r_{i1}^L	r_{i2}^L	r_{i3}^L	r_{i4}^L	$H_1(\tilde{R}_i^L)$	$H_2(\tilde{R}_i^L)$
$\tilde{\tilde{R}}_1$	0.10	0.27	0.33	0.53	1.00	1.00	0.20	0.27	0.33	0.42	0.90	0.90
$\tilde{\tilde{R}}_2$	0.16	0.33	0.38	0.51	1.00	1.00	0.25	0.33	0.37	0.43	0.90	0.90
$\tilde{\tilde{R}}_3$	0.09	0.28	0.34	0.51	1.00	1.00	0.20	0.28	0.33	0.41	0.90	0.90
$\tilde{\tilde{R}}_4$	0.03	0.11	0.14	0.25	1.00	1.00	0.07	0.11	0.13	0.19	0.90	0.90
$\tilde{\tilde{R}}_5$	0.17	0.32	0.36	0.45	1.00	1.00	0.25	0.32	0.35	0.40	0.90	0.90

Table 8. The defuzzified values of $\tilde{\tilde{S}}_i$ and $\tilde{\tilde{R}}_i$.

i	$Def(\tilde{\tilde{S}}_i)$	$Def(\tilde{\tilde{R}}_i)$
1	0.653	0.304
2	0.833	0.347
3	1.219	0.306
4	0.821	0.127
5	1.332	0.331

Table 9. The calculated values for $\tilde{\tilde{Q}}_i$ with $\nu = 0.2$.

	$\tilde{Q}_i^U$						$\tilde{Q}_i^L$					
	q_{i1}^U	q_{i2}^U	q_{i3}^U	q_{i4}^U	$H_1(\tilde{Q}_i^U)$	$H_2(\tilde{Q}_i^U)$	q_{i1}^L	q_{i2}^L	q_{i3}^L	q_{i4}^L	$H_1(\tilde{Q}_i^L)$	$H_2(\tilde{Q}_i^L)$
$\tilde{\tilde{Q}}_1$	−0.44	0.18	0.38	1.00	1.00	1.00	−0.08	0.20	0.37	0.66	0.90	0.90
$\tilde{\tilde{Q}}_2$	−0.32	0.30	0.47	0.97	1.00	1.00	0.02	0.30	0.45	0.68	0.90	0.90
$\tilde{\tilde{Q}}_3$	−0.41	0.24	0.44	1.03	1.00	1.00	−0.04	0.25	0.42	0.70	0.90	0.90
$\tilde{\tilde{Q}}_4$	−0.54	−0.06	0.09	0.57	1.00	1.00	−0.28	−0.06	0.07	0.31	0.90	0.90
$\tilde{\tilde{Q}}_5$	−0.28	0.32	0.49	0.94	1.00	1.00	0.05	0.32	0.47	0.69	0.90	0.90

Table 10. The calculated values for $\tilde{\tilde{Q}}_i$ with $\nu = 0.5$.

	$\tilde{Q}_i^U$						$\tilde{Q}_i^L$					
	q_{i1}^U	q_{i2}^U	q_{i3}^U	q_{i4}^U	$H_1(\tilde{Q}_i^U)$	$H_2(\tilde{Q}_i^U)$	q_{i1}^L	q_{i2}^L	q_{i3}^L	q_{i4}^L	$H_1(\tilde{Q}_i^L)$	$H_2(\tilde{Q}_i^L)$
$\tilde{\tilde{Q}}_1$	−0.65	0.07	0.28	1.00	1.00	1.00	−0.23	0.09	0.27	0.60	0.90	0.90
$\tilde{\tilde{Q}}_2$	−0.54	0.17	0.38	1.00	1.00	1.00	−0.14	0.18	0.35	0.63	0.90	0.90
$\tilde{\tilde{Q}}_3$	−0.55	0.19	0.42	1.13	1.00	1.00	−0.12	0.20	0.39	0.72	0.90	0.90
$\tilde{\tilde{Q}}_4$	−0.69	−0.06	0.13	0.76	1.00	1.00	−0.33	−0.05	0.11	0.41	0.90	0.90
$\tilde{\tilde{Q}}_5$	−0.46	0.26	0.47	1.09	1.00	1.00	−0.05	0.27	0.44	0.73	0.90	0.90

The results are provided in Tables 6, 7 and 8 respectively. The values of Q_i for different weights of maximum group utility (ν) is determined by using Eq. (21) and the corresponding ranking values that resulted from $\nu = 0.2$ and 0.5 are shown in Tables 9 and 10 respectively. Then the ranking order of suppliers are obtained by increasing values of Q_i and shown in Table 11.

Table 11. The Ranking of suppliers according to Q_i.

i	$\nu = 0.2$	Rank	$\nu = 0.5$	Rank
1	0.2841	2	0.1778	2
2	0.3661	4	0.2580	3
3	0.3326	3	0.2991	4
4	**0.0119**	1	**0.0346**	1
5	0.3822	5	0.3483	5

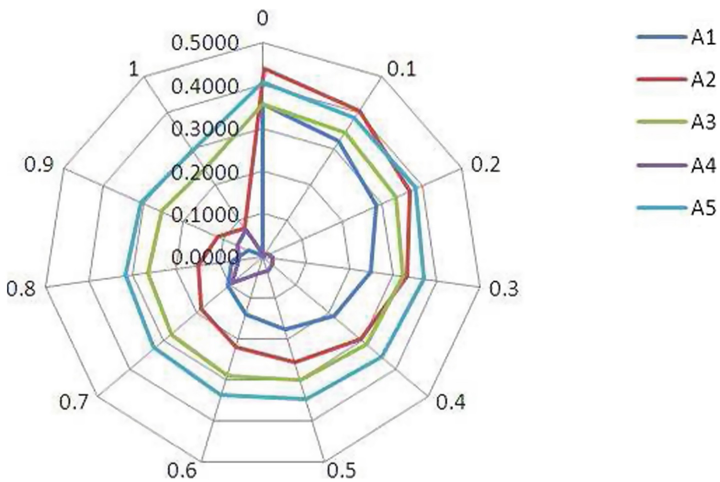

Fig. 2. Q_i values for different maximum group utilities (ν).

Table 12. The Q_i values for different maximum group utilities (ν).

	A_1	A_2	A_3	A_4	A_5
$\nu = 0.1$	0.3196	0.4021	0.3438	**0.0043**	0.3835
$\nu = 0.3$	0.2487	0.3301	0.3215	**0.0194**	0.3709
$\nu = 0.5$	0.1778	0.2580	0.2991	**0.0346**	0.3483
$\nu = 0.7$	0.1070	0.1859	0.2768	**0.0947**	0.3257
$\nu = 0.9$	**0.0361**	0.1139	0.2545	0.0649	0.3031

Results and Sensitivity Analysis
The ranking order of the suppliers is obtained by ascending order of Q_i values of the suppliers as shown in Table 11. When ν value is considered as 0.2, A_4 becomes the best green supplier with a score of 0.0119. Similarly, when ν value is considered 0.5 the best supplier is A_4 with a score of 0.0346. Thus, A_4 is considered as the best supplier for the organization who will meet all the GSCM needs in the uncertain environment. The proposed method can help DMs to achieve an acceptable compromise by assigning suitable value of ν according to their preferences. If DMs are concerned about both group utility and individual regret, then $\nu = 0.5$ would be selected; if DMs are concerned about group utility, then $0.5 < \nu < 1$ would be used; if DMs are concerned about individual regret, then $0 < \nu < 0.5$ would be utilized.

The sensitivity analysis of the results is carried out to investigate the impact of various factors of supplier evaluation. The maximum group utility value (ν) is changed from 0.1 to 0.9 increasing by 0.2 to analyze the result of the problem. The results of the sensitivity analysis are presented in Table 12 and graphically in Fig. 2. The fourth supplier A_4 has the best rankings in 4 cases out of 5 cases as shown in Fig. 2 . According to the sensitivity analysis, this paper finds that the proposed approach yields reasonable results and presents suitable outcomes to support managers in decision making.

6 Conclusions and Future Research Directions

The aim of this paper is to select suitable green suppliers for a manufacturing company considering GSCM factors. The decision making becomes more complicated as collected information are generally incomplete and uncertain. Decision makers involved in the evaluation process may express different opinions according to their expertise and knowledge. Thus, in the proposed model, IT2F linguistic variables, a new way of modeling uncertain information, is used to express the opinions of decision makers. The IT2F VIKOR method, a newly developed MCDM method, is used to handle the multiple conflicting criteria of different dimensions to select the suitable green suppliers for the company. From the results of case study it is concluded that the IT2F-VIKOR method for group evaluation with IT2F linguistic information is more suitable to select the green supplier for the organization. The proposed method is superior to other methods in ranking suppliers. An acceptable compromise solution of the maximum "group utility" of the "majority" and the minimum of the individual regret of the "opponent" are achieved by this method. It is worth to note that the TOPSIS method just provides sole rank of the alternatives, and cannot effectively reflect the experts' preferences. But by this method the decision maker's preferernces are reflected.

Following directions are suggested for the future research. In future, a suitable model could be developed that could deal with the interdependence relations among different criteria. Also, other aggregation method instead of weighted aggregation method may be used for the rankings of suppliers.

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Integrated-Optimization of Production, Preventive Maintenance and Spare Parts Inventory of Continuous Operating Series Systems

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Abstract. The abstract should summarize the contents of the paper in short terms, i.e. 150-This paper introduces an integrated-optimization model of production, preventive maintenance (PM) and spare parts inventory in order to maximize the systems' availability and minimize the both maintenance and inventory related costs while applying on the continuous operating multi-components series systems of a 210 MW coal-fired thermal power plant in eastern region of India. The PM considers the repairs and mechanical services such as cleaning, checking, adjusting etc. which restore the system in between the 'as bad as old' (ABAO) and 'as good as new' (AGAN) condition. The multi-objective functions i.e. availability and cost of maintenance and spare parts inventory under spare parts constraint have solved using Generalized Reduced Gradient (GRG) search method under the software LINGO 14. After performing several such preventive maintenances, say *j* numbers, when system reliability has reached to a minimum level then overhauling including mechanical services, repairs and bulk replacements of the components based on their condition monitoring have proposed for improving the system reliability into as good as new condition. The case study illustrates that the single-objective optimization model considering the cost function alone may not provide a cost-effective preventive maintenance interval, rather, considerations of both the availability and cost as objective functions under spare parts constraint may be a cost-effective approach.

Keywords: Integrated-optimization · Production · Preventive maintenance · Spare parts · Overhauling

1 Introduction

All production plants are expected to be operational and available for the maximum time possible in order to maximize their productions as well as profits of the organizations. The plants consist of physical assets that deteriorate with usage and time, thus, maintenance actions are required to restore the assets back to their operational condition and increase the production [1, 2]. On the other hand, the effectiveness of a maintenance program, either corrective maintenance (CM) or preventive maintenance (PM), depends

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on the availability of spare parts. Unavailability of the required spare parts in the time of maintenance may prolong the downtime of the plant and incur unnecessary costs. To ensure availability of spare parts in the time of maintenance, a general tendency is to overstock them. On the other hand, excess inventory involves extensive working capital. Thus the stock level of spare parts should be based on the maintenance policy [3]. In this regard, the production, maintenance and spare parts may be considered as the interrelated activities and to be optimized simultaneously so that the availability of the system should be maximized and both the maintenance and inventory related costs should be minimized.

There are papers that addressed the integrated optimization of maintenance and spare parts inventory, see [3–16]. Most of them [3–13] considered the age or block-based replacement policy as these two models can be easily presented in the form of a mathematical model [15]. In these models the components are replaced either at a certain age or at a fixed time interval regardless of the age. But age-based or block-based replacement before their end of operating life may not be a cost-effective approach. Therefore, Wang [14, 15] presented a delay-time based model which is neither age nor block-based, but inspection-based. In this concept, if a defect is identified in any inspection, then it is delayed for repair or replacement in the next inspection, if it is not failed during the second inspection interval, called Delay-time. Wang et al. [16] optimized jointly the preventive inspection interval and spare parts inventory considering the condition-based replacements of the components. Very few papers [17, 18] also integrated the problem of production (lot size) and preventive maintenance planning of the manufacturing plants.

The literature survey reveals that the previous integrated-optimization models considered the two activities simultaneously, i.e., preventive maintenance along with spare parts inventory or production planning, but not the three activities simultaneously. Further, the commonly used objectives on maintenance optimization are based on either cost minimization or profit maximization including the cost of maintenance, products/spare parts inventory and production. On the other hand, some papers [1, 19, 20] addressed the applicability of multi objection functions, i.e., availability maximization and maintenance cost minimization in the field of preventive maintenance interval (PMI) optimization. The availability maximization approach may be an effective way to maximize production and minimize downtime simultaneously. The minimization of cost function is also an effective way of minimizing maintenance, inventory and production related costs. Therefore, consideration of both the availability and cost functions simultaneously in the field of integrated-optimization of production, PM and spare parts inventory may be an effective approach from the view point of both saving of plant's expenditure and safety of the equipments.

Moreover, most of them optimized the PM policy based on the assumption of perfect maintenance by the uniform improvement of all maintenance activities. But there are various PM activities such as mechanical services, preventive repair and replacement having different improvement capabilities [21–26] of the system reliability. Mechanical services such as lubricating, cleaning, checking, adjusting etc. reduce the degradation of the SCs strength by improving the operating environments. Preventive repairs slow down the degradation of the subsystems and/or components (SCs) and restore the system in between the 'as bad as old' (ABAO) and 'as good as new' (AGAN) condition. Hence, both of them can be considered as the imperfect maintenance activities. However, the preventive replacements of the defective SCs restore the system in to its AGAN condition of operation. After some periodic PM including mechanical services and repairs, preventive replacement scheduling has proposed by those authors [21–26] when the system reliability reaches to a threshold.

The aim of this paper is to show the application of the integrated-optimization model on the capital intensive multi-component continuous operating series system of a coal-fired power plant in eastern region of India. Because, Coal-fired thermal power plants in India accomplish more than 65% of the required power [27]. The series systems like boiler, turbine, condenser, generator etc. of the thermal power plants are expected to be operating continuously throughout each financial year for fulfilling uninterrupted electricity demand of the domestic and commercial customers as well as increasing the plant's profit. The plant follows the PM only in the time of overhauling with the interval of 16000 to 18000 operating hours, based on the manufactures thumb rule and availability of the spare parts. The overhauling includes bulk repairs and replacements of the worn out components. The CMs is followed for the unscheduled failures between the overhauling. But, the following of the CMs during this long interval may not be cost effective as failure of any such system incurs big revenue loss. On the other hand too many PM interruptions also lead to the high revenue loss and low availability [1, 28]. Therefore the PM interruptions per year to be optimized so that both the CM and PM cost along with required spare parts inventory cost to be minimized and availability to be maximized. In this regard, this research work aspires to:

- Study the reliability characteristics of the series systems of the plant under present maintenance strategy;
- Investigate the applicability of the integrated-optimization of three activities, i.e., power generation, PM and spare parts inventory based on single objective (cost of maintenance and spare parts inventory) and multi-objectives (both the systems' availability and cost of maintenance and spare parts inventory) function;
- Decide overhauling interval including mechanical services, repairs and bulk replacements of the components based on their condition monitoring for uplifting the system reliability into as good as new condition.

2 Model Development

2.1 Model Description

The decision variables of the integrated-optimization model are availability, preventive maintenance interval (PMI), spare parts ordering interval (OI) and spare parts ordering quantity (OQ). The decision variables have optimized by maximizing the availability and minimizing the ratio of sum of maintenance and spare parts inventory costs per year under proposed PM to the same under conventional CM subjected to the spare parts constraint. The availability, preventive maintenance and spare parts inventory cost functions of the continuous operating series systems have developed considering the

PM-downtime revenue loss. After operation of each PMI the mechanical service and preventive repairs have suggested ignoring replacements. In this paper, the model of estimating the effects of mechanical service and repair actions on to the reliability as well as failure rate behavior bases on well known age reduction method [22, 24] has used. After performing several such PMs, say *j* numbers, when system reliability downs to a minimum level then overhauling including mechanical services (cleaning of external ash film and internal scaling) repairs and bulk replacements of the components based on their condition monitoring have proposed for uplifting the system reliability into as good as new condition.

2.2 Model Assumptions and Notations

Before modeling the integrated-optimization model, following assumptions are considered based on plants' practices, maintenance crew experiences and from existing literatures:

1. The PM considers the mechanical service and repairs of the components based on their condition monitoring and restore the system in between the 'as bad as old' (ABAO) and 'as good as new' (AGAN) condition.
2. The overhauling including mechanical service, repairs and replacements of the SCs based on their condition monitoring restore the system in to its AGAN condition of operation.
3. After performing several PM, say *j* times, overhauling is scheduled when system reliability reaches to a threshold.
4. All preventive activities i.e. inspection, mechanical service, repairs and replacements of the SCs of the series systems require shut down of the plant.
5. Time between failures (TBF) data follow the Weibull distribution while time to repair (TTR) data follow the lognormal distribution.

Notations:

β_i & θ_i	Shape & Scale parameters of the Weibull distribution of a subsystem i
$\lambda_i(t)$	Weibull failure rate function of a subsystem i.
s_i	Shape parameter of the lognormal distribution of a subsystem i
t_{medi}	Location parameter (median time to failure) of a subsystem i number of subsystems in the boiler
A_p	Availability of the system under periodic PM with PMI 'T'
T	The PMI, which to be optimized
$N_i(T)$	Expected number of failures during PMI 'T' of a subsystem i
N_p	Number of PM required per year
DT_p	Down time required for PM
DT_{cpi}	Sudden breakdown time of a subsystem i during PMI
T_{pi}	Mean time to preventive maintenance (MTTPM) of a subsystem i
C_{pi}	Cost of PM per hour of a subsystem i
A_c	Availability under CM of the system

A_{ci}	Availability under CM of a subsystem i (i = 1,2,…,n)
DT_{ci}	Sudden breakdown time of a subsystem i under CM
$N_i(8760)$	Expected number of failures of a subsystem i per year, i.e., 8760 operating hours, under CM
C_{PM}	Expected preventive maintenance cost (PMC) of the system per year
$(RP_i)_c$	Replacement probability of a subsystem i in a CM
C_{ui}	Unit cost of a subsystem *i*
T_{ci}	Mean time to repair (MTTR) of a subsystem *i*
C_{ci}	Cost of CM per hour of a subsystem *i*
EP	Revenue (energy price) per hour
IC_p	Inspection cost of the system at the time of PM
$R_i(T)$	Reliability of a subsystem *i* after operating PMI 'T'
C_{CM}	Expected mean corrective maintenance cost (CMC) of the system per year
C_{invp}	Expected inventory cost per extended lifecycle under PM with ordering interval 'T_o'
T_o	Ordering interval
C_o	Ordering cost per order
$N_s(T_o)$	Total number of expected spare parts required for the system during T_o for CM & PM
$N_i(T_o)$	Expected failures of a subsystem *i* during T_o
C_{hi}	Inventory carrying/holding cost of a subsystem *i*
C_{invc}	Expected inventory cost per lifecycle under CM
$(N_{sp})_{max}$	Maximum allowable spare parts per year

2.3 System Availability Under PM

Expected steady state availability (A_p) of the series system under proposed periodic PM with maintenance interval 'T' can be calculated as:

$$A_p = \frac{\text{Expected uptime of the system per year with PM}}{\text{One year operation time}}$$

$$= \frac{\left(\begin{array}{r}\text{One year operation time } - \text{ Total downtime per year of the subsystems for CM} \\ - \text{Total downtime per year of the system for PM}\end{array}\right)}{\text{One year operation time}}$$

$$\Rightarrow A_p = \frac{8760 - \sum_{i=1}^{n}[N_i(T) \times DT_{cpi} \times N_p] - [DT_p \times N_p]}{8760} \tag{1}$$

where $N_i(T)$ is the integral of the Weibull failure rate over the time T as:

$$N_i(T) = \int_0^T \lambda_i(t).dt = \int_0^T \frac{\beta_i}{\theta_i}.\left(\frac{t}{\theta_i}\right)^{\beta_i - 1}.dt = \left(\frac{T}{\theta_i}\right)^{\beta_i} \tag{2}$$

$$N_p = \frac{8760}{T + DT_p} \tag{3}$$

Therefore the Eq. (1) can be written as:

$$A_p = \frac{8760 - \sum_{i=1}^{n}\left[(T/\theta_i)^{\beta_i} \times DT_{cpi} \times \{8760/(T + DT_p)\}\right] - \left[DT_p \times \{8760/(T + DT_p)\}\right]}{8760} \tag{4}$$

2.4 System Availability Under CM

The average availability of a series system per year (8760 continuous operation hours), consisting of n number of subsystems, under conventional CM (A_c) can be calculated as:

$$A_c = \prod_{i=1}^{n} A_{ci} = \prod_{i=1}^{n} \frac{\text{Expected uptime of the subsystem } i \text{ per year with conventional CM}}{\text{One year operation time}}$$

$$= \prod_{i=1}^{n} \frac{\text{One year operation time} - \text{Downtime per year of the subsystems } i \text{ for CM}}{\text{One year operation time}}$$

$$\Rightarrow A_c = \prod_{i=1}^{n} \frac{8760 - \{DT_{ci} \times N_i(8760)\}}{8760} \tag{5}$$

where $N_i(8760)$ can be estimated following the Eq. (2) as:

$$N_i(8760) = (8760/\theta_i)^{\beta_i} \tag{6}$$

Therefore, the Eq. (5) can be rewritten as:

$$A_c = \prod_{i=1}^{n} \frac{8760 - \left\{DT_{ci} \times (8760/\theta_i)^{\beta_i}\right\}}{8760} \tag{7}$$

2.5 Maintenance Cost Under PM

The proposed PM strategy considers CM for the sudden failures during the PMI 'T' along with repairs and mechanical services in the time of PM. The CM considers the repair or replacement of the failure component based on the condition. Therefore the expected preventive maintenance cost (PMC) of a series system per year can be estimated as:

$$\begin{aligned} C_{PM} &= \sum_{i=1}^{n}\left[N_i(T) \times \{((RP_i)_c.C_{ui} + T_{ci}.C_{ci}) + (DT_{ci} \times EP)\} \times N_p\right] \\ &\quad + \left[IC_p + \sum_{i=1}^{n}\{N_i(8760).R_i(T).(T_{pi}.C_{pi})\} + (DT_p \times EP)\right] \times N_p \\ \Rightarrow C_{PM} &= \sum_{i=1}^{n}\left[(T/\theta_i)^{\beta_i} \times \{((RP_i)_c.C_{ui} + T_{ci}.C_{ci}) + (DT_{ci} \times EP)\} \times (8760/(T + DT_p))\right] \\ &\quad + \left[IC_p + \sum_{i=1}^{n}\left\{(8760/\theta_i)^{\beta_i} \times e^{-(T/\theta_i)^{\beta_i}} \times (T_{pi}.C_{pi})\right\} + (DT_p \times EP)\right] \times (8760/(T + DT_p)) \end{aligned} \tag{8}$$

2.6 Maintenance Cost Under CM

The expected mean corrective maintenance cost (CMC) per year, which is incurred by the plant, can be estimated as:

$$\mathrm{C_{CM}} = \sum_{i=1}^{n} [\{(\mathrm{C_{ui}} + \mathrm{T_{ci}.C_{ci}}) + (\mathrm{DT_{ci}} \times \mathrm{EP})\} \times \mathrm{N_i}(8760)]$$
$$\Rightarrow \mathrm{C_{CM}} = \sum_{i=1}^{n} \left[\{(\mathrm{C_{ui}} + \mathrm{T_{ci}.C_{ci}}) + (\mathrm{DT_{ci}} \times \mathrm{EP})\} \times (8760/\theta_i)^{\beta_i}\right] \tag{9}$$

2.7 Inventory Cost Under PM

The PM considers the repair or replacement of failed component in the time of CM based on condition monitoring. Therefore the spare parts to be stored for the CM during the ordering interval ($\mathrm{T_o}$). The spare parts inventory cost generally consists of (a) costs of holding the spare parts required in the time of maintenance, (b) ordering cost and (c) shortage cost due to the unavailability of the spare parts. The shortage cost can be interpreted as production loss due to the maintenance delay. In this paper the downtime production losses due to the unscheduled and scheduled downtimes are considered in the CMC and PMC respectively. Therefore shortage cost is not included in the inventory cost.

The expected number of spare parts consumption for CM during the ordering interval ($\mathrm{T_o}$) can be calculated as:

$$\mathrm{N_s(T_o)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{(RP_i)_c.N_i(T_o)} \tag{10}$$

where $\mathrm{(RP_i)_c} = 1$, if replacement is done, otherwise '0' and $\mathrm{N_i(T_o)}$ can be estimated following the Eq. (2) as:

$$\mathrm{N_i(T_o)} = (\mathrm{T_o}/\theta_i)^{\beta_i} \tag{11}$$

Then the expected inventory cost per year under PM with ordering interval '$\mathrm{T_o}$' can be calculated as:

$$\mathrm{C_{invp}} = \mathrm{N_p} \times \left[\left(\frac{\mathrm{T}}{\mathrm{T_o}}\right).\mathrm{C_o} + \frac{1}{2}\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{(RP_i)_c.N_i(T_o).T_o.C_{hi}}\right]$$
$$\Rightarrow \mathrm{C_{invp}} = \mathrm{N_p} \times \left[\left(\frac{\mathrm{T}}{\mathrm{T_o}}\right).\mathrm{C_o} + \frac{1}{2}\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{(RP_i)_c}.\left((\mathrm{T_o}/\theta_i)^{\beta_i}\right).\mathrm{T_o.C_{hi}}\right] \tag{12}$$

2.8 Inventory Cost Under CM

In this context, the expected average inventory cost per year under conventional CM (C_{invc}), assuming single order, can be estimated as:

$$C_{invc} = C_o + \sum_{i=1}^{n} (RP_i)_c.N_i(8760) \times C_{ui} + \frac{1}{2}\sum_{i=1}^{n} (RP_i)_c.N_i(8760) \times 8760 \times C_{hi}$$

$$\Rightarrow C_{invc} = C_o + \sum_{i=1}^{n} (RP_i)_c.(8760/\theta_i)^{\beta_i} \times C_{ui} + \frac{1}{2}\sum_{i=1}^{n} (RP_i)_c.(8760/\theta_i)^{\beta_i} \times 8760 \times C_{hi} \quad (13)$$

2.9 Spare Parts Constraint Function

The optimization problem may optimize the number of spares for each subsystem less than 1 (one) during the PMI. However, at least one spare part for each subsystem is to be stored during this interval. Then the minimum number of spare parts during the PMI should be at least total number of subsystems 'n'. On the other hand all these n numbers of spare parts may not be consumed during a PMI. Some of the spares may be saved in one PMI and can be used in the next PMI. Therefore there should be an allowable limit of spare parts per year which should be less than the n $\times$ N_p. With these assumptions the maintenance spare parts constraint may be written as:

$$n \le \left[\sum_{i=1}^{n} (RP_i)_c \times N_i(T_o)\right] \times \left(\frac{8760}{T_o + DT_p}\right) \le (N_{sp})_{max} \quad (14)$$

where $(N_{sp})_{max} \le n \times N_{p.}$

2.10 The Multi-objective Optimization Model

Finally, the bi-objective integrated-optimization model has mathematically formulated as follows:

$$\begin{aligned}
&\text{maximize} \quad A_p \\
&\text{minimize} \quad (C_{PM} + C_{invp})/(C_{CM} + C_{invc}) \\
&\text{subjected to:} \\
&\qquad A_p \ge A_c \\
&\qquad C_{PM} \le C_{CM} \\
&\qquad C_{invp} \le C_{invc} \\
&n \le \left[\sum_{i=1}^{n} (RP_i)_c \times N_i(T_o)\right] \times \left(\tfrac{8760}{T_o+DT_p}\right) \le (N_{sp})_{max} \\
&T \ge 0,\ T_o \ge 0, A_c \ge 0, A_p \ge 0, C_{CM} \ge 0, C_{PM} \ge 0, C_{invp} \ge 0 \,\&\, C_{invc} \ge 0
\end{aligned} \quad (15)$$

In the above model the minimization objective function is a ratio and is unit less like that of A_p. Therefore, both the objective functions can be transformed into a single objective function by assigning equal weights to the both.

In order to study the effectiveness of the single-objective and multi-objective maintenance models with or without spare parts constraint, this paper splits the above bi-objective integrated-optimization model in to four models as follows:

Model 1: Multi-objective optimization model considering system availability maximization and maintenance and inventory costs minimization under spare parts constraint.
Model 2: Multi-objective optimization model considering system availability maximization and maintenance and inventory costs minimization without spare parts constraint.
Model 3: Single-objective optimization model considering minimization of maintenance and inventory costs under spare parts constraint.
Model 4: Single-objective optimization model considering minimization of maintenance and inventory costs without spare parts constraint.

2.11 Expected Benefit

After optimization of the PMI (T^*) the expected revenue increase (RI), maintenance cost savings (MCS), inventory cost savings (INVCS) and finally, expected preventive maintenance benefit (PMB) per year can be calculated respectively as:

$$RI = (A_p - A_c) \times 8760 \times EE \tag{16}$$

$$MCS = C_{CM} - C_{PM} \tag{17}$$

$$INVCS = C_{invc} - C_{invp} \tag{18}$$

$$PMB = RI + MCS + INVCS \tag{19}$$

2.12 Overhauling Scheduling

Mechanical services improve the deteriorating environment and slow down the failure rate, i.e., in simple ward, improves the reliability curve. In this regard Tsai et al. [22] introduced a reliability improvement factor which improves the reliability curve. But in case of the power plant, mechanical services along with repairs and replacements are done in the time of overhauling. Therefore, the historical failure data from that plant already represents the failure probability after improvement of the operating environments by mechanical services. Therefore, introduction of improvement factor of mechanical service for that system reliability is not required. But, the improvement factor for preventive repair introduced by Tsai et al. [22] has used to uplift the system reliability between the ABAO and AGAN condition. The effects of various actions on

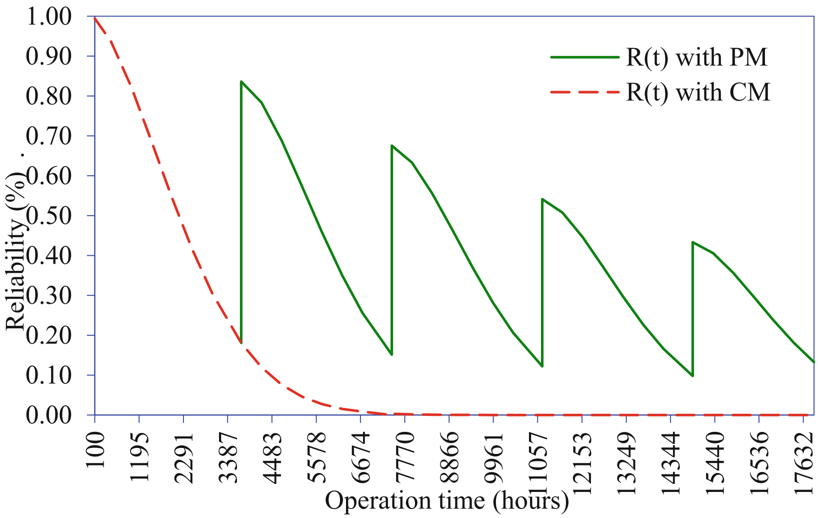

Fig. 1. Reliability under conventional CM and proposed PM.

to the reliability curve have shown in the Fig. 1. The overhauling has scheduled after j times of PM when system reliability downs below a minimum limit.

The initial reliability of a subsystem i on the j^{th} PM is also equal to the final reliability after (j-1)th PM as:

$$\mathrm{R}_{0ji} = \mathrm{R}_{f(j-1)i} = \mathrm{R}_{0(j-1)i}.\mathrm{R_i(T)} = \mathrm{R}_{0(j-1)i}.\mathrm{e}^{-\left[(\mathrm{T}/\theta_i)^{\beta_i}\right]} \tag{20}$$

where $\mathrm{R_{o(j\text{-}1)i}}$ & $\mathrm{R_{f(j\text{-}1)i}}$ are the initial and final reliabilities of the subsystem i on the (j-1)th stage of operation.

The final reliability of the subsystem i after the j^{th} PM (repairs) with improvement factor m_i ($0 \le m_i \le 1$) can be expressed as:

$$\begin{gathered}\mathrm{R}_{fji} = \mathrm{R}_{0ji} + m_i \times \left(\mathrm{R}_{0(j-1)i} - \mathrm{R}_{f(j-1)i}\right)\\ \Rightarrow \mathrm{R}_{fji} = \mathrm{R}_{0(j-1)i}.\mathrm{e}^{-\left[(\mathrm{T}/\theta_i)^{\beta_i}\right]} + m_i \times \left(\mathrm{R}_{0(j-1)i} - \mathrm{R}_{f(j-1)i}\right)\end{gathered} \tag{21}$$

where $\mathrm{R_{oi}}$ is the initial reliabilities of the new subsystem i.

The improvement factor (m_i) of the system after mechanical service and repair can be estimated as [23]:

$$m_i = \sum_{i=1}^{n} f_i(T).d_i \tag{22}$$

where $\mathrm{R_{oi}}$ is the initial reliabilitie of the new subsystem i. where $f_i(T)$ is the probability of the failure of a subsystem i at time 'T' and d_i is the percents of the failures recovered by repairing.

Therefore, the initial and final reliability (R_{fj}) of the system having n numbers of subsystems after the j^{th} PM (mechanical service and repair) can be expressed respectively as:

$$R_{0j} = \prod_{i=1}^{n} \left[R_{0(j-1)i}.e^{-\left[(T/\theta_i)^{\beta_i}\right]} \right] \tag{23}$$

$$R_{fj} = \prod_{i=1}^{n} \left[R_{0(j-1)i}.e^{-\left[(T/\theta_i)^{\beta_i}\right]} + m_i \times \left(R_{0(j-1)i} - R_{f(j-1)i} \right) \right] \tag{24}$$

3 Case Study

The proposed model is demonstrated through the series subsystems of a continuous operating coal fired thermal power plant.

3.1 Description of the Coal-Fired Boiler

The coal-fired power plant under study is of 210 MW and is situated in the eastern region of India. The boiler consists of seven series subsystems ($n = 7$) which are economizer tubes (ECOT), furnace wall tubes (FWT), primary superheater tubes (PSHT), platen superheater tubes (PLSHT), final superheater tubes (FSHT), primary reheater tubes (PRHT) and final reheater tubes (FRHT). Failure of any one of these tubes will lead to the failure of the boiler as well as of the power generation.

3.2 Data Collection, Classification and Parameters Estimation

The raw data are collected from the maintenance and purchase logbook of the power plant and classified into time between failures (TBF), time to repairs (TTR), cause of failures and cost data of the subsystems. This research work assumes the Weibull distribution of the failure data as it is very flexible and most commonly used model in failure data analysis [29]. The lognormal distribution is assumed for repair data [22]. The parameters of the Weibull distribution (θ_i & β_i) and lognormal distribution (s_i & $t_{med\,i}$) of the subsystems ($i = 1, 2, \ldots, n$) are estimated using least-square curve fitting method [30, 31]. The MTBF and MTTR of the subsystems, following the Weibull distribution and lognormal distribution respectively are calculated as:

$$\mathrm{MTBF}_i = \theta_i\, \Gamma\left(1 + \frac{1}{\beta_i}\right) \tag{25}$$

where $\Gamma(.)$ is the gamma function.

$$\mathrm{MTTR}_i = t_{med\,i} \times e^{\left(s_i^2/2\right)} \tag{26}$$

The results are shown in the Table 1, which illustrates that the subsystems (tubes) are in the increasing failure rate (IFR) region as their β_i are more than one. As a result, the boiler lifecycle becomes about 8000 h, well before their overhauling interval (16000 to 18000 h). Therefore PM is required to decrease their failure rate and saving in maintenance and inventory costs.

Table 1. Estimated reliability and maintainability parameters

Sl. No.	Sub-systems	Parameters of Weibull distribution		$MTBF_i$ (h)	Parameters of lognormal distribution		$MTTR_i$ (h)
		β_i	θ_i (h)		s_i	t_{medi} (h)	
1	ECOT	1.69	3689	3292	0.91	34.20	52
2	FWT	2.04	9160	8115	0.11	35.39	65
3	PSHT	2.7	14875	13179	0.91	53.08	80
4	PLSHT	2.43	13835	12268	0.73	49.9	65
5	FSHT	1.35	7144	6550	0.87	40.64	60
6	PRHT	2.85	11631	10365	1.09	55.44	100
7	FRHT	3.02	14609	13051	0.99	24.75	40

3.3 Optimization, Results and Discussions

The objective functions under constraints have been optimized by using the Generalized Reduced Gradient (GRG) search method of the software LINGO 14 and the Genetic Algorithm (GA) and Simulated Annealing (SA) toolbox of MATLAB® 13 for finding the PMI (T^*), optimum availability (A_p), inventory ordering interval (T_o) and the number of spare parts ($N_s(T_o)$) to be stored in the inventory. The multiple objective functions have been transformed into a single objective function by assigning equal weight to both the objective functions. The input data for the model are as: $DT_{pi} = 140$ h, $IC_p =$ Rs. 5000 INR, EP = 850000 INR per hour, $C_o =$ Rs. 1000 INR. Other input data for the subsystems are shown in the Table 2.

Table 2. Input data to the model

Subsystems (i)	θ_i (h)	β_i	T_c & T_p (h)	DT_{ci} (h)	DT_{cpi} (h)	C_{ci} (INR)	C_{pi} (INR)	C_{ui} (INR)	C_{hi} (INR)
(ECOT)	3689	1.69	52	117	87	500	500	5000	100
(FWT)	9160	2.04	65	130	100	500	500	5000	100
(PSHT)	14875	2.7	80	145	115	500	500	5000	100
(PLSHT)	13835	2.43	65	130	100	500	500	5000	100
(FSHT)	7144	1.35	60	125	95	500	500	5000	100
(PRHT)	11631	2.85	100	165	135	500	500	5000	100
(FRHT)	14609	3.02	40	105	75	500	500	5000	100

The optimization results of the GRG, GA and SA are shown in the Table 3. The Table 3 shows that the optimized results of GA & SA are same however the SA takes more time (800 iterations) than the GA (51 iterations). However, the GRG technique gives the T^* and A_p lower than the GA and SA. Therefore we have considered this value for safe calculation of the PMB. From the Table 3 it is found that the availability of the boiler has been increase from 89.42% to 92.31%. Though the increase in availability (2.92%) is shown very small, however for a multi-component system (boiler) it is not a small number and this small availability improvement helps in a large amount of revenue increase (INR 215189400) of the plant (Table 4).

Table 3. Optimization results in GRG, GA and SA

Optimization algorithm	Objective (fitness) function value	Iterations	T^* (h)	T_o (h)	A_c (%)	A_p (%)	$N_s(T_o)$ (Nos.)
GRG	−0.2246	70	3724	3689	89.42	92.31	7
GA	−0.224774	51	3726	3689	89.42	92.33	7
SA	−0.224774	800	3726	3689	89.42	92.33	7

Table 4. Comparison of the various optimization models.

Optimization Models	T* (h)	A_c (%)	A_p (%)	T_o (h)	$N_s(T_o)$ (Nos)	C_{CM} (INR)	C_{PM} (INR)	MCS (INR)
Model 1	**3724**	**89.42**	**92.31**	**3689**	**7**	**1013681**	**708708**	**304973**
Model 2	3724	89.42	92.31	313	0.12	1013681	708708	304973
Model 3	3650	89.42	92.30	3689	7	1013681	705180	308508
Model 4	**2589**	89.42	91.77	1982	2.45	1013681	676012	337669

Optimization models	C_{invc} (INR)	C_{invp} (INR)	INVCS (INR)	RI (INR)	PMB (INR)
Model 1	**81091**	**49797**	**31294**	**215189400**	**215525667**
Model 2	81091	7087080	7005989	215189400	**145488384**
Model 3	81091	49797	31294	214444800	214784602
Model 4	81091	41615	39476	174981000	175358145

Further, in order to compare the effectiveness of the multi-objective (cost and availability functions) optimization model over the single objective (cost function) optimization model, the decision variables have optimized by considering the cost function alone excluding the availability function. In both the model, the influence of the spare parts constraint on the decision variables has studied also. The observations are shown in the Table 4. The RI, MCS, INVCS and PMB have calculated using the Eqs. (16) to (19) respectively for the all models and the results have tabulated in the Table 4.

The Table 4 illustrates that the PMB is maximum (215525667 INR) while considering the Model 1 than the other models. Further, consideration of spare parts constraint (Model 1 & 3) gives more PMB than the models having no spare parts constraint (Model 2 & 4). Therefore, it is clear that the consideration of the cost function alone may not provide a cost-effective PMI, rather, considerations of both the availability and cost as objective functions with spare parts constraint may be a cost-effective approach.

Most of the previous papers [21, 22, 24] suggested to assign the replacements when the system reliability downs below 60%. But in the present case the system reliability has reached to 18.08% after operation of each PMI based on cost and availability. Therefore the assumption of minimum reliability level as 60% for replacement of a capital intensive multi-component continuous operating series system may not be practical. In the present case the overhauling decision has taken when the system reliability after j^{th} PM cannot uplift above 50%. The Fig. 1 shows the reliability under conventional CM and proposed PM. It is observed from the Fig. 6 that after 4^{th} PM or operation of 4^{th} PMI the system reliability (43.35%) cannot be improved above 50% . Therefore at that time the overhauling has suggested.

4 Conclusions and Future Scope of the Work

In this paper, a multi-objective, integrated-optimization model of production (availability), condition-based preventive maintenance including mechanical services and repairs and spare parts inventory is presented for the continuous operating series systems of a thermal power plant. The objective functions are maximization of the system availability along with minimization of the maintenance and inventory costs under the constraints of spare parts. The case study validates that the integrated-optimization of production, preventive maintenance and spare parts inventory considering cost function alone (single-objective optimization model) may not provide a cost-effective PMI, rather, consideration of both the availability and cost functions (multi-objective optimization model) under spare parts constraint may be a cost-effective approach. The model can efficiently improve the boiler's availability and reduces the maintenance and inventory costs simultaneously. As a result, the model may enhance the PMB as well as profitability of a coal-fired power plant. A overhauling scheduling based on well known age reduction method is also proposed when the system reliability after j^{th} PM cannot improve above 50%. This approach increases the life cycle of the boiler. Therefore, the proposed integrated-optimization model may be very effective for the boilers of the coal-fired thermal power plants.

Further, it is suggested that, after the plant operation of every PMI, the maintenance crew should be ready with all maintenance resources. When the plant will fail suddenly after any PMI, then the mechanical services and repairs of the defective components, based on inspection, should be performed without taking any shutdown at the time of optimized PMI. This strategy may eliminate further the PM interruptions and increase the profitability of the plants.

The proposed model to be verified further applying on the other series subsystems, i.e. turbine, condenser and generator. The basic concept of this integrated-optimization model may also be used to the series systems of other continuous operating plants like nuclear power plants, chemical plants, process plants, oil refineries etc.

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PSO Based H∞ PID Controller for a 2nd Order Time Delay System

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Abstract. The objective of the present work is to design an $H\infty$ control theory based PID controller for a 2nd order time delay process. A novel approach Particle Swarm Optimization (PSO) has been employed to estimate the parameters of the PID controller. The infinity norm of the closed-loop plant transfer function with respect to disturbance has been minimized to satisfy $H\infty$ control objective. The step response of the second order time- delay system has been studied with application of the designed controller for unity feedback. A parameter variation in the form of disturbance is applied in the processes and step response of the system has been observed for multiple cases. Simulation study reveals that the PSO based $H\infty$ PID controller is effective and robust in the face of disturbance like parametric variation of the system.

Keywords: $H\infty$ control · PID controller · Particle Swarm Optimization · Time delay system

1 Introduction

How to control a process whenever it is unstable due to a disturbance has been a great topic of interest to the researchers since long. To counteract this issue many control strategies have been developed by the researchers [1]. The conventional PID controllers are already being used for many years to control a plant and process [2, 3]. But the conventional controllers are not robust and they are unsuitable to work properly for all type of plants and disturbances. This is a serious and important issue for a closed-loop control system. In order to combat such kind of problems several approaches and tuning methods are adopted by the researchers to design PID controller for stabilization of a plant [4, 5]. In this regard classical tuning approaches like Ziegler Nichol's tuning methods are also employed in [6, 7]. Recently, design and synthesis of robust PID controller has been reported in several literatures. Among which the $H_2/H\infty$ or $H\infty$ based robust PID controller design are more popular and they are being widely used recently by the many researchers [8, 9].

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An analytical design method of PID controller based on H∞ control theory has been proposed in [10]. The design of a multivariable PID controller based on Linear Matrix Inequality (LMI) approach has been presented in [11], where the multivariable fractional-order (FO) PID controller guaranteed the stability of a closed-loop system.

Besides the above analytical approaches several heuristic methods like, Genetic Algorithm (GA), Particle Swarm Optimization (PSO) etc. are also being employed for tuning and design of PID controller. A fixed structure PSO based $H\infty$ PID controller has been designed in [12] where $H\infty$ PID controller has been designed for both SISO as well as for the MIMO system. A new tool based on PSO has been employed in [13] for synthesis of a robust PID controllers satisfying H∞ norm specifications. The method of GA is also found extensive applications in design of PID controller. In [14, 15] GA has been used for tuning of three parameters of the PID controller to achieve mixed based H_2/H-infinity control.

In the present work a fixed structure PID controller has been designed for a 2nd order delayed system. The parameters of the PID controllers are estimated through Particle Swarm Optimization (PSO) method subject to the minimization of the H∞ norm of the closed-loop plant transfer function. To the best of authors knowledge this work has not been investigated detail in existing literatures.

The entire paper is fabricated as follows: Sect. 2 has described the proposed structure of the PID controller and the study model of the time-delay 2nd order plant. Sections 3 and 4 are illustrated the H∞ control theory along with the mixed-sensitivity based approach in design of H∞ controller. The proposed optimization problem and the implementation of PSO based algorithms to design PID controller has been described details in Sects. 5 and 6 respectively. The results of simulations are evaluated in Sect. 7.

2 Proposed Structure of the Controller and the Plant

In this work to design H∞ based PID controller a conventional fixed structure PID controller has been taken into consideration. The transfers function $K(s)$ of this PID controller can be presented as follows;

$$K(s, K_P, T_I, T_D, N) = K_P\left(1 + \frac{1}{T_I s} + \frac{T_D s}{1 + \frac{T_D s}{N}}\right) \tag{1}$$

where K_P, T_I, T_D represent the proportional gain, the integral time constant and the derivative time constant respectively. This transfer functions, $K(s)$ of the PID controller is augmented by a low pass filter on the derivative part. The term T_D/N denotes the filter's time constant. All these parameters are assumed to be positive. The structure of this type of PID controller is very simple to implement, extensively used in industry framework and easy for controller tuning.

In order to investigate the performance of the proposed PID controller a nominal 2nd order SISO plant model has been considered. A time delay $e^{-\theta s}$ has been assumed to be associated with the plant. Therefore, the transfer function G(s) of the plant with time delay can be written as;

$$G(s) = \frac{ke^{-\theta s}}{s(s\tau + 1)} \tag{2}$$

where k is the static gain parameter of the process and τ is the time constant of the system and θ represents the time delay.

The delay term $e^{-\theta s}$ can be approximated using *Taylor series* expansion as $(1 - \theta s)$ and thereby the plant G(s) is reduced to a non-minimum phase transfer function. The approximated transfer function is then given by

$$G(s) = \frac{k(1 - \theta s)}{s(s\tau + 1)} \tag{3}$$

The basic block-diagram of the unity feedback control system under study is depicted in Fig. 1. All the symbols in the Fig. 1 have their usual significances.

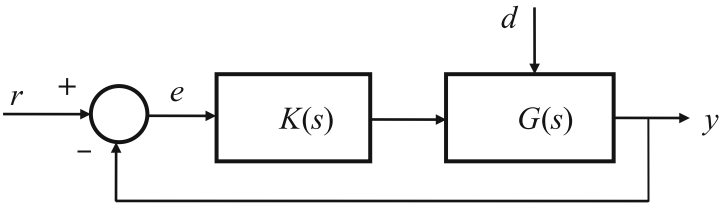


Fig. 1. Block-diagram of the proposed unity feedback control system

In this work time delay and the time constant of the plant is kept fixed at $\theta = 0.2$ and $\tau = 10$ respectively. The value of the static gain parameter is set initially $k = 2$ and thereafter changes in steps to higher values in order to introduce parametric disturbance to the plant for the purpose of simulation.

3 Basics of $H\infty$ Control Theory

H∞ Optimal control theory deals with the problem of finding a control law or the controller for a given plant such that a certain optimality criterion is achieved. It is a control problem that includes a cost functional which is a function of state and the control variables [16].

The configuration of a general Linear Time Invariant (LTI) control system model has been presented in Fig. 2. Here, $P(s)$ is plant transfer function. It is assumed that the plant $P(s)$ is mapping exogenous inputs 'd' and control inputs 'u' to the controlled outputs 'z' and the measured outputs 'y'.

$$\begin{pmatrix} z(s) \\ y(s) \end{pmatrix} = P(s) \begin{pmatrix} d(s) \\ u(s) \end{pmatrix} \tag{4}$$

The sate-feedback control law can be expressed as

$$u = K(s)y = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} y \tag{5}$$

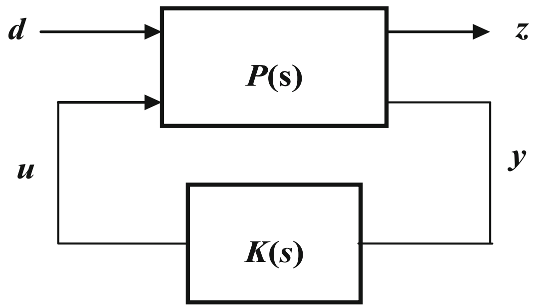


Fig. 2. General LTI control system design setup

where A_k, B_k, C_k and D_k are the parameter matrices of the controller $K(s)$. The plant $P(s)$ after input and output partitioning can be written as

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} \tag{6}$$

The closed-loop transfer function from disturbance d to the controlled output z is

$$M(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{7}$$

The control objective in $H\infty$ control theory is to minimize the $H\infty$ norm of the closed-loop plant transfer function from d to z. This is accomplished by finding a controller K (s) based on the measured information in y and produces a control signal u which counteracts the influence of d on z, thereby minimizing the closed-loop norm of the transfer function from d to z.

In practice, the problem can be thought in sub-optimal sense instead of optimal problem. The sub-optimal $H\infty$ control problem consists of finding a controller $K(s)$ such that

$$\|M(P, K)\|_\infty < \gamma \tag{8}$$

where $\gamma > 0$ is some prescribed performance level.

The above condition can be read in state-space matrix form as

$$\left\| C(sI - A)^{-1}B + D \right\|_\infty < \gamma \tag{9}$$

where A, B, C, D are the state-space matrices, they can be abstracted from $M(P, K)$.

4 Closed-Loop System and Mixed-Sensitivity Based H_∞ Control

The layout of the closed-loop plant considering a fixed-structure PID controller is presented in Fig. 3 where $G(s)$ is the open-loop time delay plant, $K(s)$ is the controller whose parameters are to be optimized. $W_1(s)$ and $W_2(s)$ are two frequency dependent weighting functions employed for getting desired characteristics of the closed-loop plant. In mixed-sensitivity based H_∞ control problem the objective is to minimize the weighted "Sensitivity" transfer function $S(s)$ $[= (I + G(s)K(s))^{-1}]$, which ensures disturbance attenuation in the plant and the weighted "Complementary Sensitivity" transfer function $K(s)S(s)$ $[= K(s)(I + G(s)K(s))^{-1}]$ that certain robustness in design and minimizes the control effort. The LTI state-space model of a system can be augmented incorporating weighted sensitivity and complementary sensitivity functions following mixed-sensitivity based H_∞- control theory and the augmented plant can be represented by the following Eqs. (10–12);

$$\dot{x}_p = A_p x_p + B_{p1} d + B_{p2} u \tag{10}$$

$$z = C_{p1} x_p + D_{\mathrm{p}11} d + D_{\mathrm{p}12} u \tag{11}$$

$$y = C_{p2} x_p + D_{p21} d + D_{p22} u \tag{12}$$

where x_p is the augmented plant state vector, 'u' is the controller output, 'y' is the measured signal influenced by the disturbance 'd'. The regulated output 'z' is used to study the performance of the plant.

The PID controller, $K(s)$ can also be constituted by the following state-space equations;

$$\dot{\hat{x}} = A_k \hat{x} + B_k y \tag{13}$$

$$u = C_k \hat{x} + D_k y \tag{14}$$

Thus, the combined state-space formulation of the closed-loop plant along with the controller $K(s)$ is given by

$$\dot{\chi} = A_{cl} \chi + B_{cl} d \tag{15}$$

$$z = C_{cl} \chi + D_{cl} d \tag{16}$$

where $\dot{\chi} = \begin{bmatrix} \dot{x}_p \\ \dot{\hat{x}} \end{bmatrix}$; $A_{cl} = \begin{bmatrix} A_p + B_{p2} D_k C_{p2} & B_{p2} C_k \\ B_k C_{p2} & A_k \end{bmatrix}$; $B_{cl} = \begin{bmatrix} B_{p1} + B_{p2} D_k D_{p21} \\ B_k D_{p21} \end{bmatrix}$;

$$C_{cl} = [\, C_{p1} + D_{p12} D_k C_{p2} \;\; D_{p12} C_k \,]; \;\; D_{cl} = D_{p11} + D_{p12} D_k D_{p21}.$$

The closed-loop transfer function between 'd' to 'z' can then be computed as

$$T_{zd} = \begin{bmatrix} W_1(s)S(s) \\ W_2(s)K(s)S(s) \end{bmatrix} = C_{cl}(sI - A_{cl})^{-1} B_{cl} + D_{cl} \tag{17}$$

It can be stated from the guidelines of mixed-sensitivity based $H\infty$ controller design that $W_1(s)$ is related to the sensitivity function $S(s)$, it should be a low-pass filter to reduce the error sensitivity in the low frequency range for output disturbance rejection whereas $W_2(s)$ should be a high-pass filter in order to guarantee the stability of the controlled

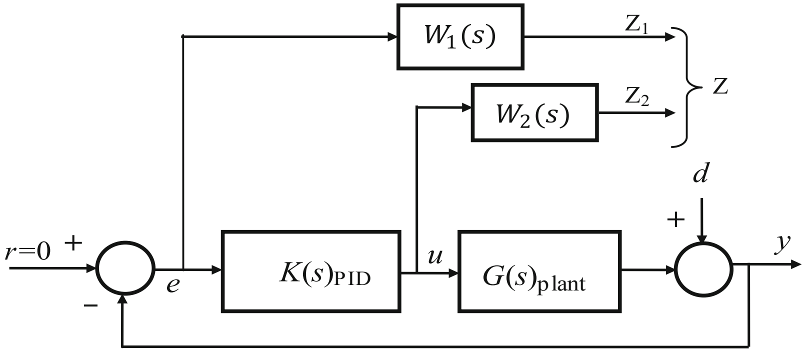


Fig. 3. Mixed-sensitivity based closed-loop plant in H∞ control theory

system under diverse operating conditions. The $H\infty$ norm of the above closed-loop transfer function $\|T_{zd}\|_\infty$ is computed in MATLAB.

5 Optimization Problem of the Present Work

The objective of the present optimization problem is to minimize $\|T_{zd}\|_\infty < \gamma$, where $\gamma > 0$ is a positive designable parameter. $\|T_{zd}\|_\infty < \gamma$, a test which involves computing the norm of T_{zd} at infinitely many frequencies and the target of minimization of $\|T_{zd}\|_\infty$ and hence 'γ' is based on well-known small-gain theorem which signifies the effect of disturbance input 'd' on to the performance output 'z' is minimum. The parameter 'γ' can be scaled to $\left(\frac{1}{\gamma}\{\|T_{zd}\|_\infty\} < 1\right)$. It is to be noted that the closed-loop plant system matrices A_{cl}, B_{cl}, C_{cl} and D_{cl} contain unknown parameters of the controller K(s). Here, $K(s)$ is the PID controller of the problem. There are three tuning parameters of the PID controller; the controller gain K_P, T_I and T_D. These parameters are to be optimized applying PSO based optimization techniques via minimization of the desired objective function ($J = \|T_{zd}\|_\infty < \gamma$). Thus, the optimization problem can then be constituted as;

$$\text{Minimize } J = \|T_{zd}\|_\infty < \gamma; \gamma > 0 \tag{18}$$

Subjected to:

$$K_P^{min} \leq K_P \leq K_{\text{P}}^{max};\ T_I^{min} \leq T_I \leq T_I^{max};\ T_D^{min} \leq T_D \leq T_D^{max}$$

The parameters of the PID controller K_P, T_I and T_D are computed in the following section applying PSO based optimization technique.

6 Controller Design Using PSO

Particle Swarm Optimization (PSO) is a bio-inspired population based stochastic optimization technique [17]. At start PSO begins to run with a randomly generated population set of variables 'N' called 'swarm of particles'. Each particle in the swarm is a different possible set of unknown parameters which are to be optimized. The objective is to search the solution space by moving the particles towards the 'best fit solution'. In every *generation*, the fitness of each particle is checked based on the user defined "objective function". The PSO algorithm continuously stores and updates the best fit variables in a swarm ('$pbest_i$', $i = 1, 2, 3,. \ldots, N$) and finally converges to a single best variable ('*gbest*'). The velocity (v_i) and position (s_i) of an i^{th} particle in the swarm are updated by the following iterative equations;

$$v_i(\mathrm{n}) = w * v_i(n-1) + ac_1 * ran_1 * (gbest - s_i(n-1)) \\ + ac_2 * ran_2 * (pbest_i - s_i(n-1)) \tag{19}$$

$$\mathrm{s_i(n) = s_i(n-1) + v_i(n)} \tag{20}$$

Here, n is the number of iteration. ac_1, ac_2 are acceleration coefficients. w is the inertia weight. ran_1 and ran_2 are two uniformly distributed random numbers. These are standard parameters of the PSO based optimization algorithm. Values and physical significances of these parameters are available in standard references [18]. In this work, different values for the PSO parameters are selected form 'PSO Toolbox' in MATLAB [19] during implementation of the algorithm.

The objective of the present work is to minimize the desired objective function ($J = \|T_{zd}\|_\infty < \gamma$) using PSO. The flowchart of the executed algorithm is presented in the Fig. 4. Here particle is constituted by a three element vector consisting of the PID controller parameters i.e.,

$$Particle : [K_P, T_I, T_D] \tag{21}$$

The population started with random values for each particle within its specified range as given in Table 1. The controller parameters are updated in each generation within this specified range. Set of basic parameters of the PSO algorithm are given in Table 2; dimension of inputs, number of iterations, particle size etc. It is to be noted that the choice of these parameters affects the performance and the speed of convergence of the algorithm.

The weights $W_1(s)$ and $W_2(s)$ are computed as $W_1(s) = \frac{1.8}{s+1.7}$; $W_2(s) = \frac{0.62s+10.4}{0.38s+1}$. The $H\infty$ norm of the closed-loop transfer function $\|T_{zd}\|_\infty$ is computed in MATLAB for the proposed second-order system with time delay. The PSO algorithm generates the best set of parameters of the PID controller by minimizing the objective function 'J' and the output results are presented in Table 3. The convergence rate of 'J' for the best solution gbest with number of particles 10 & 20 and the epochs (iterations) 100 & 200 are shown in Fig. 5(a) and (b) respectively.

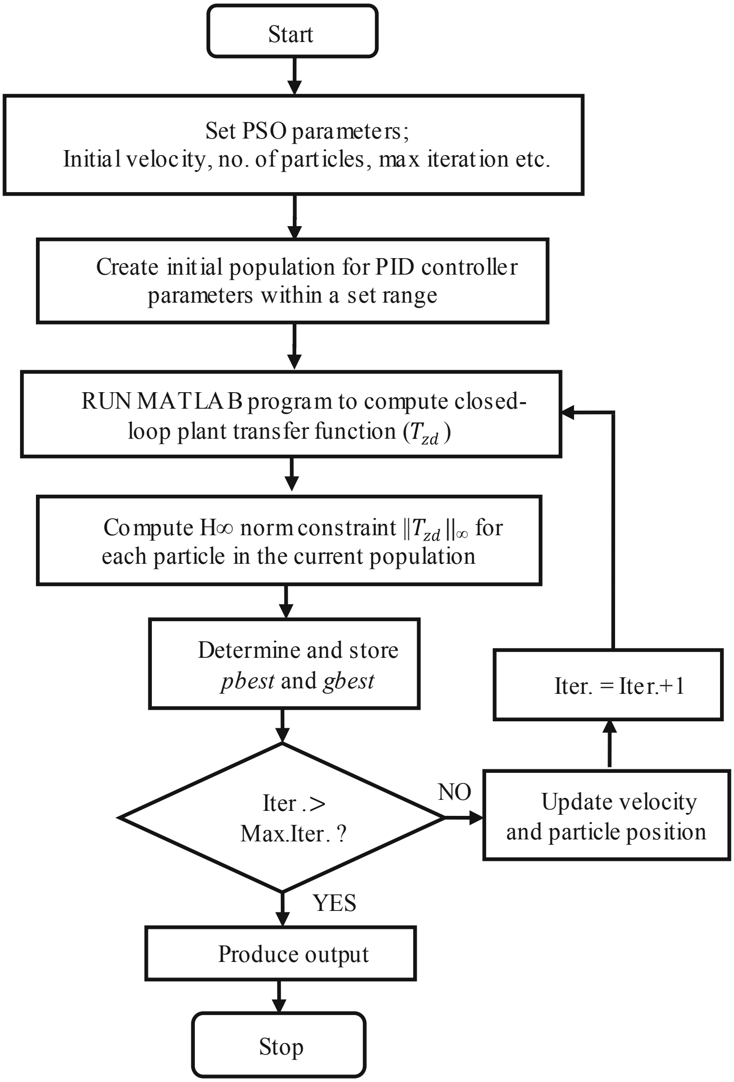


Fig. 4. Implemented flowchart to get PSO based H∞ PID controller.

Table 1. Range of PID controller parameters

Variables of the PID controller	Minimum range	Maximum range
K_P	0.01	10
T_I	1	30
T_D	0.05	5

Table 2. Standard PSO parameters

Standard Parameters of PSO	Value	Standard Parameters of PSO	Value
Particle size	10 & 20	ac_1, ac_2	2, 2
Dimension of inputs	3	w_{start}, w_{end}	0.9, 0.4
Maximum generation (epoch)	100, 200	ran_1, ran_2	0, 1
Minimum error gradient terminates run	$1 \times e^{-6}$	PSO Type	Common ‘0’

Table 3. PSO based PID controller parameters

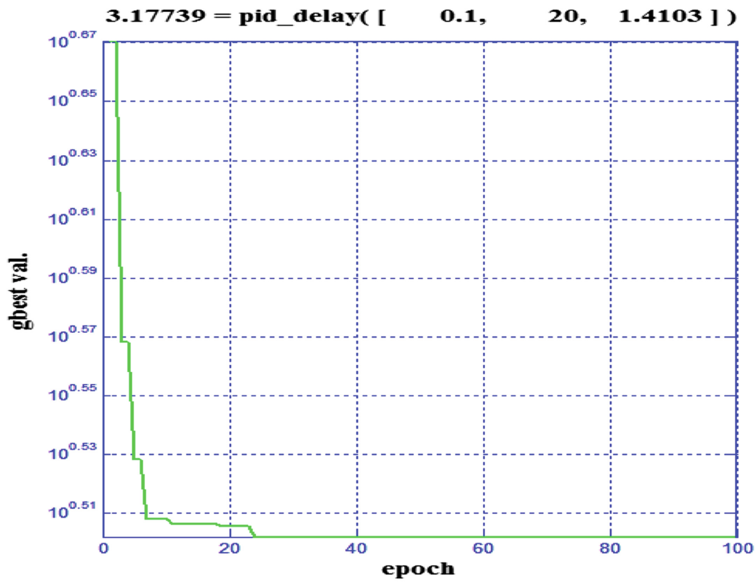
PSO based PID parameters	Maximum number of iterations (epochs)	Particle size	Value of ‘J’
$K_P = 0.1$; $T_I = 20$; $T_D = 1.4103$	100	10	3.1773
$K_P = 0.1$; $T_I = 20$; $T_D = 1.4025$	200	20	3.1648

7 Evaluation of Results

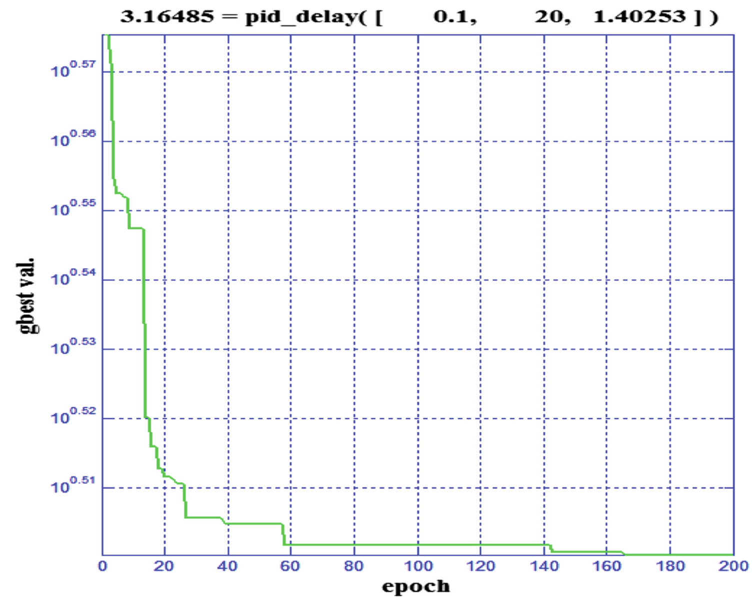
Conventional PID controller is versatile but in case of disturbance it may not be effective to show robustness in the system. The main aim of this paper is to show robustness of the plant through H∞ based PID controller in case of disturbance like parametric variation of the plant. In this work proposed PSO algorithm has been tested with different set of the particle size and iterations. The optimization program runs until generation of suitable set of PID parameters which meet desired performance of the system.

The performance of the obtained PSO based PID controller is investigated in the face of step disturbance in the proposed study system. It has been observed that the plant becomes unstable when the plant gain increases from k = 2 to k = 5. However, after application of PSO based H∞ PID controller the system becomes not only stable condition but also improves transient stability. The performance of the proposed controller is further checked for several set of system static gain, from k = 10, 20, 50,100 up to 200. It has been found that for all cases the system retain its stable and steady state operating condition with application of the designed controller. Thus it is possible to conclude that the obtained controller is not only effective to mitigate transient stability but also robust in performance. The step response of the system with and without controller for different set of the system gain has been shown in Fig. 6(a)–(c).

In Fig. 6(a) when plant is being subjected to step disturbance, it goes to marginally stable condition and with application of PSO based H∞ PID controller it become stable with minimum overshoot and quickly reaches to its steady-state within 50 s. The system gain is increased further successively from k = 5 to higher values and step response analysis of the plant has been investigated for each value of the gain. It has been noticed that the proposed PID controller brings the system to steady-state operating conditions



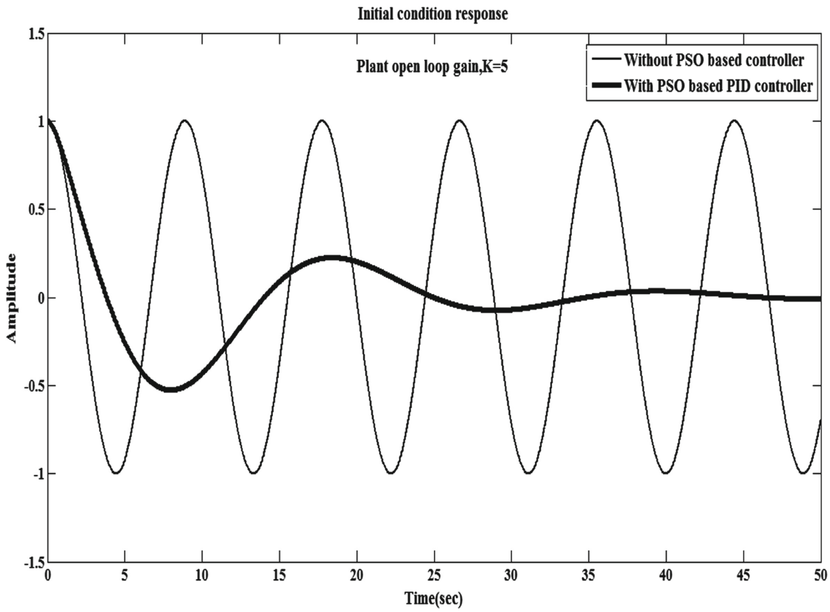
(a)



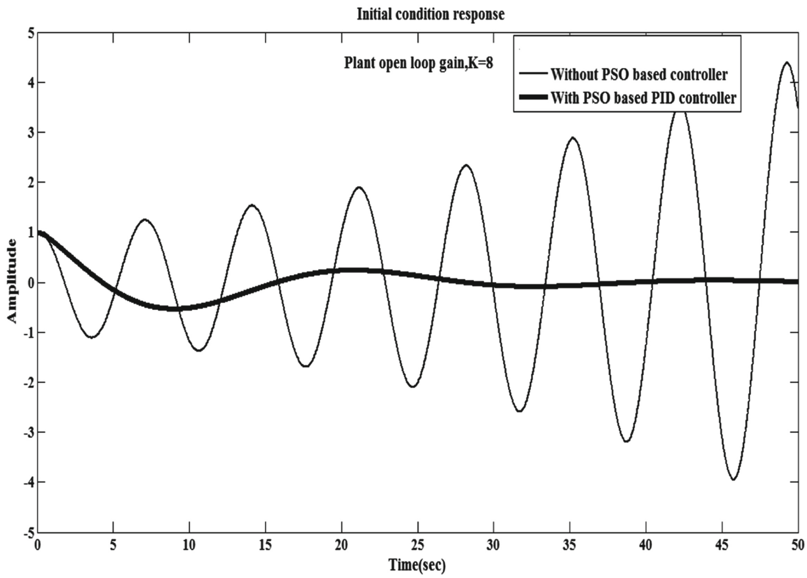
(b)

Fig. 5. (a). Convergence characteristics for 100 iterations. (b). Convergence characteristics for 200 iterations.

efficiently regardless of changes of the gain parameter of the plant G(s). The simulation

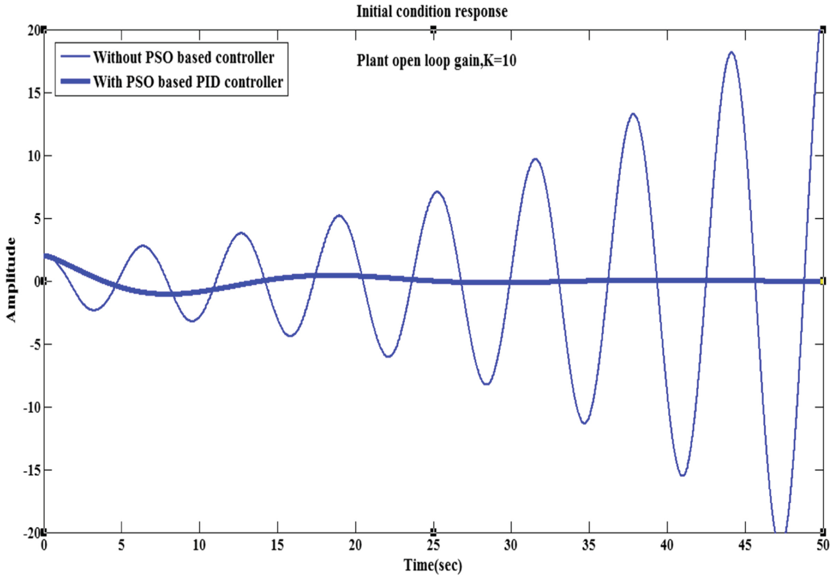


(a)



(b)

Fig. 6. (a). Response without and with PSO based PID controller (k = 5). (b). Response without and with PSO based PID controller (k = 8). (c). Response without and with PSO based PID controller (k = 10).



(c)

Fig. 6. (*continued*)

results for the system with gain $k = 8$ and for $k = 10$ are also plotted in Figs. 6(b) and (c) respectively. In view of these results it is reasonable to conclude that the PSO based H∞ PID controller exhibits good robustness characteristics as well as effective in parametric variation of the plant.

8 Conclusions

In this research work a novel procedure has been proposed which designs H∞ control theory based PID controller for a time-delay second order system. The optimal parameters of the PID controller are estimated employing Particle Swarm Optimization (PSO) technique. The PSO algorithm runs to get the parameters of the PID controller satisfying H∞ norm constraint of the closed-loop plant transfer function. The mixed-sensitivity based H∞ control theory has been adopted to get the closed-loop plant transfer function. The performance of the designed controller has been tested for several set of parameter variation of the study plant. It has been revealed that the PSO based H∞ PID controller is not only robust but also effective in mitigation of parametric variation of the plant. The advantage of this approach is that in comparison to other heuristics approaches the PSO is a simple and easy to implement and computationally efficient algorithm. The traditional methods of tuning of the PID controller can be avoided.

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Realization of Original Quantum Entanglement State from Mixing of Four Entangled Quantum States

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Abstract. Copenhagen interpretation decohere the measurement as the observer remains outside the system and the observations collapses the wave function irreversibly whereas in Everett's Relative-State Formulation the observer remains inside the system as a part of its many subsystems therefore no collapse of wave functions takes place and the subsystems remain entangled throughout, which is also reversible. In the light of Everett's Relative-State Formulation an attempt has been taken to extend the GHZ branes for four subsystems and the purpose of this essay are to analyze the state of entanglement for the subsystems four or more than that.

Keywords: Einstein-Rosen bridges · GHZ-entangled triplets · K-entangled quadruplets

1 Introduction

In a ground breaking paper written by Susskind [1], he successfully argued about the phenomena of Einstein-Rosen bridges (ERB) and Einstein Podolsky-Rosen entanglement are same thing in different manifestations, that is ER=EPR. In this context he mentioned that "Energy is conserved but entanglement is not, except under special circumstances. If two systems are distantly separated so that they can't interact, then the entanglement between them is conserved under independent local unitary transformations" [1]. A quick review of the paper [1] as follows:

Alice and Bob meet a bunch of entangled particles which is produced in vacuum. Alice and Bob shared it in half by breaking it into half of the Bell's pair each. Now Alice and Bob squeezed their potions and make it a Black hole. But from the phenomena of ER=EPR a wormhole is established between them and if they jump on both sides they cannot meet at any intermediate place through ERB. But if they do some local unitary operations between two extremely entangled states separated in the form of two black holes prior to their jump they can probably meet at some intermediate place of ERB. In other words if Alice and Bob perform some local unitary operations on two different black hole sides and readjust it, they will get back initial Alice-Bob sharing Bell pairs.

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To justify this, an illustration for two ways of thinking [1] is given to establish this aforementioned event by the "Wigner's Friend Experiment" shown in Fig. 1. To start with a cat is in half dead and half alive state. Cat is a collection of N qubits and Wigner's Friend comes in and observes the cat and records the results in his memory device which results in collapse of Wave function to one of 2^N states according to Copenhagen interpretation while they remain entangled. Now, Wigner observes them as a single system and measures the Z component of all N qubits by observing Wigner's Friend, it turns the entire system product of GHZ states [2]. Let, Wigner starts with state $|0\rangle$. If he measures his friend in the state $|0\rangle$, he continues with state $|0\rangle$ otherwise Wigner transitions to state $|1\rangle$. This is an entangled system of Wigner's Friend, cat and Wigner which is a GHZ state shown in Fig. 2, given by

$$\{|00\rangle + |11\rangle\} \otimes |0\rangle \rightarrow \{|000\rangle + |111\rangle\} \tag{1}$$

Now we can analyze the same incident in the light of Everett's Relative State Formulation.

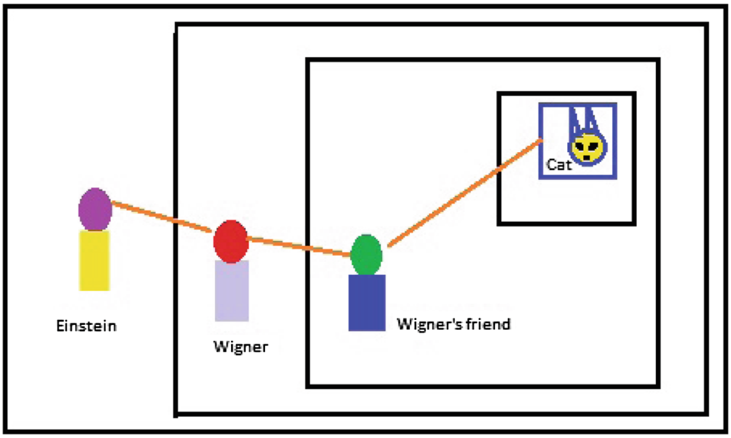


Fig. 1. Wigner's Friend Experiment [1]

Now Einstein who is outside the room observes Wigner and registers his observations on X basis (σ_x) instead of Z basis (σ_z).

According to the conventions and it is well described in [1], the quantum states in the X basis are $|L\rangle$ and $|R\rangle$ which corresponds to "Left" and "Right" instead of $|0\rangle$ and $|1\rangle$ as in Z basis.

So that,

$$\begin{aligned} |0\rangle &= |L\rangle + |R\rangle \qquad \text{Taking,} \sqrt{2} = 1 \\ |1\rangle &= |L\rangle - |R\rangle \end{aligned} \tag{2}$$

Equation (1) changes to

$$\{|000\rangle + |111\rangle\} = \{|00L\rangle + |00R\rangle\} + \{|11L\rangle - |11R\rangle\} \tag{3}$$

$$\{\overset{C\,WF}{|00\rangle} + \overset{C\,WF}{|11\rangle}\} \otimes \overset{W}{|0\rangle} \to \{\overset{C\,WF\,W}{|000\rangle} + \overset{C\,WF\,W}{|111\rangle}\}$$

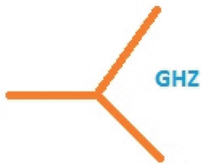


Fig. 2. "GHZ" state [1]

which gives rise to wave functions given below

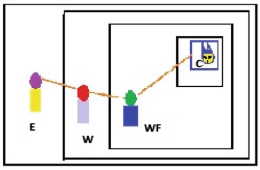
$$\begin{array}{l} \{|00\rangle - |11\rangle\} \otimes |L\rangle \\ \text{or} \\ \{|00\rangle - |11\rangle\} \otimes |R\rangle \end{array} \tag{4}$$

Figure 3 depicts the outcomes;

Einstein measures Wigner

$$|0\rangle = |L\rangle + |R\rangle$$
$$|1\rangle = |L\rangle - |R\rangle$$

Taking $\sqrt{2} = 1$



$$\{\overset{C\,WF\,W}{|000\rangle} + |111\rangle\} = \{\overset{C\,WF\,W}{|00L\rangle} + |00R\rangle\} + \{|11L\rangle - |11R\rangle\}$$

Wave function becomes

$$\{\overset{C\,WF}{|00\rangle} + |11\rangle\} \otimes \overset{W}{|L\rangle}$$

or

$$\{|00\rangle - |11\rangle\} \otimes |R\rangle$$

Fig. 3. Outcomes of Einstein measurement [1]

Now, Einstein measures the last Qubit in the X basis and his measurement yields one of two results, X = L or X = R. By this measurement, when he detects X = L, Einstein pushed back to the original entangled state $\{|00\rangle + |11\rangle\}$ which is the state before Wigner's measurement. So when X = L Einstein does nothing and when it is X = R, he simply rotates the phase of Wigner's friend and changes the state $\{|00\rangle - |11\rangle\}$ to $\{|00\rangle + |11\rangle\}$ which ends up in the maximally entangled original state.

2 Extension of the Work for One More Observer (Total 4 Observers)

When Einstein measuring the Wigner's state in the X basis, $|0\rangle = |L\rangle + |R\rangle$ or $|1\rangle = |L\rangle - |R\rangle$, if Einstein finds Wigner in the state of $|0\rangle$, he continues with state $|0\rangle$ otherwise Einstein flips to the state $|1\rangle$. This is an entangled system of Wigner, Wigner's Friend, cat and Einstein which is a new state, analogues to four-party GHZ state, let us denote as "K" state shown in Fig. 4, expressed by;

$$\{|000\rangle + |111\rangle\} \otimes |0\rangle \rightarrow \{|0000\rangle + |1111\rangle\} \tag{5}$$

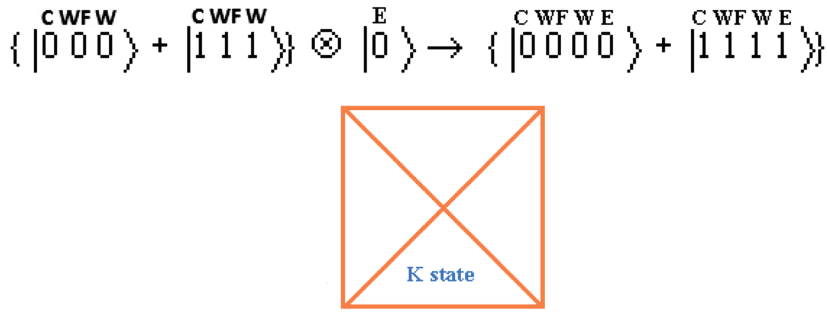


Fig. 4. "K" state

Suppose if we introduce one more observer, say Feynman (Fig. 5) in this thought experiment and try to analyze the situation for the following two cases.

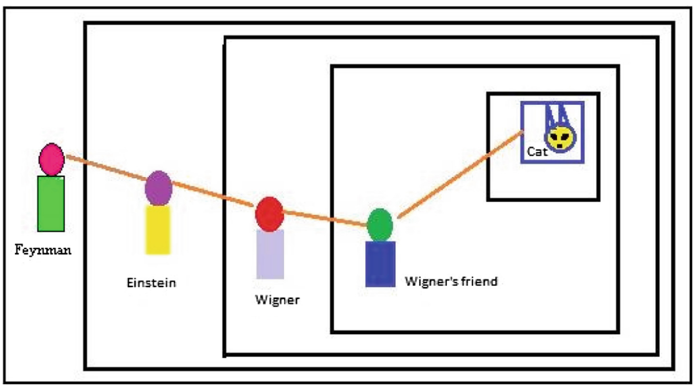


Fig. 5. Wigner's Friend Experiment with Feynman as an observer

2.1 Case I

While Einstein taking the measurements Feynman enters and stands outside the room shown in Fig. 5. He observes Einstein and registers his observations of Einstein's state on Y basis (σ_y) instead of X basis (σ_x).

The quantum states in the Y basis are $|L\rangle$ and $|R\rangle$ which corresponds to "Left" and "Right" instead of $|0\rangle$ and $|1\rangle$ as in Z basis.

But,

$$\begin{aligned} |0\rangle &= |L\rangle - |R\rangle \qquad \text{Taking, } \sqrt{2}i = 1 \\ |1\rangle &= |L\rangle + |R\rangle \end{aligned} \tag{6}$$

Applying similar logic what has been depicted earlier in Fig. 3, Feynman measures the last qubit or the Einstein's state in Y basis while Einstein himself is measuring the state of Wigner in X basis. It is a simultaneous measurement performed by Einstein and Feynman. Applying Eqs. (2) and (6) on (5) for the last two qubits in X and Y basis respectively, the following equation is obtained as follows:

$$\{|0000\rangle + |1111\rangle\} = \{|00(|L\rangle + |R\rangle)(|L\rangle - |R\rangle)\rangle + |11(|L\rangle - |R\rangle)(|L\rangle + |R\rangle)\rangle\} \tag{7}$$

This measurement yields one of four results given below. From this measurement Einstein and Feynman jointly find the possible four entangled states and the related wave functions collapse to:

$$(|00\rangle + |11\rangle) \otimes |L\rangle \otimes |L\rangle \tag{8.1}$$

$$-(|00\rangle - |11\rangle) \otimes |L\rangle \otimes |R\rangle \tag{8.2}$$

$$(|00\rangle - |11\rangle) \otimes |R\rangle \otimes |L\rangle \tag{8.3}$$

$$-(|00\rangle + |11\rangle) \otimes |R\rangle \otimes |R\rangle \tag{8.4}$$

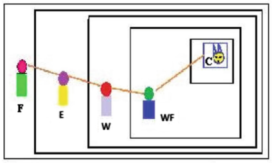
This outcome is shown in Fig. 6, where we discover that the measurement yields one of the four results given by Eqs. (8.1–8.4) depicted in Eq. 9.

$$\begin{aligned} &X \otimes Y \approx |L\rangle \otimes |L\rangle \\ &\text{or} \\ &X \otimes Y \approx |L\rangle \otimes |R\rangle \\ &\text{or} \\ &X \otimes Y \approx |R\rangle \otimes |L\rangle \\ &\text{or} \\ &X \otimes Y \approx |R\rangle \otimes |R\rangle \end{aligned} \tag{9}$$

Feynman measures Einstein

$$|0\rangle = |L\rangle - |R\rangle$$
$$|1\rangle = |L\rangle + |R\rangle$$

Taking, $\sqrt{2}i = 1$



$$|\underset{\text{C WF W E}}{0000}\rangle + |1111\rangle = |\underset{\text{C WF}}{00}\underbrace{(|L\rangle + |R\rangle)}_{\text{W}}\underbrace{(|L\rangle - |R\rangle)}_{\text{E}}\rangle + |11(|L\rangle - |R\rangle)(|L\rangle + |R\rangle)\rangle$$

Wave function becomes

$$(|\overset{\text{C WF}}{00}\rangle + |11\rangle) \otimes |\overset{\text{W}}{L}\rangle \otimes |\overset{\text{E}}{L}\rangle$$
$$-(|00\rangle - |11\rangle) \otimes |L\rangle \otimes |R\rangle$$
$$(|00\rangle - |11\rangle) \otimes |R\rangle \otimes |L\rangle$$
$$-(|00\rangle + |11\rangle) \otimes |R\rangle \otimes |R\rangle$$

Fig. 6. Outcomes of Feynman measurement

Here it is worth mentioning, the measurements were taken simultaneously by Feynman and Einstein so if any necessary changes like phase reversal etc. of any state (after getting the four evolved equally likely mixed states expressed by Eqs. 8.1–8.4) attempted by both of them simultaneously in order to restore the original state collapse that particular state due to the law of uncertainty. So only one local operation is allowed for rectification/correction at a time by any one of them to restore the initial state.

Another important assumption is, the position of Einstein is in between Feynman and Wigner and Einstein is measuring Wigner's state and Wigner is observing his friend's state, so if any local correction is needed by Einstein on these entangled states (Eqs. 8.1–8.4) to project back the original entangle state ($|00\rangle + |11\rangle$) he can perform it only for one of its internal local state. In other words Einstein can only rotate the phase of the cat or Wigner's friend. Feynman standing at the ultimate position behind Einstein can rotate only the external phase which is the overall phase of the mixed entangled state.

Equations 8.1 to 8.4 are called the "combinatorial" mixed quantum entangled states where one of the four operations expressed in Eq. 9, has been occured. The proof is given below. It is a tensor product of two basis vectors "X" and "Y" with four combinations mentioned above, is itself a vector space, together with an operation of bilinear composition.

2.1.1 Bilinearity and Its Different Manifestations

Here, X and Y be two vector spaces over the same base field F. A bilinear map is a function it takes two vector inputs $B : X \times Y \rightarrow R$ where R is a scalar and returns a output which is scalar. In this operation each individual input is linear when the other input is held constant, therefore B has a bilinear form.

It has some important properties:

i. There is a canonical isomorphism, therefore tensor product is symmetric, which mean

$$X \otimes Y \approx Y \otimes X \tag{10}$$

$$\Rightarrow L \otimes R \approx R \otimes L \tag{11}$$

ii. It is well known that for a given two vector spaces X and Y over a field “R”, so that, the tensor product P composed of X and Y is defined by $P = X \otimes Y$ which is also a vector space and whose elements and operations are performed by the following ways.

$$\begin{aligned}
&\forall x, x_1, x_2 \in X; \forall y, y_1, y_2 \in Y \\
&\text{where,} \quad |x_1\rangle = |L\rangle + |R\rangle \text{ and } |x_2\rangle = |L\rangle - |R\rangle \text{ for X basis} \\
&\qquad\qquad\qquad \text{and} \\
&\qquad |y_1\rangle = |L\rangle - |R\rangle \text{and} |y_2\rangle = |L\rangle + |R\rangle \text{ for Y basis}
\end{aligned}$$

$$(x_1, y) + (x_2, y) \simeq (x_1 + x_2, y) \simeq (2|L\rangle, y) \tag{12.1}$$

$$(x_1, y) - (x_2, y) \simeq (x_1 - x_2, y) \simeq (2|R\rangle, y) \tag{12.2}$$

$$(x, y_1) + (x, y_2) \simeq (x, y_1 + y_2) \simeq (x, 2|L\rangle) \tag{12.3}$$

$$(x, y_1) - (x, y_2) \simeq (x, y_1 - y_2) \simeq (x, -2|R\rangle) \tag{12.4}$$

From Eqs. (12.1–12.4), one can observe that performing different additive combinations of two spin possibilities $|x_1\rangle, |x_2\rangle$ and $|y_1\rangle, |y_2\rangle$ in X and Y basis respectively gives rise to a single state ($2|L\rangle$or $2|R\rangle$) with “+” or “−”sign.

Now, simple multiplication of Eqs. (12.1) and (12.3) produces $(2|L\rangle, 2|L\rangle)$ and on further normalization by a factor of 2 gives $(|L\rangle, |L\rangle)$; which can be expressed by $|L\rangle \otimes |L\rangle$.

By performing similar operations on Eqs. (12.1) and (12.4) produce $(|R\rangle, |L\rangle) \Rightarrow |R\rangle \otimes |L\rangle$, and on Eqs. (12.2) and (12.3) produce $(|L\rangle, -|R\rangle) \Rightarrow -(|L\rangle \otimes |R\rangle)$, again on Eqs. (12.2) and (12.4) produce $(|R\rangle, -|R\rangle) \Rightarrow -(|R\rangle \otimes |R\rangle)$.

Therefore the Eqs. (12.1–12.4) are analogous with Eqs. (8.1–8.4).

Equation 8.1 tells that no action is required by any of them to get the cat or Wigner’s friend state to the original entangled state $(|00\rangle + |11\rangle)$ which was the state before Wigner’s measurement.

Suppose if we denote a “+” sign for no change in overall phase of the mixed entangled state or the external phase and the second “+” sign for no change in the phase of the Wigner’s friend of Eq. 8.1. So we can define Eq. 8.1 as (+,+) state.

Equations 8.2 and 8.3 are called the “combinatorial” or “mixed” quantum entangled states where $X \otimes Y \simeq |L\rangle \otimes |R\rangle$ or $X \otimes Y \simeq |R\rangle \otimes |L\rangle$ operation has been performed. It is a tensor product of two basis vectors “X” and “Y” is itself a vector space, together with an operation of bilinear composition. Here in both the cases of Eqs. 8.2 and 8.3, Einstein has to change the state of Wigner's friend by rotating the phase of him. But in the Eq. 8.2, apart from Einstein, Feynman has to change overall phase of the mixed entangled states to get the original entangled state $(|00\rangle + |11\rangle)$. Therefore, we can denote Eq. 8.2 as $(-,-)$ state and Eq. 8.3 as $(+,-)$ state.

Finally in the Eq. 8.4, only Feynman has to change overall phase of the mixed entangled state to get the original entangled $(|00\rangle + |11\rangle)$ state. So it is $(-, +)$ state.

2.2 Case II

While Einstein taking the measurements Feynman enters and stands outside the room shown in Fig. 5 and observes Einstein and registers his observations of Einstein's state on X basis (σ_x).

Applying Eqs. (2) on (5) for the last two qubits in X, the following equation is obtained as follows:

$$\{|0000\rangle + |1111\rangle\} = \{|00(|L\rangle - |R\rangle)(|L\rangle - |R\rangle)\rangle + |11(|L\rangle + |R\rangle)(|L\rangle + |R\rangle)\rangle\} \tag{13}$$

This measurement yields one of four results given below. Again, from this measurement Einstein and Feynman jointly find the possible four entangled states and the related wave functions collapse to:

$$(|00\rangle + |11\rangle) \otimes |L\rangle \otimes |L\rangle \tag{14.1}$$

$$(|00\rangle - |11\rangle) \otimes |L\rangle \otimes |R\rangle \tag{14.2}$$

$$(|00\rangle - |11\rangle) \otimes |R\rangle \otimes |L\rangle \tag{14.3}$$

$$(|00\rangle + |11\rangle) \otimes |R\rangle \otimes |R\rangle \tag{14.4}$$

Equations 14.1 and 14.4 tell that no action is required by any of them to get the cat or Wigner's friend state to the original entangled state $(|00\rangle + |11\rangle)$ which was the state before Wigner's measurement. So these equations are of (+,+) states.

For Eqs. 14.2 and 14.3, Einstein has to change the state of Wigner's friend by rotating the phase of him, So these equations are of $(+,-)$ states.

Above all, Feynman has to change nothing of the overall phase of the mixed entangled states to get the original entangled state $(|00\rangle + |11\rangle)$.

3 Discussions

If we compare between the two cases Case I and Case II, we shall observe that the equations envovled in Case I are composed of four different quantum states (+,+), $(+,-)$, $(-, +)$ and $(-,-)$ whereas Case II consists of only two states (+,+) and $(+,-)$. But in both the cases desirable quantum state is (+,+). It implies that the probability of getting the desirable state in Case I is 0.25 while for Case II is increased to 0.50. Apparently it appears to somebody that the "Thought experiment" done for Case II (performed in X basis twice) is better than Case I for two reasons.

i. Case II has 25% more probability of getting original entangled state as an outcome than its counterpart.
ii. No change in the overall phase of any mixed entangled states to get the original entangled state. So no end correction is needed for any of the quantum states by Feynman.

But in the aspect of duality theory we can extend our understanding in the case of blackhole.

At the Institute for Advanced Study (IAS) in Princeton, New Jersey, in the United States in March 2016 Prof. Leonard Susskind gave an conceptual lecture on ER=EPR [1], where he also covered the topic "Teleportation through the wormhole" [1, 3, 4].

Prof. Susskind explained the aforementioned three observer system in context of duality theory where a collection of a large number of GHZ-entangled triplets shared between the Cat, Wigner's Friend and Wigner were squeezed individually to form a triplet of black holes [1] after undergoing some evolution with time and the geometry of the triplet's interior will consist of three tube-like regions bound together by a GHZ brane in the form of tripartite wormholes [5, 6] as shown in Fig. 7. Here GHZ states which has been constructed by the aforementioned measurement are enriched in GHZ states and can be distilled out.

According to Everett's Relative State Formulation, it is a some kind of three sided Einstein Rosen Bridge. Here, no two blackholes are entangled with each other but each one is entangled with the union of other two. It means that if Wigner and Wigner's friend cooperate in terms of reversing the measurement processes, the combination of Wigner and Wigner's friend able to send a message, which will meet with the message send from the Cat's end at the interior of GHZ brane shown in Fig. 7(a). In this context, some unitary local operations have to be done if necessary by flipping the sign of the wave function Wigner's friend state by Einstein for some bad states which represent bad wormholes and make it possible for the cat to communicate messages with Wigner's friend and Wigner at the interior of the GHZ brane.

But according to the Copenhagen interpretations, the Einstein-Rosen Bridge (ERB) or the complex Tensor network formed between the Cat and Wigner's friend due to entanglement of quantum states simply snip off when Wigner made the measurements of his friend. Therefore, the measurement is irreversible and none of them can send messages that could meet at center of ERB.

In line with three observer system similar kind of analogy can be thought for four observer system where a collection of a large number of K-entangled quadruplets shared between Cat, Wigner's Friend, Wigner and Einstein and then their four shares are squeezed individually to form a qraduplet of black holes after undergoing some evolution with time and the geometry of quaduplet's interior will consist of four tube-like regions bound together by a K brane at the form of quadripartite wormholes also shown in Fig. 7(b).

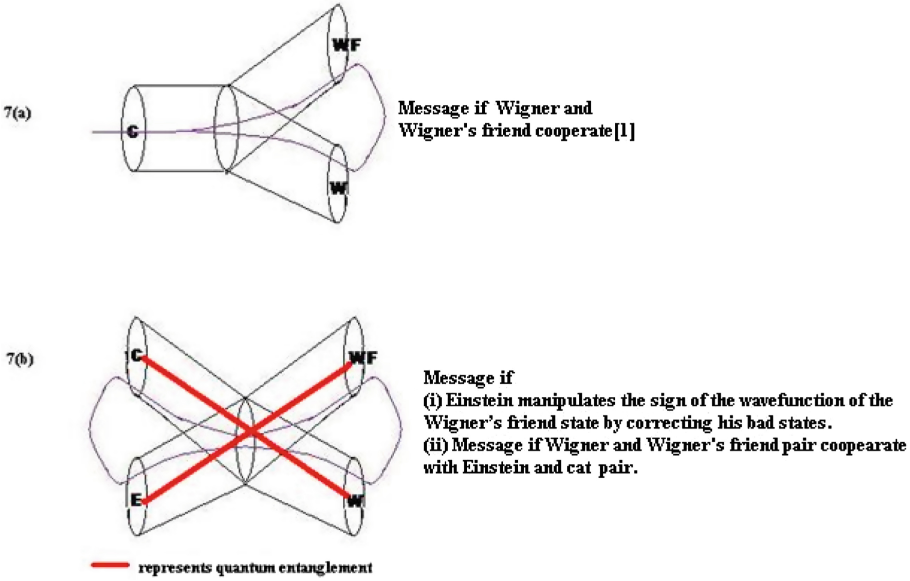


Fig. 7. Transmission of messages in between the blackholes assembly

From sets of Eqs. 8.1, 8.2, 8.3, 8.4, 14.1, 14.2, 14.3 and 14.4 and in the light of Everett's Relative State Formulation, it has been observed that some kind of four sided Einstein Rosen Bridge has been developed between the cat, Wigner's friend, Wigner and Einstein. Here, two pairs of blackholes are entangled with the other one along with the blackholes which are lying diagonally opposite to each other are also entangled. Here from the Fig. 7(b), the combination of Wigner and Wigner's friend which constitute a blackhole pair and the combination of Einstein and the cat which constitute another blackhole pair are entangled. Also Einstein is entangled with Wigner's friend and the cat is entangled with the Wigner. It means that if Wigner and Wigner's friend cooperate in terms of reversing the measurement processes and so is done by Einstein and the cat combination can able to send two messages originated from two pairs end and it will meet at the interior of K brane shown in Fig. 7(b).

4 Conclusions

i. Some unitary local opeartions have to be done if necessary by flipping the sign of the wavefunction of Wigner's friend state by Einstein. As Einstein and Wigner's friend are two highly entangled states, so Einstein can easily manipulate the sign of the wavefunction of the Wigner's friend state by choosing the appropriate sign of his wavefunction which can be done by correcting his bad states.
ii. Finally Feynman has to change overall phase of the mixed entangled state to get the original entangled state for some bad states which represent bad wormholes and make it possible for the cat and Einstein combination pair to communicate messages with Wigner and Wigner's friend combination pair at the interior of the K brane.

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Sensitivity of the WRF Model to the Parameterized Physical Process

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Abstract. The present study explores the performance of different microphysics (MP) and cumulus parameterization (CP) scheme in the simulation of an extreme rainfall event during 0000 UTC July 11 2016 - 0000 UTC July 12, 2016 at the west coast of India, Mumbai. Twelve experiments were conducted using a dynamic core of Advanced Weather Research and Forecasting (WRF) model with Three Dimensional-Variational (3D-Var) assimilation system. The Global Forecast System (GFS) analysis data were utilised with INSAT-3D sounder profile and compared with Global Satellite Mapping of Precipitation (GSMaP). The results indicate that among all experiments, WRF Single-Moment 6-class (WSM6) - the microphysics scheme has better quality in simulation of the rainfall over the Betts-Miller-Janjic (BMJ) - the cumulus parameterization scheme, and all four MP schemes with BMJ - cumulus parameterization combination gives the accurate predictive ability over the region.

Keywords: WRF model · Rainfall · Parameterization schemes · Mumbai extreme event

1 Introduction

WRF is utilized by several operative administrations for medium and short-range weather forecasting which is also an open research device. It gives so many different physics options that can be adaptably joined from multiple points of view. The model physics parameterizations are arranged separately, as follows: microphysics, surface physics, atmospheric radiation physics, planetary boundary layer physics and cumulus parameterization [1]. There are many interaction for the communications among these schemes through the model state variables (potential temperature, wind, moisture, etc.).

In case of this expansive availability of parameterizations, it is hard to characterize which combination gives a better description of a meteorological application or interest region and phenomenon. Ruiz [2] in 2007 and Ruiz and Saulo [3] in 2006 break down the WRF sensitivity to the utilization of different convective parameterizations and planetary boundary layer. These schemes work for

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medium and short range forecast. That focus on rain representation and planetary boundary layer for potential effect. The impacts of the combinations for different variables were not studied. Ruiz [4] in 2009 talked about the likelihood to find an ideal design for operational purposes of WRF parameterizations over South America. Results show that the affectability of short-range weather forecasts for different model physics is very big, but any of the combinations do not give better outcomes over the entire domain.

The WRF models are used many time for short and medium range weather forecasting by many operational services [1,5–7]. The fact of the WRF-ARW is an appropriate research tool, as it has multiple physics options and they can be combined in many different ways [8–14]. The reactivity of quantitative rainfall forecast to various changes of the Kain-Fritsch (KF) scheme and resolution at which grid spacing values the KF scheme may no longer be needed on simulated rainfall was studied by Duda [15]. Incidentally, the KF scheme [12,16,17] is frequently utilised to improve forecasts for convective parameterization for grid spacing under 20 km, believable because it has been shown that the KF scheme gives better convective parameterization than other CP schemes such as the Betts-Miller-Janjic and Grell-Devenyi schemes [13,14,18]. Lekhadiya and Jana [19] in 2018 concentrated over Jharkhand and adjoining region for an extreme rainfall event during 18–19 August 2016 using different combination of MP and CP scheme in the WRF model. They show that Lin et al. scheme and WSM6 MP scheme with a combination of Multi-scale Kain-Fritsch CP scheme give good results for rainfall.

For this study, twelve experiments have been performed by using four different MP schemes e.g., WSM 3-class scheme, WSM 5-class scheme, WSM 6-class scheme and Thompson scheme. With the combination of three different cumulus parameterization scheme e.g., Kain-Fritsch (KF), Betts-Miller-Janjic (BMJ) and Grell-Freitas (GF) Ensemble scheme. This experiment has been performed for 0000 UTC of 11^{th} July 2016 initial condition for the study in which MP and CP schemes gives accurate results for the simulation of extreme rainfall event at the west coast of India, Mumbai during 0000 UTC July 11, 2016 - 0000 UTC July 12, 2016.

2 Model Description

The National Center for Atmospheric Research (NCAR) developed the WRF model. It has so many features like a non-hydrostatic, fully compressible system of equation, hydrostatic pressure, a terrain-following with the constant pressure surface at the top level of the model of the vertical coordinate system. An Arakawa-C grid of staggered grid utilised in the model and for a time integration the third-order Runge-Kutta scheme utilised for both vertical and horizontal direction. The WRF model incorporates with so many physics options like MP, Planetary Boundary Layer (PBL), CP, shortwave and longwave radiations, land surface, shallow cumulus, surface layer, urban surface and ocean model, with multi options for each process. In the present study, the WRF-ARW model is used for comparison of different MP option and CP option. The horizontal resolution used 30 km (Fig. 1) with 36 vertical levels. The input detail of the model is given in Table 1.

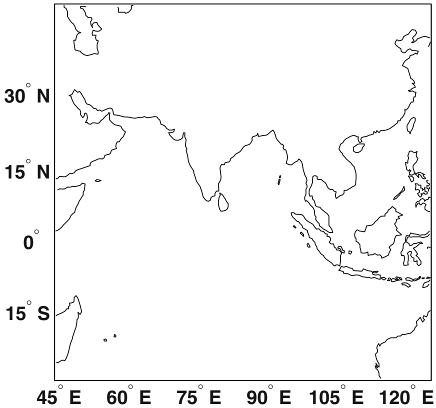

Fig. 1. Model domain utilized for the study

3 Data and Numerical Experiments Description

3.1 Data Description

The INSAT-3D retrieved temperature and humidity profiles at 43 pressure levels and data is available at L2B product from Meteorological & Oceanographic Satellite Data Archival Centre (MOSDAC) of Indian Space Research Organization (ISRO) url:www.mosdac.gov.in. In the present study, WRF-ARW model is initialized with the GFS data at National Center for Environment Prediction (NCEP) with every 6 hourly intervals of time period having the resolution of $0.5^\circ \times 0.5^\circ$ for the comparison of parameterization of an extreme rainfall event. Many land-soil and atmospheric variables are available through this dataset, from precipitation, temperatures and winds to soil moisture and environmental ozone concentration.

The GSMaP data is supported by the Japan Aerospace Exploration Agency (JAXA) and the Japan Science and Technology Agency (JST). It joins both infrared (IR) and passive microwave (PMW) sensors information from a satellite to outline precipitation at high temporal and spatial resolution. The detailed data information about the GSMaP can be obtained from Usio et al. [20] and Aonashi et al. [21]. Seto [22] in 2005 showed a comparison of the GSMaP data and other high-resolution rainfall data and also rain gauge measurements over Japan. Obtained results show that the calculation has been enhanced as far as rain classification methods over land. In this study, GSMaP's surface precipitation products called "GSMAP MVK version 6.0" was utilized for comparison.

Table 1. Model configuration and microphysics a with combination of cumulus parameterization schemes used in the study

Model	WRF model
Dynamics	Non-hydrostatic
Number of domains	1
Grid size	300×300
Horizontal resolution	30 km
Data	GFS and INSAT-3D
Integration time step	90 s
Projection of map	Mercator
Horizontal grid distribution	C-grid of Arakawa
Nesting	Oneway
Vertical coordinate	Hydrostatic pressure followed by Terrain and coordinate with 36 vertical levels
Microphysics	(1) WSM 3-class scheme
	(2) WSM 5-class scheme
	(3) WSM 6-class scheme
	(4) Thompson scheme
Longwave radiation scheme	Rapid Radiative Transport Model (RRTM) scheme
Shortwave radiation scheme	Dudhia scheme
Cumulus parameterization schemes	(1) Kain-Fritsch (KF) scheme
	(2) Betts-Miller-Janjic (BMJ) scheme
	(3) Grell-Freitas Ensemble (GF) scheme
Land surface scheme	Unified Noah land surface model
Planetary Boundary Layer (PBL) scheme	Yonsei University (YSU) scheme

3.2 Numerical Experiments Description

A detailed study of the rainfall event using the WRF-ARW model for the impact of different combinations of MP and CP schemes are carried out. Using four different MP schemes coupling with three CP schemes total 12 experiments were performed. The details of the experiments are reported in Table 2.

4 Case Study

An extreme rainfall event which occurred at the west coast of India, Mumbai during 11^{th} July 2016. From GSMaP data 24 h accumulated rainfall during 0000 UTC July 11, 2016 to 0000 UTC July 12, 2016 is shown in Fig. 2. From Fig. 2, we can see that there was 65 mm rainfall in 24 h at Mumbai and adjoining region.

Table 2. Description of the numerical experiments with their combinations

Experiment name	MP scheme	CP scheme
A1	WSM-3	KF
A2	WSM-3	BMJ
A3	WSM-3	GF
B1	WSM-5	KF
B2	WSM-5	BMJ
B3	WSM-5	GF
C1	WSM-6	KF
C2	WSM-6	BMJ
C3	WSM-6	GF
D1	TS	KF
D2	TS	BMJ
D3	TS	GF

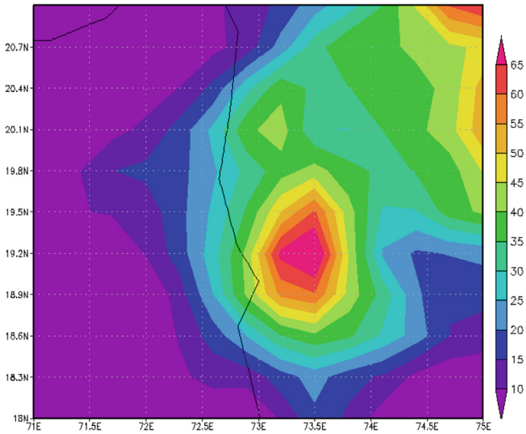

Fig. 2. 24 h accumulated rainfall (mm) map from GSMaP during 0000 UTC July 11, 2016 to 0000 UTC July 12, 2016

5 Results and Discussion

To assess the effect of the INSAT-3D observed temperature and humidity profiles at the numerical forecast, the CNT test is done to function a benchmark. The extreme rainfall event which occurred at the west coast of India, Mumbai during 0000 UTC July 11, 2016 to 0000 UTC July 12, 2016 has been predicted. The 24 h accumulated rainfall (mm) from GSMaP is plotted in Fig. 2. The comparison of different MP and CP schemes with different times like 6 h, 12 h, 18 h and 24 h done through WRF model. Table 3 shows that the 6 hourly accumulated rainfall in tabular form that we can see in Fig. 3 in the form of a bar graph.

Table 3. 6 hourly accumulated rainfall in the tabular form

Experiments	Time(hrs)			
	6(t = 2)	12(t = 3)	18(t = 4)	24(t = 5)
A1	5	9	10	15
A2	10	45	60	62
A3	10	30	42	44
B1	5	9	10	15
B2	15	45	50	63
B3	10	39	40	43
C1	5	9	10	15
C2	15	45	65	69
C3	10	35	40	45
D1	5	9	10	10
D2	5	35	60	63
D3	10	25	35	40

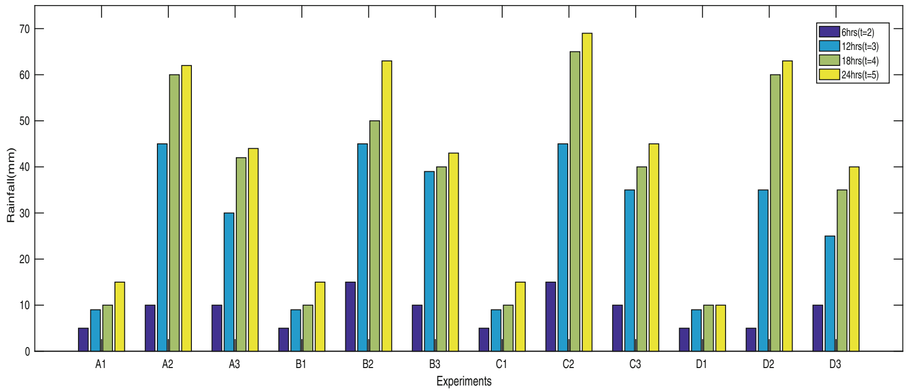

Fig. 3. 6 hourly accumulated rainfall in terms of a bar diagram

5.1 Rainfall Prediction

The rainfall was observed on July 11, 2016 (Fig. 2) at Mumbai. All MP schemes with the combination of different CP schemes (Fig. 4: A1, B1, C1 and D1; A3, B3, C3 and D3) failed to simulate rain at Mumbai and adjoining region. It shows poor rain as compared to actual. The 24 h (0000 UTC July 11, 2016 to 0000 UTC July 12, 2016) predicted accumulated rainfall from CNT and EXP are shown in Fig. 4. It also shows that the result of the experiments A1, B1, C1 and D1 shows only 15 mm, 15 mm, 15 mm and 10 mm rainfall respectively, which is very less compare to the actual (Fig. 2) rainfall measured during that time period. The result of experiments A2, B2, C2 and D2 gives the best match with actual result (Fig. 2) that is 62 mm, 63 mm, 69 mm and 63 mm rainfall respectively. The result of the experiments A3, B3, C3 and D3 gives average precipitation like 44 mm, 43 mm, 45 mm and 40 mm rainfall respectively. From Fig. 4, it could be visible that the heavy rainfall event is properly captured by way of the INSAT-3D data with the 24 h accumulated rainfall being of the order of 62 mm over Mumbai. Even though the rainfall is considerably underestimated via INSAT-3D, the region perfectly matches with the observed rainfall (Fig. 1). Rainfall prediction was improved for A2, B2, C2 and D2 by using incorporation of the INSAT-3D data and additionally the rainfall expected by using the assimilation experiment (Fig. 4; A2, B2, C2 and D2) is better matched of spatial distribution with the INSAT-3D data than CNT run.

5.2 Humidity

The spatial distribution of the 24 h (0000 UTC July 11, 2016 to 0000 UTC July 12, 2016) accumulated observed humidity (gm/kg) at 500 hPa is plotted in Fig. 5 wherein the comparison of different MP schemes with the combination of CP schemes were depicted. All over results suggest that some of the experiments shows positive impact over Mumbai and adjoining region and also some points shows degradation in specific humidity forecast with compared to CNT run.

5.3 Temperature

The spatial distribution of the 24 h (0000 UTC July 11, 2016 to 0000 UTC July 12, 2016) accumulated observed temperature (in K) at 500 hPa is shown in Fig. 6. It shows that the comparison of different MP schemes with the combination of different CP schemes were depicted. All over results suggest that the A1, B1, C1 and D1 experiments failed to simulate heavy rainfall event and A3, B3, C3 and D3 experiments show less intensity of heavy rainfall event and also A2, B2, C2 and D2 experiments shows maximum rainfall occurred in Mumbai and adjoining region. The results show that all MP schemes with a combination of BMJ-CP schemes had the essential principle lies in the relaxation of the temperature profile towards reference thermodynamics profile and precipitation is obtained as a necessary consequence from the conservation of water substance.

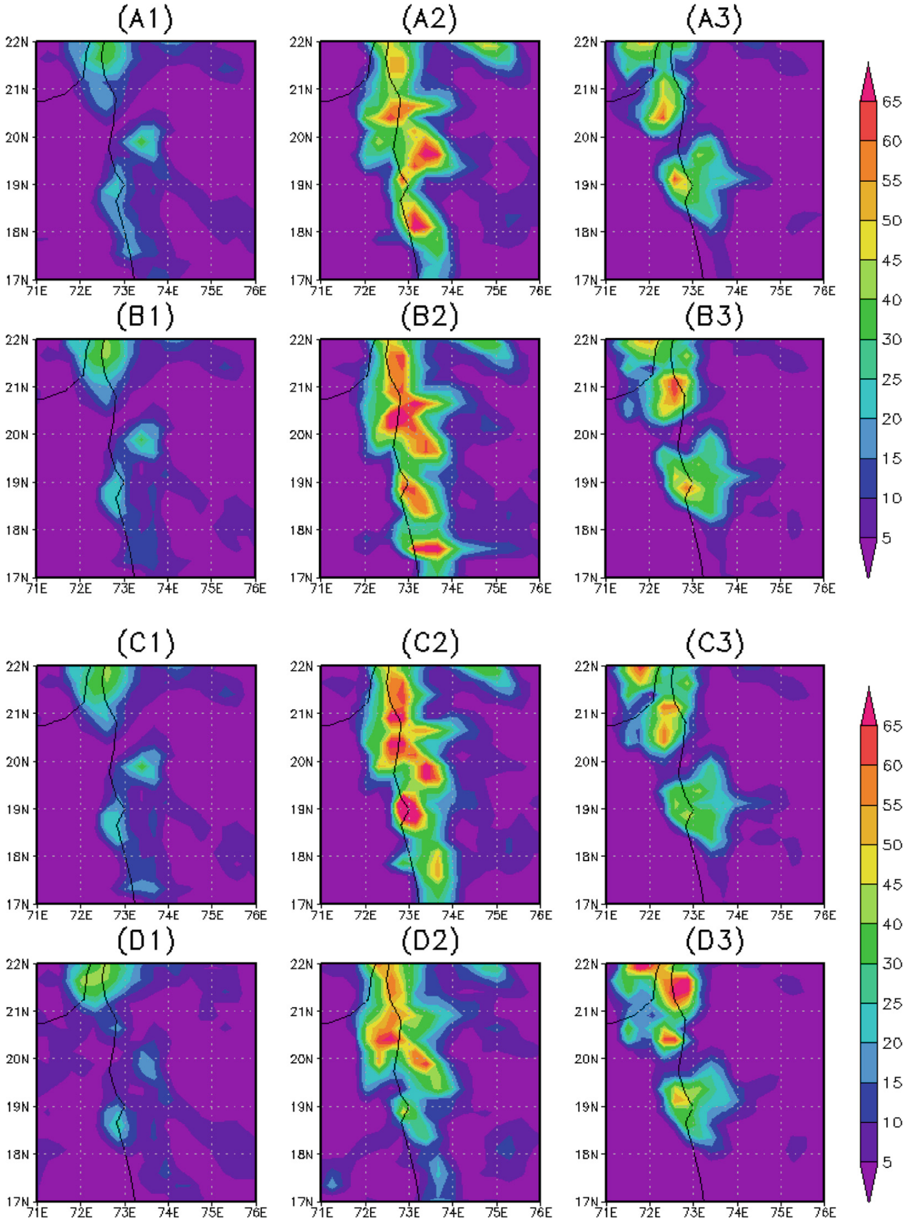

Fig. 4. 24 h accumulated observed rainfall (mm)

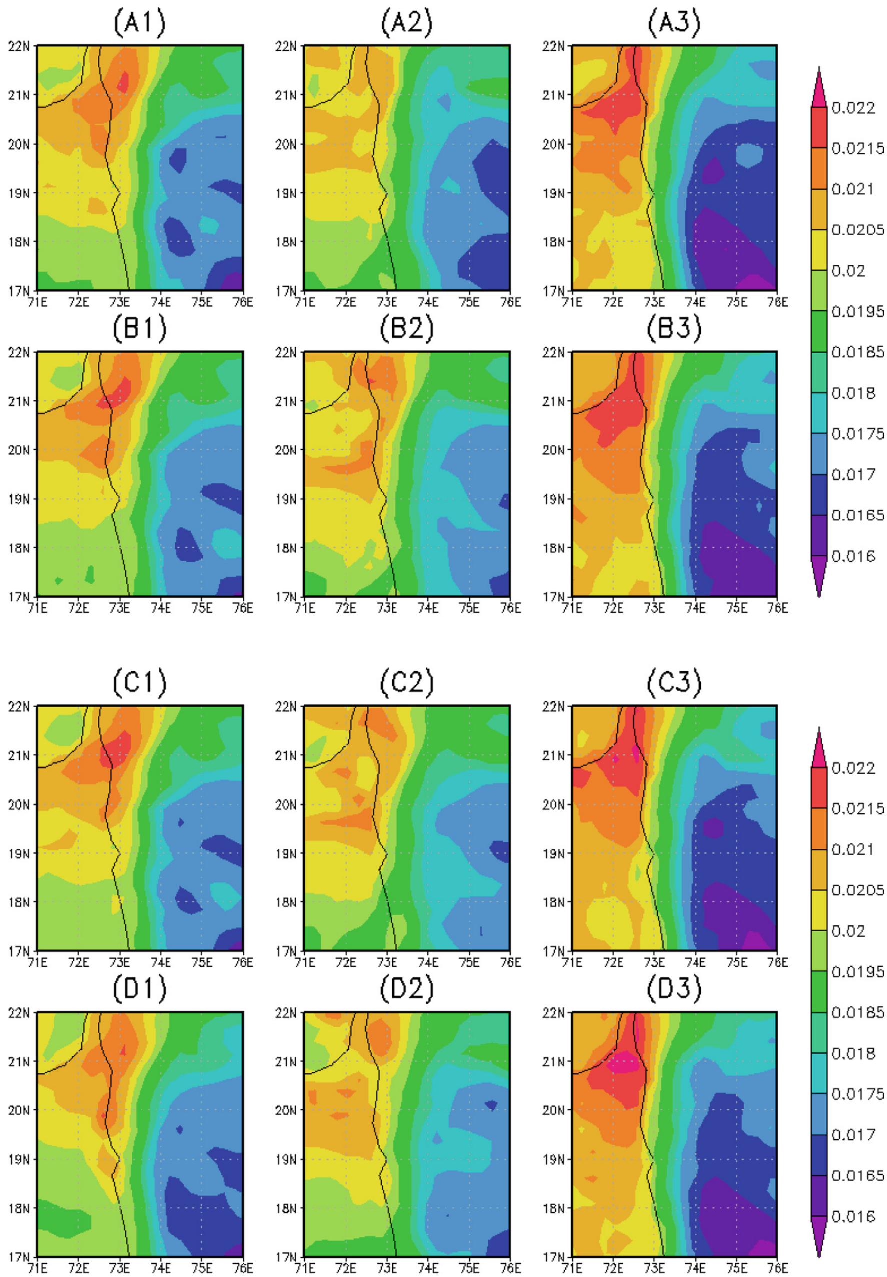

Fig. 5. 24 h accumulated observed humidity (gm/kg)

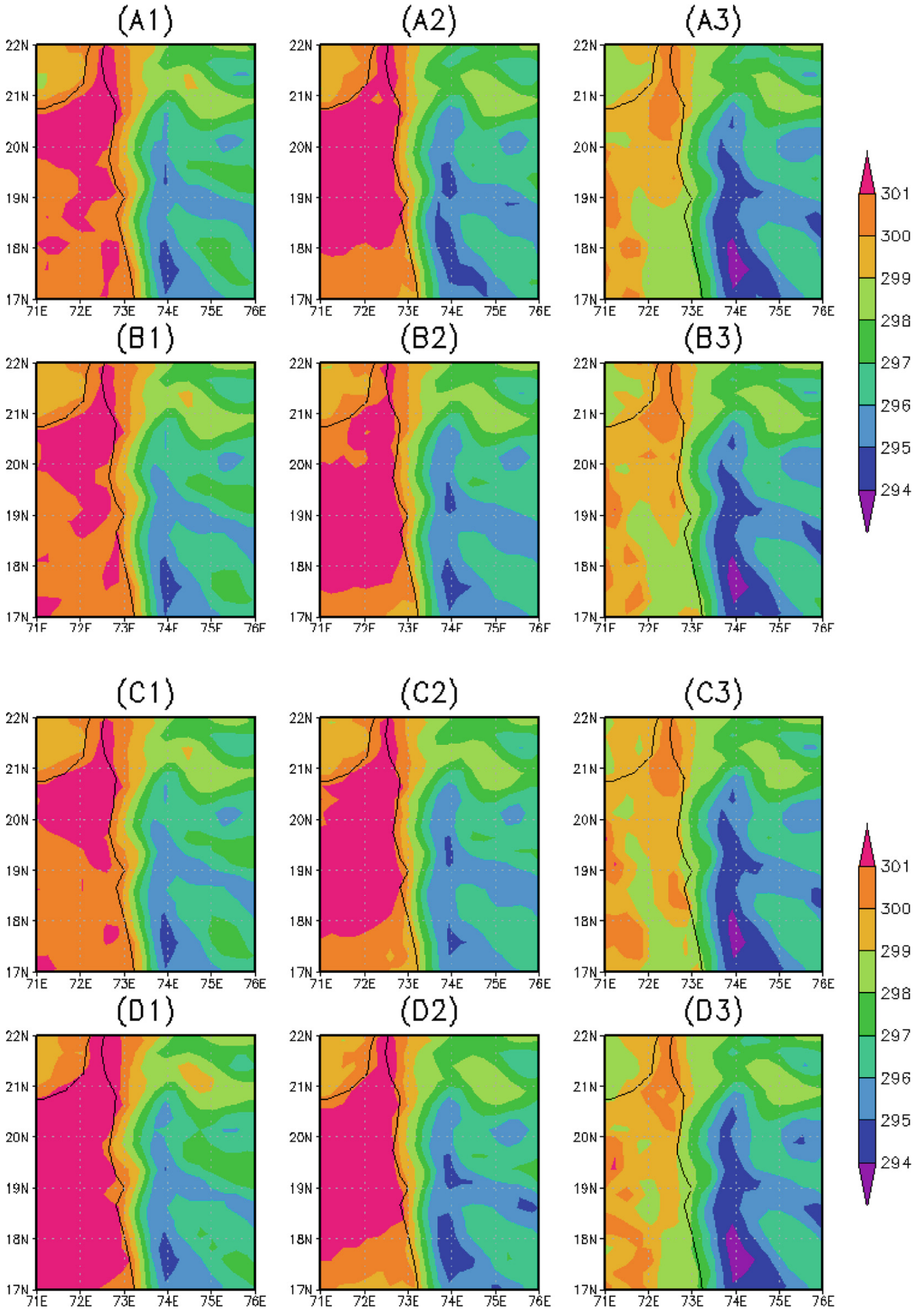

Fig. 6. 24 h accumulated observed temperature (K)

6 Conclusion

The present study utilizes the WRF model system for the performance of the different parameterization schemes at the west coast of India, Mumbai during 0000 UTC July 11, 2016 to 0000 UTC July 12, 2016. GFS analyses data are used for this study, the possible causes of this heavy rainfall event by analysing the accumulated total grid-scale precipitation features around Mumbai. From the numerical experiments, it has been observed that the experiment A1, B1, C1 and D1 do not give a satisfactory result. The same case happens for the experiment A3, B3, C3 and D3 as well. The result of experiment A2, B2, C2 and D2 which gives the best match with actual rainfall from GSMaP. So, we can conclude that any of MP scheme with BMJ-CP scheme gives a good result for precipitation. The comparison of humidity and temperature profiles shown in Figs. 5 and 6 respectively.

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A Fixed Charge Solid Transportation Problem with Possibility and Expected Value Approaches in Hybrid Uncertain Environment

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Abstract. The main investigation of the paper is to develop the restricted fixed charge solid transportation problem under a hybrid uncertain environment where fuzziness and roughness coexist. A fuzzy rough STP model is developed by integrating the classical STP, fuzzy set theory, and rough set theory, which apparently provide a way to accommodate the uncertainty. For solving the problem, we apply the fuzzy rough expected value operator and propose the possibility based STP model with fuzzy rough parameters on a rough space. A numerical example is presented to describe the fuzzy rough approach using Lingo 13.0 optimization software. Finally, a graphical presentation has also shown to describe the comparison between two proposed approaches. Some important managerial decisions are also drawn by observing the optimal result.

Keywords: Expected value · Solid transportation problem · Fuzzy rough set · Hybrid uncertain environment · Possibility value approach

1 Introduction

Haley [5] in the year 1962, was first developed the solid transportation problem (STP). STP is the process of distributing certain goods from its manufacturing center or source to its various destination points or demand centers considering the capacity of different type of vehicles respectively. In many practical situation, there are often vagueness appear in the transportation system due to lack of information about the system, insurgency in the transportation policy,

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different type of unexpected factors like as lack of evidence, fluctuation in the financial market etc. So study of such problem has a practical importance. STP is nothing but an extension of traditional transportation problem TP where conveyance constraint is added with source and demand constraints. Also in literature, there are so many works in which TP/STP considers the different uncertain environment. For example interested readers should refer to [1–4]. Das et al. [15,16] describes about type-2 fuzzy variables with different types of membership functions like Gaussian, Trapezoidal etc. Sinha et al. [17] presented a profit maximizing solid transportation problem with trapezoidal interval type-2 fuzzy number. Das et al. [10] describes broadly about rough interval approach in their paper. It is found that most of the time the uncertainty handle by a single theory related to uncertainty. Like as fuzzy set theory, stochastic process, rough set theory, randomness, uncertainty theory etc. But in real life sometimes there need a hybrid tools which can handle the uncertainty in a better way. Maity et al. in their paper, [19,20] discussed about multi-objective transportation problem under uncertain environment. Roy et al. [21,22] and [23] have discussed about multi-choice multi-objection transportation goal with interval goal. Kundu et al. [29] have done research in the field of multi-objective multi-item solid transportation problem in fuzzy environment.

Obviously rough sets, in contrast to precise sets, cannot be characterized in terms of information about their elements. With any rough set a pair of precise sets, called the lower and the upper approximation of the rough set, is associated. The lower approximation consists of all objects which surely belong to the set and the upper approximation contains all objects which possibly belong to the set. The difference between the upper and the lower approximation constitutes the boundary region of the rough set. Different researches [11] to [13] are there to present the rough set in precise manner. Approximations are fundamental concepts of rough set theory. Rough set theory has an overlap with many other theories dealing with imperfect knowledge, e.g., evidence theory, fuzzy sets, Bayesian inference and others. Nevertheless, the theory can be regarded as an independent, complementary not competing discipline, in its own rights. With this direction here in this paper we consider the hybrid uncertain environment of rough set and fuzzy set theory to crack the uncertainty that exist in the parameters of STP. To the best of our knowledge this is the first attempt to solve such type of problem in hybrid uncertain environment.

The main outline of the paper is as follows:

- We have used fuzzy rough set as hybrid set in the paper.
- One solid transportation model is presented where inputs are taken as fuzzy rough set.
- We have proposed two different approach to convert these hybrid uncertain variable into crisp form, one is possibility valued approach and the other is expected valued approach.

With this introductory part the rest of the paper organized as follows. Section 2 gives some necessary and preliminary concepts; Sect. 3 is about the mathematical model formulation with statements. Next Sect. 4 discusses the solutions approaches. A numerical experiment is presented in Sect. 5, discussion on result and managerial insights is shown in Sect. 6, last Sect. 7 draw the conclusion of this paper with some managerial discussion.

2 Preliminaries

In this section, some fundamental concepts are presented which are related to the research.

2.1 Rough Set

As a new mathematical tool Zdzislaw Pawlak in the early 1980s [6,9] introduced Rough set theory to handle vagueness and uncertainty. Rough set theory has a great advantage compare to the other theory so far introduced in mathematics. For details reader are refer to [14]. Rough set comprises of approximation and boundary regions. The definitions of approximations as well as boundary region are as follows:

- R-lower approximation of X

$$R_*(x) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq X\}$$

- R-upper approximation of X

$$R_*(x) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \phi\}$$

- R-boundary region of X

$$RN_R(X) = R^*(X) - R_*(X)$$

The presentation of rough set is shown in Fig. 1

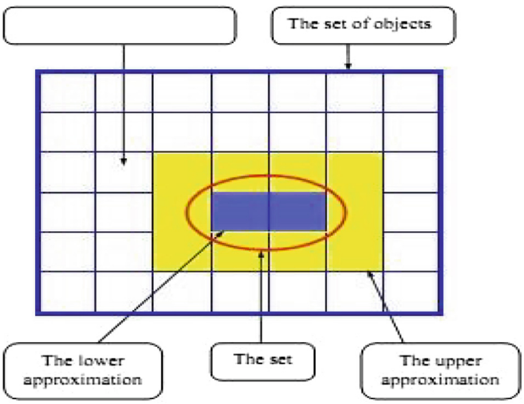

Fig. 1. Rough set

2.2 Fuzzy Set

Fuzzy sets were first proposed by Zadeh [18] in 1965. The definition of a fuzzy set is given below.

Definition: If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{x, \mu_{\tilde{A}}(x) \mid x \in X\}$$

$\mu_{\tilde{A}}(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space M. Its range is the subset of non negative real numbers whose supremum is finite. Figure 2 shows membership function of fuzzy set.

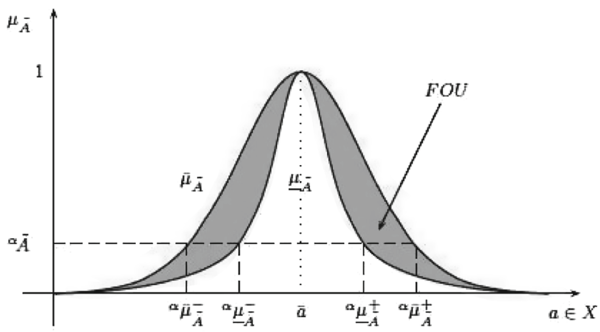

Fig. 2. Fuzzy set $\tilde{A}$

2.3 Fuzzy Rough Variable

A fuzzy rough variable is a measurable function from a rough space $(\Lambda, \Delta, A, \pi)$to the set of fuzzy variables such that $Pos\{\xi(\lambda) \in B\}$ is a measurable function λ of for any Boral set B of $\Re^n$. For details readers should refer [8].

3 Problem Description and Formulation

Here we have presented a Solid transportation problem with the help of fuzzy rough uncertain variables. The notation and assumption of the model is as follows:

(a) Notation:

- $\tilde{\tilde{c}}_{ijk}$ is the per unit transportation cost in Rs.
- $\overline{\tilde{f}^{e}}_{ijk}$ is the per unit restricted fixed charge in Rs.
- a_i represents the total available source in i^{th} source point which is denoted in Kg.

- b_j represents the total available demands in j^{th} demand point which is denoted in Kg.
- e_k represents the total conveyance capacities in deterministic forms which is denoted in Kg.
- x_{ijk} represents the total transported amount from i^{th} source to j^{th} demand point through k^{th} vehicle.

(b) Assumptions:

- The model is an unbalanced transportation where $a_i \neq b_j \neq e_k$.
- All the conveyances are fully loaded here. We have not considered partially loaded case.

In this paper we have develop a solid transportation problem where uncertainty exist in the different parameters of objective function. The Fig. 3 shows the one stage solid transportation problem from distribution center to retailer.

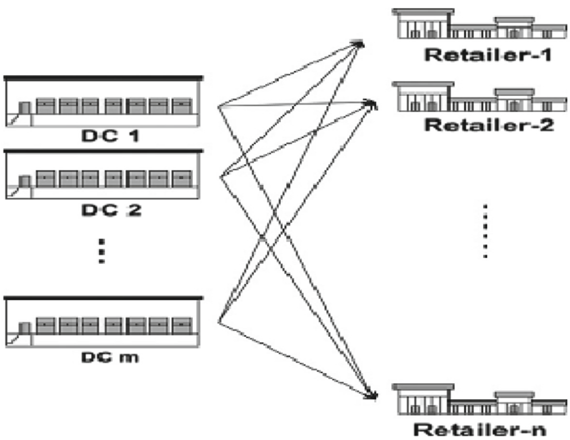

Fig. 3. One stage solid transportation problem

The main objective is to minimize the total transportation cost in uncertain environment. For the mathematical model presentation we consider m number of sources with n destinations and l conveyances and hence the model is as follows.

$$MinZ = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(\tilde{\bar{c}}_{ijk} + \tilde{\bar{f}}^{e}{}_{ijk})x_{ijk} \tag{1}$$

Subject to,

$$\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq a_i, \quad i = 1, 2, 3, \cdots, m. \tag{2}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \geq b_j, \quad j = 1, 2, 3, \cdots, n. \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k, \quad k = 1, 2, 3, \cdots, l. \tag{4}$$

where $x_{ijk} \geq 0$, $\forall$ i,j and k and it denotes the unknown quantity to be transport. Here $\bar{\tilde{c}}_{ijk}$ is unit transportation cost and $\overline{\tilde{f^e}}_{ijk}$ is the restricted fixed charge on conveyance considered in a hybrid uncertain environment of fuzzy and rough set theory. Also a_i, b_j and e_k are representing the available source, demand and conveyance capacities in deterministic form.

4 Solution Approaches

Here in the objective function the parameters are fuzzy rough variable. So direct solution of this model is not possible and hence we proposed two new approaches to solve this type of problem in hybrid uncertain environment.

4.1 Solution Using Possibility Value Approach

The possibility approach is based on the possibility measures and chance constrained programming (CCP). However, as stated in the papers [25] and [26], possibility and necessity measures lack the self-duality property, while the credibility measure is a self-dual measure. Therefore, credibility measure might be more suitable to be used to construct the fuzzy chance constraint. But we have used possibility approach, as it deals with the uncertainty that exist in the objective function and in the set of constraints by possibility measures. Applying this possibility approach to our proposed model, the model become as. Minψ
Subject to

$$Pos\big(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(\bar{\tilde{c}}_{ijk} + \overline{\tilde{f^e}}_{ijk})x_{ijk} \leq \psi\big) \geq \delta \tag{5}$$

and constraints (2)–(4) with $x_{ijk} \geq 0$, $\forall$, i,j and k.
Here $\delta \in [0, 1]$ is the predetermined threshold specified by the decision maker (DM). The model (5) is a possibility rough solid transportation problem (PRSTP). Now following steps discussed by Shiraz et al. [8], we obtain the deterministic form of the possibility constraint of model (5) and hence the model became as follows. Many studies on the transportation planning problems with fuzzy parameters, e.g., Mula et al. [27] and Sun et al. [28], found that the confidence value δ of the fuzzy chance constraint influence the best solution of the

problem. Sun et al. [28] designed a fuzzy simulation-based approach to identify the best confidence value. However, this paper just set δ to a certain number which is between 0 and 1 as per the instruction given by the decision maker.

$$\begin{cases} (\psi^*)^{inf(\alpha)} = min \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \big(c_{ijk}^{m_2-sup(\alpha)} + R^1(\delta) c_{ijk}^{\beta} x_{ijk} \\ \qquad + f_{ijk}^{e^{m_2-inf(\alpha)}} R^1(\delta) f_{ijk}^{e^{m_2-inf(\alpha)}} \big) x_{ijk} \\ (\psi^*)^{sup(\alpha)} = min \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \big(c_{ijk}^{m_2-sup(\alpha)} + R^1(\delta) c_{ijk}^{\beta} x_{ijk} \\ \qquad + f_{ijk}^{e^{m_2-inf(\alpha)}} R^1(\delta) f_{ijk}^{e^{m_2-inf(\alpha)}} \big) x_{ijk} \end{cases}$$

subject to constraints (2)–(4) with $x_{ijk} \geq 0, \forall\ i,\ j$ and k.

4.2 Solution Using Expected Value Approach

When we apply the expected value approach according to Shiraz et al. [8], then our model became,

$$\big(min \frac{1}{8} \big(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} (c_{ijk}^{a-m_1} - c_{ijk}^{b-m_1} - c_{ijk}^{c-m_1} - c_{ijk}^{d-m_1} + c_{ijk}^{a-m_2} - c_{ijk}^{b-m_2} - c_{ijk}^{c-m_2} - c_{ijk}^{d-m_2}$$

$$+ f_{ijk}^{e^{a-m_1}} - f_{ijk}^{e^{b-m_1}} - f_{ijk}^{e^{c-m_1}} - f_{ijk}^{e^{d-m_1}} + f_{ijk}^{e^{a-m_2}} - f_{ijk}^{e^{b-m_2}}$$

$$- f_{ijk}^{e^{c-m_2}} - f_{ijk}^{e^{d-m_2}})\big) + 4 \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \big(- c_{ijk}^{\alpha} \int_0^1 L(t)dt + \int_0^1 R(t)dt) \big) x_{ijk}$$

and constraints (2)–(4) with $x_{ijk} \geq 0, \forall\ i, j$ and k.
where the last terms indicating the left and right expansion of the fuzzy number respectively.

5 Numerical Experiment

For numerical experiment we have considered a STP model with two sources, two destinations and two conveyances. All the inputs of the problem have been taken in fuzzy rough set. The inputs are combination of fuzzy and rough set which means they are in hybrid form. All the inputs are taken according to expert opinion. The respective inputs for objective function in fuzzy rough form are given in Table 1. And the respective restricted fixed charge for the vehicle

Table 1. Input for different parameters of STP

$\tilde{\tilde{c}}_{111}$	$\tilde{\tilde{c}}_{121}$	$\tilde{\tilde{c}}_{211}$	$\tilde{\tilde{c}}_{112}$
$(4, [15, 26], [13, 38], 4)$	$(6, [35, 40], [23, 46], 6)$	$(2, [16, 20], [8, 22], 2)$	$(5, [8, 12], [6, 14], 5)$
$\tilde{\tilde{c}}_{221}$	$\tilde{\tilde{c}}_{212}$	$\tilde{\tilde{c}}_{221}$	$\tilde{\tilde{c}}_{222}$
$(3, [10, 12], [33, 35], 3)$	$(4, [15, 20], [23, 46], 4)$	$(5, [11, 12], [6, 12], 5)$	$(2, [9, 14], [3, 12], 2)$

Table 2. Result of the STP model obtained by proposed solution methods

Solution using possibility value approach	Solution using expected value approach
Total cost = [347, 481],	Total cost = 407
Transported amount (solution)	
$x_{111} = [2, 6]$, $x_{221} = [4, 10]$, $x_{211} = [8, 12]$, $x_{222} = [2, 7]$ and other variables are zero	$x_{112} = 14$, $x_{121} = 16$, $x_{211} = 11$, $x_{222} = 2$ and other variables are zero

1 are consider as $\overline{f}^1_{111} = (1, [5, 8], [3, 4], 1)$, $\overline{f}^1_{121} = (1, [2, 4], [1, 2], 1)$, $\overline{f}^1_{211} = (4, [2, 5], [4, 8], 4)$ and $\overline{f}^1_{221} = (3, [5, 6], [4, 5], 3)$. The sources are $a_1 = 14$, $a_2 = 11$, demands are $b_1 = 9$, $b_2 = 15$, capacities of conveyances are $e_1 = 17$, $e_2 = 15$. Source, demand and conveyance are taken in crisp form

5.1 Results

With these numerical data presented in Table 1, we solve the problem by the two proposed solution approaches and for soft computing we have used the Lingo 13.0 software. We got the optimal results in both case and the results are presented in Table 2. For this problem we choose $\alpha = \delta = 0.5$. From the Table 2, we can see that the expected value cost (407) belongs to the interval [347, 481], which we obtained using the possibility value approach.

5.2 Comparison Between Two Solution Approaches

Figure 4 shows the graphical representation of the two approaches. The result obtained in possibility value approach is in interval form. So it is difficult to draw a comparison graph between the possibility value approach and expected value approach. To overcome the difficulty, we have taken average of the values of Table 2 which are in interval form. From the above table, it can be clearly stated the result obtained in expected value approach is superior to the possibility value approach.

5.3 Numerical Experiment in Real Life Situation

In the previous Sect. 5, an example is discussed to show the proposed method. To discuss the proposed method, we have studied a numerical experiment in real life situation. For numerical experiment we have considered a STP model with two sources, two destinations and two conveyances. Two sources are Agartala and Udaipur, two destinations are Kailasahar and Belonia. Truck and tripper are two conveyance which are used here for transportation. The numerical data are collected from Tarasankar plastic company which is located at Bodhjungnagar, Agartala Tripura, a North-eastern state of India. All the inputs of the problem have been taken in fuzzy rough set. Expert opinions are taken to select the inputs. The respective inputs for objective function in fuzzy rough form are given in Table 3. And the respective restricted fixed charge for the vehicle 1 are considered

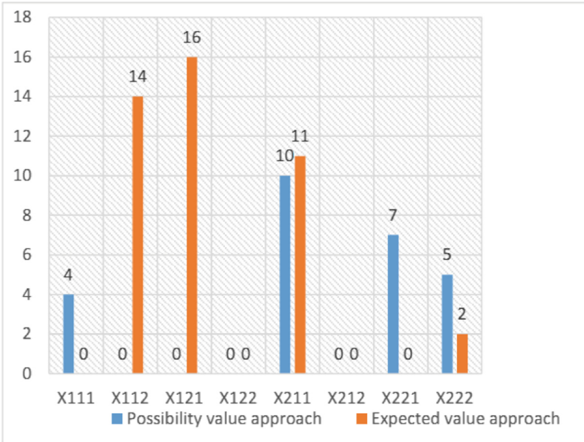

Fig. 4. Comparison between transported amounts in two different approaches

Table 3. Input for different parameters of STP for real situation

$\tilde{\tilde{c}}_{111}$	$\tilde{\tilde{c}}_{121}$	$\tilde{\tilde{c}}_{211}$	$\tilde{\tilde{c}}_{112}$
$(3, [13, 24], [13, 38], 3)$	$(5, [33, 40], [23, 46], 5)$	$(3, [16, 19], [8, 22], 3)$	$(4, [8, 10], [6, 14], 8)$
$\tilde{\tilde{c}}_{221}$	$\tilde{\tilde{c}}_{212}$	$\tilde{\tilde{c}}_{221}$	$\tilde{\tilde{c}}_{222}$
$(2, [10, 15], [33, 35], 2)$	$(2, [15, 19], [23, 46], 2)$	$(6, [11, 12], [6, 12], 6)$	$(3, [9, 14], [3, 12], 3)$

as $\overline{f}^1_{111} = (1, [5, 8], [3, 4], 1)$, $\overline{f}^1_{121} = (1, [2, 4], [1, 2], 1)$, $\overline{f}^1_{211} = (4, [2, 5], [4, 8], 4)$ and $\overline{f}^1_{221} = (3, [5, 6], [4, 5], 3)$. The sources are $a_1 = 14$, $a_2 = 11$, demands are $b_1 = 9$, $b_2 = 15$, capacities of conveyances are $e_1 = 15$, $e_2 = 17$. Source, demand and conveyance are taken in crisp form.

5.4 Results of the Real Life Problem

With these real life numerical data presented in Table 3, we solve the problem by the two proposed solution approaches and for soft computing we have used the Lingo 13.0 software. We got the optimal results in both case and the results are presented in Table 3. For this problem we choose $\alpha = \delta = 0.5$. From the Table 3, we can see that the expected value cost (410) belongs to the interval $[377, 490]$, which we obtained using the possibility value approach (Table 4).

Table 4. Result of the STP model obtained by proposed solution methods

Solution using possibility value approach	Solution using expected value approach
Total cost = [377, 490],	Total cost = 410
Transported amount (solution)	
$x_{111} = [3, 5]$, $x_{221} = [4, 8]$, $x_{211} = [8, 10]$, $x_{222} = [2, 6]$ and other variables are zero	$x_{112} = 14$, $x_{121} = 16$, $x_{211} = 11$, $x_{222} = 2$ and other variables are zero

6 Discussion on Result and Some Managerial Insights

The solution of the above model has been discussed in two different approaches. Here the input parameters are in fuzzy rough set. The discussion and managerial insights are as follows:

- Fuzzy rough set is a special case of uncertainty. Rough set is bounded by lower and upper approximation. Here fuzzy set is bounded by lower and upper approximation of the rough set. This type of variable is very critical to define.
- Here all the variables are taken according to expert's opinion. Experts and decision makes have helped us to such type of research.
- Here we obtained the optimal results using two different approaches that proposed in this paper. It can observe that the formats of solutions are of two types. One is in interval form and another is in number form.
- So it will not likely to compare this two results. But here it will very helpful for the decision maker (DM), because the inputs are same and outputs are come in two different forms.
- In such a case it becomes easy for the DM to choose the appropriate solution approach according to the requirement of the problem.
- Also form the result we see that the expected value approaches' solution is lying in the interval that obtained using the possibility approach which justify the convergence test.

7 Conclusion and Future Scope

In this paper, we have developed a STP model with fuzzy rough parameters to accommodate uncertainty, which has been solved using two different approaches and using the Lingo 13.0 software. A comparison is shown between the two approaches. Observing the result it can conclude that both the solutions are near about to each other which guaranteed the convergence of the methods. Also it noticed here the possibility approach is giving the result in interval form and the expected value approach is giving it in a number. It will help the decision maker to choose the result format according to the problem.

The concept proposed in this paper is quiet simple and easy understandable. It will helps the transportation engineering science to solve their complicated problem under uncertainty. Hence this become a great scope for future study in the transportation engineering science. The demand and capacity can be also uncertain, which is quite valuable for the future work. Beside this, in future this type of problem can be further considered as a future extension by considering some other constraints and other important factors in STP. This type of problem can also be solved using different software technique like GA, PSO etc.

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Dynamics of Prey–Predator System in Crisp and Fuzzy Environment with Special Imprecise Growth Rate, Rate of Conversion and Mortality Rate

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Abstract. In this article we analyze a generalized simple prey-predator system with fuzzy number. In this article, we proposed a new way of studying the stability. Analysis of prey−predator model in crisp and fuzzy environment with special imprecise growth rate, rate of conversion and mortality rate. We find the equilibrium points and discuss the stability of the equilibrium points under fuzziness. The local and asymptotic stability analysis of the model system have been carried out. Also the analytical finding has been supported through numerical simulations.

1 Introduction

Mathematical modeling is an important classical system, which can be used to translate real world problems into mathematical problems. Over the last century, mathematics has made a great impact to model and understand biological phenomena. Generally, biomathematical models consists of mathematical representation, treatment and modeling of the biological process using a variety of applied mathematical techniques and tools. In 1838 the Belgian mathematician Verhulst introduced the logistic equation, which is a kind of generalisation of the equation for exponential growth but with a maximum value for the population. He used data from several countries in particular Belgium, to estimate the unknown parameters. The work of Verhulst was rediscovered only in 1920s [1]. The prey-predator system played an important role in bio-mathematics. Most important thing of predator prey system is their stability analysis [2]. Ordinary differential equations also played a great role in the stability analysis and explanation of predator-prey system [3–8]. One of the important or useful tools is Differential Equation. Differential Equations becomes a successful paradigm on a real world problem where uncertainty occurs. This uncertainty can occurs in the experimental part, the data collection as well as in the initial conditions. Uncertainty which occurs could not be stable. In the study of differential equation in a fuzzy environment, the term fuzzy differential equation is used for

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referring to differential equations with fuzzy coefficients and differential equations with fuzzy initial conditions values. When the population is small, there are plenty of resources for each individual, so per capita birth rate should be high, per capita death rate should be low and the population will grow larger. Logistic models are used in studying intraspecific as well as interspecific competition and predator-prey relationship. In recent years fuzzy set theory has been applied in stability analysis and explanation of prey-predator system using non linear differential equations [9–12].

In this paper we have considered one prey and one predator species with logistic growth under fuzzy environment. We have considered the biological parameters-intrinsic growth rate, rate of conversion and mortality rate as taken triangular fuzzy number in nature. Here stability analysis have been carried out in crisp and fuzzy environment. Finally we consider a numerical example to support of our proposed approach.

2 Preliminaries

For the development of the model in fuzzy environment we need some preliminary definition and concepts which will be used later on. Zadeh [13] first introduced fuzzy sets as a mathematical way of representing vagueness in everyday life.

Definition 21 ***Fuzzy Set:*** *Let X be a non empty set. A fuzzy set A in X is characterised by its membership function$\mu_A : X \rightarrow [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \varepsilon X$. The fuzzy set A is completely determined by the set tuples $A = \{(x, \mu_A(x)) : x \varepsilon X\}$*

Definition 22 $\alpha-$***Cut and Strong*** $\alpha-$***Cut of a Fuzzy Set:*** *Let A be a given fuzzy set defined on a non empty set X and μ be its membership function. Then for any $\alpha \varepsilon [0,1]$ the $\alpha-$cut and strong $\alpha-$cut are denoted by μ^{α} and $\mu^{\alpha+}$ and defined by $\mu^{\alpha} = \{x \mid \mu_A(x) \geq \alpha, \forall x \varepsilon X\}$ and $\mu^{\alpha+} = \{x \mid \mu_A(x)\, \alpha, \forall x \varepsilon X\}$*

Definition 23 ***Fuzzy Number:*** *A fuzzy number u is a pair $(\underline{u}, \bar{u})$ of functions $\underline{u}(\alpha), \bar{u}(\alpha); 0 \leqslant \alpha \leqslant 1$ which satisfy the following requirements:*

1. *$\underline{u}(\alpha)$ is bounded monotonic increasing left continuous function*
2. *$\bar{u}(\alpha)$ is bounded monotonic increasing left continuous function*
3. *$\underline{u}(\alpha) \leqslant \bar{u}(\alpha), 0 \leqslant \alpha \leqslant 1$*

Definition 24 ***Triangular Fuzzy Number:*** *A triangular fuzzy number (TFN) is represented by $\tilde{A} = (a, b, c)$ and its membership function is defined as bellow*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, \mathit{if}\ x \leqslant a \\ \frac{x-a}{b-a}, \mathit{if}\ a \leqslant x \leqslant b \\ \frac{c-x}{c-b}, \mathit{if}\ b \leqslant x \leqslant c \\ 0, \mathit{if}\ x \geqslant c \end{cases} \tag{1}$$

Definition 25 ***Operations in Triangular Fuzzy Number:*** *For given two TFNs* $\tilde{A}_1 = (a_1, b_1, c_1)$ *and* $\tilde{A}_2 = (a_2, b_2, c_2)$ *we define addition, subtraction, multiplication and division by*

1. $\tilde{A}_1 + \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$
2. $\tilde{A}_1 - \tilde{A}_2 = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$
3. $\tilde{A}_1.\tilde{A}_2 = (min\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\}, b_1b_2, min\{a_1a_2, a_1c_2, a_2c_1, c_1c_2\})$
4. $\frac{\tilde{A}_1}{\tilde{A}_2} = \tilde{A}_1.\frac{1}{\tilde{A}_2}$ *where* $\frac{1}{\tilde{A}_2} = \Big(min\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, median\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}, max\{\frac{1}{a_2}, \frac{1}{b_2}, \frac{1}{c_2}\}\Big)$.

3 Model Formation

The mathematical expression of the prey-predator model with two species precise biological parameters is given by

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - axy \tag{2a}$$
$$\frac{dy}{dt} = axy - dy \tag{2b}$$

where x, and y denotes the population density of the prey and predator species respectively at time t. In this model $r(> 0)$ denotes the intrinsic growth rate of the prey species in the absence of predator and $d(> 0)$ is natural death rate of the predator in the absence of the prey. Finally a denotes the conversion coefficient of prey biomass into predator biomass.

4 Positive Invariance of the System

Let $Y = \begin{pmatrix} x \\ y \end{pmatrix}$, where $Y \in \mathbb{R}^2$

$Y = \begin{pmatrix} F_1(Y) \\ F_2(y) \end{pmatrix}$, i.e $Y = \begin{pmatrix} rx\left(1 - \frac{x}{k}\right) - axy \\ axy - dy \end{pmatrix}$

where $F : C_+ \to \mathbb{R}^2$ and $F \in C^\infty(\mathbb{R}^2)$ and C^∞ stands for continuously differentiable function.

The above system becomes,

$$\dot{Y} = F(Y)$$

with $Y(\theta) = (\phi_1(\theta), \phi_2(\theta)) \in C_+$ and $\phi_i(\theta) > 0$ for $i = 1, 2$.

It is easy to check in the above equation that whenever choosing $Y(\theta) = 0$ such that $Y_i = 0$, then $F_i(Y)|y_i(t) \in C_+ \geq 0$ for i = 1, 2.

Due to Lemma (Yang et al. [2,4])any solution of the above equation with $Y(\theta) \in C_+$, say $Y(t) = Y(t, Y(\theta))$ such that $Y(\theta) \in \mathbb{R}^2$ $\forall t > 0$.

5 Boundedness of the System

Theorem 1. *The solutions of the above system is bounded.*

Proof. Let us define $W(t) = x(t) + y(t)$. The time derivatives

$$\begin{aligned}\frac{dW}{dt} &= \frac{dx}{dt} + \frac{dy}{dt}\\ &= rx\left(1 - \frac{x}{k}\right) - dy\end{aligned}$$

Now

$$\begin{aligned}\frac{dW}{dt} + qW &= rx\left(1 - \frac{x}{k}\right) - dy + q(x+y)\\ &= -\frac{r}{k}\{x^2 - \frac{k}{r}(r+q)x\} + (q-d)y\\ &= \frac{k(r+q)^2}{4r} - \frac{r}{k}\left(x - \frac{k(r+q)}{2r}\right)^2 + (q-d)y\end{aligned}$$

$$if q \leq d, Then \quad \frac{dW}{dt} + qW \leq M$$

where $M = \frac{k(r+q)^2}{4r}$ (say). Therefore, $\frac{dW}{dt} + qW \leq M(constant)$, which is a linear differential equation in W. After solving we get,$W \leq \frac{M}{q} + Ce^{-qt}$, where C is an integrating constant.
At t $= 0$, W $= 0$, we have $C \geq -\frac{M}{q}$. Therefore, $W \leq \frac{M}{q}(1 - e^{-qt})$ and $W \geq 0$, since $W = x + y$, x and y both are ≥ 0. So, $0 \leq W(x(t), y(t)) \leq \frac{M}{q}(1 - e^{-qt})$, which implies that $0 \leq W(x(t), y(t)) \leq \frac{M}{q}$ as $t \to \infty$

Hence all the solution of the above system are bounded and so we can now analyze the stability of the system.

6 Dimensionless of the Logistic Model in Crisp

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - axy \tag{3a}$$

$$\frac{dy}{dt} = axy - dy \tag{3b}$$

Let $U = \frac{x}{k}$, $V = ay$. The Dimensionless system becomes

$$\frac{dU}{dt} = rU\left(1 - U\right) - UV \tag{4a}$$

$$\frac{dV}{dt} = (cU - d)V \tag{4b}$$

where $ak = c$ (say), r, d all are positive constants.

Equilibrium Points and Existence Criteria
The equilibrium points of (3a, b) are given by $E_0(0,0)$, $E_1(1,0)$, and $E_2\big(\frac{d}{c}, r(1-\frac{d}{c})\big)$. $E_0(0,0)$ is called trivial equilibrium point while $E_1(1,0)$ are called axial equilibrium point and $E_2\big(\frac{d}{c}, r(1-\frac{d}{c})\big)$ is called interior as well as planer equilibrium point. $E_0(0,0)$ and $E_1(1,0)$ exists with out any restriction. But $E_2\left(\frac{d}{c}, r(1-\frac{d}{c})\right)$ exists if $c > d$.

7 Stability Analysis of System (3a, b) in Crisp Environment

$$\frac{dU}{dt} = rU\,(1-U) - UV \qquad = F(U,V) \tag{5a}$$

$$\frac{dV}{dt} = (cU-d)V \qquad = G(U,V) \tag{5b}$$

Now, the variational matrix of system (7) is given by

$$J = \begin{pmatrix} \frac{\partial F}{\partial U} & \frac{\partial F}{\partial V} \\ \frac{\partial G}{\partial U} & \frac{\partial G}{\partial V} \end{pmatrix}$$
$$= \begin{pmatrix} r - 2rU - V & -U \\ cV & cU - d \end{pmatrix}$$

Theorem 2. *The trivial equilibrium point $E^0(0,0)$ is always unstable.*

Proof. The variational matrix is given by

$$J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix}$$

The characteristic equation is given by

$$\begin{vmatrix} r-\lambda & 0 \\ 0 & -d-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (r-\lambda)(-d-\lambda) = 0, \;\; \Rightarrow \lambda = r, -d$$

Since r and d are positive constant, hence $(0,0)$ is saddle point and hence unstable.

Theorem 3. *The axial equilibrium point $E^1(1,0)$ is stable if $c < d$ and otherwise it is unstable.*

Proof. The variational matrix is given by

$$J(1,0) = \begin{pmatrix} -r & 0 \\ 0 & c-d \end{pmatrix}$$

Similarly, from the above matrix the eigen values are $-r$, and $c-d$. Again, Since r is positive constant, and $<$ if $c-d<0$ then the point $(1,0)$ is stable and Otherwise it is saddle point and hence unstable.

Theorem 4. *The interior equilibrium point $E^*\left(\frac{d}{c}, r(1-\frac{d}{c})\right)$ is*

(i) Stable if $rd = 4c(c-d)$
(ii) Stable if $c > d$ otherwise it i unstable.
(iii) Asymptotically stable if $rd < 4c(c-d)$

Proof. The variational matrix is given by

$$J\left(\frac{d}{c}, r(1-\frac{d}{c})\right) = \begin{pmatrix} -\frac{rd}{c} & -\frac{d}{c} \\ rc(1-\frac{d}{c}) & 0 \end{pmatrix}$$

The characteristic equation is given by

$$\begin{vmatrix} -\frac{rd}{c} - \lambda & -\frac{d}{c} \\ rc(1-\frac{d}{c}) & -\lambda \end{vmatrix} = 0 \tag{6}$$

$$\Rightarrow \lambda^2 + \frac{rd}{c}\lambda + rd(1-\frac{d}{c}) = 0 \tag{7}$$

$$\Rightarrow \lambda = -\frac{rd}{2c} \pm \frac{1}{2c}\sqrt{(rd)^2 - 4cdr(c-d)} \tag{8}$$

There may arise three cases:-

I: When $rd = 4c(c-d), Then\, \lambda = -\frac{rd}{2c}$ (of multiplicity 2), The two eigen value are negative. So the interior point $E^*\left(\frac{d}{c}, r(1-\frac{d}{c})\right)$ is stable.

II: When $rd > 4c(c-d)$ then from the above relation (7) one eigen value is negative and other is negative if $\sqrt{(rd)^2 - 4cdr(c-d)} < rd$, so two eigen value is negative and hence the interior point $E^*\left(\frac{d}{c}, r(1-\frac{d}{c})\right)$ is stable otherwise it is unstable.

III: Finally, when $rd < 4c(c-d)$, Then the eigen values reduces to in the form $\lambda = -\frac{rd}{2c} \pm i\left(\frac{1}{2c}\sqrt{4cdr(c-d)-(rd)^2}\right)$, which is complex number(eigen value) but real part of the complex eigen value is negative, so the interior point $E^*\left(\frac{d}{c}, r(1-\frac{d}{c})\right)$ is Asymptotically Stable.

8 Stability Analysis of System (3a, b) in Fuzzy Environment Taking Intrinsic Growth Rate $\tilde{r}$ as a Triangular Fuzzy Number

In model (3a, b) let $\tilde{r}$ be Triangular Fuzzy Number. Let $\tilde{r} = (r_1, r_2, r_3)$ where r_1, r_2, and r_3 all are positive and, $r_1 < r_2 < r_3$
Then the system (3a, b) becomes

$$\frac{d\tilde{U}}{dt} = \tilde{r}U\,(1-U) - UV \qquad = \tilde{F}(U, V) \tag{9a}$$

$$\frac{dV}{dt} = (cU-d)V \qquad = G(U, V) \tag{9b}$$

Equilibrium Points. The equilibrium points of system (8) are given by $E_0(0, 0)$,
$E_1(1, 0)$, and $E^*\left(\frac{d}{c}, \tilde{r}(1-\frac{d}{c})\right)$. The Jacobian matrix is given by

$$J = \begin{pmatrix} \tilde{r} - 2\tilde{r}U - V & -U \\ cV & cU - d \end{pmatrix}$$

Theorem 5. *The trivial equilibrium point $E^0(0, 0)$ is always unstable.*

Proof. The jacobian matrix is given by

$$J(0, 0) = \begin{pmatrix} \tilde{r} & -0 \\ 0 & -d \end{pmatrix}$$

Here the eigen values are $\tilde{r}$ and −d. Since $\tilde{r}$ and d both are >0, so the one eigen values >0 and other eigen values <0, Therefore the point (0,0) is saddle point and hence unstable.

Theorem 6. *The axial equilibrium point $E^1(1, 0)$ is stable if $c < d$ and depends on $\tilde{a}$, otherwise it is unstable.*

Proof. The jacobian matrix is given by

$$J\big(1,0\big) = \begin{pmatrix} \tilde{r} - 2\tilde{r} & -1 \\ & \\ 0 & c-d \end{pmatrix}$$

Here the eigen values are $(\tilde{r} - 2\tilde{r})$ and $(c - d)$.
Now r_1, r_2, and r_3 all are positive and, $r_1 < r_2 < r_3$
and $\tilde{r} - 2\tilde{r} = (r_1 - 2r_3, -r_2, r_3 - 2r_1)$
Since $r_1 < r_3$, so $r_1 < 2r_3$, or $r_1 - 2r_3 < 0$ and $-r_2 < 0$, $r_3 - 2r_1 < 0$ if $r_1 < r_3 < 2r_1$ and $r_3 - 2r_1 > 0$ if $2r_1 < r_3$
Let $\tilde{a} = (r_1 - 2r_3, -r_2, r_3 - 2r_1)$, Then the jacobian matrix becomes

$$J(1,0) = \begin{pmatrix} \tilde{a} & -1 \\ & \\ 0 & c-d \end{pmatrix}$$

There are two eigen values $\tilde{a}$ and $(c - d)$.
Where $a_1 = r_1 - 2r_3, a_2 = -r_2, a_3 = r_3 - 2r_1$ There may arise four casses.

Case I: We consider the relations $r_1 < r_3 < 2r_1$ and $c - d < 0$, then $a_3 = r_3 - 2r_1 < 0$, i.e $\tilde{a} < 0$, so the two eigen values are negative. Therefore the equilibrium point (1,0) is stable.

Case II: If $2r_1 < r_3$ then the defuzzification value of $\tilde{a} = (a_1, a_2, a_3)$ is $\frac{a_1+2a_2+a_3}{2} = \frac{(r_1-2r_3)+2(-r_2)+(r_3-2r_1)}{2} = -\frac{(r_1+2r_2+r_3)}{2} < 0$ as r_1, r_2, and r_3 all are positive. Therefore in this case the point (1,0) is stable.

Case III: If $r_1 < r_3 < 2r_1$ and $c > d$, then $\tilde{a} < 0$ but $(c - d) > 0$, (1,0) point is saddle point and hence it is unstable.

Case IV: If $2r_1 < r_3$ and $c > d$, then the equilibrium point (1,0) is a saddle point and hence it is unstable.

Theorem 7. *The Interior Equilibrium Point $E^*(\frac{d}{c}, \tilde{r}(1 - \frac{d}{c}))$ is Asymptotically stable.*

The Jacobian matrix is given by

$$J\Big(\frac{d}{c}, \tilde{r}\big(1 - \frac{d}{c}\big)\Big) = \begin{pmatrix} \tilde{r} - \tilde{r} + \frac{d}{c}(\tilde{r} - 2\tilde{r}) & -\frac{d}{c} \\ & \\ \tilde{r}(c-d) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{b} & -\frac{d}{c} \\ & \\ \tilde{r}(c-d) & 0 \end{pmatrix}$$

Where $\tilde{b} = \tilde{r} - \tilde{r} + \frac{d}{c}(\tilde{r} - 2\tilde{r})$
Now $\tilde{b} = \tilde{r} - \tilde{r} + \frac{d}{c}(\tilde{r} - 2\tilde{r}) = (r_1 - r_3, 0, r_3 - r_1) + \frac{d}{c}(r_1 - 2r_3, -2r_2, r_3 - 2r_1) = (r_1 - r_3 + \frac{d}{c}(r_1 - 2r_3), -2\frac{d}{c}r_2, r_3 - r_1 + \frac{d}{c}(r_3 - 2r_1))$

The characteristic equation of the Jacobian matrix is given by

$$\begin{vmatrix} \tilde{b}-\lambda & -\frac{d}{c} \\ \tilde{r}(c-d) & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda\tilde{b} + \frac{d}{c}\tilde{r}(c-d) = 0 \tag{10}$$

Let $\tilde{C} = \frac{d}{c}\tilde{r}(c-d)$ (Say) and we consider the cases

Case I : when $(c-d) < 0$ then $\tilde{c} = \frac{d}{c}\tilde{r}(c-d) < 0$, therefore the Eq. (2) reduces to in the form

$\lambda^2 - \lambda\tilde{b} + \tilde{c} = 0 \qquad \Rightarrow \lambda = \frac{1}{2}(\tilde{b} \pm \sqrt{(\tilde{b})^2 - 4\tilde{c}})$

Since $\tilde{c} < 0$, then $(\tilde{b}^2 - 4\tilde{c}) > 0$, Let $\sqrt{\tilde{b}^2 - 4\tilde{c}} = \tilde{d}(Say) > 0$

So $\lambda = \frac{1}{2}(\tilde{b} \pm \tilde{d})$.

Here , we consider two cases

Subcase I: When $\lambda = \frac{1}{2}(\tilde{b}+\tilde{d}) = \frac{1}{2}(r_1(1+\frac{d}{c}) - r_3(1+\frac{2d}{c}), -\frac{2d}{c}r_2, r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c})) + \frac{1}{2}(d_1, d_2, d_3)$

$= \frac{1}{2}(\tilde{b}+\tilde{d}) = \frac{1}{2}(r_1(1+\frac{d}{c}) - r_3(1+\frac{2d}{c}) + d_1, -\frac{2d}{c}r_2 + d_2, r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c}) + d_3)$, Here $\tilde{d} = (d_1, d_2, d_3) > 0$

Now $\lambda = \frac{1}{2}(\tilde{b}+\tilde{d}) < 0$ if $(r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c}) + d_3) < 0$ or, $(r_3 + (r_3 - 2r_1)\frac{d}{c} - r_1 + d_3) < 0$

or $(r_3 - 2r_1)\frac{d}{c} < (r_1 - d_3 - r_3)$ or, $\frac{d}{c} > \frac{r_1 - d_3 - r_3}{r_3 - 2r_1}$ if $r_3 - 2r_1 < 0$

(We consider the relation $r_1 < r_3 < 2r_1$ i.e $r_3 - 2r_1 < 0$ and $r_1 - r_3 < 0$ then $(r_1 - d_3 - r_3) < -d_3 < 0$)

So $\lambda = \frac{1}{2}(\tilde{b}+\tilde{d}) < 0$ if $\frac{d}{c} > \frac{r_1 - d_3 - r_3}{r_3 - 2r_1}$ if $r_3 - 2r_1 < 0$

Subcase II: When $\lambda = \frac{1}{2}(\tilde{b}-\tilde{d}) = \frac{1}{2}(r_1(1+\frac{d}{c}) - r_3(1+\frac{2d}{c}), -\frac{2d}{c}r_2, r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c})) - \frac{1}{2}(d_1, d_2, d_3)$

$= \frac{1}{2}(r_1(1+\frac{d}{c}) - r_3(1+\frac{2d}{c}), -\frac{2d}{c}r_2, r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c})) + \frac{1}{2}(-d_3, -d_2, -d_1)$

$= \frac{1}{2}(r_1(1+\frac{d}{c}) - r_3(1+\frac{2d}{c}) - d_3, -d_2 - \frac{2d}{c}r_1, r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c}) - d_1)$, Here $\tilde{d} = (d_1, d_2, d_3) > 0$

Similarly

Now $\lambda = \frac{1}{2}(\tilde{b}-\tilde{d}) < 0$ if $(r_3(1+\frac{d}{c}) - r_1(1+\frac{2d}{c}) - d_1) < 0$

or $(r_3 - 2r_1)\frac{d}{c} < (r_1 + d_1 - r_3)$ or, $\frac{d}{c} > \frac{r_1 + d_1 - r_3}{r_3 - 2r_1}$ if $r_3 - 2r_1 < 0$

So $\lambda = \frac{1}{2}(\tilde{b}-\tilde{d}) < 0$ if $\frac{d}{c} > \frac{r_1 + d_1 - r_3}{r_3 - 2r_1}$ if $r_3 - 2r_1 < 0$

Thus in the both cases the eigen values are negative if

$$\frac{d}{c} > \frac{r_1 + d_1 - r_3}{r_3 - 2r_1} > \frac{r_1 - d_3 - r_3}{r_3 - 2r_1} if r_3 - 2r_1 < 0 \tag{11}$$

Finally, the interior equilibrium $(\frac{d}{c}, \tilde{r}(1-\frac{d}{c}))$ is stable if condition (7) hold.

Case II : when $(c-d) > 0$ then Eq. (6) reduces in the form $\lambda^2 - \lambda\tilde{b} + \tilde{C} = 0$ where $\tilde{C} = \frac{d}{c}\tilde{r}(c-d) > 0$. Therefore $\lambda = \frac{1}{2}(\tilde{b} \pm \tilde{d})$ where $\sqrt{\tilde{b}^2 - 4\tilde{C}} = \tilde{d} > 0$

Subcase 1: when $\tilde{b}^2 > 4\tilde{C}$ then $\tilde{d} > 0$ so the two eigenvalues are $\frac{1}{2}(\tilde{b} \pm \tilde{d})$ these eigenvalues are negative if the condition (7) hold.

Subcase 2: $\tilde{b}^2 < 4\tilde{C}$ then $\sqrt{\tilde{b}^2 - 4\tilde{C}} = \tilde{d}$ is imaginary. So the two eigenvalues are $\frac{1}{2}(\tilde{b} \pm i\sqrt{4\tilde{C} - \tilde{b}^2})$This eigen values are asymptotically stable if the real part of the eigenvalues are negative i.e. $\frac{\tilde{b}}{2} < 0$ i.e. if $\frac{d}{c} > \frac{r_1 - r_3}{r_3 - 2r_1}$ provided $r_1 < r_3 < 2r_1$.

9 Stability Analysis of System (3a, b) in Fuzzy Environment Taking the Conversion Rate $\tilde{c}$ as a Triangular Fuzzy Number

Let $\tilde{c} = (c_1, c_2, c_3)$ be a TFN where $0 < c_1 < c_2 < c_3$ then the system (4a, b) becomes

$$\frac{dU}{dt} = rU\,(1 - U) - UV \qquad = F(U, V) \tag{12a}$$

$$\frac{d\tilde{V}}{dt} = (\tilde{c}U - d)V \qquad = \tilde{G}(U, V) \tag{12b}$$

9.1 Equilibrium Points

The equilibrium points of (11) are given by $E_0(0,0)$, $E_1(1,0)$, and $E^*\left(\frac{d}{\tilde{c}}, r(1 - \frac{d}{\tilde{c}})\right)$. $E_0(0,0)$ is called trivial equilibrium point while $E_1(1,0)$ are called axial equilibrium point and $E^*\left(\frac{d}{\tilde{c}}, r(1 - \frac{d}{\tilde{c}})\right)$ is called interior equilibrium point.

Theorem 8. *The equilibrium point* $(0, 0)$ *is unstable.*

Proof. At this point the eigen values are r and $-$d i.e. one eigenvalues is (+)ve and other is ($-$)ve and so the point (0,0) is unstable.

Theorem 9. *The equilibrium point (1,0) is stable if* $c_3 < d$ *otherwise it is unstable.*

Proof. The eigen values of the jacobian matrix computed at the point (1,0) are $-r$ and $\tilde{c} - d$ hence both the eigen values will be negative if $c_3 < d$ as the intrinsic growth rate r is always positive.

Theorem 10. *The equilibrium point* $(\frac{d}{\tilde{c}}, r(1 - \frac{d}{\tilde{c}}))$ *is stable if* $r < \frac{c_1 + c_3 - 2c_1c_3}{c_2 - 2c_1}$ *provided* $c_2 > 2c_1$ *and* $\left(dC'_{min} + C''_{min}\right) > 0$ *where*
$C'_{min} = Min\{(\frac{1}{c_3} - \frac{2}{c_1})(\frac{c_1}{c_3} - 1), (\frac{1}{c_3} - \frac{2}{c_1})(\frac{c_3}{c_1} - 1), (\frac{1}{c_1} - \frac{2}{c_3})(\frac{c_1}{c_3} - 1), (\frac{1}{c_1} - \frac{2}{c_3})(\frac{c_3}{c_1} - 1)\}$
and $C''_{min} = Min\{(\frac{c_1}{c_3} - \frac{dc_1}{c_3^2}), (1 - \frac{d}{c_1}), (1 - \frac{d}{c_3}), (\frac{c_3}{c_1} - \frac{dc_3}{c_1^2})\}$

Proof. The Jacobian matrix at the equilibrium point $(\frac{d}{\tilde{c}}, r(1-\frac{d}{\tilde{c}}))$ is given by $\begin{pmatrix} \tilde{M_{11}} & \tilde{M_{12}} \\ \tilde{M_{21}} & \tilde{M_{22}} \end{pmatrix}$ where $\tilde{M_{11}} = \frac{rd}{\tilde{c}} - \frac{2rd}{\tilde{c}}$; $\tilde{M_{12}} = -\frac{d}{\tilde{c}}$; $\tilde{M_{21}} = r\tilde{c}(1-\frac{d}{\tilde{c}})$ and $\tilde{M_{22}} = \frac{\tilde{c}d}{\tilde{c}} - d$.
The characteristic equation is given by

$$\lambda^2 - (\tilde{M_{11}} + \tilde{M_{22}})\lambda + \tilde{M_{11}}\tilde{M_{22}} - \tilde{M_{12}}\tilde{M_{21}} = 0 \tag{13}$$

Therefore the eigen values of the jacobian matrix have the negative real parts if $(\tilde{M_{11}} + \tilde{M_{22}}) < 0$ and $(\tilde{M_{11}} + \tilde{M_{22}})(\tilde{M_{11}}\tilde{M_{22}} - \tilde{M_{12}}\tilde{M_{21}}) < 0$ i.e. $r < \frac{c_1+c_3-2c_1c_3}{c_2-2c_1}$ provided $c_2 > 2c_1$ and $\left(dC^{'}_{min} + C^{''}_{min}\right) > 0$ where $C^{'}_{min} = Min\{(\frac{1}{c_3}-\frac{2}{c_1})(\frac{c_1}{c_3}-1), (\frac{1}{c_3}-\frac{2}{c_1})(\frac{c_3}{c_1}-1), (\frac{1}{c_1}-\frac{2}{c_3})(\frac{c_1}{c_3}-1), (\frac{1}{c_1}-\frac{2}{c_3})(\frac{c_3}{c_1}-1)\}$ and $C^{''}_{min} = Min\{(\frac{c_1}{c_3} - \frac{dc_1}{c_3^2}), (1-\frac{d}{c_1}), (1-\frac{d}{c_3}), (\frac{c_3}{c_1} - \frac{dc_3}{c_1^2})\}$. Finally, this is the required stability condition.

10 Stability Analysis of the Logistic Model in Fuzzy Environment Taking the Mortality Rate D as a TFN

Considering the mortality rate $\tilde{d}$ as a TFN the system () becomes

$$\frac{du}{dt} = ru(1-u) - uv = F(u, v) \tag{14a}$$

$$\frac{\tilde{d}v}{dt} = (cu - \tilde{d})v = \tilde{G}(u, v) \tag{14b}$$

The equilibrium points of system (2a, b) are $(0, 0)$; $(1, 0)$ and $(\frac{\tilde{d}}{c}, r(1-\frac{\tilde{d}}{c}))$

Theorem 11. *The equilibrium point* $(0, 0)$ *is a saddle point and hence unstable.*

Proof. The jacobian matrix of the system (2a, b) computed at the trivial equilibrium point $(0, 0)$ has the eigen values r and $-\tilde{d}$. As the intrinsic growth rate r and the mortality rate $\tilde{d}$ are positive the equilibrium point $(0, 0)$ is a saddle point and hence unstable.

Theorem 12. *The equilibrium point* $(1, 0)$ *is stable if* $c < d_1$

Proof. The jacobian matrix of the system (2a, b) computed at the trivial equilibrium point $(1, 0)$ has the eigen values $-r$ and $c - \tilde{d}$. As the intrinsic growth rate r is always positive the equilibrium point $(1, 0)$ is stable if the other equilibrium point $c - \tilde{d}$ is negative if $c < d_1$.

Theorem 13. *The interior equilibrium point* $(\frac{\tilde{d}}{c}, r(1-\frac{\tilde{d}}{c}))$ *is stable if* $\frac{r}{c} < \frac{d_1-d_3}{d_3-2d_1}$ *provided* $(d_3 - 2d_1) > 0$ *and* $\left(D^{'}_{min} + D^{''}_{min}\right) > 0$*, where* $D^{'}_{min} = Min\{(d_1 - 2d_3)(d_1 - d_3), (d_1 - 2d_3)(d_3 - d_1), (d_3 - 2d_1)(d_1 - d_3), (d_3 - 2d_1)(d_3 - d_1)\}$ *and* $D^{''}_{min} = Min\{d_1(c - d_3), d_1(c - d_1), d_3(c - d_3), d_3(c - d_1)\}$

Proof. The Jacobian matrix at the equilibrium point $(\frac{\tilde{d}}{c}, r(1-\frac{\tilde{d}}{c}))$ is given by $\begin{pmatrix} \tilde{A_{11}} & \tilde{A_{12}} \\ \tilde{A_{21}} & \tilde{A_{22}} \end{pmatrix}$ where $\tilde{A_{11}} = \frac{r}{c}(\tilde{d}-2\tilde{d})$; $\tilde{A_{12}} = -\frac{\tilde{d}}{c}$; $\tilde{A_{21}} = r(c-\tilde{d})$ and $\tilde{A_{22}} = (\tilde{d}-\tilde{d})$. The characteristic equation is given by

$$\lambda^2 - (\tilde{A_{11}} + \tilde{A_{22}})\lambda + \tilde{A_{11}}\tilde{A_{22}} - \tilde{A_{12}}\tilde{A_{21}} = 0 \tag{15}$$

Therefore the eigen values of the jacobian matrix have the negative real parts if $(\tilde{A_{11}}+\tilde{A_{22}}) < 0$ and $(\tilde{A_{11}}+\tilde{A_{22}})(\tilde{A_{11}}\tilde{A_{22}}-\tilde{A_{12}}\tilde{A_{21}}) < 0$ i.e. $(\tilde{A_{11}}\tilde{A_{22}}-\tilde{A_{12}}\tilde{A_{21}}) > 0$. Now $(\tilde{A_{11}} + \tilde{A_{22}}) < 0$ and $(\tilde{A_{11}}\tilde{A_{22}} - \tilde{A_{12}}\tilde{A_{21}}) > 0$ gives the condition $\frac{r}{c} < \frac{d_1-d_3}{d_3-2d_1}$ provided $(d_3 - 2d_1) > 0$ and $\left(D'_{min} + D''_{min}\right) > 0$ respectively, where $D'_{min} = Min\{(d_1 - 2d_3)(d_1 - d_3), (d_1 - 2d_3)(d_3 - d_1), (d_3 - 2d_1)(d_1 - d_3), (d_3 - 2d_1)(d_3 - d_1)\}$ and $D''_{min} = Min\{d_1(c - d_3), d_1(c - d_1), d_3(c - d_3), d_3(c - d_1)\}$. Finally, this is the required stability condition otherwise it is unstable.

11 Numerical Simulation and Discussions

This section presents a numerical example which is depending on some artificially chosen data to support the theoretical development of the logistic growth models. A set of values of parameters is considered as follows in appropriate units. $r = 0.8$, $k = 12$, $a = 0.3$, $d = 0.8$, $ts = [0, 100]$, $z0 = [0.4, 0.8]$ and clearly, from the graph the interior equilibrium point is stable. Suppose the intrinsic growth rate (r), the conversion rate (c) and the mortality rate (d) be triangular fuzzy numbers given by $\tilde{r} = (0.5, 0.6, 0.7)$; $\tilde{c} = (0.3, 0.4, 0.5)$ and $\tilde{d} = (0.4, 0.5, 0.6)$ then the variational matrix at the trivial equilibrium point $(0, 0)$ is given by $\begin{pmatrix} (0.5, 0.6, 0.7) & 0 \\ 0 & -(0.4, 0.5, 0.6) \end{pmatrix}$ and the characteristic equation of the above variational matrix is given by

$$\lambda^2 - \{(0.9, 1.1, 1.3)\}\lambda - (2, 3, 3.5) = 0 \tag{16}$$

Here we clearly see that the coefficient of λ and the constant term are both negative. Hence the roots of the characteristic equation cannot have the negative real parts and consequently the trivial equilibrium point is unstable.
Similarly the characteristic equation of the variational matrix at the axial equilibrium point $(1, 0)$ is

$$\lambda^2 + \{(-0.3, 0.1, 0.6)\}\lambda + (-0.8, 0, 0.8) = 0 \tag{17}$$

Now defuzzifying the triangular fuzzy numbers $(-0.3, 0.1, 0.6)$ and $(-0.8, 0, 0.8)$ we get the defuzzification value $\frac{1}{8}$ and 0. Both of them are non-negative Hence the roots of the characteristic equation cannot have the negative real parts and consequently the axial equilibrium point is unstable.
and the characteristic equation of the variational matrix at the interior equilibrium point $((\frac{4}{5}, \frac{5}{4}, \frac{7}{3}), (\frac{-14}{15}, \frac{-3}{20}, \frac{7}{50}))$ is

$$\lambda^2 + \left(\frac{-97}{75}, 0, \frac{101}{30}\right)\lambda + \left(\frac{-3207}{1875}, \frac{3}{40}, \frac{13646}{5625}\right) = 0 \tag{18}$$

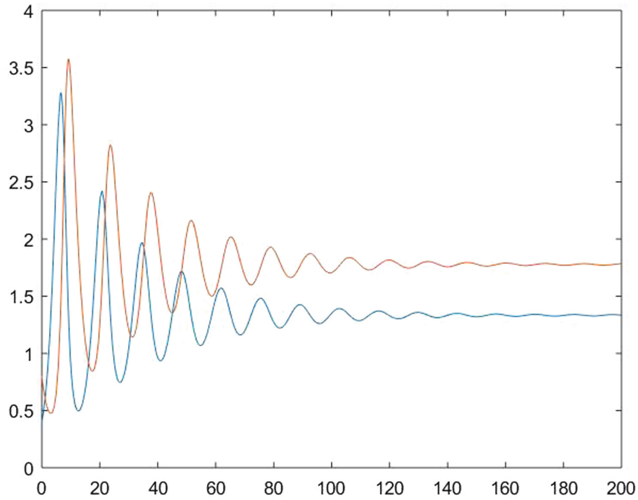

Fig. 1. Plot of predator-prey vs time solution in crisp environment

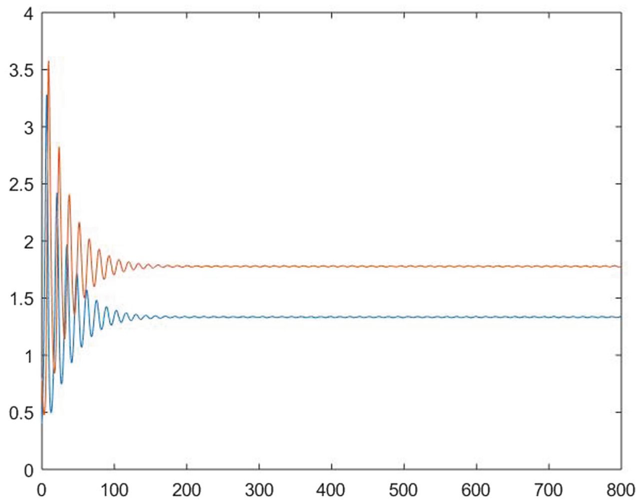

Fig. 2. Plot of predator-prey vs time solution in crisp environment for long period.

Now defuzzifying the triangular fuzzy numbers $\left(\frac{-97}{75}, 0, \frac{101}{30}\right)$ and $\left(\frac{-3207}{1875}, \frac{3}{40}, \frac{13646}{5625}\right)$ we get both the defuzzyfication values are positive. Hence the roots of the characteristic equation have the negative real parts and consequently the interior equilibrium point is stable.

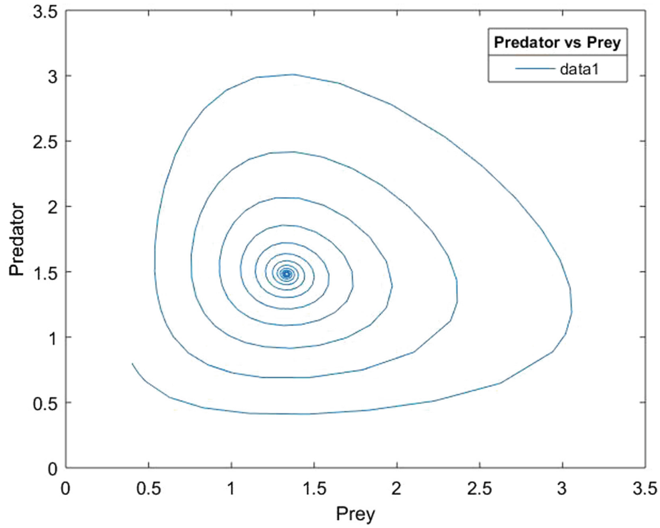

Fig. 3. Plot of predator vs prey solution in crisp environment

12 Conclusion

Prey-Predator model has different development in theoretical and practical applications in the field of biomathematics. Most of the researchers have developed the prey-predator model based on the assumption that the biological parameters are precisely known but the scenario is different in real life situation. In this chapter, we have developed a method to find the biological equilibrium points, when some biological parameters are imprecise in nature. Figures 1 and 2 depicts that predator -prey vs time solution in crisp environment. Figure 3 shows predator vs prey solution in crisp environment.

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On Partial Monotonic Behaviour of Past Entropy and Convolution of Extropy

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Abstract. Shannon [20] gave a measure of uncertainty that plays an irresistable role in the field of communication theory. Since the proposal of the Shannon entropy many entropies have been proposed later and found to be useful in different areas. Recently, Frank et al. [6] gave a complementary dual of the Shannon entropy and named it 'extropy'. The uncertainty associated with a random experiment is expected to get reduced when the interval containing the outcome gets smaller. However this result is in general not true for entropies with absolutely continuous random variables. In the present paper we define conditional cumulative past entropy and give the conditions necessary for its partial monotonic behaviour. Further, a result on convolution of extropy have been presented.

Keywords: Entropy · Extropy · Log-concave · Log-convex · Partial monotonicity

1 Introduction

The importance of measuring uncertainty in the fields of communication theory, information theory, probability and statistics was firstly identified by Shannon [20]. He coined the term entropy which measures the average amount of uncertainty when the outcome of a random experiment is revealed. Let X be a discrete (or absolutely continuous) random variable having cumulative distribution function (cdf) $F_X(x)$ and probability mass function $p_i,\ i = 1, 2 \ldots, n$ (probability density function $f_X(x)$ with support set $\mathfrak{D} = \{x \in \mathbb{R} : f_X(x) > 0\}$). The discrete and the differential forms of the Shannon entropy are given by

$$H(X) = -\sum_{i=1}^{n} p_i \ln p_i$$

and

$$H(X) = -\int_{\mathfrak{D}} f_X(x) \ln (f_X(x))\, dx$$

respectively. In order to quantify uncertainty various generalizations of the Shannon entropy have been developed in literature by many researchers. Some of them

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are presented in this paper. Uncertainty may also sometimes be related to past or future. Hence, Di Crescenzo and Longobardi [5] proposed and studied the following cumulative past entropy (CPE)

$$\xi(X) = -\int_{\mathfrak{D}} F_X(x) \ln \left(F_X(x)\right) dx. \tag{1}$$

Similar to this definition cumulative residual entropy has also been defined and studied in literature where cdf $F_X(x)$ is replaced by the survival function $\overline{F}_X(x) = 1 - F_X(x)$. Entropies play an important role in the study of likelihood-based inference principles and large deviation theory in the area of probability and statistics. Shannon, Renyi, Tsallis and Varma entropies defined subsequently find applications in data compression, signal processing, ecological diversity and as a measure of complexity and uncertainty in different areas like Physics, Electronics and Coding Theory.

Generalized entropy of order α proposed by Renyi [16] is given by

$$H_\alpha(X) = \frac{1}{1-\alpha} \ln \left(\int_{\mathfrak{D}} f_X^\alpha(x) dx \right), \ \alpha > 0, \ \alpha \neq 1.$$

Tsallis [22] defined and carried out the study for the following generalized entropy

$$S_\alpha(X) = \frac{1}{\alpha - 1} \left(1 - \int_{\mathfrak{D}} f_X^\alpha(x) dx \right), \ \alpha > 0, \ \alpha \neq 1.$$

Kapur entropy of order α and type β given by Kapur [10] is defined as

$$H_{\alpha,\beta}(X) = \frac{1}{\beta - \alpha} \ln \left(\frac{\int_{\mathfrak{D}} f_X^\alpha(x) dx}{\int_{\mathfrak{D}} f_X^\beta(x) dx} \right), \ \alpha \neq \beta, \ \alpha > 0, \ \beta > 0.$$

Varma's [24] entropy of order α and type β is given by

$$H_\alpha^\beta(X) = \frac{1}{\beta - \alpha} \ln \left(\int_{\mathfrak{D}} f_X^{\alpha+\beta-1}(x) dx \right), \quad \beta - 1 < \alpha < \beta, \ \ \beta \geq 1.$$

The details can be found in Cover and Ash [1], Thomas [4], Hooda [8], Kovacevic et al. [11], Maszczyk and Duch [12], Sadek et al. [17], Tuli [23] and Yeung [26]. In addition to these many entropies have been proposed by many researchers.

Another measure of uncertainty called 'extropy' has gained importance in the recent years. It is a compliment dual of the Shannon entropy and was defined by Frank et al. [6] as

$$J(X) = -\frac{1}{2} \int_{\mathfrak{D}} f_X^2(x) dx. \tag{2}$$

Qiu [13] studied various comparison results on extropy and also gave characterization results, monotone properties and statistical applications concerning extropy of order statistics and record values. Qiu and Jia [14] proposed and studied the concept of residual extropy to measure the residual uncertainty of a random variable. They carried out the study of residual extropy measure for order statistics. For the random variable $X_t = [X - t | X \geq t]$ it is defined as

$$J(X_t) = -\frac{1}{2\overline{F}^2_X(t)} \int_t^\infty f_X^2(x)dx, t \geq 0. \tag{3}$$

Recently, Jose and Sathar [9] carried this study for $k-$record values. Further, Raqab and Qiu [15] investigated the monotone properties and stochastic orders of ranked set samples in terms of extropy measure.

Assume X to be an absolutely continuous random variable with cdf $F_X(x)$, $x \in (-\infty, \infty)$. For any $a < b$ consider the interval $A = (a, b)$ and the event $X \in A$ then the conditional density function of X is

$$f_{X|A}(x) = \frac{f_X(x)}{F_X(b) - F_X(a)}, \quad a < x < b.$$

Hence, the conditional Shannon entropy of X given A can be given as

$$H\left(X|A\right) = -\int_a^b f_{X|A}(x) \ln\left(f_{X|A}(x)\right) dx. \tag{4}$$

The detailed study of the conditional Shannon entropy was done in the paper by Sunoj et al. [21]. Now we give the definition of partial monotonicity of a function as defined by Gupta and Bajaj [7].

Definition 1. *Let A and B be two intervals such that $B \subseteq A$, then entropy function H is said to be partially increasing if $H(X|X \in B) \leq H(X|X \in A)$. H with this property is said to be partially monotonic. When the inequality is reversed, H is said to be partially decreasing.*

Shangari and Chen [19] in their paper derived that the necessary and sufficient condition for conditional Shannon entropy $H(X|A)$ of X as well as conditional Renyi entropy of order α i.e $H_\alpha(X|A)$ of X to be a partially increasing function in the interval A is $F_X\left(x\right)$ is log-concave. Gupta and Bajaj [7] proved that for conditional Kapur entropy $H_{\alpha,\beta}(X|A)$ and the conditional Tsallis entropy $S_\alpha(X|A)$ the condition required is that $F_X\left(x\right)$ should be concave. Later, Sati and Gupta [18] studied the monotonic nature of conditional Varma entropy. The conditional Varma's entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) in A provided $F_X(x)$ is log concave and $\alpha + \beta > (<)2$. Very recently, Xia [25] studied partial monotonicity of the discrete Shannon entropy and also pointed out some errors in the results of Shangari and Chen [19] and Gupta and Bajaj [7]. Motivated by these studies, in the present paper we first propose a new conditional entropy based on CPE given by Eq. (1) and study its partial monotonic behaviour. We proved that if $\int_a^x F_X(u)du$ is log-concave and $F_X(a) = 0$ then the conditional CPE $\xi(X|A)$ of X is increasing in b when we consider the interval $A = (a, b)$. Let X be a random experiment that is repeated to measure its reproducibility or precision or both, then measure of uncertainity of the experiment is the function $U = X_1 - X_2$; where X_1 and X_2 are two independent and identically distributed copies of an experiment X. We prove that if random variables X_1 and X_2 have log concave probability functions then the conditional extropy of $U = X_1 - X_2$ given $B = \{a \leq X_1, X_2 \leq b\}$ is partially increasing function of B.

Analogous to the expression of conditional Shannon entropy, in view of Eq. (1) we define the conditional CPE of X given $A = (a, b)$ as

$$\begin{aligned}\xi(X|A) &= -\int_{\mathfrak{D}} F_{X|A}(x) \ln \left(F_{X|A}(x)\right) dx \\ &= -\int_a^b \frac{F_X(x)}{F_X(b) - F_X(a)} \ln\left(\frac{F_X(x)}{F_X(b) - F_X(a)}\right) dx. \qquad (5)\end{aligned}$$

The following lemma of Bagnoli and Bergstrom [2] about the log-concavity property will be used in deriving the main result in the upcoming section.

Lemma 1. *A twice-differentiable function, say $g(x)$ is log-concave if and only if $\{g'(x)\}^2 - g(x)\, g''(x) \geq 0$ for all x in the domain of $g(x)$. The function $g(x)$ is log-convex iff we have the reverse inequality.*

2 Monotonic Behaviour of CPE and Convolution of Extropy

Many entropies defined is over the entire support set of X. However this restriction on entropies can be relaxed. The monotonicity properties and convolution of the various conditional entropy measures viz. Shannon, Renyi, Tsallis, Kapur and Varma entropy have widely been studied in literature (see Chen [3], Gupta and Bajaj [7], Sati and Gupta [18], Shangari and Chen [19]). For an interval A, the conditional Shannon entropy $H(X|X \in A)$ defined by Eq. (4) may serve as indicator of uncertainty. One may intuitively guess that the uncertainty reduces when we have some prior information related to the random variable concerned. If one has the information that the outcome of the experiment is in an interval, it is expected that the uncertainty reduces as the interval shrinks. Also the partial monotonicity is trivially true for all discrete X (as shown in Shangari and Chen [19]). But this proposition is not true in general. Thus, many authors have provided necessary and sufficient conditions for the proposition to be true.

The following theorem provides the conditions under which the conditional CPE $\xi(X|A)$ given by Eq. (5) is increasing in b, where $A = (a, b)$.

Theorem 1. *Let X be an absolutely continuous random variable with pdf $f_X(x)$, cdf $F_X(x)$ and let A be the event $a < X < b$ with $F_X(a) = 0$. If $\int_a^x F_X(u)du$ is log-concave, then the conditional CPE*

$$\xi(X|A) = -\int_a^b \frac{F_X(x)}{F_X(b)} \left(\ln F_X(x) - \ln F_X(b)\right) dx$$

is increasing in b.

Proof. When $F_X(a) = 0$ in Eq. (5) we get

$$\begin{aligned}\xi(X|A) &= -\int_a^b \frac{F_X(x)}{F_X(b)} \left(\ln F_X(x) - \ln F_X(b)\right) dx \\ &= -\int_a^b \frac{F_X(x) \ln F_X(x)}{F_X(b)} dx + \frac{\ln F_X(b)}{F_X(b)} \int_a^b F_X(x) dx \\ &\equiv \phi(b) \quad \text{(say)}\end{aligned}$$

On differentiation,

$$\begin{aligned}\frac{d\phi(b)}{db} &= -\frac{1}{F_X^2(b)} \left(F_X^2(b) \ln F_X(b) - F_X(b) \int_a^b F_X(x) \ln F_X(x) dx\right) \\ &\quad + \frac{1}{F_X^2(b)} \left(\int_a^b F_X(x) dx\right) \left(F_X(b) - F_X(b) \ln F_X(b)\right) + \ln F_X(b) \\ &= \frac{f_X(b)}{F_X^2(b)} \psi_1(b),\end{aligned} \tag{6}$$

where

$$\psi_1(x) = \int_a^x F_X(u) \ln F_X(u) du + \left(\int_a^x F_X(u) du\right) (1 - \ln F_X(x)).$$

Now

$$\begin{aligned}\psi_1'(x) &= F_X(x) \ln F_X(x) + (1 - \ln F_X(x)) F_X(x) - \left(\int_a^x F_X(x) du\right) \frac{f_X(x)}{F_X(x)} \\ &= \frac{1}{F_X(x)} \left(F_X^2(x) - f_X(x) \int_a^x F_X(u) du\right).\end{aligned} \tag{7}$$

Note that $\int_a^x F_X(u)du$ is log-concave, therefore using Lemma 1 we obtain

$$F_X^2(x) - f_X(x) \int_a^x F_X(x) dx \geq 0, \forall x.$$

Hence $\psi_1'(x) \geq 0$ i.e $\psi_1(x)$ is an increasing function in x. Hence, for $a < b$ we have $\psi_1(a) \leq \psi_1(b)$. Observe that $\psi_1(x)|_{x=a} = 0$. Therefore, $0 = \psi_1(a) \leq \psi_1(b)$. Thus, using (6), $\frac{d\phi(b)}{db} \geq 0$ which implies $\phi(b)$ is increasing in b, i.e., $\xi(X|A)$ is increasing in b.

Our forthcoming result is on convolution of extropy given by Frank et al. [6] in Eq. (2). We replicate an experiment a number of times for the precision and reproducibility of the outcomes. A function $U = X_1 - X_2$ is the best uncertainity measure when the experiment is repeated independently under identical conditions. Here, X_1 and X_2 are outcomes of such an experiment X with common density $f_X(x)$. If the additional information of the form $B = \{\, a \leq X_1, X_2 \leq b \,\}$ is provided then uncertainty should reduce.

Let $B = \{a < X_1, X_2 < b\}$ and $F_X(x) = P(X \leq x)$. The marginal probability density function of $U = X_1 - X_2$ given the additional information in form of B is

$$g(u; a, b) = \frac{1}{(F_X(b) - F_X(a))^2} \int_{a+u}^{b} f_X(x) f_X(x-u) dx, \ \forall \ u \in [0, b-a].$$

Chen [3] proved that if the random variables X_1 and X_2 have log-concave probability density functions which take value in B, then the conditional Shannon entropy of U given B is partially monotonic in B. Sahangari and Chen [19] claimed and Gupta and Bajaj [7] proved that under the same condition on X_1 and X_2 and if $\alpha > 0$, $\alpha \neq 1$, the conditional Tsallis and Renyi entropy of U given B is partially increasing function of B. Later, Sati and Gupta [18] studied the partial monotonic behaviour of the conditional Varma's entropy of U given B.

The following lemma of Chen [3] (also see Sati and Gupta [18]) is useful in proving the next result of the section which provides the conditions for conditional extropy of U given B to be partially increasing/decreasing function of B.

Lemma 2. *1. Let X_1 and X_2 have log-concave probability density functions. If the function $\phi(u)$ is increasing in $|u|$, then $E(\phi(U)|a < X_1, X_2 < b)$ is increasing in b for any a, and decreasing in a for any b; where $U = X_1 - X_2$.*
2. If $f_X(x)$ is log-concave, then $g(u; a, b)$ is decreasing function of u on $u \in [0, b-a]$.

Theorem 2. *Let the random variables X_1 and X_2 have log-concave probability density functions and $B = \{a < X_1, X_2 < b\}$, then the conditional extropy of U given B is partially increasing function of B.*

Proof. The conditional extropy of U given B is:

$$J(U) = -\frac{1}{2} \ln \left(\int_a^b (g(u; a, b))^2 \, du \right).$$

For fixed a, if we choose for any $b_1 \leq b_2$,

$$\psi_1(u) = g(u; a, b_1) \ \text{ and } \ \psi_2(u) = g(u; a, b_2).$$

Clearly here $\psi_1(u)$ and $\psi_2(u)$ are non-negative. Also, let $p = 2$ and $q = 2$, then $p > 0, \ q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now using Hölder's inequality we have

$$\int \psi_1(u)\psi_2(u) du \leq \left(\int (\psi_1(u))^p du \right)^{1/p} \left(\int (\psi_2(u))^q du \right)^{1/q},$$

$$\text{i.e.,} \int g(u; a, b_1) g(u; a, b_2) du \leq \left(\int g^2(u; a, b_1) du \right)^{1/2} \left(\int g^2(u; a, b_2) du \right)^{1/2}. \tag{8}$$

For fixed $b > 0$, let

$$\phi_1(u) = -g(u; a, b);$$

then

$$\phi_1'(u) = -g'(u; a, b).$$

By the assumption of log concavity and using Lemma 2(2), $\phi_1(u)$ increases in u. Therefore, by Lemma 2(1) for any $a < b_1 < b_2$, we have

$$\begin{aligned} E(\phi_1(U)|a \le X_1, X_2 \le b_1) &\le E(\phi_1(U)|a \le X_1, X_2 \le b_2), \\ \Rightarrow \int g^2(u; a, b_2)du &\le \int g(u; a, b_1)g(u; a, b_2)du. \end{aligned} \tag{9}$$

From (8) and (9), we have

$$\int g^2(u; a, b_2)du \le \int g^2(u; a, b_1)du. \tag{10}$$

Therefore we have

$$-\frac{1}{2}\int g^2(u; a, b_1)du \le -\frac{1}{2}\int g^2(u; a, b_2)du,$$
$$\text{i.e., } J(U|a < X_1, X_2 < b_1) \le J(U|a < X_1, X_2 < b_2); \text{for } b_1 \le b_2.$$

Hence, the conditional extropy of U given B is increasing in b for fixed a.

Now for fixed b, if we choose for any $a_1 \le a_2$,

$$\psi_3(u) = g^2(u; a_1, b)\ g^2(u; a_2, b) \text{ and } \psi_4(u) = \left(g^2(u; a_1, b)\right)^{-1}.$$

Clearly $\psi_3(u)$ and $\psi_4(u)$ are non-negative. Also, let $p = \frac{1}{2}$ and $q = -1$, then $p < 1,\ q < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now using Hölder's inequality we have

$$\begin{aligned} &\left(\int (\psi_3(u))^p du\right)^{1/p} \left(\int (\psi_4(u))^q du\right)^{1/q} \le \int \psi_3(u)\psi_4(u)du, \\ &i.e., \left(\int g(u; a_1, b)\ g(u; a_2, b)du\right)^2 \left(\int (g(u; a_1, b))^2 du\right)^{-1} \\ &\qquad\qquad \le \int (g(u; a_2, b))^2 du, \\ &\Rightarrow \int g(u; a_1, b)\ g(u; a_2, b)du \le \left(\int (g(u; a_1, b))^2 du\right)^{\frac{1}{2}} \left(\int (g(u; a_2, b))^2 du\right)^{\frac{1}{2}}. \end{aligned} \tag{11}$$

For fixed $a_2 > 0$, let

$$\phi_2(u) = -g(u; a_1, b);$$

then,

$$\phi_2'(u) = -g'(u; a_1, b).$$

By the assumption of log concavity and using Lemma 2(2), $\phi_2(u)$ increases in u. Therefore, by Lemma 2(1) for any $a_1 < a_2 < b$, we have

$$E(\phi_2(U)|a_2 \leq X_1, X_2 \leq b) \leq E(\phi_2(U)|a_1 \leq X_1, X_2 \leq b),$$
$$\int (g(u;a_1,b))^2 du \leq \int g(u;a_1,b)g(u;a_2,b)du. \tag{12}$$

From (11) and (12), we have

$$\int (g(u;a_1,b))^2 du \leq \int (g(u;a_2,b))^2 du. \tag{13}$$

Therefore we have

$$-\frac{1}{2}\int (g(u;a_2,b))^2 du \leq -\frac{1}{2}\int (g(u;a_1,b))^2 du,$$
$$\text{i.e.,}\ \ J(U|a_2 < X_1, X_2 < b) \leq J(U|a_1 < X_1, X_2 < b); \text{for } a_1 \leq a_2.$$

Hence, the conditional extropy of U given B is decreasing in a for fixed b.

Therefore the conditional extropy of U given B is partially increasing in B.

Remark 1. From the above theorem, it can be interpreted that the conditional extropy of U given B partially increases in B; which shows its reasonability of being an compliment dual of entropy measure.

The following examples describe the densities where Theorem 2 is applicable.

Example 1. 1. Consider two independent Weibull random variables X_1 and X_2 with the common pdf

$$f_X(x;\alpha) = \alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}},\ x \geq 0,\ \alpha \geq 1, \lambda \geq 0.$$

Weibull density function is log-concave for $\alpha \geq 1$, then using Theorem 2 conditional extropy of U given interval B is partially increasing function of B.

2. X_1 and X_2 be two independent gamma random variables with common probability density function

$$f_X(x;\lambda,\alpha) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},\ x \geq 0,\ \alpha \geq 1, \lambda \geq 0.$$

Since gamma density is log-concave for $\alpha \geq 1$, then using Theorem 2 conditional extropy of U given interval B is partially increasing function of B.

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FP-Captcha: An Improved Captcha Design Scheme Based on Face Points

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Abstract. Captcha's are random significant challenges, used to differentiate between humans and bots in order to prevent unauthorized access to web services. Over a decade lots of research have been carried out for generating numerous types of Captcha's; specially using image and face detection and recognition techniques. Rather than machine, humans are outstandingly good at several things, including pattern recognition, linguistic abilities, and innovative thinking. Although, computers are rapidly improving the same, but most programs still don't recognize as well as a child. Humans are inherently capable of detecting faces in a variety of contexts; even parts of face portions are not visible yet. Modern advancement in Artificial Intelligence, Deep Learning and advanced Image Processing techniques have made the Captcha based security system unsecure and vulnerable. However, as the time progresses Captcha's are equipped with high usability, robustness and producing new unique challenges. This paper proposes a novel face point based Captcha, which employs various face points detection as its test, where user will ask to click on correct face points of all human faces presented in the Captcha challenge; which comprises of real and fake face images, with balanced noise and distortions, embedded in a composite background. Over 100 unique and random FP-Captcha's are generated and solved by 115 UG-students, with accuracy of 97.39% for correct responses. Consequently, the probability of passing a FP-Captcha test by random guessing attack was 0.00000245%. Therefore, we conclude that, FP-Captcha is secure and robust against malicious attacks and offer better human accuracy.

Keywords: Captcha · Face feature points · Random guessing attack

1 Introduction

Captcha (Completely Automated Public Turing test to tell Computers and Humans Apart) aims at preventing unwanted machines from making certain errors [16]. These are type of challenge–response tests, used in different application of computing to determine whether the users are human or not. Captcha's are mainly used for security checking to ensure only human users can pass

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through it, apart from it can protect web-sites from receiving spam mail, automated registered accounts, password hacking, finding bank information etc [17]. Design of Captcha's must be simple, easily recognizable by humans, but enough difficult for automatic programs or bots'. This can also be defined as a Reverse Tuning Test, used to submit response to distinguish between a human client from a machine, and grants or refuses permissions accordingly. Recently most web-site's equipped with Text-based Captcha's (widely deployed Captcha Scheme) are quite easy to understand for any advanced Artificial Intelligent based software [18]. By time, being broad research works are carried on the security of Captcha's, and most of Captcha models have been cracked and broken using latest sophisticated recognition techniques. From a detail study and research, we have observed that the success rate of solving a Captcha for humans reaches up to 98% on the other hand the success rate of machine solving Captcha is less than 15%.

To design a successful and efficient Captcha system; Usability, Robustness and Response time are the most important aspects and features. Few essential and important properties to design successful Captcha system are as follows.

– **Robust:** The Captcha challenge should be strong enough, that it should not vulnerable to any machine attack and no-effort attack.
– **Automated:** The Captcha challenge must be generated automatically.
– **Usable:** Human must be able to solve the Captcha challenge easily, within a limited time and achieve high success rate. Also no educational background is needed.
– **Language Independent:** A good Captcha system should be language independent.
– **Database:** Database used in Captcha design should be kept secret.

The idea of image based Captcha's are to propose with multiple variations of fundamental processes, to show users an imperfect image, and they are requested to conduct a recognition task, which proceed through algorithms. Therefore, image based Captcha's becomes a decent and favorable substitute than others, because of many benefits. But normal image recognition Captcha's are not secured enough due to huge progress and development of the deep learning algorithms. In addition, audio Captcha's and video-based Captcha's work in lower pursuit rate and higher bandwidth, caused many websites don't support these types of Captcha's. Hence, it is rather necessarily required to implement, an extensible usable Captcha.

In this research article, we proposed FP-Captcha (Face Point Captcha), where users are asked to click on human face points like eye, nose, etc. A collection of eight real and fake faces are embedded on a complex background, each equipped with different pattern and noise. If users clicked on correct face points (up to certain tolerance level) Captcha challenge will be solved, otherwise a new Captcha challenge will be generated. We compared our proposed FP-Captcha design features with other currently existing Captcha system by means of accuracy, usability and robustness. The results of our assessment shows that FP-Captcha is capable for a practical wide espousal, as well as suitable for mouse click or touch screen devices. The overall organization of the paper is as follows. In Sect. 2, we discussed the literature survey report. In Sect. 3, we described details of methodology of FP-Captcha system. In Sect. 4, we discussed about experimental result and analysis of automated attacks, and finally, in Sect. 5, we draw some conclusions briefing the contributions of our research.

2 Related Works

In this paper we limit our scope on image based Captcha schemes, which are more secure and usable than text, audio, video or puzzle based Captcha schemes. Image or picture based Captcha's mainly uses the concept of detection or recognition of object on the same, to find a solution. Humans can detect or recognize object so easily, leads the researchers to create variety of image Captcha designs. It was observed that users preferred image-based Captcha's rather than others schemes, [5] yields a high success rate to solve Captcha challenge, and are more suitable than text-based ones [9]. One of the famous image-based Captcha was Asirra [3], which questioned users to differentiate between dogs and cats, on twelve dissimilar images arbitrarily taken from an external website. ASSIRA was successfully attacked by Zhu et al. [19] for a 12-image Captcha challenge; the success rate was 10.3%. Gao et al. [5] proposed a Jigsaw puzzle based Captcha (size: m$\times$ m), which considered tiles of single images with m $\geq$ 3 and two tiles are misplaced from entire image. The users have to find corrected misplaced tiles and put them on the proper location in order to pass the Captcha challenge. The complexity of Jigsaw puzzle will increases as number of tiles will increased. Chew et al. [1] proposed Naming Captcha, where users asked to enter a word that best describes the common object among six images. IMAGINATION [2] ask users to annotate a distorted, composite image in a managed way. One out of the 8 images users have to choose to annotate, clicking near to its center. The chosen image should be matched with system generated words, produces correct response. This type of Captcha is language dependent and was attacked by Zhu et al. [19] with a overall success rate 4.95%. SEMAGE [13] Captcha challenge showed different images of animals. They could be real or cartoon like; total eight images; from where users asked to choose same animals. In Deep Captcha [9] six 3D models of real world objects were chosen and asked users to sort them by their size. ARTiFACIAL [12] was an earlier work on face detection, where an image of human face, inserted and blended in a cluttered background, and users asked to click on six face points, the four eye corners and the two mouth corners. ARTiFACIAL was successfully attacked by Li [8], with success rate of 18.0% and an average time of 1.47 s. In aiCAPTCHA [11] users asked to select an object, matched with Captcha challenge from a background consists of group of random objects. In FR-CAPTCHA [7] users asked to select one genuine face pair of same person, to correctly solve the Captcha challenge, where multiple face images with other non-face images are embedded in background with optimized distortion and noises. FaceDCAPTCHA [6] was built with human faces, cartoons and sketches blended in a multicolored background and users asked to point on all human faces. FR-Captcha and FaceDcaptcha was attacked by Gao et al. [4] with a best success rate 48% and 42% respectively. In CONSCHEME [18] users questioned to count the number of cubes to successfully pass the Captcha challenge. This was an interactive three-dimensional Captcha system, which contains lots of cubes stacked in a three-dimensional room, where walls, ceiling and floor are labeled with the same stickers. In DeRection [18] scheme, the users asked to find out all the distorted regions from a given a GIF image to pass the Captcha challenge. Users may also asked to change the frame or the image as required.

3 Proposed Methodology

Now a days designing a good Captcha is a great challenge to the researchers. Lots of study and researches are going on this concern. In the era of machine learning and deep learning, Captcha challenges can be solved and cracked by automatic programs easily. According to the performance Image Captcha is better than Text Captcha; but due to enormous image processing techniques, deep learning with convolution neural network algorithms, Captcha challenges are also becoming fragile. The designing guideline proposed in [18] Captcha must be generated and designed with few essential characteristics and properties. Captcha can't be robust and secure for infinite time. Every Captcha schemes are breakable, and attacked by improved sophisticated algorithm; it's only a matter of time.

In our scheme, we are choosing face images, along with fake face images as a object associated with different types of distortions, which can be detected by human very easily, whereas automatic programs can't able to recognize. Human can easily detect a face, distinguished from fake face images. Also we designed our Captcha scheme with optimized distortions and noises; so that, human can still able to detect it, whereas bots or automatic programs can't. Considering the various attack algorithms and techniques applied in [4]; we designed some random distortions and at least eight different noise patterns on face and fake images, in such a way, even after binarization of the image, no face component will be detected. Moreover we blended real and fake images on the preprocessed background, in a overlapped manner, made it a composite image; which is hard to detect by automatic programs. User are asked to click on face points of male or female face points. If they do correctly challenge is solved; otherwise not. The overall procedure of FP-Captcha is given in Figs. 1 and 2.

Primarily we designed FP-Captcha for general purpose computer, but it can also be suitable for any other touchscreen or mobile devices. We implement FP-Captcha based on three distinct elements. Firstly, a set of background images associated with various patterns shapes and face edges. Secondly, a set of real human faces and others are avatars, cartoons or face like images; and at last a variety of visual distortion types (e.g., various irregular patterns, various noises, intensity and luminance adjustment) with different amount of degree are applied on FP-Captcha as required.

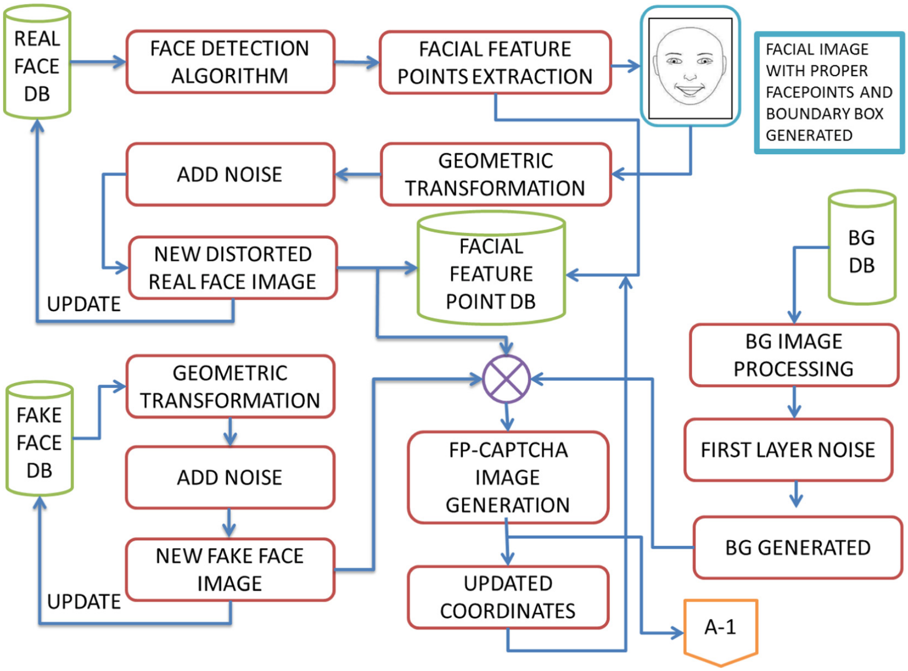

Fig. 1. Steps for generating the proposed FP-Captcha.

3.1 FP-Captcha Preprocessing

The proposed structure for FP-Captcha is described in Figs. 1 and 2. All the figures used in generation of FP-Captcha are categorized into three types; Real Human face (R), Fake face (F) and Background (B). Real face are collected from Aberdeen Data-set; and stored in R_{DB}. Background images are arbitrarily created and fake images are collected from internet resources, stored in B_{DB} and F_{DB} respectively. A separate face point database FP_{DB} is used to store real face point features. All the images are rectified and processed and undergoes several refinement to produce a new Captcha image. First, cluttered background images B_{IMG} are generated with added noise, and processed with various steps; followed by real face images R_{IMG} and fake face images F_{IMG}. Finally C_{Total} = R_{IMG} + F_{IMG} = 8 images are randomly blended with variable opacity with the background; R_{IMG} (2 to 3 images) F_{IMG} (5 to 6 images). Finally another noise layer is added to the background images to reduce the variance between foreground and background image to make it much more robust. The proposed FP-Captcha generation process [10] can be represented as,

$$C_{IMG} = f(I^{`}, \beta, d, F_p, \theta, \alpha) \tag{1}$$

where, C_{IMG} is target Captcha, $I^{\text{`}} = R_{IMG} \otimes F_{IMG} \otimes B_{IMG}$, β = Scale factor for resizing images, d represents all distortions applied, F_p = center_x and center_y, coordinates of face points, $\ominus$ is rotation angle of image, and α is opacity for blending image.

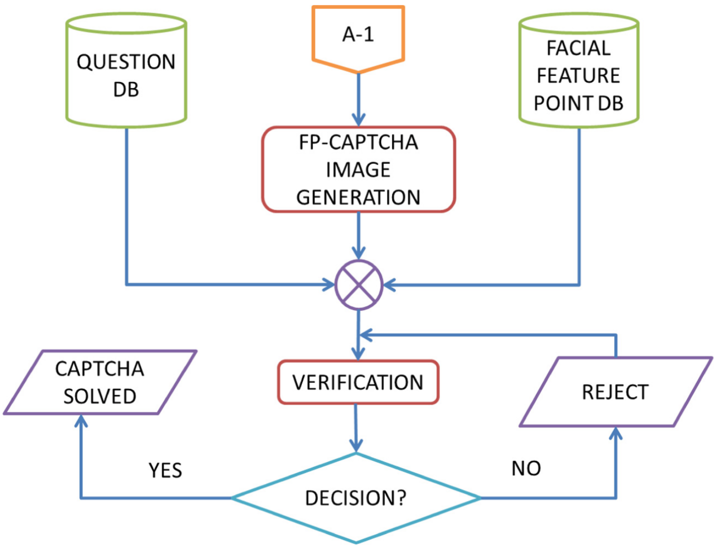

Fig. 2. Steps for generating the proposed FP-Captcha.

3.2 Background Image Processing

The proposed FP-Captcha intend to use for web application; better viewed on personal computer or tablet screen. A blank image of size 800 × 500 is generated, colored arbitrarily and randomly filled with various filled shapes of different colors, irrespective of overlapping. Next Salt-peeper, Gaussian and Speckle noise; with density parameters ranging from 0.02 to 0.09 added randomly to produce final background image of FP-Captcha. For our database we produced around fifty background images B_{IMG}, with different image processing parameters like luminescence, contrast and intensities. For our application we restrict the size of image to 800 × 500; since we arbitrarily inserted $C_{Total} = 8$ images; however size can be changed to any dimension, when total number of real and fake images R_{Total} will increased.

3.3 Real and Fake Image Processing

All the real images are collected from Aberdeen color Dataset and they are resized according to our applications. Height width of real images are chosen in such a way; so that they can undergo Cascade-Object-Detector successfully. The Cascade-Object-Detector employed with the Viola-Jones algorithm used to detect human faces, eyes, noses, mouth etc. All the information related to face points are stored in face point database (FP_{DB}). All fake images, namely some avatar, cartoon type pictures are collected from Internet and stored in face database F_{DB}. Next these images undergo with geometric transformations like rotation, followed by variable distortions and noises. To make our scheme more robust, some images chosen from R_{IMG} and F_{IMG} are rotated clockwise or anti-clock wise based on midpoint. For producing a single Captcha we used two or three real face images randomly from R_{IMG} and five to six fake images randomly from F_{IMG}. The different operations are applied on real and fake image processing are as follows.

- **Resize Operation:** All R_{IMG} and F_{IMG} images are collected, first resized and cropped according to our requirements. We used two standard methods imresize(I, scale) and imcrop(I, rect) to modify the raw image, where I is source image, scale is a resize factor(β), rect is 4-element position vector for size and position of the crop rectangle. The size of newly resized image; x_{width} varies from 170 to 178 pixels and y_{height} varies from 200 to 230 pixels.
- **Gamma Correction:** This is applied randomly on all R_{IMG} and F_{IMG}, to change the intensity of the images linearly, using standard method lin2rgb(I), where I is a linear RGB image. To adjust brightness and contrast, we used the method imadjust(I, [low_in high_in], [low_out high_out]), where these parameters are contrast limits for input and output images.

Algorithm 1. Image Resizing, Cropping and Quality Enhancing

Input: IMG, where IMG can be a real or a fake image, to be resized and enhance quality.

Output: $IMG^{‘}$**, where** $IMG^{‘}$ **is partially processed and enhanced image with removal of unnecessary portions of IMG and is in desired size, where x_width ranges from 170 to 178 pixels and y_width ranges from 200 to 230 pixels.**

procedure ImgRcq(IMG)
  for each IMG in R_{IMG} and F_{IMG} **do**
    $IMG^{‘}$ = resize($IMG^{‘}$, β) // β is a scale factor, $0 \leq \beta \leq 1$
    $IMG^{‘}$ = crop($IMG^{‘}$, rect) //rect is vector element [x,y,a,b], where (x,y) are the coordinates, and a,b represents length, breadth of cropped rectangle.
    $IMG^{‘}$ = gammargb($IMG^{‘}$) //gamma correction to increase intensity
    $IMG^{‘}$ = imgadjust($IMG^{‘}$) //improve brightness and contrast
  end for
  Return $IMG^{‘}$
end procedure

3.4 Cascade Object Detector

Only R_{IMG} images are undergo through a face cascade object detector to detect face feature points like eyes, nose, mouth etc. and store it to FP_{DB}. The Cascade Object Detector is based on Viola-Jones algorithm, fitted with several pre-trained classifiers for detecting frontal faces, profile faces, noses, eyes, and the upper body etc.

Algorithm 2. Real Image Processing; Face and Face Points Detection

Input: R_{IMG} **where** R_{IMG} **is real face image in which face and others face points needs to be detected.**

Output: center_x and center_y, calculated from bbox, where center_x and center_y are coordinates of face points. bbox is a boundary box of faceDetect.

procedure IMGFACEDETECT(R_{IMG})
  for each IMG in R_{IMG} **do**
    faceDetect = vision.CascadeObjectDetector(face-part); // face-parts are face, lips, nose, right eye, left eye etc. Viola-Jones algorithm is used for face detection
    bbox=step(faceDetect, R_{IMG});
    findcenter(bbox) returns center_x,center_y
    Return center_x,center_y // return face points
    $R^{`}_{IMG} = R_{IMG}$
    $R^{`}_{IMG}$ = imgEnhan($R^{`}_{IMG}$)
    return $R^{`}_{IMG}$
    update FP_{DB} with center_x,center_y
  end for
end procedure

Algorithm 3. Fake Image Processing

Input: F_{IMG} **where** F_{IMG} **is fake face image from** F_{DB}**.**

Output: $F^{`}_{IMG}$**, where** $F^{`}_{IMG}$ **are processed fake images stored in** F_{DB}

procedure IMGFAKE(F_{IMG})
  for each F_{IMG} in F_{DB} **do**
    $F^{`}_{IMG} = F_{IMG}$
    $F^{`}_{IMG}$ = imgEnhan($F^{`}_{IMG}$)
    $F^{`}_{IMG}$ = Distor($F^{`}_{IMG}$) // local distortion
    $F^{`}_{IMG}$ = Noise($F^{`}_{IMG}$) // local noise
    return $F^{`}_{IMG}$
    update F_{DB}
  end for
end procedure

Algorithm 4. Add distortion to real and fake image

Input: IMG, where IMG can be a real or a fake image, on which distortion to be added.

Output: $IMG^{`}$**, where** $IMG^{`}$ **is processed image with added distortions.**

procedure ADDDISTN(IMG)
  for each IMG in R_{IMG} and F_{IMG} **do**
    $IMG^{`}$ = rotate($IMG^{`}$,θ,crop) // where θ is a scale factor, $-45° < \theta < 45°$; crop make output image, same size as the input image
    $IMG^{`}$ = insertMarker($IMG^{`}$,z,c,s) // where z is any marker like star, square, plus etc.; c is the color to be filled; and s is size of marker.
    $IMG^{`}$ = insertShape($IMG^{`}$,x,l,c,o) // where x represents different shapes like polygon, circle etc; l is line-width; o is opacity ($0 \leq o \leq 1$,default=0.6)
    $IMG^{`}$ = insertNoise($IMG^{`}$,N,γ) //where N is random noise like Salt & Pepper, Gaussian, Speckle etc., with intensity $0.01 \leq \gamma \leq 0.06$.
  end for
  Return $IMG^{`}$
end procedure

Algorithm 5. Processing of Background Images

Input: void

Output: B_{IMG} **Background images stored in** B_{DB}

procedure IMGBACKGR(void)
  for $k \leftarrow 1$ to N **do** // N is number of background image required.
    B_{IMG} = zeros(800,500) //generate blank image of size 800×500
    B_{IMG} = addRandCol(color) //apply random color to background
    for $k \leftarrow 1$ to 100 **do** B_{IMG} = insertShape(x) // where x are random shapes like filled rectangle, filled polygon etc.
      B_{IMG} = Noise(B_{IMG}) // add random noise
    end for
    return B_{IMG}
    update B_{DB}.
  end for
end procedure

Viola and Jones described a fast and robust method for face detection [14], which is much faster than any other algorithm at that time. Later [15] they improved their classifier, built from computationally efficient features using AdaBoost for features selection; which radically reduces computation time while improving detection accuracy. Human faces are mostly similar regarding some feature points (all humans have eyes, nose, mouth etc.), leads to a concept Haar features to detect face from an shown in Fig. 3.

Algorithm 6. FP-Captcha Creation

Input: R'_{IMG}**, from** R_{DB}**;** F'_{IMG} **from** F_{DB}**, and** B_{IMG} **from** B_{DB} **are collected.**

Output: C_{IMG}**; where** C_{IMG} **is final FP-Captcha.**

```
procedure CREATECAPTCHA(R'_IMG, F'_IMG, B_IMG)
    for each B_IMG in B_DB do
        B_IMG is horizontally separated into two rows row_1 and row_2.
        for R'_IMG in R_DB and F'_IMG in F_DB do
            blendRandPic(R_IMG || F_IMG, x_1, y_1, row_1, α, n) // blend real or fake
              image in row_1 randomly at coordinate (x_1, y_1) respectively. α is blending opacity,
              0 ≤ α ≤ 1; n is total number of real and fake images, n ≤ 4.
            blendRandPic(R_IMG || F_IMG, x_2, y_2, row_2, α) // blend real or fake image
              in row_2 randomly at coordinate (x_2, y_2) respectively.
        end for
        return C_IMG
    end for
end procedure
```

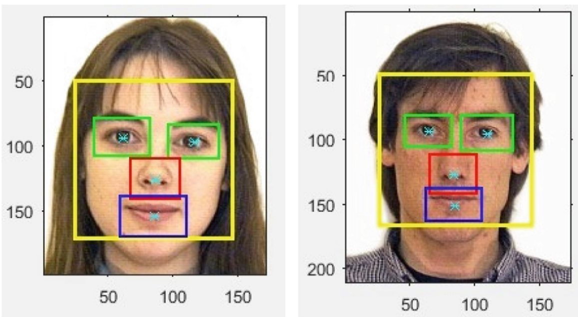

Fig. 3. Detection of face boundary box and other face feature points, denoted by $\star$.

3.5 Distortions and Noise

After first phase of processing, all R_{IMG} and F_{IMG} images, are undergo some random distortions and noises. Without distortion and noise, faces in Captcha can be easily detected by automatic programs and made Captcha vulnerable. We designed three types of distortions stated in the Table 1.

After detection of face in Sect. 3.4, R_{IMG} and F_{IMG} images are rotated randomly $0°$ to $\pm 45°$ around the center axis. The portion of images outside of the original dimension is cropped and all black pixels are replaced by pure white pixels. To find the new position of $[x', y']$ in rotated image from position $[\mathrm{x}, \mathrm{y}]$ in original image, the following operations are performed. First, center is translated from $[0, 0]$ to center of image; rotation performed by rotation matrix; again made a inverse translation of the pixels. The coordinates of new face

Table 1. Distortion Types

Distortion type	Section	Applied	Intensity
Rotation	Geometric	Locally	$0°$ to $\pm 45°$ randomly
Gamma correction	Degradation	Locally	0.2–0.5 randomly
Brightness and contrast	Degradation	Locally	0.4–0.7 randomly
Aperiodic noise	Noise	Locally	Randomly
Gaussian noise	Noise	Globally and locally	0.01–0.09 randomly
Salt & peeper	Noise	Globally and locally	0.01–0.09 randomly
Speckle	Noise	Globally and locally	0.01–0.09 randomly

points are updated in FP_{DB} synchronized with relevant Captcha questions. The geometric transformation matrices used in rotations are as follows.

$$\textit{Rotation:} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2}$$

$$\textit{Translation:} \quad x' = x + t_x \tag{3}$$

$$y' = y + t_y \tag{4}$$

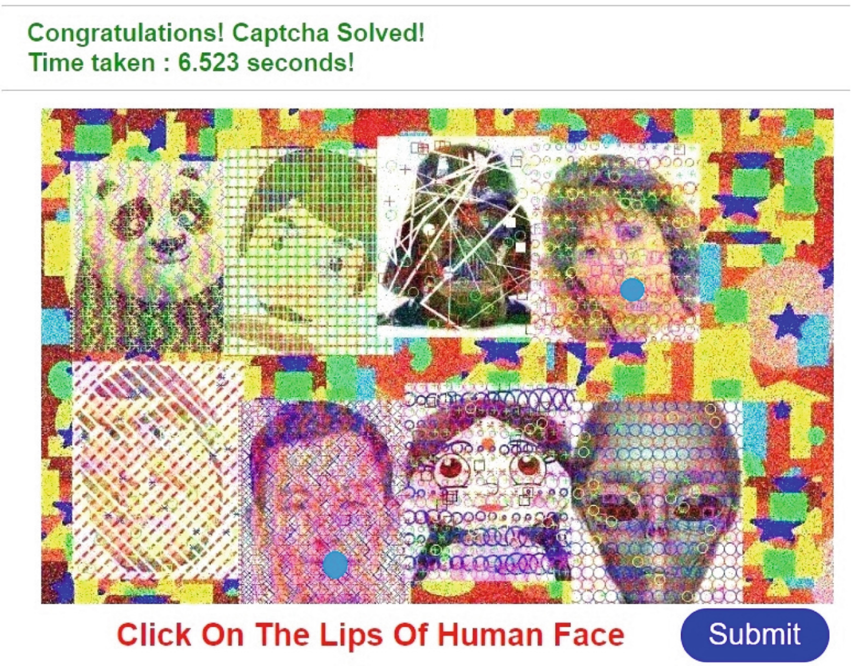

Fig. 4. FP-Captcha with two real faces.

To implement FP-Captcha, we designed eight different aperiodic patterns and added randomly to R_{IMG} and F_{IMG}. The patterns are created in such a way every images belongs to a Captcha will be different, followed by very small amount (0.01 to 0.09) of Gaussian noise, Salt Pepper noise and Speckle noise added locally and stored as R_{IMG} and F_{IMG}. The patterns are created by inserting lines, different shapes like circle, rectangle, triangle and different types of markers. Due to randomness, few R_{IMG} and F_{IMG} images are degraded and distorted very poorly are discarded.

3.6 Image Selection and Placement

After processing of R_{IMG} and F_{IMG} images, the newly created real images are denoted by $R^{'}_{IMG}$ and $F^{'}_{IMG}$ stored in respective R_{DB} and F_{DB}. From Sect. 3.2 newly created background images B_{IMG} are divided into two rows; where each row can contain four images. Now $R^{'}/IMG$ and $F^{'}/IMG$ images are placed randomly among eight positions on B_{IMG}. The images are placed using image selection algorithm, in such a way, by each row there are four images, and their boundary can overlap each other to some extent. Every $R^{'}/IMG$ and $F^{'}/IMG$ images are embedded on B_{IMG} by Alpha Blending operation, with a variable opacity (0.6 to 0.9). To determine the face points of $R^{'}/IMG$, we used a translation matrix and update FP_{DB}. After images are placed, again Gaussian noise, Salt Pep-per noise and Speckle noise are added with maximum intensity of 0.03; finally FP-Captcha was produced and shown in Fig. 4.

4 Experimental Result and Analysis

The proposed FP-Captcha has a dimension of 800 $\times$ 500, however it can be of any size. In this section, we described the verification process, evaluation scheme by human users, accuracy of proposed method, response time of users and attack analysis. Each Captcha background B_{img}, embedded with $R^{'}_{IMG}$ and $F^{'}_{IMG}$ images, correlated with four questions and stored in FP_{DB}. The FP_{DB} also contains all face feature points related to each and every R_{IMG}. The questions are generated, which randomly ask user to click either on right eye, left eye, nose or lips of all real human faces present on that Captcha. The Captcha and questions are synchronized, such that every time a user will asked to solve a new challenge. The user will asked with questions on random basis and to click on desired real face points; if they clicked properly, coordinates of that point is verified with coordinate stored in FP_{DB}. If it matches, Captcha is solved, otherwise user asked for a new challenge.

The proposed FP-Captcha has been evaluated with 115 UG students volunteers of our Institution, and performance is compared with other recent image Captcha techniques. We created eighty FP-Captcha's with different random combinations of $F^{'}_{IMG}$ and two or three $R^{'}_{IMG}$ images, equipped with different patterns, distortions and noises. Among them forty FP-Captcha's are selected for testing purpose and each volunteer asked to solve FP-Captcha twice, where first

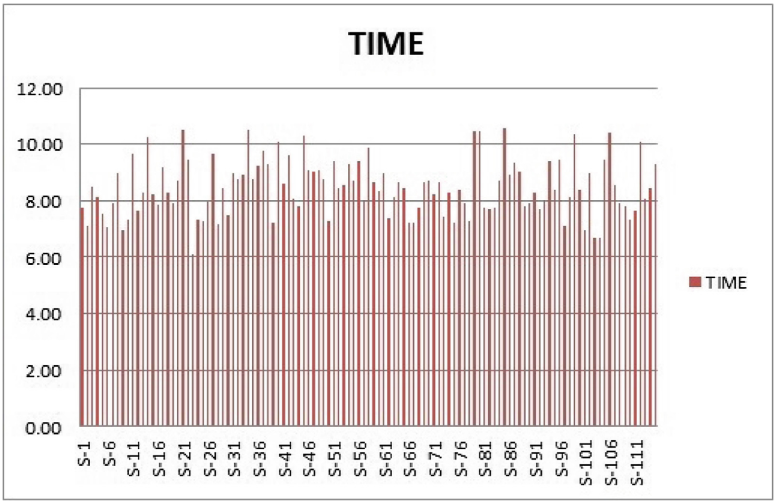

Fig. 5. 115 Student users responses to solve FP-Captcha with two real faces.

FP-Captcha contains two real face images and next FP-Captcha contains three face images. All the responses are collected, stored and overall accuracy was analyzed. We compute average accuracy of user response with the following Eq. 5.

$$Accuracy = \frac{CorrectResponses}{TotalNumberofResponses} \times 100 \tag{5}$$

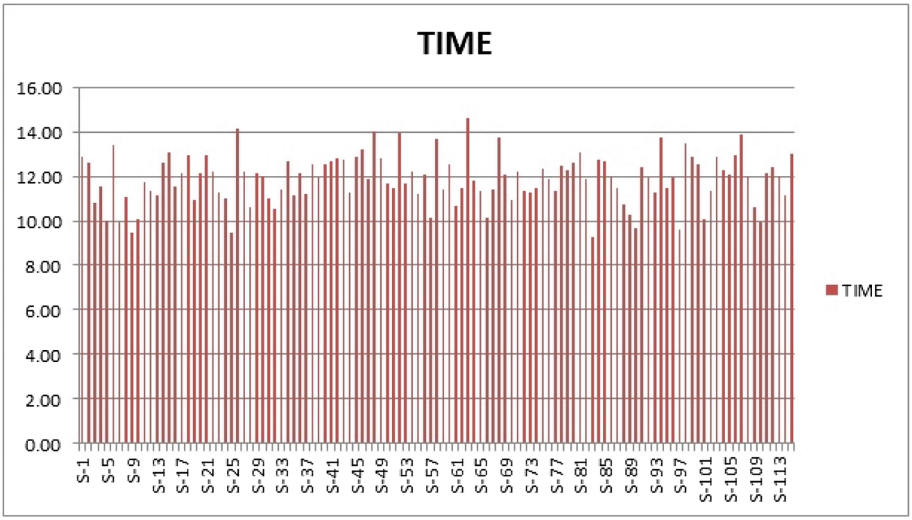

Fig. 6. 115 Student users responses to solve FP-Captcha with three real faces.

The average accuracy is calculated 98.26%, when there are only two real face images in the FP-Captcha; whereas in presence of three real face images the accuracy is 96.52%. Therefore we conclude that, the accuracy of FP-Captcha is 97.39% on average. The average time taken to solve FP-Captcha with two real images is 8.46 s (Fig. 5), whereas average time taken to solve with three images is 11.86 s. Therefore we conclude that, response time to solve FP-Captcha is 10.06 s (Fig. 6) on average (Table 2).

Table 2. Usability study results of FP-Captcha

	With two $R^{'}_{IMG}$	With three $R^{'}_{IMG}$
Solving rate	98.26%	96.52%
Average response time	8.46 s	11.86 s

We also executed various face detection test on the generated FP-Captcha images and $R^{'}_{IMG}$ images, via the Cascade detector in Matlab, Google's Cloud Vision API, Kairos Face Recognition and Microsoft Azure face detection algorithm. These recent algorithms are not able to detect the faces from FP-Captcha's, even a single face from $R^{'}_{IMG}$ images are not detected, which ensure robustness of our FP-Captcha scheme. Moreover to establish the robustness of FP-Captcha we represent probability analysis [6] of the random guessing attack, which attempts to break the FP-Captcha. To solve the Captcha, it will ask to click twice or thrice on face points, and all the clicks must be correct. In FP-Captcha, there are 800 $\times$ 500 = 400,000 pixels, and there are two cases for correct responses (for random guess, tolerance of mouse click 25 $\times$ 25 pixels). The probability for break the FP-Captcha using a random guess is calculated as follows.

- **Case: 1:** Two real face images $R^{'}_{IMG}$ in Captcha.

$$\frac{2\times(25\times25)}{400000}\times\frac{(25\times25))}{(400000-25\times25)} = 0.00000488$$

- **Case: 2:** Three real face images $R^{'}_{IMG}$ in Captcha

$$\frac{3\times(25\times25)}{400000}\times\frac{2\times(25\times25)}{(400000-2\times25\times25)}\times\frac{25\times25}{(400000-25\times25)} = 0.0000000229$$

Therefore, we conclude that, the probability of breaking the FP-Captcha with a random guess attack is 0.00000245 on average.

From Table 3, we compare our FP-Captcha scheme with other Captcha Schemes. The accuracy of our FP-Captcha is 97.39% on average, which is relatively good than others; whereas response time is much better and significantly upright than others. Also probability of random guessing attack is significantly low than others schemes; leads to our FP-Captcha more robust and usable.

Table 3. Performance analysis of various Captcha schemes

Ref. Captcha's	Accuracy	Response time	Prob. of rand. guess. attack
ARTiFACIAL [8]	99.7%	14 s	0.2
Assira [3]	83%	15 s	0.5
IMAGINATION [2]	85%	-	0.000062
CONSCHEME [18]	85.07%	15.12 s	0.0769
DeRection [18]	86.43%	12.77 s	0.0046
FaceDCAPTCHA [6]	92.47%	-	0.00237
FR-CAPTCHA [7]	94%	-	0.0069
FP-Captcha	97.39%	10.16 s	0.00000251

5 Conclusion

In this paper, we have studied and compared different types of image Captcha's, and implemented FP-Captcha, which is effective, efficient for preventing automated attacks, and human can successfully solve the challenge with an average response time 10.16 s. Our proposed FP-Captcha can be used in any platform and successfully keep computers and human apart. Moreover to improve security we used different pattern distortions for each image embedded in FP-Captcha. We are also trying to improve our distortion methods and blending method, to produce more significant challenge. Further improvement can be possible if we remove the rectangle background of face image or made that background portion transparent.

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Decision Making Under Uncertainty via Generalized Parabolic Intuitionistic Fuzzy Numbers

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Abstract. In this paper, generalized parabolic intuitionistic fuzzy number has been studied. The aim of this paper is to present arithmetic operations of generalized parabolic intuitionistic fuzzy numbers by using $(\alpha, \beta)-$cut method. In spite of these, exponential, logarithmic and n^{th} root of generalized parabolic intuitionistic fuzzy numbers have been done followed by numerical examples. Furthermore, rank of generalized parabolic intuitionistic fuzzy numbers has been evaluated based on mean and value. Finally, a decision making problem has been discussed to show its existence and applicability in real life.

Keywords: Generalized parabolic intuitionistic fuzzy number · Ranking · Decision making

1 Introduction

The classical decision making is a process having single or a group of experts usually assume that all the criteria and their weight of criteria are in terms of crisp value and ranking the alternatives without any difficulty. In most of the real life situations, the classical decision making methods fail, due to the presence of uncertainty. To reduce such uncertainty Zadeh [1] introduced fuzzy set theory (FST) as an extension of classical set theory by adding membership function (MF). It can easily deal with uncertainty in decision making problems. There are several methods based on FST have been developed and successfully applied in decision sciences. But there are some situations where FST is not sufficient. To reduce such uncertainty Atanassov [2] generalized FST to intuitionistic fuzzy set theory (IFST) by adding both MF and non-membership function (NMF). It deals with vague or imprecise data more appropriately as compared to FST. At present, intuitionistic fuzzy sets (IFSs) are being studied and used in various fields of science and technology. There are several intuitionistic fuzzy numbers (IFNs) have been developed and applied in decision science. Usually, generalized and normal triangular, trapezoidal intuitionistic fuzzy numbers are used to reduce uncertainty in decision making problems.

In literature, Atanassov et al. [3], Li et al. [4], Liu and Wang [5], Li et al. [6], Shaw and Roy [7], Shaw and Roy [8], Ban and Tuse [9], Zeng et al. [10],

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Chen et al. [11] etc. have developed several intuitionistic fuzzy decision making methods. All existing methods have been successfully applied in various decision making problems related to our real life. Recently Garg and Rani [12] introduced some generalized weighted averaging aggregation operators for aggregating the several complex intuitionistic fuzzy sets by using t-norm operations and proposed a multi-criteria decision making method based on these operators. Jose [13] presented a decision making model in multiple person decision making problems under intuitionistic fuzzy environment. Joshi and Kumar [14] described accuracy function for Interval-Valued IFSs and applied to multiple attributes group decision making problem. Fan et al. [15] proposed intuitionistic fuzzy weighted arithmetic averaging operator and intuitionistic fuzzy weighted geometric averaging operator. These operators are used to aggregate intuitionistic fuzzy information. Based on these operators, the author proposed a intuitionistic fuzzy multi attribute decision making method. Based on nature of complex situations of real life problems, there are lots of IFNs have been developed. In this paper, generalized parabolic type of IFN has been introduced and applied in decision making problems.

The structure of this work is as follows: Sect. 2 provides some basic definitions related to generalized parabolic intuitionistic fuzzy numbers (GPIFNs). Section 3 presents arithmetic operations of GPIFNs followed by numerical examples. Section 4 describes ranking of GPIFNs. Section 5 describes a decision making problem by using proposed ranking approach of GPIFNs and finally conclusion of this paper is illustrated.

2 Preliminaries

In this section some basic definitions of GPIFNs are discussed.

Definition 1. *GPIFN*
$A =< [a_1, b_1, c_1, w_1], [a_1^{'}, b_1, c_1^{'}; \eta_1] >$ *be GPFN whose MF* $\mu_A(x)$ *and NMF* $\nu_A(x)$ *are*

$$\mu_A(x) = \begin{cases} w_1(\frac{x-a_1}{b_1-a_1})^2; & x \in [a_1, b_1] \\ w_1; & x = b_1 \\ w_1(\frac{c_1-x}{c_1-b_1})^2, & x \in [b_1, c_1] \end{cases}$$

$$\nu_A(x) = \begin{cases} \{\frac{(b_1-x)+(x-a_1^{'})\sqrt{\eta_1}}{b_1-a_1^{'}}\}^2; & x \in [a_1^{'}, b_1] \\ \eta_1; & x = b_1 \\ \{\frac{(x-b_1)+(c_1^{'}-x)\sqrt{\eta_1}}{c_1^{'}-b_1}\}^2; & x \in [b_1, c_1^{'}] \end{cases}$$

Definition 2. $(\alpha, \beta)-$*cuts of a GPIFN*
The $(\alpha, \beta)-$*cut of* $A =< [a_1, b_1, c_1, w_1], [a_1^{'}, b_1, c_1^{'}; \eta_1] >$ *is defined by*

$$^{\alpha}A = \Big[a_1 + (b_1 - a_1)\sqrt{\tfrac{\alpha}{w_1}}, c_1 - (c_1 - b_1)\sqrt{\tfrac{\alpha}{w_1}}\Big],$$
$$^{\beta}A = \Big[\frac{b_1 - a_1^{'}\sqrt{\eta_1} - (b_1 - a_1^{'})\sqrt{\beta}}{1 - \sqrt{\eta_1}}, \frac{b_1 - c_1^{'}\sqrt{\eta_1} - (c_1^{'} - b_1)\sqrt{\beta}}{1 - \sqrt{\eta_1}}\Big]$$

3 Arithmetic Operations of GPIFNs

In this section discussed about arithmetic operations of GPIFNs by using $(\alpha, \beta)-$cuts method [16].

Let $A =< [a_1, b_1, c_1, w_1], [a_1^{'}, b_1, c_1^{'}; \eta_1] >$ and $B =< [a_2, b_2, c_2, w_2], [a_2^{'}, b_2, c_2^{'}; \eta_2] >$ be any two GPIFNs whose respective MF and NMF are

$$\mu_A(x) = \begin{cases} w_1(\frac{x-a_1}{b_1-a_1})^2; & x \in [a_1, b_1] \\ w_1; & x = b_1 \\ w_1(\frac{c_1-x}{c_1-b_1})^2, & x \in [b_1, c_1] \end{cases}$$

$$\nu_A(x) = \begin{cases} \{\frac{(b_1-x)+(x-a_1^{'})\sqrt{\eta_1}}{b_1-a_1^{'}}\}^2; & x \in [a_1^{'}, b_1] \\ \eta_1; & x = b_1 \\ \{\frac{(x-b_1)+(c_1^{'}-x)\sqrt{\eta_1}}{c_1^{'}-b_1}\}^2; & x \in [b_1, c_1^{'}] \end{cases}$$

and

$$\mu_B(x) = \begin{cases} w_2(\frac{x-a_2}{b_2-a_2})^2; & x \in [a_2, b_2] \\ w_2; & x = b_2 \\ w_2(\frac{c_2-x}{c_2-b_2})^2, & x \in [b_2, c_2] \end{cases}$$

$$\nu_B(x) = \begin{cases} \{\frac{(b_2-x)+(x-a_2^{'})\sqrt{\eta_2}}{b_2-a_2^{'}}\}^2; & x \in [a_2^{'}, b_2] \\ \eta_2; & x = b_2 \\ \{\frac{(x-b_2)+(c_2^{'}-x)\sqrt{\eta_2}}{c_2^{'}-b_2}\}^2; & x \in [b_2, c_2^{'}] \end{cases}$$

Where, $w_1 = min(w_1, w_2)$, $\eta_2 = max(\eta_1, \eta_2)$ and $\alpha \in [0, w_1], \beta \in [\eta_2, 1]$.

Then α-cuts and β-cuts of GPIFNs A and B are denoted by ${}^{\alpha}A$, ${}^{\alpha}B$, ${}^{\beta}A$ and ${}^{\beta}B$ given by

$$\begin{aligned} {}^{\alpha}A &= \Big[a_1 + (b_1 - a_1)\sqrt{\frac{\alpha}{w_1}}, c_1 - (c_1 - b_1)\sqrt{\frac{\alpha}{w_1}}\Big], \\ {}^{\beta}A &= \Big[\frac{b_1 - a_1^{'}\sqrt{\eta_1} - (b_1 - a_1^{'})\sqrt{\beta}}{1 - \sqrt{\eta_1}}, \frac{b_1 - c_1^{'}\sqrt{\eta_1} - (c_1^{'} - b_1)\sqrt{\beta}}{1 - \sqrt{\eta_1}}\Big] \\ {}^{\alpha}B &= \Big[a_2 + (b_2 - a_2)\sqrt{\frac{\alpha}{w_2}}, c_2 - (c_2 - b_2)\sqrt{\frac{\alpha}{w_2}}\Big], \\ {}^{\beta}B &= \Big[\frac{b_2 - a_2^{'}\sqrt{\eta_2} - (b_2 - a_2^{'})\sqrt{\beta}}{1 - \sqrt{\eta_2}}, \frac{b_2 - c_2^{'}\sqrt{\eta_2} - (c_2^{'} - b_2)\sqrt{\beta}}{1 - \sqrt{\eta_2}}\Big] \end{aligned}$$

3.1 Addition of GPIFNs

To calculate addition of two GPIFNs A and B, first we add (α, β)-cuts of A and B by using interval arithmetic as given below

$${}^{\alpha}A + {}^{\alpha}B = \Big[a_1 + a_2 + (b_1 - a_1 + b_2 - a_2)\sqrt{\frac{\alpha}{w_1}}, c_1 + c_2 - (c_1 - b_1 + c_2 - b_2)\sqrt{\frac{\alpha}{w_1}}\Big] \quad (1)$$

$$
{}^{\beta}A + {}^{\beta}B = \Big[\frac{b_1 - a_1^{'}\sqrt{\eta_2} - (b_1 - a_1^{'})\sqrt{\beta} + b_2 - a_2^{'}\sqrt{\eta_2} - (b_2 - a_2^{'})\sqrt{\beta}}{1-\sqrt{\eta_2}}, \\ \frac{b_1 - c_1^{'}\sqrt{\eta_2} + (c_1^{'} - b_1)\sqrt{\beta} + b_2 - c_2^{'}\sqrt{\eta_2} - (c_2^{'} - b_2)\sqrt{\beta}}{1-\sqrt{\eta_2}}\Big] \tag{2}
$$

To find MF and NMF, we first equate to x both 1st and 2nd components of (1) and (2),

$$
x = a_1 + a_2 + (b_1 - a_1 + b_2 - a_2)\sqrt{\frac{\alpha}{w_1}}, \quad x = \frac{b_1 - a_1^{'}\sqrt{\eta_2} - (b_1 - a_1^{'})\sqrt{\beta} + b_2 - a_2^{'}\sqrt{\eta_2} - (b_2 - a_2^{'})\sqrt{\beta}}{1-\sqrt{\eta_2}}
$$

$$
x = c_1 + c_2 - (c_1 - b_1 + c_2 - b_2)\sqrt{\frac{\alpha}{w_1}}, \quad x = \frac{b_1 - c_1^{'}\sqrt{\eta_2} + (c_1^{'} - b_1)\sqrt{\beta} + b_2 - c_2^{'}\sqrt{\eta_2} - (c_2^{'} - b_2)\sqrt{\beta}}{1-\sqrt{\eta_2}}
$$

Now, expressing α, β in terms of x and setting $\alpha \geq 0, \alpha \leq w_1$ and $\alpha \leq w_1, \alpha \geq 0$ and $\beta \leq 1, \beta \geq \eta_2$ and $\beta \geq \eta_2, \beta \leq 1$ in (1) and (2) then we have,

$$
\alpha = w_1\Big\{\frac{x - (a_1 + a_2)}{(b_1 + b_2) - (a_1 + a_2)}\Big\}^2; \quad x \in [a_1 + a_2, b_1 + b_2]
$$

$$
\alpha = w_1\Big\{\frac{(c_1 + c_2) - x}{(c_1 + c_2) - (b_1 + b_2)}\Big\}^2; \quad x \in [b_1 + b_2, c_1 + c_2]
$$

And

$$
\beta = \Big\{\frac{b_1 + b_2 - (a_1^{'} + a_2^{'})\sqrt{\eta_2} - x(1-\sqrt{\eta_2})}{(b_1 + b_2) - (a_1^{'} + a_2^{'})}\Big\}^2; \quad x \in [a_1^{'} + a_2^{'}, b_1 + b_2]
$$

$$
\beta = \Big\{\frac{x(1-\sqrt{\eta_2}) - (b_1 + b_2) + (c_1^{'} + c_2^{'})\sqrt{\eta_2}}{(c_1^{'} + c_2^{'}) - (b_1 + b_2)}\Big\}^2; \quad x \in [b_1 + b_2, c_1^{'} + c_2^{'}]
$$

Therefore, the required MF $\mu_{A+B}(x)$ and NMF $\nu_{A+B}(x)$ are

$$
\mu_{A+B}(x) = \begin{cases} w_1\{\frac{x-(a_1+a_2)}{(b_1+b_2)-(a_1+a_2)}\}^2, & x \in [a_1 + a_2, b_1 + b_2]; \\ w_1, & x = b_1 + b_2; \\ w_1\{\frac{(c_1+c_2)-x}{(c_1+c_2)-(b_1+b_2)}\}^2, & x \in [b_1 + b_2, c_1 + c_2]. \end{cases}
$$

and

$$
\nu_{A+B}(x) = \begin{cases} \{\frac{(b_1+b_2)-x(1-\sqrt{\eta_2})-(a_1^{'}+a_2^{'})\sqrt{\eta_2}}{(b_1^{'}+b_2^{'})-(a_1^{'}+a_2^{'})}\}^2, & x \in [a_1^{'} + a_2^{'}, b_1 + b_2]; \\ \eta_2, & x = b_1 + b_2; \\ \{\frac{x(1-\sqrt{\eta_2})-(b_1+b_2)+(c_1^{'}+c_2^{'})\sqrt{\eta_2}}{(c_1^{'}+c_2^{'})-(b_1+b_2)}\}^2, & x \in [b_1 + b_2, c_1^{'} + c_2^{'}]. \end{cases}
$$

3.2 Subtraction of GPIFNs

Let A and B be two GPIFNs, then the subtraction of A and B denoted by $A - B$ whose respective MF and NMF are as follows.

$$\mu_{A-B}(x)=\begin{cases} w_1\{\frac{x-(a_1-c_2)}{(b_1-b_2)-(a_1-c_2)}\}^2, & x\in[a_1-c_2,b_1-b_2];\\ w_1, & x=b_1-b_2;\\ w_1\{\frac{(c_1-a_2)-x}{(c_1-a_2)-(b_1-b_2)}\}^2, & x\in[b_1-b_2,c_1-a_2]. \end{cases}$$

and

$$\nu_{A-B}(x)=\begin{cases} \{\frac{b_1-b_2-(a_1^{'}-c_2^{'})\sqrt{\eta_2}-x(1-\sqrt{\eta_2})}{(b_1-b_2)-(a_1^{'}-c_2^{'})}\}^2, & x\in[a_1^{'}-c_2^{'},b_1-b_2];\\ \eta_2, & x=b_1-b_2;\\ \{\frac{x(1-\sqrt{\eta_2})-(b_1-b_2)+(c_1^{'}-a_2^{'})\sqrt{\eta_2}}{(c_1^{'}-a_2^{'})-(b_1-b_2)}\}^2, & x\in[b_1-b_2,c_1^{'}-a_2^{'},]. \end{cases}$$

3.3 Division of GPIFNs

Let A and B be two GPIFNs, then the division of A and B denoted by $\frac{A}{B}$ whose respective MF and NMF are as follows.

$$\mu_{\frac{A}{B}}(x)=\begin{cases} w_1\{\frac{xc_2-a_1}{(b_1-a_1)+(c_2-b_2)x}\}^2, & x\in[\frac{a_1}{c_2},\frac{b_1}{b_2}];\\ w_1, & x=\frac{b_1}{b_2};\\ w_1\{\frac{c_1-xa_2}{(c_1-b_1)+(b_2-a_2)x}\}^2, & x\in[\frac{b_1}{b_2},\frac{c_1}{a_2}]. \end{cases}$$

and

$$\nu_{\frac{A}{B}}(x)=\begin{cases} \{\frac{b_1-xb_2+(xc_2^{'}-a_1)\sqrt{\eta_2}}{(c_2^{'}-b_2)x+(b_1-a_1^{'})}\}^2, & x\in[\frac{a_1^{'}}{c_2^{'}},\frac{b_1}{b_2}];\\ \eta_2, & x=\frac{b_1}{b_2};\\ \{\frac{xb_2-b_1+(c_1^{'}-xa_2)\sqrt{\eta_2}}{(b_2-a_2^{'})x+(c_1^{'}-b_1)}\}^2, & x\in[\frac{b_1}{b_2},\frac{c_1^{'}}{a_2^{'}}]. \end{cases}$$

3.4 n^{th} root of GPIFN

Let A be a GPIFN, then n^{th} root of A denoted by $A^{\frac{1}{n}}$ whose respective MF and NMF are as follows.

$$\mu_{A^{\frac{1}{n}}}(x)=\begin{cases} w_1(\frac{x^n-a_1}{b_1-a_1})^2, & x\in[a_1^{\frac{1}{n}},b_1^{\frac{1}{n}}]\\ w_1, & x=b_1^{\frac{1}{n}}\\ w_1(\frac{c_1-x^n}{c_1-b_1})^2, & x\in[b_1^{\frac{1}{n}},c_1^{\frac{1}{n}}]. \end{cases}$$

and

$$\nu_{A^{\frac{1}{n}}}(x)=\begin{cases} \{\frac{b_1-a_1^{'}\sqrt{\eta_1}-x^n(1-\sqrt{\eta_1})}{b_1-a_1^{'}}\}^2, & x\in[a_1^{'\frac{1}{n}},b_1^{\frac{1}{n}}]\\ \eta_1, & x=b_1^{\frac{1}{n}}\\ \{\frac{x^n(1-\sqrt{\eta_1})-b_1+c_1^{'}\sqrt{\eta_1}}{c_1^{'}-b_1}\}^2, & x\in[b_1^{\frac{1}{n}},c_1^{'\frac{1}{n}}]. \end{cases}$$

3.5 Exponential of GPIFN

Let A be a GPIFN, then exponential of A denoted by e^A whose respective MF and NMF are as follows.

$$\mu_{e^A}(x) = \begin{cases} w_1\{\frac{\ln(x)-a_1}{b_1-a_1}\}^2, & x \in [e^{a_1}, e^{b_1}] \\ w_1, & x = e^{b_1} \\ w_1\{\frac{c_1-\ln(x)}{c_1-b_1}\}^2; x \in [e^{b_1}, & x \in [e^{b_1}, e^{c_1}]. \end{cases}$$

and

$$\nu_{e^A}(x) = \begin{cases} \{\frac{b_1-a_1^{'}\sqrt{\eta_1}-(1-\sqrt{\eta_1})\ln(x)}{b_1-a_1^{'}}\}^2, & x \in [e^{a_1^{'}}, e^{b_1}] \\ \eta_1, & x = e^{b_1} \\ \{\frac{(1-\sqrt{\eta_1})\ln(x)-b_1+c_1^{'}\sqrt{\eta_1}}{c_1^{'}-b_1}\}^2, & x \in [e^{b_1}, e^{c_1^{'}}]. \end{cases}$$

3.6 Logarithm of GPIFN

Let A be a GPIFN, then logarithm of A denoted by lnA whose respective MF and NMF are as follows.

$$\mu_{\ln(A)}(x) = \begin{cases} w_1\{\frac{e^x-a_1}{b_1-a_1}\}^2, & x \in [\ln a_1, \ln b_1] \\ w_1, & x = \ln b_1 \\ w_1\{\frac{c_1-e^x}{c_1-b_1}\}^2, & x \in [\ln b_1, \ln c_1]. \end{cases}$$

and

$$\nu_{\ln(A)}(x) = \begin{cases} \{\frac{b_1-a_1^{'}\sqrt{\eta_1}-(1-\sqrt{\eta_1})e^x}{b_1-a_1^{'}}\}^2, & x \in [\ln a_1^{'}, \ln b_1] \\ \eta_1, & x = \ln b_1 \\ \{\frac{(1-\sqrt{\eta_1})e^x-b_1+c_1^{'}\sqrt{\eta_1}}{c_1^{'}-b_1}\}^2, & x \in [\ln b_1, \ln c_1^{'}]. \end{cases}$$

3.7 Numerical examples

Suppose $A =< [2, 3, 4; 0.2], [1, 3, 5; 0.4] >$ and $B =< [4, 6, 8; 0.5], [3, 6, 9; 0.8] >$ be two GPIFNs whose MFs and NMFs are

$$\mu_A(x) = \begin{cases} 0.2(x-2)^2, & x \in [2, 3] \\ 0.2, & x = 3 \\ 0.2(4-x)^2, & x \in [3, 4]. \end{cases}$$

$$\nu_A(x) = \begin{cases} \{\frac{(3-x)+(x-1)\sqrt{0.4}}{2}\}^2, & x \in [1, 3] \\ 0.4, & x = 3; \\ \{\frac{(x-3)+(5-x)\sqrt{0.4}}{2}\}^2, & x \in [3, 5]. \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0.5(\frac{x-4}{2})^2, & x \in [4, 6] \\ 0.5, & x = 6 \\ 0.5(\frac{8-x}{2})^2, & x \in [6, 8]. \end{cases}$$

$$\nu_B(x) = \begin{cases} \{\frac{(6-x)+(x-3)\sqrt{0.8}}{3}\}^2, & x \in [3,6] \\ 0.8, & x = 3; \\ \{\frac{(x-6)+(9-x)\sqrt{0.8}}{3}\}^2, & x \in [6,9]. \end{cases}$$

(α,β)-cuts of P and Q as given below

$^{\alpha}A = [2+\sqrt{\frac{\alpha}{0.2}},\ 4-\sqrt{\frac{\alpha}{0.2}}]$, $\quad$ $^{\beta}P = [\frac{3-\sqrt{0.4}-2\sqrt{\beta}}{1-\sqrt{0.4}},\ \frac{3-5\sqrt{0.4}+2\sqrt{\beta}}{1-\sqrt{0.4}}]$

$^{\alpha}B = [4+2\sqrt{\frac{\alpha}{0.5}},\ 8-2\sqrt{\frac{\alpha}{0.5}}]$, $\quad$ $^{\beta}Q = [\frac{6-3\sqrt{0.8}-3\sqrt{\beta}}{1-\sqrt{0.8}},\ \frac{6-9\sqrt{0.8}+3\sqrt{\beta}}{1-\sqrt{0.8}}]$

Then addition, subtraction, division, n^{th} root, exponential and logarithm of GPIFNs are defined as given below

1) Addition of A and B

$$\mu_{A+B}(x) = \begin{cases} 0.2(\frac{x-6}{3})^2, & x \in [6,9]; \\ 0.2, & x = 9; \\ 0.2(\frac{12-x}{3})^2, & x \in [9,12]. \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \{\frac{9-4\sqrt{0.8}-x(1-\sqrt{0.8})}{5}\}^2, & x \in [4,9]; \\ 0.8, & x = 9; \\ \{\frac{x(1-\sqrt{0.8})-9+14\sqrt{0.8}}{5}\}^2, & x \in [9,14]. \end{cases}$$

2) Subtraction of A and B

$$\mu_{A-B}(x) = \begin{cases} 0.2(\frac{x+6}{3})^2, & x \in [-6,-3]; \\ 0.2, & x = -3; \\ 0.2(\frac{x}{-3})^2, & x \in [-3,0]. \end{cases}$$

$$\nu_{A-B}(x) = \begin{cases} \{\frac{8\sqrt{0.8}-3-x(1-\sqrt{0.8})}{5}\}^2, & x \in [-8,-3]; \\ 0.8, & x = -3; \\ \{\frac{3+2\sqrt{0.8}+x(1-\sqrt{0.8})}{5}\}^2, & x \in [-3,2]. \end{cases}$$

3) Division of A and B

$$\mu_{\frac{A}{B}}(x) = \begin{cases} 0.2(\frac{8x-2}{2x+1})^2, & x \in [\frac{1}{4},\frac{1}{2}]; \\ 0.2, & x = \frac{1}{2}; \\ 0.2(\frac{4-4x}{2x+1})^2, & x \in [\frac{1}{2},1]. \end{cases}$$

and

$$\nu_{\frac{A}{B}}(x) = \begin{cases} \{\frac{3-6x+(9x-1)\sqrt{0.8}}{3x+2}\}^2, & x \in [\frac{1}{9},\frac{1}{2}]; \\ 0.8, & x = \frac{1}{2}; \\ \{\frac{6x-3+(5-3x)\sqrt{0.8}}{3x+2}\}^2, & x \in [\frac{1}{2},\frac{5}{3}]. \end{cases}$$

4) n^{th} root of A

$$\mu_{A^{\frac{1}{n}}}(x) = \begin{cases} 0.2(x^n-2)^2, & x \in [2^{\frac{1}{n}},3^{\frac{1}{n}}]; \\ 0.2, & x = 3^{\frac{1}{n}}; \\ 0.2(4-x^n)^2, & x \in [3^{\frac{1}{n}},4^{\frac{1}{n}}]. \end{cases}$$

and

$$\nu_{A^{\frac{1}{n}}}(x) = \begin{cases} \{\frac{3-\sqrt{0.4}-x^n(1-\sqrt{0.4})}{2}\}^2, & x \in [1, 3^{\frac{1}{n}}]; \\ 0.4, & x = 3^{\frac{1}{n}}; \\ \{\frac{x^n(1-\sqrt{0.4})-3+5\sqrt{0.4}}{2}\}^2, & x \in [3^{\frac{1}{n}}, 5^{\frac{1}{n}}]. \end{cases}$$

5) Exponential of A

$$\mu_{e^A}(x) = \begin{cases} 0.2(\ln(x) - 2)^2, & x \in [e^2, e^3]; \\ 0.2, & x = e^3; \\ 0.2(4 - \ln(x)), & x \in [e^3, e^4]. \end{cases}$$

and

$$\nu_{e^A}(x) = \begin{cases} \{\frac{3-\sqrt{0.4}-(1-\sqrt{0.4})\ln(x)}{2}\}^2, & x \in [e^1, e^3]; \\ 0.4, & x = e^3; \\ \{\frac{(1-\sqrt{0.4})\ln(x)-3+5\sqrt{0.4}}{2}\}^2, & x \in [e^3, e^5]. \end{cases}$$

6) Logarithm of A

$$\mu_{\ln(A)}(x) = \begin{cases} 0.2(e^x - 2)^2, & x \in [\ln(2), \ln(3)]; \\ 0.2, & x = \ln(3); \\ 0.2(4 - e^x), & x \in [\ln(3), \ln(4)]. \end{cases}$$

and

$$\nu_{\ln(A)}(x) = \begin{cases} \{\frac{3-\sqrt{0.4}-(1-\sqrt{0.4})e^x}{2}\}^2, & x \in [\ln(1), \ln(3)]; \\ 0.4, & x = \ln(3); \\ \{\frac{(1-\sqrt{0.4})e^x-3+5\sqrt{0.4}}{2}\}^2, & x \in [\ln(3), \ln(5)]. \end{cases}$$

Note: From the above it is observed that addition, subtraction, division, n^{th} root, exponential and logarithm of GPIFNs is also a GPIFN.

4 Rank of GPIFNs

In this section, rank of GPIFNs has been discussed. Rank of GPIFNs has been evaluated based on mean and value of MF and NMF defined as below

Let $A =< [a_1, b_1, c_1; w_1],\ [a'_1, b_1, c'_1; \eta_1] >$ be a GPIFN whose $(\alpha, \beta)-$cut are

$$^{\alpha}A = \left[a_1 + \sqrt{\frac{\alpha}{w_1}}(b_1 - a_1), c_1 - \sqrt{\frac{\alpha}{w_1}}(c_1 - b_1)\right],$$

$$^{\beta}A = \left[\frac{b_1 - a'_1\sqrt{\eta_1} - \sqrt{\beta}(b_1 - a'_1)}{1 - \sqrt{\eta_1}}, \frac{b_1 - c'_1\sqrt{\eta_1} + \sqrt{\beta}(c'_1 - b_1)}{1 - \sqrt{\eta_1}}\right]$$

Then the mean of MF $\bar{M}$ and NMF $\bar{M}'$ of A is defined by

$$\bar{M} = \frac{\int_0^{w_1}\{a_1 + \sqrt{\frac{\alpha}{w_1}}(b_1 - a_1)\}d\alpha + \int_0^{w_1}\{c_1 - \sqrt{\frac{\alpha}{w_1}}(c_1 - b_1)\}d\alpha}{2}$$
$$= \frac{w_1}{6}(a_1 + 4b_1 + c_1)$$
$$\bar{M}' = \frac{\int_{\eta_1}^1 \frac{b_1 - a'_1\sqrt{\eta_1} - \sqrt{\beta}(b_1 - a'_1)}{1-\sqrt{\eta_1}}d\beta + \int_{\eta_1}^1 \frac{b_1 - c'_1\sqrt{\eta_1} - \sqrt{\beta}(c'_1 - b_1)}{1-\sqrt{\eta_1}}d\beta}{2}$$
$$= \frac{1}{2(1-\sqrt{\eta_1})}\Big[\frac{2}{3}(a'_1 - 2b_1 + c'_1)(1 - \eta_1{}^{\frac{3}{2}}) + \{2b_1 - (a'_1 + c'_1)\sqrt{\eta_1}\}(1 - \eta_1)\Big]$$

Then the value of MF $V_\mu(A)$ and $V_\nu(A)$ are defined as

$$V_\mu(A) = \bar{M}\Big[\int_0^{w_1}\{c_1 - \sqrt{\frac{\alpha}{w_1}}(c_1 - b_1)\}d\alpha - \int_0^{w_1}\{a_1 + \sqrt{\frac{\alpha}{w_1}}(b_1 - a_1)\}d\alpha\Big]$$
$$= \frac{w_1{}^2}{18}(c_1 - b_1)(a_1 + 4b_1 + c_1)$$
$$V_\mu(A) = \bar{M}'\Big[\int_{\eta_1}^1 \frac{b_1 - c'_1\sqrt{\eta_1} - \sqrt{\beta}(c'_1 - b_1)}{1 - \sqrt{\eta_1}}d\beta - \int_{\eta_1}^1 \frac{b_1 - a'_1\sqrt{\eta_1} - \sqrt{\beta}(b_1 - a'_1)}{1 - \sqrt{\eta_1}}d\beta\Big]$$
$$= \frac{(c'_1 - a'_1)}{2(1-\sqrt{\eta_1})^2}\Big[\frac{2}{3}(a'_1 - 2b_1 + c'_1)(1 - \eta_1{}^{\frac{3}{2}}) + \{2b_1 - (a'_1 + c'_1)\sqrt{\eta_1}\}(1 - \eta_1)\Big]$$
$$\{\frac{\eta_1{}^{\frac{2}{3}} - \eta_1{}^{\frac{1}{2}} + 2}{3}\}$$

Then the Rank or value index of A ia defined by

$$R_\lambda(A) = \lambda V_\mu(A) + (1 - \lambda)V_\nu(A); \quad \lambda \in [0, 1] \tag{3}$$

where $V_\mu(A)$ and $V_\nu(A)$ are defined above. The rank of order depend on the value of $R_\lambda(A)$.

For two GPIFNs A and B

1. Rank(A) $<$ Rank (B) if $R_\lambda(A) < R_\lambda(B)$.
2. Rank(A) $\geq$ Rank (B) if $R_\lambda(A) \geq R_\lambda(B)$.

5 Decision Making via GPIFNs

Intuitionistic fuzzy decision making is a process to choose the best option from available options. Here ranking of GPIFNs is applied to the intuitionistic fuzzy decision making problem to show its existance and applicability in our daily life.

5.1 Methodology

Suppose $B_1, B_2, B_3, ..., B_p$ be p decision makers have to choose the best alternative among n alternatives say X_1, X_2, X_3,....X_n based on m criteria say M_1, M_2, M_3,...,M_m. Computational procedure is as given below

Step I: First the decision makers choose the linguistic variables (LVs) for the importance weight of each criterion and then choose the LVs for the rating of alternatives with respect to each criterion that are expressed in terms of GPIFN.
Step II: The decision makers evaluate the importance weight of each criterion using linguistic weight variables.
Step III: Evaluate the aggregated intuitionistic fuzzy weight $\tilde{w_j}$ of criterion Mj by using the following equation

$$\tilde{w_j} = \frac{1}{p}[\tilde{w_j}^1(+)\tilde{w_j}^2(+)......(+)\tilde{w_j}^p]$$

Step IV: Using rating variables the decision makers form intuitionistic fuzzy matrix which is of the form

$$E = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & .... & M_n \end{array} \\ \begin{bmatrix} s_{11} & s_{12} & .... & s_{1n} \\ s_{21} & s_{22} & .... & s_{2n} \\ .... & .... & .... & .... \\ s_{m1} & s_{m2} & .... & s_{mn} \end{bmatrix} \end{array}$$

where s_{ij} are intuitionistic fuzzy ratings given by decision makers.
Step V: Using proposed arithmetic operations, the decision evaluate

$$\tilde{D_i} = \sum_{j=1}^{n} \tilde{D_{ij}} w_j$$

Step VI: Decision makers will choose the alternative whose value index will be maximum.

5.2 A Case Study

Suppose a group of decision makers B_1, B_2 and B_3 has been formed to conduct an interview to choose the most suitable candidate for the post of Bank PO among the three eligible candidate X_1, X_2 and X_3 on the basis of four benefit criteria M_1, M_2, M_3 and M_4.

Step I: First the decision makers choose the linguistic weighting variable (Table 1) for weight of each criteria and the linguistic rating variable (Table 2) to evaluate the ratings of alternatives with respect to each criterion.

Table 1. Linguistic variable for Weight of each Criterion

L	$< [0.13, 0.15, 0.21; 0.2], [0.11, 0.15, 0.22; 0.4] >$
M	$< [0.23, 0.28, 0.24; 0.3], [0.21, 0.28, 0.36; 0.7] >$
H	$< [0.41, 0.45, 0.52; 0.5], [0.39, 0.45, 0.55; 0.6] >$
VH	$< [0.54, 0.58, 0.65; 0.7], [0.51, 0.58, 0.68; 0.9] >$

Table 2. Linguistic variable for the ratings

P	$< [0.15, 0.19, 0.25; 0.1], [0.13, 0.19, 0.28; 0.5] >$
F	$< [0.18, 0.24, 0.32; 0.3], [0.14, 0.24, 0.34; 0.7] >$
G	$< [0.45, 0.58, 0.63; 0.5], [0.41, 0.58, 0.64; 0.7] >$
VG	$< [0.65, 0.69, 0.72; 0.7], [0.63, 0.69, 0.75; 0.9] >$

Step II: To evaluate the importance of weight of each criteria (Table 3) linguistic weighting variable are used (Table 1)

Table 3. The importance of weight of each criterion given by Decision Makers

Decision Makers →	B_1	B_2	B_3
Criterion ↓			
$M1$	M	L	M
$M2$	H	VH	H
$M3$	VH	H	VH
$M4$	VH	VH	VH

Step III: The weight of criteria are aggregated by using (Table 2) and the following equation to evaluate the aggregate fuzzy weight $\tilde{w_j}$ of the criteria Mj

$$\tilde{w_j} = \frac{1}{p}[\tilde{w_j}^1(+)\tilde{w_j}^2(+)......(+)\tilde{w_j}^k]$$

$$\tilde{w_1} = < [0.19, 0.23, 0.29; 0.27], [0.17, 0.23, 0.31; 0.7] >$$

$$\tilde{w_2} = < [0.45, 0.49, 0.56; 0.56], [0.43, 0.49, 0.59; 0.6] >$$

$$\tilde{w_3} = < [0.49, 0.53, 0.61; 0.63], [0.47, 0.53, 0.64; 0.8] >$$

$$\tilde{w_4} = < [0.54, 0.58, 0.65; 0.7], [0.51, 0.58, 0.68; 0.9] >$$

Step IV: The decision matrix R is obtained from the ratings given by decision makers.

$$R = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \end{array} \\ \begin{bmatrix} P & F & F & G \\ G & F & VG & G \\ VG & VG & G & G \end{bmatrix} \end{array}$$

Then the fuzzy decision matrix R is constructed as given below

	M_1	M_3	M_3	M_4
X_1	$< [0.15, 0.19, 0.25; 0.1]$, $[0.13, 0.190.25; 0.5] >$	$< [0.18, 0.24, 0.32; 0.3]$, $[0.14, 0.24, 0.34; 0.7] >$	$< [0.18, 0.24, 0.32; 0.3]$, $[0.14, 0.24, 0.34; 0.7] >$	$< [0.45, 0.58, 0.63; 0.5]$, $[0.41, 0.58, 0.64; 0.7] >$
X_2	$< [0.45, 0.58, 0.63; 0.5]$, $[0.41, 0.58, 0.64; 0.7] >$	$< [0.18, 0.24, 0.32; 0.3]$, $[0.14, 0.24, 0.34; 0.7] >$	$< [0.65, 0.69, 0.72; 0.7]$, $[0.63, 0.69, 0.75; 0.9] >$	$< [0.45, 0.58, 0.63; 0.5]$, $[0.41, 0.58, 0.64; 0.7] >$
X_3	$< [0.65, 0.69, 0.72; 0.7]$, $[0.63, 0.69, 0.75; 0.9] >$	$< [0.65, 0.69, 0.72; 0.7]$, $[0.63, 0.69, 0.75; 0.9] >$	$< [0.45, 0.58, 0.63; 0.5]$, $[0.41, 0.58, 0.64; 0.7] >$	$< [0.45, 0.58, 0.63; 0.5]$, $[0.41, 0.58, 0.64; 0.7] >$

Step V: To evaluate

$$\tilde{D_i} = \sum_{j=1}^{n} \tilde{D_{ij}} w_j$$

using our proposed arithmetic operations. For different values of $\lambda \in [0, 1]$ we get different value.

Table 1: For $\lambda = 0$.

$\tilde{D_i}$	Value-index $R_\lambda(A)$ for $\lambda \in [0, 1]$	Rank
$\tilde{D_1} =< [0.44, 0.63, 0.86; 0.74], [0.36, 0.63, 0.92; 1.96] >$	2.58	3^{rd}
$\tilde{D_2} =< [0.63, 0.89, 1.56; 0.97], [0.53, .89, 1.24; 2.1] >$	3.72	2^{nd}
$\tilde{D_3} =< [0.88, 1.14, 1.59; 1.25], [0.78, 1.14, 1.52; 2.36] >$	5.74	1^{st}

Table 2: For $\lambda = 0.5$.

$\tilde{D_i}$	Value-index $R_\lambda(A)$ for $\lambda \in [0, 1]$	Rank
$\tilde{D_1} =< [0.44, 0.63, 0.86; 0.73], [0.36, 0.63, 0.92; 1.96] >$	1.297	3^{rd}
$\tilde{D_2} =< [0.63, 0.89, 1.56; 0.97], [0.53, .89, 1.24; 2.1] >$	1.904	2^{nd}
$\tilde{D_3} =< [0.88, 1.14, 1.59; 1.25], [0.78, 1.14, 1.52; 2.36] >$	3.85	1^{st}

Table 3: For $\lambda = 1$

$\tilde{D_i}$	Value-index $R_\lambda(A)$ for $\lambda \in [0, 1]$	Rank
$\tilde{D_1} =< [0.44, 0.63, 0.86; 0.73], [0.36, 0.63, 0.92; 1.96] >$	0.0146	3^{rd}
$\tilde{D_2} =< [0.63, 0.89, 1.56; 0.97], [0.53, .89, 1.24; 2.1] >$	0.098	2^{nd}
$\tilde{D_3} =< [0.88, 1.14, 1.59; 1.25], [0.78, 1.14, 1.52; 2.36] >$	0.204	1^{st}

Step VI: After observing the above tables it is clear that for any value of $\lambda \in [0, 1]$ the order of $\tilde{D_1}$, $\tilde{D_2}$ and $\tilde{D_3}$ will remain same that is $\tilde{D_3} > \tilde{D_2} > \tilde{D_1}$. Hence the ranking order of three alternatives $X_3 > X_2 > X_1$ and X_3 is the best alternative among the three alternatives.

6 Conclusion

Intuitionistic fuzzy decision making is widely applicable in various areas. There are several IFNs exist and successfully applied in decision making problems. Most of the decision making problems have been done by using triangular and trapezoidal type of intuitionistic fuzzy numbers. In this paper generalized parabolic type of intuitionistic fuzzy number has been studied. Arithmetic operations of GPIFNs have been successfully done by using (α, β)-cuts method followed by numerical examples. From the arithmetic operations of GPIFNs, it is observed that the addition, subtraction, division, n^{th} root, exponential and logarithm of GPIFNs is also a GPIFN. Rank of GPIFNs has been evaluated based on mean and value. Finally, solving a decision making problem which shows its existence in our daily life.

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An Investigation of Involving Supplier and Manufacturer Based Inventory Models Under Uncertain Fuzzy Constraints

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Abstract. The study & involvement of supplier in new product manufacturing have essentially centered around the globe considering, long-term manufacture method and other successful factors. In this paper we will check a feasible and analytic solution which recognizes both long-term as well as short-term operational methods regarding suppliers context. The principle part of this paper depends on information from an alternate contextual investigation of involving a new supplier and manufacturing of new product in respected garden chemical company as well as magazine company. Our main aim is to minimize the cost using space constraint and maximize the profit using budget constraint for short-term objectives as well as long term advantages of involving a new supplier in product manufacture. Here this research contributes to the dynamic capabilities which permutes us to observed, to understand and to facilitate the problem that how organizations can adequately redesigns a maximum profit and minimize cost for models in product manufacturing from different resources prepared by suppliers.

Keywords: Supplier · Product manufacture · Minimizing cost · Maximizing profit

1 Introduction

Since many years, few reviews have demonstrated that product manufacture and improvement has turned into severe in creating or keeping up a capacity position in a business zone [5,7,13,26] and [28]. Moreover, the request for considering product manufacturing in terms of performance and cost was tough to set the platform. Classification of supplier based approaches have been foreseeing from the conventional approaches that further resulted in product quality, profit and demand of different parameters [12] and [22].

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Different ideas and constructive approaches were provided by suppliers [1,4] and [20]. In contrast, some problematic issues were faced by suppliers too [3] and [8]. The subject of research has been in a hefty portion by both internal and external performers for product improvement especially with R & D department and advertising clients. Suppliers in product improvement have to be specific & contended in reducing manufacturing time, and product costs whilst enhancing product quality.

However, inclusion of suppliers has blended consequences [3,11,14] and [30]. For instance, companies need to adjust to the most significant part and situation. Maintaining supplier's contribution to product manufacturing, it shows both long and short term effects. In connection to supplier involvement for product manufacturing, this infers the way that suppliers ought to be required in product improvement requires an examination of the conditional elements and the basic procedures to be maintained. Supplier association in new product manufacturing ventures has turned into an increasingly prevalent technique for improving product expenses and quality and manufacturing expenses and time.

Besides, there are numerous researchers who have solved examples based on above criteria including [2,15,23,24,27,29,31,32] and [10]. Moreover, Mondal et al. [19] considered inventory system that is depending on price for improving items. Further, based on price dependent demand Liang and Zhou [16] obtained results for degraded items using two warehouse inventory models. Meanwhile a new concept of advance payment was introduced by Maiti et al. [18] determined optimal order policy with selling price and uncertain lead time.

In fact, a key differentiator is the one who has the ability to portray different past and present clients. The process may have commitments with probabilistic & deterministic approaches depends upon the need of the customer hence certain parameters in uncertain fuzzy constraint must be established. Charnes constraint was first formed by Charnes et al. [6] in order to rectify uncertainty & vagueness. Further, Liu and Iwamura [17] taught fuzzy parameters that determine fuzzy, random and rough sets. To diminish the issues into crisp ones Charnes limitation is the essential source utilized by Panda et al. [21]

In this paper, we have studied the factors i.e., the requirement of product, number of suppliers. Using these factors we have studied two models over here. In the first model, we have studied how much requirement of product and purchase of items can be done so that our total cost will be minimized using space constraint. In the second model, we have studied how much we can add the products so that our profit will be maximize using budget constraint. We can further go for involving new suppliers as well as manufacturers for developing new products considering minimize the cost or maximize the profit.

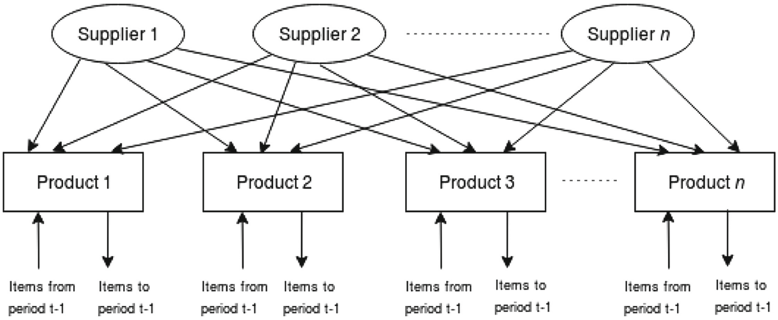
2 Assumptions

- Considering the planning horizon, demand of the product is periodic.
- As shortages are not allowed and re-ordering of the items are not permissible, all the requirements must be in a given interval of time.

- Since, the cost of transaction is based on supplier unit, it doesn't depend on products involved in terms of items and it's quantities.
- Holding Cost here is product dependent per product per period.
- As the total storage space is limited, products requires an adequate storehouse.

3 Mathematical Modelling

Based on the assumptions of the model, the figure below determines the scenario of multi-product and multi cost constraints (i.e., minimizing cost and maximizing profit) under space and budget constraints respectively.



Here in this paper we have formulated multi-product and multi-cost parameters under different constraints by below symbols.

Indies
$i = 1, \ldots, I$ indicates products, $j = 1, \ldots, J$ indicates suppliers, $t = 1, \ldots, T$ indicates periods.

General Form

The general form of the problem can be written as,
Case-I

$$Min\, Z = \sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j \qquad \text{[in case of cost minimization problem.]}$$

$$\begin{aligned} subject\ to: \quad & a_i x_i + b_j y_j \geq 0 \\ & x_i, y_j \ \geq 0 \\ & a_i, b_j \ \geq 0 \end{aligned}$$

For Cost Minimization Problem
Following notations are used for multi-product multi-cost constraints
x_i = number of units of liquid products
y_j = number of units of solid products
Z = total cost
S = space constraint

Case-II

$$Max\,Z = \sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j \qquad \text{[in case of profit maximization problem.]}$$

$$\begin{aligned} subject\ to: \quad & a_i x_i + b_j y_j \geq 0 \\ & x_i, y_j \ \geq 0 \\ & a_i, b_j \ \geq 0 \end{aligned}$$

For Profit Maximization Problem

Following notations are used for multi-product multi-profit constraints
x_i = advertising strategy of magazine 1
y_j = advertising strategy of magazine 2
Z = total profit
B = budget constraint

4 Different Types of Fuzzy Constraints

4.1 Crisp Space Constraint

For the crisp finite space constraint we have to consider it as

$$a_1 Q_1 + a_2 Q_2 \leq S \tag{1}$$

4.2 Random Space Constraint

In this consideration, the models remain same as developed above, except the space constraint of the system. Here, $\bar{S}$ is random. For this type of model we impose constraint as

$$\begin{aligned} &\Pr(\bar{S} \geq a_1 Q_1 + a_2 Q_2) \geq g, \qquad \text{where } g \in (0,1) \text{ is a specified permissible probability.} \\ &\text{or}, a_1 Q_1 + a_2 Q_2 \leq m_s + \sigma_s \Phi^{-1}(1-g), \text{(cf. Rao [24])} \end{aligned} \tag{2}$$

where m_s and σ_s are the expectation and s.d. of normal dist. random variable $\bar{S}$ respectively and $\Phi^{-1}(x)$ denotes inverse function of standard normal distribution of standard normal variate $\dfrac{\bar{S} - m_s}{\sigma_s}$.

4.3 Fuzzy Space Constraint

If the space horizon $\tilde{S}$ is fuzzy in nature, it can be expressed by the fuzzy constraint $\tilde{S} \geq a_1 Q_1 + a_2 Q_2$ determine through possibility & necessity sense (cf. Dubois and Prade [9]). The above constraint reduces to

$$\text{Pos}(\tilde{S} \geq a_1 Q_1 + a_2 Q_2) \geq \rho_1, \qquad \text{and} \qquad \text{Nes}(\tilde{S} \geq a_1 Q_1 + a_2 Q_2) \geq \rho_2$$

where ρ_1 and ρ_2 represent the degree of impreciseness. Let $\tilde{S} = (S_1, S_2, S_3)$ be TFN then, using Lemma-1 and 2 we get

$$a_1 Q_1 + a_2 Q_2 \leq \begin{cases} (1-\rho_1) S_3 + \rho_1 S_2, & \text{in possibility sense} \\ (1-\rho_2) S_2 + \rho_2 S_1, & \text{in necessity sense.} \end{cases} \tag{3}$$

4.4 Fuzzy-Random Space Constraint

In this case, the Space Constraint $\tilde{\bar{S}}$ is fuzzy-random in nature and the fuzzy-random constraint is $\tilde{\bar{S}} \geq a_1Q_1 + a_2Q_2$. It stands for the relations which are analyzed through pos. & neccess. sense (cf. Dubois and Prade [9]) along with chance the constraint. The above constraint reduces to

$$\Pr\big[\mathrm{Pos}(\tilde{\bar{S}} \geq a_1Q_1 + a_2Q_2) \geq \rho_3\big] \geq g_1, \qquad \text{and} \qquad \Pr\big[\mathrm{Nes}(\tilde{\bar{S}} \geq a_1Q_1 + a_2Q_2) \geq \rho_4\big] \geq g_2$$

where (ρ_3 and ρ_4) and (g_1 and g_2) represent the degree of impreciseness and uncertainty due to randomness respectively. Let $\tilde{\bar{S}} = (\bar{S}, S_l, S_r)$ be L-R fuzzy-random variable then, we get

$$a_1Q_1 + a_2Q_2 \leq \begin{cases} m_s + \sigma_s\Phi^{-1}(1-g_1) + R^{-1}(\rho_3)S_r, & \text{in possibility sense} \\ m_s + \sigma_s\Phi^{-1}(1-g_2) - L^{-1}(1-\rho_4)S_l, & \text{in necessity sense.} \end{cases} \tag{4}$$

where m_s and σ_s are the expectation and s.d. of normal dist. random variable $\bar{S}$ respectively and $\Phi^{-1}(x)$ denotes inverse function of standard normal distribution of standard normal variate $\frac{\bar{S}-m_s}{\sigma_s}$.

4.5 Rough Space Constraint

If the space constraint $\hat{S}$ is rough in nature, the rough constraint $\hat{S} \geq a_1Q_1+a_2Q_2$ is reduced to the crisp form as

$$Tr(\hat{S} \geq a_1Q_1 + a_2Q_2) \geq tr_1.$$

$$\text{i.e. } a_1Q_1 + a_2Q_2$$
$$\leq \begin{cases} S_4 - \frac{tr_1(S_4-S_3)}{\xi_1}, & \text{if } S_2 \leq a_1Q_1 + a_2Q_2 \leq S_4 \\ \dfrac{\xi_1(S_2-S_1) + (1-\xi_1)S_2(S_4-S_3) - tr_1(S_4-S_3)(S_2-S_1)}{\xi_1(S_2-S_1)+(1-\xi_1)(S_4-S_3)}, & \text{if } S_1 \leq a_1Q_1 + a_2Q_2 \leq S_2 \\ S_4 + \dfrac{(1-\xi_1-tr_1)(S_4-S_3)}{\xi_1}, b & \text{if } S_3 \leq a_1Q_1 + a_2Q_2 \leq S_1 \\ S_3 \end{cases} \tag{5}$$

where $\hat{S} = ([S_1, S_2][S_3, S_4])$, $0 \leq S_3 \leq S_1 \leq S_2 \leq S_4$, is a rough variable and $\xi_1 \in (0, 1)$ and $tr_1 \in [0, 1]$ is the confidence level.

4.6 Fuzzy-Rough Space Constraint

If the space Constraint $\tilde{\hat{S}}$ is fuzzy-rough in nature, the fuzzy-rough constraint $\tilde{\hat{S}} \geq a_1Q_1 + a_2Q_2$ is reduced in the following forms which are crisp in nature.

$$\mathrm{Tr}\big[\mathrm{Pos}(\tilde{\hat{S}} \geq a_1Q_1 + a_2Q_2) \geq \rho_5\big] \geq tr_2, \qquad \text{and} \qquad \mathrm{Tr}\big[\mathrm{Nes}(\tilde{\hat{S}} \geq a_1Q_1 + a_2Q_2) \geq \rho_6\big] \geq tr_2$$

The above constraints are finally reduced to the following forms.

$$\begin{cases} a_1Q_1 + a_2Q_2 \\ \leq \begin{cases} S_4 - \frac{tr_2(S_4-S_3)}{\xi_2} + (1-\rho_5)S_R, & \text{if } S_2 \leq a_1Q_1 + a_2Q_2 - (1-\rho_5)S_R \leq S_4 \\ \frac{\xi_2(S_2-S_1)+(1-\xi_2)S_2(S_4-S_3)-tr_2(S_4-S_3)(S_2-S_1)}{\xi_2(S_2-S_1)+(1-\xi_2)(S_4-S_3)} & \\ +(1-\rho_5)S_R, & \text{if } S_1 \leq a_1Q_1 + a_2Q_2 - (1-\rho_5)S_R \leq S_2 \\ S_4 + \frac{(1-\xi_2-tr_2)(S_4-S_3)}{\xi_2} + (1-\rho_5)S_R, & \text{if } S_3 \leq a_1Q_1 + a_2Q_2 - (1-\rho_5)S_R \leq S_1 \\ S_3 + (1-\rho_5)S_R \end{cases} \\ and \\ a_1Q_1 + a_2Q_2 \\ \leq \begin{cases} S_4 - \frac{tr_2(S_4-S_3)}{\xi_2} - \rho_6 S_L, & \text{if } S_2 \leq a_1Q_1 + a_2Q_2 + \rho_6 S_L \leq S_4 \\ \frac{\xi_2(S_2-S_1)+(1-\xi_2)S_2(S_4-S_3)-tr_2(S_4-S_3)(S_2-S_1)}{\xi_2(S_2-S_1)+(1-\xi_2)(S_4-S_3)} & \\ -\rho_6 S_L, & \text{if } S_1 \leq a_1Q_1 + a_2Q_2 + \rho_6 S_L \leq S_2 \\ S_4 + \frac{(1-\xi_2-tr_2)(S_4-S_3)}{\xi_2} + (1-\rho_6)S_R, & \text{if } S_3 \leq a_1Q_1 + a_2Q_2 + \rho_6 S_L \leq S_1 \\ S_3 - \rho_6 S_L \end{cases} \end{cases}$$

where $\tilde{\hat{S}} = (\hat{S} - S_L, \hat{S},\ \hat{S} + S_R)$, $\hat{S} = ([S_1, S_2][S_3, S_4])$, $0 \leq S_3 \leq S_1 \leq S_2 \leq S_4$, is a fuzzy-rough variable and $\xi_2 \in (0, 1)$ and $\rho_5, \rho_6 \in [0, 1]$, $tr_2 \in [0, 1]$ are the possibility and trust confidence levels respectively.

Note: In case of budget constraint we will consider B for *Profit Maximization Model* as we have taken S for *Cost Minimization Model*.

5 Solution Methodology

Ga is ideally used to discover ideal arrangement. Initial steps of GAs can be found in Holland [6], Michalewicz [8], and Gen et al.[4]. Further, we have delineated GA system in beneath figure to begin the pursuit GAs are instated with a population of people. The parents here are considered as chromosomes in the pursuit space. GAs utilize principally two administrators to be specific, hybrid and change to guide the populace to the worldwide ideal. Hybrid permits trading data between various arrangements (chromosomes) and change builds the assortment in the populace. After the determination and assessment of the underlying population, chromosomes are chosen on which the hybrid and change parameters are connected. Next, the new population is framed. We need to keep the process going on and unless the stopping criteria are met [1].

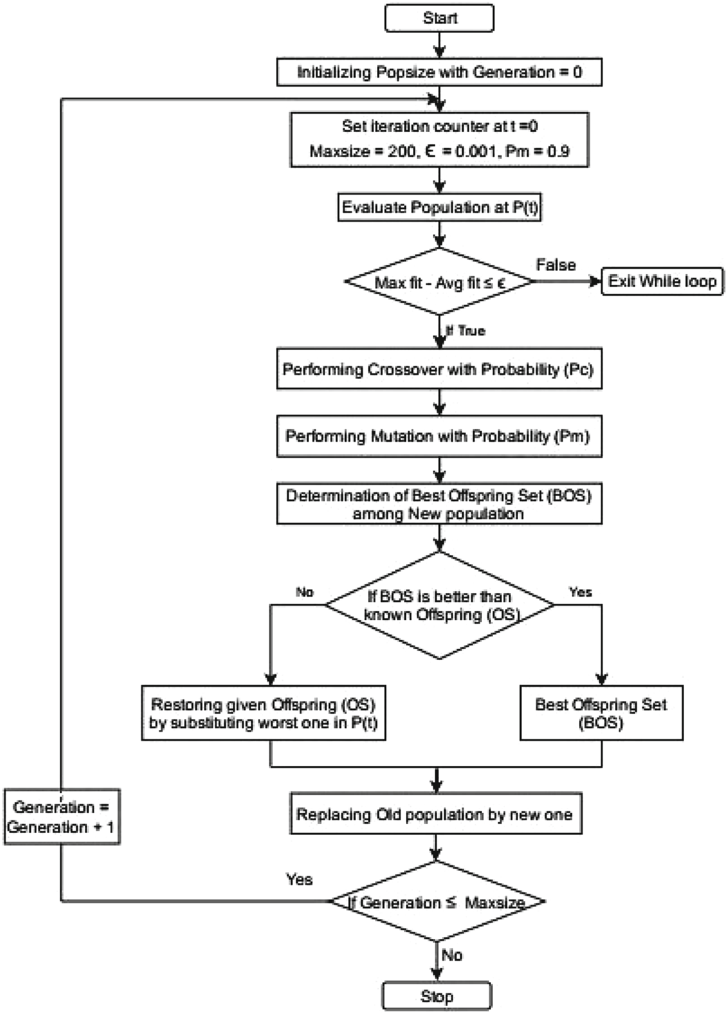


Fig. 1. Flow-chart of Genetic Algorithm (GAs)

6 Numerical Examples for Cost Minimization and Profit Maximization Models

Cost Minimization Problem

Statement: In this case, we consider a scenario with two types of product i.e., liquid & solid and ten types of chemicals. i.e., O^{2-}(oxides), CO_3^{2-}(carbonates), $COOH$(carboxyl), SO_2OH(sulphonic acid), $-SH$(thiol), H^+(hydrogen), H_3O^+(hydronium), $AGNO_3$(silver nitrate), KNO_3(potassium nitrate) and Cl_2(chlorine). Now suppose a person requires 10, 11, 12, 15, 12, 11, 10, 16, 13 and 10 units for above chemicals in his garden. Moreover, liquid product contains 1, 2, 3, 4, 5, 6, 7, 8, 9, 8 units per jar along with solid product containing 9, 10, 8, 7, 5, 6, 4, 3, 2, 1 per carton respectively for above chemicals. Now if the liquid product sells for Rs. 6 per jar and a dry product sells for Rs. 8 per carton.

Product 1	10 types of chemicals	10	11	12	15	12	11	10	16	13	10
Product 2:1	Liquid	1	2	3	4	5	6	7	8	9	8
Product 2:2	Solid	9	10	8	7	5	6	4	3	2	1

Our goal is to find how many of each should he purchase to minimize the total cost and meet the requirements?

Profit Maximization Problem

Statement: Here, The magazine company wishes to plan under advertising strategy. There are two media under consideration call them magazine 1 and magazine 2 has a reach of 6 and 8 potential customers. The cost per page of the advertising is Rs. 12 and 19 in magazine 1 and magazine 2 respectively. The company has a monthly budget of Rs. 50. There is an important requirement that the total reach for the income group under Rs. 2000 per annum should not exceed 55, 54, 58, 54, 53, 56, 54, 57, 59 potential customers respectively. The reach in magazine 1 and magazine 2 for this income group is potential customers respectively.

Magazine 1	15	13	11	18	16	17	15	14	18
Magazine 2	10	18	17	15	16	14	13	12	11

Our goal is to find how many pages should be bought in the two magazines to maximize the total reach?

7 Results via LINGO and GA

See Tables 1 and 2.

Table 1. Solution table for Model 1 (Cost Minimization)

Serial number	Decision variable	Solution through LINGO	Solution through Genetic Algorithms (GAs)
1	x	1.522	1.5445
2	y	1.272	1.2606
3	Min Z	19.308	19.3517

Table 2. Solution table for Model 2 (Profit Maximization)

Serial number	Decision variable	Solution through LINGO	Solution through Genetic Algorithms (GAs)
1	x	1.7037	1.6733
2	y	1.5555	1.5678
3	Max Z	22.6666	22.5819

8 Input and Output via Fuzzy Constraints

8.1 Input of the Following Fuzzy Constraints

See Tables 3 and 4.

Table 3. Input values for different space constraints of Model 1

Time horizon	Crisp	Random	Fuzzy (Pos& Nes Sense)	Fuzzy-Random (Pos&Nes Sense)	Rough	Fuzzy-Rough (Pos&Nes Sense)
Related Inputs For $p_1 = 2.00$ & $p_2 = 2.50$	6.0	$m_s = 5.50$ $\sigma = 0.750$ $\Phi = 0.90$	$S_1 = 5.0$ $S_2 = 3.0$ $S_3 = 4.0$ $\rho_1 = 0.80$ $\rho_2 = 0.30$	$m_S = 20,$ $\sigma_S = 5$ $S_l = 0.40,$ $S_r = 0.50$ $\rho_3 = 0.80,$ $\rho_4 = 0.20$ $L(x) = R(x)$ $= 1 - x$	$S_1 = 20,$ $S_2 = 18$ $S_3 = 15,$ $S_4 = 25$ $\xi_1 = 0.50$ $tr_1 = 0.80$	$S_1 = 20,\ S_2 = 18$ $S_3 = 15,\ S_4 = 25$ $\xi_2 = 0.40,$ $tr_2 = 0.80$ $S_L = 5,$ $S_R = 5$ $\rho_5 = 0.70,$ $\rho_6 = 0.20$

Table 4. Input values for different budget constraints of Model 2

Time horizon	Crisp	Random	Fuzzy (Pos& Nes Sense)	Fuzzy-Random (Pos& Nes Sense)	Rough	Fuzzy-Rough (Pos&Nes Sense)
Related Inputs For $p_1 = 12.0$ & $p_2 = 19.0$	$B = 50.0$	$m_b = 48$ $\sigma_b = 1.50$ $\Phi = 0.90$	$B_1 = 18$ $B_2 = 15$ $B_3 = 12$ $\rho_1 = 0.80$ $\rho_2 = 0.30$	$m_b = 20,\ \sigma_b = 5$ $B_l = 5,\ B_r = 7$ $\rho_3 = 0.80,\ \rho_4 = 0.20$ $L(x) = R(x) = 1 - x$	$B_1 = 20,$ $B_2 = 18$ $B_3 = 15,$ $B_4 = 25$ $\xi_1 = 0.50$ $tr_1 = 0.80$	$B_1 = 20,$ $B_2 = 18$ $B_3 = 15,$ $B_4 = 25$ $\xi_2 = 0.50,$ $tr_2 = 0.80$ $B_L = 5,$ $B_R = 7$ $\rho_5 = 0.20,$ $\rho_6 = 0.70$

8.2 Output of the Following Fuzzy Constraints

Fuzzy space constraints					Fuzzy Budget Constraints				
Sr. no.	Horizon	X	Y	Min Z	Sr. no.	Horizon	X	Y	Max Z
1	Crisp	1.522	1.272	19.308	1	Crisp	1.703	1.555	22.666
2	Random	1.501	1.263	19.114	2	Random	1.692	1.459	22.154
3	Fuzzy-Pos	1.431	1.249	18.784	3	Fuzzy-Pos	1.631	1.413	21.921
4	Fuzzy-Nes	1.410	1.221	18.621	4	Fuzzy-Nes	1.543	1.391	21.613
5	Fuzzy Random-Pos	1.382	1.161	18.246	5	Fuzzy Random-Pos	1.411	1.364	21.32
6	Fuzzy Random-Nes	1.342	1.142	18.069	6	Fuzzy Random-Nes	1.321	1.319	20.871
7	Rough	1.274	1.321	17.874	7	Rough	1.141	1.112	19.616
8	Fuzzy Rough-Pos	1.204	1.101	16.951	8	Fuzzy Rough-Pos	1.111	1.091	18.432
9	Fuzzy Rough-Nes	1.150	1.023	16.452	9	Fuzzy Rough-Nes	1.026	1.011	18.141

9 Conclusion and Future Work

Formulation of Genetic Algorithm and comparison of the results with LINGO has been reported in the present paper. Moreover, we have tested LINGO & GA on the set of two LPP's widely used in the literature compared with LINGO & GA can produce better performance in terms of both, efficiency and success rate for the majority of nested functions. Henceforth, the performance of the GA was investigated & compared with LINGO. Global optima for various tests functions suggests that inventory models often perform well considering traditional methods such as Genetic Algorithms and LINGO in terms of both efficiency and success rate for a large amount of data. GA is very efficient whenever the amount of data is large and more compatible with LINGO when data is small. An attempt is made to satisfy all dynamic environment requirements. Randomness can be reduced gradually, to improve the quality of the solution.

Future work will inculcate to find the solution for any number of data in terms of cost minimization and profit maximization within managing new supplier in product manufacturing. In addition, possible modifications to improve its performance will also be considered. There are some relevant works to pursue in the near future too. Firstly, we can minimize our cost maximize the profit to meet the requirements for a large amount of data in Economic Order Quantity (EOQ Model) & Economic Product Quantity (EPQ Model) respectively. Secondly, some work by using these methods we can predict the future requirements and analyse future loss, profit and manufacturer & suppliers using uncertain fuzzy constraints too. Lastly, GA can be modified to solve the multi-objective optimization problem in combination with other optimization techniques.

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A Decision Making Approach for Finding Cause of Disease Under Hesitant Fuzzy Environment

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Abstract. In medical science it is said that prevention is better than cure. Nowadays, medical science has been developed in such a way that almost all of the disease are becoming near to curable. Though, we should prevent the diseases by knowing the exact reason for disease. But, to find the exact reasons for any disease are a critical task and the cause may vary person to person. In this paper we are trying to develop a methodology using hesitant triangular fuzzy number to detect cause for diseases. In this paper, a new concept of expected value for hesitant triangular fuzzy number has been developed. Also, a numerical example has been worked out to show the usability of the methodology.

Keywords: Triangular hesitant fuzzy set · Expected value · Decision making

1 Introduction

In medical science it is always said that prevention is better than cure. This means it is always better to stay away from disease by good knowledge and take precautions, rather than spend time, energy and money for treatments after suffering from disease. But unfortunately, the detection of cause of any disease is not as simple as it seems. Since, every person is different from one another in mental and physical level. Also, every person's circumstances are different on the basis of food habit, environment, work habits etc. For this reason, detection of cause of disease for a person is not a straightforward decision making situation. And for obvious reason decision making under uncertainty arises in finding cause of disease for any person. In this paper, a methodology has been developed to detect the most possible cause of disease of any person on the basis the disease, habits, work culture food habits etc. In expressing the uncertainty hesitant triangular fuzzy number has been used for better handling of uncertainty. Prakash et al. [7] projected a decision making method with linguistic assessments based on expected value of triangular hesitant fuzzy number to solve multi-criteria decision making problem. They have applied the expected value which gives the same value for two different Hesitant Fuzzy element. In our paper, with the help of an example we have shown that our proposed expected value produces more relevant result compared to the existing one. In addition, a new concept of expected value has been introduced in this paper.

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Also, a numerical example has been worked out in which 10 disease has been considered as criteria and five causes are taken as alternatives. And finally a valid and proved conclusion has been drawn from the example and showed that utility of the proposed methodology.

2 Literature Review

Hesitant Fuzzy Set (HFS) was introduced by Torra [10] as a new extension of Fuzzy Set. Wu et al. [12] gave a notice on the multi criteria decision making (MCDM) problem in which the criteria are in dissimilar priority levels and the criteria values take the form of hesitant fuzzy linguistic numbers (HFLNs). They projected a fresh approach which is based on the generalized prioritized aggregation operator of HFLNs to explain MCDM problem. Wang et al. [11] tried to enter the classical soft sets to hesitant fuzzy soft sets which are pooled by the soft and hesitant fuzzy sets. With the help of level soft set, the hesitant fuzzy soft sets are applied to a decision making problem. Dursaun et al. [3] presented a fuzzy multi–criteria group decision making framework based on the principle of fuzzy measure and fuzzy integral for evaluating health care waste treatment alternatives for Istanbul. Yu [16] investigated multi–criteria group decision making problems where a prioritization relationship existed over the criteria under hesitant fuzzy environment. Rodriguez et al. [8] presented a summary on hesitant fuzzy sets with the aim of providing an apparent viewpoint on the different concepts, tools and trends related to hesitant fuzzy sets. Yao and Li [14] wished-for a new score function in which mean and variances are considered. They also introduced the fundamental operators such as hesitant fuzzy weighted averaging operator and hesitant weighted geometric operator to get the wide-ranging estimation provided by the decision maker on each attribute.

On the whole there are two approaches in multi-criteria decision making problems (a) multiple attribute decision making (MADM), in which decision has to be taken in discrete space and given attention on how to pick different alternatives from existing alternatives and (b) multiple objective decision making (MODM) in which decision has to be taken in continuous space and a number of objective functions are to be achieved simultaneously. The concept of fuzzy set theory was first introduced by Zadeh [18]. An approach to multi criteria decision making using fuzzy sets was given by Bellman and Zadeh [1]. Yager [13] illustrated that in fuzzy multi criteria decision making (FMCDM), the best alternatives have the highest membership grades. Saaty [9] urbanized Analytical Hierarchy Process (AHP). Fan et al. [5] anticipated a new approach to solve the MADM problems. TOPSIS method was developed by Hawang and Yoon [6], where similarity are measured to ideal solutions. Yoon and Hawang [15] projected the advance fuzzy TOPSIS procedure. Later as a consequence of the litheness and dependability of the TOPSIS procedure it is developed and used gradually. Prakash et al. [7] projected a decision making method with linguistic assessments based on expected value of triangular hesitant fuzzy number to solve multi-criteria decision making problem. Zeng [19] explained a fuzzy attribute decision making method in which the attribute weights and decision matrix elements are fuzzy variables. The traditional Hesitant Fuzzy Set has been extended to Triangular Hesitant Fuzzy set by Yu [17] in which membership grade of an element to a given set is represented by a number of possible triangular fuzzy numbers.

3 Preliminary

In this section, we discuss some concept of fuzzy set, hesitant fuzzy set, hesitant fuzzy distance measures and triangular hesitant fuzzy set.

Definition 1:
A fuzzy set is one, which assigns grades of membership between 0 and 1 to objects within its universe of discourse X. If X is a universal set then a fuzzy set A is defined by its membership function Zadeh [18]

$$\mu_A : X \to [0, 1]$$

In real life situations, it is not easy to define one membership grade for one element. In most of the situations different experts may assign different membership grades according to their level of hesitancy. For this limitation, an advanced concept of fuzzy set was proposed and is known as hesitant fuzzy set.

Definition 2:

- Triangular Fuzzy Number

A triangular fuzzy number A is denoted by (a, b, c) and defined by its membership function as:

$$\mu_A(x) = \begin{cases} 0, x < a \\ \frac{x-a}{b-a}, a \le x \le b \\ \frac{c-x}{c-b}, b \le x \le c \\ 0, x > c \end{cases}$$

Definition 3:

- Hesitant Fuzzy Set

Let X be a reference set, a hesitant fuzzy set on X is defined in terms of a function that applying to X returns a subset of $[0, 1]$. To be easily understood an expression of HFS given by Torra [10]:

$$A = \{< x, h_A(x) > | x \in X\}$$

Where $h_A(x)$ is a set of values in $[0, 1]$ which representing the membership degrees of the element $x \in X$ to the set A by different experts. In general $h_A(x)$ is called hesitant fuzzy element (HFE).

Definition 4:

- Score function

For a hesitant fuzzy elements h, score function $s(h)$ is defined as

$$s(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma$$

Where l_h is the number of elements in the hesitant fuzzy element h.

With the help of the score function comparisons are made between two HFEs as follows:

Let us consider two HFEs h_1 and h_2,, then

If $s(h_1) > s(h_2)$ then h_1 is superior to h_2, denoted by $h_1 > h_2$

If $s(h_1) = s(h_2)$ then h_1 is indifferent to h_2, denoted by $h_1 = h_2$

However, in some cases the definition of the score function does not give reliable results for comparing two HFEs.

For example, consider $h_1 = \{.1, \ .2, \ .3\}$ and $h_2 = \{.1, \ .3\}$ be two HFEs, then by of score function

$$s(h_1) = \frac{.1 + .2 + .3}{3} = .2$$

$$s(h_2) = \frac{.1 + .3}{2} = .2$$

Thus, $s(h_1) = s(h_2)$, but the HFEs are different.

Later, Chen et al. [3] defined the concept of deviation degree.

Definition 5:

- Deviation function

For a HFE h deviation degree is defined as Chen et al. [3]

$$\overline{\sigma}(h) = \left(\frac{1}{l_h}\sqrt{\sum_{\gamma \in h}(\gamma - s(h))^2}\right)^{\frac{1}{2}}$$

It is the standard variance, which reflects the deviation degree between all values in the HFE h and their mean value.

With the help of variance function Chen et al. [3] gave a new method for comparison of HFEs:

Definition 6:

Let h_1 and h_2 be two hesitant fuzzy elements, $s(h_1)$ and $s(h_2)$ be its score functions and $\overline{\sigma}(h_1)$ and $\overline{\sigma}(h_2)$ be its deviation degrees, if

- $s(h_1) > s(h_2)$ then h_1 is superior to h_2, denoted by $h_1 > h_2$

If $s(h_1) = s(h_2)$

1. If $\overline{\sigma}(h_1) = \overline{\sigma}(h_2)$ then $h_1 = h_2$
2. If $\overline{\sigma}(h_1) > \overline{\sigma}(h_2)$ then $h_1 < h_2$
3. If $\overline{\sigma}(h_1) < \overline{\sigma}(h_2)$ then $h_1 > h_2$

Definition 7:

- Triangular hesitant fuzzy set

Decision makers sometimes give imprecise and incomplete data. For this reason it will be better if we use triangular fuzzy number as it provides real information given by decision maker. To define triangular hesitant fuzzy set, let X be a finite set. Then the triangular hesitant fuzzy set is defined as

$$T = \{x, H_T(x) : x \in X\}$$

Where $H_T(x)$ is the set some triangular fuzzy numbers of the element $x \in X$. $H_T(x)$ is called the triangular hesitant fuzzy element. The triangular fuzzy number in $H_T(x)$ is denoted by h and is represented by

$$t = (a_i, b_i, c_i) \text{ where } t \in h$$

Definition 8:

- Expected value for triangular hesitant fuzzy element

The expected value for triangular hesitant fuzzy element h is defined as

$$E(h) = \frac{1}{3\#h} \sum_{t \in h} (a_i + b_i + c_i)$$

where $\#h$ is the number of triangular fuzzy number in h.

Definition 9:

- Proposed expected value for triangular hesitant fuzzy element

The proposed expected value for triangular hesitant fuzzy element h is defined as

$$E(H_i) = \frac{1}{3}(\frac{\sum a_{1i}}{n} + \frac{\sum b_{1i}}{n} + \frac{\sum c_{1i}}{n}) + (\frac{\sum a_{1i}}{n} \frac{\sum b_{1i}}{n} \frac{\sum c_{1i}}{n})^{1/3}$$

Where $H_i = (a_{1i}, b_{1i}, c_{1i}); i = 1, 2, \ldots, n$

Let $H_1 = \{(1, 2, 3), (2, 3, 4)\} \& H_2 = \{(0.5, 2, 3.5), (1, 3.5, 4.5)\}$. Then

$$E(H_1) = \frac{1}{3}.\frac{1}{2}(1 + 2 + 2 + 3 + 3 + 4) = 2.5$$

$$E(H_2) = \frac{1}{3}.\frac{1}{2}(.5 + 1 + 2 + 3.5 + 3.5 + 4.5) = 2.5$$

Though $H_1 \neq H_2$, the expected value are same.

Now calculating expected value by our proposed technique, we get

$$E(H_1) = \frac{1}{2}\left[\frac{1}{3}(\frac{1+2}{2} + \frac{2+3}{2} + \frac{3+4}{2}) + (\frac{1+2}{2}.\frac{2+3}{2}.\frac{3+4}{2})^{1/3}\right] = 3.6795$$

$$E(H_2) = \frac{1}{2}\left[\frac{1}{3}(\frac{.5+1}{2} + \frac{2+3.5}{2} + \frac{3.5+4.5}{2}) + (\frac{.5+1}{2}.\frac{2+3.5}{2}.\frac{3.5+4.5}{2})^{1/3}\right] = 3.5103$$

So, our technique produces more relevant result.

4 Methodology for Multi Criteria Decision Making Using Triangular Hesitant Fuzzy Sets

Consider $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives (cause of disease) and $C = \{C_1, C_2, \ldots, C_n\}$ be the set criteria (diseases). The decision making matrix of triangular Hesitant fuzzy set can be represented as follows.

$$H = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1n} \\ h_{21} & h_{22} & \ldots & h_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ h_{m1} & h_{m2} & \ldots & h_{mn} \end{bmatrix}$$

Here, h_{ij} is the triangular fuzzy hesitant element.

Calculating the expected value by our proposed formula:

$$E(H_i) = \frac{1}{3}\left(\frac{\sum a_{1i}}{n} + \frac{\sum b_{1i}}{n} + \frac{\sum c_{1i}}{n}\right) + \left(\frac{\sum a_{1i}}{n}\frac{\sum b_{1i}}{n}\frac{\sum c_{1i}}{n}\right)^{1/3}$$

Where $H_i = (a_{1i}, b_{1i}, c_{1i}); i = 1, 2, \ldots, n$

Each triangular hesitant fuzzy element $h_{ij}(i = 1 \ldots m, j = 1 \ldots j)$ in the triangular hesitant fuzzy decision matrix takes the form as:

$$E = \begin{bmatrix} E_{11} & E_{12} & \ldots & E_{1n} \\ E_{21} & E_{22} & \ldots & E_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ E_{m1} & E_{m2} & \ldots & E_{mn} \end{bmatrix}$$

Now, the weight vector $w = (w_1, w_2, \ldots, w_n)$ for the criteria will be evaluated by the following formula:

$$w_j = \frac{f_j(C_j)}{\sum\limits_{j=1}^{n} f_j(C_j)}$$

Here, $w_j \geq 0 \,\&\, \sum\limits_{i=1}^{n} w_i = 1 \& f_i(C_i)(j = 1 \ldots n)$ is the standard deviation of the expected values for different alternatives with respect to a criterion.

$$f_j(C_j) = \sqrt{\frac{1}{m}\sum_{j=1}^{m}(E_{ij} - \frac{1}{m}\sum_{i=1}^{n} E_{ij})^2}$$

Finally, the weighted expected value for each alternative $A_i(i = 1 \ldots m)$ can be obtained by the formula and ranking can be done easily.

$$E(A_i) = \sum_{j=1}^{n} w_j E_{ij}$$

5 A Case Example

In 1998 a study was done on police officers for the development of their mental health due to various issues they faced during their service such as death of peoples, stress on workplaces etc. A set of 10 diseases are taken for example and set of five causes are taken for our example. The set of disease are as: 1. Irritable bowel syndrome (D_1), 2. Constipation (D_2), 3. High blood pressure (D_3), 4. Headaches (D_4), 5. Cardio vascular diseases (D_5), 6. Insomnia (D_6), 7. Nausea (D_7), 8. Diabetic (D_8), 9. Arthritis (D_9), 10. Mental Fatigue (D_{10}). The alternatives are taken cause of disease as: 1. Stress (A_1), 2. Tension (A_2), 3. Depression (A_3), 4. Anexiety (A_4), 5. Anger (A_5). We are considering the following linguistic variable (Table 1) Chen [2] for the case example:

Table 1. Linguistic Variables in TFN

Linguistic variable	Linguistic values
VL	(0, 0, 0.1)
L	(0, 0.1, 0.3)
ML	(0.1, 0.3, 0.5)
M	(0.3, 0.5, 0.7)
MH	(0.5, 0.7, 0.9)
H	(0.7, 0.8, 0.9)
VH	(0.8, 0.9, 1.0)

The rating of each alternative for each criteria are given by four experts and the rating are explained by the following table (Table 2):

Table 2. Rating of Alternatives

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
A_1	H	H	H	VH	H	H	H	MH	L	VH
	M	M	MH	H	H	VH	MH	M	VL	H
	MH	MH	H	H	MH	MH	ML	M	VL	H
	ML	ML	H	MH	H	H	H	MH	L	MH
A_2	H	H	H	VH	H	M	ML	M	L	H
	MH	MH	VH	MH	M	M	L	ML	VL	H
	M	M	H	H	MH	ML	ML	L	VL	MH
	M	MH	H	M	M	M	L	ML	VL	H
A_3	H	H	VH	VH	H	VH	MH	ML	ML	VH
	H	VH	H	H	MH	H	H	M	L	VH
	H	H	VH	VH	H	VH	ML	L	L	H
	H	MH	H	VH	H	VH	H	MH	M	VH

(*continued*)

Table 2. (*continued*)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}
A_4	ML	L	H	VL	L	H	MH	VL	L	H
	H	VL	MH	M	VL	VH	H	L	VL	MH
	H	L	ML	ML	VL	H	ML	VL	L	VH
	MH	ML	M	M	L	H	M	L	VL	H
A_5	ML	L	H	H	H	L	VL	L	L	M
	MH	ML	MH	VH	MH	VL	ML	VL	VL	ML
	M	VL	VH	H	ML	ML	M	M	L	M
	L	L	H	MH	M	M	L	ML	L	MH

The linguistic variables in the above table are expressed in terms of hesitant triangular fuzzy sets as given below (Table 3):

Table 3. Linguistic Variables in HTFN

	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10
A_1	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0, 0.1, 0.3)	(0.8, 0.9, 1.0)
	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0.5, 0.7, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0, 0, 0.1)	(0.7, 0.8, 0.9)
	(0.5, 0.7, 0.9)	(0.5, 0.7, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.5, 0.7, 0.9)	(0.1, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0, 0, 0.1)	(0.7, 0.8, 0.9)
	(0.1, 0.3, 0.5)	(0.1, 0.3, 0.5)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0, 0.1, 0.3)	(0.5, 0.7, 0.9)
A_2	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.3, 0.5, 0.7)	(0.1, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)
	(0.5, 0.7, 0.9)	(0.5, 0.7, 0.9)	(0.8, 0.9, 1.0)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.1, 0.3, 0.5)	(0, 0, 0.1)	(0.7, 0.8, 0.9)
	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.1, 0.3, 0.5)	(0.1, 0.3, 0.5)	(0, 0.1, 0.3)	(0, 0, 0.1)	(0.5, 0.7, 0.9)
	(0.3, 0.5, 0.7)	(0.5, 0.7, 0.9)	(0.7, 0.8, 0.9)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.1, 0.3, 0.5)	(0, 0, 0.1)	(0.7, 0.8, 0.9)
A_3	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.5, 0.7, 0.9)	(0.1, 0.3, 0.5)	(0.1, 0.3, 0.5)	(0.8, 0.9, 1.0)
	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.8, 0.9, 1.0)
	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.1, 0.3, 0.5)	(0, 0.1, 0.3)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)
	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0.8, 0.9, 1.0)
A_4	(0.1, 0.3, 0.5)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)	(0, 0, 0.1)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0, 0, 0.1)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)
	(0.7, 0.8, 0.9)	(0, 0, 0.1)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0, 0, 0.1)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0, 0.1, 0.3)	(0, 0, 0.1)	(0.5, 0.7, 0.9)
	(0.7, 0.8, 0.9)	(0, 0.1, 0.3)	(0.1, 0.3, 0.5)	(0.1, 0.3, 0.5)	(0, 0, 0.1)	(0.7, 0.8, 0.9)	(0.1, 0.3, 0.5)	(0, 0, 0.1)	(0, 0.1, 0.3)	(0.8, 0.9, 1.0)
	(0.5, 0.7, 0.9)	(0.1, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0, 0, 0.1)	(0.7, 0.8, 0.9)
A_5	(0.1, 0.3, 0.5)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0.7, 0.8, 0.9)	(0, 0.1, 0.3)	(0, 0, 0.1)	(0, 0.1, 0.3)	(0, 0.1, 0.3)	(0.3, 0.5, 0.7)
	(0.5, 0.7, 0.9)	(0.1, 0.3, 0.5)	(0.5, 0.7, 0.9)	(0.8, 0.9, 1.0)	(0.5, 0.7, 0.9)	(0, 0, 0.1)	(0.1, 0.3, 0.5)	(0, 0, 0.1)	(0, 0, 0.1)	(0.1, 0.3, 0.5)
	(0.3, 0.5, 0.7)	(0, 0, 0.1)	(0.8, 0.9, 1.0)	(0.7, 0.8, 0.9)	(0.1, 0.3, 0.5)	(0.1, 0.3, 0.5)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.3, 0.5, 0.7)
	(0, 0.1, 0.3)	(0, 0.1, 0.3)	(0.7, 0.8, 0.9)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0.3, 0.5, 0.7)	(0, 0.1, 0.3)	(0.1, 0.3, 0.5)	(0, 0.1, 0.3)	(0.5, 0.7, 0.9)

Now, we determine the expected value by our proposed technique and get the decision making matrix:

$$
\begin{array}{c}
 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & D_8 & D_9 & D_{10}
\end{array} \\
\begin{bmatrix}
0.584217 & 0.584217 & 0.819332 & 0.836816 & 0.819332 & 0.836816 & 0.675451 & 0.58845 & 0.041667 & 0.836816 \\
0.819332 & 0.58845 & 0.87932 & 0.743729 & 0.632122 & 0.434086 & 0.187703 & 0.286999 & 0.029167 & 0.819332 \\
0.869164 & 0.836816 & 0.896579 & 0.913746 & 0.819332 & 0.913746 & 0.675451 & 0.393155 & 0.245368 & 0.913746 \\
0.675451 & 0.123936 & 0.584217 & 0.319283 & 0.041667 & 0.87932 & 0.584217 & 0.041667 & 0.041667 & 0.836816 \\
0.393155 & 0.123936 & 0.836816 & 0.836816 & 0.584217 & 0.224838 & 0.224838 & 0.224838 & 0.054167 & 0.485885
\end{bmatrix}
\end{array}
$$

Now the weight of the criteria obtained from the formula $w_j = \frac{f_j(C_j)}{\sum_{j=1}^{n} f_j(C_j)}$ where,

$$f_j(C_j) = \sqrt{\frac{1}{m}\sum_{j=1}^{m}(E_{ij} - \frac{1}{m}\sum_{i=1}^{n} E_{ij})^2}$$

we get:

$w_1 = 0.086638$, $w_2 = 0.143311866$, $w_3 = 0.057302$, $w_4 = 0.10767$, $w_5 = 0.14468$, $w_6 = 0.14054$, $w_6 = 0.14054$, $w_7 = 0.110452$, $w_8 = 0.091882$, $w_9 = 0.041508$, $w_{10} = 0.076017$.

Therefore, the weighted expected values of the alternatives (causes of disease) can be obtained by the formula $E(A_i) = \sum_{j=1}^{n} w_j E_{ij}$ as:

$$E(A_1) = 0.701551$$

$$E(A_2) = 0.548839$$

$$E(A_3) = 0.782319$$

$$E(A_4) = 0.407441$$

$$E(A_5) = 0.390674$$

Therefore, the final ranking can be done from the expected values are

$$A_3 > A_1 > A_2 > A_4 > A_5$$

This implies that the diseases are caused mostly by the cause A_3,i.e., by depression.

6 Conclusion and Discussion

In this paper, a methodology has been developed to become aware of the most possible reason of illness of any person on the basis the illness, behavior, work culture, food habits etc. In expressing the uncertainty hesitant triangular fuzzy number has been used for better handling of uncertainty. In addition, a new idea of expected value has been introduced in this paper and with the help of an example it has been shown that our proposed expected value is more relevant than the existing one. Also, a numerical example has been worked out in which 10 disease has been taken as criteria and five causes are taken as alternatives. And finally a valid and proved conclusion has been drawn from the example and showed that utility of the proposed methodology.

Our proposed technique works well when we have a large number of risks because it helps spread the impact of the risks. Sometimes we may miss the inclusion of positive risk, which may affect the final outcome. While using our proposed expected value our risk attitude should be neutral, otherwise it may affect the final calculation.

Our proposed expected value can be modified and can be applied in Multi- Criteria Decision making Problem where expected value calculation is required.

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Optimization of Multi-objective Stochastic Linear Programming Problem in Fuzzy Environment: An Iterative-Interactive Optimization Process

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Abstract. In this article, we consider Multi-Objective Stochastic Linear Programming Problem (MOSLPP) and present a general iterative-interactive optimization process (IIOP) for determining preferable Pareto optimal solution in fuzzy environment. Sakawa et al. originally presented an interactive method to optimize MOSLPP in fuzzy environment. They considered the expectation of each fuzzy objective function of MOSLPP to simultaneously attain respective goal. Accordingly, they had set initial reference membership level as unity for all cases. However, we observe in existing methods that expectations of conflicting objective functions cannot simultaneously attain respective goals in fuzzy environment. In addition, we cannot effectively specify any objective function as main objective function that has highest priority of the decision maker, in fuzzy environment in several other multi-objective optimization methods. Here in proposed IIOP, decision maker can set an objective function as main objective function. Next, we employ trade-off ratios among membership functions of expectations of objective functions in fuzzy environment to elicit corresponding reference membership levels and thereby determine preferable Pareto optimal solution to MOSLPP in fuzzy environment. We illustrate proposed IIOP through numerical application of a multi-objective supply chain management model along with numerical examples. We present managerial insights by performing sensitivity analysis of key parameters of this model. Finally, we draw conclusions.

Keywords: Fuzzy optimization · Multi-objective stochastic linear programming · Iterative-interactive optimization process · Supply chain management · Trade-off ratio

1 Introduction

Multi-objective linear programming is the process of optimizing simultaneously and systematically a collection of objective functions [37]. In order to effectively specify preferences of decision maker (DM), researchers have presented several multi-objective

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linear programming methods in literature. In real life cases, they specify fuzzy programming and stochastic programming to be amongst most popular methods for solving Multi-Objective Linear Programming Problems (MOLPP) [22, 24].

In fuzzy environment, researchers [9, 11, 19] have considered aspiration level of each objective function and thus characterized Pareto optimal solution to MOLPP. Historically, Bellman and Zadeh [3] presented decision making in fuzzy environment. Recently, Luhandjula [22] praised them for raising intellectual respectability of fuzzy optimization. Zimmermann [39] categorized various kinds of impreciseness as fuzziness and consequently presented a method for solving MOLPP in fuzzy environment. Next, Sakawa et al. [27] developed interactive method in fuzzy environment for solving MOLPP by connecting DM within solution process. In this method, DM controls search direction of optimal solution to MOLPP in fuzzy environment. Consequently, corresponding Pareto optimal solution achieve DM's imprecise aspiration levels. We find that researchers have applied various fuzzy optimization methods and there by solved many real world problems, e.g. Werners [33], Lai and Hwang [20], Wu et al. [35], Garg H. [13, 14], Garg et al. [15–17], Wei et al. [31], Salehi et al. [28], Su et al. [30], Garai et al. [10, 12] etc. Ebrahimnejad and Verdegay [7], Chakraborty et al. [4] and Garai et al. [9] studied real life based multi-objective optimization models in fuzzy environment. Also, researchers investigated MOLPP with fuzzy objective functions, interactive methods, Karush-Kuhn-Tucker optimality conditions and all that; e.g. Bahman et al. [2] developed an improved fuzzy self-adaptive learning multi-objective particle swarm optimization algorithm for dynamic economic emission dispatch problem by applying interactive fuzzy optimization method in stochastic environment. Hence from extensive literature survey that yields scores of fuzzy optimization methods with applications, we find interaction with DM to be essential for generating preferable Pareto optimal solution to MOLPP in fuzzy environment. Moreover, in numerous cases, DM decides on joining analyst during optimization process to comprehend business thoroughly [24–27]. Besides, one should note of the fact that despite having a highly preferred objective function in real life based MOLPP, DM cannot always assign weights, utility functions and all that as corresponding relative significance.

In existing interactive fuzzy optimization method, firstly developed by Sakawa and subsequently applied by fellow researchers [24–27], we observe that DM cannot effectively specify an objective function as main objective function. Also, in this method, we detect that analyst estimates same relative significance to each objective function in fuzzy environment. However, we better not expect each conflicting fuzzy objective function to attain corresponding goal concurrently [11]. Accordingly, we propose that initial reference membership level of each fuzzy objective function needs to be based on analytically derived results. We find through extensive literature survey that DM prefers an optimal solution, at which analyst presents much desirable optimal value to main objective function along with satisfactory optimal values to other objective functions to MOLPP in fuzzy environment [11]. Moreover, interaction with DM at each step of optimization process wastes precious time and money of DM. In addition, DM can remain unavailable during optimization process. One ought to consider fundamental questions like preference of never-perfect model based compromise over time of top-rank DM [33, 34].

On the other hand, it seems natural to consider that impreciseness should be expressed by a fusion of fuzziness and randomness rather than by either fuzziness or randomness in most of real life situations [25]. To effectively handle DM's ambiguous judgements in MOLPP as well as randomness of parameters involved in objective functions and constraints, Sakawa and other fellow scientists incorporated interactive fuzzy satisfying methods associated with deterministic problems into Multi-Objective Stochastic Linear Programming Problems (MOSLPP) [26, 27]. They describe how a mathematical model can be formulated for random data variations in chance constrained problems to permit violation of constraints up to specified probability levels [24]. In last few decades, researchers have introduced several stochastic programming models like expectation optimization, variance minimization, probability maximization, fractile criterion optimization together with chance constrained programming methods to derive DM's satisficing solution from set of Pareto optimal solutions [22, 24, 39].

Thus, we observe in this article that minimal yet efficient interaction with DM is more sensible and scientific [11]. Accordingly, we plan to develop an iterative-interactive optimization process (IIOP), which is based on specified main objective function and generates most preferable Pareto optimal solution to MOLPP in fuzzy environment. Here we decide to opt for expectation optimization model in stochastic environment so as to easily digest proposed IIOP without unnecessarily deviating into other details.

We organize rest of article as follows. The next section contains some preliminaries to be used later in the paper. In Sect. 3, we introduce expectation model of MOSLPP in fuzzy environment. Here we discuss existing interactive fuzzy optimization method. In Sect. 4, we determine Pareto optimal solution based on specified main objective function in fuzzy environment and presents an IIOP for finding preferable Pareto optimal solution based on specified main objective function to MOSLPP in fuzzy environment. In Sect. 5, we formulate a multi-objective supply chain management model and thereby optimize it by applying proposed IIOP in fuzzy environment. Also, we perform sensitivity analysis of vital parameters and provide key managerial insights. Finally, we draw conclusions and identify future scopes of research in Sect. 6.

2 Preliminaries

Fuzzy sets were first introduced by Zadeh [37] in 1965 as a mathematical way of representing impreciseness or vagueness in everyday life.

Definition 1 *(Fuzzy set).* A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as the following set of pairs $\tilde{A} = \{(x, \mu_A(x) : x \in X\}$. Here $\mu_A : X \to [0, 1]$ is a mapping called the membership function of the fuzzy set $\tilde{A}$ and $\mu_A(x)$ is called the membership value or degree of membership of $x \in X$ in the fuzzy set $\tilde{A}$.

Definition 2 *(Normal fuzzy set).* A fuzzy set $\tilde{A}$ of the universe of discourse X is called a normal fuzzy set implying that there exist at least one $x \in X$ such that $\mu_A(x) = 1$. Otherwise the fuzzy set is subnormal.

2.1 Multi Objective Non-linear Optimization Problem (MONLP)

A MONLP may be taken in the following form:

$$\begin{aligned} &\text{minimize/maximize } \mathrm{z}(x) = (z_1(x)\ldots z_k(x))^T \\ &\text{subject to: } x \in \mathrm{X} = \left\{x \in R^n : g_j(x) \le or = or \ge b_j,\ j = 1\ldots..m\right\} \\ &l_i \le x_i \le u_i,\ i = 1, 2\ldots\ldots n; \end{aligned} \tag{1}$$

Some basic definitions on Pareto optimal solutions are introduced below.

Definition 3 *(Complete optimal solution).* x^* is said to be a complete optimal solution to the MONLP (1) if and only if there exists $x^* \in X$ such that $f_r(x^*) \le f_r(x)$, for $r = 1, 2\ldots\ldots k$ and for all $x \in X$.

However, when the objective functions of the MONLP conflict with each other, a complete optimal solution does not always exist. Thus, the concept of Pareto optimality arises. We define it as follows:

Definition 4 *(Pareto optimal solution).* x^* is said to be a Pareto optimal solution to the MONLP (1) if and only if there does not exist another. $x \in X$ such that $f_r(x^*) \le f_r(x)$ for all $r = 1, 2\ldots\ldots k$ and $f_j(x) \ne f_j(x^*)$ for at least one $j,\ j \in \{1, 2\ldots k\}$.

Definition 5 *(Local optimal solution).* $x^* \in X$ (The feasible set of constrained decisions) is said to be a local Pareto optimal solution to the MONLP (1) if and only if x^* is Pareto optimal in $X \cap N(x^*, \delta)$ where $N(x^*, \delta)$ denotes the δ neighbourhood of x^* defined by $\{x \in R^n \| |x - x^*|| < \delta,\ \delta \in R^+\}$.

3 Expectation Model of Fuzzy MOSLPP

3.1 Formulation of MOSLPP

We assume that coefficients in objective functions and right hand side constants of constraints are random variables and thereby formulate general MOSLPP as follows:

$$\begin{aligned} &\text{minimize } (z_1(x)\ldots z_k(x))^T \\ &\text{subject to} \\ &\mathrm{A}x \le \bar{b}, \\ &x \ge 0, \end{aligned} \tag{2}$$

Here x is an n-dimensional column vector of decision variables i.e. $x = (x_1\ldots x_n)^T$. A is $m \times n$ coefficient matrix; $z_i(x),\ i = 1\ldots k$ are k conflicting objective functions, defined by $z_i(x) = \bar{c}_i x,\ \forall i = 1\ldots k; \bar{c}_i,\ \forall i = 1\ldots k$ are n-dimensional random variable row vectors with finite means $E(\bar{c}_i)$ and $n \times n$ positive definite variance covariance matrices $V_i = [v^i_{jh}] = \left[Cov\left\{c_{ij}, c_{ih}\right\}\right],\ \forall i = 1\ldots k$ and $\bar{b} = (b_1\ldots b_n)^T$ is n-dimensional column vector, both of whose elements are random variables. We assume that a_q is the q$^{\text{th}}$ row vector of A and $\bar{b}_q$ is the q$^{\text{th}}$ element of $\bar{b}$. Whereas other coefficients can also be random variables, we keep model (2) simpler by considering only $\bar{c}_i,\ \forall i =$

$1 \dots k$ and $\bar{b}$ as random variables. We apply solution methods appropriate to stochastic events. Here we employ chance constrained conditions that permit constraint violations up to specified probability limits. Also, we consider random variable $\bar{b}_q$ to possess continuous probability distribution function, as given below:

$$F_q(r) = P(\bar{b}_q \le r)$$

Hence $\forall q = 1 \dots m$, $\mathrm{P}\big(a_q x \le \bar{b}_q\big) \ge \beta_q$ implies that $a_q x \le F_q^{-1}(1-\beta_q)$. Here we define $X(\beta)$ as follows:

$$X(\beta) = \Big\{x \mid a_q x \le F_q^{-1}(1-\beta_q),\ q = 1 \dots m,\ x \ge 0\Big\}$$

We replace constraints in model (2) with chance constrained conditions and with satisfying probability levels β_i, $\forall i = 1 \dots k$. Hence, we reformulate model (2) to following chance constrained model:

$$\begin{aligned} &\text{minimize } (z_1(x) \dots z_k(x))^T \\ &\text{subject to} \\ &x \in X(\beta) \end{aligned} \tag{3}$$

3.2 Expectation Model of MOSLPP

Here we present expectation model in brief. In expectation model, DM minimizes expected values of objective functions, subject to constraints in order to deal with MOSLPP. Here we obtain MOSLPP by replacing objective functions $z_i(x) = \bar{c}_i x$, $\forall i = 1 \dots k$ with corresponding expectations as follows:

$$\begin{aligned} &\text{minimize } \big(z_1^E(x) \dots z_k^E(x)\big)^T \\ &\text{subject to} \\ &x \in X(\beta) \end{aligned} \tag{4}$$

Here $z_i^E(x) = E\big[z_i(x)\big] = E[\bar{c}_i]x$ denotes expectation of $z_i(x) = \bar{c}_i x$, $\forall i = 1 \dots k$.

3.3 Expectation Model of MOSLPP with Existing Interactive Fuzzy Optimization Method

Researchers recognize how concept of Pareto optimality is fundamental to MOSLPP (4). By considering imprecise nature inherent in our judgement to MOSLPP (4), we quantify linguistic statement by generating membership function $\mu_i(z_i^E(x))$, $i = 1 \dots k$ from predetermined goals and tolerance values of expectation $z_i^E(x)$, $i = 1 \dots k$ of corresponding objective function $z_i(x)$, $i = 1 \dots k$ in fuzzy environment [6, 24]. Consequently, we convert MOSLPP into following equivalent optimization model:

$$\begin{aligned} &\max\min\big\{\mu_i(z_i^E(x)),\ i = 1 \dots k\big\} \\ &\text{subject to} \\ &x \in X(\beta) \end{aligned} \tag{5}$$

We find that approach to model (5) is preferable, only when fuzzy decision is proper representation of fuzzy preferences of DM [6]. However, as Sakawa and other fellow researchers have described in [24], this situation occurs rarely in practice. In their interactive fuzzy optimization method, DM specifies aspiration level of achievement for membership function (in other words, reference membership level $\hat{\mu}_i$) of expectation $z_i^E(x),\ i = 1\ldots k$ of objective function $z_i(x),\ i = 1\ldots k$. Next, we obtain corresponding M-Pareto optimal solution closest to $\hat{\mu} = \left(\hat{\mu}_1,\ \hat{\mu}_2 \ldots \hat{\mu}_k\right)^T$ or even better than that by solving following min-max problem:

$$\begin{aligned}
&\min \max_{i=1\ldots k} \left(\hat{\mu}_i - \mu_i(z_i^E(x))\right) \\
&\text{subject to} \\
&x \in X(\beta).
\end{aligned} \tag{6}$$

Equivalently when we set $v = \max\limits_{i=1\ldots k} \left(\hat{\mu}_i - \mu_i(z_i^E(x))\right)$, we obtain following single objective optimization model:

$$\begin{aligned}
&\text{minimize } v \\
&\text{subject to} \\
&\hat{\mu}_i - \mu_i\left(z_i^E(x)\right) \le v,\ i = 1\ldots k, \\
&x \in X(\beta).
\end{aligned} \tag{7}$$

4 Pareto Optimal Solution Based on Specified Main Objective Function in Fuzzy Environment

Before concentrating on optimization method, here we consider a key concern of DM. In any real life based MOLPP, we detect that DM has distinctive preference for one objective function among all. In this article, we call it main objective function. However, in existing interactive fuzzy optimization method to solve MOSLPP [24–27], researchers treat all objective functions uniformly. So, there is little scope for DM to effectively specify any main objective function. Consequently, DM is unable to find desirable Pareto optimal solution to MOSLPP in fuzzy environment in many real life-based cases. We find it as a major weakness in existing interactive fuzzy optimization methods for solving MOSLPP. In this article, we consider an objective function as main objective function and subsequently develop an IIOP to solve MOSLPP in fuzzy environment.

Again, one key job in existing interactive fuzzy optimization method for solving MOSLPP is to determine reference membership levels $\hat{\mu}_1,\ \hat{\mu}_2 \ldots \hat{\mu}_k$ of expectations $z_i^E(x),\ i = 1\ldots k$ of fuzzy objective functions $z_i(x),\ i = 1\ldots k$. We find in existing interactive fuzzy optimization method [24] that initial reference membership levels $\hat{\mu}_1$, $\hat{\mu}_2 \ldots \hat{\mu}_k$ are taken as one in all cases. However, we detect that conflicting objective functions do not necessarily attain individual goals simultaneously. Hence it cannot be much judicious to concurrently consider each initial reference membership level as one. In this article, we compute initial reference membership levels analytically by employing trade-off ratios among membership functions of expectations of fuzzy objective functions as follows:

Initially we take expectation of any objective function, say $z_t^E(x)$, $t \in \{1 \ldots k\}$ arbitrarily. Next we employ chain rule and subsequently compute trade-off ratio π_{tj} between membership function $\mu_t(z_t^E(x))$ of $z_t^E(x)$ and membership functions $\mu_j(z_j^E(x))$, $j = 1 \ldots k$, $j \neq t$ of expectations of other objective functions $z_j^E(x)$, $j = 1 \ldots k$, $j \neq t$, one by one, as follows [24]:

$$\pi_{tj} = -\frac{\partial \mu_t(z_t^E(x))}{\partial \mu_j(z_j^E(x))} = -\frac{\partial \mu_t(z_t^E(x))}{\partial z_t^E(x)} \frac{\partial z_t^E(x)}{\partial z_j^E(x)} \left(\frac{\partial \mu_j(z_j^E(x))}{\partial z_j^E(x)} \right)^{-1}, \ j = 1 \ldots k, \ j \neq t$$

Thereby we derive a set of numbers $\bar{\mu}_1 \ldots \bar{\mu}_k$ by using following formula [24]:

$$\bar{\mu}_j = \left| \left(-\frac{\partial \mu_t(z_t^E(x))}{\partial \mu_j(z_j^E(x))} \right)^{-1} \right| \bar{\mu}_t, \ j = 1 \ldots k, \ j \neq t$$

We write $\bar{\mu}_t = \xi > 0$ and define $\hat{\mu}$ as $\hat{\mu} = \mathrm{m}ax\{\bar{\mu}_1 \ldots \bar{\mu}_k\}$. Thus $\hat{\mu} \neq 0$. Consequently; we set initial reference membership level $\hat{\mu}_i^{(0)}$, $i = 1 \ldots k$ of expectation of i$^{\text{th}}$ objective function as follows [24]:

$$\hat{\mu}_i^{(0)} = \bar{\mu}_i / \hat{\mu}, \forall i = 1 \ldots k \ (\because \hat{\mu} \neq 0) \tag{8}$$

In this article, we make use of these reference membership levels of expectations of fuzzy objective functions to determine M-Pareto optimal solution to generalized MOSLPP (3).

IIOP for finding Pareto optimal solution to MOSLPP in fuzzy environment:
We integrate above thoughts into a general framework and develop an IIOP to determine Pareto optimal solution to MOSLPP (2) in fuzzy environment. We have already observed that this solution is based on specified main objective function and thereby is much preferable to DM. We specify steps of proposed IIOP as follows:

Step 1: First, DM specifies satisfying probability level $(\beta_q, q = 1 \ldots m)$ to each constraint. Next, we compute individual maximum and minimum values of expectation $(z_i^E(x), i = 1 \ldots k)$ of each objective function $(z_i(x), i = 1 \ldots k)$ under given constraints. In case, we find an unbounded solution, we consider any large number as extremum. Then we supply the information to DM and thereby acquire goals and tolerance values of expectation of each objective function from DM.
Step 2: We employ goals and tolerance values to construct corresponding membership functions $\mu_i(z_i^E(x))$, $i = 1 \ldots k$. In case, DM is not available, or DM cannot specify any of these goals and tolerance values, we can consider corresponding individual optimum value.
Step 3: Next, we determine corresponding initial reference membership level $\hat{\mu}_i^{(0)}$, $i = 1 \ldots k$ analytically. In order to do this, we arbitrarily select expectation $z_t^E(x)$ of an objective function. Then we apply chain rule and compute trade-off ratios π_{tj} between its membership function $\mu_t(z_t^E(x))$ and membership functions $\mu_j(z_j^E(x))$, $j = 1 \ldots k$, $j \neq t$ of expectations $\left(z_j^E(x), \ j = 1 \ldots k, \ j \neq t\right)$ of other objective

functions as follows [24–27]:

$$\pi_{tj} = -\frac{\partial \mu_t(z_t^E(x))}{\partial \mu_j(z_j^E(x))} = -\frac{\partial \mu_t(z_t^E(x))}{\partial z_t^E(x)} \frac{\partial z_t^E(x)}{\partial z_j^E(x)} \left(\frac{\partial \mu_j(z_j^E(x))}{\partial z_j^E(x)} \right)^{-1}, \; j = 1 \ldots k, \; j \neq t.$$

Step 4: Here we derive a set of numbers $\bar{\mu}_1, \bar{\mu}_2 \ldots \bar{\mu}_k$ by using following formula:

$$\bar{\mu}_j = \left| \left(-\frac{\partial \mu_t(z_t^E(x))}{\partial \mu_j(z_j^E(x))} \right)^{-1} \right| \bar{\mu}_t, \; j = 1 \ldots k, \; j \neq t.$$

For $\bar{\mu}_t = \xi(> 0)$ and $\hat{\mu} = \max\{\bar{\mu}_1, \bar{\mu}_2 \ldots \bar{\mu}_k\}$, we determine initial reference membership levels $\hat{\mu}_i^{(0)}$, $i = 1 \ldots k$ by applying following formula in fuzzy environment:

$$\hat{\mu}_i^{(0)} = \bar{\mu}_i \big/ \hat{\mu}, \; \forall i = 1 \ldots k \; (\because \hat{\mu} \neq 0).$$

Step 5: We now request DM to specify main objective function. Let it be $z_\alpha(x)$, for some $\alpha \in \{1 \ldots k\}$. Here we employ absolute value function and thereby obtain following single objective optimization problem in fuzzy environment:

$$\begin{aligned}
&\text{minimize } v \\
&\text{subject to} \\
&\hat{\mu}_\alpha^{(0)} - \mu_\alpha(z_\alpha^E(x)) \leq v, \\
&|\hat{\mu}_i^{(0)} - \mu_i(z_i^E(x))| \leq v, \; \forall i = 1 \ldots k, \; i \neq \alpha \\
&v \geq 0, \; x \in X.
\end{aligned}$$

Let $x^{(1)}, v^{(1)}$ are optimal values of x and v respectively and $z_i^E(x^{(1)}), \mu_i(z_i^E(x^{(1)}))$, $\forall i = 1 \ldots k$ are optimal values of $z_i^E(x), \mu_i(z_i^E(x)), \forall i = 1 \ldots k$ respectively to above model after first iteration. This completes first iteration.

Step 6: Next, we determine updated reference membership levels after n iterations (n is natural number) by applying method of bisection as follows [25, 27]:

$$\forall i = 1 \ldots k, \; \hat{\mu}_i^{(n)} = \begin{cases} \left(\mu_i(z_i^E(x^{(n)})) + \hat{\mu}_i^{(n-1)} \right)/2, & \text{if } \mu_i(z_i^E(x^{(n)})) < \hat{\mu}_i^{(n-1)} \\ \left(1 + \mu_i(z_i^E(x^{(n)})) \right)/2, & \text{otherwise} \end{cases} \tag{9}$$

Step 7: By employing these updated reference membership levels $\hat{\mu}_i^{(n)}$, $\forall i = 1 \ldots k$, we solve following optimization model:

$$\begin{aligned}
&\text{minimize } v \\
&\text{subject to} \\
&\hat{\mu}_\alpha^{(n)} - \mu_\alpha(z_\alpha^E(x)) \leq v, \\
&|\hat{\mu}_i^{(n)} - \mu_i(z_i^E(x))| \leq v, \; \forall i = 1 \ldots k, \; i \neq \alpha, \\
&v \geq 0, \; x \in X.
\end{aligned}$$

Let $x^{(n+1)}$, $v^{(n+1)}$ are optimal values of x and v respectively and $z_i^E(x^{(n+1)})$, $\mu_i(z_i^E(x^{(n+1)}))$, $\forall i = 1 \ldots k$ are optimal values of $z_i^E(x)$, $\mu_i(z_i^E(x))$, $\forall i = 1 \ldots k$ respectively after $(n+1)$ iterations to above model.

Step 8: Proceeding in this fashion, we end when main objective function $z_\alpha(x)$ attains satisficing optimal values. Let us suppose that iteration stops after m steps and $x^{(m)}$, $z_j^E(x^{(m)})$, $\mu_j(z_j^E(x^{(m)}))$ are M-Pareto optimal values of x, $z_j^E(x)$, $\mu_j(z_j^E(x))$, $j = 1 \ldots k$ respectively. Now we test uniqueness of above M-Pareto optimal solution by solving following model [24, 25]:

$$\text{maximize } \sum_{i=1}^{k} \varepsilon_i$$

subject to

for minimizing type of objective functions: $z_i^E(x^{(m)}) - \varepsilon_i \geq z_i^E(x)$, $i = 1 \ldots k$,

$\big($for maximizing type of objective functions: $z_i^E(x) - \varepsilon_i \geq z_i^E(x^{(m)})$, $i = 1 \ldots k,\big)$

$\varepsilon_i \geq 0$, $i = 1 \ldots k$, $x \in X$.

Let $\bar{\varepsilon}_i$ and $\bar{x}$ are optimal values of ε_i and x respectively $\forall i = 1 \ldots k$. If $\bar{\varepsilon}_i = 0$, $\forall i = 1 \ldots k$, solution after m iterations $(x^{(m)})$ is Pareto optimal solution to given MOSLPP. Otherwise, solution $\bar{x}$ to above model is Pareto optimal solution to given MOSLPP.

Step 9: In case, DM is satisfied with Pareto optimal solution, we stop. Otherwise we request DM to revise goals, tolerance values and main objective function. Go to Step 2. IIOP is thus complete.

5 Multi-objective Supply Chain Management Model in Fuzzy Environment

We illustrate proposed IIOP by considering optimization of a multi-objective based Supply Chain Management (SCM) model. We find that ensuring competitiveness in today's globally connected marketplace is very demanding and calls for different business strategies than what businesses had employed in the past. Thus, inevitability of SCM research comes into picture. Scientists define supply chain as an integrated system or network synchronizing a series of inter-related business processes in order to achieve following:

a. acquires raw material;
b. adds value to raw materials by transforming them into finished products;
c. distributes these products to distribution centres and sell to retailers or customers;
d. facilitates flow of raw materials, finished goods, cash and information among various partners.

Again, scientists suggest that long term sustainability of SCM model requires optimization of objective functions not individually but simultaneously. Ghodsypour and O'Brien [18] developed an integrated analytical hierarchy process and linear programming approach for solving a multi-objective, single-item, single-period capacitated supplier selection and order lot-sizing problem. Xia and Wu [36] introduced a new model by using rough sets theory with a multi-objective mixed integer program to support supplier selection decisions in total business volume discounts environments. Sirias and Mehra [29] studied quantity-dependent discounts versus lead time-dependent discounts in supply chains through a simulation study. One can always look at the comprehensive review provided by Aissaoui et al. [1]. Literature review of SCM area reveals that there are considerable amount of papers integrating production and distribution decisions, e.g. [5, 8, 23]. Liang [21] developed an interactive multi-objective linear programming model for solving fuzzy multi-objective transportation problems with piecewise linear membership function.

Thus, we can find extensive research works that are based on various other aspects of sustainability of SCM, like environmental concern, customer satisfaction, brand value creation, rights of employees and all that.

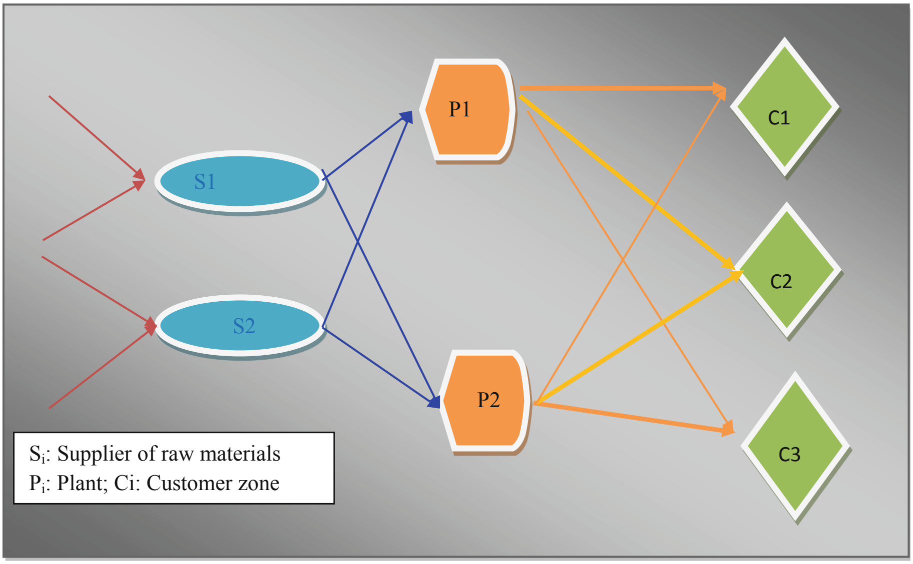

Fig. 1. Flow chart of logistics flows among suppliers, plants and customer zones of SCM model

But we find it surprising to have awfully very little research that requires optimization of several objective functions simultaneously but offer maximum priority to main objective function, without involving any weight or utility function. DM requires that there should be no deviation from goal of main objective function or this goal has to be achieved irrespective of other conflicting objective functions. In this article, we observe that one can employ trade off among several objective functions for achieving most preferable optimal value to main objective function. Hence, we can concentrate on this mathematical aspect only and thereby apply proposed IIOP to yield most preferable Pareto optimal solution to a SCM model. We propose the SCM model as follows.

Hypothesis of proposed model

In this article, in order to show effectiveness and efficiency of proposed IIOP, we consider a much simplified scenario with following assumptions:

i. There is exactly one finished product.
ii. There are two suppliers, two manufacturing plants and three customer zones. We illustrate their interconnections graphically in Fig. 1.
iii. All suppliers can supply all plants with finished product.
iv. We can ship raw materials to any of two plants for producing finished products.
v. We can transport finished products to different customer zones, based on demand.
vi. We consider only one time interval.
vii. We consider proposed model in fuzzy environment.
viii. DM specifies satisficing probability level as 0.8 to each constraint.

We describe other assumptions as and when we require them.

Description of model

In this article, we attempt to capture dynamics of single product in SCM model. We present notations of parameters along with their descriptions in Table 1. We present indices for formulation of proposed model in Table 2. We observe that all suppliers can send raw materials to any manufacturer. Again, based on demands, all manufacturers can send finished products to any distribution centre. In a more general case, some suppliers get preference over others depending on their previous performances, quality and timeliness of products delivered etc. We can have similar arrangements for manufacturing plants. On the other hand, some distribution centres can have more importance over others; and we have to ensure that these distribution centres never fall short of demand.

Table 1. Notations along with descriptions of parameters of SCM model

Entity	Notation	Description
Supplier	$L(j)$	Capacity of customer 'j' for the product
	$CS(j)$	Cost of making product by supplier 'j'
	$STC(j, k)$	Transportation cost of product from supplier 'j' to plant 'k'
Plant	$U(k)$	Capacity f plant 'k'
	$LC(k)$	Labour cost of plant 'k'
	$MC(k)$	Manufacturing cost of plant 'k'
	$IC(k)$	Inventory cost of plant 'k'
	$PTC(k, l)$	Plant transportation cost of plant 'k' to customer zone 'l'
Customer zone	$D(l)$	Demand at customer zone 'l'
	$SP(l)$	Selling price at customer zone 'l' of a product

Table 2. Indices used during formulation of SCM model

Index	Description	Maximum value
j	Supplier	2
k	Plant	2
l	Customer zone	3

Decision variables
We employ three types of decision variables in this model. They are

a. vendor shipment variables,
b. plant shipment variables and
c. inventory variables.

We present detailed list of decision variables in Table 3. We note that there are 13 non-negative decision variables in this model and all of which can take only integer values.

Table 3. Notations and descriptions of decision variables of SCM model

Variable	Description
$x_{j,k}$	Amount of product from supplier 'j' to plant 'k'
$y_{k,l}$	Amount of product from plant 'k' to customer zone 'l'
i_k	Inventory of product at plant 'k'

Formulation of mathematical model
Here we consider *three* objective functions and *four* sets of constraints as follows:

Objective functions
This section provides description of objective functions of proposed SCM model as follows:

a. We observe that single most important obligation of any SCM model is generating revenue. Hence, we plan to maximize total revenue of SCM model. For simplicity, we consider selling price at customer zone 'l' for products shipped from plant 'k' to customer zone 'l' for specified time interval and describe Total External Revenue (TER) of supply chain as follows:

$$\text{maximize } REV = \sum_k \left(\sum_l y_{k,l}.SP(l) \right)$$

On the other hand, literature survey finds that minimizing each component of operating cost is an important performance metric in SCM operations. In this article, we plan to minimize total operating cost. It consists of two components, viz. Total Transportation Costs (TTC) and Total Procurement Costs (TPC).

b. We define TTC as sum of transportation costs from supplier 'j' to plant 'k' and from plant 'k' to customer zone '*l*' for specified time interval. Therefore, for specified time interval, we describe TTC as follows:

$$\text{minimize } TTC = \sum_j \sum_k \left(x_{j,k}.s_j.STC(j,k)\right) + \sum_k \sum_l \left(y_{k,l}.PTC(k,l)\right)$$

c. Again we define TPC as total cost of procuring raw materials by supplier 'j', multiplied by the quantity shipped to plant 'k' by supplier 'j' for specified time interval. Therefore, for specified time interval, we describe TPC as follows:

$$\text{minimize } TPC = \sum_j \left(CS(j).s_j.x_{j,k}\right)$$

Constraints

Following constraints model shipment capacities of plants, capacities of suppliers, inventory-balancing cost and total operating cost:

a. Shipment capacity of supplier

For each supplier 'j' in specified time interval, combined amount of product shipped from supplier 'j' to plant 'k', whenever it is viable, is less than or equal to capacity of supplier 'j' for this product. Consequently, for each supplier 'j' in specified time interval, we can describe corresponding shipment capacity constraints as follows:

$$\sum_k s_j.x_{j,k} \leq L(j),\ \forall\, j$$

b. Shipment capacity of plant

For each plant 'k' in specified time interval, combined amount of product shipped to each of customer zones must not exceed corresponding plant capacity U_k. Consequently for each plant 'k' in specified time interval, we can describe corresponding shipment capacity constraint as follows:

$$\sum_l y_{k,l} \leq U_k,\ \forall\, k$$

c. Inventory capacity of plant

For each plant 'k' in specified time interval, combined amount of product in its inventory cannot exceed corresponding inventory capacity. Consequently, for each plant 'k' in specified time interval, we can describe corresponding inventory capacity constraint as follows:

$$i_k \leq I(k),\ \forall k.$$

d. Inventory balancing constraint

For each plant 'k' in specified time interval, combined amount of product shipped from supplier 'j' to plant 'k' is exactly equal to sum of combined amount in inventory at plant 'k' and combined amount of product shipped from plant 'k' to customer zones. Consequently, for each plant ‘k’ in specified time interval, we can describe corresponding inventory balancing constraint as follows:

$$\sum_j s_j.x_{j,k} = \left(\sum_l y_{k,l}\right) + i_k, \forall \text{ k}$$

Here variable s_j denotes whether supplier ‘j’ can supply finished product or not. In a more generic formulation of this model, we can define s_j as binary variable. However, for sake of simplicity alone, here we consider s_j to be fixed.

Again, in this article, we consider coefficients of objective functions as random variables and with associated means, as presented in Table 4 and right-hand side constants of constraints as normal random variables, as presented in Table 5.

Table 4. Expectations of coefficients of objective functions of SCM model

Cost coefficient		Expectation of cost coefficient
1^{st}Objective function:		
	Selling price/unit at customer zone 1	$\overline{155}$
	Selling price/unit at customer zone 2	$\overline{120}$
	Selling price/unit at customer zone 3	$\overline{135}$
2^{nd}Objective function:		
$STC(j, k)$	j = 1, k = 1	$\overline{40}$
	j = 1, k = 2	$\overline{26}$
	j = 2, k = 1	$\overline{36}$
	j = 2, k = 2	$\overline{28}$
$PTC(k, l)$	k = 1, l = 1	$\overline{18}$
	k = 1, l = 2	$\overline{22}$
	k = 1, l = 3	$\overline{16}$
	k = 2, l = 1	$\overline{12}$
	k = 2, l = 2	$\overline{15}$
	k = 2, l = 3	$\overline{18}$
3^{rd}Objective function:		
	j = 1, k = 1	$\overline{360}$
	j = 1, k = 2	$\overline{380}$
	j = 2, k = 1	$\overline{400}$
	j = 2, k = 2	$\overline{335}$

Table 5. Right hand side constants of constraints as normal random variables of SCM model

Right-hand side constant of constraint	Values as normal random variable
Shipment capacity of supplier 1	$\widetilde{130}$
Shipment capacity of supplier 2	$\widetilde{90}$
Shipment capacity of plant 1	$\widetilde{85}$
Shipment capacity of plant 2	$\widetilde{115}$
Inventory capacity of plant 1	$\widetilde{40}$
Inventory capacity of plant 2	$\widetilde{35}$

Table 6. Individual optimum values of expectations of objective functions of SCM model

Objective function	Individual maximum value	Individual minimum value
REV	11,310	468
TTC	85,400	4680
TPC	31,000	0

Table 7. Goals and tolerance values of fuzzy objective functions of SCM model

Objective function	Goal	Tolerance value
REV	7000	1000
TTC	38000	2500
TPC	11000	6000

5.1 Proposed IIOP on SCM Model in Fuzzy Environment

Here we apply proposed IIOP on above model to yield preferable Pareto optimal solution based on specified main objective function in fuzzy environment.

Step 1: We compute individual maximum and minimum values of expectations of objective functions $\left(z_i^E(x),\ \text{i} = 1, 2, 3\right)$ under given constraints and present in Table 6. Based on these values, DM can specify goals and tolerance values of expectations of objective functions, as presented in Table 7. We employ these goals and tolerance values of expectations of objective functions and construct corresponding membership functions in fuzzy environment.
Step 2: Next we have to determine corresponding initial reference membership level. So, we choose expectation of any one objective function, say $z_1^E(x)$ arbitrarily. By applying chain rule, we numerically determine trade-off ratios between membership function of expectation of $z_1^E(x)$ and membership functions $\mu_j(z_j^E(x)),\ j = 2, 3$ of expectations of other objective functions $z_j^E(x),\ j = 2, 3$ as follows (Table 8):

Table 8. Comparison between IIOP and existing interactive fuzzy stochastic optimization method

Pareto optimal solution by applying		Remarks
Proposed IIOP (TTC as main objective function)	Sakawa's method	
$REV^* = 6,219$ $TTC^* = 37,980$ $TPC^* = 15,240$	$REV^* = 6,396$ $TTC^* = 39,495$ $TPC^* = 14,640$	We obtain more preferable Pareto optimal solution to MOSLPP in fuzzy stochastic environment by applying proposed IIOP than existing method

$$\pi_{12} = -\frac{\partial\mu_1(z_1^E(x))}{\partial\mu_2(z_2^E(x))} = -\frac{1}{1000} * 0.0883 * \left(-\frac{1}{4000}\right)^{-1} = 0.221; \pi_{13} = -\frac{\partial\mu_1(z_1^E(x))}{\partial\mu_3(z_3^E(x))}$$
$$= -\frac{1}{1000} * 0.1256 * \left(-\frac{1}{6000}\right)^{-1} = 0.754$$

Step 3: Numerical approximation yields initial reference membership levels of expectations of objective functions in fuzzy environment as $\hat{\mu}_1^{(0)} = 0.22075$, $\hat{\mu}_2^{(0)} = 1$and $\hat{\mu}_3^{(0)} = 0.2929$.
Step 4: Here we consider that DM specifies TTC as main objective function. As per proposed IIOP, we obtain M-Pareto optimal solution and subsequently Pareto optimal solution to proposed SCM model as follows (– denotes Pareto optimality): $REV^* = 6219$, $TTC^* = 37,980$, $TPC^* = 15,240$.

Here we observe that proposed IIOP generates more desirable optimal value than corresponding specified aspiration level to expectation of main objective function *TTC* in fuzzy environment. Moreover, in proposed IIOP, optimal values to expectations of all other objective functions *REV* and *TPC* lie within specified goals and tolerance values in fuzzy environment. Thus, these optimal values are satisfactory to DM in fuzzy environment.

5.2 Sensitivity Analysis and Managerial Insights

We perform sensitivity analysis of parameters of proposed SCM model. Although proposed model has several parameters, we have page limitations. So, we consider the percentage change of key parameters only. Subsequently we present corresponding Pareto optimal solutions by altering values of following parameters:

1. shipment capacity of supplier 1
2. shipment capacity of plant 1
3. inventory capacity of plant 1

We depict the phenomenon of percentage changes of shipment capacity of supplier 1 in Table 9. We observe that when shipment capacity of supplier 1 is drastically reduced to below 50 levels, it pessimistically affects optimal values of all three objective functions.

This analysis suggests that suppliers should have yardstick shipment capacity to minimize transportation cost as well as minimize total procurement cost of supply chain. As this strategy improves optimal values of all objective functions of proposed SCM model, we find it to be win-win situation for DM as well as environment in fuzzy environment (Table 10).

Table 9. Sensitivity analysis for shipment capacity of supplier 1 to SCM model

Inventory capacity of plant 1	Optimal value of REV	Optimal value of TTC	Optimal value of TPC
30	6,022	38,740	16,380
50	6,116	38,265	15,780
90	6,219	37,980	15,240
130	6,219	37,980	15,240
170	6,219	37,980	15,240
210	6,219	37,980	15,240

Table 10. Sensitivity analysis for shipment capacity of plant 1 to SCM model

Shipment capacity of plant 1	Optimal value of REV	Optimal value of TTC	Optimal value of TPC
70	6,102	38,275	15,945
80	6,170	37,930	15,075
85	6,219	37,980	15,240
90	6,221	37,975	15,245
100	6,219	37,710	15,240
110	6,924	48,905	14,840

Next, we talk about sensitivity analysis for shipment capacity of plant 1, as given in Table 11. Here we observe that each augmentation to shipment capacity of plant 1 improves optimal value of TTC and TPC both. This is consistent with common perception. Moreover, we observe that when shipment capacity is significantly increased (at 110), optimal value of REV also increases considerably.

Next, we present sensitivity analysis for inventory capacity of plant 1, by which we find no change to Pareto optimal solution due to alterations in inventory capacity of plant 1. We present results in Table 11.

Here we keep in mind that as the motto is to illustrate proposed IIOP for determining preferable Pareto optimal solution based on main objective function to MOSLPP, we consider a much simplified SCM model in fuzzy environment. Hence, we suggest more research at this juncture to decide how inventory capacities of plants affect total revenue and total cost of SCM models in fuzzy environment. Next, we present sensitivity analysis

Table 11. Sensitivity analysis for inventory capacity of plant 1 to SCM model

Inventory capacity of plant 1	Optimal value of REV	Optimal value of TTC	Optimal value of TPC
20	6,219	37,980	15,240
30	6,219	37,980	15,240
40	6,219	37,980	15,240
50	6,219	37,980	15,240
60	6,219	37,980	15,240

for inventory capacity of plant 1, by which we find no change to Pareto optimal solution due to alterations in inventory capacity of plant 1. We present results in Table 11. Here we keep in mind that as the motto is to illustrate proposed IIOP for determining preferable Pareto optimal solution based on main objective function to MOSLPP, we consider a much simplified SCM model in fuzzy environment. Hence, we suggest more research at this juncture to decide how inventory capacities of plants affect total revenue and total cost of SCM models in fuzzy environment.

These results on sensitivity analysis present profound knowledge on proposed SSCM model. Here we recall what Oliver Wight at https://www.oliverwighteame.com has found. They find that although supply chain collaborations can reduce operating costs by up to 50% and an optimized supply chain can typically improve perfect order ratings by 17%, 66% of change initiatives fail due to lack of leadership and 75% of businesses lack the process to successfully deploy the strategy. Hence the DM has to take sequence of sensible decisions based on above sensitivity analysis results for long term sustainability of any SCM model.

5.3 Sub Cases with Different Main Objective Functions

Now we consider REV as specified main objective function. By applying proposed IIOP with same goals and tolerance values of expectations of objective functions, we find Pareto optimal solutions to SCM model in fuzzy environment, as presented in Table 12. Here we observe that optimal value of REV is more desirable than corresponding specified goal. Also, optimal values of other objective functions, viz. TTC and TPC, lie within specified goals and tolerance values and thereby are satisfactory to DM in fuzzy environment.

Again, we consider TPC as specified main objective function. By applying proposed IIOP with same goals and tolerance values of expectations of objective functions, we find Pareto optimal solutions to SCM model in fuzzy environment, as presented in Table 12.

Table 12. Pareto optimal solution for different main objective functions to SCM model

Main objective function	Pareto optimal solution	Remarks
REV	$REV^* = 6,224$ $TTC^* = 38,005$ $TPC^* = 15,235$	More preferable optimal value to expectation of main objective function REV than corresponding specified goal is obtained
TTC	$REV^* = 6,219$ $TTC^* = 37,980$ $TPC^* = 15,240$	More preferable optimal value to expectation of main objective function TTC than corresponding specified goal is obtained
TPC	$REV^* = 6,928$ $TTC^* = 47,540$ $TPC^* = 13,195$	More preferable optimal value to expectation of main objective function TPC than corresponding specified goal is obtained

Here we observe that optimal value of TPC is more desirable than corresponding specified goal. Also, optimal values of other objective functions, viz. REV and TTC, lie within specified goals and tolerance values and thereby are satisfactory to DM in fuzzy environment.

6 Conclusions

In this article, we introduce a general IIOP for determining preferable Pareto optimal solution to MOSLPP in fuzzy stochastic environment. This solution is based on main objective function, as specified by DM. We find that there cannot effectively have any main objective function in several existing crisp and fuzzy multi-objective optimization methods for solving MOSLPP. Whereas some existing methods assign weights, goals, priorities, utility functions and the like to objective functions to MOSLPP in fuzzy stochastic environment, extensive literature review hits upon various weaknesses of these approaches. However, DM specifies a main objective function and thereby analyst obtains preferable Pareto optimal solution to MOSLPP by applying proposed IIOP in fuzzy stochastic environment. Here in Sect. 5, we numerically show that optimal solution obtained by applying proposed IIOP is always Pareto optimal and more desirable to DM. In case, DM alters specified main objective function, we can determine corresponding Pareto optimal solution to MOSLPP, based on new main objective function in fuzzy stochastic environment. Even if, DM is unable to choose main objective function, or no DM is available at all, we can apply proposed IIOP in fuzzy stochastic environment.

Again, Sakawa et al. and fellow scientists had set initial reference membership levels as one in existing interactive fuzzy optimization methods for solving MOSLPP. However, in this article, we observe that expectation of each conflicting objective function attains respective goal simultaneously. Also, in this circumstance, we find that trade off ratios among corresponding membership functions play a key role. Subsequently, we derive initial reference membership levels analytically by employing trade-off ratios among membership functions of expectations of objective functions in fuzzy stochastic environment.

Numerical applications in production planning problem as well as SCM model along with sensitivity analysis of key parameters in Sect. 5 show effectiveness and efficiency of proposed IIOP for solving MOSLPP in fuzzy stochastic environment. Moreover, proposed process can save time and money of DM and hence is more sustainable in long run.

We recall that real-life decision making exercises are looked upon as tools to handle complex problems of modern society [33, 34], and these tools must be checked against real life problems always. If there are complains about efficiency of tools, analyst must re-examine and redesign the tool. When a new tool is found, analyst can be satisfied. However, she/he cannot forget to recheck it in practice and develop further [34]. Here proposed IIOP is interactive in nature and hence we consider it as an optimal tool for solving complex real-life models in imprecise environment. Finally, as scientists articulate, it is especially important for us not to be influenced by only past successes and failures in explaining multi-objective methods to DM. If we bring into the classroom only what comes out of boardroom, then inadequacy of today's DM's will be imposed on tomorrows.

Future research directions:
We locate lots of scopes for further research in proposed IIOP and proposed SCM model in future and enlist some of them as follows:

i. We can employ various single objective linear programming approaches for solving single objective model within proposed process.
ii. We can develop analogous processes for optimizing multi-objective stochastic non-linear optimization problems in fuzzy stochastic environment.
iii. We can consider other real life based imprecise environments like intuitionistic fuzzy environment, T-environment, soft environment etc. and build up analogous processes for solving MOSLPP in these environments.
iv. Again, we can augment several conditions of sustainability, like green environment, customer satisfaction and corporate social responsibility to proposed SCM model and optimize it.
v. Additionally, we can extend proposed SCM model by considering futuristic components like RFID, Internet of Things and all that to this model.
vi. We can consider large number of parameters in proposed SCM model and apply proposed IIOP on it.
vii. And last but not the least, we can apply analogous IIOPs to more comprehensive version of proposed SCM model in different imprecise environments.

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An Integrated Model of EOQ and Newsboy Problem for Substitutable Items with Space Constraints

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Abstract. This paper develops multi-items Economic Order Quantity (EOQ) inventory models for a wholesaler and several retailers. A single period newsboy type inventory problem for two substitutable items is studied with a space constraint. The demands for wholesaler and retailers are random and stock dependent respectively. To make it more realistic, it is assumed that the wholesaler sells a certain portion of the stock instantaneously to the retailers as per their initial demands. For wholesaler, if items are not completely exhausted, then these are stored and sold instantaneously at a salvage price which are lower than normal selling price. Again, if one item is exhausted but system is surplus, then under co-operation the exhausted item is substitute by the other item and excess item is sold by using salvage price. As the items are substitutable, several mutual cases between wholesaler and retailers are considered as subsystem. Retailers sell their stocks with stock dependent demands and shortages are allowed. Assuming that wholesaler and retailers under a single management, an integrated model is formulated to maximize a single objective profit function with the space constraint. A numerical results have been presented for illustration. Finally, it is solved through LINGO technique.

Keywords: Economic order quantity model · Newsboy problem · Stock dependent demand · Substitutable items · Space constraint

1 Introduction

Economic Order Quantity (EOQ) models have gained considerable attention in the last decade. In this model, uncertainties arise from factors such as market demand, product quality, competition and promotions introduce risks to the both wholesaler and retailers. In order to increase the performance of the system by sharing the risks involved, contracts that include specifications regarding the quality, quantity, whole sale prices are undertaken between the wholesaler and retailer with the purpose that such agreements would be beneficial to both parties. Most commonly studied examples of contracts are sale rebate, quantity

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flexibility, wholesale price, each of which provides the retailer with different incentives to make them order more than they would with only a wholesale price scheme.

The newsboy problem is a classical inventory problem that is very significant in terms of both theoretical and practical considerations. Gallego and Moon [3,4] defined the newsboy problem as the tool to decide the stock quantity of an item when there is a single purchasing opportunity before the start of the selling period and the demand for the item is random. The classical newsboy model assumes that if the order quantity is larger than the realized demand, a single discount is used to sell excess inventory or that excess inventory is disposed off. On the other hand, if the order quantity is less than demand, then profit is lost. Here, we introduce the two substitutable items under the capital cost. The objective is to find the optimum tradeoff between the risk of overstocking (incurring disposal cost) and of under stocking (losing profit).

In this study, we consider an integrated EOQ model and newsboy type problem with several retailers and a wholesaler, for two substitutable items. It is assumed that substitution takes place only at stock out situation. General expressions are derived for expected total profit of this model, the profit of the retailer and expected profit of the wholesaler. Some special cases regarding substitution are considers to obtain necessary condition. A single wholesaler with two substitute items sells the goods to several retailers instantaneously against their random demands. The items are sold as per the demands of the retailers and if their is any excess amount, the wholesaler adopts the push sell policy to sell that extra units to the retailers, for which an interest is charged and is paid by the wholesaler.

Several extensions to the newsboy model have been done in the literature [1,9,10] for the single item. More recently, single item newsboy problem is considered with random lot size [16]. Early extension for multiple items assumed their independent demands [13]. More recent extension to two items includes substitutability [8] without constraints. All these extension have been done in single level (i.e., single retailer/single vendor), although it has widened the application area of classical newsboy problem for multi-level. Recently two-items newsboy problem has been managed between single-retailer vs. single-vendor [7,11] for non-deteriorate items without any constraint.

The inventory models of multi-items under some resource constraints in crisp environment are available in all reference books [6,14,15]. There are also some research works [2] on deteriorating multi-item inventory models in stochastic environment. In all the problems, it is assumed that demands of the items are independent to one another. Now-a-days maintenance of inventory of perishable goods is a problem of major concern to the modern business organizations as most of the usable deteriorate with time. Consequently Goyal and Giri [5] are presented an excellent survey in modeling of deteriorating item.

In this paper, a newsboy type substitutable two-items inventory control problem is formulated. Single wholesaler with instantaneous replenishment of two substitutable items sells the goods to several retailers instantaneously against their random demands. The items are first sold as per the requirements (demands) of the retailers and if their is any excess amount, the wholesaler adopts the push-sale policy to sell that extra units to the retailers and proceeds against these amounts are paid by retailers at the end of the time period T, for which an interest is charged and paid by the wholesaler.

The retailers also follow the newsboy type process. A portion of the goods of the retailers are sold instantaneously against the random demand of the customers and other fraction, if it is there, deteriorates at a constant rate over the time T and is sold instantaneously towards the end of time period T.

Assuming wholesaler and retailers are from the same management, a single objective profit function is formulated with space constraints and solved using LINGO technique. When wholesaler and retailers are different, separate objective functions are formulated and compromise solutions. Several models under the sub-systems for the different types of uses of substitute items are formulated. Finally, the models are illustrated through numerical results.

2 Assumptions and Notations

The inventory model is developed on the basis of the following assumptions and notations.

2.1 Assumptions

The following assumptions are used for the proposed supply chain model.

1. Single wholesaler and multiple retailers are considered.
2. The model is developed for a single period.
3. Demands at wholesaler assumed to be probabilistic.
4. Demands at retailers assumed to be stock dependent.
5. At wholesaler's level, if an item is not completely exhausted instantaneously then it is stored and it is sold instantaneously at a salvage price which is lower than normal selling price.
6. Two type of raw material and finished product are considered.
7. At wholesaler's level if one item is exhausted and other item in hand, then exhausted item will be substitute by the other.
8. Shortages of goods are allowed and fully backlogged.
9. Lead time is zero.

2.2 Notations

(i) Q_i is the lot of the wholesaler, $i = 1, 2$.
(ii) D_i is the random demand of the wholesaler of i-th item.
(iii) $q_{ij}(t)$ is the lot size of i-th item of j-th retailer at time t.
(iv) Q_{ij} is the lot size of i-th item of j-th retailer at time $t = 0$.
(v) The demand at retailer level $d_i(q)$ stock dependent and d_0 constant demand when items are exhausted.

(vi) c_i unit price for i-th item.
(vii) p_i unit selling price for i-th item.
(viii) μ_i expected demand for i-th item.
(ix) l_i unit shortage cost for i-th item for wholesaler.
(x) s_i unit salvage cost for i-th item.
(xi) h_i unit holding cost for i-th item and j-th retailer.
(xii) A_{ij} ordering cost for i-th item and j-th retailer.
(xiii) c_i^3 unit shortage cost for i-th item and j-th retailer.

3 Formulation of Model

We consider a supply chain (SC) production inventory control system consisting of a single wholesaler and 2 retailers.

3.1 For Wholesaler

The wholesaler has two substitutable items with Q_1 and Q_2 quantities and the demand by the retailers of these two items are D_1 and D_2 respectively. Depending on the demand the system is some times surplus and some times shortages can be occurred. The relation between Q_1, Q_2, D_1, D_2 and shortages cases are as follows:

Model 1. $D_1 < Q_1$, $D_2 < Q_2$, both items surplus. In this case the expected profit is given by

$$\begin{aligned} TC_{W1} &= E\left[p_1 D_1 + p_2 D_2 + s_1(Q_1 - D_1) + s_2(Q_1 - D_2)\right] - c_1 Q_1 - c_2 Q_2 \\ &= p_1\mu_1 + p_2\mu_2 + s_1(Q_1 - \mu_1) + s_2(Q_1 - \mu_2) - c_1 Q_1 - c_2 Q_2 \qquad (1) \end{aligned}$$

subject to, $\quad Q_{11} + Q_{12} = Q_1,\ Q_{21} + Q_{22} = Q_2.$

Putting the values of Q_1 and Q_2, we get

$$\begin{aligned} TC_{W1} = p_1\mu_1 + p_2\mu_2 + s_1(Q_{11} + Q_{12} - \mu_1) \\ + s_2(Q_{11} + Q_{12} - \mu_2) - c_1 Q_1 - c_2(Q_{21} + Q_{22}) \qquad (2) \end{aligned}$$

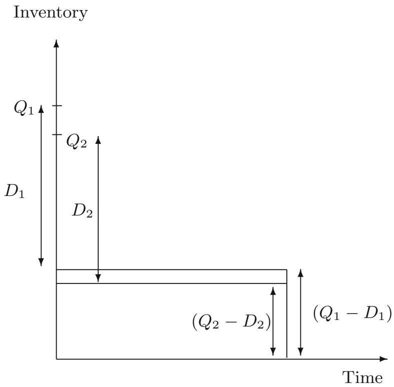
Model 2. $D_1 > Q_1$, $D_2 > Q_2$, both items shortage. In this case, the expected profit is given by

$$\begin{aligned} TC_{W2} &= E\left[p_1 Q_1 + p_2 Q_2 - l_1(D_1 - Q_1) - l_2(D_2 - Q_2)\right] - c_1 Q_1 - c_2 Q_2 \\ &= p_1 Q_1 + p_2 Q_2 - l_1(\mu_1 - Q_1) - l_2(\mu_2 - Q_2) - c_1 Q_1 - c_2 Q_2 \qquad (3) \end{aligned}$$

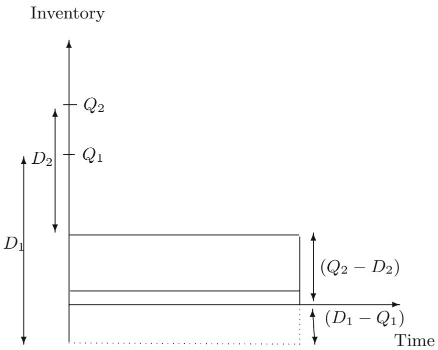
subject to, $\quad Q_{11} + Q_{12} = Q_1,\ Q_{21} + Q_{22} = Q_2.$

Putting the values of Q_1 and Q_2, we get

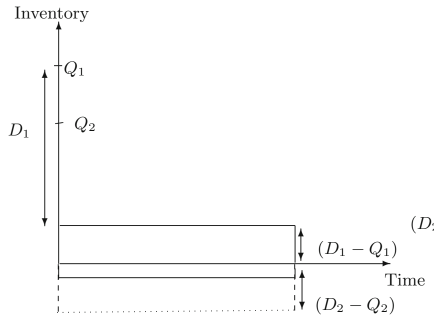
$$\begin{aligned} TC_{W2} = p_1 Q_1 + p_2 Q_2 - l_1(\mu_1 - Q_{11} - Q_{12}) \\ - l_2(\mu_2 - Q_{21} - Q_{22}) - c_1(Q_{11} + Q_{12}) - c_2(Q_{21} + Q_{22}) \qquad (4) \end{aligned}$$



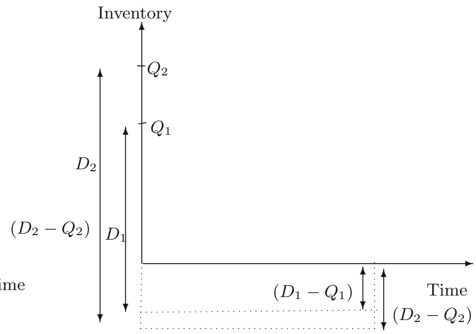
$(D_1 < Q_1)$ and $(D_2 < Q_2)$



(One item shortage but sustem surplus)



One item is surplus but system shortage



$(D_1 > Q_1)$ and $D_2 > Q_2$

Model 3. In this model, one item is shortage and system is surplus. Here shortage item will be substitutable by the other. Let $D_1 > Q_1$ and $D_2 < Q_2$. First item will be substitute by the second item.

$$\begin{aligned} TC_{W3} &= E\left[p_1Q_1 + p_2D_2 + p_2(D_1 - Q_1) + s_2((Q_2 - D_2) - (D_1 - Q_1))\right] - c_1Q_1 - c_2Q_2 \\ &= p_1Q_1 + p_2\mu_2 + p_2(\mu_1 - Q_1) + s_2((Q_2 - \mu_2) - (\mu_1 - Q_1)) - c_1Q_1 - c_2Q_2, \quad (5) \end{aligned}$$

such that, $\quad Q_{11} + Q_{12} = Q_1,\ Q_{21} + Q_{22} = Q_2.$

Putting the values of Q_1 and Q_2, we get

$$\begin{aligned} TC_{W3} = p_1(Q_{11} + Q_{12}) + p_2(Q_{21} + Q_{22}) - l_1(\mu_1 - Q_{11} - Q_{12}) \\ - l_2(\mu_2 - Q_{21} - Q_{22}) - c_1(Q_{11} + Q_{12}) - c_2(Q_{21} + Q_{22}) \end{aligned} \quad (6)$$

Model 4. One item is shortage and the system will be shortage. Here $D_1 > Q_1$ and $D_2 < Q_2$. Although, some of the shortage amount of first item will be substitute by the second item.

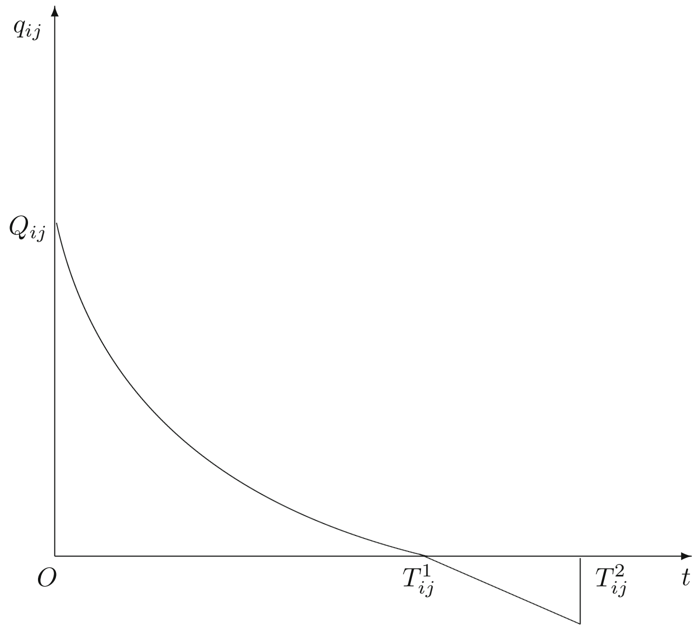
$$\begin{aligned} TC_{W4} &= E\left[p_1Q_1 + p_2D_2 + p_2(Q_1 - D_2) + l_1((D_1 - Q_1) - (Q_2 - D_2))\right] - c_1Q_1 - c_2Q_2 \\ &= p_1Q_1 + p_2\mu_2 + p_2(Q_2 - \mu_2) + l_1((\mu_1 - Q_1) - (Q_2 - \mu_2)) - c_1Q_1 - c_2Q_2, \quad (7) \end{aligned}$$

satisfying the condition, $Q_{11} + Q_{12} = Q_1$, $Q_{21} + Q_{22} = Q_2$.
Putting the values of Q_1 and Q_2, we get

$$\begin{aligned} TC_{W4} = {} & p_1(Q_{11} + Q_{12}) + p_2\mu_2 + p_2(Q_{21} + Q_{22} - \mu_2) + l_1((\mu_1 - Q_{11} - Q_{12}) \\ & - (Q_{21} + Q_{22} - \mu_2)) - c_1(Q_{11} + Q_{12}) - c_2(Q_{21} + Q_{22}) \end{aligned} \tag{8}$$

3.2 For Retailer

Let $q_{ij}(t)$ be the quantity of i-th item for j-th retailer at any time t, with stock dependent demand d_i and if T^1_{ij} be the time of shortage for the retailer j, then the governing differential equations are



$$\frac{dq_{ij}}{dt} = \begin{cases} -(d_0 + d_i q_{ij}), & 0 \le t \le T^1_{ij}; \\ -d_0, & T^1_{ij} \le t \le T^2_{ij}. \end{cases} \tag{9}$$

The inventory condition for the model are
$q_{11}(0) + q_{12}(0) = Q_1$, $q_{21}(0) + q_{22}(0) = Q_2$, $q_{ij}(T^1_{ij}) = 0$ and $q_{ij}(T^2_{ij}) = s_{ij}$, where s_{ij} is the total amount of shortage.

From the Eq.(9), we get

$$\frac{dq_{ij}}{dt} = -(d_0 + d_i q_{ij}), 0 \le t \le T^j_2$$

$$\frac{dq_{ij}}{dt} + d_i q_{ij} = -d_0 \tag{10}$$

This is a linear differential equation of q_{ij}, the integrating factor will be $e^{\int d_i dt} = e^{d_i t}$.

Multiplying Eq.(10) by integrating factor and integrating we get

$$
\begin{aligned}
q_{ij}e^{d_i t} &= \int (-d_0 e^{d_i t})dt \\
&= A - \frac{d_0}{d_i}e^{d_i t},\ A \text{ is an integrating constant.} \\
\therefore q_{ij} &= Ae^{-d_i t} - \frac{d_0}{d_i} \qquad (11)
\end{aligned}
$$

Using the condition $q_{ij}(0) = Q_{ij}$ we get

$$
\begin{aligned}
Q_{ij} &= A - \frac{d_0}{d_1} \\
\therefore A &= Q_{ij} + \frac{d_0}{d_i}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
q_{ij} &= \left(Q_{ij} + \frac{d_0}{d_i}\right) e^{-d_i t} - \frac{d_0}{d_i} \\
&= Q_{ij}e^{-d_i t} + \frac{d_0}{d_i}\left(e^{-d_i t} - 1\right),\ 0 \leq t \leq T_{ij}^1. \qquad (12)
\end{aligned}
$$

Using the condition $q_{ij}(T_{ij}^1) = 0$ we get

$$
\begin{aligned}
0 &= Q_{ij}e^{-d_i T_{ij}^1} + \frac{d_0}{d_i}\left(e^{-d_i T_{ij}^1} - 1\right) \\
\text{or, } 0 &= Q_{ij} + \frac{d_0}{d_i}\left(1 - e^{-d_i T_{ij}^1}\right) \\
\text{or, } \frac{d_0}{d_i}e^{d_i T_{ij}^1} &= Q_1 + \frac{d_0}{d_i} \\
\text{or,} e^{d_i T_{ij}^1} &= \frac{d_i}{d_0}Q_{ij} + 1 \\
\text{or,} d_i T_{ij}^1 &= \log\left(1 + \frac{d_0}{d_i}Q_{ij}\right) \\
\text{or,} T_{ij}^1 &= \frac{1}{d_i}\left[\log\left(1 + \frac{d_0}{d_i}Q_{ij}\right)\right]. \qquad (13)
\end{aligned}
$$

Again, from the second part of the Eq.(9), we get

$$
\frac{dq_{ij}}{dt} = -d_0 \text{ or, } \quad dq_{ij} = -d_0 dt.
$$

Integrating between the limit $q_{ij}(T_{ij}^1) = 0$ and $q_{ij}(T_{ij}^2) = S_{ij}$ we get

$$
S_{ij} = -d_0[T_{ij}^2 - T_{ij}^1]. \qquad (14)
$$

Holding cost

$$
\begin{aligned}
H &= h_i \int_0^{T_{ij}^2} q_{ij} dt \\
&= h_i \int_0^{T_{ij}^2} \left[Q_{ij} e^{-d_i t} + \frac{d_0}{d_i} \left(e^{d_i t} - 1 \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= h_i \left[\frac{Q_{ij} e^{-d_i t}}{-d_i} + \frac{d_0}{d_i} \left(\frac{e^{-d_i t}}{-d_i} - t \right) \right]_0^{T_{ij}^2} \\
&= h_i \left[\left\{ \frac{Q_{ij} e^{-d_i T_{ij}^2}}{-d_i} + \frac{d_0}{d_i} \left(\frac{e^{-d_i T_{ij}^2}}{-d_i} - T_{ij}^2 \right) \right\} - \left\{ \frac{Q_{ij}}{-d_i} + \frac{d_0}{d_i} \left(\frac{1}{-d_i} \right) \right\} \right]
\end{aligned}
$$

$$
= h_i \left[\frac{Q_{ij} e^{-d_i T_{ij}^2}}{-d_i} - \frac{d_0}{d_i} \left(\frac{e^{-d_i T_{ij}^2}}{d_i} + T_{ij}^2 \right) + \frac{Q_{ij}}{d_i} + \frac{d_0}{d_i^2} \right]. \tag{15}
$$

$$
\text{Shortage cost } S = \frac{1}{2} c_3^i (T_{ij}^2 - T_{ij}^1) S_{ij}. \tag{16}
$$

The total cost for the retailer due to finished goods can be expressed as sum of the setup cost, holding cost and shortage cost as follows

$$
TC_R = A_i + h_i \left[\frac{Q_{ij} e^{-d_i T_{ij}^2}}{-d_i} - \frac{d_0}{d_i} \left(\frac{e^{-d_i T_{ij}^2}}{d_i} + T_{ij}^2 \right) + \frac{Q_{ij}}{d_i} + \frac{d_0}{d_i^2} \right] + \frac{1}{2} c_3^i (T_{ij}^2 - T_{ij}^1) S_{ij} \tag{17}
$$

Integrated Model

Assume that the wholesaler and retailer form a single management system. An integrated model is formulated to maximize a single objective profit function

$$
\begin{aligned}
\max TP &= TC_{w1} + TC_{w2} + TC_{w3} + TC_{w4} + TP_R \qquad (18) \\
\text{subject to,} \quad & a_1(Q_{11} + Q_{12}) \geq \lambda A \\
& a_2(Q_{21} + Q_{22}) \geq (1 - \lambda) A \\
& c_1(Q_{11} + Q_{21}) \leq B_1 \\
& c_2(Q_{12} + Q_{22}) \leq B_2,
\end{aligned}
$$

where a_i denote space requirement for one unit of ith item and $0 \leq \lambda \leq 1$. The minimum space available for the wholesaler is A units. The maximum investment for the i-th retailer is B_i.

4 Numerical Example

In this section, the author consider an example to illustrate the proposed model. Here author consider an inventory model consisting of single wholesaler and two retailers.

Table 1. A set of input data for wholesaler.

Items (i)	Unit price (c_i)\$	Selling price (p_i)\$	Salvage value (s_i)\$	Shortage cost (l_i)\$
1	55	80	60	5
2	60	85	65	6

The wholesaler contains two substitutable items with expected demand $\mu_1 = 3.0$ units, $\mu_2 = 2.5$ units and minimum storage space 400 sq. units. The other relevant data for wholesaler are

For both the retailers there is a constant demand $d_0 = 11$ units and consider a cycle for both retailer is $T = 10$ units. Other relevant data are

Table 2. A set of input data for Retailer-1 and Retailer-2.

Retailer	Stock dependent demand (d_i)	Holdingcost (h_i)\$	Shortagecost (c_{3i})\$	Setup cost (A_i)\$	Budget (B_i)\$
1	0.30	2.0	2.0	25	14000
2	0.28	3.0	2.5	35	12000

The author use the LINGO software package to obtain the optimal solution for this integrated model. The optimal solutions are tabulated

Table 3. An optimal result for this integrated model.

Members	Quantity		Shortage cost \$		Profit \$
	Item-1	Item-2	Item-1	Item-2	
Wholesaler	148	166	...	...	22308
Retailer-1	83	91	66.46	62.75	2134
Retailer-2	65	75	72.5	67.7	1392

5 Discussion

In this integrated model, present more profit for wholesaler with minimum storage space and retailers with maximum investment. This is because, if substitutions of items are not allowed, then the classical model gives optimal values of the decisions variables which gives maximum profit for one retailer which may not give the maximum profit for for the other retailer. Hence, solutions are obtained for multi-objective optimization problem. Also, some solutions for the compromise solutions for the optimal profit for both wholesaler and retailers are presented.

6 Practical Implementation

In our daily life situations, we use some seasonal goods. For this, there are small markets connected with a wholesale market. A wholesaler receives the products, say two types of same items with different companies from suppliers and immediately sells these quantities to the retailers of the small markets and if any excess amount in hand, then sells these amount with lower price. The wholesaler follows newsboy type problem and on the other hand retailers follow EOQ model type. This model consisting of a wholesaler and several retailers. The wholesaler considered for two substitutable items. The wholesaler sells instantaneously and disposes those to retailers and adopting push sale, if required. The wholesaler may allow either shortage or sale surplus amount with salvage cost. The retailers purchase the products from wholesaler and sell to the customers according to EOQ model with stock dependent demands. The retailers may allow for shortage for each item. The retailers have limitation on maximum investment. The members of the system decide how much quantities to be ordered in order to have maximum profit.

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Determination of Centre of Origin in Gunshot Analysis Using Triangular Fuzzy Number

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Abstract. In any bloodstain pattern analysis determination of Centre of origin is very important study for crime scene reconstruction. In this paper we will use triangular fuzzy number for the location of the bloodstains found on the crime scene and direction cosines of the path followed by the bloodstains. A method will be proposed for determination of point of intersection of the trajectories of blood drops (Centre of origin).

Keywords: Triangular fuzzy number · Point of origin · Bloodstain pattern analysis

1 Introduction

The increased criminal activity now a day's causes heavy work load on criminal investigators. Therefore it is important to have efficient decision support system for crime scene reconstruction. In this paper we are making an attempt to construct a methodology using triangular fuzzy number to determine Centre of origin where someone was shot and the eye-witness is absent. Some important definitions are described briefly which will be used for our study:

- Area of origin:

  It is the area from where the blood flow was originated in three dimensional axes.

- Angle of impact:

  The acute angle formed between the direction of a blood drop and the plane of the surface it strikes.

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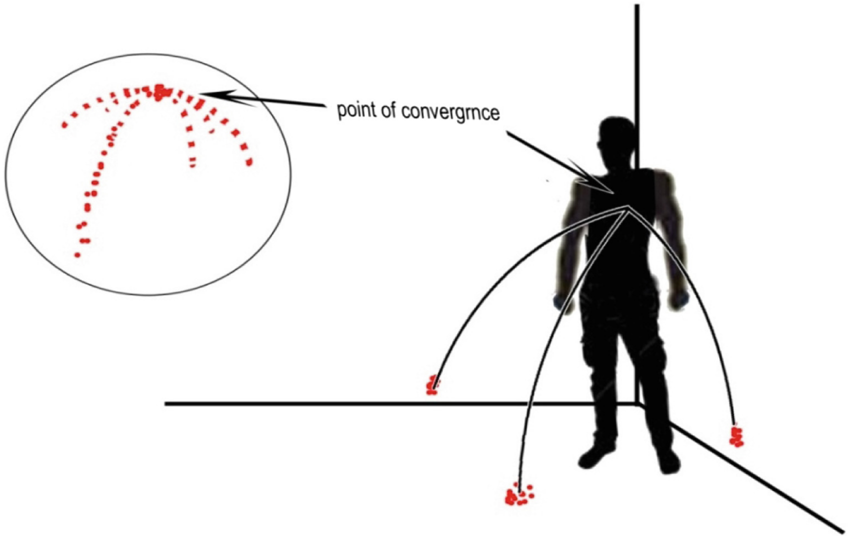

Fig. 1. Diagram for Point of convergence

In this paper we will focus only on the determination of point of origin. From the bloodstains found on the crime scene we can get the angle of impact i.e. the acute angle between the direction of flow of blood and the plane where the blood drop stroke. Also we get the direction of the path of flow.

The shape of the bloodstains in any crime scene depends only upon the impact forces and the angle at which the blood strikes the surface. For the calculation of impact angle the length to width ratio is mostly used principle from last few decades. We calculate the angle of impact by the formula $arc\,\sin^{-1}\frac{L}{W}$. The description is shown in the figure. (James et al. 2005).

But the angle of impact we found is not as simple as it seems. The shape of the bloodstain also depends on the characteristics of the surface of impacts, i.e. its texture, absorptive properties and thickness. If the surface is rougher than the shape of the stains are irregular and not appropriate for interpretation. Similarly in absorptive surface a stain becomes more compact than usual. The tail of the bloodstain gives us the impact of direction of the trajectory of the blood drops. This is also uncertain in nature as for the properties of the surfaces on which it was deposited. The movement of the victim or activities and in some cases the insects also causes disruption of the shape of bloodstains in the crime scene. Also some insect eats bloodstain which may causes error (James et al. 2005). And we encounter significant errors average of 5%, in calculating angle of impacts and determination of direction of trajectory of blood drops in bloodstain pattern analysis (Wilis et al. 2001; Rowe 2006). That is why the angle of impact and the directionality calculated from the crime scene is not certain in nature.

We face one major problem is to choose the appropriate stains for interpretation. The blood drops travelling with higher energy and linearly produces more appropriate bloodstains than blood drops with lower energy. Also the stains should be as long as possible. The more circular the stain the more error in calculating the impact angle as

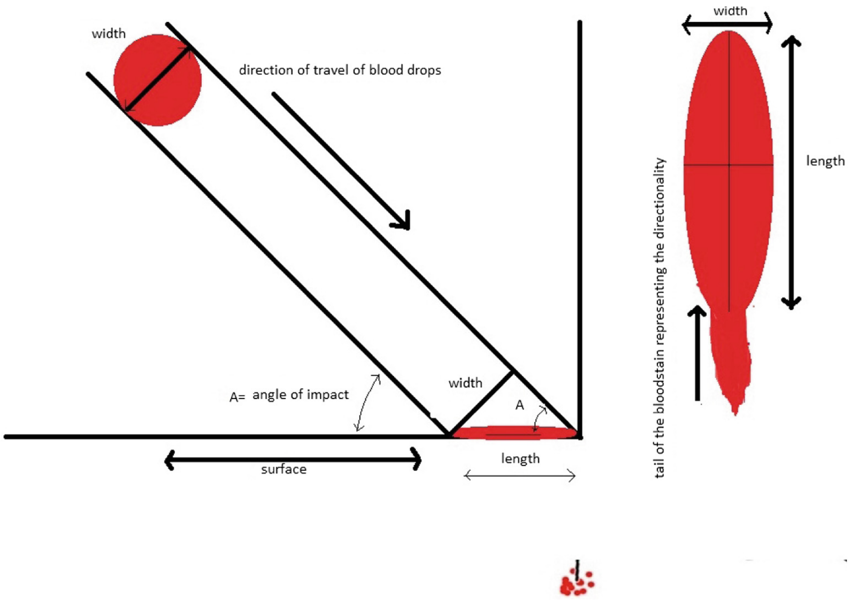

Fig. 2. Directionality of bloodstains and angle of impact

well as blood trajectory (James et al. 2005). In gunshot injury the blood spatters are in very high velocity and possesses higher energy, which results in flight of blood drops more linear than other injuries.

Determination of point of origin is nothing but a matter of geometrical fact of determining the path of blood flow and to find the approximately common point of intersection. The locations of bloodstains in any crime scene are uncertain in nature in terms of height and distance due to movement of the victim or the messy situation may arise at the time of crime. That is why we will use triangular fuzzy number for each coordinates of the bloodstains found on the crime scene. Also the angle of impact is not free from errors and for direction cosines of the path of blood flow we will use triangular fuzzy number.

2 Literature Review

Till now various methods are used for Determination of area of origin. In tangent method 1st by direction of bloodstains we determine the angle of impact and point of origin of bloodshed in 2D. Then by Tan function we determine the height of the area of origin and evaluate the trajectory of blood as the hypotenuse of a right angled triangle. Although, this method is valid for fast moving blood drops, like in gunshot injuries. In string method the trajectories of blood drops are considered as straight lines. Then fix string at the position of the bloodstains and pulling them away from the surface of impact on which bloodstains split. The area of origin is estimated where most of the strings intersect (James et al. 2005). The virtual string method is a just the traditional string method interpreted with computer software. Buck et al. 2011 used the non-contact methods digital photogrammetry, tachymetry and laser scanning combined with CAD and photogrammetry software. (Camana 2013) proposed a probabilistic approach for determination of area of orgine and gave a probability density function for point of convergence in 3D.

3 Mathematical Preliminaries

Direction cosines: Let a straight line makes an angle α, β, γ with the x-axis, y-axis and z-axis then $(cos\alpha, cos\beta, cos\gamma)$ are called the direction cosines of the straight line.

The equation of line whose direction cosines are l, m, n and passing through the points (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Fuzzy set Zadeh (1965); Zimmermann (1985):

A fuzzy set is one which assigns grades of membership between 0 and 1 to objects within its universe of discourse. If X is a universal set then a fuzzy set A is defined by, its membership function

$$\mu_A : X \to [0, 1]$$

Fuzzy number: A fuzzy set A on R is said to be a fuzzy number if the following conditions are satisfied:-

- A must be a normal fuzzy set
- $^{\alpha} A$ must be a closed interval for every $\alpha \in [0, 1]$
- The support of A, must be bounded

Triangular fuzzy number: A triangular fuzzy number A can be defined as a triplet (a, b, c) whose membership function is given by

$$A(x) = \begin{cases} \frac{(x-a)}{b-a} \; for \; a \le x \le b \\ \frac{c-x}{c-b} \; for \; b \le x \le c \\ = 0 \; otherwise \end{cases}$$

Let $A \, and \, B$ be two triangular fuzzy numbers parameterized by the triplet (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively, then operations of triangular fuzzy number is given by

$$A + B = (a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$
$$A - B = (a_1, b_1, c_1) - (a_2, b_2, c_2) = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$$
$$A \times B = (a_1, b_1, c_1) \times (a_2, b_2, c_2) = (a_1 a_2, b_1 b_2, c_1 c_2)$$
$$A \div B = (a_1, b_1, c_1) \div (a_2, b_2, c_2) = \left(a_{1/} a_2, b_{/1} b_2, c_{1/} c_2\right)$$
$k(a_1, b_1, c_1) = (ka_1, kb_1, kc_1)$ where k is a real number.

Let $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ be two triangular fuzzy numbers then the distance between them is given by

$$d(A, B) = \sqrt{\frac{1}{3}\left\{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2\right\}}.$$

4 Methodology

Let us take p number of bloodstains in a group of bloodstains under consideration. From each blood droplets or bloodstains we can have angle of impacts i.e. the angle from which the flow strikes that surface where blood was found. Let the angles be θ_1, θ_2, $\ldots \theta_p$. Obviously we get the direction cosines of the path of the bloodstains. Let the direction cosines for corresponding bloodstains be

$$(l_1, m_1, n_1), (l_2, m_2, n_2) \cdots \left(l_p, m_p, n_p\right)$$

The location of p bloodstains in three dimensions is

$$(x_1, y_1, z_1), (x, y_2, z_2) \cdots \left(x_p, y_p, z_p\right).$$

As mentioned earlier the angle of impacts as well as the direction cosines and the location of bloodstains are fuzzy in nature so we will consider the direction cosines and each coordinates of each points as triangular fuzzy number denoted by

$$\left(\tilde{l}_1, \tilde{m}_1, \tilde{n}_1\right), \left(\tilde{l}_2, \tilde{m}_2, \tilde{n}_2\right) \cdots \left(\tilde{l}_p, \tilde{m}_p, \tilde{n}_p\right)$$

$$\text{and } (\tilde{x}_1, \tilde{y}_1, \tilde{z}_1), (\tilde{x}_2, \tilde{y}_2, \tilde{z}_2) \cdots \left(\tilde{x}_p, \tilde{y}_p, \tilde{z}_p\right)$$

where each $\tilde{l}_i$, $\tilde{m}_i$, $\tilde{n}_i$ and $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_i$ $i = 1, 2, ..p$; are triangular fuzzy number.

We know that the equation of line whose direction cosines are l, m, n and passing through the points (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

The equation of lines followed by the bloodstains whose angle of impacts is θ_1, θ_2, $\cdots \theta_p$ respectively are

$$\frac{x - \tilde{x}_1}{\tilde{l}_1} = \frac{y - \tilde{y}_1}{\tilde{m}_1} = \frac{z - \tilde{z}_1}{\tilde{n}_1} \tag{1}$$

$$\frac{x - \tilde{x}_2}{\tilde{l}_2} = \frac{y - \tilde{y}_2}{\tilde{m}_2} = \frac{z - \tilde{z}_2}{\tilde{n}_2} \tag{2}$$

$$\cdots$$

$$\frac{x - \tilde{x}_p}{\tilde{l}_p} = \frac{y - \tilde{y}_p}{\tilde{m}_p} = \frac{z - \tilde{z}_p}{\tilde{n}_p} \; (p)$$

where each $\tilde{l}_i, \tilde{m}_i, \tilde{n}_i$ and $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i$ $i = 1, 2, ..p$; are triangular fuzzy number instead of real crisp number.

To find the point of origin we have to find the common intersection point of these p lines.

Let us consider the path of i$^{\text{th}}$ and j^{th} line is

$$\frac{x - \tilde{x}_i}{\tilde{l}_i} = \frac{y - \tilde{y}_i}{\tilde{m}_i} = \frac{z - \tilde{z}_i}{\tilde{n}_i} = \tilde{r}_i$$
$$\frac{x - \tilde{x}_j}{\tilde{l}_j} = \frac{y - \tilde{y}_j}{\tilde{m}_j} = \frac{z - \tilde{z}_j}{\tilde{n}_j} = \tilde{r}_j$$

where $1 \le i \ne j \le p$

Therefore the coordinate of any point on the i^{th} line are $P_i(\tilde{x}_i + \tilde{l}_i\tilde{r}_i,\ \tilde{y}_i + \tilde{m}_i\tilde{r}_i,\ \tilde{z}_i + \tilde{n}_i\tilde{r}_i)$ and the coordinate of any point on the j$^{\text{th}}$ line are $P_j(\tilde{x}_j + \tilde{l}_j\tilde{r}_j,\ \tilde{y}_j + \tilde{m}_j\tilde{r}_j,\ \tilde{z}_j + \tilde{n}_j\tilde{r}_j)$. As the i^{th} line and j$^{\text{th}}$ line originate from the same point therefore for some $\tilde{r}_j$ and $\tilde{r}_j$ the two points P_j and P_j coincides. So we have

$$\tilde{x}_i + \tilde{l}_i\tilde{r}_i = \tilde{x}_j + \tilde{l}_j\tilde{r}_j,\ \ \tilde{y}_i + \tilde{m}_i\tilde{r}_i = \tilde{y}_j + \tilde{m}_j\tilde{r}_j,\ \ \tilde{z}_i + \tilde{n}_i\tilde{r}_i = \tilde{z}_j + \tilde{n}_j\tilde{r}_j$$

Or

$$\tilde{l}_i\tilde{r}_i - \tilde{l}_j\tilde{r}_j + (\tilde{x}_i - \tilde{x}_j) = 0$$

$$\tilde{m}_i\tilde{r}_i - \tilde{m}_j\tilde{r}_j + (\tilde{y}_i - \tilde{y}_j) = 0$$

$$\tilde{n}_i\tilde{r}_i - \tilde{n}_j\tilde{r}_j + (\tilde{z}_i - \tilde{z}_j) = 0$$

From 1st two relations we get

$$\tilde{r}_i = \frac{\tilde{m}_j(\tilde{x}_i - \tilde{x}_j) - \tilde{l}_j(\tilde{y}_i - \tilde{y}_j)}{\tilde{l}_j\tilde{m}_i - \tilde{l}_i\tilde{m}_j}$$
$$\tilde{r}_j = \frac{\tilde{m}_i(\tilde{x}_i - \tilde{x}_j) - \tilde{l}_i(\tilde{y}_i - \tilde{y}_j)}{\tilde{l}_j\tilde{m}_j - \tilde{l}_i\tilde{m}_j}$$

and these values will satisfy the 3rd relation. Substituting these $\tilde{r}_j$ or $\tilde{r}_j$ in the point we get the coordinate of the point of intersection of the i^{th} line and j$^{\text{th}}$ line.

1st we fix $i = 1$, then for $j = 2, 3, \ldots p$ we get $\tilde{r}_1, \tilde{r}_2, \ldots \tilde{r}_p$ and $p - 1$ number of points of intersection between 1st and 2nd path, 1st and 3rd path and so on, say

$$\tilde{P}_1^1\left(\tilde{X}_1^1, \tilde{Y}_1^1, \tilde{Z}_1^1\right), \tilde{P}_2^1\left(\tilde{X}_2^1, \tilde{Y}_2^1, \tilde{Z}_2^1\right), \ldots, \tilde{P}_{p-1}^1\left(\tilde{X}_{p-1}^1, \tilde{Y}_{p-1}^1, \tilde{Z}_{p-1}^1\right)$$

where each $\tilde{X}_i^1, \tilde{Y}_i^1$ and $\tilde{Z}_i^1$; $i = 1, 2 \ldots p - 1,$ is triangular fuzzy numbers defined by the triplets

$$\tilde{X}_i^1 = \left(X_{1i}^1, X_{2i}^1, X_{3i}^1\right), \tilde{Y}_i^1 = \left(Y_{1i}^1, Y_{2i}^1, Y_{3i}^1\right), \tilde{Z}_i^1 = \left(Z_{1i}^1, Z_{2i}^1, Z_{3i}^1\right).$$

Their point of intersection will be

$$\begin{aligned}
\tilde{P}^1 &= \left(\tilde{P}_1^1 + \tilde{P}_2^1 + \ldots + \tilde{P}_{p-1}^1\right)/(p-1)\\
&= \left(\left(\tilde{X}_1^1, \tilde{Y}_1^1, \tilde{Z}_1^1\right), + \tilde{P}_2^1\left(\tilde{X}_2^1, \tilde{Y}_2^1, \tilde{Z}_2^1\right) + \ldots + \left(\tilde{X}_{p-1}^1, \tilde{Y}_{p-1}^1, \tilde{Z}_{p-1}^1\right)\right)/(p-1)\\
&= \left(\tilde{X}_1^1 + \tilde{X}_2^1 + \ldots \tilde{X}_{p-1}^1, \tilde{Y}_1^1 + \tilde{Y}_2^1 + \ldots \tilde{Y}_{p-1}^1, \tilde{Z}_1^1 + \tilde{Z}_2^1 + \ldots \tilde{Z}_{p-1}^1\right)/(p-1)\\
&= \left(\frac{\tilde{X}_1^1 + \tilde{X}_2^1 + \ldots \tilde{X}_{p-1}^1}{p-1}, \frac{\tilde{Y}_1^1 + \tilde{Y}_2^1 + \ldots \tilde{Y}_{p-1}^1}{p-1}, \frac{\tilde{Y}_1^1 + \tilde{Y}_2^1 + \ldots \tilde{Y}_{p-1}^1}{p-1}\right)\\
&= \left(\frac{\left(\sum_i^{p-1} X_{1i}^1, \sum_i^{p-1} X_{2i}^1, \sum_i^{p-1} X_{3i}^1\right)}{p-1}, \frac{\left(\sum_i^{p-1} Y_{1i}^1, \sum_i^{p-1} Y_{2i}^1, \sum_i^{p-1} Y_{3i}^1\right)}{p-1}, \frac{\left(\sum_i^{p-1} Z_{1i}^1, \sum_i^{p-1} Z_{2i}^1, \sum_i^{p-1} Z_{3i}^1\right)}{p-1}\right)\\
&= \left(\left(\frac{\sum_i^{p-1} X_{1i}^1}{p-1}, \frac{\sum_i^{p-1} X_{2i}^1}{p-1}, \frac{\sum_i^{p-1} X_{3i}^1}{p-1}\right), \left(\frac{\sum_i^{p-1} Y_{1i}^1}{p-1}, \frac{\sum_i^{p-1} Y_{2i}^1}{p-1}, \frac{\sum_i^{p-1} Y}{p-1}\right), \left(\frac{\sum_i^{p-1} Z_{3i}^1}{p-1}, \frac{\sum_i^{p-1} Z_{3i}^1}{p-1}, \frac{\sum_i^{p-1} Z_{3i}^1}{p-1}\right)\right)\\
&= \left(\tilde{X}_1, \tilde{Y}_1, \tilde{Z}_1\right)(Say)
\end{aligned}$$

As each of the coordinates is triangular fuzzy number so the operations will follow the basic addition, multiplication and division of triangular fuzzy operations mentioned earlier.

Similarly we fix $i = 2$, then for $j = 1, 3, ..p$ again we get different values of $\tilde{r}_1$, $\tilde{r}_2$, $\ldots \tilde{r}_p$ and $p-1$ number of points of intersection between 2nd and 1st path, 2nd and 3rd path and so on, say $\tilde{P}_1^2(\tilde{X}_1^2, \tilde{Y}_1^2, \tilde{Z}_1^2)$, $\tilde{P}_2^2(\tilde{X}_2^2, \tilde{Y}_2^2, \tilde{Z}_2^2)$, $\ldots \tilde{P}_{p-1}^2(\tilde{X}_{p-1}^2, \tilde{Y}_{p-1}^2, \tilde{Z}_{p-1}^2)$.

Their average point of intersection will be

$$\begin{aligned}
\tilde{P}^2 &= \left(\tilde{P}_1^1 + \tilde{P}_2^2 + \ldots + \tilde{P}_{p-1}^2\right)/(p-1)\\
&= \left(\left(\tilde{X}_1^2, \tilde{Y}_1^2, \tilde{Z}_1^1\right), + \tilde{P}_2^2\left(\tilde{X}_2^2, \tilde{Y}_2^2, \tilde{Z}_2^2\right) + \ldots + \left(\tilde{X}_{p-1}^2, \tilde{Y}_{p-1}^2, \tilde{Z}_{p-1}^2\right)\right)/(p-1)\\
&= \left(\tilde{X}_1^2 + \tilde{X}_2^2 + \ldots \tilde{X}_{p-1}^2, \tilde{Y}_1^2 + \tilde{Y}_2^2 + \ldots \tilde{Y}_{p-1}^2, \tilde{Z}_1^2 + \tilde{Z}_2^2 + \ldots \tilde{Z}_{p-1}^2\right)/(p-1)\\
&= \left(\frac{\tilde{X}_1^2 + \tilde{X}_2^2 + \ldots \tilde{X}_{p-1}^2}{p-1}, \frac{\tilde{Y}_1^2 + \tilde{Y}_2^2 + \ldots \tilde{Y}_{p-1}^2}{p-1}, \frac{\tilde{Z}_1^2 + \tilde{Z}_2^2 + \ldots \tilde{Z}_{p-1}^2}{p-1}\right)\\
&= \left(\frac{\left(\sum_i^{p-1} X_{1i}^2, \sum_i^{p-1} X_{2i}^2, \sum_i^{p-1} X_{3i}^2\right)}{p-1}, \frac{\left(\sum_i^{p-1} Y_{1i}^2, \sum_i^{p-1} Y_{2i}^2, \sum_i^{p-1} Y_{3i}^2\right)}{p-1}, \frac{\left(\sum_i^{p-1} Z_{1i}^2, \sum_i^{p-1} Z_{2i}^2, \sum_i^{p-1} Z_{3i}^2\right)}{p-1}\right)
\end{aligned}$$

$$= \left(\left(\frac{\sum_i^{p-1} X_{1i}^2}{p-1}, \frac{\sum_i^{p-1} X_{2i}^2}{p-1}, \frac{\sum_i^{p-1} X_{3i}^2}{p-1} \right), \left(\frac{\sum_i^{p-1} Y_{1i}^2}{p-1}, \frac{\sum_i^{p-1} Y_{2i}^2}{p-1}, \frac{\sum_i^{p-1} Y_{3i}^2}{p-1} \right), \left(\frac{\sum_i^{p-1} Z_{3i}^2}{p-1}, \frac{\sum_i^{p-1} Z_{3i}^2}{p-1}, \frac{\sum_i^{p-1} Z_{3i}^2}{p-1} \right) \right)$$

$$= \left(\tilde{X}_2, \tilde{Y}_2, \tilde{Z}_2 \right) (Say)$$

Proceeding in this way we get all the aggregated point of intersections as $\tilde{P}^1$, $\tilde{P}^2$, $\ldots \tilde{P}^{p-1}$ and our Centre of origin is

$$\begin{aligned} P &= \left(\tilde{P}_1 + \tilde{P}^2 + \ldots + \tilde{P}^{p-1} \right) / (p-1) \\ &= \frac{\left(\left(\tilde{X}_1, \tilde{Y}_1, \tilde{Z}_1 \right) + \left(\tilde{X}_2, \tilde{Y}_2, \tilde{Z}_2 \right) + \ldots + \left(\tilde{X}_{p-1}, \tilde{Y}_{p-1}, \tilde{Z}_{p-1} \right) \right)}{p-1} \\ &= \left(\frac{\tilde{X}_1 + \tilde{X}_2 + \ldots \tilde{X}_{p-1}}{p-1}, \frac{\tilde{Y}_1 + \tilde{Y}_2 + \ldots \tilde{Y}_{p-1}}{p-1}, \frac{\tilde{Z}_1 + \tilde{Z}_2 + \ldots \tilde{Z}_{p-1}}{p-1} \right) \\ &= \left(\tilde{X}, \tilde{Y}, \tilde{Z} \right) = ((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3))(Say) \end{aligned}$$

Now we find the membership grade of each of the $p-1$ point's $\tilde{P}^1, \tilde{P}^2, \ldots, \tilde{P}^{p-1}$ points.

We consider an ideal point of convergence with membership grade $((1, 1, 1), (1, 1, 1), (1, 1, 1))$

Thus distance from the most ideal points to our calculated point of convergence are less as we tends to the appropriate points.

We define the membership grade of the points are as

$$\begin{aligned} \mu\left(\tilde{P}^i \right) &= 1 - \sqrt{\frac{1}{3} \left\{ d^2 \left(\frac{(x_1, x_2, x_3)}{\tilde{X}_i}, (1, 1, 1) \right) + d^2 \left(\frac{(y_1, y_2, y_3)}{\tilde{Y}_i}, (1, 1, 1) \right) + d^2 \left(\frac{(z_1, z_2, z_3)}{\tilde{Z}_i}, (1, 1, 1) \right) \right\}} \\ &= 1 - \sqrt{\frac{1}{3} \left\{ d^2 \left(\frac{(x_1, x_2, x_3)}{(x_{1i}, x_{2i}, x_{3i})}, (1, 1, 1) \right) + d^2 \left(\frac{(y_1, y_2, y_3)}{(y_{1i}, y_{2i}, y_{3i})}, (1, 1, 1) \right) + d^2 \left(\frac{(z_1, z_2, z_3)}{(z_{1i}, z_{2i}, z_{3i})}, (1, 1, 1) \right) \right\}} \\ &= 1 - \sqrt{\frac{1}{3} \left\{ d^2 \left(\left(\frac{x_1}{x_{1i}}, \frac{x_2}{x_{2i}}, \frac{x_3}{x_{3i}} \right), (1, 1, 1) \right) + d^2 \left(\left(\frac{y_1}{y_{1i}}, \frac{y_2}{y_{2i}}, \frac{y_3}{y_{3i}} \right), (1, 1, 1) \right) + d^2 \left(\left(\frac{z_1}{z_{1i}}, \frac{z_2}{z_{2i}}, \frac{z_3}{z_{3i}} \right), (1, 1, 1) \right) \right\}} \end{aligned}$$

As we get the membership grades of each points we get the most favorable points where each coordinate are also fuzzy triangular number and the crisp value can have by Centre of gravity method.

5 Conclusion and Discussion

In this paper we proposed to use triangular fuzzy number for the location of bloodstains in three dimensional axes and direction cosines. Then we discussed a method for determining average intersection of the path of flow of bloodstains i.e. Centre of origin.

One major advantage of this method over tangent and string method is that it considers the uncertainty of both the angle of impact and directionality of the blood flow better. In this method the area of origin is determined in 3D where as in other methods we have to determine the point of convergence in 2D first then proceed to determine area of origin in 3D. Also the consideration of intersection of each pair of trajectories and taking of their averages will causes more less errors.

One limitation of this method is that, this method is applicable only for one Centre of origin. If blood spatter was originated from more than one point then this method does not differentiate between two points. In that case, this will be on the investigator to determine what number of wounds or Centre of origins are there and choose appropriate bloodstains for corresponding Centre of origins under study.

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An Advanced Distance Measure for Intuitionistic Fuzzy Sets and Its Application in Decision Making

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Abstract. The intuitionistic fuzzy set (IFS), which is the generalization of fuzzy set, is a very useful tool to deal with vagueness and ambiguity. In the application of IFSs, distance measure is an imperative device in decision sciences. Though a lot of distance measures have been defined in literature, many of them fail to satisfy some axioms of distance measure or encounter some counterintuitive cases. In this paper, a novel distance measure for IFSs is developed which is based on matrix norm and an increasing function; and accordingly, a comparative study is also carried out to demonstrate the efficiency and validity of the proposed novel distance measure. Furthermore, some numerical examples of pattern recognition and medical decision making are performed under this setting.

Keywords: Uncertainty · Intuitionistic fuzzy sets · Distance measure · Medical decision making

1 Introduction

In 1986 Atanassov [1] developed the theory of IFS, which is the extension of Zadeh's [2] fuzzy set theory (FST). IFS allows to assign each element a membership degree, a non-membership degree and a hesitation degree, whereas FST assigns to each element only a membership degree and the complement of the membership degree is addressed as the non-membership degree. Consequently, IFS has been treated as a more effective tool in dealing with the uncertainty than the FST. Now a days, researchers have paid more attention on IFS for the uncertainty modelling problem and applied in a wide range of areas, such as, decision making, medical diagnosis, fuzzy optimization, pattern recognition etc. Distance measures and similarity measures of IFSs have become an important tool in decision analysis, pattern recognition, medical diagnosis and machine learning. Burillo and Bustince [3] developed some distance measures between IFSs incorporating the membership and non-membership functions. Szmidt and Kacprzyk [4] defined four new distances which were based on the geometric interpretation of IFSs. Later, Grzegorzewski [5] proposed some distance measures based on Hausdorff metric. Chen [6] gave some counterintuitive examples which were not properly explained by Grzegorzewski's distance measure. Xu [7] presented some weighted distances based on geometric distance model. Zhang and Yu [8] introduced two new distance measures and

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compared the advantages and disadvantages of the two approaches. Boran and Akay [9] put forwarded two new types of distance measures and discussed the relation between distance and similarity measures. Further, Wang and Xin [10], Park et al. [11], Yang and Chiclana [12] developed several distance and similarity measures for IFS. Song et al. [13] developed distance measure by defining the similarity measure for IFSs. Hatzimichailidis et al. [14] proposed distance between IFSs by introducing the matrix norms and fuzzy implications. Recently, Luo et al. [15] developed a distance measure based on Hatzimichailidis's distance measure. However, there may show inconsistent results if we adopt different distance measures in practical applications. Therefore, defining more new distance measures with various aspects is always beneficial to get more efficient distance measure. This motivates us to define an improved and efficient distance measure.

The detail works of the paper are shortened as follows. Section 2 contains some relevant preliminary definitions. In Sect. 3 some existing distance measures are reviewed briefly. In Sect. 4, a novel distance measure is proposed by using Frobenius norm of matrix and an increasing function. Numerical comparison, pattern recognition problem and medical decision problem are analyzed to show the efficiency of the proposed method and compared the result with various existing results in Sect. 5. Result and conclusion are drawn in Sect. 6. Lastly, a concrete and brief conclusion is presented in Sect. 7.

2 Preliminary

In this section, some relevant definitions and necessary background of IFS theory (Dutta et al., [16, 26], Atannasov, [1], and Garge, [17]) are reviewed.

Definition: 2.1. Fuzzy set is a set in which every element has degree of membership of belonging in it. Mathematically, let X be a universal set. Then the fuzzy subset A of X is defined by its membership function $\mu_A\,;\;X \to [0\,,\,1]$, which assign a real number $\mu_A(x)$ in the interval [0, 1], to each element $x \in A$, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A. i.e., $A = \{\langle x_i,\; \mu_A(x_i)\rangle : x_i \in X\}$.

Definition 2.2. A Intuitionistic fuzzy set A on a universe of discourse X is of the form $A = \{(x,\; \mu_A(x)\,,\; v_A(x)\,;\; x \in X)\}$, where $\mu_A(x) \in [0\,,\,1]$ is called the "degree of membership of x in A", $v_A(x) \in [0\,,\,1]$ is called the "degree of non-membership of x in A", and $v_A(x)$ and $v_A(x)$ satisfy the condition that $0 \le\; \mu_A(x) + \mu_A(x) \le 1$. The amount $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is called hesitancy of x which is reflection of lack of commitment or uncertainty associated with the membership or non-membership or both in A.

Let A and B be two IFSs given as $A = \{\langle x, \mu_A(x), v_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), v_B(x)\rangle : x \in X\}$. Denote $A^c = \{\langle x, v_A(x), \mu_A(x)\rangle : x \in X\}$.

$$\begin{aligned}
&A \subseteq B \Leftrightarrow\; (\forall x \in X)\,(\mu_A(x) \le \mu_B(x)\;\&\; v_A(x) \ge v_B(x));\\
&A = B \Leftrightarrow\; (\forall x \in X)\,(\mu_A(x) = \mu_B(x)\;\&\; v_A(x) = v_B(x));\\
&A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x)\,,\; v_A(x) \wedge v_B(x)\rangle : x \in X\};\\
&A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x)\,,\; v_A(x) \vee v_B(x)\rangle : x \in X\}
\end{aligned}$$

Definition 2.3. Let $M^{n\times n}$ be a metric space, $\|U\| : M^{n\times n} \to R$ is a function that satisfies the following conditions:

(i) $\|U\| \geq 0$, $\|U\| = 0$ if and only if $U = 0$ for all $U \in M^{n\times n}$
(ii) $\|aU\| = |a|\,\|U\|$, $a \in M$ for all $U \in M^{n\times n}$
(iii) $\|U + V\| \leq \|U\| + \|V\|$ for all $U, V \in M^{n\times n}$
(iv) $\|UV\| \leq \|U\|\,\|V\|$, for all U, $V \in M^{n\times n}$, then $\|U\|$ is a norm of matrix U.

Definition 2.4. A metric (distance) in a set X is a real function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

(i) $d(x, y) \geq 0$, for all $x, y \in X$
(ii) $d(x, y) = 0 \Leftrightarrow x = y$, for all $x, y \in X$
(iii) $d(x, y) = d(y, x)$, for all $x, y \in X$
(iv) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$

Let X be a nonempty set and A, B and C are IFSs on X. Then the function is a metric if it satisfies the following axioms:

$D1.$ $0 \leq d(A, B) \leq 1$
$D2.$ $d(A, B) = 0 \Leftrightarrow A = B$
$D3.$ $d(A, B) = d(B, A)$
$D4.$ if A ⊆ B ⊆ C, then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$

Definition 2.5. The Frobenius norm is the Euclidean norm on $F^{m\times n}$ and is define from the Frobenius inner product on the spaces of all matrices. This norm is defined as $\|A\| = \sqrt{\sum_{i=1}^{m}\sum_{i=1}^{n}\left|a_{ij}\right|^2} = \sqrt{trace(A^*A)}$ where A is a $m \times n$ matrix and A^* is the conjugate transpose of A.

3 Some Existing Distance Measures

In this section, some existing distance measures for IFSs are reviewed.

The Hamming distance $d_H(A, B)$:

$$d_H(A, B) = \frac{1}{2}\sum_{i=1}^{n}[|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)|]$$

The normalised Hamming distance $d_{nH}(A, B)$:

$$d_{nH}(A, B) = \frac{1}{2n}\sum_{i=1}^{n}[|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)|]$$

The Euclidean distance $d_e(A, B)$:

$$d_e(A, B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2\right]}$$

The normalised Euclidean distance $d_{ne}(A, B)$:

$$d_{ne}(A, B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2}$$

Wang and Xin proposed approach

$$d_W(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)|}{4} + \frac{\max\{|\mu_A(x_i) - \mu_B(x_i)|,\ |v_A(x_i) - v_B(x_i)|\}}{2}\right]$$

Szmidt and Kacprzyk [4, 18] further modified these distances for IFS by considering the three parameters of IFS: degree of membership $\mu_A(x)$, degree of non-membership $v_A(x_i)$ and degree of hesitancy $\pi_A(x_i)$ given by:

$$d^I(A, B) = \tfrac{1}{2}\sum_{i=1}^{n}\left[\,|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_A(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|\,\right]$$

$$d^{II}(A, B) = \tfrac{1}{2n}\sum_{i=1}^{n}\left[\,|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|\,\right]$$

$$d^{III}(A, B) = \sqrt{\tfrac{1}{2}\sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2\right]}$$

$$d^{IV}(A, B) = \sqrt{\tfrac{1}{2n}\sum_{i=1}^{n}\left[(\mu_A(x_i) - \mu_B(x_i))^2 + (v_A(x_i) - v_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2\right]}$$

$$d^V(A, B) = \tfrac{1}{n}\sum_{i=1}^{n}\left[\frac{|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|}{4} + \frac{\max(|\mu_A(x_i) - \mu_B(x_i)|,\ |v_A(x_i) - v_B(x_i)|,\ |\pi_A(x_i) - \pi_B(x_i)|)}{2}\right]$$

Yang and Francisco's distance measure

$$d^{VII}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_A(x_i) - \mu_B(x_i)|,\ |v_A(x_i) - v_B(x_i)|, |\pi_A(x_i) - \pi_B(x_i)|\}$$

Grzegorzewski's distance measure

$$d^{iv}(A, B) = \frac{1}{n}\left[\sum_{i=1}^{n}\max\{|\mu_A(x_i) - \mu_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}\right]$$

Jin Han Park's distance measure

$$d^{VIII} = \frac{1}{4n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|)$$
$$+\ 2\max\{|\mu_A(x_i) - \mu_B(x_i)|, |v_A(x_i) - v_B(x_i)|, |\pi_A(x_i) - \pi_B(x_i)|\}$$

Song's distance measure

$$d^{IX}(A, B) = 1 - \frac{1}{3n}\sum_{i=1}^{n}\left(\begin{array}{c} 2\sqrt{\mu_A(x_i)\mu_B(x_i)} + 2\sqrt{v_A(x_i)v_B(x_i)} + \sqrt{\pi_A(x_i)\pi_B(x_i)} + \\ \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))} + \sqrt{(1-v_A(x_i))(1-v_B(x_i))} \end{array}\right)$$

Xu's [19] distance measure

$$d^X_{IFSs}(A, B) = \frac{1}{2}\sqrt{\left(\Delta^{AB}_{\mu}\right)^2 + \left(\Delta^{AB}_{v}\right)^2 + \left(\Delta^{AB}_{\pi\mu}\right)^2 + \left(\Delta^{AB}_{\pi\mu}\right)^2},$$

where

$$\Delta^{AB}_{\mu} = \mu_A - \mu_B,\ \Delta^{AB}_{v} = v_A - v_B,\ \Delta^{AB}_{\pi\mu} = Assign^{\pi\mu}_A - Assign^{\pi\mu}_B,$$
$$\Delta^{AB}_{\pi v} = Assign^{\pi v}_A - Assign^{\pi v}_B$$

Hatzimichailidis's distance measure

$$d_T(A, B; \sigma_T) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

where

$$\sigma_T(a, b) = \frac{b}{a \vee b}$$

$$d_R(A, B; \sigma_R) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

Where $\sigma_R(a, b) = 1 - a + ab$

$$d_L(A, B; \sigma_L) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

where,

$$\sigma_L(a, b) = \min(1,\ 1 - a + ab)$$

$$d_{KD}(A, B; \sigma_{KD}) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

where

$$\sigma_{KD}(a, b) = \max(1 - a,\ b)$$

$$d_M(A, B; \sigma_M) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

where

$$\sigma_M(a, b) = \max(a,\ b)$$

$$d_{LA}(A, B; \sigma_{LA}) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

Where,

$$\sigma_{LA}(a, b) = abd_G(A, B; \sigma_G) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\|}{2n},$$

where $\sigma_G(a, b) = 1$ for $a \le b$, for $a > b$

Minxia Luo and Ruirui Zhao's distance measure

$$d_f(A, B;\ f) = \frac{\|\Pi(\mu_A) - \Pi(\mu_B)\| + \|\Pi(v_A) - \Pi(v_B)\| + \|\Pi(\pi_A) - \Pi(\pi_B)\|}{2n},$$
$$\left\|\prod\right\| = \sqrt{\lambda \max},$$

λ is the largest non-negative eigenvalue of positive definite Hermitian matrix $\prod^T \prod$

4 Novel Distance Measure Between IFSs

Hatzimichailidis [14] proposed a new method to define the distance measure between IFSs based on matrix norm and fuzzy implication. Luo et al. [15] have developed a distance followed by the work of Hatzimichailidis.

In this section, we shall propose a novel distance measure between IFSs by using Frobenius matrix norm and an increasing function. Furthermore, it is also shown that the proposed distance satisfies the axiomatic definition of the distance measure. Moreover, numerical analyses are exhibited to show the applicability of the proposed distance measure.

Consider a finite universe of discourse $X = \{x_1, x_2, \ldots\ldots, x_n\}$. Let $A = \{\langle x, \mu_A(x), v_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), v_B(x)\rangle : x \in X\}$ be two IFSs and $g : [0, 1] \times [0, 1] \to [0, 1]$ be an increasing function. Then the distance between the two IFSs A and B is given by

$$d_g(A, B) = \max\left\{\frac{\left\|\prod(\mu_A) - \prod(\mu_B)\right\|_F}{n}, \frac{\left\|\prod(v_A) - \prod(v_B)\right\|_F}{n}\right\},$$

Where, $\prod(\mu_A) = [g(\mu_A(x_i)\ ,\ \mu_A(x_j))\,]_{n\times n}$, $\prod(v_A) = [g(v_A(x_i)\ ,\ v_A(x_j))]_{n\times n}$ and $\|\prod\|_F = \sqrt{trace(\prod^T \prod)}$.

Also, $g(x, y) = \sin(x + y)\frac{\pi}{4}$ such that $g(0, 0) = 0$ and $g(1, 1) = 1$

Theorem 4.1. d_g is a distance measure between the IFSs A and B.

Proof: (i) Clearly, $d_g \ge 0$. $0 \le g\big(\mu_A(x_i)\ ,\ \mu_A\big(x_j\big)\big) \le 1$ for $\mu_A(x_i)\,,\ \mu_A\big(x_j\big) \in [0, 1]$ and $0 \le g\big(v_A(x_i)\ ,\ v_A\big(x_j\big)\big) \le 1$ for $v_A(x_i)\,,\ v_A\big(x_j\big) \in [0, 1]$ therefore, $0 \le \frac{\|\prod(\mu_A) - \prod(\mu_B)\|_F}{n} \le 1$ and similarly $0 \le \frac{\|\prod(v_A) - \prod(v_B)\|_F}{n} \le 1$. Hence, $0 \le d_g \le 1$.

(ii) Consider $A = B$ then $\mu_A(x_i) = \mu_B(x_i)$ & $v_A(x_i) = v_B(x_i)$. Thus we have, $\left\| \prod(\mu_A) - \prod(\mu_B) \right\|_F = 0$ and $\left\| \prod(v_A) - \prod(v_B) \right\|_F = 0$.
Hence, $d_g = 0$.
Conversely, let $d_g = 0$, then $\left\| \prod(\mu_A) - \prod(\mu_B) \right\|_F = 0$ and $\left\| \prod(v_A) - \prod(v_B) \right\|_F = 0$.
Now $\left\| \prod(\mu_A) - \prod(\mu_B) \right\|_F = 0$

$$\begin{aligned}
&\Rightarrow \prod(\mu_A) - \prod(\mu_B) = 0 \\
&\Rightarrow \prod(\mu_A) = \prod(\mu_B) \\
&\Rightarrow \left[g\left(\mu_A(x_i), \mu_A\left(x_j\right)\right) \right]_{n \times n} = \left[g\left(\mu_B(x_i), \mu_B\left(x_j\right)\right) \right]_{n \times n} \\
&\Rightarrow g\left(\mu_A(x_i), \mu_A\left(x_j\right)\right) = g\left(\mu_B(x_i), \mu_B\left(x_j\right)\right)
\end{aligned}$$

$\Rightarrow \mu_A(x_i) = \mu_B(x_i),\ \mu_A\left(x_j\right) = \mu_B\left(x_j\right)$ (As g is an increasing function)
Similarly, $\left\| \prod(v_A) - \prod(v_B) \right\|_F = 0$

$$\Rightarrow v_A(x_i) = v_B(x_i),\ v_A\left(x_j\right) = v_B\left(x_j\right)$$

Thus, $d_g = 0$ implies that $A = B$. Hence, $d_g = 0$ iff $A = B$.

(iii) Form the definition of norm obviously

$$\left\| \prod(\mu_A) - \prod(\mu_B) \right\|_F = \left\| \prod(\mu_B) - \prod(\mu_A) \right\|_F \text{ and}$$

$\left\| \prod(v_A) - \prod(v_B) \right\|_F = \left\| \prod(v_B) - \prod(v_A) \right\|_F$. Therefore, $d_g\,(A, B) = d_g\,(B, A)$.

(iv) Let $C = \{\langle x, \mu_C(x), v_C(x) \rangle : x \in X\}$ be an IFSs. Then from the Definition 2.4

$$\begin{aligned}
\left\| \frac{\prod(\mu_A) - \prod(\mu_B)}{n} \right\|_F &= \left\| \frac{\prod(\mu_A) - \prod(\mu_C) + \prod(\mu_C) - \prod(\mu_B)}{n} \right\|_F \\
&\le \left\| \frac{\prod(\mu_A) - \prod(\mu_C)}{n} \right\|_F + \left\| \frac{\prod(\mu_B) - \prod(\mu_C)}{n} \right\|_F \\
\text{and} \left\| \frac{\prod(v_A) - \prod(v_B)}{n} \right\|_F &= \left\| \frac{\prod(v_A) - \prod(v_C) + \prod(v_C) - \prod(v_B)}{n} \right\|_F \\
&\le \left\| \frac{\prod(v_A) - \prod(v_C)}{n} \right\|_F + \left\| \frac{\prod(v_B) - \prod(v_C)}{n} \right\|_F
\end{aligned}$$

$$\begin{aligned}
\text{Thus, } \max &\left\{ \frac{\left\| \prod(\mu_A) - \prod(\mu_B) \right\|_F}{n}, \frac{\left\| \prod(v_A) - \prod(v_B) \right\|_F}{n} \right\} \\
\le \max &\left\{ \frac{\left\| \prod(\mu_A) - \prod(\mu_C) \right\|_F}{n}, \frac{\left\| \prod(v_A) - \prod(v_C) \right\|_F}{n} \right\} \\
+ \max &\left\{ \frac{\left\| \prod(\mu_B) - \prod(\mu_C) \right\|_F}{n}, \frac{\left\| \prod(v_B) - \prod(v_C) \right\|_F}{n} \right\}
\end{aligned}$$

Hence, $d_g\,(A, B) \le d_g\,(A, C) + d_g\,(C, B)$.
Therefore, d_g satisfies all the axioms of metric.

5 Numerical Analysis

Consider the pairs of IFSs A and B as shown in the Table 1. Different distance measures fail to measure the correct distance between these two IFSs.

Table 1. Comparison of different distance measures with some counterintuitive examples of IFSs

A	$\left\{\begin{matrix}\langle 0.3, 0.2\rangle,\\ \langle 0.4, 0.3\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle 0.3, 0.2\rangle,\\ \langle 0.4, 0.3\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle 0.5, 0.4\rangle,\\ \langle 0.4, 0.3\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle 0.5, 0.4\rangle,\\ \langle 0.4, 0.3\rangle\end{matrix}\right\}$
B	$\left\{\begin{matrix}\langle 0.15, 0.25\rangle,\\ \langle 0.25, 0.35\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle 0.16, 0.26\rangle,\\ \langle 0.26, 0.36\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle 0.4, 0.4\rangle,\\ \langle 0.5, 0.4\rangle\end{matrix}\right\}$	$\left\{\begin{matrix}\langle 0.6, 0.3\rangle,\\ \langle 0.3, 0.2\rangle\end{matrix}\right\}$
d_T	**0.05**	**0.05**	0.11	0.10
d_R	**0.05**	**0.05**	**0.05**	**0.05**
d_L	**5.55 × 10^{-17}**	**5.55 × 10^{-17}**	**5.55 × 10^{-17}**	**5.55 × 10^{-17}**
d_{KD}	**0.10**	**0.10**	**0.09**	**0.09**
d_M	**0.10**	**0.10**	0.07	0.09
d_{LA}	**0.06**	**0.06**	0.05	**0.06**
d_G	**0.05**	**0.05**	0.33	**0.05**
d^{IV}	0.13	0.12	**0.14**	**0.14**
d^{VI}	0.15	0.14	**0.10**	**0.10**
d^{VII}	**0.15**	**0.15**	**0.15**	**0.15**
d^{VIII}	**0.15**	0.14	**0.15**	**0.15**
d^{IX}	**0.01**	**0.01**	**0.01**	0.03
d_f	0.20	0.19	0.14	0.18
d_g	0.2132	0.1983	0.0844	0.1396

In Table 1, it is seen that many existing distance measures are not able to measure the correct distance between two IFSs A and B. But, the proposed distance measure d_g can identify the correct distances in such situations. Though the distance measures of d_f and d_g are in the same order but the obtained degree of distances by d_g are more confident for these pairs of IFSs rather than d_f.

5.1 Application to Pattern Recognition Problem

In this section, we will discuss the classification of the pattern recognition problems by using the proposed distance measure.

Example 1 [20]. Consider three known patterns represented by the IFSs A_1, A_2 and A_3 in the universe of discourse $X = \{x_1, x_2, x_3, x_4\}$ respectively, which are given by

$$A_1 = \left\{ \begin{array}{l} \langle x_1, 0.8, 0.1 : x_1 \in X \rangle, \langle x_2, 0.5, 0.3 : x_2 \in X \rangle, \langle x_3, 0.5, 0.5 : x_3 \in X \rangle, \\ \langle x_4, 0.6, 0.1 : x_4 \in X \rangle \end{array} \right\}$$
$$A_2 = \left\{ \begin{array}{l} \langle x_1, 0.5, 0.4 : x_1 \in X \rangle, \langle x_2, 0.4, 0.2 : x_2 \in X \rangle, \langle x_3, 0, 0 : x_3 \in X \rangle, \\ \langle x_4, 0.3, 0.4 : x_4 \in X \rangle \end{array} \right\}$$
$$A_3 = \left\{ \begin{array}{l} \langle x_1, 0.6, 0.3 : x_1 \in X \rangle, \langle x_2, 0.7, 0.2 : x_2 \in X \rangle, \langle x_3, 0.6, 0.1 : x_3 \in X \rangle, \\ \langle x_4, 0.2, 0.5 : x_4 \in X \rangle \end{array} \right\}$$

Suppose, there is an another unknown pattern B which is given by the IFS

$$B = \left\{ \begin{array}{l} \langle x_1, 0.7, 0.2 : x_1 \in X \rangle, \langle x_2, 0.5, 0.2 : x_2 \in X \rangle, \langle x_3, 1, 0 : x_3 \in X \rangle, \\ \langle x_4, 0.4, 0.3 : x_4 \in X \rangle \end{array} \right\}.$$

Our target is to classify the unknown pattern B into one of the pattern A_1, A_2 and A_3.

The details of the degrees of the distance measures $d(A_1, B)$, $d(A_2, B)$ and $d(A_3, B)$ by different existing distance measures are discussed in Table 2.

Table 2. Distance measures and classification of unknown patterns

Distances	$d(A_1, B)$	$d(A_2, B)$	$d(A_3, B)$
d_g	0.2950	0.4693	0.1705
d^{II}	**0.2250**	0.3500	**0.2250**
d^{VII}	**0.2250**	0.3500	**0.2250**
d_W	0.2188	0.2813	0.1937

From Table 2, it is observed that the distance measures d_g and d_W can properly identify the unknown pattern. Therefore, the unknown pattern belongs to class 3.

Example 2. This example has been taken from [15]. Consider the set of three given patterns

$$P_1 = \{\langle x_1, 15.0, 0.25 : x_1 \in X \rangle, \langle x_2, 0.25, 0.35 : x_2 \in X \rangle, \langle x_3, 0.35, 0.45 : x_3 \in X \rangle\}$$
$$P_2 = \{\langle x_1, 0.05, 0.15 : x_1 \in X \rangle, \langle x_2, 0.15, 0.25 : x_2 \in X \rangle, \langle x_3, 0.25, 0.35 : x_3 \in X \rangle\}$$
$$P_3 = \{\langle x_1, 0.16, 0.26 : x_1 \in X \rangle, \langle x_2, 0.26, 0.36 : x_2 \in X \rangle, \langle x_3, 0.36, 0.46 : x_3 \in X \rangle\}$$

There is an unknown pattern $S = \{\langle x_1, 0.3, 0.2 : x_1 \in X \rangle, \langle x_2, 0.4, 0.3 : x_2 \in X \rangle, \langle x_3, 0.5, 0.4 : x_3 \in X \rangle\}$. The classification of the unknown pattern and different distance measures along with the proposed distance measure are discussed details in Table 3.

Table 3. Classification of pattern by using distance measures

Distances	$d(P_1, S)$	$d(P_2, S)$	$d(P_3, S)$	Classification results
d^{II}	0.15	0.30	0.14	P_3
d^{IV}	0.13	0.28	0.12	P_3
d^{VI}	0.15	0.25	0.14	P_3
d^{VII}	0.15	0.30	0.14	P_3
d^{VIII}	0.15	0.30	0.14	P_3
d_T	0.05	0.12	0.05	Cannot classified
d_R	0.04	0.07	0.04	Cannot classified
d_L	3.70×10^{-17}	3.70×10^{-17}	3.70×10^{-17}	Cannot classified
d_{KD}	0.10	0.15	0.10	Cannot classified
d_M	0.10	0.15	0.10	Cannot classified
d_{LA}	0.07	0.08	0.07	Cannot classified
d_G	0.05	0.08	0.05	Cannot classified
d^{IX}	0.01	0.05	0.01	Cannot classified
d_f	0.20	0.40	0.19	P_3
d_g	0.21	0.35	0.19	P_3

In Table 3, the distance measures d^{II}, d^{IV}, d^{VI}, d^{VII}, d^{VIII}, d_f and the proposed distance measure d_g can properly classify the unknown pattern and belongs to the class 3. Moreover, it can be easily seen that the proposed distance measure gives more confident results than the other distance measures. Therefore, the distance measure d_g will be more helpful in pattern recognition problems.

5.2 Application in Medical Decision Analysis

In this section, we shall apply the proposed distance measure to a decision making problem to diagnosis the symptoms of a patient based on intuitionistic fuzzy relation which is discussed in [15, 19–25].

Let P = {Al, Bob, Joe, Ted} be the set of patients, S = {temperature, headache, stomach pain, cough, chest pain} be the set of symptoms, D = {viral fever, Malaria, typhoid, stomach problem, chest problem} be the set of diseases. Our intention is to carry out the right decision for each patient $p_i, i = 1, 2, 3, 4$ from the set of symptoms $s_j, j = 1, 2, 3, 4, 5$ for each disease $d_k, k = 1, 2, 3, 4, 5$.

The symptom-disease intuitionistic fuzzy relation $S \to D$ and patient-symptom intuitionistic fuzzy relation $P \to S$ are given in Table 4 and in Table 5 respectively.

Table 4. symptoms characteristic for the diagnoses

$S \to D$	Viral Fever	Malaria	Typhoid	Stomach Pain	Chest problem
Temperature	(0.4, 0.0)	(0.7, 0.0)	(0.3, 0.3)	(0.1, 0.7)	(0.1, 0.8)
Headache	(0.3, 0.5)	(0.2, 0.6)	(0.6, 0.1)	(0.2, 0.4)	(0.0, 0.8)
Stomach pain	(0.1, 0.7)	(0.0, 0.9)	(0.2, 0.7)	(0.8, 0.0)	(0.2, 0.8)
Cough	(0.4, 0.3)	(0.7, 0.0)	(0.2, 0.6)	(0.2, 0.7)	(0.2, 0.8)
Chest Pain	(0.1, 0.7)	(0.1, 0.8)	(0.1, 0.9)	(0.2, 0.7)	(0.8, 0.1)

Table 5. Symptoms characteristic for the patients

$P \to S$	Temperature	Headache	Stomach pain	Cough	Chest pain
Al	(0.8, 0.1)	(0.6, 0.1)	(0.2, 0.8)	(0.6, 0.1)	(0.1, 0.6)
Bob	(0.0, 0.8)	(0.4, 0.4)	(0.6, 0.1)	(0.1, 0.7)	(0.1, 0.8)
Joe	(0.8, 0.1)	(0.8, 0.1)	(0.0, 0.6)	(0.2, 0.7)	(0.0, 0.5)
Ted	(0.6, 0.1)	(0.5, 0.4)	(0.3, 0.4)	(0.7, 0.2)	(0.3, 0.4)

Using the proposed distance measure d_g, different distances between patients and diseases are calculated in Table 6.

Table 6. Distance measures between patients and diseases

$P \to D$	Viral fever	Malaria	Typhoid	Stomach pain	Chest problem
Al	0.259	**0.232**	0.282	0.487	0.542
Bob	0.401	0.485	0.297	**0.149**	0.386
Joe	0.261	0.362	**0.221**	0.487	0.543
Ted	0.282	**0.254**	0.332	0.399	0.460

Table 7. Comparison of the previous result with our results

	Al	Bob	Joe	Ted
The diagnosed result in [19]	Viral Fever	Stomach Problem	Typhoid	Viral Fever
The diagnosed result in [20]	Viral Fever	Stomach Problem	Typhoid	Viral Fever
The diagnosed result in [21]	Malaria	Stomach Problem	Typhoid	Malaria
The diagnosed result in [22]	Malaria	Stomach Problem	Malaria	Malaria
The diagnosed result in [23]	Malaria	Stomach Problem	Typhoid	Viral Fever
The diagnosed result in [24]	Malaria	Stomach Problem	Typhoid	Viral Fever
The diagnosed result in [18]	Viral Fever	Stomach Problem	Typhoid	Malaria
The diagnosed result in [25]	Malaria	Stomach Problem	Typhoid	Viral Fever
The diagnosed result in [15]	Malaria	Stomach Problem	Typhoid	Malaria
The diagnosed result by using proposed distance measure	Malaria	Stomach Problem	Typhoid	Malaria

6 Result and Discussion

The principle of minimum distance degree indicates that lesser the distance measure of alternative signifies a proper diagnosis. The obtained results are highlighted by bold character in Table 6. From the Table 6, the proper diagnosis are Al suffers from Malaria, Bob suffer from Stomach pain, Joe suffers from Typhoid and Ted suffers from Malaria, which coincides with the results [15] and [26]. Various studies have been done so far in medical diagnosis using fuzzy set, interval valued fuzzy set, IFS, interval valued intuitionistic fuzzy set etc. It is seen that IFS gained more popularity in medical diagnosis because of its interesting concept of membership and non-membership degree. The applicability and adeptness are shown by studying pattern recognition problems, medical decision making analysis via the proposed distance measure.

7 Conclusion

Although several distance measures between IFSs have been proposed to deal with uncertainty in information systems, most of them have encountered some counter-intuitive cases. In this research, we have developed a novel distance measure between IFSs based on matrix norm and increasing function which overcome the counterintuitive cases. It is observed from the above discussed numerical results that the proposed distance measure is more efficient, technically sound and produce more confident results in different applications. Researcher may apply this distance measure in various fields.

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Generalized Type-2 Intuitionistic Fuzzy Approaches for Allocation and Redistribution of Resources in the Disaster Operation

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Abstract. The aftermath of unpredictable events like disaster urges for logistic deployment with humanitarian aid to provide relief to the people in the affected areas. All the information about the requirements of the victims is also difficult to measure in the calamity. This research investigates the proper plan to response in the anarchic situation by introducing a mathematical model for solid transportation problem for disaster relief authority in uncertain environment with generalized type-2 intuitionistic fuzzy number, so that it can execute the relief operation in such catastrophe with uncertain information about demand. The objective of the mathematical model is to minimize the total transportation cost. The model is designed in two phases: the first phase initiates the transportation of humanitarian items from some central distribution center arranged by disaster management authority, non-governmental organization, humanitarian aid agencies etc. to some relief center where people can obtain help with proper relief. Through this research, one fact is noticed that after a fixed time interval, some items are finished in some relief center according to the consume rate of the items and some other relief centers are available with that items. So, this research has introduced a redistribution phase to mitigate the wastage of items as well as quick response with minimization of transportation cost. A numerical example is presented to show the efficiency of the model performance and solved using LINGO optimization solver.

Keywords: Disaster management · Solid transportation problem · Generalized type-2 intuitionistic fuzzy number · Redistribution · LINGO

1 Introduction

Disaster may be natural or anthropogenic, always causes damages in the society and put it in a very grievous condition. These agonizing events suddenly strike without or with little warning to the society. After striking of disaster,

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the demolished society have to be proper cared and some relief authority, non-governmental organization (NGO), aid. agencies have performed their active association by providing humanitarian items and recover them quickly from the sudden shock to put back to the normal situation. There are many more historical records which reveals that several natural disasters like earthquake, flood, landslide, hurricane, drought, volcanic eruption, fire, tsunami etc. and human caused disaster like terrorist attack, nuclear exploitation etc. causes lots of damages and kills thousands of people every year. Massive earthquake in Nepal (2015), Haiti earthquake (2010), excessive flood in Assam and Kerela (2018), worst flood in Chennai, drought in Ethiopia (2016) are some deadliest disaster causing worsen ambiance of millions of people desolating their lives and property. Regarding this condition of society, it is very urgent to contribute relief govern by well planned management to execute their operation for disaster relief [1–3]. To deal this smoothly, the concept of solid transportation problem (STP) is beneficial for the authority. Basically STP which is introduced by Haley [4] is an extension of transportation problem (TP), first introduced by Hitchcoke [5]. In the literature, lots of article review TP in uncertain environment reviewed from the literature [6–8]. Recently Jana et al. [9] used generalized type-2 intuitionistic fuzzy and its application to the TP. The intuitionistic fuzzy is applied in operation management in [10]. It is also recognized that interest of STP in uncertain environment are growing day by day [11–14].

After the catastrophe, the society face immeasurable damages, which is the cause of incalculable demand. All the information about the requirement can not predict exactly at that crisis time. But it is necessary to provide humanitarian items to the affected people although it is not exactly predictable. In the literature, there are lots of paper contributing their work for humanitarian logistic in disaster by proper decision making considering uncertainty in [15–18]. It is a true fact that most of the decision take place in the real world in which goals and constraints are not clear or precisely defined to a crisp value and hence uncertainty arises [19]. The decision maker (DM) have confused sometimes in the catastrophic situation and faced a big challenge in conveying the items properly so that people would be benefited. So consideration of uncertainty in the mathematical model is the best to handle the chaotic environment.

In this paper, we have introduced a mathematical model for STP for conveying humanitarian items in disaster operation for large-scale recovery. The objective of the model is minimization of the total transportation cost. To this context, consider central distribution centers (CDCs) are the main source from where humanitarian items are transported to the relief centers (RCs) established very near to the affected areas (AAs) in the first phase immediately after the disaster. Each AA is serviced by one particular RC at a time. But after some fixed time interval it has been noticed that some RCs have excess amount and some are lack of some particular humanitarian items based on the consumed rate of resources. Consume rate of resources implies the utilization of some resources in that particular araes. The low consumption of items basically indicates that it starts to recover from the effect of disaster and less amount of humanitarian items are used by the people in such areas. Thus sufficient amount of humanitarian items are available in such RCs than consumption rate. But at the same time, some of the AAs still face the worse effect of disaster and demand for more humanitarian items. In this situation RCs with excess of items are able to deliver the humanitarian items to the RCs where it is unavailable. Regarding this situation, the authority have started to redistribute the items among the RCs according to the availability of items in RCs. The purpose of initialization of redistribution is to mitigate the wastage and to quick response with that particular items which is inaccessible in a particular RCs with minimum cost. The fact of redistribution in crisis operation have been seen in the literature [20]. Since disastrous situation can not predict properly about the exact demand, so this model is considered in uncertain environment. This uncertainty of the model is handled by using generalized type-2 intuitionistic fuzzy number (GT2IFN). Type-2 intuitionistic fuzzy set is advanced over type-1 intuitionistic fuzzy set as the membership function are themselves in fuzzy. This model may help DM to make their proper decision in the allocation of resource in the catastrophe by considering redistribution.

2 Preliminaries

In this section, we first discuss some basic ideas on intuitionistic type-2 fuzzy sets with definition and notation for convenience of explaining general concepts.

Definition 2.1: Generalized intuitionistic fuzzy number(GIFN): An intuitionistic fuzzy number $\widetilde{D}^I = \{< x, \mu_{\widetilde{D}}, \nu_{\widetilde{D}} >\}$ of the real line R is called GIFN, if the following hold:

(i) there exists x $\in$ R, $\mu_{\widetilde{D}}(n) = \omega$, $\nu_{\widetilde{D}} = 0$, $0 < \omega \leq 1$
(ii) $\mu_{\widetilde{D}}$ is continuous mapping from R to the close interval [0, ω] and x $\in$ R, the relation $0 \leq \mu_{\widetilde{D}}(\mathrm{x}) + \nu_{\widetilde{D}}(\mathrm{x}) \leq \omega$ holds.

The membership function of $\widetilde{D}^I$ is of the following form

$$\mu_{\widetilde{D}}(x) = \begin{cases} \omega g_1(x), & m - \alpha \leq x \leq m; \\ \omega, & x = m; \\ \omega j_1(x), & m \leq x \leq m + \beta; \\ 0 & otherwise \end{cases}$$

The non-membership function of $\widetilde{D}^I$ is of the following form

$$\nu_{\widetilde{D}}(x) = \begin{cases} \omega g_2(x), & m - \alpha^{'} \leq x \leq m, 0 \leq \omega(g_1(x) + g_2(x)) \leq \omega; \\ 0, & x = m; \\ \omega j_2(x), & m \leq x \leq m + \beta^{'}, 0 \leq \omega(j_1(x) + j_2(x)) \leq \omega; \\ \omega, & otherwise \end{cases}$$

In this equation g_1(x) and j_1(x) are strictly increasing and decreasing function in [m-α, m] and [m, m+β] respectively, and g_1(x) and j_1(x) are strictly decreasing and increasing function of in [m-$\alpha^{'}$, m] and [m, m+$\beta^{'}$] respectively, where n is the mean value of $\widetilde{D}^I$. The left and right spreads of membership function $\mu_{\widetilde{D}}$(x) are called α and β. The left and right spread of non-membership function $\nu_{\widetilde{D}}$(x) are called $\alpha^{'}$ and $\beta^{'}$.

Definition 2.2: Generalized trapezoidal intuitionistic type-2 fuzzy number (GTIT2FN): Let $\Theta \in$ {L,U} and $d_1^{'\Theta} \leq d_1^{\Theta} \leq d_2^{\Theta} \leq d_3^{\Theta} \leq d_4^{\Theta} \leq d_4^{'\Theta}$. A GTIT2FN $\widetilde{D}^I$= [$\mu^L_{\widetilde{D}}$(x), $\mu^U_{\widetilde{D}}$(x), $\nu^L_{\widetilde{D}}$(x), $\nu^U_{\widetilde{D}}$(x)] in R written as (d_1^{Θ}, d_2^{Θ}, d_3^{Θ}, d_4^{Θ}; ω^{Θ})($d_1^{'\Theta}$, $d_2^{'\Theta}$, $d_3^{'\Theta}$, $d_4^{'\Theta}$; ω^{Θ}) has a membership function

$$\mu_{\widetilde{D}}(x) = \begin{cases} \omega^{\Theta}\frac{x-d_1^{\Theta}}{d_2^{\Theta}-d_1^{\Theta}}, & d_1^{\Theta} \leq x \leq d_1^{\Theta} \\ \omega^{\Theta}, & d_2^{\Theta} \leq x \leq d_3^{\Theta} \\ \omega^{\Theta}\frac{d_4^{\Theta}-x}{d_4^{\Theta}-d_3^{\Theta}}, & d_3^{\Theta} \leq x \leq d_4^{\Theta} \\ 0, & otherwise; \end{cases}$$

and non-membership function

$$\nu_{\widetilde{D}}(x) = \begin{cases} \omega^{\Theta}\frac{d_2-x}{d_2^{\Theta}-d_1^{'\Theta}}, & d_1^{'\Theta} \leq x \leq d_2^{\Theta}; \\ 0, & d_2^{\Theta} \leq x \leq d_3^{\Theta}; \\ \omega^{\Theta}\frac{x-d_3^{\Theta}}{d_4^{'\Theta}-d_3^{\Theta}}, & d_3^{\Theta} \leq x \leq d_4^{'\Theta} \\ \omega^{\Theta}, & otherwise \end{cases}$$

Definition 2.3: α **- cut set:** A α- cut set of $\widetilde{D}^I$=(d_1^{Θ}, d_2, d_3, d_4, ω)($d_1^{'}$, d_2, d_3, $d_4^{'\Theta}$, ω) is a crisp subset of R that is defined as follows: D_α= $\{x : \mu_{\widetilde{D}^I}(x) \geq \alpha\}$=[$D_1(\alpha)$, $D_2(\alpha)$]
= [d_1^{Θ}+ $\frac{\alpha}{\omega}$($d_1 - d_1^{\Theta}$), $d_4 -$ $\frac{\alpha}{\omega}$($d_4 - d_3$)]

Definition 2.4: β **- cut set:** A β-cut set of $\widetilde{D}^I$=(d_1^{Θ}, d_2, d_3, d_4, ω)($d_1^{'}$, d_2, d_3, $d_4^{'\Theta}$, ω) is a crisp subset of R that is defined as follows: D_β=$\{x : \nu_{\widetilde{D}^I}(x) \leq \beta\}$=[$D_1^{'}(\beta)$, $D_2^{'}(\beta)$]
= [$d_1 -$ $\frac{\beta}{\omega}$(d_2-$d_1^{'}$), d_3+ $\frac{\beta}{\omega}$($d_4^{'\Theta} - d_3$)]

The arithmetic operation of GT2IFN are refer in [9].

2.1 Defuzzfication Rule for Generalized Type-2 Intuitionistic Fuzzy Number:

There are many method for defuzzification among them we adopt mean of interval method to find the values of membership and non-membership function of GT2IFN as in [9].

Mean Interaval Method [9]

The $(\alpha, \beta)-$cut of the GT2IFN is given by $D_{\alpha,\beta} = \{[D_1(\alpha), D_2(\alpha)]; [D_1^{'}(\beta), D_2^{'}(\beta)], \alpha + \beta \leq \omega, \alpha, \beta \in [0, \omega]\}$

where $D_1(\alpha) = d_1 + \frac{\alpha^{\Theta}}{\omega^{\Theta}}(d_2^{\Theta} - d_1)$, $D_2(\alpha) = d_4^{\Theta} - \frac{\alpha^{\Theta}}{\omega^{\Theta}}(d_4^{\Theta} - d_3^{\Theta})$, $D_1^{'}(\beta) = d_2^{\Theta} - \frac{\beta}{\omega}(d_2^{\Theta} - d_1^{'\Theta}), D_2^{'}(\beta) = d_3^{\Theta} + \frac{\beta}{\omega}(d_4^{'\Theta} - d_3^{\Theta})$. Now by mean of interval method the representation of the membership function is

$$R_{\mu}(\widetilde{D}^I) = \frac{1}{2}\int_0^{\omega}(D_1(\alpha) + D_2(\alpha))d\alpha$$
$$= \frac{1}{2}\int_0^{\omega}[d_1 + d_4^{\Theta} + \frac{\alpha^{\Theta}}{\omega^{\Theta}}\{(d_2^{\Theta} - d_1) - (d_4^{\Theta} - d_3^{\Theta})\}]d\alpha$$
$$= \frac{1}{2}[d_1\omega + d_4^{\Theta}\omega + \frac{\omega}{2}\{(d_2^{\Theta} - d_1) - (d_4^{\Theta} - d_3^{\Theta})\}]$$
$$= \frac{\omega(d_1 + d_2^{\Theta} + d_3^{\Theta} + d_4^{\Theta})}{4}$$

Now, by the mean of interval method, the presentation of the non-membership function is

$$R_{\nu}(\widetilde{D}^I) = \frac{1}{2}\int_0^{\omega}(D_1(\beta) + D_2(\beta))d\beta$$
$$= \frac{1}{2}\int_0^{\omega}[d_2^{\Theta} + d_3^{\Theta} - \frac{\beta}{\omega}\{(d_2^{\Theta} - d_1^{'\Theta}) - (d_4^{'\Theta} - d_3^{\Theta})\}]d\beta$$
$$= \frac{1}{2}[d_2^{\Theta}\omega + d_3^{\Theta}\omega - \frac{\omega}{2}\{(d_2^{\Theta} - d_1^{'\Theta}) - (d_4^{'\Theta} - d_3^{\Theta})\}]$$
$$= \frac{\omega(d_1^{'\Theta} + d_2^{\Theta} + d_3^{\Theta} + d_4^{'\Theta})}{4}$$

Let $\widetilde{D}^I = ((d_1, d_2^{\Theta}, d_3^{\Theta}, d_4; \omega_1)(d_1^{'\Theta}, d_2^{\Theta}, d_3^{\Theta}, d_4^{'\Theta}; \omega_1)$ *and* $\widetilde{E}^I = ((e_1^{\Theta}, e_2^{\Theta}, e_3^{\Theta}, e_4^{\Theta}; \omega_2^{\Theta})(e_1^{'\Theta}, e_2^{\Theta}, e_3^{\Theta}, e_4^{'\Theta}; \omega_2^{\Theta})$ be two GTIFNs, then from [21]

(i) $\widetilde{D} \prec \widetilde{E}$ iff $H(\widetilde{D}^I) < H(\widetilde{E}^I)$
(ii) $\widetilde{D} \succ \widetilde{E}$ iff $H(\widetilde{D}^I) > H(\widetilde{E}^I)$
(iii) $\widetilde{D} = \widetilde{E}$ iff $H(\widetilde{D}^I) = H(\widetilde{E}^I)$

Where, $H(\widetilde{D}^I) = \frac{R_{\mu}(\widetilde{D}^I) + R_{\mu}(\widetilde{E}^I)}{2} = \frac{\omega(d_1 + 2d_2^{\Theta} + 2d_3^{\Theta} + d_4^{\Theta} + d_1^{'\Theta} + d_4^{'\Theta})}{8}$

$H(\widetilde{E}^I) = \frac{R_{\nu}(\widetilde{D}^I) + R_{\nu}(\widetilde{E}^I)}{2} = \frac{\omega(e_1^{\Theta} + 2e_2^{\Theta} + 2e_3^{\Theta} + e_4^{\Theta} + e_1^{'\Theta} + e_4^{'\Theta})}{8}$

Where $\omega^{\Theta} = min(\omega_1^{\Theta}, \omega_2^{\Theta})$. With the help of the above formula we have optimized the introduced model for STP for disaster response in the GT2IFN environment.

3 Concept of the Model

When disaster strikes and damages a large society, living people in that region get affected and urges for relief. Consider a set of CDCs from where humanitarian items (e.g food, medicine, clothes, drinking water etc.) are delivered to a set of RCs set up very near to the AAs. The different types of humanitarian items are shipped to RCs from some CDCs using particular types of conveyances (e.g. truck, dumper etc.). But after a certain time interval, it is observed by the authority that some items are unavailable in some RCs due to the more consumption of those particular items in those particular RCs according to the needs of people based on degree of devastation of the disaster. At the same time those items have some excess amount in another RCs i.e. consumption of those items are minimum in these RCs which becomes sufficient with that items. In this situation authority make decision regarding the emergency condition, the RCs which have excess amount of items redistribute to the RCs where items are lack with the same vehicles used in the previous phase of transportation. As one

RC is assigned for providing relief with humanitarian items to one AAs, so when there is an unavailability of humanitarian items in the RCs, consequently other RCs which are available with that items are become active to redistribute that item to bare RCs. The carrying charge in the redistribution phase is less than that of CDCs to RCs. Total transportation cost of this redistribution phase is bounded by a budget constraint. In this research, the above mention fact is formulated with a mathematical model to minimize the total cost of transportation considering the parameters are in GT2IFN (Fig. 1).

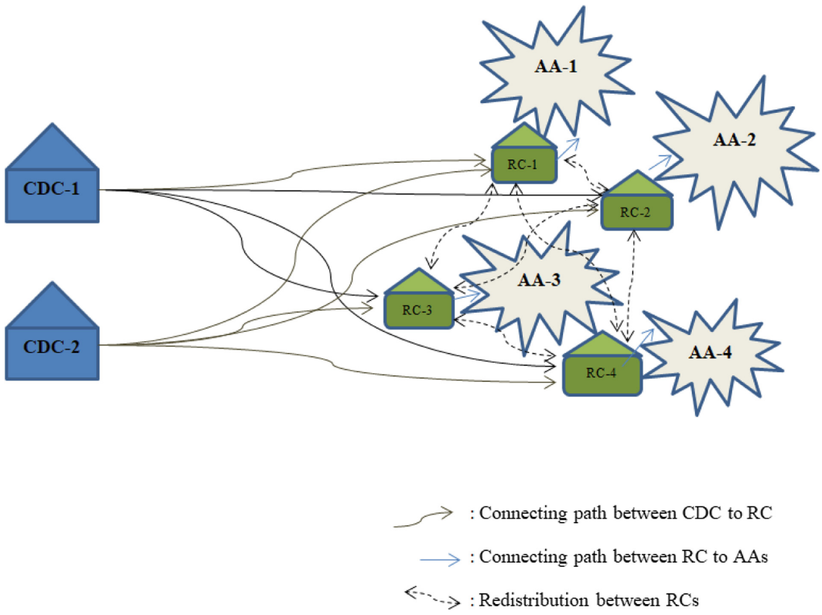

Fig. 1. Distribution process in disaster operation

Assumption for the Model

- The mathematical model is an unbalanced STP.
- CDC transported different types of humanitarian items to the RC where people can obtain relief.
- Redistribution started after a fixed time period which is determined specifically according to the devastation for the model.
- Redistribution is not considered between same two RCs.
- Vehicles are same for both the phases of distribution of humanitarian items.
- Budget is restricted for the redistribution process.

Indices

- I: Set of CDC indexed by i.
- J: Set of RC indexed by j.
- K: Set of conveyance indexed by k.
- P: Set of humanitarian items indexed by p.
- $J^{'}$: Set of RC with the requirement of p-th relief materials in the redistribution phase indexed $j^{'}$.

Parameter

$\widetilde{c}^p_{ijk}$: Transportation cost for shipping the p-th humanitarian items from i-th CDCs to j-th RCs through k-th conveyance.
$\widetilde{q}^p_{jj'k}$: Transportation cost for transferring the p-th humanitarian items from occupied j-th RCs to empty $j^{'}$-th RCs through k-th conveyance in redistribution phase.
$\widetilde{a}^p_i$: Availability of p-th humanitarian items at i-th CDC
$\widetilde{b}^p_j$: Demand of the p-th humanitarian items at the j-th RCs.
$\widetilde{e}^p_k$: Capacity of the k-th conveyance to convey p-th humanitarian items.
$\widetilde{b}^{Ec}_j$: Consumed amount of p-th humanitarian items in the j-th RC after a period of time.
$\widetilde{b}^{Ep}_{j'}$: Required amount of p-th humanitarian items in the $j^{'}$-th RC after a period of time.
$\widetilde{B}$: Budget for redistribution phase.

Decision Variables

- x^p_{ijk}: Amount of p-th humanitarian items shipped from i-th CDC to j-th RCs through k-th conveyance.
- $y^p_{jj'k}$: Amount of p-th humanitarian items transferring from j-th RCs to $j^{'}$-th RCs through k-th conveyance in the phase of redistribution.

Mathematical Model. The mathematical model is designed in two phase of distribution minimizing the total transportation cost, where first phase is considered for the allocation of humanitarian items from CDCs to RCs and second phase is considered for redistribution of humanitarian items among the RCs. The model is presented below:

$$Min\ Z_1 = \sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{p=1}^{P}\left(\sum_{i=1}^{I}\widetilde{c}^p_{ijk}x^p_{ijk} + \sum_{j'=1}^{J'}\widetilde{q}^p_{jj'k}y^p_{jj'k}\right) \tag{1}$$

subject to

$$\sum_{j=1}^{J}\sum_{k=1}^{K} x_{ijk}^{p} \leq \widetilde{a}_{i}^{p} \quad \forall i, p \tag{2}$$

$$\sum_{i=1}^{I}\sum_{k=1}^{K} x_{ijk}^{p} \geq \widetilde{b}_{j}^{p} \quad \forall j, p \tag{3}$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J} x_{ijk}^{p} \leq \widetilde{e}_{k}^{p} \quad \forall k, p \tag{4}$$

$$\sum_{j=1}^{J}\sum_{k=1}^{K} y_{jj'k}^{p} = \widetilde{b}_{j'}^{Ep} \quad \forall j', p \; and \; j \neq j' \tag{5}$$

$$\widetilde{b}_{j'}^{Ep} = \widetilde{b}_{j}^{p} - \widetilde{b}_{j}^{Cp} \quad \forall j \neq j' \tag{6}$$

$$\sum_{j=1}^{J}\sum_{j'=1}^{J'}\sum_{k=1}^{K}\sum_{p=1}^{P} \widetilde{q}_{jj'k}^{p} y_{jj'k}^{p} \leq B \quad \forall j \neq j' \tag{7}$$

$$\sum_{j=1}^{j}\sum_{j'=1}^{j'} y_{jj'k}^{p} \leq \widetilde{e}_{k}^{p} \quad \forall k, p \tag{8}$$

$$x_{ijk}^{p}, y_{jj'k}^{p} \in R^{+}, \tag{9}$$

The objective function (1) minimizes the total transportation cost of the disaster operation including the transportation cost for shipping humanitarian items from CDCs to RCs in the first phase and transportation cost due to redistribution among the RCs in the second phase. The constraint (2) defines that shipment from CDC to RC can not exceed the availability of humanitarian items in the CDC. The constraint (3) defines that transporting amount of humanitarian items are able to fulfill the demand of RCs. The constraint (4) is the capacity of conveyance to transport humanitarian items. The constraint (5) is define the redistribution of the humanitarian items after a fixed time interval according to the necessity of RCs by the other RCs which have available amount of items. The constraint (6) define to measure the amount of humanitarian items redistributed to the RC where items are finished after the consumption by the AAs from the RCs where items are excess. The constraint (7) is the budget cost for redistribution. The constraint (8) is the capacity of conveyance for redistribution phase which is same as the previous stage. The constraint (9) is the non-negativity restriction.

4 Solution Methodology

The mathematical model introduced in the Sect. 3 is nothing but a STP for the disaster relief operation. The parameters of the model are in the GT2IFN. The steps are given below:

Step-1: Construct the parameter in the form GT2IFN.
Step-2: Convert the uncertain parameter to the crisp parameter by the method in the Sect. 2.1.
Step-3: Find the optimal solution by solving the STP using LINGO optimization solver.
Step-4: Find the fuzzy optimal value of the objective function by putting the optimal solution.

5 Computational Studies and Numerical Example

The model performance is analyzed with a numerical example. Numerical values are considered arbitrarily relevant for the problem. Numerical set up is constructed here is based on considering 2 CDCs, 4 RCs, 2 humanitarian items, 2 types of conveyance. Moreover the fixed time interval is 8 days after which redistribution process is started. Consider the RC-1 and RC-4 are unavailable with the humanitarian item-1 (i.e p = 1) and RC-3 is unavailable with the humanitarian item-2 (i.e p = 2) after 8 days. Total budget is also predefined for the redistribution phase so that cost would be minimized. If the budget is crossed it should be better to transport humanitarian items from CDCs to RCs directly. For confined the cost in a minimum range, budget constraint for redistribution is defined in the mathematical model.

5.1 Input for the Mathematical Model

The objective function is constructed in two parts indicating the transportation cost both for the first phase of transportation from CDC to RC and for the redistribution phase determined by the addition of two terms: product of per unit transportation cost ($\widetilde{c}^{p}_{ijk}$) with the amount of humanitarian items shipped from CDCs to AAs (x^{p}_{ijk}) and product of per unit transportation cost ($\widetilde{q}^{p}_{jj'k}$)with the amount of humanitarian items among the RCs ($y^{p}_{jj'k}$). The input for the objective function and the constraints are given in the Table 1.

5.2 Result Analysis

The mathematical model presented in the Sect. 3 is coded in LINGO optimization solver which is based on the generalized reduced gradient (GRG) algorithm. The optimal result obtained for mathematical model minimizing cost is 1271.34. The allocation of humanitarian items from CDCs to RCs is shown in the Table 2. These allocation is exhibited by a graphical representation in the Fig. 2 which fulfills the total demand of the AAs in the first phase.

But after a period of 8 days, it has been noticed that effect of disaster is control to some extent in the AA-2 i.e it is recovered from the continuous demolition. Therefore people in that area do not desire more humanitarian items,

Table 1. Input of the objective function and constraint for the model

Transportation cost of humanitarian items from CDCs to RCs				
$k = 1$				
	$i = 1$		$i = 2$	
j	$p = 1$	$p = 2$	$p = 1$	$p = 2$
1	(3,4,6,7;0.9) (2,4,6,8; 0.9)	(4,4.5,5.5,7;1) (3.5,4.5,5.5,7.5;1)	(6,6.2,6.5,7;0.8) (5.8,6.2,6.5,7.5;0.8)	(7,7.3,7.5,8;0.8) (6.8,7.3,7.5,8.5;0.8)
2	(7.2,7.6,8,8.5;0.9) (7,7.6,8,9;0.9)	(6,7,7.2,7.5;0.9) (5.8,7,7.2,8;0.9)	(9.2,9.5,9.8,10.2;0.6) (9,9.5,9.8,10.6;0.6)	(10,10.2,10.5,11;0.5) (9.8,10.2,10.5,11.5;0.5)
3	(6.5,6.8,7,7.5;0.9) (6.2,6.8,7,8;0.9)	(7,7.2,7.5,8;0.6) (6.8,7.2,7.5,8.5;0.6)	(7,7.5,8,8.2;1) (6.8,7.5,8,8.8;1)	(10,10.3,10.6,11.2;0.7) (9.7,10.3,10.6,11.5;0.7)
4	(7.8,8,8.2,8.5;0.8) (7.5,8,8.2,9;0.8)	(7.5,8,8.3,8.5;0.9) (7.2,8,8.3,9;0.9)	(7,7.5,8,8.5;0.8) (6.8,7.5,8,9;0.8)	(6,6.8,7,7.5;0.9) (5.8,6.8,7,8;0.9)
$k = 2$				
	$i = 1$		$i = 2$	
j	$p = 1$	$p = 2$	$p = 1$	$p = 2$
1	(6,6.5,7,7.5;0.8) (5,6.5,7,8;0.8)	(7,7.5,8,8.2;0.7) (6.8,7.5,8,8.5;0.7)	(8,8.2,8.5,9;0.6) (7.8,8.2,8.5,9.5;0.6)	(9,9.2,9.5,10;0.7) (8.8,9.2,9.5,10.5;0.7)
2	(6.2,6.5,6.9,7.2;1) (6,6.5,6.9,7.5;1)	(5.1,5.3,5.5,6;1) (5.,5.3,5.5,6.5;1)	(9.8,10,10.6,11.5;0.6) (9.5,10,10.6,12;0.6)	(8,8.5,9,9.5;0.9) (7.8,8.5,9,10;0.9)
3	(7.5,7.8,8,8.2;0.7) (7.2,7.8,8,8.5;0.7)	(8,8.2,8.5,9;0.9) (7.2,8.2,8.5,9.2;0.9)	(11,11.2,11.5,12;0.5) (10.5,11.2,11.5,12.5;0.5)	(10,10.5,11,11.5;0.6) (9,10.5,11,12;0.6)
4	(6,6.5,7,7.5;0.8) (5.8,6.5,7,7.8;0.8)	(7.5,7.8,8,8.2;0.9) (7.2,7.8,8,8.5;0.9)	(9,9.5,10,10.2;0.6) (8.8,9.5,10,10.5;0.6)	(8,8.5,9,9.5;0.7) (7.5,8.5,9,10;0.7)
Capacity of CDC				
i	$p = 1$		$p = 2$	
1	(200,250,280,310;0.35)(190,250,280,350;0.35)		(150,200,220,250;0.38)(120,200,220,270;0.38)	
2	(110,150,180,210;0.55)(80,150,180,260;0.55)		(120,150,180,210;0.6)(110,150,180,230;0.6)	
Conveyance capacity				
k	$p = 1$		$p = 2$	
1	(50,55,60,65;0.9)(48,55,60,70;0.9)		(60,65,68,70;0.85)(55,65,68,75;0.85)	
2	(70,75,80,89;0.76)(65,75,80,95;0.76)		(58,62,65,70;0.78)(55,62,65,75;0.78)	
Demand of RCs				
p	j=1	j=2	j=3	j=4
1	(30,34,37,40;0.8) (20,34,37,35;0.8)	(30,40,45,50;0.6) (25,40,45,55;0.6)	(25,32,40,45;0.8) (20,32,40,50;0.8)	(20,30,32,38;1) (18,30,32,40;1)
2	(40,45,50,55;0.6) (38,45,50,60;0.6)	(28,38,42,45;0.7) (25,38,42,50;0.7)	(22,25,30,32;0.8) (18,25,30,40;0.8)	(25,27,30,32;0.9) (20,27,30,35;0.9)
Demand of RCs after a fixed period of interval				
p	$j'=1$	$j'=2$	$j'=3$	$j'=4$
1	(10,15,20,25;0.7) (9,15,20,30;0.7)	-	-	(10,12,15,18;1) (8,12,15,20;1)
2	-	-	(11,15,18,22;0.9) (9,15,18,28;0.9)	-
Budget for redistribution: (2000,3000,4000,5000;0.9)(1500,3000,4000,5500;0.9)				
Transportation cost for the redistribution phase				
$k = 1,\ p = 1$				
j	$j' = 1$	$j' = 2$	$j' = 3$	$j' = 4$
1	–	(3,3.2,3.5,4;0.8) (2.8,3.2,3.5,4.2;0.8)	(5,5.5,5.8,6;0.4) (4.8,5.5,5.8,6.5;0.4)	(3.5,4,4.2,4.8;0.7) (3,5,4,4.2,5;0.7)
2	(3,3.5,3.6,3.8;0.9) (2.5,3.5,3.6,4.2;0.9)	–	(2,2.8,3,3.2;0.8) (1.8,2.8,3,3.5;0.8)	(6.5,6.8,7,7.5;0.3) (6,6.8,7,8;0.3)
3	(4,4.5,4.8,5;0.6) (3.8,4.5,4.8,5.2;0.6)	(3.5,4,4.2,5;0.5) (3,4,4.2,5.5;0.5)	–	(3,3.2,3.5,4;0.8) (2.8,3.2,3.5,4.2;0.8)
4	(3,3.5,3.6,3.8;0.9) (2.5,3.5,3.6,4.2;0.9)	(3,3.5,3.9,4.3;0.8) (2.5,3.5,3.9,4.8;0.8)	(3,3.5,3.8,4.1;0.9) (2.9,3.5,3.8,4.5;0.9)	–

Table 1. (*continued*)

Transportation cost of humanitarian items from CDCs to RCs				
$k = 1,\ p = 2$				
j	$j' = 1$	$j' = 2$	$j' = 3$	$j' = 4$
1	–	(4,4.2,4.5,5;0.7) (3.8,4.2,4.5,5.5;0.7)	(6,6.2,6.5,7;0.5) (5.8,6.2,6.5,7.5;0.5)	(4,4.2,4.5,4.8;0.6) (3.8,4.2,4.5,5.2;0.6)
2	(2.5,3,3.5,4;0.8) (2,3,3.5,4.5;0.8)	–	(3,3.5,3.8,4;0.7) (2.8,3.5,3.8,4.5;0.7)	(4,4.2,4.5,5;0.8) (3.5,4.2,4.5,5.5;0.8)
3	(4.2,4.6,4.8,5;0.6) (3.8,4.6,4.8,5.2;0.6)	(2,2.5,3,3.5;0.9) (1.8,2.5,3,3.8;0.9)	–	(4.2,4.5,4.8,5.2;0.6) (4,4.5,4.8,5.5;0.6)
4	(3,3.5,3.8,4;0.7) (2.8,3.5,3.8,4.5;0.7)	(2.5,3,3.5,4;0.8) (2,3,3.5,4.5;0.8)	(3,3.5,3.9,4.3;0.8) (2.5,3.5,3.9,4.8;0.8)	–
$k = 2,\ p = 1$				
j	$j' = 1$	$j' = 2$	$j' = 3$	$j' = 4$
1	–	(4.2,4.5,4.8,5.2;0.6) (4,4.5,4.8,5.5;0.6)	(4,4.5,5,5.2;0.8) (3.7,4.5,5,5.5;0.8)	(6,6.5,6.8,7.1;0.5) (5.5,6.5,6.8,7.8;0.5)
2	(1.5,2,2.5,3;0.9) (1.2,2,2.5,4;0.9)	–	(4,4.2,4.5,4.8;0.8) (3.5,4.2,4.5,5.2;0.8)	(3,3.5,4,4.2;0.6) (2.8,3.5,4,4.5;0.6)
3	(3.8,4,4.2,4.5;0.8) (3.5,4,4.2,4.5.2;0.8)	(0.5,1.5,1.8,2;1) (0.3,1.5,1.8,2.5;1)	–	(1.5,2,2.5,3;0.9) (1.2,2,2.5,4;0.9)
4	(6,6.5,7,7.2;0.4) (5.5,6.5,7,7.5;0.4)	(2,2.8,3,3.2;0.8) (1.8,2.8,3,3.5;0.8)	(6,6.5,6.8,7.1;0.5) (5.5,6.5,6.8,7.8;0.5)	–
$k = 2,\ p = 2$				
j	$j' = 1$	$j' = 2$	$j' = 3$	$j' = 4$
1	–	(3,3.5,3.8,4.1;0.9) (2.9,3.5,3.8,4.5;0.9)	(3,3.5,3.9,4.3;0.8) (2.5,3.5,3.9,4.8;0.8)	(4,4.2,4.5,5;0.7) (3.8,4.2,4.5,5.2;0.7)
2	(1.2,1.5,2.2,2.5;1) (1,1.5,2.2,3;1)	–	(6,6.5,7,7.2;0.4) (5.5,6.5,7,7.5;0.4)	(3.2,3.8,4,4.5;0.6) (3,3.8,4,4.8;0.6)
3	(3,3.5,4,4.2;0.9) (2.8,3.5,4,4.5;0.9)	(3,3.5,3.9,4.3;0.8) (2.5,3.5,3.9,4.8;0.8)	–	(1.5,2,2.5,3;0.9) (1.2,2,2.5,4;0.9)
4	(3.8,4,4.2,4.5;0.8) (3.5,4,4.2,4.5.2;0.8)	(1.2,1.5,2.2,2.5;1) (1,1.5,2.2,3;1)	(4,4.2,4.5,5;0.7) (3.8,4.2,4.5,5.2;0.7)	–

In table, index j indicates the number of RCs, i for CDCs, k for conveyances used, p for humanitarian item.

consequently less amount items are consumed in RC-2, which is 5.4 (for p = 1 in j = 2) and 11.65 (for p = 2 in j = 2) which is less than the previous demand in the first stage from the Table 1 after defuzzified. At the same time, remaining three AAs are not recovered their normal situation before disaster. There are lots of people in those areas still demand humanitarian items for getting relief. In this regard the authority have concerned about the quick response of humanitarian items to the AA-1, AA-3 and AA-4 and low wastage of items in RC-2 with minimization of cost as well. In this regard response authority have decided to redistribute the excess items from RC-2 to RC-1, RC-3 and RC-4. The redistributed amount of humanitarian item-1 is 10.6 (for RC-1) and 8.75 (for RC-4) and humanitarian item-2 is 15.3 (for RC-3) as shown in the Table 2. The shipment after redistribution of humanitarian item-1 from RC-2 to RC-1 and RC-4 and humanitarian item-2 to RC-3 is portrait in the Fig. 3.

Table 2. Allocation of humaniatarian items

Distribution of humaniatarian items from CDCs to AAs								
$k = 1$					$k = 2$			
	$i = 1$		$i = 2$		$i = 1$		$i = 2$	
j	$p = 1$	$p = 2$	$p = 1$	$p = 2$	$p = 1$	$p = 2$	$p = 1$	$p = 2$
1	28.5	6.7375	-	-	-	21.9875	-	-
2	-	-	23.5875	26.95	-	-	1.1625	-
3	-	22.2	-	-	28.4	-	-	-
4	-	-	-	-	30	-	-	25.425
Redistribution of humaniatarian items								
	$j' = 1$		$j' = 2$		$j' = 3$		$j' = 4$	
j	$p = 1$	$p = 2$	$p = 1$	$p = 2$	$p = 1$	$p = 2$	$p = 1$	$p = 2$
1	-		-	-	-		-	-
2	10.6	-	-	-	-	15.3	8.75	-
3	-		-	-		-	-	-
4	-	-	-	-		-	-	

In table, index j indicates the number of RCs, i for CDCs, k for conveyances used, p for humanitarian item.

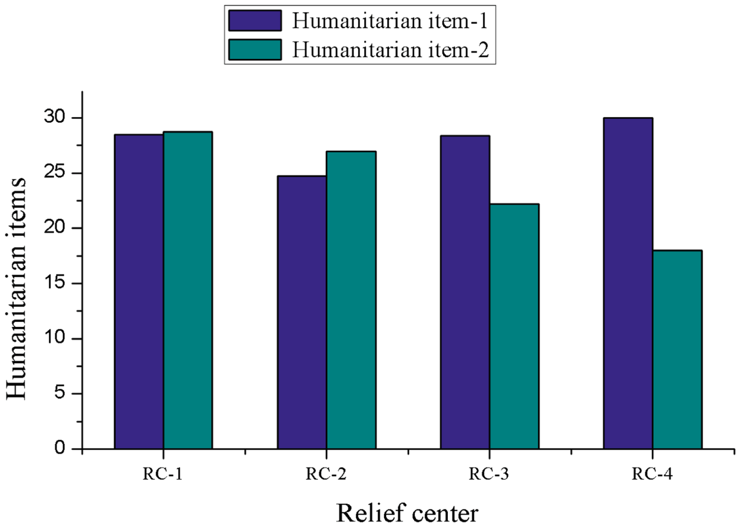

Fig. 2. Distribution of humanitarian items from CDC to AA

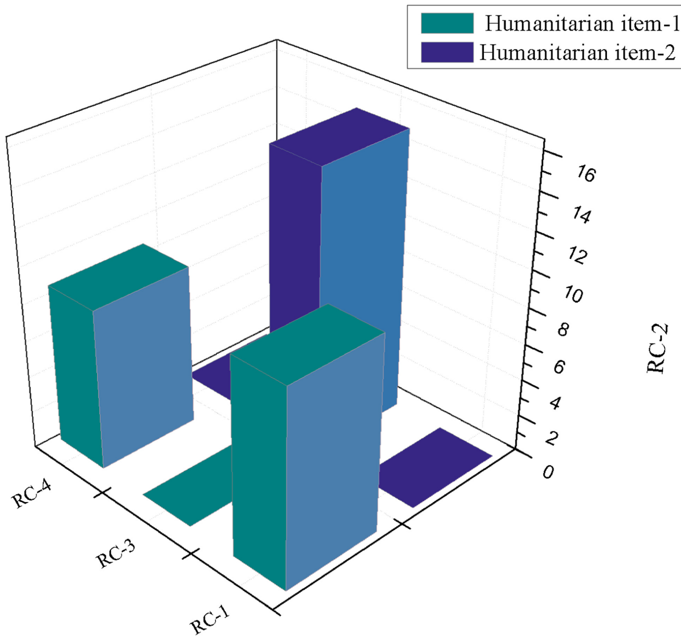

Fig. 3. Redistribution of humanitarian items among RCs

6 Conclusion and Future Scope

Through the research a mathematical model is investigated for STP in calamity for supplying humanitarian items to the affected people minimizing the total transportation cost. This contribution of the paper is basically for disaster management authority for making decision to execute their relief operation for providing humanitarian aid to the demolished society. The mathematical model introduced in the paper is constructed in two phases: first phase is for supplying humanitarian items from some central distribution center to relief center where people can get help for surviving in disaster. But with the recovery from the attack of disaster in some particular affected areas after a fixed time interval, people lowers to consume humanitarian items from their assign relief centers. Consequently the relief centers are become available with humanitarian items with sufficient amount. During the same time, some areas are not able to recover from the demolition of disaster and humanitarian items are still required in those areas. This fact initiate the redistribution phase regarding the quick response and mitigate the wastage as well minimize the transportation cost. Through this model optimal allocation of the humanitarian items in both the phases are completed with the fulfillment of the demand. The proficiency of the model is explained with a numerical example and solved with LINGO optimization solver showing the smooth functioning of the model.

There are various scope of the research for future extension. The uncertainty can be measure with different tools such as unceratain theory, also stochastic programming also applicable for future purpose. The model can be solved also taking time minimization as objective function in future research.

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Application of Artificial Intelligence on Behavioral Finance

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Abstract. Nowadays there is a concern in everyone's mind regarding the changes that are going to happen very soon or we can say the changes that are already happening in today's world. Yes, I am talking about the 4th industrial revolution that is going to take place very soon due to which many new jobs will be created whereas, on the other hand many existing jobs will get disappeared. Everyone is talking about Artificial Intelligence and its pros and cons but we wanted to think in other aspects too regarding artificial intelligence. So, in this research paper, we put the spotlight on Application of Artificial Intelligence on Behavioral Finance. By this we not only get to know about artificial intelligence and its merits and demerits but also, get to know about its applications on behavioral finance.

In this paper we tried to explain the meaning of artificial intelligence as well as its advantages and disadvantages and thereafter, the meaning of behavioral finance is considered so as to put clarity over the topic. Artificial intelligence is also known as machine learning as it is related to work which is going to be done by the machines rather than manually by human beings.

Further, we included the interaction of behavioral finance and artificial intelligence (AI) in the future when AI will be full-fledged used for the working and its working for the economy and future developments. With the use of artificial intelligence in behavioral finance, the outcomes will be more accurate as with the use of machines the psychological biasness can be reduced and accuracy will be increased. This paper helps to understand the working of artificial intelligence with behavioral finance and its applications so as to get ready for the future beforehand.

Keywords: Behavioral finance · Artificial Intelligence (AI) · Industrial revolution · Economical effects · Customer investing behaviour

1 Introduction

In present dynamic world, the concept of artificial intelligence is gained the attention of many people, i.e., AI, is not only the topic of interest among the employed people but it is a matter of concern and interest among the future leaders of the world. All the people across the world have their own concern about the growing perspective of AI, concerns of some individuals are concerned their existing jobs, on the other hand some

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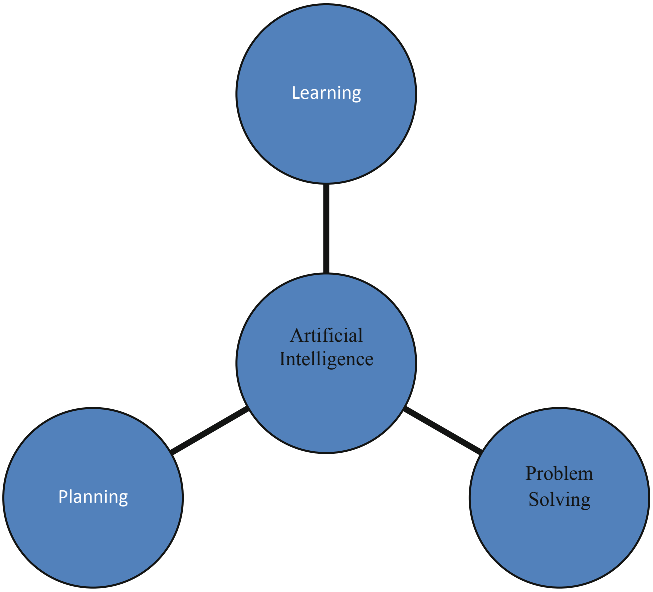
have concern regarding their future jobs So, it can be inferred that AI has become a topic of great concern. In a way it is a future and we have to accommodate according to it.

The main focus of this research paper is not confined about the discussion of pros and cons of AI, but authors are curious to explore the unexplored dimension of artificial intelligence. Many researches are conducted in the domain of artificial intelligence but very less research is conducted to understand the impact of AI on behavioural finance and understanding its application in behavioural finance. Initially, we need to understand the concept of Artificial Intelligence; it's a kind of replacement of human beings not in each and every aspect but in few aspects. Artificial Intelligence is a development of computer systems which are able to perform all activities or tasks which require human intelligence and can act as a replacement of human beings and the tasks which involve human touch. Artificial area is an area of computer science that emphasis the creation of intelligent machines that work and react like humans.



Behavioral political economy includes behavioral finance as its sub domain. It initiates various theories which are based on psychology; these theories are based on the clarification of stock exchange anomalies. The objective is to highlight and identify the role of folks in financial decisions. Under the behavioural finance, we have perceived the information flow and the attributes of intermediaries are constantly stimulating the investor's decisions and as well as market results. It is the combination of different concepts like, mental accounting, herd behaviour, anchoring, and high self-rating. Mental accounting we can say that it is a mind exercise to find the best options for our investment specifically. Herd behaviour explains that folks mean to copy the money behaviours of the bulk, or herd. Anchoring defines that enclosing a disbursement level to an exact reference, like disbursement extra money on what's gave the impression to be a stronger item of consumer goods. At the end, high- self rating define as an ability of one person based on personality to rank him/herself higher as compared to others. For instance, an investor might appreciate himself as an investment leader or best investment advisor when its investment decision proves to be good however can revoke his contribution to an investment not proves good.

The impact of artificial intelligence on finance can be observed in various ways:

Personalized financial services- Artificial Intelligence expands the gamut money services by suggesting square measure known as client financial services. Client monetary services keep the customers and their distinctive demands at the core of their extremely optimized offerings.

The reduced cost of Artificial Intelligence in finance- AI in finance is a machine-driven processes and this has drastically reduced the value of serving customers. Whereas AI, on one hand, has reduced the value of economic services, on the contrary, it has created finance extraordinarily convenient to avail.

Business acceleration- Business acceleration refers to the different ways a company is using AI to expedite knowledge-based activities to boost potency and performance, like monetary establishments making investment methods for his or her investors. Whereas this kind of activity is usually viewed as a chance to scale back prices through the automation of internal processes, it ought to even be thought-about in terms of the firm's ability to remodel the client expertise.

The Future- AI in finance is all concerning continuous learning and re-learning of patterns, data, and developments within the money world. AI offers the pliability to create upon this system or line of monetary product and services. It implies that company needs to focus and they have to improvise upon their offerings. Once introduced, AI can keep the money services updated and prepared to face the market. AI in finance is, therefore, invaluably causative to the money trade. Over time, AI isn't solely reaching to revolutionize the money trade however it is slowly and gradually becoming the trade itself.

Over all Artificial intelligence is beneficial for the economy and has brought a new and easy work structure for people engaged in finance with various options available to them. Behavioral Finance and artificial intelligence both work side by side with each other providing advantages to each other.

The best example of Artificial intelligence is "mobile keypad". In mobile keypad if the similar word is used frequently by the individual than automatically the word is saved for the future correspondence which makes it easy to type or quickly respond. Artificial intelligence derives value based on business outcomes around customer experience, cost reduction and revenue generation. AI is simply the latest advanced analytical technology that might help achieve the desired outcomes.

2 Literature Review

Geetha and Vimala (2012) in their paper explored the role of AI in asset valuation, risk management securities trading and monitoring and customer relationship management (CRM). They also explored the pros and cons of AI in asset valuation and risk management. The study identified the benefits of AI techniques that they are helpful in reducing the risk of frauds and they techniques are knowledge-based, machine learning, and natural language processing.

Divya (2015) in their paper proposed an Artificial Intelligence design for classifying Big Data and also explored the uses of AI in data management and decision making.

Russell and Dewey (2015) in their study explored the benefits of Artificial Intelligence (AI) and the various uses in which this technology can be put to use. They also explored the AI's economic impact.

Vempati (2016) in their study identified the challenges faced by AI technology as well as the future scope of AI technology in Indian scenario. Researcher also recommended ways to improve AI adoption in India by developing a deliberate strategy by Government of India.

Kashiwagi (2015) in their study explored the use of Artificial Intelligence (AI) in finance sector. Researcher in their study identified uses of AI like Text mining, voice recognition of financial reports, Anomaly detection through pattern recognition, Market analysis through data mining, Formulation of investment strategies.

Bentley and Brundage (2018) in their study clarified the myths regarding the threats of Artificial Intelligence and promoted the bright future of AI technology in various fields.

Hammond (2015) in their study explored the history and future scope of Behavioral Finance. Researcher also explored the sentiment shift in investment behaviour of people since 1980s as well as explained the sub topics of behaviour finance.

Smith and McGuire (2006) in their study explored the history of Artificial Intelligence (AI). Researcher explored the Turing Test, History of AI applied to Chess, Expert Systems AI Winter and its lessons, Japan's Fifth Generation Computer System project in detail to examine the evolution of Artificial Intelligence.

Camerer (2017) in their research describes the interaction between Artificial Intelligence (AI) and behavioral economics. Researcher also explored the differences in machine learning and AI as well as ways in which AI can overcome the human limitations. Researcher used examples of various exploratory researches to describe the interaction between Artificial Intelligence (AI) and behavioral economics.

Barbara and Grosz (2016) in their report explored the history of Artificial Intelligence (AI) for over hundred years as well as report explored the future of AI technology in changing the lives of people in future. The report explored the uses of AI in year 2030.

Kowalski he was explain the various benefits of Artificial intelligence in his research. And also the tools and techniques of artificial intelligence give the contribution in different domains like theory for decision making, philosophy etc.

Chella and Ignazio (2006) in their study has studied the interaction between AI and robotics including the history of uses of AI in robotics and future scope of AI in robotics.

To understand the meaning of behaviour finance, you should understand the origin of behaviour finance. Shiller (2003) guide the new comers in the field and give a detail explanation related to evolution of behavioral finance. There were number of theories formed to explain the fluctuation in stock prices and these theories also explain the issue related to volatility in stock prices.

Chen and Lai (2013), they give more focus on that expected return affected by company structure. They took 352 sample of Taiwanese companies, these companies were introduced a change in their standard industrial classification because of their business nature redesign. So they were found the impact of reclassification had negligible impact on the stock prices.

Doviak (2015) they took financial planner view to explain the term behavioral finance. They were explaining in their study that application of behavioral strategies not for every individual. They suggest before implementing the strategies, you should carefully analyze the client's capabilities or tendencies it would help to increase success in planning field.

3 Research Methodology

3.1 Type of Research

The focus of this research paper is to understand and explore the linkage and application of artificial intelligence on behavioral finance. To conduct the study descriptive research was done.

3.2 Objectives of Study

1. To find the impact of application of artificial intelligence on Behavioural Finance.
2. To Measure future impact of interaction between artificial intelligence and behavioral finance.
3. To find out impact of artificial intelligence on future jobs.

4 Rationale of Study

1. Firstly, this study will help to explore the area which is still unexplored and not much research is conducted in this area. For example much discussion has been done to understand the uses of artificial intelligence and how it makes the life easy and comfortable for many individuals. Instead, the challenges that will be faced by many individuals and companies in future are still less discussed and researched.
2. Moreover, there is a myth among the individuals that "Artificial Intelligence kills the future jobs". This is in fact not true as we have to upgrade as an individual according to the changing dynamics of the market. Upgrading the individual will make sure that we will be able to survive and retain ourselves in changing technological environment.
3. Thirdly, Artificial intelligence creates lot of jobs opportunities in future but only for those who will be able to serve in highly technological environment.

4.1 Data Collection Tool

To conduct the study both primary and secondary data has been used. Data has been analyzed by the help of ANOVA, Regression Analysis, Correlation.

4.2 Area of the Study

To conduct the study the target respondents were randomly selected form the area of National Capital Region (NCR).

4.3 Research Approach

To conduct the study primary method is used and questionnaire has been used to collect the data.

4.4 Sampling Technique and Sample Size

Convenience sampling method is be used and sample size is 200.

4.5 Research Instrument

Data is collected through structured questionnaire. Secondary data is collected from, web sites, E-book, Journals etc.

4.6 Hypothesis

H1: Awareness level about AI is positively related with the demographics (such as age, gender, qualification, occupation and marital status) of the respondents.
H2: Opinion about need of AI in investing behaviour is positively related to the demographics (such as age, gender, qualification, occupation and marital status) of the respondents.
H3: Usefulness of AI for B2B companies are positively linked to the demographics (such as age, gender, qualification, occupation and marital status) of the respondent.

5 Data Collection and Analysis

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	17.411	5	3.482	3.753	.003[b]
	Residual	180.009	194	.928		
	Total	197.420	199			

a. Dependent Variable: Do you find the need of AI in customer investing behavior ?

b. Predictors:(Constant), QUALIFICATION, OCCUPATION, AGE, MARITAL STATUS, GENDER

Coefficients[a]

Model		Unstandardized Coefficients B	Std. Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	1.575	.537		2.934	.004
	AGE	-.107	.081	-.098	-1.315	.190
	GENDER	.397	.175	.195	2.273	.024
	MARITAL STATUS	.439	.160	.218	2.743	.007
	OCCUPATION	-.080	.064	-.095	-1.241	.216
	QUALIFICATION	.162	.094	.143	1.719	.087

a. Dependent Variable: Do you find the need of AI in customer investing behavior ?

The above table shows that there exists significant difference between respondent's opinion about need of AI in customer investing behaviour and demographics of respondents. Thus, the H2 is accepted.

Correlations

		Do you find the need of AI in customer investing behavior ?	AGE	GENDER	MARITAL STATUS	OCCUPATION	QUALIFICATION
Pearson Correlation	Do you find the need of AI in customer investing behavior ?	1.000	-.137	.039	.217	-.057	.100
	AGE	-.137	1.000	-.160	-.136	.144	.248
	GENDER	.039	-.160	1.000	-.388	.296	-.406
	MARITAL STATUS	.217	-.136	-.388	1.000	-.129	.344
	OCCUPATION	-.057	.144	.296	-.129	1.000	.159
	QUALIFICATION	.100	.248	-.406	.344	.159	1.000
Sig. (1-tailed)	Do you find the need of AI in customer investing behavior ?	.	.027	.291	.001	.211	.079
	AGE	.027	.	.012	.028	.021	.000
	GENDER	.291	.012	.	.000	.000	.000
	MARITAL STATUS	.001	.028	.000	.	.034	.000
	OCCUPATION	.211	.021	.000	.034	.	.012
	QUALIFICATION	.079	.000	.000	.000	.012	.
N	Do you find the need of AI in customer investing behavior ?	200	200	200	200	200	200
	AGE	200	200	200	200	200	200
	GENDER	200	200	200	200	200	200
	MARITAL STATUS	200	200	200	200	200	200
	OCCUPATION	200	200	200	200	200	200
	QUALIFICATION	200	200	200	200	200	200

The above table shows that there exists a negative correlation between opinion of respondents about need of AI in investing behaviour and age of respondents.

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	25.653	5	5.131	2.786	.019[b]
	Residual	357.222	194	1.841		
	Total	382.875	199			

a. Dependent Variable: Is AI useful for B2B companies ?

b. Predictors: (Constant), QUALIFICATION, OCCUPATION, AGE, MARITAL STATUS, GENDER

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	2.945	.756		3.894	.000
	AGE	-.155	.114	-.102	-1.354	.177
	GENDER	.552	.246	.194	2.239	.026
	MARITAL STATUS	.122	.225	.044	.541	.589
	OCCUPATION	-.251	.090	-.216	-2.780	.006
	QUALIFICATION	.239	.133	.152	1.801	.073

a. Dependent Variable: Is AI useful for B2B companies ?

The above table shows that there exists significant difference between respondent's opinion about usefulness of AI for B2B companies and demographics of respondents. Thus, we accept H1 hypothesis and reject H0.

Correlations

		Is AI useful for B2B companies ?	AGE	GENDER	MARITAL STATUS	OCCUPATION	QUALIFICATION
Pearson Correlation	Is AI useful for B2B companies ?	1.000	-.132	.068	.062	-.154	.029
	AGE	-.132	1.000	-.160	-.136	.144	.248
	GENDER	.068	-.160	1.000	-.388	.296	-.406
	MARITAL STATUS	.062	-.136	-.388	1.000	-.129	.344
	OCCUPATION	-.154	.144	.296	-.129	1.000	.159
	QUALIFICATION	.029	.248	-.406	.344	.159	1.000
Sig. (1-tailed)	Is AI useful for B2B companies ?	.	.031	.170	.191	.014	.343
	AGE	.031	.	.012	.028	.021	.000
	GENDER	.170	.012	.	.000	.000	.000
	MARITAL STATUS	.191	.028	.000	.	.034	.000
	OCCUPATION	.014	.021	.000	.034	.	.012
	QUALIFICATION	.343	.000	.000	.000	.012	.
N	Is AI useful for B2B companies ?	200	200	200	200	200	200
	AGE	200	200	200	200	200	200
	GENDER	200	200	200	200	200	200
	MARITAL STATUS	200	200	200	200	200	200
	OCCUPATION	200	200	200	200	200	200
	QUALIFICATION	200	200	200	200	200	200

The above table shows that there exists a negative correlation between respondent's opinion about usefulness of AI for B2B companies and age of respondents.

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	5.708	5	1.142	6.860	.000[b]
	Residual	32.287	194	.166		
	Total	37.995	199			

a. Dependent Variable: Do you hear this terminology "Artificial Intelligence" ?

b. Predictors: (Constant), QUALIFICATION, OCCUPATION, AGE, MARITAL STATUS, GENDER

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	1.495	.227		6.575	.000
	AGE	-.058	.034	-.121	-1.691	.093
	GENDER	-.114	.074	-.128	-1.542	.125
	MARITAL STATUS	.036	.068	.041	.532	.595
	OCCUPATION	.031	.027	.085	1.151	.251
	QUALIFICATION	.149	.040	.301	3.737	.000

a. Dependent Variable: Do you hear this terminology "Artificial Intelligence" ?

The above table shows that there exists significant difference between respondent's awareness about AI and demographics of respondents. Thus, we accept H1 hypothesis and reject H0.

Correlations

		Do you hear this terminology "Artificial Intelligence" ?	AGE	GENDER	MARITAL STATUS	OCCUPATION	QUALIFICATION
Pearson Correlation	Do you hear this terminology "Artificial Intelligence" ?	1.000	-.020	-.221	.199	.073	.350
	AGE	-.020	1.000	-.160	-.136	.144	.248
	GENDER	-.221	-.160	1.000	-.388	.296	-.406
	MARITAL STATUS	.199	-.136	-.388	1.000	-.129	.344
	OCCUPATION	.073	.144	.296	-.129	1.000	.159
	QUALIFICATION	.350	.248	-.406	.344	.159	1.000
Sig. (1-tailed)	Do you hear this terminology "Artificial Intelligence" ?	.	.392	.001	.002	.154	.000
	AGE	.392	.	.012	.028	.021	.000
	GENDER	.001	.012	.	.000	.000	.000
	MARITAL STATUS	.002	.028	.000	.	.034	.000
	OCCUPATION	.154	.021	.000	.034	.	.012
	QUALIFICATION	.000	.000	.000	.000	.012	.
N	Do you hear this terminology "Artificial Intelligence" ?	200	200	200	200	200	200
	AGE	200	200	200	200	200	200
	GENDER	200	200	200	200	200	200
	MARITAL STATUS	200	200	200	200	200	200
	OCCUPATION	200	200	200	200	200	200
	QUALIFICATION	200	200	200	200	200	200

The above table shows that there exists a negative correlation between respondent's awareness about AI and age of respondents.

ANOVA[a]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	39.011	5	7.802	6.765	.000[b]
	Residual	223.744	194	1.153		
	Total	262.755	199			

a. Dependent Variable: Human behavior is affected by any technological changes or innovations ?

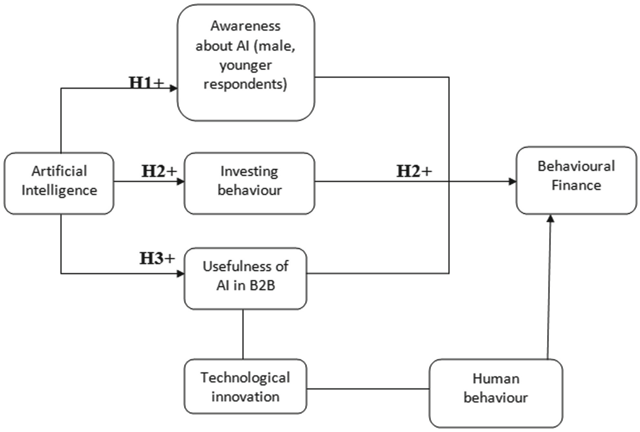
b. Predictors: (Constant), QUALIFICATION, OCCUPATION, AGE, MARITAL STATUS, GENDER

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients		
		B	Std. Error	Beta	t	Sig.
1	(Constant)	.526	.598		.879	.381
	AGE	.031	.091	.025	.342	.733
	GENDER	.226	.195	.096	1.158	.248
	MARITAL STATUS	.751	.178	.323	4.209	.000
	OCCUPATION	.081	.072	.084	1.128	.261
	QUALIFICATION	.210	.105	.162	2.004	.046

a. Dependent Variable: Human behavior is affected by any technological changes or innovations ?

The above table shows that there exists significant difference between respondent's opinion about technological innovations and its impact on human behaviour and demographics of respondents.



Proposed Model for Application of AI in Behavioural Finance

6 Conclusions and Recommendations

- Results indicate a positive relation between respondent's opinion about need of AI in customer investing behaviour and demographics of respondents.
- Results shows that the males have more positive opinion about need of AI in customer investing behaviour.
- There exists significant difference between respondent's opinion about technological innovations and its impact on human behaviour and demographics of respondents.
- There exists significant difference between respondent's awareness about AI and demographics of respondents.
- The result shows that young respondents are more aware about Artificial Intelligence than elder respondents.
- There exists a negative correlation between respondent's opinion about usefulness of AI for B2B companies and age of respondents.
- There exists a negative correlation between respondent's awareness about AI and age of respondents.
- There exists significant difference between respondent's opinion about usefulness of AI for B2B companies and demographics of respondents.
- There is need to spread more awareness about uses of AI in investing behaviour assistance.

Limitations of study & future scope of study

- The sample size could be increased to give more realistic view.
- Researcher has used ANOVA as a tool of analysis which has its own limitations.
- Data collection methods like interview method can be used to make response more accurate.

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A Comparative Study on PID and Interval Type 2 Fuzzy Logic Controllers in a Polypropylene Chemical Plant in India via Simulation

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Abstract. Polypropylene (PP) underwent phenomenal growth in production and use throughout the world during the latter half of the 20th century. In this paper, we have developed a comparative revision for simulation models proportional-integral-derivative (PID) and Fuzzy type-2 controllers (FT2) during PP production. During PP production, the temperature is maintained by men or workers. That work will be fulfilled by PID and FT2 controllers. The proposed models consist of two inputs (temperature and flow-rate) parameters and two outputs (cold water valve, Hot water valve), which are controlled by PID and FT2 controllers. A proportional study has been made first time in the field of PP production and statistical data analysis is performed with some sensitivity analyses.

Keywords: Proportional-integral-derivative (PID) controller · Fuzzy inference system · Interval type-2 fuzzy logic · Polypropylene · Matlab simulink

1 Introduction

The conventional proportional-integral-derivative (PID) controllers are still the most widely used control structure in most of the industrial processes. This is mainly because PID controllers have simple control structures, affordable price, and effectiveness for linear systems [26,27], Miccio and Cosenza [35], Heredia-Molinero et al. [36]. Due to their linear structure, the conventional PID controllers are usually not effective if the system to be controlled has a high level of complexity, such as, time delay, high order, modeling nonlinearities, vague systems without precise mathematical models, and structural uncertainties [28]. For these reasons, many researchers have attempted to combine a conventional PID controller with a fuzzy logic controller (FLC) in order to achieve a better system performance over the conventional PID controller (cf. Harrag and Messalti [37]).

Type-1 fuzzy logic systems (T1-FLSs) have successfully applied in many fields [19–23]. The T1-FLS introduced to handle the uncertainties in real systems. But it has established to be limited in handling the uncertainties of the fuzzy sets. So,

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the type-2 fuzzy logic systems (T2-FLSs), in which type-2 fuzzy sets (T2-FSs) act as antecedent sets or consequent sets, was introduced to overcome the limitations of T1-FLS [24]. On the other hand, the type-2 fuzzy sets (T2-FSs) that were introduced by Zadeh in 1975 are able to model such uncertainties because their membership functions are themselves fuzzy; they are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set [33]. The concept of a T2-FS is an extension of the concept of ordinary fuzzy sets (type-1 fuzzy sets; T1-FSs). A T2-FS is characterized by a fuzzy membership function (i.e., the membership grade for each element of this set is a fuzzy set in $[0, 1]$), unlike a T1-FS where the membership grade is a crisp number in $[0, 1]$ [33]. Therefore, a T2-FS provides additional degrees of freedom that make it possible to model and handle the uncertainties directly [30]. The FLSs that are described with at least one T2-FS are called T2- FLSs [25]. It has also been shown that T2-FSs are much more powerful to handle uncertainties and nonlinearities directly [30]. The major problem with the T2-FLS is that the computations are much more problematical compared to the T1-FLSs. Therefore, Mendel [25] has been anticipated a special type of T2-FS called interval T2-FSs (IT2- FSs) in which the output of interval type-2 fuzzy is uncertain with an interval. The interval T2-FLSs (IT2-FLSs) has been successfully implemented in controller design [1–16, 18, 18], Safari et al. [32]. These efforts spotlight on the controllers that have been used as black box controllers in that their input-output mathematical relations are unknown.

Polypropylene (PP) is a thermoplastic polymer used in a wide variety of applications including packaging (cf. Dey and Jana [31]) and labeling, textiles, stationery, plastic parts and reusable containers of different types, laboratory equipment, loudspeakers, automotive components, and polymer banknotes, etc. An addition polymer made from the monomer PP, it is rugged and unusually resistant to many chemical solvents, bases and acids. During the PP production, the petrochemical plant wishes to maintain the temperature around 20 °C and flow rate of PP on 0.4. But in practical, these two inputs parameters changes up or down automatically by the nature of plant. Since these inputs changes abruptly, so output puts parameters will be changed automatically. These events can be done by PID controlled (cf. Li et al. [34]) which is less cost effective and less accurate than that of Fuzzy controller. This idea motivates to formulate such type of models.

Inspire of these above mentioned development of the field of PIF and Fuzzy controller, we have first time developed as:

1. New modelling method which combines the petrochemical and computer part is proposed.
2. PID and Fuzzy type-2 controllers have been developed in a petrochemical plant during polypropylene production.
3. Effective parameters controlling the models of polypropylene have been identified.
4. We optimize the input and output parameters of the simulation models.
5. Statistical data analysis is performed with some sensitivity analyses.

In this paper, first time, we have set up two different types of controllers in a petrochemical plant in India. With these controllers, input and output parameters will be changed automatically. It is soon that the Fuzzy controller will give better result that of PID controller. The statistical data analysis has been shown for validate of the proposed models.

2 Model Descriptions for Petrochemical Plant Setup

2.1 Technical Description

The configuration of Catalyst Dispersion Drum (CDD) system investigated in this study is shown in Fig. 1. The high-yield catalyst (titanium catalyst supported on $MgCl_2$) due to its possibility of furnishing high yields in the product of polypropylene (PP), moderate quantities are fed to the reaction.

The traditional systems to feed catalyst under the form of suspension in a solvent appear to be inefficacious for such above quantities, since the catalyst may transfer easily. Moreover, they would introduce undesired quantities of solvent in the process.

The problem has been solved by suspending the catalyst, via hot process, in Vaseline oil and grease and then by cooling this suspension so that it can be fed to the reaction under form of gelatinous paste in which the micro granular catalyst particle remain trapped.

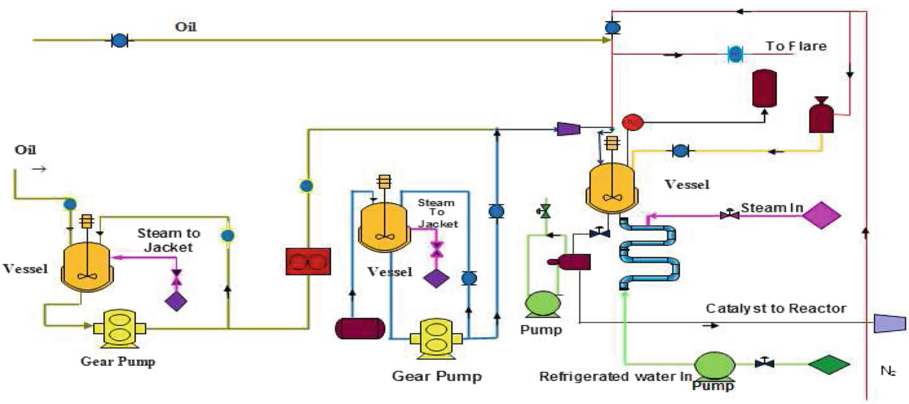

Fig. 1. Systematic diagram of Catalyst Dispersion Drum.

Grease is brought in the plant drums and transferred in the grease drum by a gear pump. The grease must be melted steam heated box before being pumped. The grease, stored in the drum must be stirred by the agitator and maintained liquid at 70–100° ready to be used in another drum where the catalyst will be suspended. The moisture free oil is pumped is pumped from the oil storage drum to oil mixing drum and is stirred by another agitator to maintain its temperature at 70–100°. Both the grease and oil vessels are refilled when level comes down and kept under nitrogen blanketing to avoid ingress of oxygen and moisture of atmosphere. In catalyst dispersion drum catalyst mud is prepared as follows (in Fig. 2):

1. Required amount of oil is fed from oil drum by a gear pump through a filter and a mass flow meter.
2. The catalyst powder, present in a sealed drum, under an inert atmosphere, is discharged in the catalyst dispersion drum. In the catalyst dispersion drum an agitator mixes catalyst and oil. The catalyst is sensitive to air, water contamination, so precautions must be taken.
3. The required amount of melted grease is transferred to catalyst dispersion drum by a gear pump through a filter and a mass flow meter.

The agitator mixes catalyst, oil and grease. During this operation the temperature is maintained at 70° by means of refrigerated water circulation and heated with steam. When catalyst suspension is homogenized it is necessary to cool the mass to 10°. Then, the catalyst mud is hydraulically transferred to catalyst loading syringe and from that to reactor.

2.2 Notations and Abbreviations

The following notations and abbreviations are used to describe the proposed model.

(i) $MFIS$ = Mamdani's fuzzy inference system.
(ii) R^2 = coefficient of determination.
(iii) $RMSE$ = root mean square error.
(iv) MAE = mean absolute error.
(v) $MAPE$ = mean absolute percentage error.
(vi) y_{pred} = predicted value.
(vii) $\overline{y}_{pred}$ = mean of the predicted values.
(viii) y_{obs} = observed value from model.
(ix) MF = membership function.

2.3 Type-2 Fuzzy Sets

A type-2 fuzzy set expresses the non-deterministic truth degree with imprecision and uncertainty for an element that belongs to a set. A type-2 fuzzy set (in Fig. 3) (cf. Castillo and Melin [33]) denoted by $\tilde{\tilde{A}}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u)$ where $x \in X, \forall u \in J_x^u \subseteq [0,1]$ and $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$ defined in Eq. (1)

$$\begin{aligned}\tilde{\tilde{A}} &= \{(x, \mu_{\tilde{A}}(x)) | x \in X\}\\ \tilde{\tilde{A}} &= \{(x, u, \mu_{\tilde{A}}(x,u)) | x \in X, \forall u \in J_x^u \subseteq [0,1]\}\end{aligned} \tag{1}$$

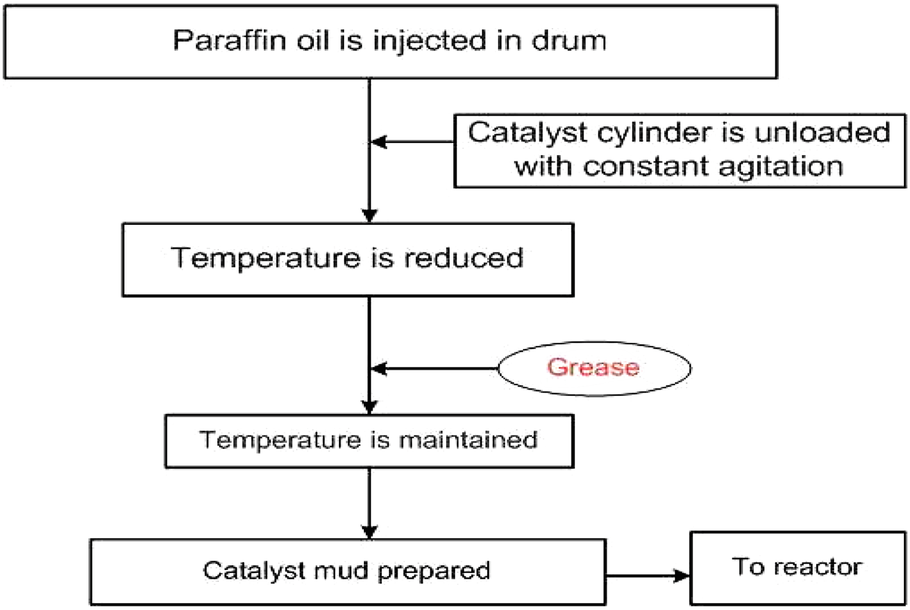

Fig. 2. Steps for the formation of catalyst dispersion drum catalyst

If $\tilde{\tilde{A}}$ is fuzzy type 2 (FT2) (in Fig. 4) continuous variable, it is denoted in Eq. (2)

$$\tilde{\tilde{A}} = \left\{ \int_{x \in X} \left[\int_{u \in J_x^u} f_x(u)/u \right] /x \right\} \tag{2}$$

where $\int\int$ denotes the union of x and u. If A is FT2 discrete, then it is denoted by Eq. (3)

$$\tilde{\tilde{A}} = \left\{ \sum_{x \in X} \mu_{\tilde{\tilde{A}}}(x)/x \right\} = \left\{ \sum_{i=1}^{N} \left[\sum_{k=1}^{M_i} f_{x_i}(u_k)/u_{ik} \right] /x_i \right\} \tag{3}$$

where $\sum\sum$ denotes the union of x and u. If $f_x(u) = 1, \forall u \in [\underline{J}_x^u, \overline{J}_x^u] \subseteq [0, 1]$, the type-2 membership function $\mu_{\tilde{\tilde{A}}}(x, u)$ is expressed by one type-1 inferior membership function, $\underline{J}_x^u = \mu_A(x)$ and one type-1 superior, $\overline{J}_x^u = \mu_A(x)$, then it is called an interval type-2 fuzzy set denoted by Eqs. (4) and (5).

$$\tilde{\tilde{A}} = \left\{ (x, u, 1) | \forall x \in X, \forall u \in [\underline{\mu}_A(x), \overline{\mu}_A(x)] \subseteq [0, 1] \right\} \tag{4}$$

or

$$\tilde{\tilde{A}} = \left\{ \int_{x\in X} \left[\int_{u\in[\underline{J}_x^u, \overline{J}_x^u]\subseteq[0,1]} 1/u \right] /x \right\} = \left\{ \int_{x\in X} \left[\int_{u\in[\underline{\mu}_A(x), \overline{\mu}_A(x)]\subseteq[0,1]} 1/u \right] /x \right\} \tag{5}$$

If $\tilde{\tilde{A}}$ is a type-2 fuzzy Singleton, the membership function is denoted and defined by equation

$$\mu_{\tilde{\tilde{A}}}(x) = \begin{cases} 1/1, \text{ si } x = x' \\ 1/0, \text{ si } x \neq x' \end{cases} \tag{6}$$

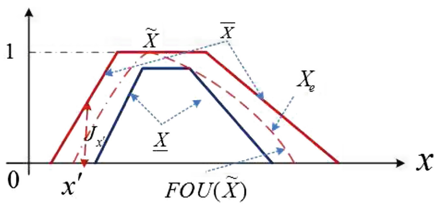

Fig. 3. Interval type 2 set

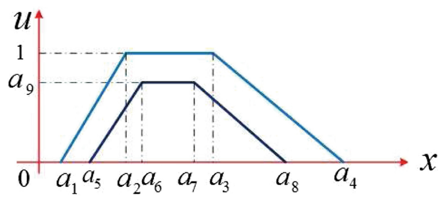

Fig. 4. The trapezoidal IT2 FS

Definition 1. *A type-1 fuzzy set X is comprised of a domain D_X of real numbers (also called the universe of discourse of X) together with a membership function (MF) $\mu_x : D_X \to [0, 1]$, i.e.*

$$X = \int_{D_x} \mu_x(x)/x \tag{7}$$

Here $\int$ denotes the collection of all points $x \in D_X$ with associated membership grade $\mu_x(x)$.

Definition 2 *(Mendel [25]). An IT2 FS $\tilde{X}$ is characterized by its MF $\mu_x(x, u)$, i.e*

$$\tilde{X} = \int_{x\in D_x} \int_{u\in J_x\subseteq[0,1]} \mu_x(x,u)/(x,u) = \int_{x\in D_x} \int_{u\in J_x\subseteq[0,1]} 1/(x,u) = \int_{x\in D_x} \left[\int_{u\in J_x\subseteq[0,1]} 1/u \right] /x \tag{8}$$

where x, called the primary variable, has domain $D_{\tilde{X}}$: $u \in [0,1]$, called the secondary variable, has domain $J_x \subseteq [0,1]$ at each $x \in D_{\tilde{X}}$; J_x is also called the support of the secondary MF, and is defined below in (25); and, the amplitude of $\mu_{\tilde{X}}(x,u)$, called a secondary grade of $\tilde{X}$, equals 1 for $\forall x \in D_{\tilde{X}}$ and $\forall u \in J_x \subseteq [0,1]$.

For general type-2 FSs $\mu_X(x,u)$ can be any number in $[0,1]$, and it varies as x and/or u vary.

Definition 3. *The uncertainty about $\tilde{X}$ is suggested by the union of all its primary memberships, which is said the footprint of uncertainty (FOU) of $\tilde{X}$, i.e.,*

$$FOU(\tilde{X}) = \bigcup_{\forall x \in D_{\tilde{X}}} J_x = \left\{(x,u) : u \in J_x \subseteq [0,1]\right\} \tag{9}$$

The size of an FOU is directly related to the uncertainty that is conveyed by an IT2 FS. So, an FOU with more area is more uncertain than one with less area.

Definition 4. *The upper membership function (UMF) and lower membership function (LMF) of $\tilde{X}$ are two T1 MFs X and X that bound the FOU.*

$$J_x = [\mu_{\underline{X}}(x), \mu_{\overline{X}}(x)] \tag{10}$$

Using (10), FOU ($\tilde{X}$) can also be expressed as

$$FOU(\tilde{X}) = \bigcup_{x \in D_x} [\mu_{\underline{X}}(x), \mu_{\overline{X}}(x)] \tag{11}$$

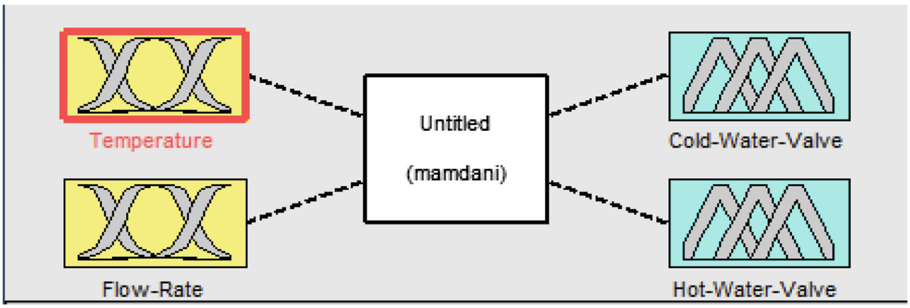

Fig. 5. Mamdani FIS for Catalyst Dispersion Drum.

2.4 Fuzzy Controller Rule Base

Since each variable of the fuzzy controller (two inputs temperature (in Fig. 5) and flow rate of PP and two outputs cold and hot water valves) has 3 and 5 membership functions respectively, 10 rules are required to generate a fuzzy output. The rule base of the fuzzy logic controller is given in figure. Fuzzy rules play most important role in the performance of fuzzy logic controllers and therefore, in this paper the rules are investigated extensively by studying the dynamic behavior of the system in the petrochemical plant. The firing strength of the fuzzy control rules are obtained by using Mamdani interface system (Figs. 6, 7, 8, 9 and 10).

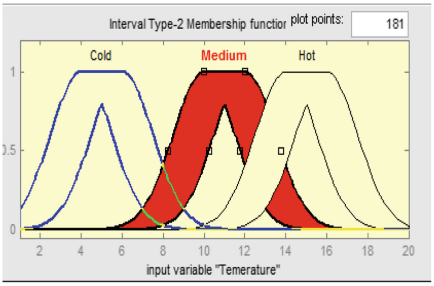

Fig. 6. Membership function of the input parameter ‘Temperature’.

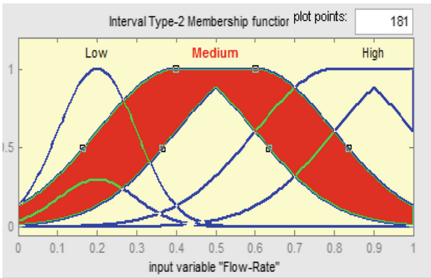

Fig. 7. Membership function of the input parameter ‘Flow rate’.

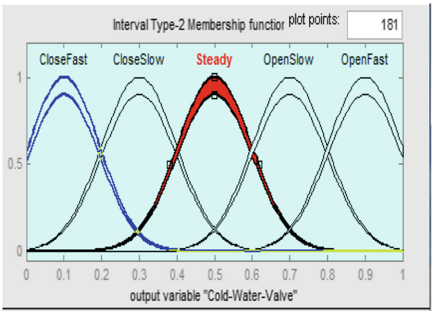

Fig. 8. Membership function of the output parameter ‘Cold water valve’.

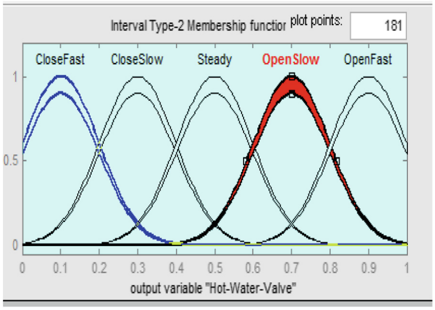

Fig. 9. Membership function of the output parameter ‘Hot water valve’.

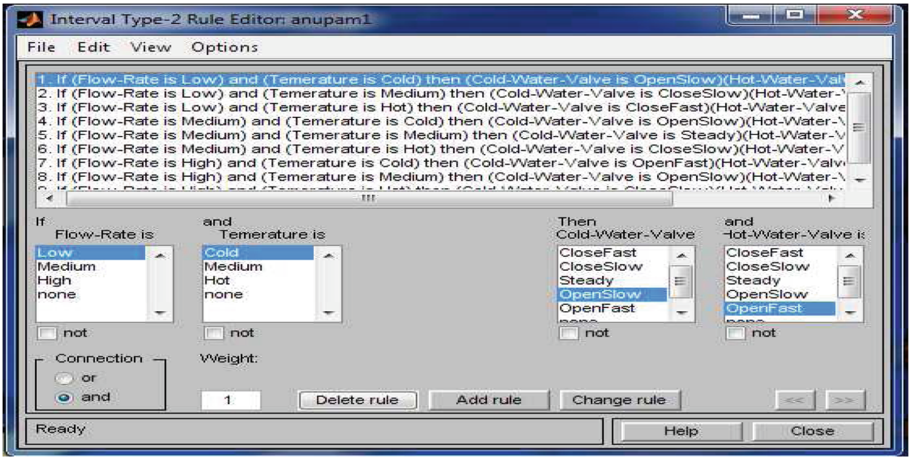

Fig. 10. Fuzzy logic rules for the FIS.

2.5 Defuzzification

The output of the FIS is a fuzzy value and therefore it must be converted to a real value. The process of conversion of a fuzzy value to a real deterministic value with which the physical system can deal is known as defuzzification. The very popular centre of gravity method of defuzzification is used to determine the required real value control output for the PP production system.

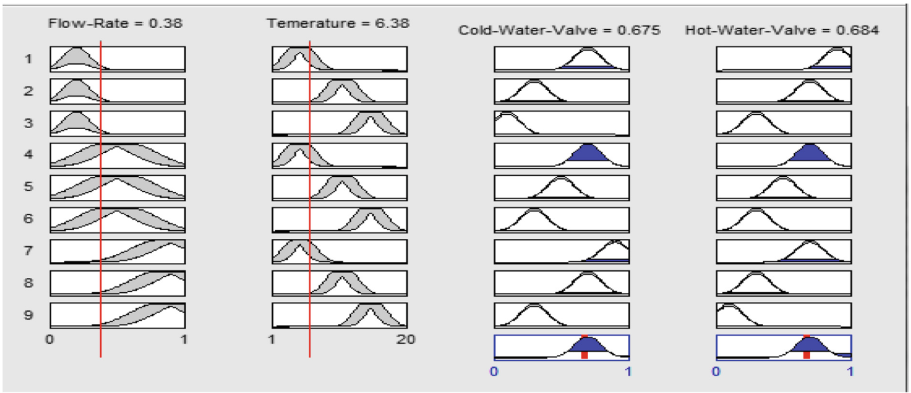

Fig. 11. Determination of outputs from the FIS.

2.6 Simulink Design Optimization CDD for PID and FT2

Here we optimize the input and output parameters of the Simulink models in Figs. 11 and 12 to meet design requirements, Simulink Design Optimization software automatically converts the requirements into a constrained optimization problem and then solves the problem using optimization techniques. The constrained optimization problem iteratively simulates the Simulink models, compares the results of the simulations with the constraint objectives, and uses optimization methods to adjust tuned parameters to better meet the objectives. We have collected 300 data set (in Table 2) for two different inputs temperature and flow-rate of PP from a renowned petrochemical in India. Then with the help of clustering, we have separated the low, medium, and height value for each input data. Then we have formulated Mamdami's T2FIS for Catalyst Dispersions Drum and also we have formulated 10 interval Fuzzy logic rules in Matlab 14.0. These FIS system generates the output controller by which decision maker can control their desire label of input and outputs. Again we have setup two different types of catalyst dispersions drum for PID and interval T2FIS controller, which are depicted in Figs. 14 and 15 respectively. These two controllers generate the desired output. The output results for temperature in both controllers are depicted in Figs. 16 and 17 respectively. From these representation, it is cleared that FIS controller give a continuous flow than that of PID controller (Fig. 13).

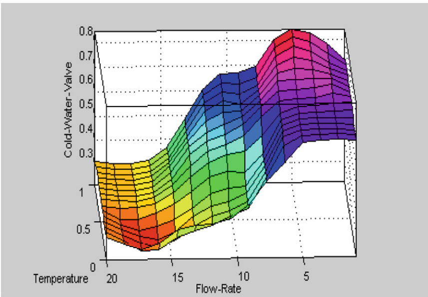

Fig. 12. Membership function of the input parameter 'Temperature'.

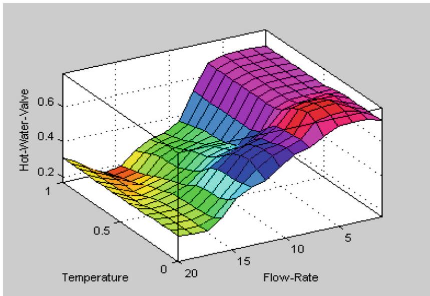

Fig. 13. Membership function of the input parameter 'Flow rate'.

3 Results and Discussion

The performance of PID controller was then compared in simulation with the one of type-2 FIS. All the simulation results confirmed the robustness and the effective control action of each fuzzy controller, with evident advantages for the type-2 FIS.

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{pred_i} - y_{obs_i})^2} \tag{12}$$

In addition, the determination coefficient (R^2) can be calculated using Eq. (13) that is given by:

$$R^2 = 1 - \frac{\sum_{i=1}^{n} (y_{pred_i} - y_{obs_i})^2}{\sum_{i=1}^{n} y_{obs_i}^2} \tag{13}$$

Mean absolute percentage error (MAPE) measures the average of the squares of the errors. The smaller values of MAPE ensure the better performance of the proposed models. The MAPE is calculated by the following Eq. (14):

$$MAPE = \frac{1}{n} \sum_{i=1}^{n} \frac{|(y_{pred_i} - y_{obs_i})|}{y_{pred_i}} \times 100\% \tag{14}$$

However, the performance and efficiency of the proposed models is also analyzed using mean absolute error (MAE), which is defined by the Eq. (15):

$$MAE = \frac{1}{n} \sum_{i=1}^{n} |y_{pred_i} - y_{obs_i}| \tag{15}$$

where n is the number of data patterns in the data set, y_{pred_i} indicates the predicted value of one data point i and y_{obs_i} is the observed value of one data point i. The important aspect of the present research work is that the decision maker can control both input and output data by using Fuzzy logic controller.

It is very much evident from Table 1 that the RMSE, MAE, MAPE, and R^2 are minimum for T2FIS for temperature and flow rate than that of PID controller. On the basis of statistical data analysis of temperature and flow-rate, we can elect the T2FLC controller as the best fitted controller.

Table 1. Statistical data analysis of PID and type-2 fuzzy controllers

Outputs	Approach	Statistics			
		RMSE	MAE	MAPE	R^2
Temparature	PID	2.573180853	0.937440479	22.08818642	2.208818642
	TYPE-II	1.006350759	0.9906198	8.596941373	0.859694137
Flow rate	PID	0.294516394	0.766004922	6.209448169	0.248377927
	TYPE-II	0.093023879	0.961531856	2.098499493	0.08393998

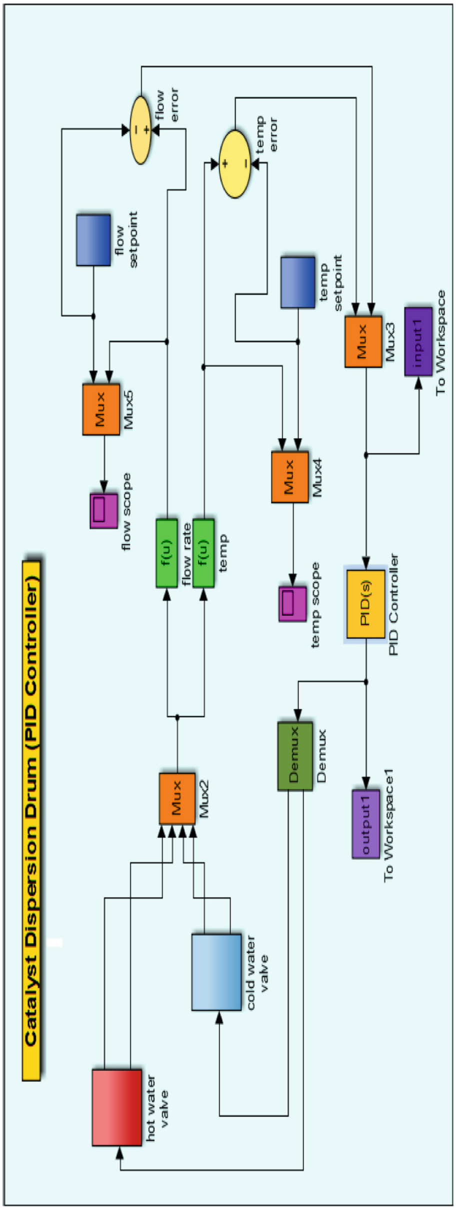

Fig. 14. Simulation model CDD using PID controller.

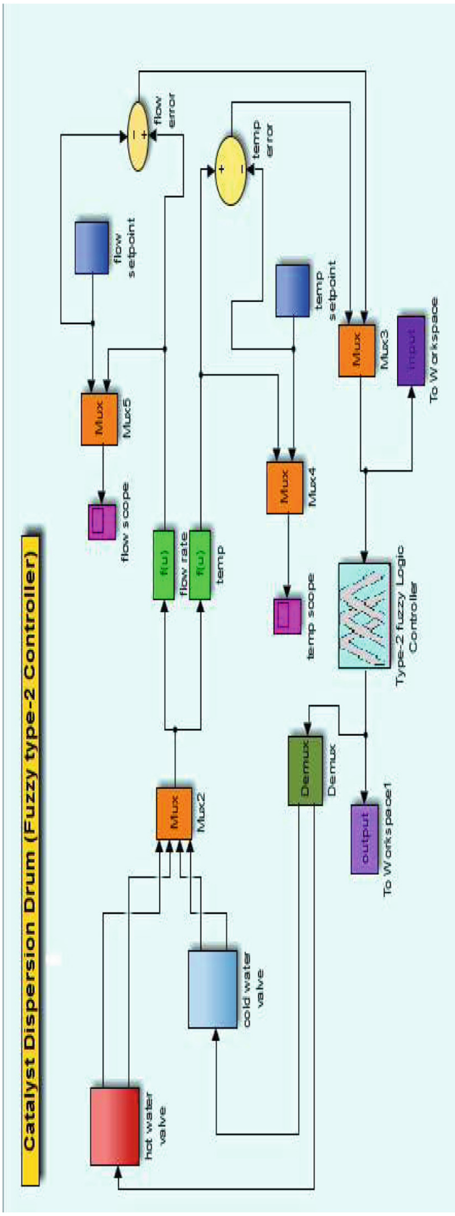

Fig. 15. Simulation model CDD using FIS controller.

The application of interval type-2 and PID FLCs to a dynamical system during the polypropylene production has been analyzed by simulations. All the simulation results demonstrate the effectiveness of type-2 FLC in achieving a very high control performance of position of different valves and allowing a faster and more precise control of the process, both for set point tracking and disturbance rejection, with less amount of overshoot as compared to the PID controller.

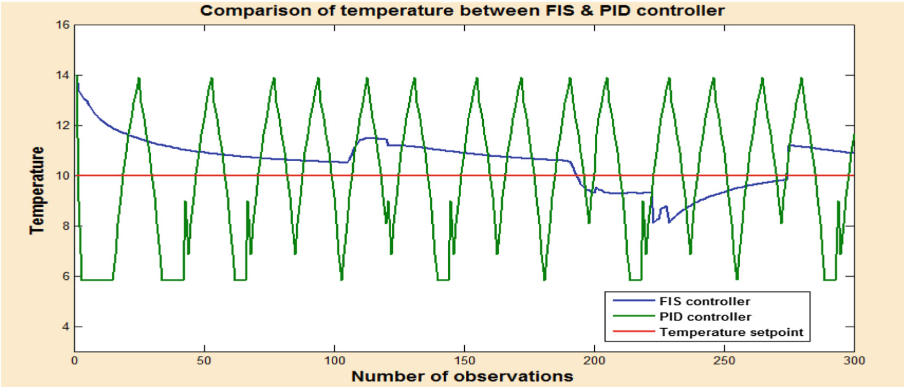

Fig. 16. Comparison of temperature between FIS and PID controller.

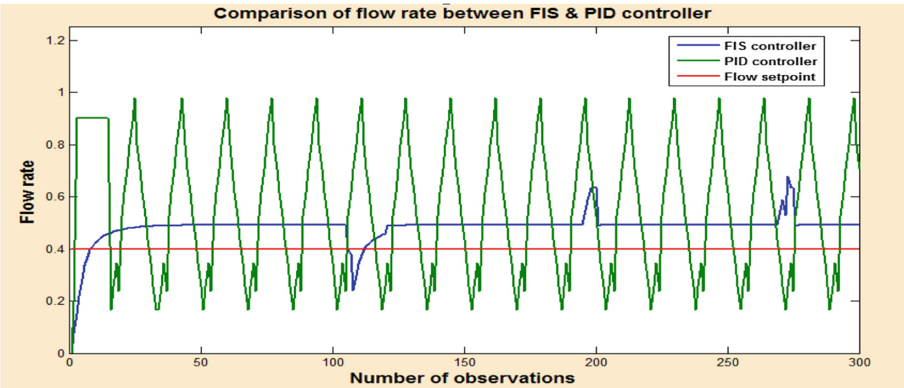

Fig. 17. Comparison of flow rate between FIS and PID controller.

Table 2. Collected data form chemical plant for Temperature & Flow-rate

Temperature	Flow-rate	Temperature	Flow-rate	Temperature	Flow-rate
13.3687002	0.077519203	11.25021211	0.487172564	10.75713012	0.491654598
13.16498712	0.149125322	11.22394188	0.487581524	10.7477266	0.491682032
13.06046399	0.218729154	11.19885161	0.487953114	10.7385643	0.49170721
12.96262372	0.283761999	11.17485587	0.488291046	10.72963542	0.491730318
12.79499626	0.335579596	11.15184428	0.48859845	10.7209325	0.491751525
12.61162069	0.371861908	11.12972867	0.488878306	10.7124499	0.491770988
12.46209077	0.395275504	11.10848102	0.489134627	10.70418214	0.49178885
12.34279975	0.41144999	11.08805256	0.489369757	10.69612261	0.491805244
12.24350404	0.423513074	11.06839232	0.489585459	10.68826499	0.491820289
12.15806035	0.43300491	11.04945412	0.489783348	10.6806032	0.491834096
12.08237402	0.440767679	11.0311961	0.489964903	10.67313141	0.491846768
12.01343892	0.447324424	11.01358272	0.490131477	10.66584402	0.491858397
11.9501716	0.452924749	10.99657998	0.490284313	10.65873566	0.49186907
11.89171681	0.457744258	10.98015571	0.490424547	10.65180116	0.491878865
11.83732027	0.461904398	10.96428029	0.490553221	10.64503553	0.491887853
11.78638299	0.465514371	10.94892643	0.490671293	10.63843398	0.491896102
11.73844878	0.468659665	10.93406891	0.490779637	10.6319919	0.491903673
11.6931071	0.471410884	10.91968435	0.490879057	10.62570483	0.49191062
11.65002134	0.473815566	10.90575105	0.490970289	10.61956848	0.491916996
11.60895637	0.4759178	10.89224879	0.49105401	10.61357871	0.491922847
11.56971693	0.477755851	10.87915873	0.491130839	10.60773153	0.491928216
11.53213754	0.479362359	10.86645492	0.491201343	10.60202307	0.491933143
11.49607185	0.480765646	10.85411179	0.491266045	10.59644961	0.491937665
11.46137943	0.481990864	10.84211462	0.491325424	10.59100754	0.491941815
11.42798308	0.483064124	10.8304495	0.491379922	10.58569337	0.491945623
11.39579716	0.484004592	10.81910404	0.49142994	10.58050337	0.491949118
11.36471924	0.484826796	10.80806654	0.491475846	10.5754308	0.491952325
11.33467779	0.485543928	10.79732592	0.491517979	10.57047174	0.491955268
11.30561321	0.486167028	10.78687168	0.491556647	10.56562337	0.491957969
11.30561321	0.486167028	10.77669387	0.491592135	10.56088298	0.491960447
10.55624792	0.491962721	10.53871074	0.491970094	10.52266627	0.491975323
10.55171562	0.491964809	10.53456527	0.491971574	10.51887207	0.491976372
10.54728359	0.491966724	10.5305108	0.491972933	10.51516058	0.491977335
10.54294942	0.491968481	10.52654517	0.491974179	10.51152987	0.491978219

4 Conclusions and Future Research Work

In this investigation, we have developed PID and Fuzzy type-2 controllers in a petrochemical plant during polypropylene (PP) production. The controller consists of two inputs (Temperature and Flow-rate of PP) and two outputs (cold water valve, Hot water valve). During the production time these inputs and outputs are controlled by PID and Fuzzy type-2. In general the production engineers have to opt for a trial and error process to set the input parameters for attaining the PID controller for PP. But if the proposed fuzzy mathematical model is applied in petrochemical plant, it will surely benefit the engineers to set the input parameters promptly to achieve the desired quality of PP. The performance of PID controller was then compared in simulation with the one of type-2 FIS. All the simulation results confirmed the robustness and the effective control action of each fuzzy controller, with evident advantages for the type-2 FIS. By statistical data analysis, we have developed a comparative study on three types of controller. We have successfully shown that type-2 fuzzy give better results than that of PID, Type-2 Fuzzy controllers respectively. For further research work, we will utilize and integrate other intelligent methods, such as interpretative structural modeling method, fuzzy artificial neural network, etc., to evaluate the scale of efficiency of our current study. Furthermore, the method used for the India petrochemical plant is quite general. Hence the product quality of other petrochemical plants and chemical industries can be evaluated and benefited by this proposed method also. The present analysis can be used for other products where the relationships between the inputs and outputs for the concern product are not previously known. The values obtained from both adaptive Neuro-fuzzy inference systems (ANFIS) and MLR models were close to the experimental results.

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A Hybrid Framework Based on PSO and Neutrosophic Set for Document Level Sentiment Analysis

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Abstract. Opinion mining for document level has undergone different testing process in the last few years. A lot of research work has contributed for analysis of such reviews to extract the sentiment associated with the texts. Fuzzy technologies and machine learning are the most explored area for sentiment analysis. In this paper, Particle Swarm Optimization (PSO), an Evolutionary algorithm has been hybridized with Neutrosophic Set concept to generate a ternary classifier. PSO has a simple and robust method for finding out global optima on huge sample points. The sentiment of a large text is classified as a ternary value: Positive, Negative or Neutral. This method is suitable to classify large sized text. This hybridization has not been dealt in the literature. Large sized positive, negative and neutral text have been generated from existing review comments of products and plots of movies. This method will be highly useful for classifying documents like review comments on research papers from different researchers, political analysis on a subject or police verification on any news for sentiment polarity finding and many more.

Keywords: Evolutionary algorithm · Neutrosophic set · Opinion mining · Particle swarm optimization · Sentiment analysis

1 Introduction

In today's scenario, seller and buyer relationship is dependent on the reviews of the product available in different social networking sites. Users are now smart enough to compare the products and buy the best one comparing the opinions available for that particular product.

The field of opinion mining is now matured enough and researchers have applied different techniques for polarity detection (classification) of a given text (Tripathy et al. 2017; Pu et al. 2017; Liu et al. 2017; Bouazizi and Ohtsuki 2017). There are broadly two types of classification of text: subjective and objective. Subjectivity classification of a given text gives an overview of how much subjective or objective the text is. An

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objective sentence declares some fact while a subjective sentence expresses personal views, feelings and beliefs. A subjective sentence cannot determine any opinion polarity while an objective sentence gives some value to determine positive or negative sentiment score.

Sentiment analysis can be divided into three levels e.g. Document level, Sentence level and Aspect level. At document level and sentence level the text provides polarities on a basis of whole document or sentence respectively. At aspect level, the text provides positive polarity for a definite aspect where as it provides negative polarity for other aspects.

In this paper, the proposed method concerns about document level analysis. In document level sentiment analysis (Tripathy et al. 2017), has shown comparison of machine learning algorithms on single instance of review to detect the opinion of the text. Bouazizi and Ohtsuki (2017), developed a multiclass and ternary classifier for Twitter dataset.

Neutrosophic classifier has the power to capture the indeterminacy beyond the crisp set of truth or false. Further, in order to identify neutral classes Neutrosophic set has been used. Application of nature inspired algorithms in sentiment analysis is in their infancy still now as well as Fuzzy logic or the extension of fuzzy logic such as Neutrosophic logic, needs more exposure on this area.

The work is organized as follows: Sect. 2 discusses the motivation of the proposed methodology. Section 3 discusses the PSO and Neutrosophic Set in sentiment analysis. Section 4 describes the proposed methodology followed by conclusion and future work in Sect. 5.

2 Motivation

In crisp logic, there are two extreme values as True and False within which the data is usually understood. In real world applications the parameters defining the domain are not bounded within crisp values only.

The conditions like usually, few, likely, not likely, mostly, unlikely, probably, nearly cannot be completely understood by truth or falsehood. To capture such situations Fuzzy Logic (Zadeh 1965) was introduced.

Fuzzy logic applied to a variable allowed values between 0.0 to 1.0. For intermediate value of a variable x say 0.2 implies that x is true for 20% of the time. A wide application of fuzzy logic in automation, artificial intelligence, expert systems, vision and speech processing, sentiment analysis has been found (Kia et al. 2009; Cheung et al. 2005; Samui et al. 2017; Ali et al. 2017).

Neutrosophiclogic (Smarandache 2016) is an extension of fuzzy logic which gives a value of indeterminacy besides truth and falsehood. The inherent neutral or indeterminate value due to system imbalance can be measured and decision needs to be taken by human expert or rule based systems in such situation. Similar to fuzzy system this logic holds operations on the tuple values of the variables like union, intersection, inference etc.

In Neutrosophic logic sum of the components need not necessarily to be 1 but any value between −0 to 3+. It can define paradoxes where variables can be truth or false at the same time. It is not possible in classic fuzzy system.

3 PSO and Neutrosophic Set

Sentiment analysis is a part of Natural Language Processing, where for a given piece of text the process identifies the underlying sentiment of the writer. In the course of understanding the review of products or movies sentiment analysis took an important part in the research. Many algorithms have been tried to generate correct analysis on the short text dataset or on the basis of sentence classification. In the previous research the algorithms mostly have been applied on the text, based on feature selection. Feature selection requires pre-processing of the given text to make it fit for the classification. Some sentiment analysis paper has been found based on the score of word's polarity defined in SentiWordNet. The earlier research generally classifies a text or a sentence as a positive or negative sentiment based piece of literature. Now, three way classifications: Positive, Negative and Neutral sentiment, replaces binary sentiment classification. This Ternary classification has a great correlation with Neutrosophic Set. In Neutrosophic Set, a proposition is defined with a percentage of Truth, Indeterminacy and Falsity. Each word in the literature is also associated with a score of positive, negative or neutral sentiment. Now if a mapping can be generated for each word's positive, neutral or negative value with a Neutrosophic Set value then it will be powerful to give a resultant score of sentiment as a ternary classifier. This shift of Binary classification to Ternary set is the strength of NeutrosophicSet in sentiment classification. Moreover, the load of pre-processing of the words can be eliminated as a direct score of word sentiment either positive, indeterminacy or negative is generated. The mapping of word's sentiment value to Neutrosophic Set value enhances the power of sentiment analysis. Moreover, Particle Swarm Optimization as evolutionary algorithm has chosen as it has been seen from the literature that it is simpler in computation as well as efficient in terms of time and accuracy than other evolutionary group members. The relationship between Neutrosophic Set, classical logic and fuzzy logic is shown in Fig. 1 (Ashbacher 2002).

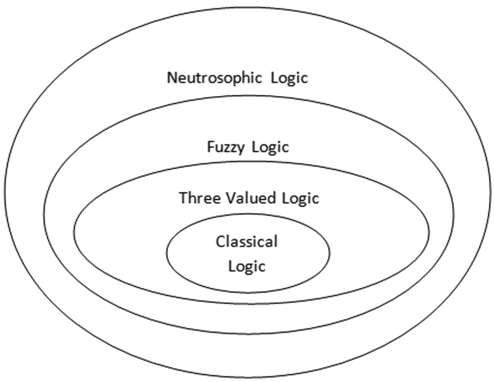

Fig. 1. Neutrosophic logic diagram

4 Proposed Framework

Neutrosophic logic gives liberty to express a word to be beyond a duel classes i.e. positive or negative. It enhances the freedom of polarity detection by adding a neutral value to it.

For document sentiment classification, fuzzy polarity detection has been used and tested on sentence or document level (Dragoni et al. 2015). The extension to the fuzzy qualifier is Neutrosophic qualifier. In the proposed methodology, SentiWordnet 3.0 for extracting the word's Neutrosophic value i.e. positive, negative and neutral is used. These three dimensional values are then used for document classification.

The Neutrosophic values of document are used to create a histogram. The probability score of the histogram is then used for calculations of fitness function as each Neutrosophic value is now considered as birds in the swarm. The swarm of points will lead to a best fitted point by using PSO. The value received after optimization is then fitted to the rule based Neutrosophic classifier to classify the document.

Figure 2 describes the flowchart of the proposed methodology which is used for classifying positive, negative and neutral document. In the first stage, the text is tokenized by Standford parser. For each verb, noun, adjective and adverb a positive and negative score from SentiwordNet 3.0 is computed. This is mapped with Neutrosophic value Truth, false and Indeterminacy with Positive, Negative and Indeterminacy (1-positive-negative) score.

Particle Swarm Optimization is applied on this Neutrosophic set value. The number of valid words for which a score can be generated from SentiWordNet is the total number of swarm in the search space. The position values of the swarm are initialized as the Neutrosophic score of the word in a 3D space. For '*x*',' *y*' and '*z*' direction the positions are set with Positive, Indeterminacy and Negative values. Optimization applied on the 3 directions using PSO. In the 3D space where Positive is *'x'* axis, Indeterminacy is *'y'* axis and Negative is *'z'* axis, histogram have been calculated for all the three direction. Using this histogram each swarm fitness value is calculated and velocity is updated in the three directions. Total fitness is generated using a combination of Positive, Indeterminacy and Negative fitness. As the maximum iteration reached the optimized values have got for each the swarm component. This optimized value having a score of Positive, Indeterminacy and Negative value will decide the polarity of the document.

Documents for testing from Blitzer and Subjective dataset are generated. In Blitzer dataset there are review comments of the products for 25 categories. Positive and Negative comments have different folders. 100 such sentences for each positive and negative documents are taken. For Neutral document the Subjective dataset is chosen. The Subjective dataset has a file describing the plot of the movies, which can be considered as a neutral document. These positive, negative and neutral documents are of large size. Thus the number of swarms that is generated for each document is suitable for having a converging value in PSO. Particle Swarm Optimization is a simple and efficient algorithm for such large number of data-points, which will converge to an optimized value.

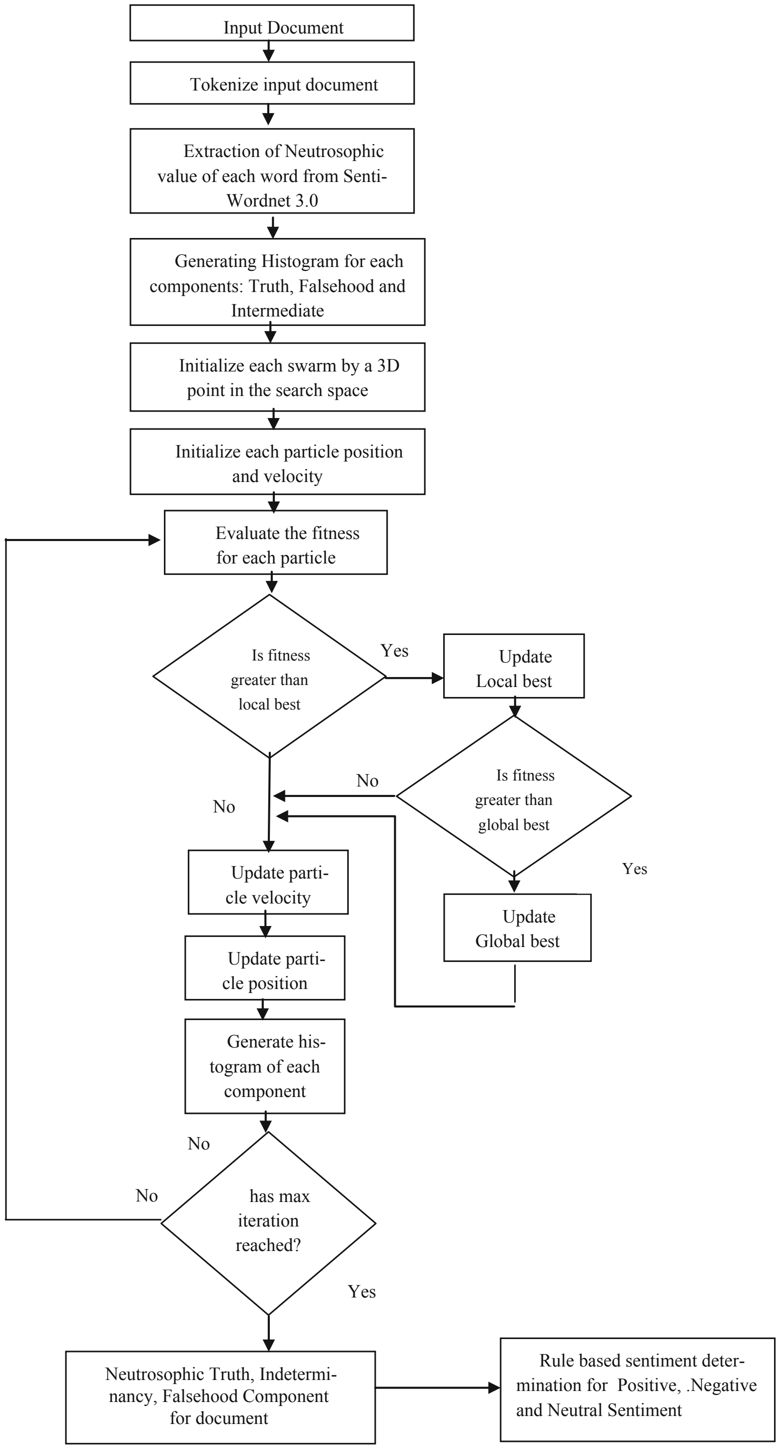

Fig. 2. Flowchart of the proposed methodology

5 Conclusion and Future Work

In the present work, a proposal based on particle swarm optimization and Neutrosophic set is presented. The framework is designed to understand the sentiment analysis of large sized text. The method is novel as it incorporates the advantages of Neutrosophic set with swarm intelligence.

For large size documents of Blitzer review dataset swarm size is large enough and giving a suitable result. But as the document size is large, number of taken document is less. So there is a need for large size documents. The proposed method can be validated with other real time datasets to investigate the accuracy rate and rate of misclassification. Future research can be applied using various meta-heuristic algorithms besides Particle swarm optimization for comparison with the result. Moreover opinion generation on other review comments such as research paper review can be explored. Accuracy improvement and more opinion classes will be the other considerations.

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Numerical Study on Electrokinetic Flow Through Periodically Modulated Soft Nanochannel

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Abstract. A numerical study has been made on the electroosmotic flow (EOF) through a polyelectrolyte layer coated periodically surface modulated soft nanochannel. We have considered the wall of nanochannel have negative charged when charge density of polyelectrolyte layer (PEL) is positive. A transformation is considered to map the physical region into the rectangular computational region. The governing equations for the electrokinetic flow are Laplace and Poisson equation for the distribution of external and induces potential; the Nernst-Planck equations for ion species and the modified Navier-Stokes equations for flow field. These equations are solved numerically over the staggered system in control volume approach. We have investigated the influence of surface modulation, heterogeneity, electrolyte concentration, PEL thickness and charged density in our study. The combined effects for geometrical modulation and charge heterogeneity in soft channel wall are found prominent. The increasing of the ionic concentration decreases the net charge and therefore, increase the flow rate. The EOF velocity increases with the ionic concentration for zero-scaled PEL charged density. The increase of PEL charged density decreases of the EOF velocity for fixed ion concentration. There is no circulation for low ionic concentration and it appears only for high ionic concentration. The effects of the induced potential are prominent for opposite PEL charged density. The increase of the softness parameter decreases the EOF velocity.

Keywords: Electroosmotic flow · Polyelectrolyte layer · Nernst-Plank equation · Softness parameter · Ion selectivity

1 Introduction

Due to growing interest in nanofabrication techniques for active control of ions transport and fluid flow, nanochannel have attracted significant attention over the past decade. Electroosmosis is the movement of solvent together with solute

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under a applied external electric field. The electroosmotic flow is determined by many factors, such as the strength of electric field, the electrolytes concentration, the surface charge of the channel, temperature, pressure, viscosity etc. The EDL thickness is defined by the Debye length. In nanofluidic devices, many interesting features happens when the characteristic length is in order to the thickness of electric double layer (EDL). Helmholtz-Smoluchowski velocity [1] occurs in the very thin EDL condition using slip EOF model. The electroosmotic flow depends upon the surface charge of the softchannel wall and this surface charge also depends on surface prosperities of soft channel wall. In most analysis of EOF, the distribution of ions are considered to obey the Boltzmann distribution and consequently Poisson-Boltzmann distribution for the EDL potential. However, the convective transport of ions play an important roles for the structure of non-homogeneous EDL.

The numerical investigation of the EOF in microchannels with sinusoidal roughness have been carried out by Yang and Liu [2]. The electrokinetic transport and mixing have been studied by Bhattacharyya and Bera [3] through a rough microchannel. Chen et al. [4] numerically investigated the EOF effects due to potential heterogeneity in a cylindrical microchannel. Kang et al. [5] studied the EOF in a annulus carrying high surface potential. Berg and Findlay [6] established analytical solution for electrokinetic flow in a circular microchannel. Ohshima [7] established analytic solution for Poisson-Boltzmann equation in a charged cylindrical narrow channel. Bera and Bhattacharyya [8] studied the electrokinetic effects on solute transport in a modulated microchannel with potential heterogeneity.

Polymer coatings are employed in many ways in EOF such as to control the velocity and chemical interactions using bio-molecule separations technology. When a nanochannel is coated with polyelectrolyte layer is often denoted as soft channel. The effect of surface charge density on electrokinetic flow through polyelectrolyte layer coated nanopore have been studied by Bera and Bhattacharyya [9]. Tessier and Slater [10] analysis EOF strength in an electrolyte confined through grafted polymer chains. EOF through grafting polymers into two parallel nanofluidic channel has been studied by Cao et al. [11]. EOF in porous cylindrical nanopore and annular geometry are investigated by Wu et al. [12] using linear Poisson-Boltzmann equation. A numerical simulation using dissipative particle dynamics (DPD) have been performed by Cao et al. [13] in a polyelectrolyte coated nanopore. Zuo et al. [14] studied the EOF suppression using polyampholyte brushes into two parallel plates in molecular dynamics simulations.

The present study deals with the non-linear effects for surface modulation and potential heterogeneity on the EOF through a Polyelectrolyte coated soft nanochannel. In comparison of our previous study [9], here we consider both the potential heterogeneity and surface modulation periodically in the soft channel. The Nernst-Plank equation is taking for considering of the above effects. The present study considers with the Brinkman modified Navier-Stoke equations for fluid flow in the interior and exterior in the PEL. The Laplace and the Poisson equation give the external and induced potential distribution respectively. We solved these governing equations numerically in a coupled manner.

2 Mathematical Model

We have considered a periodically modulated rectangular soft nanochannel of height h which filled with an incompressible Newtonian electrolyte (Fig. 1). Both walls of the soft nanochannel are geometrically modulated by placing obstacles periodically with amplitude h_p and length l_p. The wall of the nanochannel bears a constant negative ζ potential while the potential along the modulated heterogeneous portion ζ_p. A PEL is embedded in the both side of the nanochannel wall with thickness a_s. Here, PEL is considered ion-penetrable, homogeneously structured, uniform thickness a_s with fixed charge density $\rho_{fix} \cong (eZ\sigma_s/a_s)$. Here, e and Z are respectively the elementary charge and the valence of PEL and σ_s beings surface charged density of PEL. The distribution of applied electric potential is governed by the Laplace equation and electric field is generated by the placing electrodes at upstream and downstream of the soft nanochannel. The potential and length scale are non-dimensionalized by $\phi_0(= RT/F)$ and channel height h respectively. Here, gas constant is R, the absolute temperature is T and the Faraday's constant is F. The non-dimensional applied potential equation can be written in the form as

$$\nabla^2 \psi = 0 \tag{1}$$

We considered that the channel walls are electrically insulated i.e., $\nabla\psi.\mathbf{n} = 0$, here $\mathbf{n}$ is the unit outward normal vector of the wall surface. At the upstream and downstream of the softchannel, external potential ψ linearly vary with x i.e., $\psi = -\Lambda x$. The non-dimensional number $\Lambda(= E_0 h/\phi_0)$ represents the applied electric field strength.

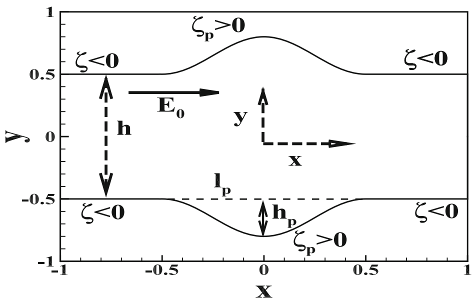


Fig. 1. Schematic diagrams of the polyelectrolyte layer coated periodically modulated soft nanochannel.

The electric field $\mathbf{E}$ $(= \mathbf{E}_x, \mathbf{E}_y)$ is evaluated by the uniform applied electric field E_0 and induced field. The total potential Φ^* is the sum of external and induced potential as $\Phi^* = \psi^*(x, y) + \phi^*(x, y)$, here induced potential is ϕ^*. The poisson equation can be expressed in the following form as

$$\nabla \cdot (\epsilon_e \mathbf{E}) = -\epsilon_e \nabla^2 \phi = \rho_e + r\rho_{fix} \tag{2}$$

The charge density is $\rho_e = \sum_i z_i e n_i$; z_i is the valance and n_i is number concentration. We assume a symmetric monovalent electrolyte of valance $z_i = \pm 1$. Here, permittivity of the medium is ϵ_e and r is a unit region function, assume 1 within polyelectrolyte layer and 0 outside it.

The Ions distribution is given by the Nernst-Planck equation and is expressed as

$$\frac{\partial n_i}{\partial t} + \nabla \cdot -D_i \nabla n_i + n_i \omega_i z_i F \mathbf{E} + n_i \mathbf{q} = 0 \tag{3}$$

Where, diffusivity of the ions is D_i, mobility is ω_i and Faraday's constant is F. The velocity field $\mathbf{q}(= u, v)$ has two component u and v along x and y directions respectively.

The electroosmotic flow is governed by the Brinkman modified Navier-Stokes equation for the interior and exterior of the PEL as

$$\rho \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} \right] = -\nabla p + \mu \nabla^2 \mathbf{q} + \rho_e \mathbf{E} - r\gamma \mathbf{q} \tag{4}$$

$$\nabla \cdot \mathbf{q} = 0 \tag{5}$$

Here, density is ρ and viscosity is μ and γ is the factor of hydrodynamics friction in the polyelectrolyte layer. Periodic boundary conditions are considered at the upstream and downstream of flow region. No-slip condition for velocity are taken on the channel wall. The ions are impenetrable in channel wall i.e., $N_i \cdot n = 0$ and contain constant ζ-potential while the potential of the modulated region is ζ_p.

2.1 Coordinate Transformation

We have used the following transformation to map the physical region into a rectangular computational region [8],

$$Y = \frac{y}{y_0(x)} \tag{6}$$

where, the shape of the obstacle is $y_0(x)$ with amplitude h_p and length l_p defined by

$$y_0(x) = \begin{cases} 1 + \frac{h_p}{2}\{1 + \cos(\frac{\pi x}{x_0})\} & -x_0 \le x \le x_0 \\ 1 & \text{elsewhere} \end{cases}$$

Using the above transformation (Eq. 6), the governing equations (Eqs. 1–5) are transformed into a rectangular domain. The concentration is scaled by the bulk number concentration n_0. Here, velocity field $\mathbf{q}$ is non-dimensionalized by the Helmholtz-Smoluchowski velocity U_{HS} $(= \epsilon_e E_0 \phi_0 / \mu)$, pressure p and time t are non-dimensionalized by $\mu U_{HS}/h$ and h/U_{HS} respectively. The EDL thickness

(λ) is defined as $\lambda = \sqrt{\epsilon_e k_B T/\Sigma_i (z_i e)^2 n_{i0}}$ and $\kappa h = h/\lambda$. The non-dimensional charge density $Q_{fix} = \rho_{fix} h^2/\epsilon_e \phi_0$. The scaled softness parameter β can be defined as $\beta = h/\lambda_0^{-1}$, here λ_0^{-1} is the softness degree in the PEL. We denote the scaled cation concentration by g and anion concentration by f. The Reynolds number is defined as $Re = U_{HS} h/\nu$, Schmidt number $Sc = \nu/D_i$, Peclet number $Pe = Re \cdot Sc$.

The non-dimensional governing equations for the external potential (ψ), induced potential (ϕ), ions(g, f) and velocity components (u, v) are given by

$$\frac{\partial^2 \psi}{\partial x^2} + Y\left[\frac{2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 - \frac{1}{y_0^2}\frac{\partial^2 y_0}{\partial x^2}\right]\frac{\partial \psi}{\partial Y} - \frac{2Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial^2 \psi}{\partial x \partial Y} + \left[\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right]\frac{\partial^2 \psi}{\partial Y^2} = 0 \quad (7)$$

$$\begin{aligned}&\frac{\partial^2 \phi}{\partial x^2} + Y\left[\frac{2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 - \frac{1}{y_0^2}\frac{\partial^2 y_0}{\partial x^2}\right]\\ &\frac{\partial \phi}{\partial Y} - \frac{2Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial^2 \phi}{\partial x \partial Y} + \left[\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right]\frac{\partial^2 \phi}{\partial Y^2} = -\frac{(\kappa h)^2}{2}(g-f) - hQ_{fix}\end{aligned} \quad (8)$$

$$\begin{aligned}&Pe\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} - Y\left[\frac{2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 - \frac{1}{y_0^2}\frac{\partial^2 y_0}{\partial x^2}\right]\frac{\partial g}{\partial Y} + \frac{2Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial^2 g}{\partial x \partial Y} - \left[\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right]\frac{\partial^2 g}{\partial Y^2}\\ &= \frac{(\kappa h)^2}{2} g(g-f) + Pe\left[\frac{\partial (gu)}{\partial x} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial (gu)}{\partial Y} + \frac{1}{y_0}\frac{\partial (gv)}{\partial Y}\right] + \left[\frac{\partial g}{\partial x}\frac{\partial \phi}{\partial x} + \left\{\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right\}\frac{\partial g}{\partial Y}\frac{\partial \phi}{\partial Y}\right]\\ &+hgQ_{fix} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\left[\frac{\partial g}{\partial x}\frac{\partial \phi}{\partial Y} + \frac{\partial g}{\partial Y}\frac{\partial \phi}{\partial x}\right] + \left[\frac{\partial g}{\partial x}\frac{\partial \psi}{\partial x} + \left\{\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right\}\frac{\partial g}{\partial Y}\frac{\partial \psi}{\partial Y}\right]\\ &-\frac{Y}{y_0}\frac{\partial y_0}{\partial x}\left[\frac{\partial g}{\partial x}\frac{\partial \psi}{\partial Y} + \frac{\partial g}{\partial Y}\frac{\partial \psi}{\partial x}\right]\end{aligned} \quad (9)$$

$$\begin{aligned}&Pe\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - Y\left[\frac{2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 - \frac{1}{y_0^2}\frac{\partial^2 y_0}{\partial x^2}\right]\frac{\partial f}{\partial Y} - \frac{2Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial^2 f}{\partial x \partial Y} - \left[\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right]\frac{\partial^2 f}{\partial Y^2}\\ &= -\frac{(\kappa h)^2}{2} f(g-f) + Pe\left[\frac{\partial (fu)}{\partial x} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial (fu)}{\partial Y} + \frac{1}{y_0}\frac{\partial (fv)}{\partial Y}\right]\\ &-\left[\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x} + \left\{\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right\}\frac{\partial f}{\partial Y}\frac{\partial \phi}{\partial Y}\right] - hfQ_{fix} + \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\left[\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial Y} + \frac{\partial f}{\partial Y}\frac{\partial \phi}{\partial x}\right]\\ &-\left[\frac{\partial f}{\partial x}\frac{\partial \psi}{\partial x} + \left\{\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right\}\frac{\partial f}{\partial Y}\frac{\partial \psi}{\partial Y}\right] + \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\left[\frac{\partial f}{\partial x}\frac{\partial \psi}{\partial Y} + \frac{\partial f}{\partial Y}\frac{\partial \psi}{\partial x}\right]\end{aligned} \quad (10)$$

$$\frac{\partial u}{\partial x} + \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial u}{\partial Y} + \frac{1}{y_0}\frac{\partial v}{\partial Y} = 0 \quad (11)$$

$$\begin{aligned}&Re\frac{\partial u}{\partial t} + Re\left[\frac{\partial (u^2)}{\partial x} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial (u^2)}{\partial Y} + \frac{1}{y_0}\frac{\partial (uv)}{\partial Y}\right] = -\left[\frac{\partial p}{\partial x} + \frac{Y}{y_0}\frac{\partial p}{\partial Y}\right]\\ &-(g-f)\frac{(\kappa h)^2}{2}\left[\left(\frac{\partial \phi}{\partial x} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial \phi}{\partial Y}\right) + \left(\frac{\partial \psi}{\partial x} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial \psi}{\partial Y}\right)\right]\end{aligned} \quad (12)$$

$$+\frac{\partial^2 u}{\partial x^2} + Y\left[\frac{2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 - \frac{1}{y_0^2}\frac{\partial^2 y_0}{\partial x^2}\right]\frac{\partial u}{\partial Y} - \frac{2Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial^2 u}{\partial x \partial Y} + \left[\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right]\frac{\partial^2 u}{\partial Y^2} - h\beta^2 u$$

$$Re\frac{\partial v}{\partial t} + Re\left[\frac{\partial (uv)}{\partial x} - \frac{Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial (uv)}{\partial Y} + \frac{1}{y_0}\frac{\partial (v^2)}{\partial Y}\right] = -\frac{Y}{y_0}\frac{\partial p}{\partial Y} - (g-f)\frac{(\kappa h)^2}{2}\frac{1}{y_0}\left[\frac{\partial \phi}{\partial Y} + \frac{\partial \psi}{\partial Y}\right]$$
$$+\frac{\partial^2 v}{\partial x^2} + Y\left[\frac{2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 - \frac{1}{y_0^2}\frac{\partial^2 y_0}{\partial x^2}\right]\frac{\partial v}{\partial Y} - \frac{2Y}{y_0}\frac{\partial y_0}{\partial x}\frac{\partial^2 v}{\partial x \partial Y} + \left[\frac{Y^2}{y_0^2}\left(\frac{\partial y_0}{\partial x}\right)^2 + \frac{1}{y_0^2}\right]\frac{\partial^2 v}{\partial Y^2} - h\beta^2 v \tag{13}$$

3 Numerical Methods

These coupled set of governing non-linear equations (Eqs. 7–13) for fluid flow, potential and ionic species concentration are solved numerically through finite volume method on a staggered grid system in a transformed domain. In the staggered grid system, the scalar quantities are evaluated at each cell center and the velocity components are evaluated at the midpoint of the cell sides to which they are normal. The discretized form of the governing equations is obtained by integrating the governing equations over each control volumes. Different control volumes are used to integrate different equations. In order to capture the sharp change in variable values accurately, we use the higher-order upwind scheme, QUICK (Quadratic Upwind Interpolation Convective Kinematics, [15]) to discretize the convective and electromigration terms in both concentration and Navier-Stokes equations. The QUICK scheme uses a quadratic interpolation/extrapolation between the three nodal values of variables to estimate its value at the interface of the control volume. The upwind scheme imparts stability to the numerical solution in the region where a steep gradient in variables occur. An implicit first-order scheme is used for discretising the time derivative terms. The resulting discretized equations are solved iteratively through the pressure correction based iterative algorithm SIMPLE [16]. The iteration starts by assuming the induced electric potential ϕ at every cell center.

We have taken uniform grid along x axis and non-uniform along y-axis and δt was considered as 0.0002. We have taken a non-uniform grid size where δy is assumed to vary between 0.005 to 0.01 with δx is either 0.02 (for grid 1) or $\delta x =$ 0.01 (for grid 2). In grid 3, we considered $\delta x = 0.01$ and $0.0025 \leq \delta y \leq 0.005$. Figure 2(a, b) show that the results obtained by grid 2 and grid 3 agree fairly well with each other and these results are in close agreement with the result due to Mirbozorgi et al. [17]. Thus, we find that grid 2 is optimal.

4 Results and Discussions

In this study, we considered the nanochannel height $h = 10\,\text{nm}$ with ζ-potential of the wall is -1 while the potential along modulated region is $\zeta_p = 0.1$ and non-dimensional PEL charged density is varying -10 to 10. The values of softness of the PEL λ_s^{-1} varies between 0.1 to 10 nm [18,19] and the corresponding softness parameter (β) lies from 1 to 10 when height of the channel is 10nm. We considered that the diffusion coefficient for both the ions are same as $D_+ =$

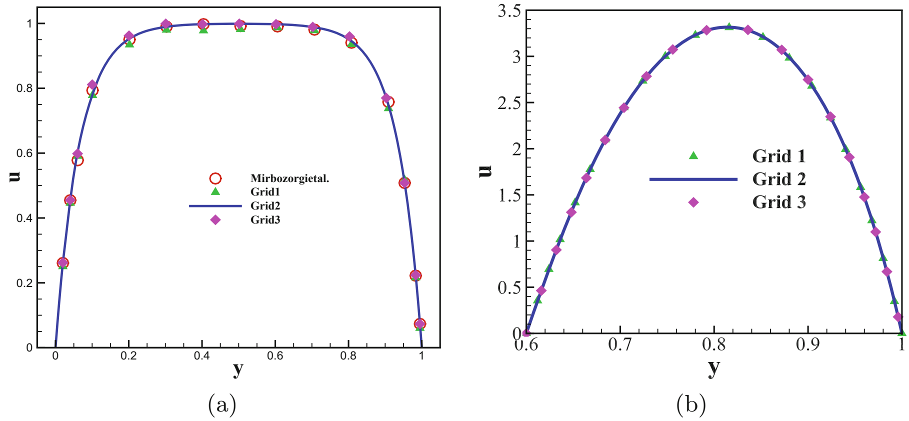
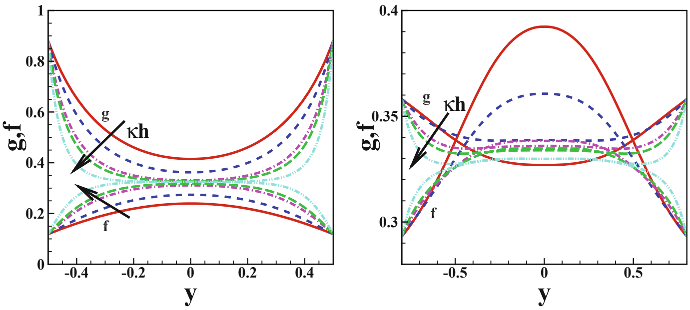


Fig. 2. Comparison of our numerical results of (a) axial velocity u in homogeneous channel with Mirbozorgi et al. [17] and (b) axial velocity u at $x = 0$ when $h_P = 0.3$, $\zeta_p = 1.0$ and $\zeta = -1$. Here, soft channel height $h = 10\,\mu\text{m}$ when $\kappa h = 21.74$, $Q_{fix} = 0$.

$D_- = 1.3 \times 10^{-10}\,\text{m}^2/\text{sec}$. We presented the solutions for different ionic strength of electrolyte solution, PEL scaled charged density and softness parameter. Here, we considered the height of the nanochannel is 20 nm, modulated length $l_p = 1$ nm, amplitude $h_p = 3$ nm, potential of the rigid nanochannel wall $\zeta = -1$, heterogeneous potential of the modulated wall $\zeta_p = 0.1$ and applied external electric field $E_0 = 10^6\,\text{V/m}$.

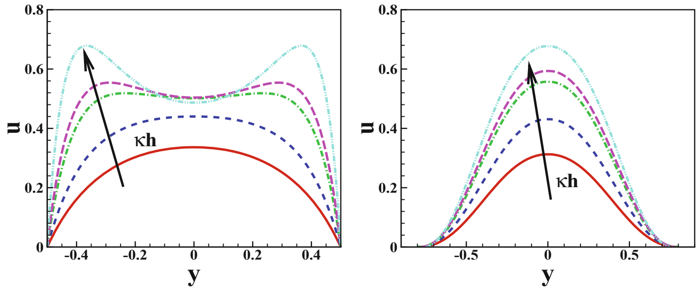
The distribution of ionic concentration (g, f) are shown in Figs. 3(a) and (b) at $x = -1$ and $x = 0$ respectively for zero non-dimensional charged ($Q_{fix} = 0$) in PEL for different κh when the PEL thickness $a_s = 3$ nm, amplitude $h_p = 0.3$ and softness parameter $\beta = 1$. Here we considered that the potential of the rigid microchannel wall $\zeta = -1$ while heterogeneous surface potential of modulated region $\zeta_p = 0.1$. Therefore, the magnitude of the anions flux inside the nanopore is significantly larger than that in the bulk, resulting in an enrichment of ionic concentrations on the anode side of the nanopore. κh increases with the increase of ionic concentration for fixed channel height. For zero PEL charged density, increase of the ionic concentration decrease of the net charged density i.e., $\rho_e (= g - f)$ of the ion.

Figures 4(a) and (b) show the distribution of axial velocity u for the positions at $x = -1$ and $x = 0$ respectively for different values of κh when PEL thickness $a_s = 3$ nm and scaled charged density $Q_{fix} = 0$, wave amplitude $h_p = 0.3$, softness parameter $\beta = 1$, heterogeneous surface potential in modulated region $\zeta_p = 0.1$ and potential of the microchannel wall $\zeta = -1$. The parameter κh increases with ionic concentration for fixed height. For strong ionic concentration, Debye layer length becomes thin and leads to a strong shielding effect. Therefore, more counter ions confined in the thin EDL and hence increase of axial velocity. The softness parameter mainly effects the hydrodynamics fields of the nanopore.



(a) (b)

Fig. 3. Variation of anion f and cation g at (a) $x = -1$, and (b) $x = 0$ for different values of κh when non-dimensional charged in PEL $Q_{fix} = 0$. Here, softness parameter softness parameter $\beta = 1$, channel height $h = 10\,\text{nm}$, PEL thickness $a_s = 3\,\text{nm}$, wave amplitude $h_p = 0.3$, Heterogeneous potential $\zeta_P = 0.1$; wall potential $\zeta = -1$ and electric field $E_0 = 10^6\,\text{V/m}$. Arrow indicate the increasing order of $\kappa h (= 10, 15, 25, 30, 50)$



(a) (b)

Fig. 4. Variation of axial velocity profile u at (a) $x = -1$, and (b) $x = 0$ for different κh when non-dimensional charged in PEL $Q_{fix} = 0$. Here, softness parameter $\beta = 1$, channel height $h = 10\,\text{nm}$, PEL thickness $a_s = 3\,\text{nm}$, wave amplitude $h_p = 0.3$, heterogeneous potential $\zeta_p = 0.1$; wall potential $\zeta = -1$ and applied electric field $E_0 = 10^6\,\text{V/m}$. Arrow indicate the increasing order of $\kappa h (= 10, 15, 25, 30, 50)$

The streamlines profile for different ionic concentration are shown in Figs. 5(a–c). Since, nanopore wall is negatively charged and charged density of PEL is positive, hence more anions are electrostatically attracted into the soft-layer, whereas more cations are repelled out. It is clear from the Figs. 5(a–c) that the increase of ionic concentration increase of circulation strength. The streamline patterns shows that the occupance of flow separation and formation of the vortex above the modulated region. Linear model can not able to explain fluid flow separation and appearance of vertical flow.

Figures 6(a) and (b) show the distribution of u velocity profile at (a) $x = -1$ and (b) $x = 0$ respectively for different scaled charged density

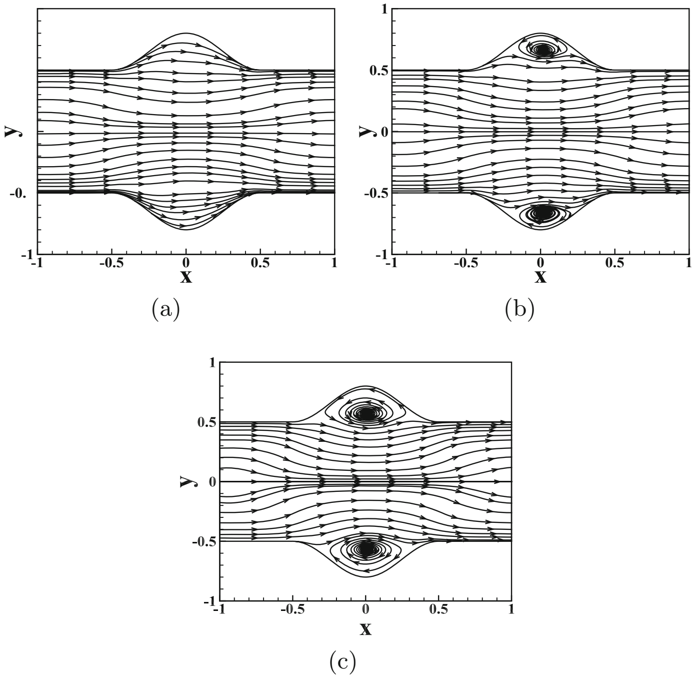


Fig. 5. Streamline profile for different values of κh (a) $\kappa h = 10$, (b) $\kappa h = 15$ (c) $\kappa h = 30$, when PEL non-dimensional charged $Q_{fix} = 10$. Here, softness parameter $\beta = 1$, channel height $h = 10\,\text{nm}$, PEL thickness $a_s = 3\,\text{nm}$, heterogeneous potential $\zeta_p = -0.1$; wall potential $\zeta = -1$ and applied electric field $E_0 = 10^6\,\text{V/m}$.

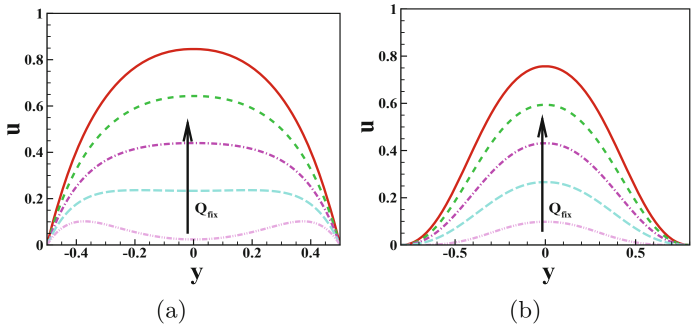
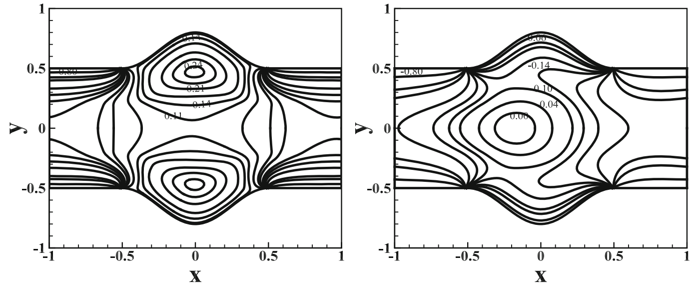
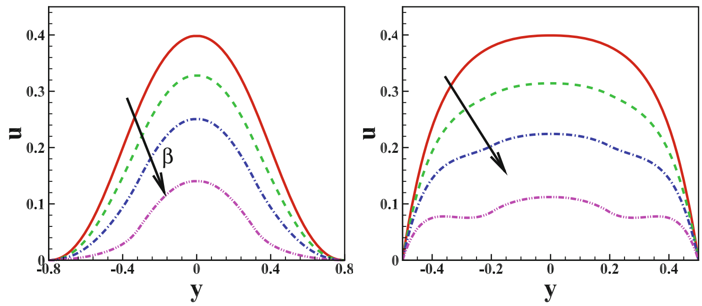


Fig. 6. Variation of axial velocity u at (a) $x = -1$, and (b) $x = 0$ for different non-dimensional charded Q_{fix} when $\kappa h = 15$. Here, softness parameter $\beta = 1$, channel height $h = 10$nm, PEL thickness $a_s = 3\,\text{nm}$, heterogeneous potential $\zeta_p = 0.1$; wall potential $\zeta = -1$ and external electric field $E_0 = 10^6\,\text{V/m}$. Arrow indicate the decreasing order of $Q_{fix}(= 10, 5, 0, -5, -10)$



(a) (b)

Fig. 7. Distribution of induce potential ϕ for different (a) $Q_{fix} = 10$, and (b) $Q_{fix} = -10$ when $\kappa h = 15$. Here, softness parameter $\beta = 1$, channel height $h = 10\,\mathrm{nm}$, PEL thickness $a_s = 3\,\mathrm{nm}$, heterogeneous $\zeta_P = 0.1$; wall potential $\zeta = -1$ and external imposed electric filed $E_0 = 10^6\,\mathrm{V/m}$.



(a) (b)

Fig. 8. Variation of axial velocity u at (a) $x = -1$, and (b) $x = 0$ for different softness parameter β. Here, softness parameter $Q_{fix} = 1$, channel height $h = 10\,\mathrm{nm}$, PEL thickness $a_s = 3\,\mathrm{nm}$, heterogeneous potential $\zeta_P = 0.1$; wall potential $\zeta = -1$ and applied electric field $E_0 = 10^6\,\mathrm{V/m}$. Arrow indicate the decreasing order of $\beta(= 1, 3, 5, 10)$.

$Q_{fix}(= 10, 5, 0, -5, -10)$ when the $\kappa h = 15$. The ζ potential of the nanopore wall is negative and PEL scaled charged is positive. Since wall is negatively charged, the increase of positive charged density in PEL decrease the EOF velocity. It is clear form the Figs. 6(a) and (b) that the u velocity increase with the decrease positive scaled charged density.

The distribution of induced potential ϕ are shown in Figs. 7(a) and (b) for fixed scaled charged density $Q_{fix}(= 1)$ at $x = -1$ and $x = 0$ respectively. Since the nanopore wall is negatively charged and the decrease of the positive charged density in PEL actually confined more counterions in EDL and decrease the net effective wall's charge. Therefore increase the u velocity. But for the heterogeneous portion, both the nanopore wall surface charge and charge density in PEL are in same sign and therefore axial velocity always increases with the decrease of the PEL scaled charged density.

In Figs. 8(a) and (b), we have shown the effects of the softness parameter on the distribution of the axial u velocity at (a) $x = -1$ and (b) $x = 0$ when PEL charged density $Q_{fix} = 1$. Softness parameter of PEL effects the hydrodynamics field of the nanopore while the conductance is not affected significantly by the flow field. It is clear from the Figs. 8(a) and (b) that the increase of the value of the softness parameter decreases the axial velocity for both homogeneous and heterogeneous portion of the naopore. The softness degree of PEL mainly affects the hydrodynamic field inside the nanopore while the conductance is not affected significantly by the flow field.

5 Conclusions

A numerical study has been performed on the electroosmotic flow (EOF) through polyelectrolyte layer coated periodically surface modulated softchannel with heterogeneous potential. A transformation is considered to map the physical region into the rectangular computational region. The governing non-linear coupled equations are numerically solved into transformed region. Pressure correction based algorithm is adapted for computing the flow field. The influence of the geometric modulation of the surface, the bulk electrolyte concentration, the geometry of the nanopore and PEL charged density are considered in this study. The combined effects for surface modulation and potential heterogeneity in charged wall are found prominent in this study. The EOF velocity increase with the bulk ionic concentration of the solution. The axial velocity increases with the increase of ionic concentration for zero-scaled charged density in PEL. The increase of charged density decreases of the axial velocity for fixed ion concentration. Circulation occur only for high ionic concentration in the heterogeneous portion. The induced potential is prominent for opposite charge density in PEL. The increment of values in softness parameter decrease the average flow rate.

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FGP Approach Based on Stanojevic's Normalization Technique for Multi-level Multi-objective Linear Fractional Programming Problem with Fuzzy Parameters

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Abstract. This paper aims to present a Fuzzy Goal Programming (FGP) method taking the help of Taylor series approximation and normalization technique due to Stanojevic to solve multi-level multi-objective linear fractional programming problem with fuzzy parameters (MLMOLFPP-FP). Firstly, a crisp model of the problem is developed using level sets followed by the construction of membership functions which are non-linear in nature. These are then linearized using first order Taylor series approximation and normalization technique [1]. The normalization technique ensures that the obtained linear membership functions have their range within the permissible limit of [0, 1]. The compromise solution for each level is calculated through FGP method. Each level decision maker imposes some preference bounds on the decision variable associated with him/her to avoid decision deadlock. Finally, the original MLMOLFPP-FP is reduced into a linear programming problem (LPP) through FGP technique where the highest degree of the membership goals is attained by minimizing the negative deviational variables. Euclidean distance function helps us to select the best FGP model from the two FGP models described to solve the MLMOLFPP-FP.

Keywords: Multi-objective programming · Multi-level programming · Fractional programming · Fuzzy numbers · Fuzzy goal programming

1 Introduction

Multi-level programming problems (MLPPs) occur mostly in hierarchical decision making organizations commonly found in supply chains, agriculture sector, business organizations, economic organizations etc. where many levels of decision makers (DMs) are involved. Various researchers have studied MLPP and

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proposed different methods to solve them. Some of them are interactive fuzzy programming by Sakawa et al. [2], Fuzzy mathematical approach by Sinha [3]. FGP approach has been extensively used by various researchers to solve MLPP and multi-level multi-objective programming problem (MLMOPP) [4–6].

MLMOPP which have the objectives in the form of ratio of linear functions is called multi-level multi-objective linear fractional programming problem (MLMOLFPP). In recent times, Lachhwani [7] have introduced modification on the FGP approach by Baky [3] for MLMOLFPP. Lachhwani and Poonia [8] used FGP approach to solve multi-level linear fractional programming problem. Nayak and Ojha have formulated MLMOLFPP with interval parameters [9].

In formulating a practical life problem as an MLMOLFPP, various ambiguities exist which makes it impossible to set the coefficients of the problem as crisp numbers. So it becomes more convenient to represent them as fuzzy numbers. MLMOLFPP with fuzzy parameters has been approached in an interactive way by Osman et al. [10]. Parametric form of the MLMOLFPP having the coefficients of the constraints as fuzzy numbers was solved in [11].

This paper presents two alternative FGP models to solve MLMOLFPP-FP. The fuzzy numbers are at first changed into crisp numbers using level sets. The membership function for each fractional objective function is constructed considering its individual optimal solution. These are then linearized using first order Taylor series approximation followed by normalization as proposed in [1]. The compromise solution for each level is calculated through FGP. Each level DM proposes some preference bounds on the concerned decision variable so that a decision deadlock scenario does not occur. Finally the original MLMOLFPP-FP is changed into a LPP through FGP methods and its optimal solution is obtained. The best FGP method from the two alternatives is chosen through the deciding metric- Euclidean distance function.

The rest of the paper is organized as follows: Sect. 2 deals with some basic notions which are used in the article. Section 3 formulates MLMOLFPP-FP and the proposed method is elaborated in Sect. 4. In Sect. 5, a numerical example illustrates the proposed approach and the paper ends with the conclusion in Sect. 6.

2 Preliminaries

Basic ideas of fuzzy sets, fuzzy numbers and different operations which can be operated on them are found extensively in [12,13].

Let $\tilde{B}$ be a fuzzy set on the universal set W and $w \in W$. Then $\mu_{\tilde{B}} : W \to [0, 1]$ denotes the membership function representing the degree of membership of w in $\tilde{B}$.

Definition 1: The α level set or α-cut of a fuzzy set $\tilde{B}$ is defined as the crisp set $B_\alpha = \{w : \mu_{\tilde{B}}(w) \geq \alpha\}$.

Definition 2: $\tilde{B}$ represents a triangular fuzzy number (TFN) if it is written as $\tilde{B} = \left(b^{(1)}, b^{(2)}, b^{(3)}\right)$ with $b^{(1)} \leq b^{(2)} \leq b^{(3)}$ and the membership function for $\tilde{B}$ is defined as follows:

$$\mu_{\tilde{B}}(w) = \begin{cases} \frac{w-b^{(1)}}{b^{(2)}-b^{(1)}}, & b^{(1)} \leq w \leq b^{(2)} \\ \frac{b^{(3)}-w}{b^{(3)}-b^{(2)}}, & b^{(2)} \leq w \leq b^{(3)} \\ 0 \quad , & otherwise \end{cases}$$

The α-cut of TFN $\tilde{B} = \left(b^{(1)}, b^{(2)}, b^{(3)}\right)$ is expressed as an interval $B_\alpha = [B_\alpha^L, B_\alpha^U] = [b^{(1)} + (b^{(2)} - b^{(1)})\alpha, b^{(3)} - (b^{(3)} - b^{(2)})\alpha]$, $\forall \alpha \in [0, 1]$ and $B_\alpha^L \leq B_\alpha^U$. α- cuts are very important in the study of fuzzy numbers as we can completely and uniquely represent a fuzzy number by its α-cuts [14]. Since α- cuts are closed intervals on the real line, arithmetic operations can be defined on them, which in turn refer to the arithmetic operations on the corresponding fuzzy number.

Definition 3: A TFN $\tilde{B} = \left(b^{(1)}, b^{(2)}, b^{(3)}\right)$ is called non-negative if $b^{(1)} \geq 0$. and non-zero positive if $b^{(1)} > 0$. Let $F(R^+)$ and $F(R^{++})$ denote the set of all non-negative fuzzy numbers and the set of all non-zero positive fuzzy numbers respectively.

3 Formulation of Multi-level Multi-objective Linear Fractional Programming Problem with Fuzzy Parameters (MLMOLFPP-FP)

In MLMOLFPP-FP, a decision maker exists at each level of decision making and multiple fractional objective functions have to be optimized at each level. Here we consider the coefficients of the objective functions and the constraints as TFNs. We consider here a decision making structure consisting of $q(>2)$ levels. The k-th level decision maker DM_k controls the decision variable w_k where $w = (w_1, w_2, ..., w_k, ..., w_q), w \in R^q$. A MLMOLFPP-FP of maximization type can be written as [2,3,7]:

1st Level:

$$\max_{w_1} \ \tilde{Z}_1(w) = \max_{w_1} (\tilde{Z}_{11}(w), \tilde{Z}_{12}(w), ..., \tilde{Z}_{1p_1}(w)) \tag{1}$$

2nd Level:

$$\max_{w_2} \ \tilde{Z}_2(w) = \max_{w_2} (\tilde{Z}_{21}(w), \tilde{Z}_{22}(w), ..., \tilde{Z}_{2p_2}(w)) \tag{2}$$

.
.
.

q-th Level:

$$\max_{w_q} \ \tilde{Z}_q(w) = \max_{w_q} (\tilde{Z}_{q1}(w), \tilde{Z}_{q2}(w), ..., \tilde{Z}_{qp_q}(w)) \tag{3}$$

Subject to

$$w \in G = \left\{ w \in R^q | \tilde{B}w \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} \ \tilde{a} \, , \, w \geq 0, \tilde{a} \in R^p \right\} \tag{4}$$

where

$$\tilde{Z}_{kr}(w) = \frac{\tilde{u}_1^{kr}w_1 + \tilde{u}_2^{kr}w_2 + ... + \tilde{u}_q^{kr}w_q + \tilde{\alpha}^{kr}}{\tilde{v}_1^{kr}w_1 + \tilde{v}_2^{kr}w_2 + ... + \tilde{v}_q^{kr}w_q + \tilde{\beta}^{kr}} = \frac{\tilde{N}_{kr}(w)}{\tilde{D}_{kr}(w)} \qquad k = 1, 2, ..., q, \; r = 1, 2, ..., p_k \tag{5}$$

The elements of the vectors $\tilde{u}_t^{kr}, \tilde{v}_t^{kr}$, $(t = 1, 2, ..., q)$, $\tilde{a}$ and the constants $\tilde{\alpha}^{kr}, \tilde{\beta}^{kr}$ are expressed as TFNs. $\tilde{B}$ is a matrix of size $p \times q$ representing the fuzzy coefficients in the constraints. It is also assumed that $\tilde{D}_{kr}(w) > 0 \forall\, w \in G$.

4 Proposed Methodology

4.1 Crisp Model Formulation of the MLMOLFPP-FP

An equivalent crisp form of the MLMOLFPP-FP problem (1)–(5) can be obtained by substituting each fuzzy number with its corresponding α-cut, for a particular value of α. It is assumed that $\tilde{N}_{kr}(w) \in F(R^+)$ and $\tilde{D}_{kr}(w) \in F(R^{++})$. In maximization (minimization) type objective function, each coefficient of the numerator is replaced by the upper (lower) bound of the corresponding α-cut [15] and each coefficient of the denominator is replaced by the lower (upper) bound of the corresponding α-cut. Hence, maximization type objective function can be written as follows:

$$(Z_{kr}(w))_\alpha = \frac{(u_1^{kr})_\alpha^U w_1 + (u_2^{kr})_\alpha^U w_2 + ... + (u_q^{kr})_\alpha^U w_q + (\alpha^{kr})_\alpha^U}{(v_1^{kr})_\alpha^L w_1 + (v_2^{kr})_\alpha^L w_2 + ... + (v_q^{kr})_\alpha^L w_q + (\beta^{kr})_\alpha^L} = \frac{(N_{kr}(w))_\alpha^U}{(D_{kr}(w))_\alpha^L} \qquad k = 1, 2, ..., q, \; r = 1, 2, ..., p_k \tag{6}$$

Inequality constraints in Eq. (4) given by

$$\sum_{j=1}^{q} \tilde{B}_{kj} w_j \leq \tilde{a}_k, \; k = 1, 2, ..., l_1, \tag{7}$$

$$\sum_{j=1}^{q} \tilde{B}_{kj} w_j \geq \tilde{a}_k, \; k = l_2 + 1, \; l_2 + 2, ..., p \tag{8}$$

can be substituted by the following set of constraints

$$\sum_{j=1}^{q} (B_{kj})_\alpha^L \, w_j \leq (a_k)_\alpha^U, \; k = 1, 2, ..., l_1, \tag{9}$$

$$\sum_{j=1}^{q} (B_{kj})_\alpha^U \, w_j \geq (a_k)_\alpha^L, \; k = l_2 + 1, \; l_2 + 2, ..., p \tag{10}$$

To ascertain the compromise solution of the problem, the upper bounds $(a_k)^U_\alpha$, $(B_{kj})^U_\alpha$ and the lower bounds $(B_{kj})^L_\alpha$, $(a_k)^L_\alpha$ of the coefficients and constants are set to have the largest feasible region [10].

Equality constraints in Eq. (4)

$$\sum_{j=1}^{q} \tilde{B}_{kj} w_j = \tilde{a}_k,\ k = l_1+1, l_1+2, ..., l_2 \tag{11}$$

is to be replaced by two alternative constraints [16]:

$$\sum_{j=1}^{q} (\tilde{B}_{kj})^L_\alpha w_j \le (\tilde{a}_k)^U_\alpha,\ k = l_1+1,\ l_1+2, ..., l_2, \tag{12}$$

$$\sum_{j=1}^{q} (\tilde{B}_{kj})^U_\alpha w_j \ge (\tilde{a}_k)^L_\alpha,\ k = l_1+1,\ l_1+2, ..., l_2 \tag{13}$$

Using Eqs. (6)–(13), the MLMOLFPP-FP problem (1)–(5) changes into a crisp MLMOLFPP. The newly formed set of constraints presented by Eqs. (9), (10), (12) and (13) are together denoted by G_α.

4.2 Construction of Membership Functions of MLMOLFPP

Each objective function of the obtained crisp MLMOLFPP is solved individually subject to the constraints to obtain their individual maximum and minimum values. A fuzzy goal is formed from each crisp objective function by associating an aspiration level to it.

Let $Z^B_{kr} = \max_{w \in G_\alpha} (Z_{kr}(w))_\alpha$ and $Z^W_{kr} = \min_{w \in G_\alpha} (Z_{kr}(w))_\alpha$, $k = 1, 2, ..., q,\ r = 1, 2, ..., p_k$ represent the maximum (or best) and the minimum (or worst) value respectively of kr-th objective function.

Hence we construct the fuzzy goals as: $\tilde{Z}_{kr}(w) \gtrsim Z^B_{kr}$ *and* $\tilde{Z}_{kr}(w) \lesssim Z^W_{kr}$ $k = 1, 2, ..., q,\ r = 1, 2, ..., p_k$ for problems where the objective function is to be maximized and minimized respectively.

$\lesssim$ and $\gtrsim$ indicate that fuzziness is associated with the aspiration levels and are termed as 'essentially less than' and 'essentially more than' as Zimmerman [13] proposed.

Membership function corresponding to the kr-th fuzzy objective goal can be described as:

$$\mu(Z_{kr}(w))_\alpha = \begin{cases} 1, & if\ (Z_{kr}(w))_\alpha \ge Z^B_{kr} \\ \frac{(Z_{kr}(w))_\alpha - Z^W_{kr}}{Z^B_{kr} - Z^W_{kr}}, & if\ Z^W_{kr} \le (Z_{kr}(w))_\alpha \le Z^B_{kr} \\ 0, & if\ (Z_{kr}(w))_\alpha \le Z^W_{kr} \end{cases} \tag{14}$$
$$k = 1, 2, ..., q,\ \ r = 1, 2, ..., p_k$$

Here, Z^B_{kr} and Z^W_{kr} give the upper tolerance and lower tolerance limit respectively for the kr-th objective goal. For each kr, $(Z_{kr}(w))_\alpha$ possesses continuous partial derivatives of order $(q+1)$ or less over the feasible region G_α. So the associated membership function $\mu(Z_{kr}(w))_\alpha$ possesses similar properties in the feasible area G_α.

4.3 Linearization of Membership Functions and Selection of Compromise Solution for Each Level

Linearization of the First Level Membership Functions: Let $\max\limits_{w \in G_\alpha} \mu(Z_{1r}(w))_\alpha$ be obtained at ${w_{1r}}^* = (w_1^{1r}, w_2^{1r}, ..., w_q^{1r})$ $r = 1, 2, ..., p_1$. The nonlinear membership function, $\mu(Z_{1r}(w))_\alpha$, is linearized using first order Taylor series approximation. The new membership function $\hat{\mu}(Z_{1r}(w))_\alpha$ is formulated as:

$$\begin{aligned}\mu(Z_{1r}(w))_\alpha &\cong \mu(Z_{1r}(w_{1r}^*))_\alpha + \sum_{t=1}^{q}(w_t - w_t^{1r})\left(\frac{\partial}{\partial w_t}\mu(Z_{1r}(w))_\alpha\right)_{at\, w = {w_{1r}}^*} \\ &= \hat{\mu}(Z_{1r}(w))_\alpha\end{aligned} \tag{15}$$

The function $\hat{\mu}(Z_{1r}(w))_\alpha$ may not satisfy the properties of a membership function since its range may not lie within the permissible range of $[0, 1]$. To handle this difficulty the concept of normalizing the function was $\hat{\mu}(Z_{1r}(w))_\alpha$ introduced by Stanojevic [1], so as to include their range within $[0, 1]$ while retaining their linearity and the position of the maximum point.

Normalization of $\hat{\mu}(Z_{1r}(w))_\alpha$ is done in the following way [1]:

$$\bar{\mu}(Z_{1r}(w))_\alpha = \frac{\hat{\mu}(Z_{1r}(w))_\alpha - g_{1r}}{h_{1r} - g_{1r}} \tag{16}$$

where g_{1r} and h_{1r} are the minimal and maximal values of $\hat{\mu}(Z_{1r}(w))_\alpha$ over the feasible region G_α.

Fuzzy Goal Programming (FGP) Formulation to Obtain Compromise Solution for the First Level: It is known that a membership function can attain the highest value of one. So for the membership function $\bar{\mu}(Z_{1r}(w))_\alpha$, the flexible membership goal along with the aspiration level unity can be written as:

$$\bar{\mu}(Z_{1r}(w))_\alpha + {D_{1r}}^- - {D_{1r}}^+ = 1, \quad r = 1, 2, ..., p_1 \tag{17}$$

${D_{1r}}^+$ and ${D_{1r}}^-$ denotes the positive and negative deviational variables respectively.

Equation (17) can be further written as follows [17]:

$$\bar{\mu}(Z_{1r}(w))_\alpha + {D_{1r}}^- = 1, \; r = 1, 2, ..., p_1$$

The first level compromise solution is obtained through the following FGP model:

$$\begin{aligned}
&\min\ \theta_1 \\
&\text{subject to} \\
&w \in G_\alpha \\
&\bar{\mu}(Z_{1r}(w))_\alpha + {D_{1r}}^- = 1 \\
&\theta_1 \geq {D_{1r}}^- \\
&0 \leq {D_{1r}}^- \leq 1 \\
&r = 1, 2, ..., p_1
\end{aligned} \tag{18}$$

Let the compromise solution for the first level be $w^{1*} = (w_1^1, w_2^1, ..., w_q^1)$. Similarly the compromise solutions for all other levels can be found out.

4.4 Preference Bounds on the Decision Vectors

In an MLPP, the goals of different level decision makers (DMs) are often conflicting in nature. So in order to get maximum benefit for their organization, cooperation among DMs of all levels is necessary. Hence each DM incorporates some relaxation which is expressed in the form of preference bounds on the decision variable under his/her control.

Let the compromise solution for the k^{th} level be $w^{k*} = (w_1^k, w_2^k, ...,$ $w_k^k, ..., w_q^k)$. The k^{th} level DM has the variable w_k^k under his/her control. Let t_k^L and t_k^U $(t_k^L \neq t_k^U)$ be the lower and upper preference bounds on w_k^k set by the k^{th} level DM. Hence we can write

$$w_k^k - t_k^L \leq w_k^k \leq w_k^k + t_k^U, \qquad k = 1, 2, ..., q \tag{19}$$

4.5 FGP Approach for MLMOLFPP

Formulation of two FGP models to solve MLMOLFPP is presented below: FGP model-1

$$\begin{aligned}
&\min\ \zeta \\
&\text{subject to} \\
&\bar{\mu}(Z_{kr}(w))_\alpha + {D_{kr}}^- = 1 \\
&w_k^k - t_k^L \leq w_k^k \leq w_k^k + t_k^U, \\
&\zeta \geq {D_{kr}}^- \\
&0 \leq {D_{kr}}^- \leq 1 \\
&w \in G_\alpha \\
&k = 1, 2, ..., q \\
&r = 1, 2, ..., p_k
\end{aligned} \tag{20}$$

FGP model-2

$$\begin{aligned}
&\min\ \varpi = \sum_{k=1}^{q} \sum_{r=1}^{p_k} D_{kr}^{-} \\
&\text{subject to} \\
&\bar{\mu}(Z_{kr}(w))_{\alpha} + D_{kr}{}^{-} = 1 \\
&w_k^k - t_k^L \leq w_k^k \leq w_k^k + t_k^U, \\
&0 \leq D_{kr}{}^{-} \leq 1 \\
&w \in G_{\alpha} \\
&k = 1, 2, ..., q \\
&r = 1, 2, ..., p_k
\end{aligned} \tag{21}$$

In general we obtain distinct solutions from the two FGP models. The model which provides the best solution can be identified with the help of Euclidean distance function which is defined below [18,19]:

$$D = \left[\sum_{k=1}^{q} \sum_{r=1}^{p_k} \{1 - \mu(Z_{kr}(w))_{\alpha}\}^2 \right]^{1/2} \tag{22}$$

The solution which provides minimum D is taken as the optimal compromise solution.

5 Numerical Illustration

We recall the problem taken by Osman et al. [10] to illustrate the efficiency of our proposed method.

$$\begin{aligned}
&[\text{1st Level}] \\
&\max_{x_1} \left(\tilde{f}_{11} = \frac{\tilde{1}x_1 + \tilde{4}x_2 - \tilde{1}x_3 - \tilde{1}}{\tilde{2}x_1 + \tilde{3}x_2 + \tilde{1}x_3 + \tilde{2}},\ \tilde{f}_{12} = \frac{\tilde{2}x_1 - \tilde{1}x_2 - \tilde{3}x_3 + \tilde{4}}{\tilde{2}x_1 - \tilde{1}x_2 + \tilde{1}x_3 + \tilde{5}} \right) \\
&[\text{2nd Level}] \\
&\max_{x_2} \left(\tilde{f}_{21} = \frac{-\tilde{3}x_1 + \tilde{2}x_2 - \tilde{2}x_3}{\tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}x_3 + \tilde{3}},\ \tilde{f}_{22} = \frac{\tilde{7}x_1 + \tilde{2}x_2 - \tilde{1}x_3 - \tilde{1}}{\tilde{5}x_1 + \tilde{2}x_2 + \tilde{1}x_3 + \tilde{1}} \right) \\
&[\text{3rd Level}] \\
&\max_{x_3} \left(\tilde{f}_{31} = \frac{-\tilde{1}x_1 - \tilde{1}x_2 - \tilde{1}x_3 + \tilde{4}}{\tilde{1}x_1 - \tilde{2}x_2 + \tilde{10}x_3 + \tilde{6}},\ \tilde{f}_{32} = \frac{\tilde{2}x_1 + \tilde{1}x_2 + \tilde{1}x_3 + \tilde{4}}{-\tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}x_3 + \tilde{10}} \right)
\end{aligned}$$

Subject to

$$\begin{aligned}
&\tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}x_3 \leq \tilde{5},\ \tilde{1}x_1 + \tilde{1}x_2 - \tilde{1}x_3 \leq \tilde{2},\ \tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}x_3 \geq \tilde{1}, \\
&-\tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}x_3 \leq \tilde{1},\ \tilde{1}x_1 - \tilde{1}x_2 + \tilde{1}x_3 \leq \tilde{4},\ \tilde{1}x_1 + \tilde{2}x_3 \leq \tilde{4}, \\
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.
\end{aligned}$$

The fuzzy parameters are taken as TFNs as follows:

$$\begin{aligned}
&\tilde{1} = (0.5, 1, 1.5),\ \tilde{2} = (1, 2, 3),\ \tilde{3} = (2, 3, 4),\ \tilde{4} = (3, 4, 5) \\
&\tilde{5} = (3, 5, 6),\ \tilde{6} = (5, 6, 6),\ \tilde{7} = (4, 7, 8),\ \tilde{10} = (8, 10, 10)
\end{aligned}$$

Assuming that $\alpha = 0.8$ is considered by all the level DMs, the crisp form of the

MLMOLFPP-FP using Eqs. (6)–(13) is obtained as:

$$
\begin{aligned}
&[\text{1st Level}]\\
&\max_{x_1}\left(f_{11}=\tfrac{1.1x_1+4.2x_2-0.9x_3-0.9}{1.8x_1+2.8x_2+0.9x_3+1.8},\ f_{12}=\tfrac{2.2x_1-0.9x_2-2.8x_3+4.2}{1.8x_1-1.1x_2+0.9x_3+4.6}\right)\\
&[\text{2nd Level}]\\
&\max_{x_2}\left(f_{21}=\tfrac{-2.8x_1+2.2x_2-1.8x_3}{0.9x_1+0.9x_2+0.9x_3+2.8},\ f_{22}=\tfrac{7.2x_1+2.2x_2-0.9x_3-0.9}{4.6x_1+1.8x_2+0.9x_3+0.9}\right)\\
&[\text{3rd Level}]\\
&\max_{x_3}\left(f_{31}=\tfrac{-0.9x_1-0.9x_2-0.9x_3+4.2}{0.9x_1-2.2x_2+9.6x_3+5.8},\ f_{32}=\tfrac{2.2x_1+1.1x_2+1.1x_3+4.2}{-1.1x_1+0.9x_2+0.9x_3+9.6}\right)
\end{aligned}
$$

Subject to

$$
\begin{aligned}
&0.9x_1+0.9x_2+0.9x_3\le 5.2,\ \ 0.9x_1+0.9x_2-1.1x_3\le 2.2,\ \ 1.1x_1+1.1x_2+1.1x_3\ge 0.9,\\
&-1.1x_1+0.9x_2+0.9x_3\le 1.1,\ \ 0.9x_1-1.1x_2+0.9x_3\le 4.2,\ \ 0.9x_1+1.8x_3\le 4.2,\\
&x_1\ge 0,\ \ x_2\ge 0,\ \ x_3\ge 0.
\end{aligned}
$$

The maximum and minimum values for each objective function are presented in Table 1.

Table 1. Maximum and minimum values for each objective function

Objective function	Max value	Point at which max occurred	Min value	Point at which min occurred
f_{11}	0.947	(0.55, 1.894, 0)	−0.689	(0, 0, 1.222)
f_{12}	1.064	(2.445, 0, 0)	−0.0014	(0.645, 0, 2.011)
f_{21}	0.689	(0, 1.22, 0)	−1.64	(3.287, 0, 0.6896)
f_{22}	1.375	(2.444, 0, 0)	−1	(0, 0, 1.222)
f_{31}	0.9964	(0, 1.222, 0)	−0.0469	(1.922, 2.2, 1.372)
f_{32}	1.845	(3.287, 0, 0.689)	0.493	(0, 0.818, 0)

We construct the linear membership functions using Eqs. (14, 15, and 16).

Solving model (18) for each level, we obtain the compromise solutions for the first, second and third levels respectively as follows:

$$
\begin{aligned}
&x_1=0.56388,\ x_2=1.8806,\ \ x_3=0.0\\
&x_1=0.6038,\ x_2=1.8406,\ \ x_3=0.0\\
&x_1=2.38619,\ \ x_2=0.05825,\ \ x_3=0.0
\end{aligned}
$$

The preference bounds decided by the DMs on the decision variables are: $0\le x_1\le 1$ (preferred by 1st level DM), $1\le x_2\le 2$ (preferred by 2nd level DM), $0\le x_3\le 0.5$ (preferred by 3rd level DM).

Using FGP model-1, we obtain the solution of the MLMOLFPP as $x_1 = 1, x_2 = 1, x_3 = 0$ and using FGP model-2 we obtain $x_1 = 0.55, x_2 =$

1.8944, $x_3 = 0$. Calculating the values of the objective functions and using Eq. (22), it can be found that FGP model-2 gives better result than FGP model-1. Comparison between FGP model-2 and other existing method is presented in Table 2.

Table 2. Comparison between our proposed method and other existing method

Objective function	FGP model-2	Osman et al. [10]
f_{11}	0.9465	0.9457
f_{12}	1.0567	1.056
f_{21}	0.5255	0.524
f_{22}	1.0567	1.056
f_{31}	0.9401	0.9378
f_{32}	0.7	0.7

It can be observed from Table 2 that our method produces better results than the results obtained in [10]. Also, the method proposed in this article is simple and computationally easy to implement.

6 Conclusion

Two FGP models are presented to solve MLMOLFPP-FP. The membership functions of the objectives of the crisp MLMOLFPP are linearized using first order Taylor series approximation and normalization technique. Compromise solution for each level is obtained and finally the compromise solution of the actual MLMOLFPP is obtained through FGP models. Euclidean distance function helps us to select the best compromise solution. The efficiency and applicability of the method is illustrated through a numerical example.

The proposed approach can be extended to solve multi-objective multi-level linear plus linear fractional programming problem with fuzzy parameters. The proposed concept can be beneficial to solve practical life problems in business organizations, transportation problems, inventory planning, agriculture etc.

Conflict of Interest. The authors declare no conflict of interest.

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A Single Period Fuzzy Production Inventory Control Model with Exponential Time and Stock Dependent Fuzzy Demand

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Abstract. In this paper, we construct a fuzzy inventory system with imperfect production control. Here we introduce the rate of change of inventory level as a fuzzy dynamical equation (FDE). We consider a exponentially time dependent initial demand function and the total demand function is advertisement dependent and depends also on the inventory level of the system. To make the system more realistic we consider the inventory level and the demand functions are fuzzy in nature. Here we use "e" and "g" operator to defuzing the fuzzy dynamical system (FDS) and to test the optimality condition of the system. Here we consider the effect of inflation rate on the objective function which is for economical profit function in maximizing form. In this model, we also consider a function for the carbon emission which is made by several reasons in inventory system. Here we introduce the imperfect rate is reliability and time dependent. In this article, we consider the production cost as reliability dependent. Further, some numerical experiments are performed and graphical representation of the results are also depicted to illustrate the model.

Keywords: Fuzzy dynamical system · Imperfect production · Exponential time and stock dependent demand · Quasi-level-wise system · Inflation rate · Fuzzy carbon emission

1 Introduction

In a production inventory system demand is a very important feature. In so many production systems like electronic goods, fashionable cloths, automobile sector etc. it is not possible to assume the demand of product is constant. In that type of industry it is very much fluctuating in the demand rate. Some times that type

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of products experience a period of rising demand and again some times demand of some products may decline. This type of experience of demand prompted many researchers [7,8,17] to develop with time varying demand pattern. Most of the continuous-time inventory models have been developed considering either linearly increasing/decreasing demand or exponentially increasing/decreasing demand patterns. Hollier and Mak [4] have first considered the exponentially decreasing demand for an inventory model. In recent time so many researchers [7,8,17] have proposed the inventory control model with the exponentially time dependent demand. But now-a-days the demand of a product not only depends on the time, it has also a greater dependency on the advertisement as well as the availability of the product. very recent Khatua et al. [11] developed a inventory model with advertisement and stock dependent demand. In this paper, we first time consider the demand function as a summation of fuzzy exponential time dependent and advertisement and stock dependent fuzzy demand.

In every production sector there are some product produced as imperfect production. Nowhere the all product produced as a perfect product. Imperfect product means some of the defective product will be reworked and the remaining products will be treated as a waste. There are several reasons behind the imperfect production during the production time like machinery fault, labor, quality of the raw materials etc. In this field, there are so many research papers in which this type of production has been discussed [5,13,14,18]. Here we consider the imperfect production rate is dependent on time and reliability of the product.

In recent times, it is not possible to dismiss the effects of rising prices. First time it was used by Buzacott [2] who developed an economic order quantity (EOQ) model with an inflation subject to different types of pricing policies. Many other researchers like Yang et al. [20], Yadav et al. [21], etc. have considered effect of inflation in their model.

Now-a-days, besides economic criteria (cost minimization or profit maximization), the societal impact of a production system for a long period has become a major topic in research and industrial application. The environmental concerns are becoming more and more relevant for firms due to more stringent various rules and regulations on carbon policies imposed by government and growing customer's awareness of the social welfare. A turn of countries have implemented carbon taxes for every unit of carbon emission whatever be the amount- low or high. Some countries have introduced carbon cap and trade scheme for their manufactures. So many researchers [1,3,6,9,10,19,22,23] have been developed their production model with the carbon emission and CE tax. In this paper, we first time introduce a function of carbon emission (CE) which is also fuzzy in nature.

In this paper, we solved the fuzzy differential equations (FDE) with the help of quasi-level-wise system which is first time introduced by Najariyan and Farahi [16] and then developed the system by Mazandarani and Najariyan [15]. After that the method is applied by Khatua et al. [12] to control a fuzzy dynamical system.

So first time we have developed the following methods for the multi item production inventory model:

- We introduced a function of carbon emission(CE) which is also fuzzy in nature.
- We considered the demand function as a summation of fuzzy exponential time dependent and advertisement and stock dependent fuzzy demand.
- We used quasi-level-wise system to control a fuzzy production inventory system with fuzzy carbon emission.

2 Frame

In this paper, we developed a mathematical model of imperfect fuzzy production inventory control model on finite time horizon with exponential time dependent demand. Also in this section we discussed about some assumptions and notations.

2.1 Assumptions

- The system runs in a single period.
- Inflation effect on the system.
- Shortage are not allowed.
- Demand rate is dependent on advertisement as well as stock.
- Initial demand is exponential time dependent.
- Inventory level and demand rate are in fuzzy nature.
- The system runs in finite time horizon.
- the inventory system involves only one item.

2.2 Notations

Variables. $u(t)$: production rate with respect to time t (decision variable);
$\tilde{x}(t)$: fuzzy stock rate with respect to time t;
$\tilde{d}(t)$: fuzzy exponential time dependent demand rate with respect to time t which is also fuzzy stock and advertisement dependent;

Parameters. $C_u(r,t)$: reliability dependent production cost;
$C_d(r)$: reliability dependent development cost;
$\gamma(t) = \delta e^{(1-r)t}$: reliability dependent imperfect rate;
S: selling price;
S_a: salvage price;
r: reliability of product where $0 < r < 1$;
θ: rate of rework where $0 < \theta < 1$;
δ: positive constant where $0 < \delta < 1$;
C_{rw}: raw material cost;
$C_d(r) = M + Le^{k\frac{r-r_{min}}{r_{max}-r}}$: reliability dependent development cost;
C_w: wear-tear cost; α: positive constant where $0 < \alpha < 1$;

P_c: penalty given to Govt. for the carbon emission;
C_{re}: rework cost;
C_v: advertisement cost;
q: catchability rate of advertisement;
v: constant advertisement rate;
A_s: amount of carbon emission for the set-up;
A_u: amount of carbon emission for the production;
A_{re}: amount of carbon emission for rework;
A_{wa}: amount of carbon emission for wastage;
ρ: inflation rate;
L: labour and energy costs;
r_{min}: minimum reliability of product;
r_{max}: maximum reliability of product;

2.3 Model Formulation

Demand is not an absolute term. It depends on so many variables and parameters. On that point is besides one more element for creating a demand function in inventory system. That is the nature of the function. And in this case for making the system more realistic, we adopt the demand function is ambiguous in nature. We put in the logistic regression function in the demand function for incubating the real facts on the arrangement. And then here we construct the fuzzy demand function as follows:

$$\tilde{d}(t) = \tilde{d}_0 \frac{e^t}{1+e^t} + \frac{v}{1+v}\tilde{x}(t) \tag{1}$$

There is the positive effect of the inventory level on the demand function.

The stock level of a production scheme in reality is also depends on the nature. For this ground here we present the inventory level fuzzy in nature as follows:

$$\tilde{\dot{x}}(t) = u(t) - \tilde{d}(t) - (1-\theta)\gamma(t)u(t) \tag{2}$$

In this paper, our production system produced imperfect products. Here the production is made on time and reliability dependent imperfect rate. In this system some portion of imperfect items is made over.

Here, we assume the unit production cost as follows:

$$C_u(r,t) = C_{rw} + \frac{C_d(r)}{u(t)} + C_w(u(t))^{\alpha} \tag{3}$$

Here we find out that in the third term the wear-tear cost proportional to the confirming force of the production rate. This means the increment of the wear-tear cost depends on the increasing of producing items.

In a production firm, carbon is emitted due to set-up, holding of inventory, production, rework and wastage of units. The total quantity of emitted carbon as follows:

$$\tilde{c}_e = A_s + A_u u(t) + A_{re}\theta\gamma(t)u(t) + A_{wa}(1-\theta)\gamma(t)u(t) + A_x\tilde{x}(t) \tag{4}$$

Here emitted carbon function is also fuzzy in nature because of the dependency on the inventory level of the product.

The objective function is then evaluated for these individuals follows:

$$\begin{aligned}\mathrm{Max}\tilde{J}(\tilde{x}(t),\tilde{d}(t),u(t),t) = \int_0^T & e^{-\rho t}[S\tilde{d}(t) - C_u(r,t)u(t) - C_h\tilde{x}(t) - P_c\tilde{c}_e \\ & - C_r\theta\gamma(t)u(t) - C_v e^q v]dt + S_a\tilde{x}(T)\end{aligned} \quad (5)$$

Subject to (2). The goal is to determine the average profit to maximize the objective function. Related inventory system of optimal control of a imperfectness in the quality of production formed the differential equation, to be determined optimal solutions in the form of the average profit so as to maximize the objective function.

3 Optimal Control Policy

We construct the quasi-level-wise system by using "e" and "g" operators (based on [12,15,16]), the Eq. (1) becomes as follows:

$$e(\underline{D}(t) + i\overline{D}(t)) = e(\underline{D}_0 + i\overline{D}_0)\frac{e^t}{1+e^t} + \frac{v}{1+v}(e(\underline{X}(t) + i\overline{X}(t))) \quad (6)$$

where $\underline{x}(t) = min\{\underline{X}(t),\overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t),\overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t),\overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0,\overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0,\overline{D}_0\}$.

Next we separate the equations as follows:

$$\begin{cases}\underline{D}(t) = \underline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\underline{X}(t)\\ \overline{D}(t) = \overline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\overline{X}(t)\end{cases} \quad (7)$$

where $\underline{x}(t) = min\{\underline{X}(t),\overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t),\overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t),\overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0,\overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0,\overline{D}_0\}$

In the similar manner as the system (6), we get the Eq. (2) as follows:

$$e(\dot{\underline{X}}(t) + i\dot{\overline{X}}(t)) = (1-(1-\theta)\delta e^{(1-r)t})u(t) - g(\underline{D}(t) + i\overline{D}(t)) \quad (8)$$

where $\underline{x}(t) = min\{\underline{X}(t),\overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t),\overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t),\overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0,\overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0,\overline{D}_0\}$

After the separation we have,

$$\begin{cases}\dot{\underline{X}}(t) = (1-(1-\theta)\delta e^{(1-r)t})u(t) - \overline{D}(t)\\ \dot{\overline{X}}(t) = (1-(1-\theta)\delta e^{(1-r)t})u(t) - \underline{D}(t)\end{cases} \quad (9)$$

where $\underline{x}(t) = min\{\underline{X}(t),\overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t),\overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t),\overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0,\overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0,\overline{D}_0\}$ Using the Eq. (7) we get,

$$\begin{cases}\dot{\underline{X}}(t) = (1-(1-\theta)\delta e^{(1-r)t})u(t) - (\overline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\overline{X}(t))\\ \dot{\overline{X}}(t) = (1-(1-\theta)\delta e^{(1-r)t})u(t) - (\underline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\underline{X}(t))\end{cases} \quad (10)$$

where $\underline{x}(t) = min\{\underline{X}(t), \overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t), \overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t), \overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0, \overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0, \overline{D}_0\}$

The quasi-level-wise system of Eq. (4), as follows:

$$e(\underline{C}_e + \overline{C}_e) = A_s + A_u u(t) + A_{re}\theta\gamma(t)u(t) + A_{wa}(1-\theta)\gamma(t)u(t) + A_x e(\underline{X}(t)) + i\overline{X}(t))) \quad (11)$$

where $\underline{x}(t) = min\{\underline{X}(t), \overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t), \overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t), \overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0, \overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0, \overline{D}_0\}$

Separating the system - (11), we get

$$\begin{cases} \underline{C}_e = A_s + A_u u(t) + A_{re}\theta\gamma(t)u(t) + A_{wa}(1-\theta)\gamma(t)u(t) + A_x\underline{X}(t)) \\ \overline{C}_e = A_s + A_u u(t) + A_{re}\theta\gamma(t)u(t) + A_{wa}(1-\theta)\gamma(t)u(t) + A_x\overline{X}(t)) \end{cases} \quad (12)$$

where $\underline{x}(t) = min\{\underline{X}(t), \overline{X}(t)\}, \overline{x}(t) = max\{\underline{X}(t), \overline{X}(t)\}$ and $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\}, \overline{d}(t) = max\{\underline{D}(t), \overline{D}(t)\}$, $\underline{d}_0 = min\{\underline{D}_0, \overline{D}_0\}, \overline{d}_0 = max\{\underline{D}_0, \overline{D}_0\}$

Now we construct the Hamiltonian of the given system as follows:

$$\begin{aligned} &H(\underline{D}(t), \overline{D}(t), \underline{X}(t), \overline{X}(t), u(t), t) = S(e(\underline{D}(t) + i\overline{D}(t))) \quad (13) \\ &-C_h(g(\underline{X}(t) + i\overline{X}(t))) - C_u(r,t)u(t) - P_c(g(\underline{C}_e + i\overline{C}_e)) - C_{re}\theta\delta e^{(1-r)t}u(t) - C_v e^q v \\ &+\phi(t)((1-(1-\theta)\delta e^{(1-r)t})u(t) - (g(\underline{D}(t) + i\overline{D}(t)))) \end{aligned}$$

Now using the Eqs. (7), (3), (12), the Eq. (13) becomes as follows:

$$\begin{aligned} &H(\underline{X}(t), \overline{X}(t), u(t), t) = S\Big[((\underline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\underline{X}(t)) \quad (14) \\ &+ i(\overline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\overline{X}(t))) - C_h(\overline{X}(t) + i\underline{X}(t)) - (C_{rw} + \frac{C_d(r)}{u(t)} + C_w(u(t))^\alpha)u(t) \\ &- P_c((A_s + A_u u(t) + A_{re}\theta\gamma(t)u(t) + A_{wa}(1-\theta)\gamma(t)u(t) + A_x\overline{X}(t)) \\ &+ i(A_s + A_u u(t) + A_{re}\theta\gamma(t)u(t) + A_{wa}(1-\theta)\gamma(t)u(t) + A_x\underline{X}(t))) - C_{re}\theta\delta e^{(1-r)t}u(t) \\ &- C_v e^q v + \phi(t)((1-(1-\theta)\delta e^{(1-r)t})u(t) - ((\overline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\overline{X}(t)) \\ &+ i(\underline{D}_0\frac{e^t}{1+e^t} + \frac{v}{1+v}\underline{X}(t)))\Big] \end{aligned}$$

Using the Pontryagin's maximal principle, we know that

$$\begin{aligned} &\frac{\partial H}{\partial u(t)} = 0 \\ &\Rightarrow -C_{rw} - C_w(\alpha+1)(u(t))^\alpha - P_c(A_u + A_{re}\theta\delta e^{(1-r)t} + A_{wa}(1-\theta)\delta e^{(1-r)t}) \\ &\quad - C_{re}\theta\delta e^{(1-r)t} + \phi(t)(1-(1-\theta)\delta e^{(1-r)t}) = 0 \\ &\Rightarrow u(t) = \Bigg[\frac{1}{C_w(1+\alpha)}\Bigg(\phi(t)(1-(1-\theta)\delta e^{(1-r)t}) - (C_{rw} + P_c(A_u \\ &\quad + A_{re}\theta\delta e^{(1-r)t} + A_{wa}(1-\theta)\delta e^{(1-r)t}) + C_{re}\theta\delta e^{(1-r)t})\Bigg)\Bigg]^{1/\alpha} \quad (15) \end{aligned}$$

And

$$\dot{\phi}(t) = \rho\phi(t) - \left(\frac{\partial H}{\partial \overline{X}(t)} + \frac{\partial H}{\partial \underline{X}(t)}\right)$$
$$= \rho\phi(t) - (-2C_h - 2P_cA_x - \frac{2v}{1+v}\phi(t)) \tag{16}$$

Therefore we get the differential equation as

$$\dot{\phi}(t) - (\rho + 2\frac{v}{1+v})\phi(t) = 2(C_h + P_cA_x) \text{ with } \phi(T) = 0 \tag{17}$$

Solving the Eq. (17) we get,

$$\phi(t) = \frac{2}{k}(C_h + P_cA_x)[e^{-k(T-t)} - 1] \text{ where } k = \rho + \frac{2v}{1+v} \tag{18}$$

Putting the value of this $\phi(t)$ in the Eq. (15), we will get the analytic expression of $u(t)$ which is optimum production. And for the input data(given in the next section) we will get the optimum value of stock level and demand of items numerically and graphically.

4 Numerical Results

In this section, we give optimal values of variables and the objective functions corresponding the numerical values of parameters given in Table 1. Also here we also give the optimal results of the variables in Table 2.

So using the values of Table 1, we get the values of variables as follows: $\underline{X}(t) = 1403.6, \overline{X}(t) = 665.5, \underline{D}(t) = 473.3, \overline{D}(t) = 227.2$. But from the properties of Quasi-level-wise system, we have the optimal values of the inventory level are $\underline{x}(t) = min\{\underline{X}(t), \overline{X}(t)\} = 665.5$ and $\overline{x}(t) = max\{\underline{X}(t), \overline{X}(t)\} = 1403.6$ and optimal values of demand of the product are $\underline{d}(t) = min\{\underline{D}(t), \overline{D}(t)\} = 227.2$ and $\overline{d}(t) = max\{\underline{D}(t), \overline{D}(t)\} = 473.3$. From these values we conclude that the optimal value of inventory level($x(t)$) lies in the interval $[665.5, 1403.6]$ and the

Table 1. Values of Parameters

Parameters	Values	Parameters	Values	Parameters	Values	Parameters	Values
ρ	0.35	α	0.5	β	0.7	δ	0.03
r_{min}	0.8	r_{max}	1	r	0.91	v	0.5
C_{rw}	2	C_w	0.3	C_{re}	2	C_h	1
C_v	2.5	P_c	0.6	C_m	50	C_l	0.7
A_s	1.5	A_u	0.5	A_{re}	0.25	A_{wa}	0.1
A_x	0.2	k_1	0.02	S	75	S_a	50
$\underline{d}_0$	5.5	$\overline{d}_0$	7.5	q	0.65		

Table 2. Optimal values of variables

T	1	3	5	7	9	11	15
$\underline{x}^*(t)$	85.4	246.6	358.0	442.6	515.0	577.9	614.1
$\overline{x}^*(t)$	86.9	254.3	378.6	488.3	609.7	768.1	1352.3
$\underline{d}^*(t)$	32.5	87.5	124.8	153.0	177.2	198.1	210.2
$\overline{d}^*(t)$	33.0	90.0	131.7	168.3	208.7	261.5	456.26
$u^*(t)$	186.9	207	231	265.3	306.8	358.3	330.3

optimal value of demand of the product $(d(t))$ lies in the interval $[227.2, 473.3]$. Optimal value of the production rate is 330. Corresponding the above optimal values of the variables we get an interval of the objective function *i.e* for the profit function which is $[\$650.12, \$38865.6]$.

5 Graphical Discussion

Here we discuss the results of the system corresponding the numerical values (Given in Table 1) graphically.

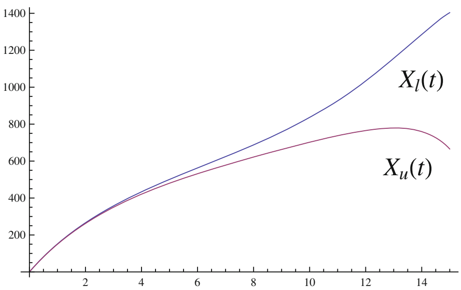

Fig. 1. Inventory level with respect to time t

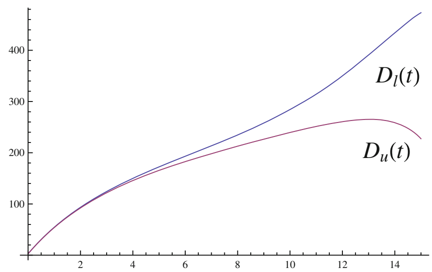

Fig. 2. Demand rate with respect to time t

In Fig. 1 a proper explanation regarding inventory level with respect to time has been found. The graph includes two main function one with blue color and other with red. These higher and lower estimated plot is due to fuzzy algorithm. The area encapsulated between the two desired curve is the original active region with which fuzzy system will take the decisions as per concerned rules set. The Fig. 2 indicates the actual values of demand rate that a fuzzy system will decide with respect to time. The higher plotting is with blue curve and the lower demand aspect is the red one. Same ways the Fig. 3 also ensures the actual prospects of production process with respect to time indeed. But the curve indicates that with an increase in inventory level demand rises then simultaneously production

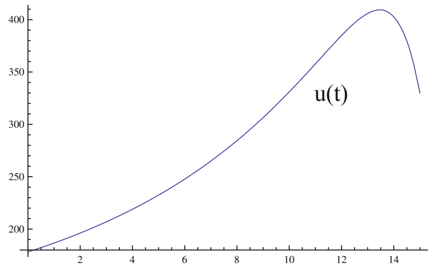

Fig. 3. Production rate with respect to time t

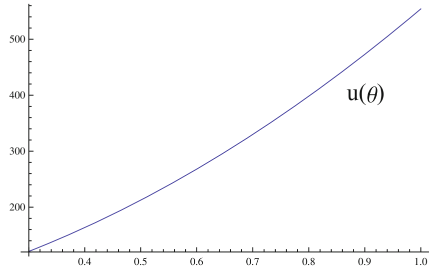

Fig. 4. Production rate with respect to θ

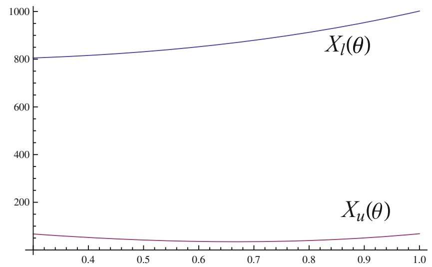

Fig. 5. Inventory level with respect to θ

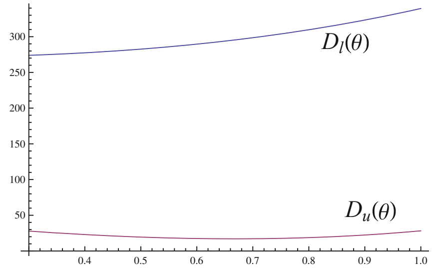

Fig. 6. Demand rate with respect to θ

also increases with respect to time to meet the entire need. After a certain period of time production falls due to certain issues. These discussion is only possible by exploring an example from real world system (Figs. 4, 5 and 6).

Here we also discussed about the production rate, stock rate and the demand rate of the product with respect to the percentage of rework. And here we see that, with the increment of rework on the imperfect production, all the variables are gradually increased.

Now these three parameters setup nowadays is very common among various automobile industries. In practical scenario let us take an example of motorcycle. Now with built in perception based incorporated inventory level with the up gradation of designs and mileage in any vehicles normally customers will move on to the system for buying it and thus demand will increase which will lead it's production process. But after a certain period of time if industries will think of launching new model then suddenly it will put a bend in the curve of it's production. Again if customers will think of some changes in the system then also it will effect the entire production process as well.

6 Conclusion

In this study, we control the production of a fuzzy production inventory system where the demand function is exponentially time dependent as well as stock dependent. Reason behind the assumption of the demand function is the fluctuation of the demand of the product with respect the time. Considering the state variables as fuzzy the proposed inventory model is represented as a fuzzy differential system. To transform the system into equivalent deterministic one we use the "e" and "g" operators which is very recent and effective technique with respect to interval mathematics. Moreover, the proposed method can be extended for Here we given the range of the optimal inventory level and the demand of the product. Also given the profit value in an interval. In reality this model appropriate for the production system like garments, automobiles company. In that type of company they produce the imperfect items and the demand of the items always fluctuated with the time.

The proposed model can be extended in type-2 fuzzy environment.

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An Approach to Develop a Dedicated Micro AI Processor for an Intelligent Fault Protection Scheme

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Abstract. In Electrical Engineering the term protection is a vital issue since long time ago. The numerous numbers of papers have been found on application of intelligent system for different protection schemes. Many of the papers have been found on simulation results only. In some cases it has been found that to interface with the real world input a PC is mandatory in case of intelligent system operation. In this paper a discussion has been drawn in favour of developing one intelligent breaker under MATLAB environment. The same has been programmed under Arduino IDE for designing the corresponding micro processor to test its outcome for authenticating the system with a simple processor in place of huge setup followed by incorporating PCs or Laptops nowadays.

Keywords: Fuzzy logic controller · Arduino · Intelligent breaker · Intelligent system · MATLAB

1 Introduction

The intelligent protection scheme has been carried out since long time before. Protection ideas are provoked by performing various simulation results. Some recent works on intelligent circuit breaker are discussed as follows. Fast current protection algorithm for intelligent high voltage circuit breaker [14] have been proposed by introducing adaptive algorithm that can change the step size of the length of changeable window in the filtering algorithm. Transient stability enhancement by fuzzy logic-controlled SMES considering coordination with optimal reclosing of circuit breakers [2] has been explained with the control scheme

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of SMES which is based on a pulse width modulation (PWM) voltage source converter (VSC) and a two-quadrant dc-dc chopper using gate-turn-off (GTO) thyristor. Investigation on controlling techniques of moving contact behaviours for vacuum circuit breaker based on fuzzy control [12] has been implemented by introducing adaptive fuzzy controlling algorithm. An intelligent control approach has been discussed for designing a low voltage DC Breaker [1] using solid-state elements as its power components and an embedded processor for implementing control algorithms that command triggering of thyristor. Development of historical database for SF6 extra high voltage circuit breaker with intelligent monitoring system [13] by incorporating a simple device expert database has been developed to assist fault detection. The SQLite, a lightweight relational database is adopted as the backend database in the historical database system. Fuzzy theory application in the evaluation of high voltage circuit breaker state [1] is based on a weight method to deal with the sudden change of the state. Other work like automated monitoring and analysis of circuit breaker operation [10] is done based on a record of waveforms taken from the circuit breaker control circuit by using a portable recorder and manually forcing an operation of the breaker. Distributed load control using FLC based MCB [4] is designed to ensure both over current protection and low voltage protection as well to any electrical loads by introducing four priority based ports. Here both inputs and outputs are taken as four in numbers. Fuzzy control of a novel Magnetic Force Actuator in 40.5 kV $SF6$ Circuit Breaker [3] has been explored with the dynamic characteristics of the magnetic force actuator which is solved by combining the four-order Runge-kutta and finite element methods together. A novel fuzzy control of the intensity of head lights for night driving [5] using optimal illumination in the event of different road and weather conditions. Application of fuzzy decision is developed for city illumination systems [12] based on microprocessor with a new intelligent control scheme. Genetic Algorithm Based Design of Fuzzy Logic Power System Stabilizers in Multimachine Power System [7] has been proposed with fuzzy expert system. Generator speed deviation and acceleration are chosen as input signals to fuzzy logic power system stabilizer. An intelligent under frequency load shedding scheme has been implemented for islanded distribution network [11] by incorporating mini-hydro operating system in islanded mode. The proposed under frequency load shedding scheme consists of a fuzzy logic load shedding controller (FLLSC) with load shedding controller module (LSCM). In this paper a discussion is made on the best way to introduce intelligent processor for designing an intelligent breaker against over current fault protection scheme. The entire coding has been carried out under Arduino environment and implemented using UNO and Duemilanove as well. This work shows that the simulation scope outcome is very close to the microcontroller based AI processor incorporated intelligent breaker. The signal is captured using MATLAB based DSO developed using M File coding.

2 Methodology

In the proposed methodology a brief discussion has been made on block diagram, soft computing technique, hardware module development, Matlab Based Plotting. The entire work has been carried out to understand the output characteristic of any fuzzy based system by introducing a single dedicated micro processor.

2.1 Block Diagram

In this system FIS is developed under MATLAB environment. Concerned FIS is then converted into equivalent microcontroller accessible code. The program is designed by introducing Arduino sketch. In Fig. 1 it has been shown what exactly is happening within the system. Microcontroller is the hardware module but it is loaded with a program based on fuzzy algorithm. The concerned algorithm is initially developed under MATLAB environment by designing one FIS with certain rules set within the FIS periphery. Successful compilation of the sketch confirmed the probability for uploading the FIS into the concerned ATMEL chip.

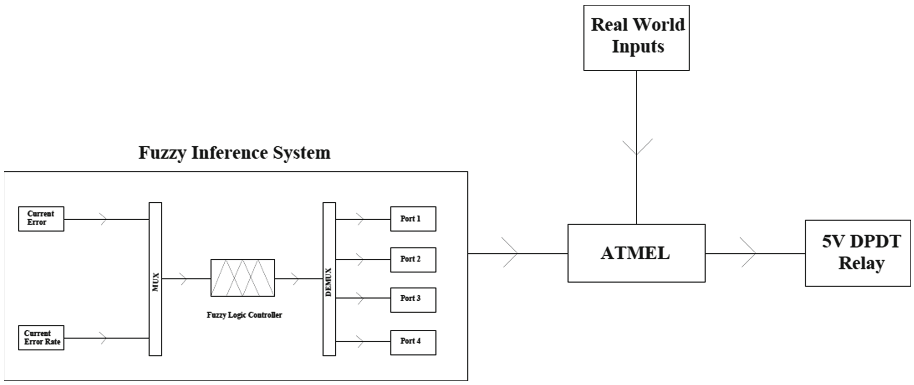

Fig. 1. Block diagram of intelligent breaker.

In this paper a new trend of breaker has been introduced to promote a new generation of protection scheme by providing over current protection. Here the controller is designed to work on the basis of two inputs belonging to current domain. The inputs are error and error rate of the current. During real time data analysis currents are fed to the system using differentiator and recorded using a scope designed using MATLAB.

2.2 Software Implementation

In this work for designing the software portion initially MATLAB is introduced for developing the concerned FIS for a particular FIS based MCB.

Simulation block is also confirmed under this environment. As it has been developed under MATLAB environment now the possible or supported hardware module selected here in this work is Microcontroller which has been discussed under the next immediate section. To run or set this program one library has been developed under Arduino environment. To obtain the plot a script file is developed under MATLAB environment. The purpose of developing script file is to capture the waveform characteristic of the hardware module. Here the purpose of developing this MATLAB based scope is to capture one shot only. The model shown in Fig. 2 describes the actual setup of the intelligent breaker designed under Matlab environment. The switching device used here is IGBT. The external signal fed to the concerned switch is from a FLC black box developed under Matlab platform.

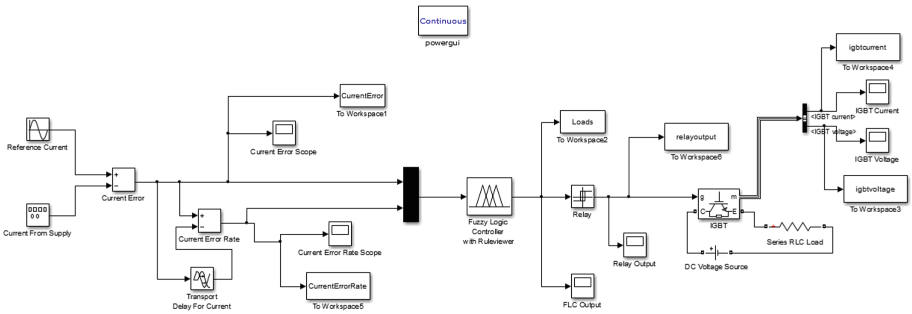

Fig. 2. Model developed under Matlab environment.

2.3 Hardware Module Development

Under this section discussion has been drawn about the hardware module that has been followed to obtain the outputs after defuzzyfication. Initially introducing a handmade microcontroller board with 8 MHz internal clock by implementing ATMEL chip AT mega 328 - PU. As this clock speed is not enough to control the entire AI process and that is why later decided to change the clock speed by adding one external crystal of 16 MHz. But the new clock speed is not possible to implement externally in the same board with 8 MHz internal clock. Another new AT mega 328P – PU has been introduced to boot load by initially connecting an oscillator of 16 MHz to the board. Now the problem is with the interfacing or downloading processes by using MATLAB environment. Recommended option for hardware module is taken into account by introducing AT mega 328P – PU. The purpose of taking AT mega 328P – PU as hardware module is for its low power consumption and compatibility with MATLAB hardware support package installed within the system availability. For communication purpose in place of Future Technology Devices International (FTDI) connector an alternate USB Transistor Transistor Logic (TTL) serial connector has been followed. The advantage of selecting six pin configured USB TTL connector is only for its male to female USB to USB connecting cable. These cables are enough hard and needs less care. Whereas FTDI bears RS – 232 port with nine pin serial data transferring cable. This type of cable is not preferable as its need more care or otherwise soldered portion of its concerned pins might introduce problem by intruding troubleshooting. The Fig. 3 shows a clear picture of entire hardware setup of the concerned work with which a dedicated micro AI processor developed for designing an intelligent breaker. In this picture board A is designed with the scope using Matlab as plotter. And board B is designed with the entire Fuzzy algorithm based intelligent breaker for controlling the tripping mechanism against over current fault.

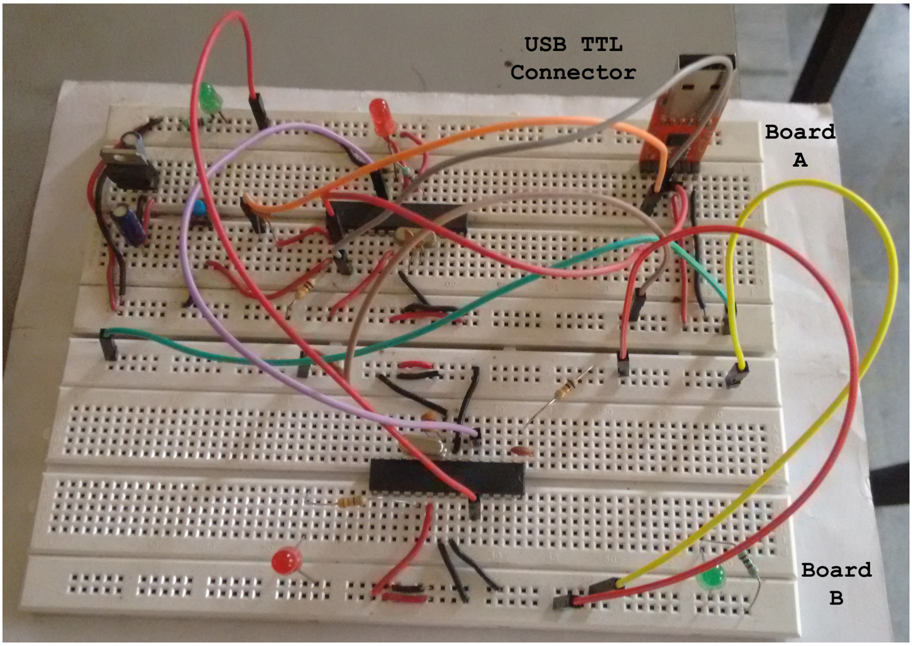

Fig. 3. Hardware setup.

2.4 Matlab Based Plotting

In this work microcontroller has been introduced from ATMEL group for setting up a hardware module. A program has been developed under Arduino platform to transfer the captured signal to the computer via serial communication. Thus the concerned program is loaded into the microcontroller board. And the input is taken as analog pin A0 fed from the output pin of another microcontroller board loaded with FIS based breaker sketch. Under MATLAB environment another M file has been introduced with the one shot capturing features compatibility. In this code file handling concept has been introduced to read the Arduino signal by MATLAB environment via the serial communication module. Here the communication module has been followed is USB TTL connector. In Fig. 4 it is clearly showing crisp output voltage with blue lines. For digital voltage axis actually 0 to 1023 bits are taken as axis levels. The serial port axis is marked as per counts received via serial communication. The led connected to the USB TTL connector blinks very fast and independent of any changes of the system. This is happening because of baud rate set here is 9600 which in turn transferring data of 10 bits at a time as 8 bits along with one start bit and stop bit as well. For serial communication at 9600 baud rate one bit initially takes 104 μ sec to transfer.

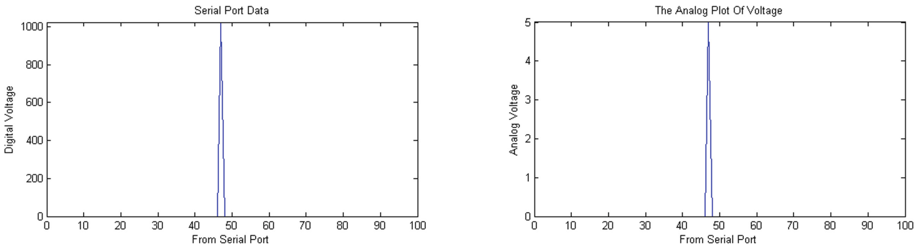

Fig. 4. Matlab plotting of output from Hardware Module.

3 Results and Discussion

The entire work has been carried out to analyse the output of the hardware module for confirming the future of AI processor based intelligent breaker in protection system. The scope result shown in Fig. 4 is obtained from Matlab simulation. A scope is designed for obtaining waveform of the hardware module and thus compared with the first one. The waveform obtained is almost same as that of Matlab result. The waveform obtained under Matlab environment is like a triangular spike. This spike is visible when there is a noise in the system but still as the system is not stable thus it is not confirmed to state that whether the developed system will be able to stand against fault currents. The stability of the system checked by running the program under Arduino environment. Uncertain blinking of the Leds ensured that the system is not stable. Initially the hardware failed to respond as it was not able to upload concerned programme. But after final compilation and run system gives some response and finally captured using the scope for an analysis of its desired waveform. The plotting of desired datas obtained from the simulation for load is given in Fig. 5. This figure shows some percentage of loads tripped due to high current intruded into the system. These results are obtained due to some sudden decision taken by the black box designed and incorporated within the hardware module.

The regression plot shows in Fig. 6 the proper best fit curve obtained after the scanning of datas under Matlab platform using scatter command. Since the

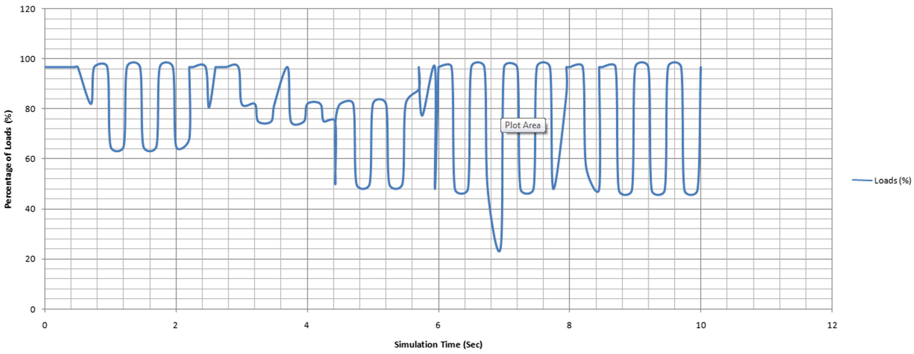

Fig. 5. Load Curves obtained from Matlab.

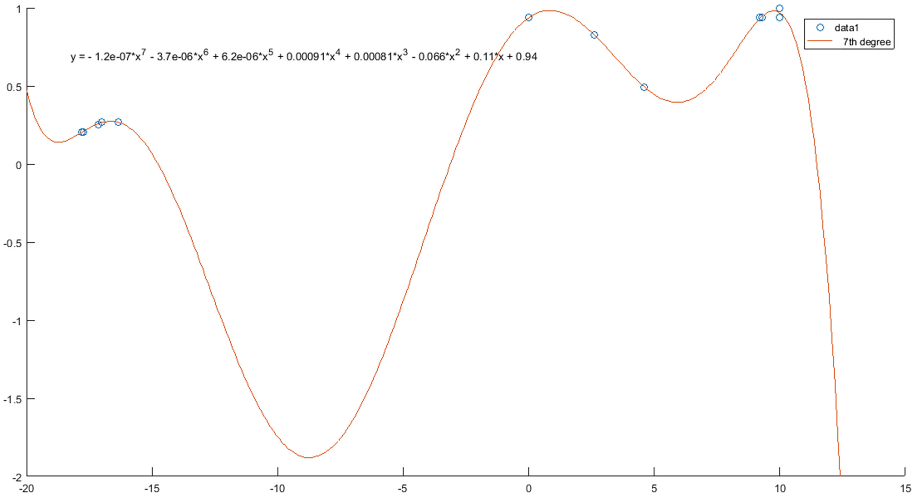

Fig. 6. Best fitting curves.

coefficient of 7th, 6th and 5th degree variables are almost tends to zero so the required polynomial equation obtained is a 4th degree equation as:

$$y = 0.00091x^4 + 0.00081x^3 - 0.066x^2 + 0.11x + 0.94$$

.

4 Conclusion

In this work a discussion has been drawn on how to develop a hardware module based on Fuzzy algorithm. The proper analysis has been done by using scope with respect to simulation result. It is very much evident that this paper ensures that it is possible to design a dedicated processor for certain objectives working under Fuzzy algorithm. This paper only described the way of developing Fuzzy system using a kind of hardware module to develop one Intelligent system using Fuzzy Logic Controller. But the response is only taken into account to verify the probability of developing one dedicated hardware module with AI processor which might bring some changes in Engineering sections and some other section of science as well. Experimental results validates the concerned idea of developing Fuzzy based protection relaying in Electrical Engineering domain.

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Two-Echelon Supply Chain Model in an Imperfect Production with Stochastic Demand Considering the Rework of the Defective Items

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Abstract. This article presents a two-echelon supply chain model consisting of a single manufacturer and a single retailer. Here, the manufacturer's production system is imperfect and produces a certain proportion of defective products with the perfect products. The manufacturer starts reworking the defective products at the end of the main production. It is assumed that the customer's demand to the retailer is stochastic in nature and the retailer's demand to the manufacturer depends on the customer's demand for the product to the retailer. Finally, we have derived a cost function of the system. Numerical examples have been presented to clarify the applicability of the proposed model and sensitivity analysis has been presented to study the effect of the change of the parameters on the optimal decision variables.

Keywords: Supply chain · Imperfect production · Stochastic demand · Rework

1 Introduction

The customers' demand for any product is the most crucial factor in any inventory management system. When reviewing the past literature, we observe that many inventory models have been formulated assuming the demand as constant [1, 2]. But, in the day to day dealing it is noticed that the demand for any physical goods hardly remains constant, we cannot estimate the demand pattern for any product in prior. To deal with such situation researchers developed supply chain models considering the demand as stochastic in nature. Mateen, Chatterjee and Mitra [3] presented a single vendor and multiple retailer vendor-managed model considering the customers demand as stochastic. Govindan [4] considers time-dependent stochastic demand while developing inventory model. In this model, they compared the performance of the traditional inventory system with the vendor managed inventory system and showed the difference between this two models.

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Sadeghian, Langaroudi and Zoraghi [5] developed a three-stage supply chain model where the demand is probabilistic and follows the normal density function. Guo and Li [6] derived an integrated multi-echelon supply chain model considering the stochastic demand. Their aim was to select suppliers and allocating orders among them so as to maximize the total profit.

The most common problem associated with any production system is the imperfect production. When manufacturing of the products is going on in the factory, we observed that a fraction of the items produced are defectives. So, the imperfect production inventory system is the area of interest to many researchers, e.g. Chiu, Gong and Wee [13], Pal, Sana and Chaudhuri [12], Pal and Mahapatra [11], Pal, Mahapatra and Samanta [10], Rosenblatt and Lee [9], Sana [8], Sarkar, Sana and Chaudhuri [14] and Taheri-Tolgari, Mirzazadeh and Jolai [7].

Again, many researchers have considered the rework of the defective products while developing their models, among them the work done by [15–19] are worth mentioning.

In this article, we have presented a pricing model in a two-echelon supply chain consisting of a single manufacturer, and a single retailer. Due to the imperfect production system, the manufacturer produces an admixture of the perfect and the imperfect products. He reworks all the defective products to make these perfect and sells the perfect products to the retailer. Here, we have assumed that the retailer's demand for the product to the manufacturer depends on the customers' demand to the retailer. Since, we have considered the customers demand for the product to the retailer as stochastic hence the retailer's demand to the manufacturer is also stochastic.

The rest of this paper has been sequenced in the following order. In the Sect. 2 we have given the notations used to develop the model and also the assumptions based on which we have formulated the model. In the Sect. 3, we have formulated the mathematical model. We have discussed the solution methodology in Sect. 4. We have presented some numerical examples in Sect. 5 to clarify the applicability of this model and in Sect. 6 we have presented the sensitivity analysis to show the effect of the parameters on the optimum decision variables. In the Sect. 7 we have drawn an overall conclusion.

2 Notations and Assumptions

Notations

$I_{1m}(t)$	Inventory level of perfect items at time t of the manufacturer
$I_{1r}(t)$	Inventory level at time t of the retailer
x	Demand rate of the customer to the retailer, which is stochastic in nature with probability density function (x)
D_r	Demand rate of the retailer to the manufacturer, which is dependent on the customer's demand to the retailer i.e. $D_r = kx$, k is a constant
p_m	Production rate of the manufacturer
p_1	Reworking rate of the defective items of the manufacturer
A_m	The set-up cost of the manufacturer
A_r	The ordering cost of the retailer
h_m	The holding cost per unit perfect item per unit time of the manufacturer

h_r	The holding cost per unit item per unit time of the retailer
C_m	Inspection cost per unit item of the manufacturer
C_p	Production cost per unit item of the manufacturer
r_m	Reworking cost per unit defective items of the manufacturer
S_m	Selling price per unit item of the manufacturer, i.e. purchasing cost per unit item of the retailer
T	Cycle time of the supply chain
TC_m	Total cost of the manufacturer
TC_r	Total cost of the retailer
TC	Total cost of the entire supply chain system

Assumptions
The model is based on the following assumptions –

(i) The demand rate of the retailer to the manufacturer depends on the demand of the customer to the retailer. Here the customers' demand to the retailer is x, which is a random variable. So, the retailer's demand to the manufacturer has been taken as kx where k is a constant.

(ii) In practice, it is observed that the demand for certain products decreases gradually with time. In the day to day dealing we observe that when the customers buy a product, they always find for the better options in the same price range. In today's highly competitive market various companies take that opportunity and try to make the products better than the previous models of these products. So, when a product launches in the market, we notice that initially, the demand for the product is high, but it decreases gradually with time due to the arrival of the new model of these products in the market. For instance, the demand for certain model the mobile phone is noticed to follow a decreasing trend with time. Keeping in mind this fact, the probability density function $f(x)$ of the demand x has been taken as an exponential distribution function as -

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & elsewhere \end{cases}$$

Where λ is the parameter of the distribution.

(iii) The manufacturer starts reworking the defective items after the termination of the main production process and reworks all the defective products before reaching the inventory level of the good quality items at the zero level, i.e. $t_1 \leq t_2 \leq t_3$.

3 Mathematical Model

In this section we have formulated the mathematical model for both the manufacturer and the retailer. A pictorial representation of this model is depicted in Fig. 1.

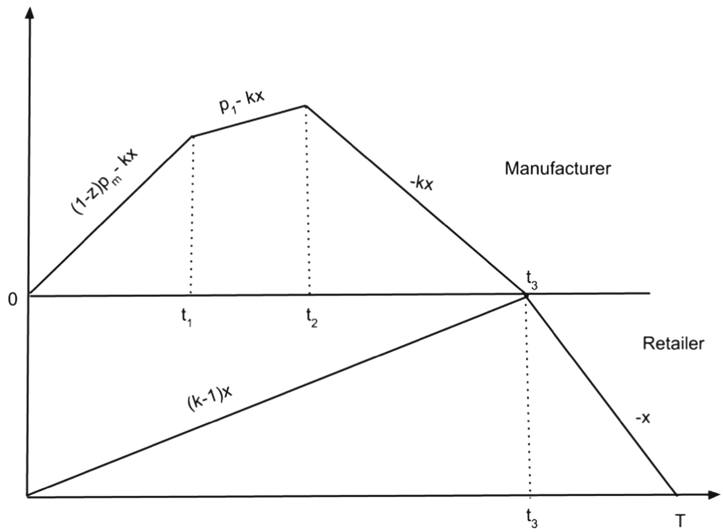

Fig. 1. A pictorial representation of this supply chain model.

3.1 Manufacturer's Model

The manufacturer starts producing the items at a rate p_m at time $t = 0$ and continues up to time $t = t_1$. Due to the imperfect production process, it produces a certain proportion of defective products with the perfect products. He starts reworking the defective products at the end of the main production process at a rate p_1 and finishes the reworking process at $t = t_2$. The inventory level reaches to the zero level at time $t = t_3$. The inventory level of the manufacturer has been described by the following differential equations –

$$\frac{dI_{1m}(t)}{dt} = \begin{cases} (1-z)p_m - kx, & 0 \le t \le t_1 \\ p_1 - kx, & t_1 \le t \le t_2 \\ -kx, & t_2 \le t \le t_3 \end{cases} \tag{1}$$

With the boundary conditions –

$$I_{1m}(0) = 0, I_{1m}(t_1) = \{(1-z)p_m - kx\}t_1, and\ I_{1m}(t_3) = 0$$

Here,

$$\begin{aligned} zp_m t_1 &= p_1(t_2 - t_1) \\ \Rightarrow t_2 &= \frac{(zp_m + p_1)t_1}{p_1} \end{aligned} \tag{2}$$

Now, solving (1) we have –

$$I_{1m}(t) = \begin{cases} \{(1-z)p_m - kx\}t, & 0 \le t \le t_1 \\ (p_1 - kx)t + \{(1-z)p_m - p_1\}t_1, & t_1 \le t \le t_2 \\ -kx(t - t_3), & t_2 \le t \le t_3 \end{cases} \tag{3}$$

From the continuity condition of $I_{1m}(t) at\ t = t_2$ we have –

$$\begin{gathered}(p_1 - kx)t_2 + \{(1-z)p_m - p_1\}t_1 = -kx(t_2 - t_3)\\ \Rightarrow p_1 t_2 + +\{(1-z)p_m - p_1\}t_1 = kxt_3\\ \Rightarrow t_3 = \tfrac{p_m t_1}{kx}(using(2))\\ \Rightarrow t_3 = \tfrac{\lambda p_m t_1}{k}\end{gathered} \tag{4}$$

using the expected value of demand $x = \frac{1}{\lambda}$.
Now, set up cost of the manufacturer $= A_m$
Production cost $= C_p p_m t_1$
Reworking cost $= r_m p_1(t_2 - t_1) = r_m z p_m t_1$ (using (2))
Holding cost –

$$\begin{aligned}
&= h_m \int_0^\infty \left[\int_0^{t_1} I_{1m}(t)dt + \int_{t_1}^{t_2} I_{1m}(t)dt + \int_{t_2}^{t_3} I_{1m}(t)dt\right] f(x)dx\\
&= h_m \int_0^\infty \left[\int_0^{t_1}\{(1-z)p_m - kx\}tdt\right.\\
&\quad \left. + \int_{t_1}^{t_2}\{(p_1 - kx)t + \{(1-z)p_m - p_1\}t_1\}dt + \int_{t_2}^{t_3} -kx(t - t_3)dt\right] f(x)dx\\
&= \frac{h_m}{2}\int_0^\infty \left[\{(1-z)p_m - kx\}t_1^2 + (p_1 - kx)\left(t_2^2 - t_1^2\right) + 2\{(1-z)p_m - p_1\}t_1(t_2 - t_1)\right.\\
&\quad \left. + kx(t_2 - t_3)^2\}\right] f(x)dx\\
&= \frac{h_m}{2}\left[(1-z)p_m\left(2t_1t_2 - t_1^2\right) + p_1(t_2 - t_1)^2 + \frac{k}{\lambda}t_3(t_3 - 2t_2)\right]\\
&= \frac{h_m p_m}{2p_1}\left[(1-z)(2zp_m - p_1) + z^2 p_m + \frac{\lambda p_m p_1}{k} - 2(zp_m + p_1)\right]t_1^2
\end{aligned}$$

Therefore, total cost of the manufacturer –

$$\begin{aligned}TC_m = A_m &+ C_p p_m t_1 + r_m z p_m t_1\\ &+ \frac{h_m p_m}{2p_1}\left[(1-z)(2zp_m - p_1) + z^2 p_m + \frac{\lambda p_m p_1}{k}\right.\\ &-2(zp_m + p_1)]t_1^2\end{aligned} \tag{5}$$

3.2 Retailer's Model

The retailer starts receiving the items at time $t = 0$ and continues up to time $t = t_3$. The retailer's inventory level depletes due to customer's demand and reaches to the zero level at time $t = T$. The retailer's inventory level is described by the following differential equations –

$$\frac{dI_{1r}(t)}{dt} = \begin{cases}(k-1)x, & 0 \le t \le t_3\\ -x, & t_3 \le t \le T\end{cases} \tag{6}$$

With boundary conditions $I_{1r}(0) = 0$, and $I_{1r}(T) = 0$.

Solving (6) we have –

$$I_{1r}(t) = \begin{cases} (k-1)xt, & 0 \le t \le t_3 \\ -x(t-T), & t_3 \le t \le T \end{cases} \tag{7}$$

Now, from the continuity condition of $I_{1r}(t) at\ t = t_3$ we have –

$$(k-1)xt_3 = -x(t_3 - T)$$
$$\Rightarrow T = kt_3 = \frac{p_m t_1}{x} = \lambda p_m t_1 \tag{8}$$

(using the expected value of x, i.e. $\frac{1}{\lambda}$)
Ordering cost $= A_r$
Purchasing cost $= S_m p_m t_1$

Holding cost –

$$= h_r \int_0^\infty \left[\int_0^{t_3} I_{1r}(t)dt + \int_{t_3}^T I_{1r}(t)dt\right] f(x)dx$$
$$= h_r \int_0^\infty \left[\int_0^{t_3} (k-1)xtdt + \int_{t_3}^T -x(t-T)dt\right] f(x)dx$$
$$= \frac{h_r}{2} \int_0^\infty \left[(k-1)xt_3^2 + x(t_3 - T)^2\right] f(x)dx$$
$$= \frac{h_r}{2\lambda}\left[(k-1)t_3^2 + (t_3 - T)^2\right]$$

(using the expected value of x, i.e. $\frac{1}{\lambda}$)

$$= \frac{h_r}{2\lambda}\left[(k-1)t_3^2 + (k-1)^2 t_3^2\right]$$

(using (8))

$$= \frac{h_r}{2k}(k-1)\lambda p_m^2 t_1^2$$

(using (8))

Therefore, total cost of the retailer –

$$TC_r = A_r + S_m p_m t_1 + \frac{h_r}{2k}(k-1)\lambda p_m^2 t_1^2 \tag{9}$$

Therefore, total cost of the system –

$$TC = \frac{1}{T}[TC_m + TC_r]$$
$$\Rightarrow TC = \frac{1}{T}\Bigg[(A_m + A_r) + \left(S_m + C_p + r_m z\right)p_m t_1 + \frac{h_r}{2k}(k-1)\lambda p_m^2 t_1^2 + \frac{h_m p_m}{2p_1}\left\{(1-z)(2zp_m - p_1) + z^2 p_m + \frac{\lambda p_m p_1}{k} - 2(zp_m + p_1)\right\}t_1^2\Bigg]$$
$$\Rightarrow TC = \frac{(A_m + A_r)}{\lambda p_m t_1} + \frac{\left(S_m + C_p + r_m z\right)}{\lambda} + \frac{h_r}{2k}(k-1)p_m t_1 + \frac{h_m}{2\lambda p_1}\left\{(1-z)(2zp_m - p_1) + z^2 p_m + \frac{\lambda p_m p_1}{k} - 2(zp_m + p_1)\right\}t_1 \tag{10}$$

(using (8))

4 Solution Methodology

Here, the production time is the decision variable. In the following theorem it has shown that the cost function of the system is convex with respect to the production time of the manufacturer.

Theorem: The cost function TC is convex with respect to the production time t_1.

Differentiating the Eq. (10) with respect to t_1 we have –

$$\frac{\partial(TC)}{\partial t_1} = -\frac{(A_m + A_r)}{\lambda p_m t_1^2} + \frac{h_r}{2k}(k-1)p_m + \frac{h_m}{2\lambda p_1}\left\{(1-z)(2zp_m - p_1) + z^2 p_m + \frac{\lambda p_m p_1}{k} - 2(zp_m + p_1)\right\}$$

And,

$$\frac{\partial^2(TC)}{\partial t_1^2} = \frac{2(A_m + A_r)}{\lambda p_m t_1^3} > 0.$$

Therefore, the cost function TC is convex with respect to t_1. ■

We can get the optimum value of the decision variable t_1, say t_1^* from the equation –

$$\frac{\partial(TC)}{\partial t_1} = 0$$

i.e.,

$$-\frac{(A_m + A_r)}{\lambda p_m t_1^2} + \frac{h_r}{2k}(k-1)p_m + \frac{h_m}{2\lambda p_1}\left\{(1-z)(2zp_m - p_1) + z^2 p_m + \frac{\lambda p_m p_1}{k} - 2(zp_m + p_1)\right\} = 0 \quad (11)$$

Once we get the optimum value of t_1 we get the optimum values of t_3 *and* T say t_3^* and T^* from the equations –

$$t_3^* = \frac{\lambda p_m t_1^*}{k} \quad (12)$$

$$T^* = \lambda p_m t_1^* \quad (13)$$

5 Numerical Examples

Example-1: We have used the following numerical values of the variables to illustrate the applicability of this present model.

$A_m = 150$, $A_r = 80$, $\lambda = 0.02$, $p_m = 140$, $S_m = 50$, $C_p = 15$, $r_m = 4$, $z = 0.1$, $h_r = 5$, $h_m = 4$, $k = 1.4$ *and* $p_1 = 40$.

We get the optimum values of t_1, t_3 *and* T as $t_1^* = 3.55$, $t_3^* = 7.11$ *and* $T^* = 9.95$ and the optimum total cost of the system is $TC = 3316.21$.

6 Sensitivity Analysis

In this section, we have studied the effect of the change in the parameters on the optimum decision variables. We have used the values of the variables from example-1 (Tables 1, 2, 3, 4, 5 and 6).

Table 1. Change in decision variables with the change in z

Changing variable	Change in variable	t_1^*	t_3^*	T^*	TC
z	0.06	4.16	8.32	11.66	3301.46
	0.07	3.94	7.88	11.04	3305.67
	0.08	3.78	7.55	10.57	3309.50
	0.09	3.65	7.30	10.22	3313.00
	0.10	3.55	7.11	9.95	3316.21

Table 2. Change in decision variables with the change in k

Changing variable	Change in variable	t_1^*	t_3^*	T^*	TC
k	1.3	5.56	11.98	15.58	3299.53
	1.4	3.55	7.11	9.95	3316.21
	1.5	2.89	5.40	8.09	3326.84
	1.6	2.53	4.44	7.11	3334.72
	1.7	2.32	3.81	6.48	3340.96

Table 3. Change in decision variables with the change in A_m

Changing variable	Change in variable	t_1^*	t_3^*	T^*	TC
A_m	130	3.40	6.79	9.51	3314.16
	150	3.55	7.11	9.95	3316.21
	170	3.71	7.41	10.38	3318.18
	190	3.85	7.70	10.78	3320.07
	210	3.99	7.98	11.18	3321.89

Table 4. Change in decision variables with the change in A_r

Changing variable	Change in variable	t_1^*	t_3^*	T^*	TC
A_r	60	3.40	6.79	9.51	3314.16
	80	3.55	7.11	9.95	3316.21
	100	3.71	7.41	10.38	3318.18
	120	3.85	7.70	10.78	3320.07
	140	3.99	7.98	11.18	3321.89

Table 5. Change in decision variables with the change in h_m

Changing variable	Change in variable	t_1^*	t_3^*	T^*	TC
h_m	2.0	1.24	2.48	3.48	3402.27
	2.5	1.40	2.81	3.94	3386.86
	3.0	1.66	3.32	4.64	3369.08
	3.5	2.12	4.25	5.95	3347.30
	4.0	3.55	7.11	9.95	3316.21

Table 6. Change in decision variables with the change in h_r

Changing variable	Change in variable	t_1^*	t_3^*	T^*	TC
h_r	5.0	3.55	7.11	9.95	3316.21
	5.5	2.23	4.46	6.25	3343.63
	6.0	1.76	3.52	4.93	3363.31
	6.5	1.50	3.00	4.20	3379.51
	7.0	1.33	3.66	3.72	3393.61

7 Observations

It is observed from the table that –

- The total cost of the system (TC) increases with the increase in the defective rate (z) as well as with the increase in the parameter k, i.e. increase in the retailer's demand to the manufacturer. Such situations are obvious, because if the production system produces more defective products, the manufacturer will have to invest more on the reworking of these products. Again, if the retailer's demand to the manufacturer increases then the manufacturer will have to produce more products to meet the retailer's demand

and consequently the total cost of the manufacturer increases. The production time decreases for both the cases.

- With the increase in the set-up cost of the manufacturer and as well as the ordering cost of the retailer, the total cost of the system and the production time both increases.
- The production time of the manufacturer increases with the increases in the holding cost if the manufacturer, but the total cost of the system decreases.
- As the retailer's holding cost increases, we find a decreasing trend in the production time of the manufacturer, although the total cost of the system increases in this case.

8 Conclusions

This article presents a manufacturer-retailer supply chain model, where the production process of the manufacturer has been considered as imperfect and produces an admixture of perfect and imperfect quality items. The manufacturer screens all the products after production and reworks all the defective products after the termination of the main production process. He sells all the products to the retailer at a rate which depends on the demand rate of the customer to the retailer. Since the customers' demand to the retailer has been considered as stochastic in nature, the retailer's demand to the manufacturer is also stochastic, the probability density function of which have been taken as the exponential distribution function. We derived the cost functions of these two supply chain players as well as the whole supply chain system and presented the numerical examples to clarify the applicability of the proposed model.

This model can be extended in several ways. Most importantly, we can consider the imperfect inspection process of the manufacturer. Furthermore, we can consider the learning effect of the employees or can add one or more members of the supply chain to extend this proposed model.

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Posynomial Geometric Programming in EOQ Model with Interval Neutrosophic Number

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Abstract. In this article, a posynomial geometric programming in EOQ model with interval neutrosophic number is presented. Here the cost, the constraint coefficients, and the right-hand sides are taken as imprecise neutrosophic interval number. In this method, the interval posynomial geometric programming problem has been transformed into equivalent parametric posynomial geometric programming problem by using (α, β, γ)-cuts of neutrosophic numbers. As an application of our method a real life problem has been considered to explain the ability and efficiency of the proposed method in neutrosophic environment. Also the result have been analyzed and justified by giving proper figures and tables. The ability of calculating the bounds of the objective value developed in this paper might help lead to more realistic modeling efforts in engineering optimization areas.

Keywords: Economic order quantity · Neutrosophic number · Neutrosophic posynomial geometric programming

1 Introduction

Geometric programming is one of the most important class of optimization techniques that enables one to model a large variety of real-world problems, mostly in the field of engineering design. Its striking structural properties and its elegant theoretical basis have led to a plenty of interesting applications as well as construction of several useful results. It has numerous applications in several areas such as circuit design [11,24] manufacturing system design [10,13] inventory management [9,16,18,22,29] project management [31], maximization of long run and short run profit [2], generalized geometric programming problem with non positive variables [35] and goal programming model [2] etc. The most remarkable property of this technique is that a problem involving highly nonlinear constraints can be transformed equivalently into a problem with only linear

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constraints applying strong duality theorem for geometric programming problems. The basic theories of geometric programming was introduced by Dufffin, Peterson and Zener [13] in order to solve wide range of engineering problems. Geometric programming derives its name due to its involved relation with geometrical concepts, specially geometric inequality and their properties that relate sums and products of positive numbers. Several researchers Avriel and Dembo [4], Beightler and Philips [6], Duffin et al. [13], Fang et al. [14], Kyparisis [21], Kortanek and No [19], Kortanek et al. [20], Maranas and Floudas [25], Peterson [27], Rajgopal [28], Yang and Bricker [36], Zhu et al. [38] developed efficient and effective algorithms for solving the geometric programming problems when the objective and constraint coefficients are known.

In many real life scenarios, due to some difficulties and the human errors actual data may not be collected precisely and hence involved constants coefficients can not be presented in a precise manner. To overcome this situation, the idea of impreciseness in GP, i.e. fuzzy geometric programming was proposed by Cao [8]. On the other hand, fuzzy set theory has been widely developed and recently several modifications have appeared. Atanassov presented Intuitionistic Fuzzy (IF) set theory, where we consider non-membership function along with membership function of imprecise information. Whereas Atanassov and Gargov [3] listed optimization in IF environment as an open problem, Angelov [1] developed optimization technique in IF environment. After that several researchers had worked using FGP and IFGP in different fields. Smarandache [32,33] introduced Neutrosophic (NS) Set, by combining nature with philosophy. It is the study of neutralities as an extension of dialectics. Interestingly, whereas IF sets can only handle incomplete information but failed in case of indeterminacy, NS set can manipulate both incomplete and imprecise information [32]. We characterize NS set by membership function (or, truth membership degree), hesitancy function (or, indeterminacy membership degree) and non-membership function (or, falsity membership degree). In NS environment, decision maker maximizes degree of membership function, minimizes both degree of indeterminacy and degree of non-membership function. Whereas we find application of NS in different directions of research, in this article, we concentrate on optimization in NS environment. The Interval Neutrosophic Set (INS) can represent uncertain, imprecise, incomplete and inconsistent information which exists in real world. SVN number is an extension of fuzzy numbers and IF numbers. Single valued fuzzy number is a special case of Single Valued Neutrosophic Set (SVNS) and is of importance for decision making problems. Ye [37] and Biswas et al. [7] studied the concept of trapezoidal neutrosophic fuzzy number as a generalized representation of trapezoidal fuzzy numbers, trapezoidal IF numbers, triangular fuzzy numbers and triangular IF numbers and applied them for dealing with multi-attribute decision making (MADM) problems. Deli and Subas [12] studied the ranking of single valued neutrosophic trapezoidal numbers and applied the concept to solve MADM problems. Liang et al. [23] presented a multi-criteria decision-making method based on single-valued trapezoidal neutrosophic preference relations with complete weight information. Tian et al. [34] multi-criteria decision-making method based on a cross-entropy with interval NSs, Sahin and Liu [30] considered multiple attribute decision-making problems with the single-valued neutrosophic information, Peng et al. [26] presented a new outranking approach for multi-criteria decision-making (MCDM) problems is developed in the context of a simplified neutrosophic environment. Basset et al. [5] presented a real life based LPP problem with neutrosophic environment. Garg and nancy [15] presented NLP method for multi-criteria decision making problems under interval NS. Although we have performed extensive literature review and have found different types of multi-criteria group decision making problems in neutrosophic environment, and various inventory optimization models has been solved in crisp, fuzzy and IF environments. But, based on our knowledge, the work of the same kind of problem has been noticeably absent in the literature in the context of neutrosophic environment. Consequently, we devise the solution procedure to solve the above said problem in neutrosophic environment.

The rest of the paper is organized as follows: Sect. 2 is assigned of some definitions and properties of NS, In Sect. 3 we discussed on posynomial geometric programming with neutrosophic number coefficients, In Sect. 4 we illustrate the numerical example. In the end, we have some conclusions in Sect. 5.

2 Mathematical Preliminaries

2.1 Notations and Assumptions

The notations and their meanings used frequently in this article have been described below (Table 1).

Table 1. Notations and their explanations

D	Demand per unit time, which is constant
H(t)	Holding cost per unit item, which is time (t) depended
I(t)	Inventory level at any time, $t \geq 0$
P(D, S)	Unit demand (D) and set-up cost (S) dependent production cost
Q	Production quantity per batch
S	Et-up cost per unit time
T	Period of cycle
TAC(D, S, Q)	Total average cost per unit time
W	Total storage capacity area
w_0	Capacity area per unit quantity

To specify scopes of study and to further simplify the proposed EOQ model, we hypothesize the following assumptions:

(i) the proposed EOQ model involves only one item;
(ii) replenishment occurs instantaneously at infinite rate;
(iii) lead time is negligible;

(iv) demand rate is constant;
(v) holding cost is time dependent ($H(t)$ = "at");
(vi) upgrade to modern machineries involves higher costs, which is considered to be a part of the set-up cost. But these machineries have higher production rates and other advantages. So, large scale production can bring down the unit production cost and it is generally adopted when demand is high. Hence, the unit production costs become inversely proportional to the set-up costs and inversely related to the rate of demand, i.e.,

$$P(D, S) = \theta D^{-x} S^{-1}, \qquad \theta, x \in R^+;$$

(vii) shortage is not allowed.

2.2 Neutrosophic Set

Let X be an universal set. A NS set $A \in X$ is defined by $A = \{(x, \mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X\}$. Here $\mu_A(x) : X \to]0^-, 1^+[$, $\sigma_A(x) : X \to]0^-, 1^+[$ and $\nu_A(x) : X \to]0^-, 1^+[$ are called membership function, hesitancy function and non-membership function of A, respectively, satisfy the condition $0^- \leq sup\ \mu_A(x) + sup\ \sigma_A(x) + sup\ \nu_A(x) \leq 3^+, \forall x \in X$.

2.3 Single Valued NS Set

Let X be an universal set. A SVNS $A \in X$ is defined by $A = \{(x, \mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X\}$. Here and thereafter $\mu_A(x) : X \to [0, 1]$, $\sigma_A(x) : X \to [0, 1]$, $\nu_A(x) : X \to [0, 1]$ are membership function, hesitancy function and non-membership function of A, respectively, satisfy the condition $0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3, \forall x \in X$.

2.4 Generalized Trapezoidal Neutrosophic Number (GTrNN)

A single valued trapezoidal neutrosophic number

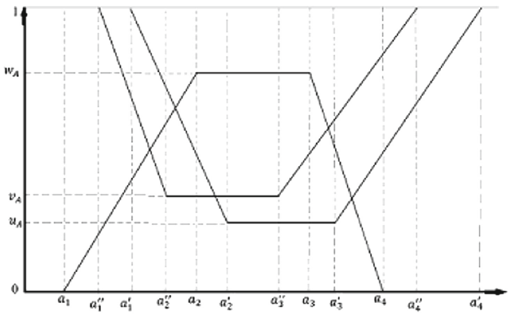
$A = \left\{< (a_1, a_2, a_3, a_4; w_A), (a'_1, a'_2, a'_3, a'_4; u_A), (a''_1, a''_2, a''_3, a''_4; v_A) > w_A, u_A, v_A \in [0, 1]\right\}$

is a special NS on the set of real numbers R and defined by

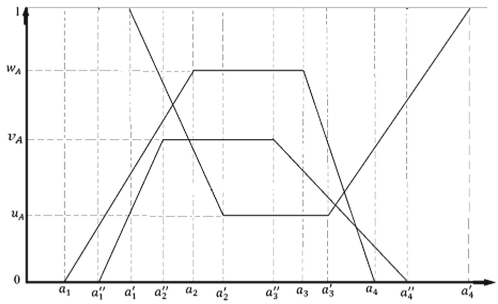
$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} w_A & \text{if} \quad a_1 \leq x \leqslant a_2 \\ w_A & \text{if} \quad a_2 \leqslant x \leqslant a_3 \\ \frac{a_4-x}{a_4-a_3} w_A & \text{if} \quad a_3 \leqslant x \leqslant a_4 \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_A(x) = \begin{cases} \frac{(a'_2-x)+u_A(x-a'_1)}{a'_2-a'_1} & \text{if} \quad a'_1 \leq x \leqslant a'_2 \\ u_A & \text{if} \quad a'_2 \leqslant x \leqslant a'_3 \\ \frac{(x-a'_3)+u_A(a'_4-x)}{a'_4-a'_3} & \text{if} \quad a'_3 \leqslant x \leqslant a'_4 \\ 0 & \text{otherwise.} \end{cases}$$

$$\sigma_A(x)=\begin{cases}\frac{(a_2''-x)+v_A(x-a_1'')}{a_2''-a_1''} & \text{if} \quad a_1''\le x\leqslant a_2''\\ v_A & \text{if} \quad a_2''\leqslant x\leqslant a_3''\\ \frac{(x-a_3'')+v_A(a_4''-x)}{a_4''-a_3''} & \text{if} \quad a_3''\leqslant x\leqslant a_4''\\ 0 & \text{otherwise.}\end{cases} \quad or \quad \sigma_A(x)=\begin{cases}\frac{x-a_1''}{a_2''-a_1''}v_A & \text{if} \quad a_1''\le x\leqslant a_2''\\ v_A & \text{if} \quad a_2''\leqslant x\leqslant a_3''\\ \frac{a_4''-x}{a_4''-a_3''}v_A & \text{if} \quad a_3''\leqslant x\leqslant a_4''\\ 0 & \text{otherwise.}\end{cases}$$



(a) When indeterminacy function like non-membership function



(b) When indeterminacy function like membership function

Fig. 1. Generalized trapezoidal neutrosophic number under different support

2.5 Basic Properties

The operational relation between two SVNSs $A = <(a_1, a_2, a_3, a_4); w_A, u_A, v_A>$ and $B = <(b_1, b_2, b_3, b_4); w_B, u_B, v_B>$ defined by

(1) $A \cup B = \{max(\mu_A(x), \mu_B(x)), min(\sigma_A(x), \sigma_B(x)), min(\nu_A(x), \nu_B(x)) : x \in X\}$

(2) $A \cap B = \{min(\mu_A(x), \mu_B(x)), max(\sigma_A(x), \sigma_B(x)), max(\nu_A(x), \nu_B(x)) : x \in X\}$

(3) $A + B = <(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); w_A \wedge w_B, u_A \vee u_B, v_A \vee v_B>$

(4) $A - B = <(a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); w_A \wedge w_B, u_A \vee u_B, v_A \vee v_B>$

2.6 (α, β, γ)-Cut of GTrNN

A (α, β, γ) -cut of GTrNN $A = \{(a_1, a_2, a_3; w_A), (a_1', a_2', a_3'; u_A); (a_1'', a_2'', a_3''; v_A) : w_A, u_A, v_A \in [0, 1]\}$ is defined by $A_{<\alpha,\beta,\gamma>} = \{x : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta, \sigma_A(x) \leq \gamma, x \in R\}$ which satisfies the conditions $0 \leq \alpha \leq w_A, u_A \leq \beta \leq 1, v_A \leq \gamma \leq 1$ *or* $0 \leq \gamma \leq v_A$ *and* $0 \leq \alpha + \beta + \gamma \leq 3$. Clearly, any $< \alpha, \beta, \gamma >$-cut set $A_{<\alpha,\beta,\gamma>}$ of a GTrNN A is crisp subset of a real number set R, which is defined by
$\hat{A}_\alpha = \left[\frac{(w_A-\alpha)a_1+a_2\alpha}{w_A}, \frac{(w_A-\alpha)a_4+a_3\alpha}{w_A}\right]$, $\hat{A}_\beta = \left[\frac{(1-\beta)a_2'+(\beta-u_A)a_1'}{1-u_A}, \frac{(1-\beta)a_3'+(\beta-u_A)a_4'}{1-u_A}\right]$,
$\hat{A}_\gamma = \left[\frac{(1-\gamma)a_2''+(\gamma-v_A)a_1''}{1-v_A}, \frac{(1-\gamma)a_3''+(\gamma-v_A)a_4''}{1-v_A}\right] \Big/ \left[\frac{(v_A-\gamma)a_1''+a_2''\gamma}{v_A}, \frac{(v_A-\gamma)a_4''+a_3''\gamma}{v_A}\right]$.

3 Theorem

Let $A = \{\{< (a_1, a_2, a_3, a_4; w_A), (a_1', a_2', a_3', a_4'; u_A), (a_1'', a_2'', a_3'', a_4''; v_A) > w_A, u_A, v_A \in [0, 1]\}$ be a GTrNN. For any $0 \leq \alpha \leq w_A, u_A \leq \beta \leq 1, v_A \leq \gamma \leq 1$ *or* $0 \leq \gamma \leq v_A$ where $0 \leq \alpha + \beta + \gamma \leq 3$ then

$$A_{<\alpha,\beta,\gamma>} = A_\alpha \cap A_\beta \cap A_\gamma \tag{3.1}$$

is hold.

Proof: See Kheiri and Cao (2016) [17]

3.1 Lemma

Let $a_1, a_2, w_1, w_2 > 0$, $A = [a_1, a_2]$ be a closed interval with weights w_1, w_2, from mathematical point of view any interval can represent by a function. Then their weighted arithmetic mean is as follows

$$WAM_A(\rho) = \frac{w_1a_1 + w_2a_2}{w_1 + w_2} = a_1(1 - \rho) + a_2\rho$$

and their weighted geometric mean is as follows

$$WGM_A(\rho) = (a_1^{w_1}a_2^{w_2})^{\frac{1}{(w_1+w_2)}} = a_1^{(1-\rho)}a_2^{\rho}$$

also their weighted harmonic mean is as follows

$$WHM_A(\rho) = \frac{w_1 + w_2}{\frac{w_1}{a_1} + \frac{w_2}{a_2}} = \frac{a_1 a_2}{(1-\rho)a_2 + \rho a_1}$$

Where $\rho = \frac{w_2}{w_1+w_2}$, $\rho \in [0,1]$, the choice of the parameter ρ reflects some attitude on the part of the decision maker. $WAM_A(\rho), WGM_A(\rho), WHM_A(\rho)$ are strictly monotone increasing continuous function. Since sum of two continuous function is also a continuous function and

$$\frac{d(WAM_A(\rho))}{d\rho} = a_2 - a_1 \geq 0,$$

$$\frac{d(WGM_A(\rho))}{d\rho} = \frac{\rho(1-\rho)}{a_1^{\rho} a_2^{(1-\rho)}} \geq 0,$$

$$\frac{d(WHM_A(\rho))}{d\rho} = \frac{(a_2 - a_1)a_1 a_2}{\{(1-\rho)a_2 + \rho a_1\}^2} \geq 0 \text{ for } \rho \in [0,1].$$

3.2 Posynomial Geometric Programming with Neutrosophic Coefficients

The standard posynomial geometric programming problem with neutrosophic number coefficients is of the following form

$$\min\ \tilde{\mathrm{g}_0}(\mathrm{x}) = \sum_{\mathrm{k}=1}^{\mathrm{j}_0} \tilde{\mathrm{c}}_{0\mathrm{k}}^{\mathrm{n}} \prod_{\mathrm{l}=1}^{\mathrm{m}} \mathrm{x}_{\mathrm{l}}^{\gamma 0\mathrm{kl}}$$

$$\text{such that,}\ \ \tilde{g}_i(x) = \sum_{k=1}^{j_i} \tilde{c}_{ik}^{n} \prod_{l=1}^{m} x_l^{\gamma ikl} \leq B_i^n,\ (1 \leq i \leq p)$$

by using (α, β, γ)-cut of the neutrosophic coefficients and parameter b_i and according to Eq. 3.1 the model reduces to

$$\min\ \sum_{\mathrm{k}=1}^{\mathrm{j}_0} [\mathrm{c}_{\mathrm{L}_{0\mathrm{k}}}, \mathrm{c}_{\mathrm{R}_{0\mathrm{k}}}] \prod_{\mathrm{l}=1}^{\mathrm{m}} \mathrm{x}_{\mathrm{l}}^{\gamma 0\mathrm{kl}}$$

$$\text{such that,}\ \sum_{k=1}^{j_i} [c_{L_{ik}}, c_{R_{ik}}] \prod_{l=1}^{m} x_l^{\gamma ikl} \leq [b_{L_i}, b_{R_i}],\ (1 \leq i \leq p)$$

$$\text{where,}\ \mathrm{c}_{\mathrm{L}_{0\mathrm{k}}} = \max\left\{\mathrm{c}_{\mathrm{L}_{0\mathrm{k}}}(\alpha), \mathrm{c}_{\mathrm{L}_{0\mathrm{k}}}(\beta)\right\},\ \mathrm{c}_{\mathrm{R}_{0\mathrm{k}}} = \min\left\{\mathrm{c}_{\mathrm{R}_{0\mathrm{k}}}(\alpha), \mathrm{c}_{\mathrm{R}_{0\mathrm{k}}}(\beta)\right\},$$
$$c_{L_{ik}} = \max\left\{\mathrm{c}_{\mathrm{L}_{\mathrm{ik}}}(\alpha), \mathrm{c}_{\mathrm{L}_{\mathrm{ik}}}(\beta)\right\},$$

$$c_{R_{ik}} = \min\left\{\mathrm{c}_{\mathrm{R}_{\mathrm{ik}}}(\alpha), \mathrm{c}_{\mathrm{R}_{\mathrm{ik}}}(\beta)\right\},\ \mathrm{b}_{\mathrm{L}_{\mathrm{i}}} = \max\left\{\mathrm{b}_{\mathrm{L}_{\mathrm{i}}}(\alpha), \mathrm{b}_{\mathrm{L}_{\mathrm{i}}}(\beta)\right\},$$
$$b_{R_i} = \min\left\{\mathrm{b}_{\mathrm{R}_{\mathrm{i}}}(\alpha), \mathrm{b}_{\mathrm{R}_{\mathrm{i}}}(\beta)\right\}, (1 \leq \mathrm{i} \leq \mathrm{p}).$$

3.3 Theorem

The interval posynomial geometric programming problem

$$\min \sum_{k=1}^{j_0}[c_{L_{0k}}, c_{R_{0k}}]\prod_{l=1}^{m} x_l^{\gamma 0kl}$$

$$\text{such that, } \sum_{k=1}^{j_i}[c_{L_{ik}}, c_{R_{ik}}]\prod_{l=1}^{m} x_l^{\gamma ikl} \le [b_{L_i}, b_{R_i}],\ (1 \le i \le p). \tag{3.2}$$

The given posynomial geometric programming problem with interval coefficients can be transformed into the following parametric forms

$$\textbf{Case 1}: \ \min\ g_0(x;\rho) = \sum_{k=1}^{j_0}(c_{L_{0k}}(1-\rho) + c_{R_{0k}}\rho)\prod_{l=1}^{m} x_l^{\gamma 0kl}$$

$$g_i(x;\rho) = \sum_{k=1}^{j_i}(c_{L_{ik}}(1-\rho) + c_{R_{ik}}\rho)\prod_{l=1}^{m} x_l^{\gamma ikl} \le (b_{L_i}(1-\rho) + b_{R_i}\rho), (1 \le i \le p) \tag{3.3}$$

$$\textbf{Case 2}: \ \min\ g_0(x;\rho) = \sum_{k=1}^{j_0}(c_{L_{0k}})^{(1-\rho)}(c_{R_{0k}})^{\rho}\prod_{l=1}^{m} x_l^{\gamma 0kl}$$

$$g_i(x;\rho) = \sum_{k=1}^{j_i}(c_{L_{ik}})^{(1-\rho)}(c_{R_{ik}})^{\rho}\prod_{l=1}^{m} x_l^{\gamma ikl} \le (b_{L_i})^{(1-\rho)}(b_{R_i})^{\rho},\ (1 \le i \le p) \tag{3.4}$$

$$\textbf{Case 3}: \ \min\ g_0(x;\rho) = \sum_{k=1}^{j_0}\frac{(c_{L_{0k}})(c_{R_{0k}})}{(1-\rho)(c_{R_{0k}}) + \rho(c_{L_{0k}})}\prod_{l=1}^{m} x_l^{\gamma 0kl}$$

$$g_i(x;\rho) = \sum_{k=1}^{j_i}\frac{(c_{L_{ik}})(c_{R_{ik}})}{(1-\rho)(c_{R_{ik}}) + \rho(c_{L_{ik}})}\prod_{l=1}^{m} x_l^{\gamma ikl} \le \frac{(b_{L_i})(b_{R_i})}{(1-\rho)(b_{R_i}) + \rho(b_{L_i})}, (1 \le i \le p) \tag{3.5}$$

Proof
Let Q_1 and Q_2 be the sets of all feasible solutions of (3.2) and (3.3), respectively. Then $x \in Q_1$ iff

$$\sum_{k=1}^{j_i}[c_{L_{ik}}, c_{R_{ik}}]\prod_{l=1}^{m} x_l^{\gamma ikl} \le [b_{L_i}, b_{R_i}]. \tag{3.6}$$

Then for any k, we take $d_k \in [c_{L_{ik}}, c_{R_{ik}}]$ and $q \in [b_{L_i}, b_{R_i}]$ the above problem is substitute to the following problem

$$d_k\prod_{l=1}^{m} x_l^{\gamma ikl} \le q \tag{3.7}$$

we know that any interval number $[b_{L_i}, b_{R_i}]$ and $[c_{L_{ik}}, c_{R_{ik}}]$ can be represented as a function given by

$WAM_{A_B}(\rho) = b_{L_i}(1-\rho)+b_{R_i}\rho$, $WAM_{A_C}(\rho) = c_{L_{ik}}(1-\rho)+c_{R_{ik}}\rho$ for $\rho \in [0,1]$.

According to Lemma (Sect. 3.1) above functions are continuous and strictly monotone increasing. We obtain $d_k \in WAM_{A_C}(\rho)$ and $WAM_{A_B}(\rho)$, then for $\rho \in [0,1]$ problem reduces to

$$\sum_{k=1}^{j_i}(c_{L_{ik}}(1-\rho)+c_{R_{ik}}\rho)\prod_{l=1}^{m}x_l^{\gamma ikl} \leq (b_{L_i}(1-\rho)+b_{R_i}\rho) \tag{3.8}$$

Then $x \in Q_1$, hence $Q_1 = Q_2$. Now suppose $x_0 = (x_{01}, x_{02,\dots,x_{0n}})^T$ to be an optimal feasible solution to (3.2) then for all $x \in Q_1$ we have

$$g_0(x) \geq g_0(x_0)$$

$$\sum_{k=1}^{j_0}[c_{L_{0k}}, c_{R_{0k}}]\prod_{l=1}^{m}x_l^{\gamma 0kl} \geq \sum_{k=1}^{j_0}[c_{L_{0k}}, c_{R_{0k}}]\prod_{l=1}^{m}x_{0l}^{\gamma 0kl}$$

we take $\psi_k = [c_{L_{0k}}, c_{R_{0k}}]$ for any k the above problem becomes

$$\sum_{k=1}^{j_0}\psi_k\prod_{l=1}^{m}x_l^{\gamma 0kl} \geq \sum_{k=1}^{j_0}\psi_k\prod_{l=1}^{m}x_{0l}^{\gamma 0kl} \tag{3.9}$$

from the definition, the interval valued function $D = [c_{L_{0k}}, c_{R_{0k}}]$ is obtained

$$h_D(\rho) = c_{L_{0k}}(1-\rho) + c_{R_{0k}}\rho \text{ for } \rho \in [0,1]$$

Then $\psi \in h_D(\rho)$, problem (3.9) reduces to

$$\sum_{k=1}^{j_0}(c_{L_{0k}}(1-\rho)+c_{R_{0k}}\rho)\prod_{l=1}^{m}x_l^{\gamma 0kl} \geq \sum_{k=1}^{j_0}(c_{L_{0k}}(1-\rho)+c_{R_{0k}}\rho)\prod_{l=1}^{m}x_{0l}^{\gamma 0kl}. \tag{3.10}$$

We conclude that x_0 is an optimal feasible solution to (3.3). Problem (3.2) called perturbed PGP, that the constraints need some amendment to be standard PGP, we turn the inner programme of model (3.2) to the following standard posynomial geometric programming form

$$\min\ \mathrm{g_0(x;\rho)} = \sum_{\mathrm{k=1}}^{\mathrm{j_0}}(\mathrm{c_{L_{0k}}}(1-\rho)+\mathrm{c_{R_{0k}}}\rho)\prod_{\mathrm{l=1}}^{\mathrm{m}}\mathrm{x_l^{\gamma 0kl}}$$

$$\text{such that,}\quad g_i(x;\rho) = \sum_{k=1}^{j_i}\frac{(c_{L_{ik}}(1-\rho)+c_{R_{ik}}\rho)}{(b_{L_i}(1-\rho)+b_{R_i}\rho)}\prod_{l=1}^{m}x_l^{\gamma ikl} \leq 1,\ x>0, (1 \leq i \leq p). \tag{3.11}$$

We derive standard PGPP problem, and can solve by the dual problem of the PGP. For $\rho = 0$ and $\rho = 1$ the lower bond and upper bound of the interval value of the parameter is used to find the optimal solution respectively. Although, one can gain the intermediate optimal result by using a proper value of ρ. Using the same technique we can prove the theorem for their G.M and H.M combinations.

4 Mathematical Model Formulation

A manufacturing company produces machines PBA_{597}. The inventory carrying cost for the machines is $Rs.105$ per unit per year. The production cost of this machine varies inversely with the demand and set-up cost. From the past experiences, we can consider the production cost of the machine PBA_{597} at about $120D^{-0.75}S^{-1}$, where D is the demand rate and S is the set-up cost. The company has storage space area per unit time (w_0) and total storage space area (W) as 100 sq. ft. and 2000 sq. ft. respectively. The task is to determine the optimal demand rate (D), set-up cost (S), production quantity (Q) and hence optimal TAC of the production system.

Here the mathematical model is of the form

$$\text{Min } TAC(D,S,Q) = \frac{SD}{Q} + \frac{105Q^2}{6D} + 120D^{-0.75}S^{-1} \tag{4.1}$$
$$\text{subject to,} \quad S(Q) \equiv 100Q \leqslant 2000,$$
$$D,S,Q > 0.$$

The optimal solution is $S^* = 0.034$, $D^* = 4047.477, Q = 20$ and optimal objective value is $TAC(D,S,Q) = 15.565$. Kheiri and Cao [17] considered two-bar truss and optimal box design problem in intuitionistic fuzzy environment. Based on this methodology we are motivated to solve the above mentioned problem using the same methodology with considering appropriate modifications involved. The above said problem can reformulated in neutrosophic environment as the following

$$\text{Min } TAC(D,S,Q) = \frac{SD}{Q} + \frac{AQ^2}{6D} + \tilde{\theta}D^{1-x}S^{-1} \tag{4.2}$$
$$\text{subject to,} \quad S(Q) \equiv \tilde{w_0}Q \leqslant \tilde{W},$$
$$D,S,Q > 0.$$

where $\tilde{A} = <(87,102,108,133;0.7),(83,103,107,137;0.25),(83,101,109,136,0.4)>$
$\tilde{\theta} = <(97,117,123,153;0.65),(103,118,122,157;0.30),(99,116,124,156;0.45)>$
$\tilde{w_0} = <(88,98,102,122;0.8),(85,97,103,121;0.1),(80,95,105,130;0.3)>$
$\tilde{W} = <(1850,1950,2050,2200;0.75),(1910,1980,2020,2110;0.2),(1880,1960,2030,2150;0.5)>$

(α,β,γ)-cut of $A, \tilde{\theta}, \tilde{w_0}, \tilde{W}$ are

$\tilde{A}_\alpha = [a_L(\alpha), a_R(\alpha)] = [87 + 21.43\alpha, 133 - 35.71\alpha], \tilde{A}_\beta = [a_L(\beta), a_R(\beta)] = [109.67 - 26.67\beta, 97 + 40\beta]$,
$\tilde{A}_\gamma = [a_L(\gamma), a_R(\gamma)] = [113 - 30\gamma, 91 + 45\gamma]$ *or* $[83 + 45\gamma, 136 - 67.5\gamma]$.

$\tilde{\theta}_\alpha = [\theta_L(\alpha), \theta_R(\alpha)] = [97 + 30.77\alpha, 153 - 46.15\alpha], \tilde{\theta}_\beta = [\theta_L(\beta), \theta_R(\beta)] = [124.43 - 21.43\beta, 107 + 50\beta]$,

$\tilde{\theta}_\gamma = [\theta_L(\gamma), \theta_R(\gamma)] = [129.91 - 30.91\gamma, 97.82 + 58.18\gamma]$ *or* $[99 + 37.78\gamma, 156 - 71.11\gamma]$.

$\tilde{w_0}_\alpha = [w_{0_L}(\alpha), w_{0_R}(\alpha)] = [88 + 12.5\alpha, 122 - 25\alpha], \tilde{w_0}_\beta = [w_{0_L}(\beta), w_{0_R}(\beta)] = [98.33 - 13.33\beta, 101 + 20\beta]$,
$\tilde{w_0}_\gamma = [w_{0_L}(\gamma), w_{0_R}(\gamma)] = [101.43 - 21.43\gamma, 94.29 + 35.71\gamma]$ *or* $[80 + 50\gamma, 130 - 83.33\gamma]$.

$\tilde{W}_\alpha = [W_L(\alpha), W_R(\alpha)] = [1850+133.33\alpha, 2200-200\alpha], \tilde{W}_\beta = [W_L(\beta), W_R(\beta)] = [1997.5 - 87.5\beta, 1997.5 + 112.5\beta], \tilde{W}_\gamma = [W_L(\gamma), W_R(\gamma)] = [2040 - 160\gamma, 1910 + 240\gamma]$ *or* $[1880 + 160\gamma, 2150 - 240\gamma]$.

Taken $\alpha = 0.60, \beta = 0.35, \gamma = 0.51$ [since $0 \le \alpha \le w_A, u_A \le \beta \le 1, v_A \le \gamma \le 1$] (consider the indeterminacy function like non-membership function) $\alpha = 0.60, \beta = 0.35, \gamma = 0.25$ [since $0 \le \alpha \le w_A, u_A \le \beta \le 1, 0 \le \gamma \le v_A$](consider the indeterminacy function like membership function)

$$\begin{aligned}
\tilde{A}_{<\alpha,\beta,\gamma>} &= [\tilde{A}_\alpha \cap \tilde{A}_\beta \cap \tilde{A}_\gamma] = [100.34, 111],\\
\tilde{\theta}_{<\alpha,\beta,\gamma>} &= [\tilde{\theta}_\alpha \cap \tilde{\theta}_\beta \cap \tilde{\theta}_\gamma] = [116.93, 124.5]\\
\tilde{w_0}_{<\alpha,\beta,\gamma>} &= [\tilde{w_0}_\alpha \cap \tilde{w_0}_\beta \cap \tilde{w_0}_\gamma] = [95.5, 107],\\
\tilde{W}_{<\alpha,\beta,\gamma>} &= [\tilde{W}_\alpha \cap \tilde{W}_\beta \cap \tilde{W}_\gamma] = [1966.88, 2032.40]
\end{aligned}$$

We solve the proposed model by applying GP. Here we find that the degree of difficulty (DD) is 0. By applying Duffin, Peterson and zener theorem [13] of GP on Eq. (4.2), we obtain the DGPP as follows

$$\text{Max d(w)} = \left(\frac{1}{w_{01}}\right)^{w_{01}} \left(\frac{a(\rho)}{6w_{02}}\right)^{w_{02}} \left(\frac{\theta(\rho)}{w_{03}}\right)^{w_{03}} \left(\frac{w_0(\rho)}{W(\rho)w_{11}}\right)^{w_{11}} {w_{11}}^{w_{11}}$$

with normality, orthogonality and positivity conditions are respectively

$$\begin{aligned}
&w_{01} + w_{02} + w_{03} = 1,\\
&w_{01} - w_{02} + (1 - x)w_{03} = 0,\\
&w_{01} - w_{03} = 0,\\
&-w_{01} + 2w_{02} + w_{11} = 0,\\
&w_{01}, w_{02}, w_{03}, w_{11} \ge 0.
\end{aligned}$$

Hence in crisp environment, we find the optimal solution as follows

$$w_{01}^* = w_{03}^* = \frac{1}{4-x}, \quad w_{02}^* = \frac{2-x}{4-x}, \quad w_{11}^* = \frac{2x-3}{4-x}.$$

Thus the optimal values of the primal variables are as follows

$$D = \left\{ \frac{1}{\theta(\rho)} \left(\frac{a(\rho)}{6(2-x)} \right)^2 \left(\frac{W(\rho)}{w_0(\rho)} \right)^5 \right\}^{\frac{1}{4-x}},$$

$$S = \left\{ \left(\frac{\theta(\rho)W(\rho)}{w_0(\rho)} \right)^2 \left(\frac{6w_0(\rho)^3(2-x)}{a(\rho)W(\rho)^3} \right)^x \right\}^{\frac{1}{4-x}} \text{ and } Q = \frac{W(\rho)}{w_0(\rho)}.$$

with optimal TAC as follows

$$TAC^*(D^*, S^*, Q^*) = (4-x) \left\{ \theta(\rho) \left(\frac{w_0(\rho)}{W(\rho)} \right)^{(2x-3)} \left(\frac{a(\rho)}{6(2-x)} \right)^{(2-x)} \right\}^{\frac{1}{4-x}}.$$

$$\text{Where, } a(\rho) = \begin{cases} 100.34(1-\rho) + 111\rho, & \text{when we consider their arithmetic mean} \\ (100.34)^{(1-\rho)}(111)^{\rho}, & \text{when we consider their geometric mean} \\ \frac{(100.34*111)}{((1-\rho)111+\rho 100.34)}, & \text{when we consider their harmonic mean} \end{cases}$$

$$\theta(\rho) = \begin{cases} 116.93(1-\rho) + 124.5\rho, & \text{when we consider their arithmetic mean} \\ (116.93)^{(1-\rho)}(124.5)^{\rho}, & \text{when we consider their geometric mean} \\ \frac{116.93*124.5}{(1-\rho)124.5+\rho 116.93}, & \text{when we consider their harmonic mean} \end{cases}$$

$$W(\rho) = \begin{cases} 1966.88(1-\rho) + 2032.40\rho, & \text{when we consider their arithmetic mean} \\ (1966.88)^{(1-\rho)}(2032.40)^{\rho}, & \text{when we consider their geometric mean} \\ \frac{1966.88*2032.40}{(1-\rho)2032.40+\rho 1966.88}, & \text{when we consider their harmonic mean} \end{cases}$$

$$w_0(\rho) = \begin{cases} 95.5(1-\rho) + 107\rho, & \text{when we consider their arithmetic mean} \\ (95.5)^{(1-\rho)}(107)^{\rho}, & \text{when we consider their geometric mean} \\ \frac{95.5*107}{(1-\rho)107+\rho 95.5}, & \text{when we consider their harmonic mean} \end{cases}$$

Numerical solution of this problem are presented in following Tables 2, 3 and 4

Table 2. Optimal solutions in arithmetic mean combination

ρ	Set-up cost (S)	Demand (D)	Production quantity (Q)	Total average cost TAC (D, S, Q)
0	0.033	4197.427	20.596	15.210
0.2	0.034	4093.147	20.245	15.392
0.4	0.034	3994.730	19.911	15.571
0.6	0.035	3901.718	19.592	15.750
0.8	0.036	3813.697	19.286	15.928
1	0.036	3730.291	18.994	16.104

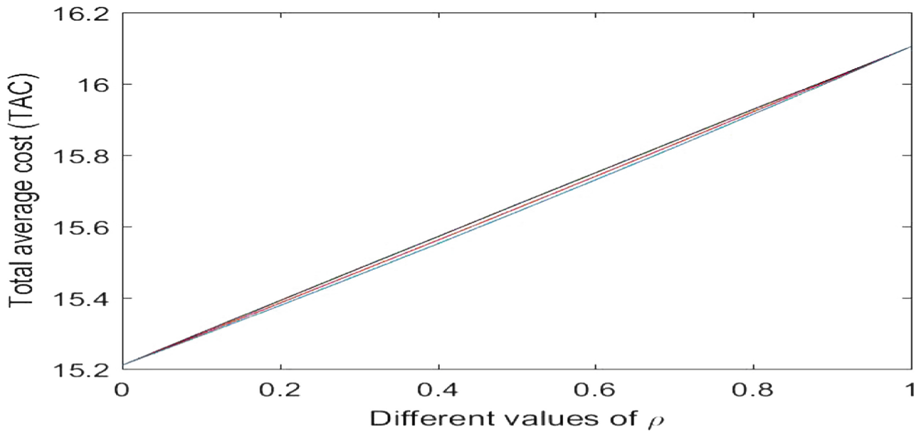


Fig. 2. This figure depicts the total average cost in AM, GM and HM

Table 3. Optimal solutions in geometric mean combination

ρ	Set-up cost (S)	Demand (D)	Production quantity (Q)	Total average cost TAC (D, S, Q)
0	0.033	4197.427	20.596	15.210
0.2	0.034	4099.540	20.265	15.385
0.4	0.034	4003.935	19.940	15.561
0.6	0.035	3910.560	19.619	15.740
0.8	0.036	3819.362	19.304	15.921
1	0.036	3730.291	18.994	16.104

Table 4. Optimal solutions in harmonic mean combination

ρ	Set-up cost (S)	Demand (D)	Production quantity (Q)	Total average cost TAC (D, S, Q)
0	0.033	4197.427	20.596	15.210
0.2	0.034	4105.630	20.284	15.378
0.4	0.034	4013.008	19.968	15.551
0.6	0.035	3919.570	19.648	15.730
0.8	0.036	3825.327	19.323	15.914
1	0.036	3730.291	18.994	16.104

5 Conclusion

In this research article, a solution procedure for PGPP with generalized trapezoidal neutrosophic number as coefficients have been presented. In this approach, we used (α, β, γ)-cut of the neutrosophic numbers turn into interval numbers, which transforms an interval number into the parametric form using their arithmetic mean, geometric mean and harmonic mean combination, and show different values of $\rho \in [0, 1]$ the changes of objective values. The choice of the parameter ρ reflects some attitude on the part of the decision maker. In this case the classical inequality on A.M, G.M, H.M i.e $A.M \geq G.M \geq H.M$ holds good. The total average cost and set-up cost in neutrosophic environment increases proportionally with that of the value of ρ while demand and production quantity deteriorates with the value of ρ. One can easily note that we get better approximation when we use harmonic mean combination rather than others, the fact has been illustrated in corresponding tables and the figure.

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Study on Non-autonomous Version of a Food Chain Model with Strong Allee Effect in Prey Species

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Abstract. A tri-trophic food chain model with strong allee effect in prey species considering the rate parameters to be time dependent has been analyzed in this article. The energy flow is from bottom level to higher trophic level. By utilizing the Mawhin's coincidence degree theorem and then by constructing a suitable Lyapunov function it has been shown that the non-autonomous system has a globally attractive positive periodic solution if some sufficient conditions are satisfied. Finally, the paper ends with a conclusion.

Keywords: Food chain model · Strong Allee effect · Coincidence degree theorem · Periodic solution · Global attractor

1 Introduction

In the study of ecology the foremost factor is predator-prey interaction, which actually configures the energy/biomass flow from one trophic level to higher trophic levels, and modulates the corresponding population size. Predator population may have either direct or indirect impact on prey population. The effects of prey responses to predators on the dynamics of the predator-prey interactions is very significant from biological point of view. Researchers have postulated many hypothesis to functionally signify the coexistence the interacting population in different ambience [1,13,14]. The pioneering of Lotka-Volterra [15,20] first suggested a model that described the relationship between prey-predator species. After that researchers have proposed many complex prey-predator models over the past few decades due to it's significance in real life problems, where the prey-predator dynamics has been extensively studied [16,17]. In several cases the famous logistic growth function has been well accepted for describing individual population growth. Although logistic growth function has been widely used, but there are lot of evidences where the populations exhibits a reverse pattern when population density is low. [2–4] in various real life scenarios. This phenomenon that at low population dencities the population growth positively dependent on the density is known as the Allee effect [2,18]. The Allee effect are mainly of two types, totally depends on how strong the per capita growth rate is

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depleted at low population densities. These two types are called the strong Allee effect [21,23,25] or critical depensation [5–7], and the weak Allee effect [18,22] or noncritical depensation [5–7]. In the present work, we put our focus on a food web system where the prey is vulnerable to strong Allee effect.

Now the natural fact is that environmental and biological parameters can not supposed to be constant throughout a time period as they are subject to fluctuate throughout a period, the seasonal effects are regarded as important selective forces on a system in a periodic environment [11,12,19,24].

The Allee threshold, all the parameters related to metabolism of the species also varies due to change in temperature, light intensity and all other facts which are supposed to be changed with season. What seasonal effects make change in the dynamics of the food web system when there is strong Allee effect in the prey species is still an unexplored area. This motivates us to consider the non-autonomous version of the proposed food chain model and by using the Gaines and Mawhins coincidence degree theorem [9] and by defining a suitable Lyapunov function, we have shown that depending on some necessary conditions the non-autonomous system has a positive periodic solution is global attractive.

We have organized the rest of the paper as below. In Sect. 2, detail assumptions and the model formulation are given, in Sects. 3 and 4, we tried to find out the necessary conditions for existence of global attractor of the non-autonomous system, in Sect. 5 numerical applications are shown to validate our findings and the finally the paper ends with a conclusion in Sect. 6.

2 Model Formulation (Seasonal Effects)

We consider the rate parameters of the non-autonomous system to be periodically varying we consider the following non-autonomous model which is given below,

$$\begin{cases} \frac{dx}{dt} = r(t)x(t)\left(1 - \frac{x(t)}{k(t)}\right)\left(\frac{x(t)}{k_0(t)} - 1\right) - \alpha(t)x(t)y(t) \\ \frac{dy}{dt} = \beta(t)x(t)y(t) - \frac{c(t)y(t)z(t)}{b+y(t)} - d_1(t)y(t), \\ \frac{dz}{dt} = \frac{d(t)y(t)z(t)}{b+y(t)} - d_2(t)z(t), \end{cases} \tag{1}$$

Where, $x(t)$, $y(t)$, and $z(t)$ are density of the prey species, middle predator and top predator respectively at time 't'. We assume that the middle predator consumes the prey species according to law of mass action and the top predator predator the middle predator according to Holling Type-II functional response.

We can have a clear view about the biological interpretations of the associated parameters of the above model from the Table 1 given below,

We assume that the carrying capacity (k) is dependent on light intensity. The parameters related to controlling the other factors of the system, ($r, k, k_0, \alpha, \beta, c$, d, d_1, d_2) are temperature dependent.

Therefore, $r, k, k_0, \alpha, \beta, c, d, d_1, d_2 \in C(R, R^+)$, $R^+ = (0, +\infty)$ are θ-periodic functions; b is a positive constant.

Table 1. The biological meaning of the related parameters are given below

Parameters	Description of parameters
r	Intrinsic growth rate of prey species
k	Environmental carrying capacity beyond which prey population can't grow
k_0	Allee threshold
α	Consumption rate of middle predator
β	Conversion efficiency of middle predator
c	Searching efficiency of the top predator for middle predator
d	Conversion efficiency of top predator in interaction with the middle predator
b	Half saturation constant
d_1	Natural mortality rate of middle predator
d_2	Natural mortality rate of top predator

It is to be noted whereas in summer the temperature is very high but the daily average light strength in winter is maximum. To avoid complexity we simply neglect this phase shifting and subsume the seasonal effect by taken into consideration the periodic rate parameters with a period of one year.

3 Existence of Positive Periodic Solution

To establish sufficient conditions for existence of positive periodic solution of our non-autonomous system (1) we first briefly stated a few concepts [8,9,26,27] which is fundamental for this section. Suppose X and Y be euclidean Banach spaces, $L : DomL \subset X \to Y$ be a linear operator, $N : X \to Y$ be a continuous mapping. The operator L is said to be a Fredholm operator with index zero if $dimKerL = codimImL < \infty$ and $ImL \in Y$ is closed. For the operator L to be Fredholm operator with index zero it is necessary that there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL$, $ImL = KerQ = Im(I - Q)$, then the restriction L_p of L to $DomL \cap kerP : (I - Q)X \to ImL$ is invertible. It is to be noted that the inverse of L_p is denoted by K_p. Now suppose Ω denotes an open bounded subset of the Banach space X, the mapping N will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$ then there exists an isomorphism $J : ImQ \to KerL$.

Lemma 1 (Continuation Theorem) [9]**.** *Let $Q \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be $L-$ compact on $\overline{\Omega}$. Then $Lx = Nx$ has at least one solution in $\Omega \cap DomL$ if the following conditions hold* *(a) for each* $\lambda \in (0,1)$*,* $x \in \delta\Omega \cap DomL, Lx \neq \lambda Nx$*;*

(b) for each $x \in \delta\Omega \cap KerL, QNx \neq 0$*;*

(c) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$*.*

Lemma 2. *Suppose that a function f is integral, uniformly continuous and non-negative on* $[0,\infty)$*, then* $lim_{t\to\infty} f(t) = 0$ *[10].*

Definition 1. *Let* $x(t)$ *is any solution of (1) and* $\bar{x}(t)$ *is a* θ *periodic solution of the non-autonomous system satisfying*

$$\lim_{t\to\infty} |\bar{x}(t) - x(t)| = 0,$$

then the solution $\bar{x}(t)$ *is to be globally attractive.*

For convenience we denote,

$$\bar{f} = \frac{1}{\theta}\int_0^\theta f(t)dt, f^L = \min_{t\in[0,\theta]} f(t), f^H = \max_{t\in[0,\theta]} f(t).$$

Theorem 1. *The non-autonomous system (1) possess at least one positive* θ *periodic solution if the following conditions hold:*
$[E_1]$ $[\beta(t)e^{L_1} - d_1(t)]_L > 0$ *where* L_1 *is defined in the proof.*

Proof. Suppose $(x(t), y(t), z(t))$ is an arbitrary positive solution of the system (1). We make change of variables
$w_1(t) = \ln x(t)$,
$w_2(t) = \ln y(t)$,
$w_3(t) = \ln z(t)$,
Then Eq. (1) becomes,

$$\begin{cases} \frac{dw_1}{dt} = r(t)\left(1 - \frac{e^{w_1(t)}}{k(t)}\right)\left(\frac{e^{w_1(t)}}{k_0(t)-1}\right) - \alpha(t)e^{w_2(t)}, \\ \frac{dw_2}{dt} = \beta(t)e^{w_1(t)} - \frac{c_1(t)e^{w_3(t)}}{b+e^{w_2(t)}} - d_1(t) \\ \frac{dw_3}{dt} = \frac{d(t)e^{w_2(t)}}{b+e^{w_2(t)}} - d_2(t), \end{cases} \tag{2}$$

It is obvious, that if the system (2) has one θ-periodic solution $(w_1^*(t), w_2^*(t), w_3^*(t))^T$, we can conclude that $Z^* = (x^*(t), y^*(t), z^*(t))^T = (exp(w_1^*(t)), exp(w_2^*(t)), exp(w_3^*(t)))^T$ is a positive $\theta-$ periodic solution of the system of the system (1). To complete the proof it is required to verify that the system (2) has at least one $\theta-$ periodic solution.

Set $X = Y = \{(w_1(t), w_2(t), w_3(t))^T \in C(R, R^3) : w_i(t+\theta) = w_i, i = 1, 2, 3\}$. and

$$\| ((w_1(t), w_2(t), w_3(t))^T \| = \sum_{i=1}^{3} \max_{t\in[0,\theta]} |w_i(t)|,$$

where ||.|| denotes the general Euclidean norm. Here X and Y both becomes Banach spaces when they are endowed with the above defined norm.

Let

$$L : DomL \cap X, L\left(w_1(t), w_2(t), w_3(t)\right)^T = \left(\frac{dw_1(t)}{dt}, \frac{dw_2(t)}{dt}, \frac{dw_3(t)}{dt}\right)^T$$

where, $DomL = \left(w_1(t), w_2(t), w_3(t)\right)^T \in C^1(R, R^3)$ and $N : X \to X$,

$$N\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} r(t)\left(1 - \frac{e^{w_1(t)}}{k(t)}\right)\left(\frac{e^{w_1(t)}}{k_0(t)-1}\right) - \alpha(t)e^{w_2(t)} \\ \beta(t)e^{w_1(t)} - \frac{c_1(t)e^{w_3(t)}}{b+e^{w_2(t)}} - d_1(t) \\ \frac{d(t)e^{w_2(t)}}{b+e^{w_2(t)}} - d_2(t) \end{pmatrix}$$

Define,

$$P\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = Q\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\theta}\int_0^\theta w_1(t) \\ \frac{1}{\theta}\int_0^\theta w_2(t) \\ \frac{1}{\theta}\int_0^\theta w_3(t) \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in X = Y.$$

Obviously, we have,

$$KerL = \{x|x \in X, x = h, h \in R^3\},$$
$$ImL = \{y|y \in Y, \int_0^\theta y(t)dt = 0\}.$$

Note 1. We have $dimkerL = 3$. From the first isomorphism theorem it can be easily proved that $codimImL$, i.e; $dim\frac{Y}{ImL}$ is also 3. Therefore, $dimKerL = codimImL = 3$.

As ImL is closed in Y, L is a Fredholm operator of index zero. It can easily be shown that P and Q are continuous projectors (Idempotent operators) s.t,

$$ImP = KerL, \ KerQ = ImL = Im(I - Q).$$

Moreover, one can verify that the inverse K_P of L_P has the following form $K_P : ImL \to DomL \cap KerP$, $K_P(y) = \int_0^t y(s)ds - \frac{1}{\theta}\int_0^\theta \int_0^t y(s)dsdt.$

Consequently, $QN : X \to Y$ and $K_P(I - Q)N : X \to X$ lead

$$QNx = \begin{pmatrix} \frac{1}{\theta}\int_0^\theta [r(t)(1 - \frac{e^{w_1(t)}}{k(t)})(\frac{e^{w_1(t)}}{k_0(t)-1}) - \alpha(t)e^{w_2(t)}] \\ \frac{1}{\theta}\int_0^\theta [\beta(t)e^{w_1(t)} - \frac{c_1(t)e^{w_3(t)}}{b+e^{w_2(t)}} - d_1(t)] \\ \frac{1}{\theta}\int_0^\theta [\frac{d(t)e^{w_2(t)}}{b+e^{w_2(t)}} - d_2(t)] \end{pmatrix}$$

$$K_P(I_Q)Nx = \int_0^t Nx(s)ds - \frac{0}{\theta}\int_0^\theta \int_0^t Nx(s)dsdt - (\frac{t}{\theta} - \frac{1}{2})\int_0^\theta Nx(s)ds.$$

From the Lebesgue theorem it follows that $K_P(I_Q)N$ and QN are continuous furthermore, $QN(\bar{\Omega})$ and $\overline{K_P(I - Q)N(\bar{\Omega})}$ both are relatively compact for any open bounded set $\Omega \subset X$. Consequently, N is $L-$ compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

For applying Lemma 1, it is necessary to have a suitable open bounded subset Ω.

From the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{cases} \frac{dw_1}{dt} = \lambda \left[r(t) \left(1 - \frac{e^{w_1(t)}}{k(t)}\right) \left(\frac{e^{w_1(t)}}{k_0(t)-1}\right) - \alpha(t) e^{w_2(t)} \right], \\ \frac{dw_2}{dt} = \lambda \left[\beta(t) e^{w_1(t)} - \frac{c_1(t) e^{w_3(t)}}{b+e^{w_2(t)}} - d_1(t) \right], \\ \frac{dw_3}{dt} = \lambda \left[\frac{d(t) e^{w_2(t)}}{b+e^{w_2(t)}} - d_2(t) \right] \end{cases} \tag{3}$$

Suppose that, $(w_1(t), w_2(t), w_3(t))^T \in X$, is a solution of (3) for a certain $\lambda \in (0,1)$. Then we have from (3),

$$r(t) \left(1 - \frac{e^{w_1(t)}}{k(t)}\right) \left(\frac{e^{w_1(t)}}{k_0(t) - 1}\right) = \alpha(t) e^{w_2(t)} \tag{4}$$

$$\beta(t) e^{w_1(t)} = \frac{c(t) e^{w_3(t)}}{b + e^{w_2(t)}} + d_1(t) \tag{5}$$

$$\frac{d(t) e^{w_2(t)}}{b + e^{w_2(t)}} = d_2(t) \tag{6}$$

From Eq. (4) we have,

$$\begin{aligned} & r(t)[\frac{e^{w_1(t)}}{k_0} + \frac{e^{w_1(t)}}{k(t)}] = 1 + \frac{e^{2w_1(t)}}{k(t)k_0(t)} + \alpha e^{w_2(t)} \\ & \Rightarrow r(t) e^{w_1(t)} = \frac{k(t)k_0(t)}{k_0(t) + k(t)} + \frac{2e^{w_1(t)}}{k_0(t) + k(t)} + \frac{\alpha e^{w_2(t)}}{k_0(t) + k(t)} \\ & \Rightarrow r(t) e^{w_1(t)} \geq \frac{k(t)k_0(t)}{k_0(t) + k(t)} \\ & \Rightarrow w_1(t) \geq \ln \frac{k(t)k_0(t)}{r(t)(k_0(t) + k(t))} = L_1 \end{aligned}$$

and,

$$\begin{aligned} r(t) \geq \frac{e^{w_1(t)}}{k(t) + k_0(t)} & \Rightarrow e^{w_1(t)} \leq r(t)(k(t) + k_0(t)) \\ & \Rightarrow w_1(t) \leq \ln r(t)(k(t) + k_0(t)) \leq H_1 \end{aligned}$$

From Eq. (6) we have,

$$\begin{aligned} d(t) e^{w_2(t)} \geq b d_2(t) \Rightarrow e^{w_2(t)} & \geq \frac{b d_2(t)}{d(t)} \\ \Rightarrow w_2(t) & \geq \ln \frac{b d_2(t)}{d(t)} = L_2 \end{aligned}$$

Again from (4)

$$e^{-w_1(t)+w_2(t)} \leq \frac{r(t)(k(t)+k_0(t))}{\alpha},$$
$$\Rightarrow e^{w_2(t)} \leq \frac{r(t)(k(t)+k_0(t))}{\alpha e^{-w_1(t)}},$$
$$\Rightarrow w_2(t) \leq \ln[\frac{r(t)(k(t)+k_0(t))e^{H_1}}{\alpha}] = H_2$$

Now from (6) we have,

$$c(t)e^{w_3(t)} \leq \beta(t)e^{w_1(t)}(b+e^{w_2(t)})$$
$$\Rightarrow e^{w_3(t)} \leq \frac{\beta(t)}{c(t)}e^{w_1(t)}(b+e^{w_2(t)})$$
$$\Rightarrow e^{w_3(t)} \leq \frac{\beta(t)}{c(t)}e^{H_1}(b+e^{H_2}).$$
$$\Rightarrow w_3(t) \leq \ln\frac{\beta(t)}{c(t)}e^{H_1}(b+e^{H_2}) = H_3.$$

Again from (5), we have,

$$\frac{c(t)e^{w_3(t)}}{b+e^{w_2(t)}} = \beta(t)e^{w_1(t)} - d_1(t)$$

From condition $[E_1]$ it follows that $[\beta(t)e^{L_1} - d_1(t)] > 0$, then we have,

$$\Rightarrow w_3(t) \geq \ln\frac{b+e^{L_2}}{c(t)}[\beta(t)e^{L_1} - d_1(t)] = L_3.$$

So, we have, $|w_1(t)| \leq H_1$, $|w_2(t)| \leq H_2$, $|w_3(t)| \leq H_3$. Clearly, each $H_i'(i = 1,2,3)$ is independent of λ. Denote $\tilde{H} = H_1 + H_2 + H_3 + \epsilon$, where ϵ is large enough so that each solution $(w_1^*(t), w_2^*(t), w_3^*(t))^T$ (if the system possess at least one solution) of the system of algebraic equation,

$$\begin{cases} \bar{r}(1-\frac{e^{w_1}}{\bar{k}})(\frac{e^{w_1}}{\bar{k}_0-1}) - \bar{\alpha}e^{w_2} = 0, \\ \bar{\beta}e^{w_1} - \frac{\bar{c}_1 e^{w_3}}{b+e^{w_2}} - \bar{d}_1 = 0 \\ \frac{\bar{d}e^{w_2}}{b+e^{w_2}} - \bar{d}_2 = 0. \end{cases} \tag{7}$$

satisfies $\|(w_1^*, w_2^*, w_3^*)^T\| < \tilde{H}$ provided that the system (7) has at least one solution.

Now we set $\Omega = \{(w_1(t), w_2(t), w_3(t))^T \in X : \|(w_1(t), w_2(t), w_3(t))^T\| < \tilde{H}\}$. It is clear that the condition (a) of Lemma 1 is satisfied. When $(w_1(t), w_2(t), w_3(t))^T \in \delta\Omega \cap KerL$, $(w_1(t), w_2(t), w_3(t))^T$ is a constant vector in R^3 with $|w_1| + |w_2| + |w_3| = \tilde{H}$. If the system (7) has at least one solution, then we get,

$$QN \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \bar{r}(1 - \frac{e^{w_1}}{\bar{k}})(\frac{e^{w_1}}{\bar{k}_0 - 1}) - \bar{\alpha} e^{w_2} \\ \bar{\beta} e^{w_1} - \frac{\bar{c}_1 e^{w_3}}{b + e^{w_2}} - \bar{d}_1 \\ \frac{\bar{d} e^{w_2}}{b + e^{w_2}} - \bar{d}_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, the condition (b) in Lemma 1 is satisfied.

As we are going to prove the condition (c) in Lemma 1, we first define a homomorphism mapping $J : ImQ \to KerL$, by $(w_1, w_2, w_3)^T \Rightarrow (w_1, w_2, w_3)^T$, by a standard and straightforward calculation we have

$$deg(JQN(w_1, w_2, w_3)^T, \Omega \cap KerL, (0,0,0)^T) = \sum_{z_i^* \in QN^{-1}(0)} sgnJQN(z_i^*)$$

$$= \sum_{(w_{1_i}^*, w_{2_i}^*, w_{3_i}^*) \in QN^{-1}(0)} detG(w_{1_i}^*, w_{2_i}^*, w_{3_i}^*) \prod_{i=1}^{n} w_{1_i}^* \prod_{i=1}^{n} w_{2_i}^* \prod_{i=1}^{n} w_{3_i}^* \neq 0$$

Then the condition (c) of Lemma 1 is also satisfied. By Lemma 1 the non-autonomous system has at least one positive θ-periodic solution on $\Omega \cap DomL$. This completes the proof.

4 Global Attractivity

Theorem 2. *If the assumption* $[E_1]$ *and the following conditions hold:*

$$[E_2] \quad 0 < x(0), y(0), z(0) < +\infty;$$

$$[E_3] \quad [\mu_1 \frac{2r(t)e^{L_1}}{k(t)k_0(t)} - \mu_1 \frac{r(t)}{k_0(t)} - \mu_1 \frac{r(t)}{k(t)} - \mu_2 \beta(t)]^L_{t \in [0,\theta]} > 0;$$

$$[E_4] \quad [\mu_1 \alpha(t) - \mu_2 \frac{c(t)e^{H_1}}{(b + e^{L_2})^2} + \mu_3 \frac{d(t)e^{L_2}}{(b + e^{H_2})^2}]^L_{t \in [0,\theta]} > 0;$$

$$[E_5] \quad [\mu_2 \frac{c(t)}{b + e^{H_2}}]^L_{t \in [0,\theta]} > 0;$$

then the system (1) has only one positive and globally attractive θ*- periodic solution.*

Proof. In Theorem 1 we have established that the non-autonomous system (1) has at least one positive θ–periodic solution $(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$ and also we have,

$$e^{L_1} \leq \tilde{x}(t) \leq e^{H_1}, e^{L_2} \leq \tilde{y}(t) \leq e^{H_2}, e^{L_3} \leq \tilde{z}(t) \leq e^{H_3}.$$

Let $(x(t), y(t), z(t))$ be any positive periodic solution of (1). Let

$$V(t) = \mu_1|\ln x(t) - \ln \tilde{x}(t)| + \mu_2|\ln y(t) - \ln \tilde{y}(t)| + \mu_3|\ln z(t) - \ln \tilde{z}(t)|. \quad (8)$$

We calculate the Dini derivative of (8) along the solution of (1) we get,

$$D^+V(t) = \mu_1 sgn(x(t) - \tilde{x}(t))(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)}) + \mu_2 sgn(y(t) - \tilde{y}(t))(\frac{\dot{y}(t)}{y(t)} - \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)})$$
$$+ \mu_3 sgn(z(t) - \tilde{z}(t))(\frac{\dot{z}(t)}{z(t)} - \frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)})$$

Now,

$$\mu_1 sgn(x(t) - \tilde{x}(t))(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)}),$$
$$= \mu_1 sgn(x(t) - \tilde{x}(t))[\frac{r(t)}{k_0(t)(x(t) - \tilde{x}(t))} - \frac{r(t)}{k(t)k_0(t)}(x^2(t) - \tilde{x}^2(t))$$
$$+ \frac{r(t)}{k(t)(x(t) - \tilde{x}(t)) - \alpha(y(t) - \tilde{y}(t))}],$$
$$= -\mu_1 \frac{r(t)}{k(t)}|x(t) - \tilde{x}(t)| + \mu_1 k_1(t)|y(t) - \tilde{y}(t)|,$$
$$\leq \mu_1(\frac{r(t)}{k_0(t)} + \frac{r(t)}{k(t)} - \frac{r(t)}{k(t)k_0(t)(x(t) - \tilde{x}(t))})|x(t) - \tilde{x}(t)| + \mu_1\alpha(t)|y(t) - \tilde{y}(t)|. \quad (9)$$

and,

$$\mu_2 sgn(y(t) - \tilde{y}(t))(\frac{\dot{y}(t)}{y(t)} - \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)})$$
$$= \mu_2 sgn(y(t) - \tilde{y}(t))[\beta(t)x(t) - \frac{c(t)z(t)}{b + y(t)} - \beta(t)\tilde{x}(t) + \frac{c(t)\tilde{z}(t)}{b + \tilde{y}(t)}]$$
$$= \mu sgn(y(t) - \tilde{y}(t))[\beta(t)(x(t) - \tilde{x}(t)) - \frac{c(t)z(t)}{b + y(t)} + \frac{c(t)\tilde{z}(t)}{b + y(t)} - \frac{c(t)\tilde{z}(t)}{b + y(t)} + \frac{c(t)\tilde{z}(t)}{b + \tilde{y}(t)}]$$
$$\leq \mu_2\beta(t)|x(t) - \tilde{x}(t)| - \frac{\mu_2 c(t)}{b + y(t)}|z(t) - \tilde{z}(t)| + \frac{\mu_2 c(t)\tilde{z}(t)}{(b + y(t)(b + \tilde{y}(t)))}|y(t) - \tilde{y}(t)|. \quad (10)$$

and

$$\mu_3 sgn(z(t) - \tilde{z}(t))(\frac{\dot{z}(t)}{z(t)} - \frac{\dot{\tilde{z}}(t)}{\tilde{z}(t)})$$
$$= \mu_3 sgn(z(t) - \tilde{z}(t))[\frac{d(t)y(t)}{b + y(t)} - \frac{d(t)\tilde{y}(t)}{b + \tilde{y}(t)}]$$
$$= \mu_3 sgn(z(t) - \tilde{z}(t))[\frac{d(t)y(t)}{b + y(t)} - \frac{d(t)\tilde{y}(t)}{b + y(t)} + \frac{d(t)\tilde{y}(t)}{b + y(t)} - \frac{d(t)\tilde{y}(t)}{b + \tilde{y}(t)}]$$
$$\leq \frac{\mu_3 d(t)}{b + y(t)}|y(t) - \tilde{y}(t)| - \frac{\mu_3 d(t)\tilde{y}(t)}{(b + y(t))(b + \tilde{y}(t))}|y(t) - \tilde{y}(t)| \quad (11)$$

Combining (9), (10), (11) we get,

$$D^+V(t) \le -\left[\mu_1 \frac{r(t)(x(t)+\tilde{x}(t))}{k(t)k_0(t)} - \mu_1 \frac{r(t)}{k_0(t)} - \mu_1 \frac{r(t)}{k(t)} - \mu_2\beta(t)\right] |x(t)-\tilde{x}(t)|$$
$$- \left[\mu_1\alpha(t) - \mu_2 \frac{c(t)\tilde{z}(t)}{(b+y(t))(b+\tilde{y}(t))} + \mu_3 \frac{d(t)\tilde{y}(t)}{(b+y(t))(b+\tilde{y}(t))}\right] |y(t)-\tilde{y}(t)|$$
$$- \left[\mu_2 \frac{c(t)}{b+y(t)}\right] |z(t)-\tilde{z}(t)|$$

Therefore,

$$D^+V(t) \le -\delta_1|x(t)-\tilde{x}(t)| - \delta_2|y(t)-\tilde{y}(t)| - \delta_3|z(t)-\tilde{z}(t)|.$$

Where,

$$\delta_1 = [\mu_1 \frac{2r(t)e^{L_1}}{k(t)k_0(t)} - \mu_1 \frac{r(t)}{k_0(t)} - \mu_1 \frac{r(t)}{k(t)} - \mu_2\beta(t)];$$
$$\delta_2 = [\mu_1\alpha(t) - \mu_2 \frac{c(t)e^{H_1}}{(b+e^{L_2})^2} + \mu_3 \frac{d(t)e^{L_2}}{(b+e^{H_2})^2}];$$
$$\delta_3 = [\mu_2 \frac{c(t)}{b+e^{H_2}}]$$

If the conditions $[E_2]-[E_5]$ hold then clearly V(t) is a non-decreasing on $[0,\infty)$. Now we integrate the above inequality from 0 to t and also noticing the condition $[E_2]$, we observe that,

$$V(t) + \delta_1 \int_0^t |x(t)-\tilde{x}(t)| + \delta_2 \int_0^t |y(t)-\tilde{y}(t)| + \delta_3 \int_0^t |z(t)-\tilde{z}(t)| \le V(0) < +\infty, for\, all\, t > 0$$

From Lemma 2, we get,

$$\lim_{t\to\infty} |x(t)-\tilde{x}(t)| = 0, \quad \lim_{t\to\infty} |y(t)-\tilde{y}(t)| = 0, \quad \lim_{t\to\infty} |z(t)-\tilde{z}(t)| = 0.$$

From the above, we can claim that the periodic solution $(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$ is globally attractive.

5 Application

To capture the seasonal effect we consider the rate parameters to be sinusoidal function. We set the parameters values as $r(t) = 0.7 + 0.05sin(t), k(t) = 10 - 0.5sin(t), k_0(t) = 2 + 0.05sin(t), \alpha(t) = 0.06 - 0.02sin(t), \beta(t) = 0.03 + 0.02cos(t), c(t) = 0.7 + 0.1sin(t), b = 5, d(t) = 0.02 - 0.01cos(t), d_1(t) = 0.01 + 0.005sin(t), d_2(t) = 0.01 - 0.005cos(t)$.

Take an easy calculation, we have $H_1 = 2.15466, H_2 = 7.5282, H_3 = 7.07152, L_1 = 0.78977, L_2 = 0.91629, L_3 = -2.91198$. One can observe that the condition $[E_1]$, that is $[\beta(t)e^{L_1} - d_1(t)]_L = 0.0117461 > 0$ is satisfied. So, by Theorem 2 we can claim that the system (1) has only one global attractive 2π positive periodic solution.

To verify our findings we choose different initial values and the corresponding integral curves are shown in Fig. 1.

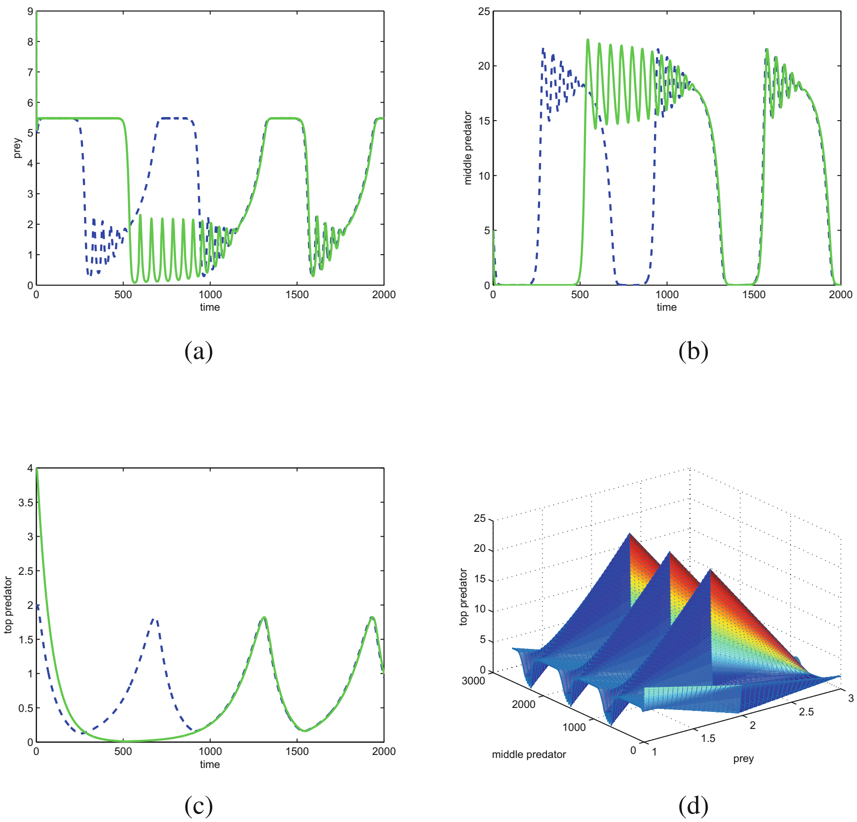

Fig. 1. Existence of unique periodic solution for the non-autonomous system (1). This figure illustrates the fact that there exists a positive periodic solution for the non-autonomous system, it also depicts that the periodic solution is unique as different solution initiates from different point ultimately converges to a unique periodic solution. The surface plot also exhibits the same.

6 Concluding Remark

We have provided the necessary conditions under which the non-autonomous system (1) possess a unique positive periodic solution in a periodically varying environment. It is to be noted that synchronous solutions often occur as a result of seasonal effects on population model. For numerical experiments, we have considered periodic rate parameters to be sinusoidal function with a period of one year to encompass seasonal pattern on the model system (see Fig. 1). It has been noticed that the non-autonomous system (1) has a globally attractive positive 2π-periodic solution (see Fig. 1). It is to be noted that the oscillation of the periodic solution is not affected by the Allee effect in prey species but it may affect the survival of the species. It can be easily seen the density of top predator changes in accordance with that of middle predator and middle predator also alters its direction of path following that of prey species.

If we incorporate harvesting on particular species in a periodic environment then the above method can be useful for determining the optimal harvesting policy. We left it for our future endeavor.

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Novel Multi-objective Green Supply Chain Model with CO_2 Emission Cost in Fuzzy Environment via Soft Computing Technique

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Abstract. In this investigation, a green supply chain (GSC) model has been developed for multi-objective optimization industrial problem. For this purpose, total cost of supply chain and total time including loading and unloading time of materials or products for transportation is minimized. The CO_2 emission cost has been considered for environmental protection concern. Due to vagueness of various data of the proposed model, the parameters of the model have been taken as triangular fuzzy numbers. Expected value technique is used to remove the fuzziness of different parameters. A novel step method has been applied to convert the multi-objectives to a single one. The converted crisp model has been solved by Generalized Reduced Gradient (GRG) technique using LINGO 17.0. A practical numerical experiment with some sensitivity analyses has been carried out to validate the proposed model.

Keywords: Green supply chain · Triangular fuzzy number · CO_2 emission cost · Expected value · Step method

1 Introduction

Recently, various types of Governmental rules and regulations for environmental protection has come into force. So every member of a supply chain tries to control environmental pollution in every footstep of their business policy. Global warming is a serious problem in the present time. It is known to all that a huge amount of carbon emission causes global warming (Xu et al. [9]). Not only the global warming but also consumption of natural resources such as water

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consumption, energy consumption and consumption of various non-renewable resources may extremely affect our next generation. Many of the above consumptions can be controlled by green supply chain. Various initiatives has been taken by the Government of various countries to reduce the environmental pollution. For example in 1970, the Clean Air Act is passed in U.S. Congress and tailpipe emissions standards have been recognized. In present time, the Governmental regulations are stricter and tighten about environmental issues and public awareness is increasing rapidly about environmental consciousness. So, the companies try to make their supply chain green by reducing carbon emission, energy consumption, water consumption, non-renewable resource consumption etc. as well as they try to minimize their various costs and time of their transportation.

So, research on green supply chain has been increasing rapidly. Sarkis [12] proposed a structure of strategic decision for GSC management. Elhedhli and Merrick [13] modeled a GSC network taking the relation between CO_2 emissions and vehicle weight. Tanimizua and Amanoa [11] develop a numerical method to reduce carbon dioxide emission considering the profit balance of suppliers. Li et al. [14] extended the GSC model in the context of dual channel. Uygun and Dede [15] evaluated the performance of GSC model by a decision making technique. Zhu and He [10] shows how supply chains' decision about greenness of product is influenced by various factors. Till now research work on GSC model is going on [cf. Liu and Yi [16]; Yang and Xiao [17]; Basiri and Heydar [18] etc.].

Most of the values of the parameters in a GSC model are vague in nature. This vagueness occurs due to fluctuation of various parameters in business market, lack of advanced knowledge about some parameters, vagueness in the language of experts' opinion etc. After the introduction of fuzzy set concept by Zadeh [19] various supply chain model has been studied by taking several parameters as fuzzy in nature. Yang and Xiao [17] considered GSC model in fuzzy uncertainties. GSC model has been developed by using fuzzy mathematical programming by Pishvaee and Razm [20]. Research works by taking this type of imprecise parameters have been done by several researchers such as Shen et al. [21], Soleimani et al. [22] etc.

The step method, also known as STEM method is applied to solve multi-objective crisp linear programming problem by Teargny et al. [3]. The step method is extended to study a general multi-objective nonlinear programming problem by Shimizu [5]. Further development has been done by several researchers such as Sawaragi et al. [4]; Nijkamp and Spronk [6]; Spronk and Telgen [7]; Choo and Atkins [8] etc.

In spite of the above studies, there are some lacunas in the construction and analysis of GSC model. To cover these research gaps a multi-objective GSC model is considered. Here total cost of supply chain including CO_2 emission cost and total time including loading and unloading time of materials or products for transportation is minimized. The parameters of the model is taken as triangular fuzzy number for vagueness of various data of the proposed model. The fuzziness of parameters is removed by expected value technique and step method is used to convert the multi-objectives to a single one. The converted crisp model is solved by GRG technique. The main objectives of the proposed work is

- In the GSC model, a very few researchers consider the cost of production, component disposal, Chemical waste treatment, waste water, solid waste and non-renewable resource consumption and also CO_2 emission cost.
- In the proposed model, the transportation time as well as the loading and unloading time is considered to analyse the model more practically.
- The step method is used to solve the multi-objective optimization problem.
- A comparative study of cost and time with respect to various tolerance levels of time is considered.

Next of the article is organized as follows. Section 1 gives summery of the work as well as the new contribution. Section 2 gives some preliminary ideas related to the work. Notations and mathematical formulation of proposed GSC model is presented in Sect. 3. Solution procedure of the model is given in Sect. 4. Numerical experiment is done in Sect. 5 to illustrate the proposed model. The experiment's result is discussed in Sect. 6. The paper is concluded in Sect. 7.

2 Preliminaries

2.1 Fuzzy Set

Fuzzy Set: Let U be the universe which is a space of objects. Then a fuzzy set $\tilde{A}$ on U is a set or ordered pairs (Mendel [1]),

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x) : \mu_{\tilde{A}}(x) \in [0,1], \forall x \in U\},$$

where $\mu_{\tilde{A}}(x)$ is called membership function.

Fuzzy Number: A convex normalized fuzzy set on the universe $\Re$ is called a fuzzy number. Let $\tilde{A}$ be a fuzzy set on the real line $\Re$ with membership function $\mu_{\tilde{A}} : \Re \rightarrow [0,1]$. Then $\tilde{A}$ is called a fuzzy number if it satisfies the following two conditions:

1. There exists exactly one interval $I \subseteq \Re$ such that $\mu_{\tilde{A}}(x) = 1, \forall\, x \in I$.
2. The membership function $\mu_{\tilde{A}}(x)$ is piecewise continuous on $\Re$.

Triangular Fuzzy Number (TFN): A fuzzy number $\tilde{A}$ is called a triangular fuzzy number (denoted by (a_1, a_2, a_3) where $a_1, a_2, a_3 \in \Re$) if its membership function $\mu_{\tilde{A}} : \Re \rightarrow [0,1]$ is as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{for } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & \text{for } a_2 < x \leq a_3 \\ 0, & \text{otherwise} \end{cases}$$

Credibility Measure: The credibility measure (Liu and Liu [2]) of the fuzzy event $\{\tilde{A} \in X\}$ where $\tilde{A}$ is fuzzy variable and $X \subset \Re$ is defined as

$$Cr\{\tilde{A} \in X\} = \frac{1}{2}(Pos\{\tilde{A} \in X\} + Nec\{\tilde{A} \in X\})$$

where $Pos\{\tilde{A} \in X\}$ and $Nec\{\tilde{A} \in X\}$ are respectively the possibility and necessity measure of the fuzzy event $\{\tilde{A} \in X\}$ which are defined as,

$$Pos\{\tilde{A} \in X\} = \sup_{x \in X} \mu_{\tilde{A}}(x) \quad \& \quad Nec\{\tilde{A} \in X\} = 1 - \sup_{x \in X^c} \mu_{\tilde{A}}(x)$$

Fuzzy Expectation: The expected value of a normalized fuzzy variable $\tilde{A}$ is denoted by E($\tilde{A}$) and is defined (Liu and Liu [2]) as

$$E(\tilde{A}) = \int_0^{\infty} Cr(\tilde{A} \geq r)dr - \int_{-\infty}^{0} Cr(\tilde{A} \leq r)dr$$

provided that at least one of the above two integrals is finite. If $\tilde{A} = (a_1, a_2, a_3)$ then according to the above mentioned definition the expected value of $\tilde{A}$ is

$$E(\tilde{A}) = \frac{1}{4}(a_1 + 2a_2 + a_3) \tag{1}$$

2.2 Step Method

Step method, also known as STEM method, is applied to solve multi-objective crisp linear programming problem [3].

The step method stands on the technique of norm ideal point method and its solution procedure consists of two stages, the analysis stage and decision stage. In the analysis stage the problem is solved by analyzer with the help of norm ideal point method and the solutions are provided to the decision makers. In the decision stage, the decision makers adjust the tolerance level to the satisfied object to get better value of dissatisfied object obtained in the analysis stage and then supply this information to analyzer for resolving the problem. By repeated application of this process the decision makers finally reach to a final satisfactory solution.

Step method is extended to solve a general multi-objective nonlinear programming problem by Shimizu [5]. Further development is done by several researchers such as Sawaragi et al. [4], Nijkamp and Spronk [6], Spronk and Telgen [7], Choo and Atkins [8] etc.

Let a multi-objective programming problem is as follows:

$$\begin{cases} \min g(x) = (g_1(x), g_2(x), \cdots, g_m(x)) \\ \text{s.t. } x \in Z, \end{cases} \tag{2}$$

where $x = (x_1, x_2, \cdots, x_n)$ and $Z = \{x \in R^n | Ax = b, x \geq 0\}$. Let the optimal solution of the problem $\min\limits_{x \in Z} g_i(x)$ is x^i and calculate the value of each objective function $g_k(x)$ at x^i, then we find m^2 objective function value,

$$g_{ki} = g_k(x^i), \quad k, i = 1, 2, \cdots, m.$$

Let $g_k^* = g_k(x^k)$, $g^* = (g_1^*, g_2^*, \cdots, g_m^*)^T$ and g_k^* is a ideal point of the problem (2). Calculate the maximum value of the objective function $g_k(x)$ at each optimal point x^i

$$g_k^{max} = \max_{1 \leq i \leq m} g_{ki}, \quad k = 1, 2, \cdots, m.$$

To illustrate it more clearly, it is listed in Table 1.

Table 1. Payoff table

g_k	x^1	$\cdots$	x^i	$\cdots$	x^m	g_k^{max}
g_1	$g_{11} = g_1^*$	$\cdots$	g_{1i}	$\cdots$	g_{1m}	g_1^{max}
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
g_i	g_{i1}	$\cdots$	$g_{ii} = g_i^*$	$\cdots$	g_{im}	g_i^{max}
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
g_m	g_{m1}	$\cdots$	g_{mi}		$g_{mm} = g_m^*$	g_m^{max}

Following Table 1, it is required to find x so that the difference between $g(x)$ and g^* is minimum, i.e., each objective is closer to ideal point. Consider the following problem,

$$\min_{x \in Z} \max_{1 \leq k \leq m} \omega_k |g_k(x) - g_k^*| = \min_{x \in Z} \max_{1 \leq k \leq m} \omega_k |\sum_{j=1}^{n} c_{kj} x_j - g_k^*|, \tag{3}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$ be the weight vector and ω_k, the kth weight, is calculated as follows:

$$\beta_k = \begin{cases} \frac{g_k^{max} - g_k^*}{g_k^{max}} \frac{1}{\|c_k\|}, \; g_k^{max} > 0 \\ \frac{g_k^* - g_k^{max}}{g_k^{max}} \frac{1}{\|c_k\|}, \; g_k^{max} \leq 0 \end{cases} \quad k = 1, 2, \cdots, m. \tag{4}$$

$$\omega_k = \beta_k / \sum_{k=1}^{m} \beta_k, \quad k = 1, 2, \cdots, m. \tag{5}$$

where $\|c_k\| = \sqrt{\sum\limits_{j=1}^{n} c_{kj}^2}$. Now the equivalent form of problem (2) is

$$\begin{cases} \min \mu \\ \text{s.t.} \begin{cases} \omega_k \Big(\sum\limits_{j=1}^{n} c_{kj} x_j - g_k^* \Big) \leq \mu, \quad k = 1, 2, \cdots, m \\ \mu \geq 0, \quad x \in Z. \end{cases} \end{cases} \tag{6}$$

Let us consider the optimal solution of the problem (6) is $(\tilde{\xi}, \tilde{\mu})^T$. In order to verify if $\tilde{\xi}$ is satisfied, the decision maker should compare $g_k(\tilde{\xi})$ with the ideal point g_k^*, $k = 1, 2, \cdots, m$. Let the decision maker has been satisfied with $g_s(\tilde{\xi})$, but dissatisfied with $g_t(\tilde{\xi})$, the following constraint is added in the next step to improve the objective value g_t,

$$g_t(x) \leq g_t(\tilde{\xi}).$$

For the satisfied object g_s, the tolerance level θ_s is added,

$$g_s(x) \leq g_s(\tilde{\xi}) + \theta_s.$$

Now in problem (6), we substitute Z by the following constraint set,

$$Z^1 = \{x \in Z | g_s(x) \leq g_s(\tilde{\xi}) + \theta_s, g_t(x) \leq g_t(\tilde{\xi})\},$$

and the objective g_s is deleted (by putting $\omega_s = 0$), then the new problem is resolved to find better solutions.

The gist of step method is as follows:

1. Consider a multi-objective programming problem:
$$\begin{cases} \min g(x) = (g_1(x), g_2(x), \cdots, g_m(x)) \\ \text{s.t. } x \in Z, \end{cases} \tag{7}$$
where $x = (x_1, x_2, \cdots, x_n)$ and $Z = \{x \in R^n | Ax = b, x \geq 0\}$.
2. Calculate minimize value of each objective neglecting others.
$$g_k(x^k) = \min_{x \in Z} g_k(x), \quad k = 1, 2, \cdots, m.$$
If all x^k are equal i.e. $x^1 = x^2 = \cdots = x^m$, then optimal solution is $x^* = x^1 = x^2 = \cdots = x^m$ and stop the process.
3. Calculate the value of every objective $g_k(x)$ at each minimum point x^k and get m^2 objective values $g_{ki} = g_k(x^i)$, $(k, i = 1, 2, \cdots, m)$. Formulate Table 1 and find the following value:
$$g_k^* = g_{kk}, \quad g_k^{max} = \max_{1 \leq i \leq m} g_{ki}, \quad k = 1, 2, \cdots, m.$$
3. Take first constraint set $Z^1 = Z$.
4. Calculate the weights $\omega_1, \omega_2, \cdots, \omega_m$ with help of Eqs. (4) and (5).
5. Then problem (7) is equivalent to:
$$\begin{cases} \min \mu \\ \text{s.t.} \begin{cases} \omega_k \Big(\sum_{j=1}^{n} c_{kj} x_j - g_k^* \Big) \leq \mu, \quad k = 1, 2, \cdots, m \\ \mu \geq 0, \quad x \in Z^i. \end{cases} \end{cases} \tag{8}$$
6. Solve problem (8) and find optimal solution as $(\xi^i, \mu^i)^T$.

7. compare between $g_k(\xi^i)(k = 1, 2, \cdots, m)$ and the ideal value g_k^*. Three cases may arises:
   (i) If the decision maker is satisfied with all $g_k(\xi^i)$, then final output is $\tilde{\xi} = \xi^i$ and end the process.
   (ii) If the decision maker is dissatisfied with all $g_k(\xi^i)$, then final output does not exist and end the process.
   (iii) If the decision maker is satisfied with $g_{s_l}(1 \leq s_l \leq m, l < m)$, turn to 8.
8. The decision maker adjust tolerance level $\delta_{s_l} > 0$ to the object g_{s_l} and formulate the following new constraint set,

$$Z^{i+1} = \{x \in Z^i | g_{s_l}(x) \leq g_{s_l}(\xi^i) + \delta_{s_l}, g_k(x) \leq g_k(\xi^i), k \neq s_l\}.$$

Let $\delta_{s_l} = 0$, $i = i + 1$ and go to 4.

3 Multi-objective Green Supply Chain Model

3.1 Notations

The structure of the GSC model consists of suppliers, producers, distributors and collectors. To present the model mathematically the following notations are used which includes indices, variables and parameters.

Indices:

(i) i : the location of suppliers $(i = 1, 2, \cdots, I)$.
(ii) j : the location of producers $(j = 1, 2, \cdots, J)$.
(iii) k : the location of distributors $(k = 1, 2, \cdots, K)$.
(iv) m : the location of collectors. $(m = 1, 2, \cdots, M)$.
(v) r : a required raw materials $(r = 1, 2, \cdots, R)$.
(vi) s : a product manufactured by producers $(s = 1, 2, \cdots, S)$.

Variables:

(i) x_{rij}^{SP} : the quantity of raw material r purchased and transported from supplier i to producer j.
(ii) x_{sjk}^{PD} : the quantity of product s transported from producer j to distributor k.
(iii) x_{skm}^{DC} : the amount of product s shipped from distributor k to collector m.
(iv) y_{rij}^{SP} : $0-1$ variable, takes value 1 if raw material r transported from supplier i to producer j, otherwise takes value 0. It is represented mathematically as follows:

$$y_{rij}^{SP} = \begin{cases} 1, & \text{if } x_{rij}^{SP} > 0 \\ 0, & \text{otherwise} \end{cases}$$

(v) y_{sjk}^{PD} : $0-1$ variable, takes value 1 if product s is transported from producer j to distributor k, otherwise takes value 0. It is represented mathematically as follows:

$$y_{sjk}^{PD} = \begin{cases} 1, & \text{if } x_{sjk}^{PD} > 0 \\ 0, & \text{otherwise} \end{cases}$$

(vi) y^{DC}_{skm} : 0−1 variable, takes value 1 if product s is transported from distributor k to collector m, otherwise takes value 0. It is represented mathematically as follows:

$$y^{DC}_{skm} = \begin{cases} 1, & \text{if } x^{DC}_{skm} > 0 \\ 0, & \text{otherwise} \end{cases}$$

Parameters:

(i) $\tilde{C}^{P}_{sj}$: production cost per unit of product s manufactured by producer j.
(ii) $\tilde{C}^{D}_{sj}$: cost of component disposal per unit of product s manufactured by producer j.
(iii) $\tilde{C}^{Ch}_{sj}$: chemical waste treatment cost per unit of product s manufactured by producer j.
(iv) $\tilde{C}^{Wt}_{sj}$: waste water cost per unit of product s manufactured by producer j.
(v) $\tilde{C}^{Sl}_{sj}$: solid waste cost per unit of product s manufactured by producer j.
(vi) $\tilde{C}^{AEPD}_{sjk}$: cost paid due to CO_2 emission per unit of product s shipped from producer j to distributor k.
(vii) $\tilde{C}^{AEDC}_{skm}$: cost paid due to CO_2 emission per unit of product s shipped from distributor k to collector m.
(viii) $\tilde{C}^{E}_{rij}$: cost paid due to energy consumption for one unit of raw material r shipped from supplier i to producer j.
(ix) $\tilde{C}^{N}_{sj}$: cost paid due to non-renewable resource consumption for one unit of product s is manufactured by producer j.
(x) $\tilde{D}_{sm}$: demand for product s in collector m.
(xiii) $\tilde{R}_{rs}$: requirement of raw material r to produce one unit of product s.
(xiv) $\tilde{A}_{ri}$: capacity limit of supplier i to supply raw material r.
(xv) $\tilde{B}_{sj}$: production capacity limit of producer j for the product s.
(xvi) $\tilde{L}_{sk}$: capacity limit of distributor k to distribute product s.
(xvi) $\tilde{t}^{SP}_{rij}$: required time for transporting raw material r from supplier i to producer j.
(xvi) $\tilde{t}^{PD}_{sjk}$: required time for transporting product s from producer j to distributor k.
(xvi) $\tilde{t}^{DC}_{skm}$: required time for transporting product s from distributor k to collector m.
(xvi) $\tilde{LU}^{SP}_{rij}$: loading and unloading time of one unit raw material r from supplier i to producer j.
(xvi) $\tilde{LU}^{PD}_{sjk}$: loading and unloading time of one unit of product s from producer j to distributor k.
(xvi) $\tilde{LU}^{DC}_{skm}$: loading and unloading time of one unit of product s from distributor k to collector m.

Note: $\tilde{a}$ represents a triangular fuzzy number, i.e. $\tilde{a} = (a_1, a_2, a_3)$ for all above mentioned fuzzy variables.

3.2 Formulation of Multi-objective GSC Model in Fuzzy Environment

Based on the above-mentioned notations, the GSC model under fuzzy environment is formulated as follows:

A GSC model is considered where the suppliers send raw materials to the producer where the various products is produced and then the products are transferred to various distributors and finally the collectors collect the products from the distributors. This supply chain is shown pictorially in Fig. 1.

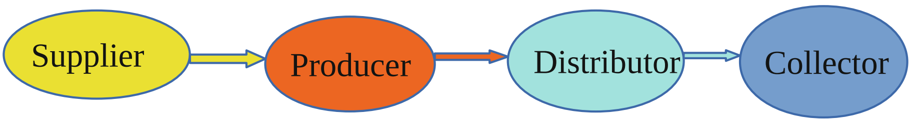

Fig. 1. Pictorial representation of proposed supply chain model

Cost is one of the main important objects of a supply chain. So every decision maker of GSC model seeks to minimize the total cost consisting of various several costs such as production cost, component disposal cost, chemical waste treatment cost, cost of solid waste and waste water cost paid for energy consumption and non-renewable resource consumption cost. Besides these costs, there are another important costs which should be paid due to the CO_2 emission at the time of transportation. Considering the above mentioned costs we consider the total cost as an objective function of the GSC model which is to be minimized. The objective function (9), consists of four terms, present the total cost $(\tilde{g}_1)$. The first term represents the cost of production, component disposal, chemical waste treatment, waste water, solid waste and non-renewable resource consumption. The second and third term represents total CO_2 emission due to transport the product from producer to distributors and from distributors to the collectors respectively. The fourth term represents the energy consumption for transporting the raw materials from supplier to the producer.

$$\begin{aligned} \text{Min } \tilde{g}_1 = & \sum_{s=1}^{S}\sum_{j=1}^{J}\{(\tilde{C}_{sj}^{P} + \tilde{C}_{sj}^{D} + \tilde{C}_{sj}^{Ch} + \tilde{C}_{sj}^{Wt} + \tilde{C}_{sj}^{Sl} + \tilde{C}_{sj}^{N}) \times \sum_{k=1}^{K} x_{sjk}^{PD}\} \\ & + \sum_{s=1}^{S}\sum_{j=1}^{J}\sum_{k=1}^{K} \tilde{C}_{sjk}^{AEPD} x_{sjk}^{PD} + \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{m=1}^{M} \tilde{C}_{skm}^{AEDC} x_{skm}^{DC} + \sum_{r=1}^{R}\sum_{i=1}^{I}\sum_{j=1}^{J} \tilde{C}_{rij}^{E} x_{rij}^{SP} \quad (9) \end{aligned}$$

The objective function (10), consists of three terms, minimizes the total transporting time $(\tilde{g}_2)$ of the GSC model. Each term consists of two parts. The first part represents transportation time for unit amount of item or products and the second part represents the loading and unloading time of materials or products in the corresponding cases.

$$\text{Min } \tilde{g_2} = \sum_{r=1}^{R}\sum_{i=1}^{I}\sum_{j=1}^{J}(\tilde{t}_{rij}^{SP} y_{rij}^{SP} + \tilde{LU}_{rij}^{SP} x_{rij}^{SP}) + \sum_{s=1}^{S}\sum_{j=1}^{J}\sum_{k=1}^{K}(\tilde{t}_{sjk}^{PD} y_{sjk}^{PD} + \tilde{LU}_{sjk}^{PD} x_{sjk}^{PD})$$
$$+ \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{m=1}^{M}(\tilde{t}_{skm}^{DC} y_{skm}^{DC} + \tilde{LU}_{skm}^{DC} x_{skm}^{DC}) \tag{10}$$

The above two objective functions is to be minimized subject to the following constraints. The constraint (11) represent the balance of raw materials. If $\tilde{R}_{rs}$ units of raw material r is needed to produce one unit of product s, then this constraint gives balance of raw materials.

$$\tilde{R}_{rs} \times \sum_{k=1}^{K} x_{sjk}^{PD} = \sum_{i=1}^{I} x_{rij}^{SP} \qquad \forall s, r, j \tag{11}$$

The constraint (12) shows the transported amount of raw material r from i^{th} supplier to various producers does not exceed the capacity limit to supply that material by the supplier.

$$\sum_{j=1}^{J} x_{rij}^{SP} \leq \tilde{A}_{ri} \qquad \forall r, i \tag{12}$$

The constraint (13) focus on the fact that transported amount of product s by the manufacturer j to the different distributors is less than or equal to the production capacity limit B_{sj} to produce product s by the j-th producer.

$$\sum_{j=1}^{J} x_{sjk}^{PD} \leq \tilde{B}_{sj} \qquad \forall s, j \tag{13}$$

The constraint (14) represents that the maximum capacity of k-th distributor to transport the product s is L_{sk} and the total transported amount to various collectors does not exceed this amount.

$$\sum_{m=1}^{M} x_{skm}^{DC} \leq \tilde{L}_{sk} \qquad \forall s, k \tag{14}$$

The constraint (15) presents fulfillment of demand of the various collectors.

$$\sum_{k=1}^{K}\sum_{m=1}^{M} x_{skm}^{DC} \leq \sum_{m=1}^{M} \tilde{D}_{sm} \qquad \forall s \tag{15}$$

The constraint (16) gives the equality of product input quantity and product output quantity.

$$\sum_{j=1}^{J} x_{sjk}^{PD} = \sum_{m=1}^{M} x_{skm}^{DC} \qquad \forall s, k. \tag{16}$$

Here all the parameters are taken as triangular fuzzy number. The objective function $\tilde{g}_1$ i.e. (9) consists of production cost $\tilde{C}^P_{sj} = ({}^1C^P_{sj}, {}^2C^P_{sj}, {}^3C^P_{sj})$, component disposal cost $\tilde{C}^D_{sj} = ({}^1C^D_{sj}, {}^2C^D_{sj}, {}^3C^D_{sj})$, chemical waste treatment cost $\tilde{C}^{Ch}_{sj} = ({}^1C^{Ch}_{sj}, {}^2C^{Ch}_{sj}, {}^3C^{Ch}_{sj})$, waste water cost $\tilde{C}^{Wt}_{sj} = ({}^1C^{Wt}_{sj}, {}^2C^{Wt}_{sj}, {}^3C^{Wt}_{sj})$, solid waste cost $\tilde{C}^{Sl}_{sj} = ({}^1C^{Sl}_{sj}, {}^2C^{Sl}_{sj}, {}^3C^{Sl}_{sj})$, non-renewable resource consumption cost $\tilde{C}^N_{sj} = ({}^1C^N_{sj}, {}^2C^N_{sj}, {}^3C^N_{sj})$, CO_2 emission cost for transportation from producers to distributors $\tilde{C}^{AEPD}_{sjk} = ({}^1C^{AEPD}_{sjk}, {}^2C^{AEPD}_{sjk}, {}^3C^{AEPD}_{sjk})$, CO_2 emission cost for transportation from distributors to collectors $\tilde{C}^{AEDC}_{skm} = ({}^1C^{AEDC}_{skm}, {}^2C^{AEDC}_{skm}, {}^3C^{AEDC}_{skm})$ and cost of energy consumption $\tilde{C}^E_{rij} = ({}^1C^E_{rij}, {}^2C^E_{rij}, {}^3C^E_{rij})$.

The objective function $\tilde{g}_2$ i.e. (10) consists of transportation time of rth material from ith supplier to jth producer $\tilde{t}^{SP}_{rij} = ({}^1t^{SP}_{rij}, {}^2t^{SP}_{rij}, {}^3t^{SP}_{rij})$, loading and unloading time of one unit of rth material transported from ith supplier to jth producer $\tilde{LU}^{SP}_{rij} = ({}^1LU^{SP}_{rij}, {}^2LU^{SP}_{rij}, {}^3LU^{SP}_{rij})$, transportation time of product s from jth producer to kth distributor $\tilde{t}^{PD}_{sjk} = ({}^1t^{PD}_{sjk}, {}^2t^{PD}_{sjk}, {}^3t^{PD}_{sjk})$, loading and unloading time of one unit of product s transported from jth producer to kth distributor $\tilde{LU}^{PD}_{sjk} = ({}^1LU^{PD}_{sjk}, {}^2LU^{PD}_{sjk}, {}^3LU^{PD}_{sjk})$, transportation time of product s from kth distributor to mth collector $\tilde{t}^{DC}_{skm} = ({}^1t^{DC}_{skm}, {}^2t^{DC}_{skm}, {}^3t^{DC}_{skm})$ and loading and unloading time of one unit of product s from kth distributor to mth collector $\tilde{LU}^{DC}_{skm} = ({}^1LU^{DC}_{skm}, {}^2LU^{DC}_{skm}, {}^3LU^{DC}_{skm})$.

In the constraints there are material requirement cost $\tilde{R}_{rs} = ({}^1R_{rs}, {}^2R_{rs}, {}^3R_{rs})$, supplier capacity limit $\tilde{A}_{ri} = ({}^1A_{ri}, {}^2A_{ri}, {}^3A_{ri})$, production capacity limit $\tilde{B}_{sj} = ({}^1B_{sj}, {}^2B_{sj}, {}^3B_{sj})$, distributor capacity limit $\tilde{L}_{sk} = ({}^1L_{sk}, {}^2B_{sk}, {}^3B_{sk})$ and demand of collectors $\tilde{D}_{sm} = ({}^1D_{sm}, {}^2D_{sm}, {}^3D_{sm})$.

4 Solution Procedure

With the help of fuzzy expectation discussed in Sect. 2 and using the Eq. (1) the equivalent crisp form of objective function (9) is found as follows:

$$\begin{aligned}
\text{Min } g_1 = & \sum_{s=1}^{S}\sum_{j=1}^{J}\Big\{\Big(\frac{({}^1C^P_{sj} + 2\times {}^2C^P_{sj} + {}^3C^P_{sj})}{4} + \frac{({}^1C^D_{sj} + 2\times {}^2C^D_{sj} + {}^3C^D_{sj})}{4} \\
& + \frac{({}^1C^{Ch}_{sj} + 2\times {}^2C^{Ch}_{sj} + {}^3C^{Ch}_{sj})}{4} + \frac{({}^1C^{Wt}_{sj} + 2\times {}^2C^{Wt}_{sj} + {}^3C^{Wt}_{sj})}{4} \\
& + \frac{({}^1C^{Sl}_{sj} + 2\times {}^2C^{Sl}_{sj} + {}^3C^{Sl}_{sj})}{4} + \frac{({}^1C^N_{sj} + 2\times {}^2C^N_{sj} + {}^3C^N_{sj})}{4}\Big) \times \sum_{k=1}^{K} x^{PD}_{sjk}\Big\} \\
& + \sum_{s=1}^{S}\sum_{j=1}^{J}\sum_{k=1}^{K} \frac{({}^1C^{AEPD}_{sjk} + 2\times {}^2C^{AEPD}_{sjk} + {}^3C^{AEPD}_{sjk})}{4} x^{PD}_{sjk} \\
& + \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{m=1}^{M} \frac{({}^1C^{AEDC}_{skm} + 2\times {}^2C^{AEDC}_{skm} + {}^3C^{AEDC}_{skm})}{4} x^{DC}_{skm} \\
& + \sum_{r=1}^{R}\sum_{i=1}^{I}\sum_{j=1}^{J} \frac{({}^1C^E_{rij} + 2\times {}^2C^E_{rij} + {}^3C^E_{rij})}{4} x^{SP}_{rij}
\end{aligned} \tag{17}$$

and the objective function (10) takes the form

$$
\begin{aligned}
\text{Min } g_2 = {} & \sum_{r=1}^{R}\sum_{i=1}^{I}\sum_{j=1}^{J}\Big(\frac{({}^1t_{rij}^{SP} + 2\times{}^2t_{rij}^{SP} + {}^3t_{rij}^{SP})}{4} y_{rij}^{SP} \\
& + \frac{({}^1LU_{rij}^{SP} + 2\times{}^2LU_{rij}^{SP} + {}^3LU_{rij}^{SP})}{4} x_{rij}^{SP}\Big) \\
& + \sum_{s=1}^{S}\sum_{j=1}^{J}\sum_{k=1}^{K}\Big(\frac{({}^1t_{sjk}^{PD} + 2\times{}^2t_{sjk}^{PD} + {}^3t_{sjk}^{PD})}{4} y_{sjk}^{PD} \\
& + \frac{({}^1LU_{sjk}^{PD} + 2\times{}^2LU_{sjk}^{PD} + {}^3LU_{sjk}^{PD})}{4} x_{sjk}^{PD}\Big) \\
& + \sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{m=1}^{M}\Big(\frac{({}^1t_{skm}^{DC} + 2\times{}^2t_{skm}^{DC} + {}^3t_{skm}^{DC})}{4} y_{skm}^{DC} \\
& + \frac{({}^1LU_{skm}^{DC} + 2\times{}^2LU_{skm}^{DC} + {}^3LU_{skm}^{DC})}{4} x_{skm}^{DC}\Big)
\end{aligned} \tag{18}
$$

and the constraints from Eqs. (11) to (16) can be written its deterministic form as

$$
\begin{aligned}
& \frac{({}^1R_{rs} + 2\times{}^2R_{rs} + {}^3R_{rs})}{4} \times \sum_{k=1}^{K} x_{sjk}^{PD} = \sum_{i=1}^{I} x_{rij}^{SP} \qquad \forall s, r, j \\
& \sum_{j=1}^{J} x_{rij}^{SP} \le \frac{({}^1A_{ri} + 2\times{}^2A_{ri} + {}^3A_{ri})}{4} \qquad \forall r, i \\
& \sum_{j=1}^{J} x_{sjk}^{PD} \le \frac{({}^1B_{sj} + 2\times{}^2B_{sj} + {}^3B_{sj})}{4} \qquad \forall s, j \\
& \sum_{m=1}^{M} x_{skm}^{DC} \le \frac{({}^1L_{sk} + 2\times{}^2L_{sk} + {}^3L_{sk})}{4} \qquad \forall s, k \\
& \sum_{k=1}^{K}\sum_{m=1}^{M} x_{skm}^{DC} \le \sum_{m=1}^{M} \frac{({}^1D_{sm} + 2\times{}^2D_{sm} + {}^3D_{sm})}{4} \qquad \forall s \\
& \sum_{j=1}^{J} x_{sjk}^{PD} = \sum_{m=1}^{M} x_{skm}^{DC} \qquad \forall s, k.
\end{aligned} \tag{19}
$$

The above deterministic multi-objective optimization problem from Eqs. (16) to (17) together with a set of constraints in Eq. (18) has been solved using step method which is discussed in Sect. 3.

5 Numerical Experiment

To demonstrate the feasibility of the proposed model, a numerical experiment is carried out for GSC model. The GSC model consists of four suppliers ($I = 4$), three producers ($J = 3$), two distributors ($K = 2$) and four collectors ($M = 4$). There are four different types of raw materials ($R = 4$) and producer produces two different types of products ($S = 2$). The values of various types of parameters are given in Tables 2, 3, 4, 5, 6, 7 and 8.

Table 2. Cost of unit production, component disposal, chemical waste treatment, waste water, solid waste, non-renewable resource consumption

	j/s	1	2	3		1	2	3
$\tilde{C}^{P}_{sj}$	1	(365, 370, 375)	(290, 300, 310)	(314, 320, 326)	$\tilde{C}^{D}_{sj}$	(43.6, 45, 46.4)	(51.3, 55, 58.7)	(60.4, 65, 69.6)
	2	(145.6, 150, 154.4)	(178.2, 200, 221.8)	(200.4, 205, 209.6)		(94.1, 95, 95.9)	(106.8, 110, 113.2)	(81.3, 88, 94.7)
$\tilde{C}^{Ch}_{sj}$	1	(171.5, 175, 178.5)	(206.8, 215, 223.2)	(206.2, 210, 213.8)	$\tilde{C}^{Wt}_{sj}$	(13.8, 18, 22.2)	(22.5, 25, 27.5)	(9.7, 13, 16.3)
	2	(161.3, 167, 172.7)	(174.9, 178, 181.1)	(191.7, 195, 198.3)		(11.3, 14, 16.7)	(12.7, 15, 17.3)	(16.1, 18, 19.9)
$\tilde{C}^{Sl}_{sj}$	1	(44, 47, 50)	(41.3, 44, 46.7)	(40.7, 43, 45.3)	$\tilde{C}^{N}_{sj}$	(61, 65, 69)	(68, 71, 74)	61, 63, 65)
	2	(28.5, 30,31 .5)	(32.8, 35, 37.2)	(28.6, 31, 33.4)		(68, 74, 80)	(65, 72, 79)	(71, 75, 79)

Table 3. Cost paid due to air emission per unit of product shipped from producers to distributors and distributors to collectors

	s	1		2	
	k	1	2	1	2
$\tilde{C}^{AEPD}_{sjk}$	$j = 1$	(55.3, 60, 64.7)	(41.8, 45, 48.2)	(56.8, 62, 67.2)	(45.6, 51, 56.4)
	$j = 2$	(78.3, 82, 85.7)	(61.2, 64, 66.8)	(81.4, 83, 84.6)	(65.1, 74, 82.9)
	$j = 3$	(83.4, 87, 90.6)	(71.6, 72, 72.4)	(72.5, 77, 81.5)	(91.7, 100, 108.3)
$\tilde{C}^{AEDC}_{skm}$	$m = 1$	(91.6, 93, 94.4)	(71.7, 75, 78.3)	(51.9, 54, 56.1)	(81.6, 88, 94.4)
	$m = 2$	(61.4, 68, 74.6)	(55, 62, 69)	(73.8, 78, 82.2)	(71.2, 79, 86.8)
	$m = 3$	(44.8, 52, 59.2)	(75.6, 82, 88.4)	(91.7, 95, 98.3)	(52.5, 58, 63.5)
	$m = 4$	(70.2, 71, 71.8)	(90.7, 93, 95.3)	(51.1, 58, 64.9)	(69.3, 73, 76.7)

Table 4. Energy consumption cost for one unit of raw material shipped from supplier to producer

		j/i	1	2	3		1	2	3
$\tilde{C}^{E}_{rij}$	$r = 1$	1	(280, 300, 320)	(290, 315,340)	(310, 325, 340)	$r = 2$	(165, 175, 185)	(208, 215, 222)	(162, 165, 168)
		2	(305, 310, 315)	(305, 308, 311)	(318, 320, 322)		(178, 190, 202)	(190, 195, 200)	(210, 220, 230)
		3	(322, 335, 348)	(332, 340, 348)	(305, 315, 325)		(205, 210, 215)	(213, 215, 217)	(207, 210, 213)
		4	(308, 320, 332)	(315, 325, 335)	(280, 300, 320)		(178, 185, 192)	(195, 200, 205)	(209, 215, 221)
	$r = 3$	1	(222, 230, 238)	(242, 245, 248)	(248, 260, 272)	$r = 4$	(372, 380, 388)	(356, 360, 364)	(362, 365, 368)
		2	(228, 235, 242)	(256, 260, 264)	(236, 240, 244)		(382, 390, 398)	(396, 400, 404)	(351, 365, 379)
		3	(265, 270, 275)	(245, 255, 265)	(248, 270, 292)		(344, 350, 356)	(378, 385, 392)	(365, 370, 375)
		4	(250, 255, 260)	(278, 280, 282)	(232, 240, 248)		(367, 375, 383)	(355, 360, 365)	(417, 420, 423)

Table 5. Required time for transporting and loading-unloading raw material r from supplier i to producer j.

		j/i	1	2	3		1	2	3
$\tilde{t}^{SP}_{rij}$	$r=1$	1	(2.1, 2.5, 2.9)	(1.2, 1.5, 1.8)	(1, 1.1, 1.2)	$r=2$	(2.2, 2.4, 2.6)	(2, 2.1, 2.2)	(2.4, 2.5, 2.6)
		2	(2.2, 2.3, 2.4)	(1.4, 1.6, 1.8)	(1.5, 1.6, 1.7)		(3.1, 3.4, 3.7)	(1.5, 1.7, 1.9)	(1.6, 1.8, 2)
		3	(1.4, 1.5, 1.6)	(1.9, 2, 2.1)	(1.7, 1.9, 2.1)		(1.5, 1.6, 1.7)	(1.21, 1.23, 1.25)	(2, 2.1, 2.2)
		4	(1.21, 1.25, 1.29)	(1.2, 1.4, 1.6)	(1.2, 1.3, 1.4)		(1.3, 1.5, 1.7)	(1.1, 1.2, 1.3)	(3.7, 3.9, 4.1)
	$r=3$	1	(2.1, 2.2, 2.3)	(2, 2.1, 2.2)	(1.9, 2, 2.1)	$r=4$	(1.9, 2.1, 2.3)	(1.8, 2, 2.2)	(2, 2.3, 2.6)
		2	(2.4, 2.5, 2.6)	(1.3, 1.5, 1.7)	(2.3, 2.4, 2.5)		(1.2, 1.5, 1.8)	(1.7, 1.9, 2.1)	(1.1, 1.4, 1.7)
		3	(.9, 1.1, 1.3)	(2, 2.1, 2.2)	(1.3, 1.7, 2.1)		(1.1, 1.3, 1.5)	(.8, 1, 1.2)	(1.2, 1.6, 2)
		4	(1.1, 1.2, 1.3)	(1.1, 1.3, 1.5)	(1, 1.2, 1.4)		(3, 3.1, 3.2)	(1.7, 1.9, 2.1)	(1.1, 1.4, 1.7)
$\tilde{LU}^{PD}_{sjk}$	$r=1$	1	(0.011, 0.015, 0.019)	(0.019, 0.021, 0.023)	(0.020, 0.021, 0.022)	$r=2$	(0.020, 0.021, 0.022)	(0.011, 0.012, 0.013)	(0.021, 0.023, 0.025)
		2	(0.9, 1, 1.1)	(0.01, 0.04, 0.07)	(0.01, 0.03, 0.05)		(0.012, 0.014, 0.016)	(0.010, 0.012, 0.014)	(0.011, 0.014, 0.017)
		3	(0.021, 0.023, 0.025)	(0.011, 0.012, 0.013)	(0.010, 0.011, 0.012)		(0.022, 0.025, 0.028)	(0.011, 0.014, 0.017)	(0.010, 0.013, 0.016)
		4	(0.009, 0.01, 0.011)	(0.01, 0.02, 0.03)	(0.02, 0.04, 0.06)		(0.011, 0.015, 0.019)	(0.010, 0.011, 0.012)	(0.010, 0.013, 0.016)
	$r=3$	1	(0.030, 0.031, 0.032)	(0.01, 0.02, 0.03)	(0.010, 0.012, 0.014)	$r=4$	(0.010, 0.011, 0.012)	(0.017, 0.019, 0.021)	(0.013, 0.015, 0.017)
		2	(0.010, 0.011, 0.012)	(0.011, 0.014, 0.017)	(0.015, 0.017, 0.019)		(0.013, 0.014, 0.015)	(0.018, 0.019, 0.020)	(0.015, 0.017, 0.019)
		3	(0.012, 0.018, 0.024)	(0.011, 0.012, 0.013)	(0.013, 0.014, 0.015)		(0.011, 0.012, 0.013)	(0.014, 0.015, 0.016)	(0.014, 0.017, 0.020)
		4	(0.011, 0.013, 0.015)	(0.014, 0.015, 0.016)	(0.017, 0.018, 0.019)		(0.012, 0.013, 0.014)	(0.010, 0.011, 0.012)	(0.012, 0.014, 0.016)

Table 6. Required time for transporting and loading-unloading product s from producer j to distributor k.

	s	1		2	
	k/j	1	2	1	2
$\tilde{t}^{PD}_{sjk}$	1	(2.6, 2.8, 3)	(1.8, 2, 2.2)	(2.3, 2.5, 2.7)	(2.1, 2.3, 2.5)
	2	(2.1, 2.3, 2.5)	(1.9, 2.1, 2.3)	(2.1, 2.5, 2.9)	(1.9, 2.1, 2.3)
	3	(1.9, 2, 2.1)	(2.4, 2.6, 2.8)	(1.4, 1.6, 1.8)	(1.5, 1.9, 2.3)
$\tilde{LU}^{PD}_{sjk}$	1	(0.022, 0.025, 0.028)	(0.020, 0.021, 0.022)	(0.01, 0.02, 0.03)	(0.019, 0.02, 0.021)
	2	(0.022, 0.023, 0.024)	(0.023, 0.025, 0.027)	(0.021, 0.023, 0.025)	(0.01, 0.02, 0.03)
	3	(0.024, 0.026, 0.028)	(0.025, 0.027, 0.029)	(0.020, 0.021, 0.022)	(0.023, 0.025, 0.027)

Table 7. Required time for transporting and loading-unloading product s from distributor k to collector m.

	s	1		2	
	k/m	1	2	1	2
$\tilde{t}^{DC}_{skm}$	1	(2, 2.1, 2.2)	(2.2, 2.4, 2.6)	(4.1, 4.5, 4.9)	(4, 4.3, 4.6)
	2	(1.1, 1.2, 1.3)	(2.1, 2.3, 2.5)	(3, 3.2, 3.4)	(5.8, 6.2, 6.6)
	3	(1.2, 1.5, 1.8)	(1, 1.3, 1.6)	(1.1, 1.5, 1.9)	(1.8, 2.1, 2.4)
	4	(1.8, 2.2, 2.6)	(2, 2.2, 2.4)	(2.2, 2.6, 3)	(2, 2.2, 2.4)
$\tilde{LU}^{DC}_{skm}$	1	(0.07, 0.08, 0.09)	(0.01, 0.03, 0.05)	(0.04, 0.06, 0.08)	(0.02, 0.04, 0.06)
	2	(0.023, 0.024, 0.025)	(0.01, 0.02, 0.03)	(0.03, 0.05, 0.07)	(0.04, 0.07, 0.1)
	3	(0.5, 0.7, 0.9)	(0.009, 0.01, 0.011)	(0.05, 0.08, 0.11)	(0.08, 0.09, 0.1)
	4	(0.1, 0.2, 0.3)	(0.03, 0.04, 0.05)	(0.07, 0.09, 0.11)	(0.04, 0.05, 0.06)

Table 8. Other parameters

	i/r	1	2	3	4	s	1	2
$\tilde{A}_{ri}$	1	(1500, 1600, 1700)	(2100, 2110, 2120)	(2225, 2250, 2275)	(0, 0, 0)	$\tilde{R}_{rs}$	(0,0,0)	(1.5, 2, 2.5)
	2	(0, 0, 0)	(6650, 6700, 6750)	(5530, 5600, 5670)	(4125, 4150, 4175)		(1.3, 2, 2.7)	(2.1, 3, 3.9)
	3	(2245, 2250, 2255)	(1980, 2000, 2020)	(0, 0, 0)	(2430, 2450, 2470)		(.2, 1, 1.8)	(.5, 1, 1.5)
	4	(2235, 2250, 2265)	(0, 0, 0)	(2050, 2100,2150)	(2630, 2650, 2670)		(1.8, 2, 2.2)	(0, 0, 0)
	m/s	1	2	3	4	k	1	2
$\tilde{D}_{sm}$	1	(940, 950, 960)	(800, 820, 840)	(640, 660, 680)	(1000, 1010, 1020)	$\tilde{L}_{sk}$	(1940, 1990, 2040)	(1760, 1780, 1800)
	2	(560, 600, 640)	(665, 670, 675)	(772, 780, 788)	(928, 930, 932)		(1440, 1470, 1500)	(1632, 1640, 1648)
	j/s	1	2	3				
$\tilde{B}_{sj}$	1	(1150, 1200, 1250)	(1025, 1050, 1075)	(1220, 1250, 1280)				
	2	(950, 1000, 1050)	(920, 950, 980)	(1120, 1200, 1280)				

The above parametric values have been used in the equivalent crisp model i.e. in the objective functions (17), (18) and in the constraints (19) and then step method (discussed in Sect. 2) is applied to crisp model by the following way.
Step 1: After calculation of every single objective function with the help of constraints, Table 9 i.e. payoff table has been obtained.

Table 9. Payoff table for two objectives

g	For Min g_1	For Min g_2	max
g_1	$11895810 = g_1^*$	12254120	$12254120 = g_1^{max}$
g_2	2260.506	$791.64 = g_2^*$	$2260.506 = g_2^{max}$

Step 2: The values of β_i and w_i $(i = 1, 2)$ following Eqs. (4) and (5) are given in Table 10.

Table 10. Values of β_i and w_i

β_1	β_2	ω_1	ω_2
.00000899736	0.513765119	.0000175123	0.999982488

Then the following single objective auxiliary equation (20), with the help of constraints (19) gives the solution of the proposed model.

$$\begin{cases} \min \mu \\ \text{s.t.} \begin{cases} \omega_1 \Big(g_1 - 11895810 \Big) \leq \mu, \\ \omega_2 \Big(g_2 - 791.64 \Big) \leq \mu, \\ \mu \geq 0, \\ \text{and the constraints of equation (19).} \end{cases} \end{cases} \tag{20}$$

Solving the above Eq. (20) the values of g_1 i.e total cost and the value of g_2 i.e. total time is obtained. Here $g_1 = 12058620$, and $g_2 = 794.49$.
Now if the decision maker (DM) is not satisfied with any of the objective then further investigation is needed. Here the value of g_2 (i.e. 794.49) is close to $g_2^* = 791.64$ where as the difference between the value of g_1 (i.e. 12058620) and g_1^* is too large with compare to the earlier. So if the DM is dissatisfied with the objective value g_1, then this value may be improved in the next step.
Step 3: If the tolerance level $\delta_s = 5$ is taken for the satisfied objective g_2 then the following equation is to be solved to improve the dissatisfied objective g_1 taking the weight (ω_2) of satisfied objective as zero.

$$\begin{cases} \min \mu \\ \text{s.t.} \begin{cases} \omega_1\Big(g_1 - 11895810\Big) \leq \mu, \\ \mu \geq 0, \\ g_1 \leq 12058620, \\ g_2 \leq 794.4925 + 5, \\ \text{and the constraints of equation (19).} \end{cases} \end{cases} \tag{21}$$

Then we get $g_1 = 12042520$, and $g_2 = 800.49$. If the DM again is dissatisfied with the value of g_1, then this process will be continued until the DM is satisfied with both the objective values. The values of g_1 and g_2, taking various tolerance level for the satisfied objective g_2 is given in the Table 11.

Table 11. Values of g_1 and g_2 for various tolerance level of g_2

Tolerance level for TIME (g_2)	COST (g_1)	TIME (g_2)	Tolerance level for TIME (g_2)	COST (g_1)	TIME (g_2)	Tolerance level for TIME (g_2)	COST (g_1)	TIME (g_2)
2	12049720	796.49	18	12011220	816.32	34	11985680	838.16
4	12045590	799.12	20	12007210	818.47	36	11980460	841.45
6	12040120	801.29	22	12003220	821.35	38	11976460	842.17
8	12035320	802.35	24	11999220	824.19	40	11972460	844.35
10	12030520	805.41	26	11995220	825.52	42	11968460	846.28
12	12025720	811.18	28	11991220	827.38	44	11967240	848.19
14	12024280	812.31	30	11987220	830.61	46	11965800	851.21
16	12015220	814.11	32	11982910	835.45	48	11959070	855.36

6 Discussion

From Table 9, the optimal value of each single objective is obtained which can be used as ideal point of the problem. Here the ideal point of this experiment is (11895810, 791.64). Then the corresponding weight of each objective function is calculated which are given in Table 10. Then by solving the single objective auxiliary equation (20), the value of g_1 and g_2 is obtained which are respectively 12058620 and 794.49. Now as the value of g_1 is not close to corresponding value in ideal point, so the DM gives various tolerance level to the satisfied objective i.e. to the objective function g_2 and value of g_1 is found which are relatively close to the value of g_1 in ideal point. The various values of g_1 corresponding to various tolerance level of g_2 is given in Table 11. From Table 11 it is clear that as the tolerance level for the objective function g_2 increases gradually, the obtained value of g_1 becomes closer to the value of g_1 in the ideal point. In Table 11, when the tolerance level of g_2 is given as 48, the corresponding value of g_1 is 11959070 and of g_2 is 855.36 which is closed to the ideal point.

7 Conclusion

In this study, a bi-objective GSC model with carbon emission cost is investigated. Here a brief review of fuzzy concept and various operations on fuzzy sets are presented in preliminaries section. CO_2 emission cost is considered in this model to fit the model with practical environmental issue. In a business market the members of a supply chain not only tries to minimize the cost but also they have a target to complete the job in shortest possible time. So focus has been given on time function besides the cost function. In this model transportation time for various materials as well as the loading and unloading time of materials in various depots has taken into account. Step method has been introduced to convert the multi-objective into single objective and a series of experiment has been done by taking various tolerance levels to the satisfied objective functions. To justify the model in light of this vagueness concept the parameters of model are taken as triangular fuzzy numbers. This work can enable to draw attention of decision makers (DM) about total time as well as total cost. The DM can enable to transport the various materials and products appropriately by saving valuable time and controlling environmental pollution.

This work may be extended in various directions. In this model deterioration of various raw materials and products is not considered. Inclusion of deterioration in this model will give a more practical reflection of business policy. The model can also be extended by taking an initial budget for the business process. The model can also be considered in type-2 fuzzy environment to recover the vagueness more appropriately.

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A Multi-item Imperfect Optimal Production Problem Through Chebyshev Approximation

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Abstract. This paper is developed for multi-item inventory model with finite time horizon. The inventory levels, productions and demands are continuous function of time. Initial and end-pints stocks and demands are known. Here production rate is unknown and considered as a control variable and stock level is taken as a state variable. It is formulated to optimize the production rate so that total cost is minimum. Once the problem is formulated as an optimal control problem i.e. in the form of an integral, using El-Gendi's [2] method. Here optimal control problem is solved by numerical approach. This technique is based on the expansion of the control variable in Chebyshev series with unknown coefficients. There is a constraint on the total space termed as space constraint. For simulation, the non-linear optimization technique Generalised Reduced Gradient Method (LINGO 11.0) [5] have been used. The optimum results are illustrated both numerically and analytically.

Keywords: Optimal control problem · Chebyshev polynomial · Finite time horizon

1 Introduction

For the numerical solution of optimal control problems described by constraints in terms of differential equations and objective function in terms of integral form, spectral methods have been used. The optimal control problems are mainly solved by dynamic programming, Variational Principle and Pontryngin's maximum principle method. Several methods have taken the advantage of the properties of the Chebyshev polynomials. Recently, an alternative algorithm for such

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problems and for such advantage has been presented by El-Gindy et al. [3]. This technique is based on the expansion of the control variable in Chebyshev series with unknown coefficients.

Once the problem is formulated as an optimal control problem i.e. in the form of an integral, using El-Gendi's [2] method, Chebyshev spectral approximations for these integrals are obtained. Recently, Chebyshev technique for the solution of optimal control problems with nonlinear programming methods is developed due to Mezzadri and Galligani [11] and application of Chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems is derived due to Kafasha et al. [8].

Chebyshev approximation is used for the highest order derivatives and then successive integrations for approximations are propagated for lower order derivatives. A system of algebraic equations with unknown coefficients replaces, then, the system dynamics and the optimal control problem is transformed into a parameter optimization problem subject to algebraic constraints.

Harris [7], first developed economical order quantity (EOQ) model and then it has been extended by some authors to form the model more practical (cf. Naddor [12], Worell and Hall [18] and others). Normally, above problems are considered as optimal control problems.

Dynamic models of production-inventory/inventory systems have been considered and solved by several researchers (cf. Padmanabhan and Vrat [13], Bendaya and Rauof [1], Hariga and Benkherouf [6], Sana et al. [16], Maity and Maiti [9,10], Roul et al. [14,15] and others). In these models, demand and/or production are assumed to be continuous functions of time.

In this paper, both productions and demands depend on time. Taking holding and production costs, the above production-inventory models are considered as an optimal control problem and solved using Chebyshev approximations (cf. El-Gindy et al. [3]and Fox and Parker [4]) and GRG technique. Model is formulated with single types of production function-quadratic with respect to time t. Here, the reduced constrained optimization problem can be solved directly by applying a constrained optimization technique or first transforming to an unconstrained one by using optimization technique.

For simulation, the non-linear optimization technique Generalised Reduced Gradient Method (LINGO 11.0) [5] have been used. The model has been illustrated by numerical data. The optimum results i.e., optimal demand and production functions, stock levels including costs for different models are presented in tabular from.

The models are developed for the following assumptions and notations.

2 Assumptions

(i) defective rates are known and constant,
(ii) shortages are not allowed,
(iii) the stock level, productions and demands are continuous function of time,
(iv) initial stocks and end stocks are known,
(v) this is a multi-item inventory model with finite time horizon.

3 Notations

n = total number of items.
S_p = total space. For i^{th} item $(i = 1, 2, ..., n)$
$d_i(t) = (a_i + b_i t)$ for $0 \le t \le t_1$ and $(a_i + b_i t_1)$ for $t_1 \le t \le T$ where a_i, b_i are known.
$u_i(t) = (\sum_{k=0}^{m} u_{ki} T_k(t))$, production rate at time t, where $u_{0i}, u_{1i}, ..., u_{mi}$ are unknown co-efficient and $T_k(t)$ are the k^{th} Chebyshev polynomial terms.
$x_i(t)$ = the inventory level at time t.
δ_i = defective parameter.
C_{ui} = production cost per unit item per unit time.
h_i = holding cost per unit item per unit time.
a_{ui} = storage area per unit item.

4 Model Formulation

A Multi-item production-inventory system with dynamic demands are considered. Here, the items are produced at a variable rate and defective at a constant rate. Demand of the items are time dependent. The differential equation for items representing above system during a fixed time-horizon, T are

$$\text{Minimize } J = \sum_{i=1}^{n} \int_{0}^{T} [h_i x_i(t) + C_{ui} u_i(t)] dt \tag{1}$$

subject to the constraints,

$$\frac{dx_i(t)}{dt} = \begin{cases} (1-\delta_i) u_i(t) - (a_i + b_i t) & \text{if } 0 \le t < t_1 \\ (1-\delta_i) u_i(t) - (a_i + b_i t_1) & \text{if } t_1 \le t \le T \end{cases} \tag{2}$$

with the space constraints

$$\sum_{i=1}^{n} x_i(t) a_{ui} \le S_p \tag{3}$$

The above Eqs. (1–3) are known as optimal control problem with state variables $x_i(t)$ and control variables $u_i(t)$ (Fig. 1).

4.1 Solution Methodology

Consider the transformation $\xi = \frac{t}{T}$ for J to convert the problem from t-interval $[0,\ t_1]$ into ξ-interval $[0,\ 1]$. Thus, the Eqs. (1)–(3) reduced to the form

$$J = \sum_{i=1}^{n} T \int_{0}^{1} [\, h_i x_i(\xi) + C_{ui} u_i(\xi)] d\xi \tag{4}$$

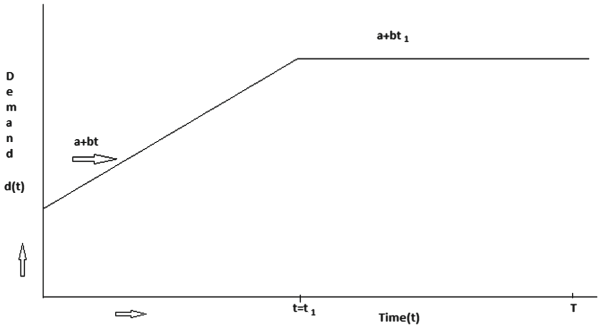

Fig. 1. Time Vs Demand functions.

subject to the constraints,

$$\frac{d}{d\xi}\Big(x_i(\xi)\Big) = \begin{cases} T\big[(1-\delta_i)u_i(\xi) - (a_i + b_i\xi T)\big] & \text{if } 0 \le \xi < \frac{t_1}{T} \\ T\big[(1-\delta_i)u_i(\xi) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \le \xi \le 1 \end{cases} \tag{5}$$

$$\sum_{i=1}^{n} x_i(\xi)a_{ui} \le S_p \;\text{ with }\; x_i(0) = 0, \quad x_i(1) = 100; \tag{6}$$

Let us consider

$$u_i(\xi) = \sum_{k=0}^{m} u_{ki}\, T_k(\xi) \tag{7}$$

$$\frac{d}{d\xi}\Big(x_i(\xi)\Big) = \phi_i(\xi) \tag{8}$$

$$x_i(\xi_s) = \sum_{j=0}^{l} b_{sj}\phi_i(\xi_j) = b_{s0}\phi_i(\xi_0) + b_{s1}\phi_i(\xi_1) + b_{s2}\phi_i(\xi_2) + \cdots + b_{sl}\phi_i(\xi_l) \tag{9}$$

$$\xi_s = \frac{1}{2}\Big(1 - \cos\frac{s\pi}{l}\Big),\; s = 0, 1, ..., l.; \qquad \sum_{j=0}^{l} b_{sj} = (1 - \cos\frac{s\pi}{l}) \tag{10}$$

where $s = 0, 1, ..., l$ and $i = 1, 2, ..., n.$ and b_{sj} are the elements of a matrix given by El-Gendi [2] method and $u_{00},\ u_{01}, ..., u_{mn}$ are unknown constants and $T_k(\xi)$, (k = 0, 1, 2..., m) is k^{th} Chebyshev polynomial

Then we have for $s = 0, 1, 2..., l$ from (10)

$$\sum_{j=0}^{l} b_{0j} = 0; \sum_{j=0}^{l} b_{1j} = (1 - \cos\frac{\pi}{l}); \sum_{j=0}^{l} b_{2j} = (1 - \cos\frac{2\pi}{l}); \cdots \sum_{j=0}^{l} b_{lj} = 2 \tag{11}$$

Also, we have

$$\xi_0 = 0; \quad \xi_1 = \frac{1}{2}\Big(1 - \cos\frac{\pi}{l}\Big); \quad \xi_2 = \frac{1}{2}\Big(1 - \cos\frac{2\pi}{l}\Big); \quad \ldots \quad \xi_l = 1 \tag{12}$$

Therefore from (5) and (8), we have

$$\phi_i(\xi_s) = \begin{cases} T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_s) - (a_i + b_i\xi_s T)\big] & \text{if } 0 \le \xi_s \le \frac{t_1}{T} \\ T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_s) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \le \xi_s \le T \end{cases} \tag{13}$$

Using (13), we can write for $s = 0, 1, 2..., l$

$$\phi_i(\xi_0) = \begin{cases} T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_0) - (a_i + b_i\xi_0 T)\big] & \text{if } 0 \le \xi_0 \le \frac{t_1}{T} \\ T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_0) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \le \xi_0 \le T \end{cases} \tag{14}$$

$$\phi_i(\xi_1) = \begin{cases} T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_1) - (a_i + b_i\xi_1 T)\big] & \text{if } 0 \le \xi_2 \le \frac{t_1}{T} \\ T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_1) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \le \xi_1 \le T \end{cases} \tag{15}$$

$$\phi_i(\xi_2) = \begin{cases} T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_2) - (a_i + b_i\xi_2 T)\big] & \text{if } 0 \le \xi_l \le \frac{t_1}{T} \\ T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_2) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \le \xi_2 \le T \end{cases} \tag{16}$$

$$\cdots$$

$$\phi_i(\xi_l) = \begin{cases} T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_l) - (a_i + b_i\xi_l T)\big] & \text{if } 0 \le \xi_l \le \frac{t_1}{T} \\ T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_l) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \le \xi_l \le T \end{cases} \tag{17}$$

The Eq. (9) can be written as

$$x_i(\xi_s) = \begin{cases} \quad b_{s0}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_0) - (a_i + b_i\xi_0 T)\big] \\ +b_{s1}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_1) - (a_i + b_i\xi_1 T)\big] \\ +b_{s2}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_2) - (a_i + b_i\xi_2 T)\big] \\ +b_{s3}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_3) - (a_i + b_i\xi_3 T)\big] \\ +\cdots + b_{sl}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_l) - (a_i + b_i\xi_l T)\big] \\ \qquad \text{if } 0 \le \xi_0, \xi_1, \xi_2, \xi_3 \ldots \xi_l \le \frac{t_1}{T} \\ b_{s0}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_0) - (a_i + b_i t_1)\big] \\ +b_{s1}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_1) - (a_i + b_i t_1)\big] \\ +b_{s2}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_2) - (a_i + b_i t_1)\big] \\ +b_{s3}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_3) - (a_i + b_i t_1)\big] \\ +\cdots + b_{sl}T\big[(1-\delta_i)\sum_{k=0}^{m} u_{ki}T_k(\xi_l) - (a_i + b_i t_1)\big] \\ \qquad \text{if } \frac{t_1}{T} \le \xi_0, \xi_1, \xi_2, \xi_3 \ldots \xi_l \le T \end{cases} \tag{18}$$

The Production level can be written as $u_i(\xi_s) =$

$$\sum_{k=0}^{m} u_{ki}\, T_k(\xi_s) = u_{0i}\, T_0(\xi_s) + u_{1i}\, T_1(\xi_s) + u_{2i}\, T_2(\xi_s) + \cdots + u_{mi}\, T_m(\xi_s) \tag{19}$$

where $T_0(\xi_s) = 1; T_1(\xi_s) = \xi_s$; $T_2(\xi_s) = 2\xi_s^2 - 1$; $\ldots$ $T_m(\xi_s) = 2\xi_s T_{m-1}(\xi_s) - T_{m-2}(\xi_s)$ for $m \ge 2$ and where $s = 0, 1, 2, 3, ..., l$.

The Eq. (4) can be approximated as follows, using the El-Gindi [3] technique and substituting the values of above Eqs. 14, 15, 16, 17, 18 and 19 into 4, 5 and 6 we have

$$J(u_{00}, u_{01}, .., u_{mn}) = \sum_{i=1}^{n} \left[T \sum_{j=0}^{l} b_{lj} \left(h_i x_i(\xi_s) + C_{ui} \sum_{k=0}^{m} u_{ki} T_k(\xi_s) \right) \right] \quad (20)$$

subject to the constraints,

$$\frac{d}{d\xi}\left(x_i(\xi_s) \right) = \begin{cases} T\left[(1-\delta_i) u_i(\xi_s) - (a_i + b_i \xi_s T) \right] & \text{if } 0 \le \xi_s < \frac{t_1}{T} \\ T\left[(1-\delta_i) u_i(\xi_s) - (a_i + b_i t_1) \right] & \text{if } \frac{t_1}{T} \le \xi_s \le 1 \end{cases} \quad (21)$$

$$\sum_{i=1}^{n} x_i(\xi_s) a_{ui} \le S_p \ \text{ with } \ x_i(0) = 0 \text{ and } \ x_i(1) = 100 \quad (22)$$

Therefore, the problem reduced to

$$J(u_{00}, u_{01}, .., u_{mn}) = \sum_{i=1}^{n} \left[T \sum_{j=0}^{l} b_{lj} \left(h_i \sum_{s=0}^{l} b_{js} \phi_i(\xi_s) + C_{ui} \sum_{k=0}^{m} u_{ki} T_k(\xi_j) \right) \right] \quad (23)$$

subject to the constraints,

$$\frac{d}{d\xi}\left(x_i(\xi_s) \right) = \begin{cases} T\left[(1-\delta_i) u_i(\xi_s) - (a_i + b_i \xi_s T) \right] & \text{if } 0 \le \xi_s < \frac{t_1}{T} \\ T\left[(1-\delta_i) u_i(\xi_s) - (a_i + b_i t_1) \right] & \text{if } \frac{t_1}{T} \le \xi_s \le 1 \end{cases} \quad (24)$$

$$\sum_{i=1}^{n} \sum_{j=0}^{l} \sum_{s=0}^{l} b_{js} \phi_i(\xi_s) a_{ui} \le S_p \ \text{ with } \ x_i(0) = 0 \ \text{ and } \ x_i(1) = 100; \quad (25)$$

The optimal control problem(23–25) is the reduced parameter optimization problem subject to algebraic constraints.

4.2 Particular Case: When $L = 2,\ N = 2$ and $m = 2$

Let $l = 2,\ n = 2$ and $m = 2$.
Hence $u_i(\xi_s) = u_{0i} + u_{1i}\xi_s + u_{2i}(2\xi_s^2 - 1)$ where $T_0(\xi_s) = 1,\ T_1(\xi_s) = \xi_s$ and $T_2(\xi_s) = (2\xi_s^2 - 1)$
Then the stock level becomes

$$x_i(\xi_s) = \begin{cases} b_{s0} T\left[(1-\delta_i) \sum_{k=0}^{2} u_{ki} T_k(\xi_0) - (a_i + b_i \xi_0 T) \right] \\ + b_{s1} T\left[(1-\delta_i) \sum_{k=0}^{2} u_{ki} T_k(\xi_1) - (a_i + b_i \xi_1 T) \right] \\ + b_{s2} T\left[(1-\delta_i) \sum_{k=0}^{2} u_{ki} T_k(\xi_2) - (a_i + b_i \xi_2 T) \right] \\ \qquad \text{if } 0 \le \xi_0, \xi_1, \xi_2 \le \frac{t_1}{T} \\ b_{s0} T\left[(1-\delta_i) \sum_{k=0}^{2} u_{ki} T_k(\xi_0) - (a_i + b_i t_1) \right] \\ + b_{s1} T\left[(1-\delta_i) \sum_{k=0}^{2} u_{ki} T_k(\xi_1) - (a_i + b_i t_1) \right] \\ + b_{s2} T\left[(1-\delta_i) \sum_{k=0}^{2} u_{ki} T_k(\xi_2) - (a_i + b_i t_1) \right] \\ \qquad \text{if } \frac{t_1}{T} \le \xi_0, \xi_1, \xi_2 \le T \end{cases} \quad (26)$$

If $0 \le \xi_0, \xi_1, \xi_2 \le \frac{t_1}{T}$, the stock level becomes

$$\begin{aligned} x_i(\xi_s) = b_{s0}T\big[(1-\delta_i)\big(u_{0i}T_0(\xi_0)+u_{1i}T_1(\xi_0)+u_{2i}T_2(\xi_0)\big)-(a_i+b_i\xi_0 T)\big] \\ +b_{s1}T\big[(1-\delta_i)\big(u_{0i}T_0(\xi_1)+u_{1i}T_1(\xi_1)+u_{2i}T_2(\xi_1)\big)-(a_i+b_i\xi_1 T)\big] \\ +b_{s2}T\big[(1-\delta_i)\big(u_{0i}T_0(\xi_2)+u_{1i}T_1(\xi_2)+u_{2i}T_2(\xi_2)\big)-(a_i+b_i\xi_2 T)\big] \end{aligned} \tag{27}$$

If $\frac{t_1}{T} \le \xi_0, \xi_1, \xi_2 \le T$, the stock level becomes

$$\begin{aligned} x_i(\xi_s) = b_{s0}T\big[(1-\delta_i)\big(u_{0i}T_0(\xi_0)+u_{1i}T_1(\xi_0)+u_{2i}T_2(\xi_0)\big)-(a_i+b_i t_1)\big] \\ +b_{s1}T\big[(1-\delta_i)\big(u_{0i}T_0(\xi_1)+u_{1i}T_1(\xi_1)+u_{1i}T_2(\xi_1)\big)-(a_i+b_i t_1)\big] \\ +b_{s2}T\big[(1-\delta_i)\big(u_{0i}T_0(\xi_2)+u_{1i}T_1(\xi_2)+u_{2i}T_2(\xi_2)\big)-(a_i+b_i t_1)\big] \end{aligned} \tag{28}$$

Here the boundary conditions are $x_i(\xi_0) = x_i(0) = 0$ for $s = 0$ and $x_i(\xi_2) = x_i(1) = 100$ for $s = 2$

If $s = 0$, $\quad x_i(\xi_s) = x_i(\xi_0) = x_i(0) = 0$,

for $0 \le \xi_0, \xi_1, \xi_2 \le \dfrac{t_1}{T}$, then we have

$$\begin{aligned} 0 &= b_{00}T\left[(1-\delta_i)\left(u_{0i}-u_{2i}\right)-a_1\right] \\ &+ b_{01}T\left[(1-\delta_i)\left(u_{0i}+\frac{1}{2}u_{1i}-\frac{1}{2}u_{2i}\right)-(a_1+\frac{1}{2}b_1T)\right] \\ &+ b_{02}T\left[(1-\delta_i)\left(u_{0i}+u_{1i}+u_{2i}\right)-(a_1+b_1T)\right] \end{aligned} \tag{29}$$

If $s = 0$, $\quad x_i(\xi_s) = x_i(\xi_0) = x_i(0) = 0$, $\quad$ for $\dfrac{t_1}{T} \le \xi_0, \xi_1, \xi_2 \le T$, $\quad$ then we have

$$\begin{aligned} 0 &= b_{00}T\left[(1-\delta_i)\left(u_{0i}-u_{2i}\right)-(a_1+b_1t_1)\right] \\ &+ b_{01}T\left[(1-\delta_i)\left(u_{0i}+\frac{1}{2}u_{1i}-\frac{1}{2}u_{1i}\right)-(a_1+b_1t_1)\right] \\ &+ b_{02}T\left[(1-\delta_i)\left(u_{0i}+u_{11}+u_{2i}\right)-(a_1+b_1t_1)\right] \end{aligned} \tag{30}$$

If $s = 2$, $\quad x_i(\xi_s) = x_i(\xi_2) = x_i(1) = 100$, $\quad$ for $0 \le \xi_0, \xi_1, \xi_2 \le \dfrac{t_1}{T}$,

then we have

$$\begin{aligned} 100 &= b_{20}T\left[(1-\delta_i)\left(u_{0i}-u_{2i}\right)-a_1\right] \\ &+ b_{21}T\left[(1-\delta_1)\left(u_{0i}+\frac{1}{2}u_{1i}-\frac{1}{2}u_{2i}\right)-(a_1+\frac{1}{2}b_1T)\right] \\ &+ b_{22}T\left[(1-\delta_1)\left(u_{0i}+u_{1i}+u_{2i}\right)-(a_1+b_1T)\right] \end{aligned} \tag{31}$$

$$\text{If } s = 2, \quad x_i(\xi_s) = x_i(\xi_2) = x_i(1) = 100, \text{ for } \frac{t_1}{T} \leq \xi_0, \xi_1, \xi_2 \leq T,$$

then we have

$$\begin{aligned}100 &= b_{20}T\Big[(1-\delta_i)\Big(u_{0i} - u_{2i}\Big) - (a_1 + b_1 t_1)\Big] \\ &\quad + b_{21}T\Big[(1-\delta_i)\Big(u_{0i} + \frac{1}{2}u_{1i} - \frac{1}{2}u_{1i}\Big) - (a_1 + b_1 t_1)\Big] \\ &\quad + b_{22}T\Big[(1-\delta_i)\Big(u_{0i} + u_{1i} + u_{2i}\Big) - (a_1 + b_1 t_1)\Big] \qquad (32)\end{aligned}$$

If $0 \leq \xi_0 \leq \dfrac{t_1}{T}$

$$\phi_i(\xi_0) = \Big((1-\delta_i)\, u_{0i} - a_i - a_i T - (1-\delta_i)\, u_{2i}\Big)$$

$$\phi_i(\xi_1) = \Big((1-\delta_i)\, u_{0i} + (1-\delta_i)\, \frac{u_{1i}}{2} - (a_i + \frac{1}{2}b_i T) - (1-\delta_i)\, \frac{u_{2i}}{2} - (a_i + b_i T)\Big)$$

$$\phi_i(\xi_2) = \Big((1-\delta_i)\, u_{0i} + (1-\delta_i)\, u_{1i} + (1-\delta_i)\, u_{2i} - 3(a_i + b_i T)\Big)$$

If $\dfrac{t_1}{T} \leq \xi_0 \leq T,$

$$\phi_i(\xi_0) = T\Big[(1-\delta_1)\, u_{0i} - (1-\delta_1)\, u_{2i} - 3(a_1 + b_1 t_1)\Big]$$

$$\phi_i(\xi_1) = T\Big[(1-\delta_i)\, u_{0i} + (1-\delta_i)\, \frac{1}{2}u_{1i} + (1-\delta_i)\, u_{2i} - 3(a_1 + b_1 t_1)\Big]$$

$$\phi_i(\xi_2) = T\Big[(1-\delta_i)\, u_{0i} + (1-\delta_i)\, u_{1i} + (1-\delta_i)\, u_{2i} - 3(a_1 + b_1 t_1)\Big]$$

Now, we have

$$J(u_{0i}, u_{1i}, u_{2i}) = \sum_{i=1}^{n}\Big[T\sum_{j=0}^{2} b_{2j}\Big(h_i \sum_{s=0}^{2} b_{js}\phi_i(\xi_s) + C_{ui}\sum_{k=0}^{m} u_{ki}T_k(\xi_j)\Big)\Big] \quad (33)$$

subject to the constraints,

$$\frac{d}{d\xi}\Big(x_i(\xi_s)\Big) = \begin{cases} T\big[(1-\delta_i)u_i(\xi_s) - (a_i + b_i\xi_s T)\big] & \text{if } 0 \leq \xi_s < \frac{t_1}{T} \\ T\big[(1-\delta_i)u_i(\xi_s) - (a_i + b_i t_1)\big] & \text{if } \frac{t_1}{T} \leq \xi_s \leq 1 \end{cases} \quad (34)$$

$$\sum_{i=1}^{n}\sum_{j=0}^{2}\sum_{s=0}^{2} b_{js}\phi_i(\xi_s)a_{ui} \leq S_p \text{ with } x_i(0) = 0,\, x_i(1) = 100 \qquad (35)$$

Then $J(u_{0i}, u_{1i}, u_{2i})$ can be expressed as the sum of $J_1(u_{0i}, u_{1i}, u_{2i})$ for $0 \le \xi_s < \frac{t_1}{T}$ and $J_2(u_{0i}, u_{1i}, u_{2i})$ for $\frac{t_1}{T} \le \xi_s < T$

$$
\begin{aligned}
J(u_{0i}, u_{1i}, u_{2i}) &= J_1(u_{0i}, u_{1i}, u_{2i}) \quad \text{for} \quad 0 \le \xi_s < \frac{t_1}{T} \\
&+ J_2(u_{0i}, u_{1i}, u_{2i}) \quad \text{for} \quad \frac{t_1}{T} \le \xi_s < 1
\end{aligned} \tag{36}
$$

Therefore $\quad J_1(u_{0i}, u_{1i}, u_{2i})$

$$
\begin{aligned}
&= \sum_{i=1}^{n} T^2 h_i \Bigg[\Big(b_{20}b_{00} + b_{21}b_{10} + b_{22}b_{20}\Big)\Big((1-\delta_i)\,u_{0i} - a_i - a_i T - (1-\delta_i)\,u_{2i}\Big) \\
&+ \Big(b_{20}b_{01} + b_{21}b_{11} + b_{22}b_{21}\Big)\Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,\frac{u_{1i}}{2} \\
&- (a_i + \frac{1}{2}b_i T) - (1-\delta_i)\,\frac{u_{2i}}{2} - (a_i + b_i T)\Big) + \Big(b_{20}b_{02} + b_{21}b_{12} + b_{22}b_{22}\Big) \\
&\quad \Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,u_{1i} + (1-\delta_i)\,u_{2i} - 3(a_i + b_i T)\Big)\Bigg] \\
&+ \sum_{i=1}^{n} \Bigg[T\,b_{20}\Big(C_{ui}u_{0i} - C_{ui}u_{2i}\Big) + T\,b_{21}\Big(C_{ui}u_{0i} + \frac{1}{2}C_{ui}u_{1i} - \frac{1}{2}C_{ui}u_{2i}\Big) \\
&+ Tb_{22}\Big(C_{ui}u_{0i} + C_{ui}u_{1i} + C_{ui}u_{2i}\Big)\Bigg]
\end{aligned} \tag{37}
$$

subject to the constraints,

$$
\begin{aligned}
&\sum_{i=1}^{n} a_{ui}T\Bigg[\Big(b_{00} + b_{10} + b_{20}\Big)\Big((1-\delta_i)\,u_{0i} - a_i - a_i T - (1-\delta_i)\,u_{2i}\Big) \\
&+ \Big(b_{01} + b_{11} + b_{21}\Big) \\
&\times \Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,\frac{u_{1i}}{2} - (a_i + \frac{1}{2}b_i T) - (1-\delta_i)\,\frac{u_{2i}}{2} - (a_i + b_i T)\Big) \\
&+ \Big(b_{02} + b_{12} + b_{22}\Big)\Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,u_{1i} + (1-\delta_i)\,u_{2i} - 3(a_i + b_i T)\Big)\Bigg] \le S_p
\end{aligned} \tag{38}
$$

Now, the problem reduces to minimize J_1 given by (37) subject to space constraint (38). The reduced problem contains unknowns u_{0i}, u_{1i}, u_{2i} which are coefficients of production. Using a gradient based non-linear optimization method-GRG(LINGO-11.0)(cf. Gabriel and Ragsdell [5]), u_{0i}, u_{1i}, u_{2i} are obtained, satisfying Space Constraint (38) in equality sense.

Therefore $J_2(u_{0i}, u_{1i}, u_{2i})$

$$
\begin{aligned}
&= \sum_{i=1}^{n} T^2 h_i \Bigg[\Big(b_{20}b_{00} + b_{21}b_{10} + b_{22}b_{20}\Big)\Big((1-\delta_1)\,u_{0i} - (1-\delta_1)\,u_{2i} - 3(a_1 + b_1 t_1)\Big)\Bigg] \\
&+ \Big(b_{20}b_{01} + b_{21}b_{11} + b_{22}b_{21}\Big)\Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,\frac{1}{2}u_{1i} + (1-\delta_i)\,u_{2i} \\
&\qquad -3(a_1 + b_1 t_1)\Big) \\
&+ \Big(b_{20}b_{02} + b_{21}b_{12} + b_{22}b_{22}\Big)\Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,u_{1i} + (1-\delta_i)\,u_{2i} \\
&\qquad -3(a_1 + b_1 t_1)\Big)\Bigg] \\
&+ \sum_{i=1}^{n} \Bigg[T\, b_{20}\Big(C_{ui}u_{0i} - C_{ui}u_{2i}\Big) + T\, b_{21}\Big(C_{ui}u_{0i} + \frac{1}{2}C_{ui}u_{1i} - \frac{1}{2}C_{ui}u_{2i}\Big) \\
&+ T b_{22}\Big(C_{ui}u_{0i} + C_{ui}u_{1i} + C_{ui}u_{2i}\Big)\Bigg]
\end{aligned} \tag{39}
$$

subject to the constraints,

$$
\begin{aligned}
&\sum_{i=1}^{n} a_{ui} T \Bigg[\Big(b_{00} + b_{10} + b_{20}\Big)\Big((1-\delta_1)\,u_{0i} - (1-\delta_1)\,u_{2i} - 3(a_1 + b_1 t_1)\Big) \\
&+ \Big(b_{01} + b_{11} + b_{21}\Big)\Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,\frac{1}{2}u_{1i} + (1-\delta_i)\,u_{2i} - 3(a_1 + b_1 t_1)\Big) \\
&+ \Big(b_{02} + b_{12} + b_{22}\Big)\Big((1-\delta_i)\,u_{0i} + (1-\delta_i)\,u_{1i} + (1-\delta_i)\,u_{2i} - 3(a_1 + b_1 t_1)\Big)\Bigg] \le S_p
\end{aligned} \tag{40}
$$

Now, the problem reduces to minimize J_2 given by (39) subject to space constraint (40). The reduced problem contains unknowns u_{0i}, u_{1i}, u_{2i} which are coefficients of production. Using a gradient based non-linear optimization method-GRG(LINGO-11.0)(cf. Gabriel and Ragsdell [5]), u_{0i}, u_{1i}, u_{2i} are obtained, satisfying Space Constraint (40) in equality sense.

5 Numerical Experiment

5.1 Input Data:

To illustrate the models, we consider the following data for $n = 2$;
$a_{u1} = 0.3$, $a_{u2} = 0.27$; $h_1 = 0.5, h_2 = 0.45$; $T = 12$; $\delta_1 = 0.010$, $\delta_2 = 0.011$, $a_1 = 5$, $a_2 = 6$; $b_1 = 2$, $b_2 = 2.1$, $S_p = 500$; $b_{00} = 0.0$, $b_{01} = 0.0$, $b_{02} = 0.0$; $b_{10} = 0.66; b_{11} = 0.20; b_{12} = 0.14; b_{20} = 0.33; b_{21} = 1.34; b_{22} = 0.33; cu_1 = 10;$

5.2 Results

For the above input data and using GRG (cf. Gabriel and Ragsdell [5]), the optimal values of production and objective function are obtained. Optimal production and optimum objective value for this model are $u_1(t) = 46 + 12.3t + 44(2t^2 - 1)$, $u_2(t) = 47.28 + 11.60t + 42.71(2t^2 - 1)$ and \$516.95.

$$x_1(t) = \begin{cases} (1-\delta_1)\big(46t + 12.3\frac{t^2}{2} + 44(\frac{2t^3}{3} - 1)\big) - \big(a_1 t + b_1 \frac{t^2}{2}\big) \text{ if } 0 \le t \le t_1 \\ (1-\delta_1)\big(46t + 12.3\frac{t^2}{2} + 44(\frac{2t^3}{3} - 1)\big) - \big(a_1 t + b_1 t t_1\big) \text{ if } t_1 \le t \le T \end{cases} \quad (41)$$

$$x_2(t) = \begin{cases} (1-\delta_2)\big(47.28t + 11.60\frac{t^2}{2} + 42.71(\frac{2t^3}{3} - 1)\big) - \big(a_2 t + b_2 \frac{t^2}{2}\big) \text{ if } 0 \le t \le t_1 \\ (1-\delta_2)\big(47.28t + 11.60\frac{t^2}{2} + 42.71(\frac{2t^3}{3} - 1)\big) - \big(a_2 t + b_2 t t_1\big) \text{if } t_1 \le t \le T \end{cases} \quad (42)$$

Table 1. Optimum results

Particular Case	u_{01}	u_{11}	u_{21}	u_{02}	u_{12}	u_{22}	$J\$$
$l = 2, m = 2, n = 2$	45.00	9.90	50.00	42.92	10.00	51.78	914.32
	41.50	11.91	43.00	40.38	12.86	48.19	668.34
	42.00	11.30	45.23	39.35	12.13	43.45	704.62
	46.00	**12.58**	**44.25**	**47.28**	**11.60**	**42.71**	**516.95**
	43.86	9.98	44.35	41.78	10.20	42.25	898.67
	46.27	8.24	45.56	45.38	9.25	32.20	1686.33
	48.62	7.38	35.68	44.32	9.70	50.02	1095.25

For the given data, it is observed from Table 1 that total cost is minimum for the model with quadratic production function. Here optimal control production inventory problems are analyzed by considering the whole finite time horizon as a single cycle and the optimum minimum cost (\$516.95).

6 Practical Implication

In this investigation, multi-item finite time horizon production-inventory model is considered. The items are produced for the some time to satisfy the demand. But once an optimum stock level is reached, then the production is stopped. In real-life situation, the production is started when the stock is nil and this will be continued for certain period keep in mind the business/marketing policy. This type of relation among production, stock and demand are observed for the items like oil, fruit, rice, etc.

7 Conclusion

This model has proposed an optimum production-inventory policy for a multi-item production inventory system with deteriorating (/defective) units, space capacity and budget/investment constraints and dynamic demand. For the first time, a multi-item dynamic system with space constraints has been formulated and solved by Chebyshev approximation and GRG techniques. The formulation and analysis presented here can be extended to other production-inventory problems with different types of demand, advertisement, deterioration/defectiveness, price discount, etc.

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Production Dependent Agricultural 3D Transportation Problem with Maximization of Annual Net Profit in Generalized Intuitionistic Fuzzy Environment

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Abstract. In this paper, we have presented a 3D transportation problem in generalized intuitionistic fuzzy environment in order to maximize the profit. Here, annual net profit as an objective function has been introduced in the transportation problem. Furthermore, as the production rate is not same for the whole year for the agricultural products, we have introduced average month wise production rate. As a result the availability varies month wise. Also, the purchasing price at each source and the selling price at each destination varies month wise because of the different month wise average production rates. At the same time, the demand at each destination varies due to the variation of selling price. This paper treats all costs, demands, supplies, capacities as generalized intuitionistic fuzzy in nature. The reduced deterministic models has been obtained on implementation of a defuzzification approach by using the accuracy function. The model has been illustrated by numerical example and also the optimal results has been presented by solving the reduced deterministic problem using generalized reduced gradient (GRG) method by lingo software.

Keywords: 3D transportation problem · Profit maximization · Generalized intuitionistic fuzzy number · Accuracy function

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1 Introduction

The traditional transportation problem which is introduced by Hitchcock [3] is an optimization problem. It minimizes the transportation cost of a product from some sources to some destinations involving two kinds of constraints taken into consideration, i.e., availability constraint and demand constraint. The traditional transportation problem (2D-TPs) has received a great interest in the literature. Many researchers have studied this problem from different perspective and obtained important results such as [4,5,7,9,19,20]. But sometimes we have to deal with one more constraint besides the above two constraints, which is conveyance constraint. A suitable mode of transportation is to be determined at each source to attain minimum total cost, or maximum profit. As a result the solid transportation problem (STPs/3D-TPs) which considers three constraints that is, source constraint, destination constraint and conveyance constraint was introduced by Haley [6] in 1962. The solid transportation problem(STP) is an extension of the traditional transportation problem(TP). Subsequently, Several researchers have carried out investigations on STP and numerous models have been considered. For example, Bhatia [10] offered an algorithm to solve a STP with indefinite quadratic objective function. Again Pandian et al. [8] offered a new approach to find an optimal solution of the STP. Ojha et al. [21] considered a STP with fixed charge, vehicle cost, price discounted varying charge and solved it using genetic algorithm.

Generally in TP or STP, depending on the decision maker (DM) there may be different objectives to optimize. The different objectives may be minimization of total transportation costs, maximization of profit, minimization of breakability, total delivery time, etc. Several researchers have concentrated their studies on the above said objectives. Extensive research works have been done by numerous researchers on the objective of maximization of profit. But still now annual net profit as an objective function is not introduced in the TP. But in practical problems, this is of great importance. The production rate of some agricultural products are not same for the whole year due to seasonal change. As a result the availability of the items varies. Again depending on the availability, the purchasing price at the source and selling price at the destination also varies. At the same time depending on the selling price demand of some items varies. This phenomenon has been ignored by the researchers in the domain of TP or STP.

In the classical TP, it is considered that the problem is well-defined and can be described precisely. However, In real life, this is not applicable because of the inherent uncertainty and impression of the available data. Due to the complex environment in the society during the transportation activities, various relevant parameters in the solid transportation problem are often treated as uncertain variables to handle the practical conditions. For example, if someone needs to make a transportation plan, the availability at each source, the demand at each destination are required to be estimated by professional judgments or probability statistics because of no precise information. As a result, it is not appropriate to consider the relevant parameters as constant numbers. Currently, the researchers are giving more importance in modeling and solving the TPs where the parameters are considered in imprecise environment such as random environment, fuzzy environment, etc. The fuzzy set theory was introduced by Zadeh in 1965 [22] which offers the preliminaries for modeling the problems in such conditions. Again The fuzzy programming approach was introduced by Zimmermann [2] for solving linear programming problem with several objective functions. The fuzzy programming approach was offered by Zimmermann [2] to solve linear programming problem with several objective functions. Plentiful works on TP has been carried out in imprecise environment. For example, Li et al. [23] developed a fuzzy approach to the multiobjective transportation problem. A multi-objective STP in imprecise environments was presented and solved by Pramanik et al. [11]. Again Pramanik et al. [13] developed a multi-objective solid transportation problem in fuzzy, bi-fuzzy environment via genetic algorithm. Pramanik et al. [12] also considered a fixed charge multi-objective STP in random fuzzy environment and got the solution.

Nowadays different generalizations and modifications of the ordinary fuzzy set theory concept have appeared. For instance, an extension of the ordinary fuzzy set theory namely the intuitionistic fuzzy set theory was introduced by Atanassov [14]. Such a concept generalizes the concept of ordinary fuzzy sets by taking into consideration not only the degree of membership of the elements to a given set, but also the degree of non-membership along with the degree of hesitation. Thus, the intuitionistic fuzzy set (IFS) concept proposes a richer representation to grip the present uncertainty as compared to the ordinary fuzzy sets concept. Farhadinia et al. [24] developed new similarity measures of generalized intuitionistic fuzzy numbers and generalized interval-valued fuzzy numbers from similarity measures of generalized fuzzy numbers. Seikh et al. [25] developed general definition of triangular intuitionistic fuzzy number by removing the normality in the definition of intuitionistic fuzzy number. Chakraborty et al. [16] presented a new approach to solve multi-objective multi-choice multi-item Atanassov's intuitionistic fuzzy TP using chance operator. Chakraborty et al. [15] used the expected value of intuitionistic fuzzy number (IFN) to solve multi-objective, multi-item intuitionistic fuzzy STP for damageable items. Again Chakraborty et al. [17] also developed arithmetic operations on generalized intuitionistic fuzzy number and solved TP. Jana [18] presented some arithmetic operations on type-2 intuitionistic fuzzy environment and solved a transportation problem.

The main contributions of this paper are summarized as follows:

- A production dependent agricultural 3D transportation problem in generalized intuitionistic fuzzy environment is formulated.
- Annual net profit as an objective function is introduced in the proposed transportation problem.
- we introduce average month wise production rate.
- The reduced deterministic models are obtained on implementation of a defuzzification approach by using the accuracy function.
- The model is illustrated by numerical example and optimal results are presented in tabular forms.
- Reduced crisp problem is solved by LINGO-14.0.

The paper is organized as follows. Section 2 reviews some basic definitions and theorems related to GIFNs. In Sect. 3, notations for the proposed model are listed. In Sect. 4 we formulate the model. In Sect. 5 equivalent crisp models are obtained. The numerical illustration is given in Sect. 6 and a set of optimal solutions are presented here. The paper ends with the conclusion.

2 Preliminaries

In this section, we evoke some basic definitions and results which will be used in the next sections of this paper.

Definition 1. *An intuitionistic fuzzy set (IFS) is a generalization of the ordinary fuzzy sets which is characterized by a membership function and a non-membership function. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a collection of some objects, then an intuitionistic fuzzy set A^I in X is defined as the form of an ordered triplet $A^I = \{\langle x_i, \mu_{A^I}(x_i), \nu_{A^I}(x_i)\rangle / x_i \in X\}$,*
where $\mu_{A^I}(x_i) : X \to [0,1]$ is called the membership function or grade of membership of x_i in $\tilde{A}$ and $\nu_{A^I}(x_i) : X \to [0,1]$ is called the non-membership function or grade of non-membership of x_i in $\tilde{A}$ satisfying the condition $0 \leq \mu_{A^I}(x_i) + \nu_{A^I}(x_i) \leq 1$. $\pi_{A^I}(x) = 1 - \mu_{A^I}(x) - \nu_{A^I}(x)$ represents the degree of hesitation or the degree of indeterminacy of x_i being in A^I in X and $0 \leq \pi_{A^I} \leq 1$.

Definition 2. *The α-cut of an IFS A^I is denoted as A^I_α and is given by:*
$A^I_\alpha = \{x \in X : \mu_{A^I}(x) \geq \alpha\}, \forall \alpha \in [0,1]$

Definition 3. *The β-cut of an IFS A^I is denoted as A^I_β and is given by:*
$A^I_\beta = \{x \in X : \nu_{A^I}(x) \leq \beta\}, \forall \beta \in [0,1]$

Definition 4. *The (α,β)-cut of an IFS A^I is denoted as $A^I_{\alpha,\beta}$ and is given by:*
$A^I_{\alpha,\beta} = \{x \in X : \mu_{A^I}(x) \geq \alpha, \nu_{A^I}(x) \leq \beta; \alpha + \beta \leq 1\}, \forall \alpha, \beta \in [0,1]$

Definition 5. *An IFS $A^I = \{\langle x, \mu_{A^I}(x), \nu_{A^I}(x)\rangle / x \in \Re\}$ is called an intuitionistic fuzzy number (IFN) if the following holds*
(i) There exists $m \in \Re$ such that $\mu_{A^I}(m) = 1$ and $\nu_{A^I}(m) = 0$ (m is called the mean value of A^I)
(ii) The membership function μ_{A^I} and non-membership function ν_{A^I} are piecewise continuous functions from R to the closed interval [0, 1] and $0 \leq \mu_{A^I}(x) + \nu_{A^I}(x) \leq 1, \forall x \in \Re$. μ_{A^I}, ν_{A^I} are of the following forms:

$$\mu_{A^I}(x) = \begin{cases} .f_1(x), & for\ m - l < x < m \\ 1, & for\ x = m \\ f_2(x), & for\ m < x < m + r \\ 0, & otherwise \end{cases}$$

and

$$\nu_{A^I}(x) = \begin{cases} g_1(x), & for\ m - l' < x < m; 0 \leq f_1(x) + g_1(x) \leq 1 \\ 0, & for\ x = m \\ g_2(x), & for\ m < x < m + r'; 0 \leq f_2(x) + g_2(x) \leq 1 \\ 1, & otherwise \end{cases}$$

Here, f_1 and f_2 are piecewise continuous, strictly increasing and strictly decreasing functions in $(m-l, m)$ and $(m, m+r)$ respectively. Again g_1 and g_2 are piecewise continuous, strictly decreasing and strictly increasing functions in $(m-l', m)$ and $(m, m+r')$ respectively. l and r are the left and right spreads of membership function μ_{A^I} respectively. Again l' and r' are the left and right spreads of non-membership function ν_{A^I} respectively. The IFN A^I is represented by $(m; l, r; l', r')$.

Definition 6. *An IFN A^I is called Trapezoidal Intuitionistic Fuzzy Number(TIFN) if its membership function μ_{A^I} and non-membership function ν_{A^I} are as follows:*

$$\mu_{A^I}(x) = \begin{cases} \frac{x-r_1}{r_2-r_1}, & \textit{for } r_1 \leq x \leq r_2 \\ 1, & \textit{for } r_2 \leq x \leq r_3 \\ \frac{r_4-x}{r_4-r_3}, & \textit{for } r_3 \leq x \leq r_4 \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\nu_{A^I}(x) = \begin{cases} \frac{r_2-x}{r_2-r_1'}, & \textit{for } r_1' \leq x \leq r_2 \\ 0, & \textit{for } r_2 \leq x \leq r_3 \\ \frac{x-r_3}{r_4'-r_3}, & \textit{for } r_3 \leq x \leq r_4' \\ 1, & \textit{otherwise} \end{cases}$$

where $r_1' \leq r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_4'$. The TIFN A^I in $\Re$ is represented as $(r_1, r_2, r_3, r_4; r_1', r_2, r_3, r_4')$ with its membership function μ_{A^I} and non-membership function ν_{A^I}.

Definition 7. *A generalized Trapezoidal Intuitionistic Fuzzy Number (GTIFN) A^I is said to be an intuitionistic fuzzy set on $\Re$ if its membership function μ_{A^I} and non-membership function ν_{A^I} are as follows:*

$$\mu_{A^I}(x) = \begin{cases} w\frac{x-r_1}{r_2-r_1}, & \textit{for } r_1 \leq x \leq r_2 \\ w, & \textit{for } r_2 \leq x \leq r_3 \\ w\frac{r_4-x}{r_4-r_3}, & \textit{for } r_3 \leq x \leq r_4 \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\nu_{A^I}(x) = \begin{cases} \frac{(r_2-x)+u(x-r_1')}{(r_2-r_1')}, & \textit{for } r_1' \leq x \leq r_2 \\ u, & \textit{for } r_2 \leq x \leq r_3 \\ \frac{(x-r_3)+u(r_4'-x)}{(r_4'-r_3)}, & \textit{for } r_3 \leq x \leq r_4' \\ 1, & \textit{otherwise} \end{cases}$$

where $r_1' \leq r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_4'$ and $w, u \in [0, 1]$ so that $w + u \leq 1$. Here, w represents the maximum degree of the membership function and u represents the minimum degree of the non-membership function. The GTIFN A^I in $\Re$ is represented as $(r_1, r_2, r_3, r_4; r_1', r_2, r_3, r_4'; w, u)$ with its membership function μ_{A^I} and non-membership function ν_{A^I}.

Definition 8. *The LR-type representation of a GTIFN* $A^I = (r_1, r_2, r_3, r_4; r_1', r_2, r_3, r_4'; w, u)$ *is given by* $A^I = (r_2, r_3; l_{\mu_r}, r_{\mu_r}; l_{\nu r}, r_{\nu r}; w, u)$, *where* $l_{\mu_r} = r_2 - r_1, r_{\mu_r} = r_4 - r_3, l_{\nu r} = r_2 - r_1'$ *and* $r_{\nu r} = r_4' - r_3$ *and its membership function* μ_{A^I} *and non-membership function* ν_{A^I} *is defined by*

$$\mu_{A^I}(x) = \begin{cases} w\frac{x-r_1}{l_{\mu_r}}, & \textit{for } r_2 - l_{\mu_r} \le x \le r_2 \\ w, & \textit{for } r_2 \le x \le r_3 \\ w\frac{r_4-x}{r_{\mu_r}}, & \textit{for } r_3 \le x \le r_3 + r_{\mu_r} \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\nu_{A^I}(x) = \begin{cases} \frac{(1-u)(r_2-x)+u(x-r_1')}{l_{\nu r}}, & \textit{for } r_2 - l_{\nu r} \le x \le r_2 \\ u, & \textit{for } r_2 \le x \le r_3 \\ \frac{(1-u)(x-r_3)+ur_{\nu r}}{l_{\nu r}}, & \textit{for } r_3 \le x \le r_3 + r_{\nu r} \\ 1, & \textit{otherwise} \end{cases}$$

Definition 9 [25]**.** *For two GTIFNs* $X^I = (a_1, a_2; l_{\mu_a}, r_{\mu_a}; l_{\nu a}, r_{\nu a}; w_a, u_a)$, $Y^I = (b_1, b_2; l_{\mu_b}, r_{\mu_b}; l_{\nu b}, r_{\nu b}; w_b, u_b)$ *and for a real* $\alpha > 0$, *the arithmetic operations are defined as follows:*

Addition
$X^I + Y^I = (a_1 + b_1, a_2 + b_2; l_{\mu_a} + l_{\mu_b}, r_{\mu_a} + r_{\mu_b}; l_{\nu a} + l_{\nu b}, a_{\nu a} + r_{\mu_b}; min(w_a + w_b), max(u_a + u_b))$

Subtraction
$X^I - Y^I = (a_1 - b_1, a_2 - b_2; l_{\mu_a} + l_{\mu_b}, r_{\mu_a} + r_{\mu_b}; l_{\nu a} + l_{\nu b}, a_{\nu a} + r_{\mu_b}; min(w_a + w_b), max(u_a + u_b))$

Scalar Multiplication
$\alpha X^I = (\alpha a_1, \alpha a_2; \alpha l_{\mu_a}, \alpha r_{\mu_a}; \alpha l_{\nu a}, \alpha r_{\nu a}; w_a, u_a)$

Theorem 1. *The* α*-cut and* β*-cut of a GTIFN* $A^I = (r_1, r_2, r_3, r_4; r_1', r_2, r_3, r_4'; w, u)$ *are given by* $A^I_\alpha = [r_1 + (r_2 - r_1)\frac{\alpha}{w}, r_4 - (r_4 - r_3)\frac{\alpha}{w}]$ *and* $A^I_\beta = [\frac{r_2 - ur_1' - (r_2 - r_1')\beta}{(1-u)}, \frac{r_3 - ur_4' + (r_4' - r_3)\beta}{(1-u)}]$, $\forall \alpha \in (0, w]\, and\, \beta \in [u, 1]$

Proof. For $\alpha \in (0, w]$,

$$\begin{aligned} \mu_{A^I}(x) \ge \alpha \; &\Rightarrow w(\frac{x - r_1}{r_2 - r_1}) \ge \alpha,\; w(\frac{r_4 - x}{r_4 - r_3}) \ge \alpha \\ &\Rightarrow x \ge r_1 + (r_2 - r_1)\frac{\alpha}{w},\; x \le r_4 - (r_4 - r_3)\frac{\alpha}{w} \\ &\Rightarrow r_1 + (r_2 - r_1)\frac{\alpha}{w} \le x \le r_4 - (r_4 - r_3)\frac{\alpha}{w} \\ &\Rightarrow A^I_\alpha = [r_1 + (r_2 - r_1)\frac{\alpha}{w}, r_4 - (r_4 - r_3)\frac{\alpha}{w}] \end{aligned}$$

Now for $\beta \in [u, 1]$,

$$\begin{aligned}
\nu_{A^I}(x) \leq \beta \; &\Rightarrow \frac{(r_2 - x) + u(x - r'_1)}{r_2 - r'_1} \leq \beta, \; \frac{(x - r_3) + (r'_4 - x)u}{r'_4 - r_3} \leq \beta \\
&\Rightarrow x \geq \frac{r_2 - ur'_1 - (r_2 - r'_1)\beta}{(1-u)}, \; x \leq \frac{r_3 - ur'_4 + (r'_4 - r_3)\beta}{(1-u)} \\
&\Rightarrow \frac{r_2 - ur'_1 - (r_2 - r'_1)\beta}{(1-u)} \leq x \leq \frac{r_3 - ur'_4 + (r'_4 - r_3)\beta}{(1-u)} \\
&\Rightarrow A^I_\beta = [\frac{r_2 - ur'_1 - (r_2 - r'_1)\beta}{(1-u)}, \frac{r_3 - ur'_4 + (r'_4 - r_3)\beta}{(1-u)}]
\end{aligned}$$

Hence it is proved.

Theorem 2. *The (α, β)-cut of a GTIFN $A^I = (r_1, r_2, r_3, r_4; r'_1, r_2, r_3, r'_4; w, u)$ is given by $A^I_{\alpha,\beta} = [r_1 + (r_2 - r_1)\frac{\alpha}{w}, r_4 - (r_4 - r_3)\frac{\alpha}{w}] \cap [\frac{r_2 - ur'_1 - (r_2 - r'_1)\beta}{(1-u)}, \frac{r_3 - ur'_4 + (r'_4 - r_3)\beta}{(1-u)}]$, $\forall \alpha \in (0, w]$, $\beta \in [u, 1]$ and $\alpha + \beta \leq 1$*

Proof. For $\alpha \in (0, w]$, the α-cut of a GTIFN $A^I = (r_1, r_2, r_3, r_4; r'_1, r_2, r_3, r'_4; w, u)$ is given by $A^I_\alpha = [r_1 + (r_2 - r_1)\frac{\alpha}{w}, r_4 - (r_4 - r_3)\frac{\alpha}{w}]$.
For $\beta \in [u, 1]$, the β-cut of the GTIFN $A^I = (r_1, r_2, r_3, r_4; r'_1, r_2, r_3, r'_4; w, u)$ is given by $A^I_\beta = [\frac{r_2 - ur'_1 - (r_2 - r'_1)\beta}{(1-u)}, \frac{r_3 - ur'_4 + (r'_4 - r_3)\beta}{(1-u)}]$.
So by Definition 4, $A^I_{\alpha,\beta} = [r_1 + (r_2 - r_1)\frac{\alpha}{w}, r_4 - (r_4 - r_3)\frac{\alpha}{w}] \cap [\frac{r_2 - ur'_1 - (r_2 - r'_1)\beta}{(1-u)}, \frac{r_3 - ur'_4 + (r'_4 - r_3)\beta}{(1-u)}]$, $\forall \alpha \in (0, w]$, $\beta \in [u, 1]$ and $\alpha + \beta \leq 1$.
Hence it is proved.

Definition 10 [1]**.** *Let the (α, β)-cut of a GTIFN is given by*

$$A^I_{\alpha,\beta} = [A_1(\alpha), A_2(\alpha)] \cap [A'_1(\beta), A'_2(\beta)]; \alpha + \beta \leq 1, \forall \alpha \in (0, w] \, and \, \beta \in [u, 1];$$

Then by Mean of (α, β)-cut method, the representation of membership function is

$$R_\mu(A^I) = \frac{1}{2} \int_0^w \Big[A_1(\alpha) + A_2(\alpha) \Big] d\alpha.$$

Again by Mean of (α, β)-cut method the representation of non-membership function is

$$R_\nu(A^I) = \frac{1}{2} \int_u^1 \Big[A'_1(\beta) + A'_2(\beta) \Big] d\beta.$$

The accuracy function of A^I is denoted by $f(A^I)$ and defined by

$$f(A^I) = \frac{R_\mu(A^I) + R_\nu(A^I)}{2},$$

to defuzzify the given numbers as deterministic one.

Theorem 3. *Let* $A^I = (r_1, r_2, r_3, r_4; r_1', r_2, r_3, r_4'; w, u)$ *be a GTIFN. Then its accuracy function is given by* $f(A^I) = \frac{1}{8}\Big[\Big\{(r_1 + r_2 + r_3 + r_4)w\Big\} + \Big\{(r_1' + r_2 + r_3 + r_4')(1-u)\Big\}\Big]$.

Proof. By definition, the accuracy function of A^I is defined by

$$
\begin{aligned}
f(A^I) &= \frac{R_\mu(A^I) + R_\nu(A^I)}{2} \\
&= \frac{\frac{1}{2}\int_0^w \Big[A_1(\alpha) + A_2(\alpha)\Big] d\alpha + \frac{1}{2}\int_u^1 \Big[A'_1(\beta) + A'_2(\beta)\Big] d\beta}{2} \\
&= \frac{1}{4}\Bigg[\int_0^w \Bigg\{\Bigg(r_1 + (r_2 - r_1)\frac{\alpha}{w}\Bigg) + \Bigg(r_4 - (r_4 - r_3)\frac{\alpha}{w}\Bigg)\Bigg\} d\alpha \\
&\quad + \int_u^1 \Bigg\{\Bigg(\frac{r_2 - ur_1' - (r_2 - r_1')\beta}{(1-u)}\Bigg) + \Bigg(\frac{r_3 - ur_4' + (r_4' - r_3)\beta}{(1-u)}\Bigg)\Bigg\} d\beta\Bigg] \\
&= \frac{1}{4}\Bigg[\Bigg\{(r_1 + r_2)\frac{w}{2} + (r_3 + r_4)\frac{w}{2}\Bigg\} + \Bigg\{(r_1' + r_2)\frac{(1-u)}{2} + (r_3 + r_4')\frac{(1-u)}{2}\Bigg\}\Bigg]
\end{aligned}
$$

Hence,

$$
f(A^I) = \frac{1}{8}\Bigg[\Bigg\{(r_1 + r_2 + r_3 + r_4)w\Bigg\} + \Bigg\{(r_1' + r_2 + r_3 + r_4')(1-u)\Bigg\}\Bigg] \quad (1)
$$

Hence it is proved.

3 Notations

To formulate the solid transportation model the following notations are used:

(i) m = number of sources.
(ii) n = number of destinations.
(iii) K = number of different types of conveyances i.e., different modes of the transportation problem.
(iv) p_i^I = the purchasing price of the product at the i^{th} origin with respect to full production rate.
(v) s_j^I = the selling price of the product at the j^{th} destination with respect to full production rate.
(vi) c_{ijk}^I = the unit transportation cost of the product from i^{th} source to j^{th} destination by k^{th} conveyance.
(vii) a_i^I = the amount of the product available at the i^{th} origin with respect to full production rate.
(viii) b_j^I = the demand of the product at the j^{th} destination with respect to full production rate.
(ix) h_j^I = the holding cost of the product at the j^{th} destination for the whole year.

(x) e_k = capacity of a single vehicle of k^{th} conveyance.
(xi) β_t = average production rate during t^{th} month.
(xii) P_{it}^I = the average purchasing price of the product at the i^{th} origin during t^{th} month.
(xiii) S_{jt}^I = the average selling price of the product at the j^{th} destination during t^{th} month.
(xiv) A_{it}^I = the average amount of the product available at the i^{th} origin during t^{th} month.
(xv) B_{jt}^I = the average demand of the product at the j^{th} destination during t^{th} month.
(xvi) x_{ijkqt} = the decision variable which is the amount of the product to be transported from i^{th} source to j^{th} destination by k^{th} conveyance during t^{th} month.

4 Model Formulation

We assume that there are m origins (or sources) O_i $(i = 1, 2, \cdots, m)$, n destinations(or demands) $D_j(j = 1, 2, \cdots, n)$, K conveyances $E_k(k = 1, 2, \cdots, K)$ i.e., different modes of transport may be trucks, cargo flights, goods trains, ships, etc. In this model, the objective is to maximize the annual net profit incurred by the transportation activities. The production rate of some items are not same for the whole year due to seasonal change. As a result the availability of the items at the origin varies. Again depending on the availability, the purchasing price at the source and selling price at the destination also varies. At the same time depending on the selling price demand of the item varies. Moreover at present situation, it is not so simple for anyone to estimate the exact amount of related parameters. Practically in application of TP, decision makers may face different uncertainties such as availability of the item in the sources, demands in the destinations, unit transportation cost etc. due to various unmanageable factors. These unmanageable factors in a TP may be as follows: (i) there exists uncertainty regarding the product availability at a source due to the time factors, (ii) there exists some sort of vagueness about the total demand of a newly launched product to the market, (iii) there may not be a decision maker who exactly knows the unit transportation cost of the first time transportation operation.

Considering this fact, we take the problem under generalized intuitionistic fuzzy environment, which aims to make service strategies on the tactical planning level. So, we formulate the problem assuming that parameters are all generalized intuitionistic fuzzy variables.

The objective of the problem is to maximize the annual net profit, which is as follows:

$$\max \quad Z^I = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{t=1}^{12}\left[(S_{jt}^I - P_{it}^I - c_{ijk}^I)\ \ x_{ijkt}\right] - \sum_{j=1}^{n} y_j h_j^I$$

Here the average purchasing price of the product at the i^{th} origin during t^{th} month i.e. P_{it}^I and the average selling price of the product at the j^{th} destination during that month i.e. S_{jt}^I is inversely proportional to average production rate during the same month i.e. β_t. So

$$\begin{aligned} P_{it}^I \propto \frac{1}{\beta_t} \ \ and \ \ S_{jt}^I \ &\propto \ \frac{1}{\beta_t} \\ \Rightarrow P_{it}^I = p_i^I \frac{1}{\beta_t} \ \ and \ \ S_{jt}^I \ &= \ s_j^I \frac{1}{\beta_t} \end{aligned}$$

Again if the product is transported to j^{th} destination for any time, then only the holding cost will be taken into consideration. So, we introduce a binary relation as

$$y_j = \begin{cases} 1, \text{ if } \sum\limits_{i=1}^{m} \sum\limits_{k=1}^{K} \sum\limits_{t=1}^{12} x_{ijkt} > 0; \\ 0, \text{ if } \sum\limits_{i=1}^{m} \sum\limits_{k=1}^{K} \sum\limits_{t=1}^{12} x_{ijkqt} = 0 \end{cases}$$

Now, the constraints are the supply constraints, demand constraints and capacity constraints. As the quantity of a product from a source cannot exceed the supply capacity, so we have

$$\sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijkt} \leq A_{it}^I, i = 1, 2, 3, \cdots, m; t = 1, 2, \cdots, 12$$

Here, the average amount of the product available at the i^{th} origin during t^{th} month i.e. A_{it}^I depends on the average production rate during that month i.e. β_t. So

$$\begin{aligned} A_{it}^I &\propto \beta_t \\ \Rightarrow A_{it}^I &= a_i^I \beta_t \end{aligned}$$

Again the quantity of a product transported to a destination should not be less than its demand, that is

$$\sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijkt} \geq B_{jt}^I, j = 1, 2, 3, \cdots, n; t = 1, 2, \cdots, 12$$

Here, we consider that the average demand of the product at the j^{th} destination during t^{th} month i.e. B_{jt}^I depends only on the average selling price of the product at the j^{th} destination during that month i. e. S_{jt}^I. B_{jt}^I is inversely proportional to S_{jt}^I. Again since S_{jt}^I is inversely proportional to average production rate during the same month i.e. β_t. So we can consider that B_{jt}^I is directly proportional to β_t. So we have

$$\begin{aligned} B_{jt}^I &\propto \beta_t \\ \Rightarrow B_{jt}^I &= b_j^I \beta_t \end{aligned}$$

The third constraint requires that the total amount of product transported from different sources to different destinations by conveyance k are not greater than the transportation capacity of total number of k^{th} conveyance available at each source. So we have

$$\sum_{j=1}^{n}\sum_{t=1}^{12} x_{ijkt} \geq N_{ik}e_k, i = 1,2,3,\cdots,m; k = 1,2,\cdots,K$$

It is usual to have the nonnegativity of decision variable x_{ijkt}, that is

$$x_{ijkt} \geq 0, i = 1,2,3; j = 1,2,3,\cdots,n; k = 1,2,\cdots,K; t = 1,2,\cdots,12$$

So, the solid transportation problem can be written as:

$$\max \quad Z^I = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{t=1}^{12}\left[(S_{jt}^I - P_{it}^I - c_{ijk}^I)\ x_{ijkt}\right] - \sum_{j=1}^{n} y_j h_j^I \tag{2}$$

where

$$P_{it}^I = \frac{p_i^I}{\beta_t}\ ,\ S_{jt}^I \ = \ \frac{s_j^I}{\beta_t}\ and \tag{3}$$

$$y_j = \begin{cases} 1, \text{ if } \sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K}\sum\limits_{t=1}^{12} x_{ijkt} > 0; \\ 0, \text{ if } \sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K}\sum\limits_{t=1}^{12} x_{ijkqt} = 0 \end{cases} \tag{4}$$

subject to

$$\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijkt} \leq A_{it}^I, i = 1,2,3,\cdots,m; t = 1,2,\cdots,12 \tag{5}$$

$$where A_{it}^I = a_i^I \beta_t \tag{6}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijkt} \geq B_{jt}^I, j = 1,2,3,\cdots,n; t = 1,2,\cdots,12 \tag{7}$$

$$where B_{jt}^I = b_j^I \beta_t \tag{8}$$

$$\sum_{j=1}^{n}\sum_{t=1}^{12} x_{ijkt} \geq N_{ik}e_k, i = 1,2,3,\cdots,m; k = 1,2,\cdots,K \tag{9}$$

$$x_{ijkt} \geq 0, i = 1,2,3; j = 1,2,3,\cdots,n; k = 1,2,\cdots,K; t = 1,2,\cdots,12 \tag{10}$$

5 Equivalent Crisp Problem

The above STP is under imprecise market supplies, demands, capacities, costs. Here, S_{jt}^I P_{it}^I, c_{ijkt}^I, s_j^I, p_i^I, h_j^I, A_{it}^I, B_{jt}^I, a_i^I and b_j^I are considered as GTIFNs,

these can be denoted as:
$S^I_{jt} = (S_{jt_1}, S_{jt_2}, S_{jt_3}, S_{jt_4}; S_{jt}{'}_1, S_{jt_2}, S_{jt_3}, S_{jt}{'}_4; w_S, u_S)$,
$P^I_{it} = (P_{it1}, P_{it2}, P_{it3}, P_{it4}; P_{it}{'}_1, P_{it2}, P_{it3}, P_{it}{'}_4; w_P, u_P)$,
$c^I_{ijkt} = (c_{ijkt_1}, c_{ijkt_2}, c_{ijkt_3}, c_{ijkt_4}; c_{ijkt}{'}_1, c_{ijkt_2}, c_{ijkt_3}, c_{ijkt}{'}_4; w_c, u_c)$,
$s^I_j = (s_{j_1}, s_{j_2}, s_{j_3}, s_{j_4}; s_{j}{'}_1, s_{j_2}, s_{j_3}, s_{j}{'}_4; w_s, u_s)$,
$p^I_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4}; p_{i}{'}_1, p_{i2}, p_{i3}, p_{i}{'}_4; w_p, u_p)$,
$h^I_j = (h_{j_1}, h_{j_2}, h_{j_3}, h_{j_4}; h_{j}{'}_1, h_{j_2}, h_{j_3}, h_{j}{'}_4; w_h, u_h)$,
$A^I_{it} = (A_{it1}, A_{it2}, A_{it3}, A_{it4}; A_{it}{'}_1, A_{it2}, A_{it3}, A_{it}{'}_4; w_A, u_A)$,
$B^I_{jt} = (B_{jt_1}, B_{jt_2}, B_{jt_3}, B_{jt_4}; B_{jt}{'}_1, B_{jt_2}, B_{jt_3}, B_{jt}{'}_4; w_B, u_B)$,
$a^I_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}; a_{i}{'}_1, a_{i2}, a_{i3}, a_{i}{'}_4; w_a, u_a)$,
$b^I_j = (b_{j_1}, b_{j_2}, b_{j_3}, b_{j_4}; b_{j}{'}_1, b_{j_2}, b_{j_3}, b_{j}{'}_4; w_b, u_b)$ respectively. The steps to solve the above model in Eqs. (2)–(10) are as follows:
Step 1: Substitute the values of S^I_{jt} P^I_{it}, c^I_{ijkt}, s^I_j, p^I_i, h^I_j, A^I_{it} given above in Eqs. (2)–(10).
Step 2: Now use the arithmetic operation presented in Definition 9.
Step 3: Then convert to crisp linear programming using Eq. (1).
Step 4: Find the optimal value and corresponding x_{ijkt} by Generalized Reduced Gradient technique using LINGO 12.0 software.

6 Numerical Experiment

In this segment, the following example of a Solid transportation problem to illustrate the efficiency and effectiveness of the proposed approach is considered. The selling prices, purchasing costs of different items, availabilities of these items in the corresponding origins, demands in the destinations, capacity of each vehicle, unit transportation costs of different items, are assumed as GTIFNs which are as follows:

6.1 Input Data

In this experiment we assume that there are two sources, two destinations, two types of conveyances i.e., $m = 2$, $n = 2$, and $K = 2$. Since the annual profit is calculated, we take a time parameter which depends on the month i.e., $T = 12$ The availabilities of the item in the origins and the demands of items in the destinations with respect to full production rate, the capacity of conveyances are given in the Table 1.

Now, the purchasing Price, selling price of the item with respect to full production rate and unit transportation cost of the item by different conveyances are given in the Tables 2, 3 and 4. Yearly holding cost in the two destinations are also given in the Table 5.

Table 1. Availabilities, demands and capacities

a_1^I $(80, 90, 100, 108; 75, 90, 100, 111; 0.8, 0.1)$	a_2^I $(100, 115, 125, 130; 90, 115, 125, 132; 0.7, 0.1)$
b_1^I $(24, 28, 30, 35; 20, 28, 30, 36; 0.8, 0.1)$	b_2^I $(38, 40, 42, 44; 32, 40, 42, 48; 0.85, 0.1)$
e_1^I $(19, 22, 27, 30; 14, 22, 27, 32; 0.7, 0.2)$	e_2^I $(15, 20, 26, 29; 12, 20, 26, 30; 0.65, 0.3)$

Table 2. Purchasing prices

p_1^I	p_2^I
$(52, 55, 58, 60; 50, 55, 58, 62; 0.8, 0.2)$	$(61, 66, 70, 71; 58, 66, 70, 72; 0.7, 0.2)$

Table 3. Selling prices

s_1^I	s_2^I
$(168, 170, 175, 178; 162, 170, 175, 180; 0.85, 0.05)$	$(175, 176, 180, 182; 170, 176, 180, 185; 0.88, 0.04)$

Table 4. unit transportation costs

i	j	c_{ij1}^I	c_{ij2}^I
1	1	$(3, 4, 6, 7; 2, 4, 6, 8; 0.5, 0.3)$	$(4, 5, 7, 8; 3, 5, 7, 9; 0.6, 0.3)$
	2	$(4, 6, 7, 8; 3, 6, 7, 9; 0.6, 0.3)$	$(5, 6, 8, 9; 4, 6, 8, 10; 0.7, 0.2)$
2	1	$(4, 6, 7, 8; 3, 6, 7, 9; 0.7, 0.1)$	$(2, 3, 4, 5; 1, 3, 4, 6; 0.5, 0.3)$
	2	$(4, 5, 7, 8; 3, 5, 7, 9; 0.6, 0.3)$	$(5, 6, 8, 9; 2, 6, 8, 10; 0.7, 0.2)$

Table 5. Holding costs

h_1^I	h_2^I
$(105, 115, 128, 132; 92, 115, 128, 137; 0.7, 0.1)$	$(88, 95, 105, 115; 80, 95, 105, 118; 0.8, 0.1)$

Here, the capacity of the two types of conveyances are 18 and 15. The number of Type 1 conveyance at the first and second sources are 5 and 2 respectively. Again the number of Type 2 conveyance at the first and second sources are 8 and 6 respectively. Now the month wise production rate are given in the Table 6.

Table 6. Month wise average production rate

$\beta_1 = 1.00$	$\beta_2 = 0.90$	$\beta_3 = 0.75$	$\beta_4 = 0.5$	$\beta_5 = 0.40$	$\beta_6 = 0.30$
$\beta_7 = 3.00$	$\beta_8 = 0.40$	$\beta_9 = 0.50$	$\beta_{10} = 0.70$	$\beta_{11} = 0.90$	$\beta_{12} = 1.00$

6.2 Optimum Results

The solution techniques used here are the Generalized Reduced Gradient (GRG) technique (using LINGO-14.0 solver).

Solving the deterministic optimization problem for the above data the maximum profit we have obtained is 233319.00. The received amount of the product in different destinations from different origins by different conveyances are listed month wise in Tables 7 and 8.

Table 7. Received amount with respect to different production rate

i	j	k	*January*	*February*	*March*	*April*	*May*	*June*
1	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	0.00	0.00	0.00	0.00	0.00	0.00
	2	1	80.10	72.09	60.08	40.05	32.04	24.03
		2	0.00	0.00	0.00	0.00	0.00	0.00
2	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	24.53	22.07	18.39	12.26	9.81	7.36
	2	1	36.15	36.15	36.15	34.29	27.43	20.57
		2	32.42	25.57	15.28	0	0	0

Table 8. .

i	j	k	*July*	*August*	*September*	*October*	*November*	*December*
1	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	0.00	0.00	0.00	0.00	0.00	0.00
	2	1	24.03	32.04	40.05	56.07	72.09	80.10
		2	0.00	0.00	0.00	0.00	0.00	0.00
2	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	7.36	9.81	12.26	17.17	22.07	24.52
	2	1	20.57	27.43	34.29	36.15	36.15	36.15
		2	0	0	0	11.85	25.57	32.42

Again we have solved the deterministic optimization problem for the above data except the production rate. With respect to full production rate for each month, i.e., 1, the maximum profit we have obtained is 230233.20. The received amount of the product in different destinations from different origins by different conveyances are listed month wise in Tables 9 and 10.

Table 9. Received amount with respect to full production rate

i	j	k	*January*	*February*	*March*	*April*	*May*	*June*
1	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	0.00	0.00	0.00	0.00	0.00	0.00
	2	1	80.10	80.10	80.10	80.10	80.10	80.10
		2	0.00	0.00	0.00	0.00	0.00	0.00
2	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	24.53	24.53	24.53	24.53	24.53	24.53
	2	1	36.15	36.15	36.15	36.15	36.15	36.15
		2	32.43	32.43	32.43	32.43	32.43	32.43

Table 10. .

i	j	k	*July*	*August*	*September*	*October*	*November*	*December*
1	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	0.00	0.00	0.00	0.00	0.00	0.00
	2	1	80.10	80.10	80.10	80.10	80.10	80.10
		2	0.00	0.00	0.00	0.00	0.00	0.00
2	1	1	0.00	0.00	0.00	0.00	0.00	0.00
		2	24.53	24.53	24.53	24.53	24.53	24.53
	2	1	36.15	36.15	36.15	36.15	36.15	36.15
		2	32.43	32.43	32.43	32.43	32.43	32.43

6.3 Discussion

In this section, we see that the maximum annual profit is 233319.00 when the production rate is different for different months. Tables 7 and 8 show the received amount from different sources to different destinations. Again with respect to full production rate the maximum annual profit is 230233.20 and for this case the received amount is shown in Tables 9 and 10. Here we observe that the annual profit with respect to varying production rate is more than that with respect to full production rate. The selling price inversely depends on the production rate i.e., when the production rate is low the selling price at the destination is high. Again the purchasing price at the source also inversely depends on the production rate i.e., when the production rate is low the purchasing price is high. But for years it has been observed that the agricultural product is purchased from the producer (farmer) by the middleman at the source at not so high cost as compared to the selling price at the destination. As a result the middleman maximize the profit in case of low production rate which is happening in this proposed model also. Again in the Tables 9 and 10 we can observe that the received amount at a particular source from a particular destination is same for every month when the production rate is full throughout the year which is a natural phenomena.

7 Conclusions

In this paper, we have focused on dealing a production dependent agricultural 3D-TP with the relevant parameters in generalized intuitionistic fuzzy environment. We have introduced the annual net profit as an objective function in the TP. Due to seasonal change the production rate of some agricultural product is not same during the whole year. As a result the availability and the purchasing price of the product at the source, selling price at the destination vary. Again depending on the selling price at the destination, demand of some items varies. So in this model the month wise production rate has been introduced, which is more realistic. Here the parameters are in generalized intuitionistic fuzzy environment. We have implemented a defuzzification approach by using the accuracy function to get the reduced deterministic model. The model has been illustrated by some numerical examples and optimal results have been shown in tabular forms. In future, this approach can be used in different realistic problems such as supply chain network design, transportation problems including space constraints, price discounts on the basis of amount of transported units.

Conflict of Interest. The authors have no conflict of interest for the publication of this paper.

Compliance with Ethical Standards. We declared that Research work done by self finance. No institutional fund has been provided.

Ethical Approval. The authors declared that this article does not contain any studies with human participants or animals performed by any of the authors.

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A Simple Arithmetic Calculator to Solve Single Sentence Mathematical Word Problems

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Abstract. In the recent past, solving mathematical word problems of varying complexities has gained popularity among researchers, for which Natural Language Processing (NLP) and Understanding (NLU) techniques, Machine Learning, Deep learning and several artificial intelligence (AI) based approaches are widely used. The paper presents a work aiming to solve single-sentence mathematical or arithmetic word problems or queries given in natural language (English). The work involves identification of problem types, simplification of the given word problem to extract the operators and the operands, mapping the operators to appropriate operands to create the mathematical expression and solve the query to generate the final result. We developed a dataset consisting of 430 diverse arithmetic word problems of different complexity levels. Our natural language calculator was evaluated on this dataset and it resulted in 86.28% accuracy.

1 Introduction

Solving mathematical word problem (MWP) has been a longstanding research topic for researchers in cognitive science, psychology and education. This research problem also caught the attention of the researchers in AI since 1960s. We are mainly motivated by the 'Google calculator'[1] integrated with the Google search engine which is able to understand a single-line natural language word problem input and capable of processing arithmetic problems like "*one minus two plus three plus minus four plus five*" and displays the answer along with a virtual calculator interface with the processed equation. We tried the Google calculator with other types of single line mathematical problems in English randomly consisting of incomplete phrase, complete sentence, question sentence starting with 'Wh' words, etc. Google calculator displays the answer with their virtual calculator application, however, it often fails as well. For example, for the sample input like "*Find the sum of product of one and two and product of three and*

[1] http://www.googleguide.com/help/calculator.html.

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four", Google calculator dose not show any answer. The inability of the Google calculator to solve such simple MWPs inspired us to create our own system and dataset to check the reliability and accuracy of our application in comparison to the existing virtual calculator of Google. Our system beats some of the functionalities of the Google virtual calculator in terms of variety and complexity of the input problems on the said dataset. To the best of our knowledge, the developed dataset is the first published dataset on single-line arithmetic word problems with basic operations (+, −, *, /). We only consider basic operations (+, −, *, /), not the "% or mod or root" operations, or any scientific calculation features or money conversion calculation functionalities available with the Google search engine calculator. We also assumed that the input text must have an explicit clue(s) about the operation(s) like 'add', 'product', 'sum' from the fixed set of clues as this is also one of the assumptions of Google calculator. The system is rule-based which works on a few simple assumptions and implements a state-diagram and a stack used for postfix-infix based expression evaluation leading to the final answer generation.

The input problems are categorized in two types – the first Category consisting of only numerical numbers in decimal format with operators (may be in words) and the second category consisting of only natural language text (cf. Sect. 3). The system after getting the input goes through several pipelined processes to generate the answer. The problem sentences have to go through a preprocessing phase to simplify and normalize them in a standard format (cf. Sect. 3.2). The second category sentences have to go through additional modification steps to bring them down to the first category (cf. Sect. 3.3). All the types of problems are then solved based on postfix expression evaluation using stack to generate the final answers (cf. Sect. 3.3).

We did not find any standard dataset which consists of single sentence arithmetic word problems or queries. We created a dataset consisting of 430 problems of varying complexities. The dataset is based on basic operations and the assumption mentioned earlier. The major contributions in the work reported in this paper are the following.

– Intelligent sentence preprocessing and conversion using NLP tools to generate mathematical expressions from natural language texts.
– Development of a dataset consisting of 430 problems, which is the first of its kind, having diverse problem types.
– State of the art performance outperforming Google Calculator on the mentioned dataset.

In Sect. 2 we describe some relevant works which are less, specifically for this kind of attempt. Then we describe our system architecture and information processing (rule-based) in detail in Sect. 3. Evaluation and error analysis of the proposed system are reported in Sect. 4, followed by conclusions in Sect. 5.

2 Related Work

MWP Solving has been a long standing research problem for researchers in cognitive science, elementary education and children psychology [13,15]. In 1964, [3] first tried to solve a fixed set of MWPs with the help of computer systems. His system, named 'STUDENT', was the first attempt to automatically solve algebra word problems using prepositional logics (LISP 1.5 program manual[2], MIT press). To the best of our knowledge, no researcher works on single line word problems except "Google". All the researches are towards to solve multi-line word problems. So, we did not find any significant research work to relate with us which are exactly of same objective. However, Multi-line MWP solving has a rich literature [13,14]. In the decade of 1980 some advancements were done

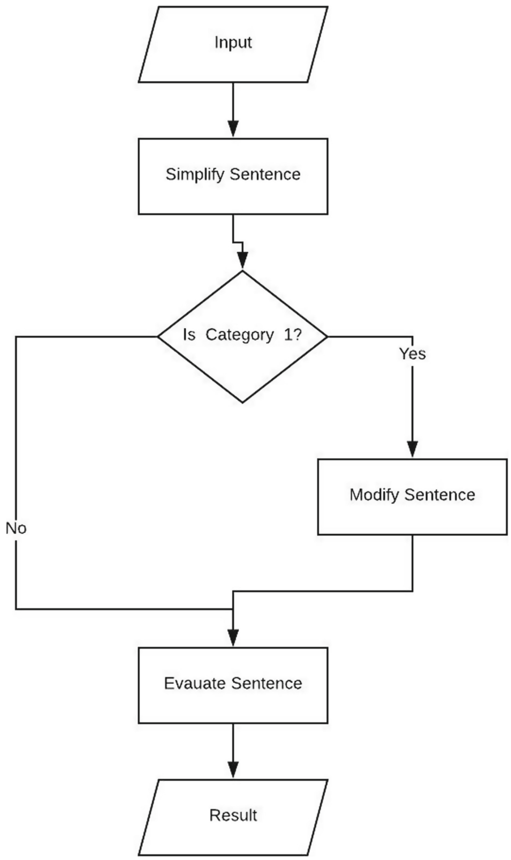

Fig. 1. System Work flow.

[2] http://www.softwarepreservation.org/projects/LISP/book/LISP%201.5%20Programmers%20Manual.pdf.

by [8,15]. 'WORDPRO' [6], 'ARITHPRO' [5] were based on human cognition behavior and propositional schema. [4] proposed a model 'CHIPS' (Concrete Human-like Inferential Problem Solver)–which is a simulator based on children psychological pattern to solve word problems. 'CHIPS' simulates a list structures to represent a word problem as schema. [8] used the text-comprehension and problem-solving techniques to solve arithmetic word problems. 'WORDPRO' [6], 'ARITHPRO' [5] were closely simulates human cognition science theories to understand input texts to generate the answers using computer.

[2] developed a 'ROBUST' simulator and solved free-format multi-step arithmetic word problems containing irrelevant information. Some other relevant research works in recent years are – Equation template based [10], Equation expression tree based [9,16], Entity, object based [7,12], and Tag-based [11]. They all tried to solve multi-line MWPs of different types. Whereas, our objective is to solve the single-sentence or phrase or only complete question sentence with mathematical queries.

3 System Architecture and Algorithms

3.1 System Work Flow

The system architecture is described in Fig. 1. The given arbitrary problem is simplified such that the simplified sentence can be broadly classified into one of the following two categories.

- **Category 0:** The sentence consists of only mathematical operators (+, −, *, /), operands (in numerical form) and brackets (left or right), e.g., $3 + (4 * 5)$.
- **Category 1:** The sentence consists of at least one preposition (e.g., of) and no Category 0 sub-sentences. E.g., "*sum of 3 and 4*" or "*sum of 5 and product of 3 and 4*", etc.

It is quite evident that "Category 0" sentences are quite simple and easy to solve. Sentence simplification (cf. Sect. 3.2) is done for both the categories. However, "Category 1" sentences are more complex and require further modification (cf. Sect. 3.3). After processing the sentence we evaluate the resultant mathematical expression to find the result.

During the modification phase, we map the operators to their corresponding operands and reduce the "Category 1" sentence to the corresponding "Category 0" equivalent sentence. This involves inserting brackets in correct places based on operator precedence, followed by replacing every coordinating conjunction (specifically, "*and*") by an appropriate operator. Operator precedence is specified in Table 1. The modules are described in the following two subsections.

Table 1. Operator precedence table

Operator	Precedence
*	3 (Highest)
/	3
+	2
−	2
(	1 (Lowest)

3.2 Sentence Simplification

The work-flow of the sentence simplification model is described in Fig. 2. We refer to the sentence containing an MWP as a query.

Rephrasing the Query: In the first step, the query is first tokenized and parts-of-speech (POS) tagged. We traverse the sentence from left to right and perform the following operations on each word token.

1. If the word token belongs to the list of **Marked Verbs**, we replace the word by some special word that conveys the mathematical operation entailed by the original word. The list of all **Marked Verbs** and their corresponding replacements are given in Table 2. After replacing the **Marked Verbs**, the word '*of*' is inserted after every replacement word. For example, if the input

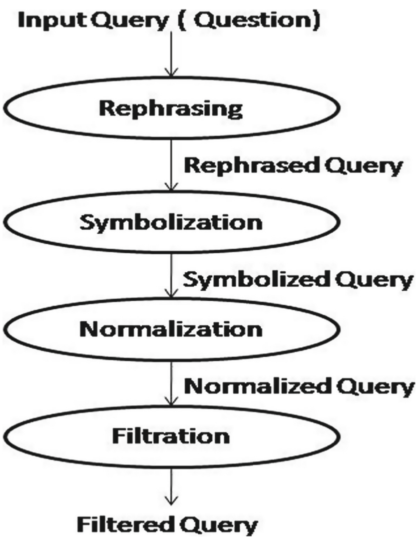

Fig. 2. Flowchart of the sentence simplification module.

sentence is "*add two and three*", it will be rephrased to "*addition of two and three*.
2. There are certain other words which are also replaced whenever they are encountered. All these words and the corresponding replacements are given in Table 3. For example, if the input sentence is "*multiply two, three, four*", it is modified to "*multiplication of two and three and four*".
3. Word with POS tag **CD** (the POS tag for cardinal numbers) are replaced by their numerical values. For example, if the word is '*two*', it is simply replaced by '*2*'.

After these changes are made to the query sentence, we refer to the newly constructed sentence as **Rephrased Query**.

Table 2. Marked verbs and their replacements.

Marked verbs	Replacements
add, adding	addition
subtract, subtracting	subtraction
multiply, multiplying	multiplication
divide, dividing	division

Table 3. Words and corresponding tokens to be added in the query.

Words	Replacements
point	['point', 'CD']
.(dot)	['point', 'CD']
,(comma)	['and', 'CC']
minus	['minus', 'NN']
-(hyphen)	['-', 'SYM']
after	Ignore
with	['by', 'IN']

Symbolization of (Adding Actual Operator To) Rephrased Query: The second step is to replace every word that specifies a mathematical operation by its corresponding operator and form a **Symbolized Query**. These words, their corresponding mathematical operations and operators are given in Table 4.

We create Symbolized Query list in the following manner.

1. Traverse through each word of the **Rephrased Query** and replace any word given in (the first column of) Table 4 by the corresponding Operator. E.g. '*addition*' is replaced by '+'. This is used to transform any phrase like "*addition of 5 and 6*" to "*+ of 5 and 6*".

Table 4. Words and their corresponding Mathematical Operations and Operators. † Subtraction (Type 1) is subtraction operation of the form op1 - op2. E.g. 1 minus 2. ‡ Subtraction (Type 2) is subtraction operation of the form op2 - op1. E.g. Subtract 1 from 2.

Words	Operation	Operator
add, sum, addition, plus, added, +	Addition	+
difference, minus, −	Subtraction (Type 1)†	−
subtract, subtraction, subtracted	Subtraction (Type 2)‡	−
multiply, into, multiplication, times, product, multiplied, *	Multiplication	*
divide, division, divided, /	Division	/

2. Traverse through the newly created **Symbolized Query** list. If any '*' or '/' is followed by '*by*' or '*to*', then delete 'by' or 'to'. This step is used to transform any phrase like "*6 divided by 3*" to "6/3".
3. Traverse through the **Symbolized Query** again. If the word '*minus*' is not preceded but followed by a number, then delete the word '*minus*' and replace it by its negative equivalent. This step is used to transform any phrase like "*minus 3*" to '−3'.

Normalization of Symbolized Query (into Either Category 1 or Category 0): The system now normalizes the **Symbolized Query** to get **Normalized Query**. The input text is either a "Category 0" or a "Category 1" sentence, or can be of any arbitrary type that are composed of both Category 0 and Category 1 sub-parts. Therefore, we need to perform some normalizations to ensure that it falls under either "Category 0" or "Category 1" sentence. Category 1 sentences are those that have at least one preposition and no Category 0 sub-parts, and Category 0 sentences are those that do not have any prepositions. If we look at the inherent characteristics of these sentences, we can come to the following conclusion. Keeping the above observation in mind, we can take any arbitrary sentence and proceed as given below.

– If at least one preposition is present, we can normalize such a sentence to Category 1 sentence. This involves removing any other Category 0 sub-part.
– If no preposition is present, it means that the input sentence is already a Category 0 sentence.

Normalization of any arbitrary type sentence to a Category 1 sentence essentially involves replacements of certain phrases by their correct counterparts which is illustrated in Table 5.

Making the normalizations mentioned in Table 5 transforms any arbitrary sentence to Category 1 sentence. E.g. an arbitrary sentence like "*sum of 1 and (*

Table 5. Original phrase in Symbolized Query and the corresponding normalized phrase in Normalized Query with examples. *op1 : operand 1, op2 : operand 2, sym : symbol, sym1 : symbol 1, sym2 : symbol 2, pre : preposition (specifically- by, from, to)*

Original phrase	Normalized phrase	Example
op1 sym op2	sym of op1 and op2	Original Phrase: + of 1 and 2 + 3 Normalized Phrase: + of 1 and + of 2 and 3
op1 sym1 sym2	sym1 of op1 and sym2	Original Phrase: 2 * + of 1 and 3 Normalized Phrase: * of 2 and + of 1 and 3
op1 sym (	sym of op1 and (	Original Phrase: * of 2 + (* of 1 and 3) and 4 Normalized Phrase: * of + of 2 and (* of 1 and 3) and 4
op1 sym pre	sym op1 and	Original Phrase: 7 + to 5 Normalized Phrase: + of 7 and 5

*1 * 2)*" is transformed to "*+ of 1 and (* of 1 and 2)*" undergoing all previous preprocessing steps. However, if the arbitrary sentence is a Category 0 sentence, **Normalized Query** remains the same as **Symbolized Query**.

Filtration of Normalized Query by Removing Irrelevant Words: The input sentence may consist of unnecessary words and symbols which hold no importance in evaluation of the expressions. E.g., words like '*the*', '*what*', '*find*', '*John*', '*can*', '*please*', '*&*', '*$*', '*#*', etc. have no role in equation generation or evaluation. Hence, we remove such words. After removing such words from the **Normalized Query**, we get the corresponding **Filtered Query**. E.g. "*Can you tell me what is the sum of one and five?*". In this sentence, '*Can*', '*tell*', '*me*', '*what*', '*is*', '*the*', '*?*' are unnecessary. After the "Normalization of Symbolized

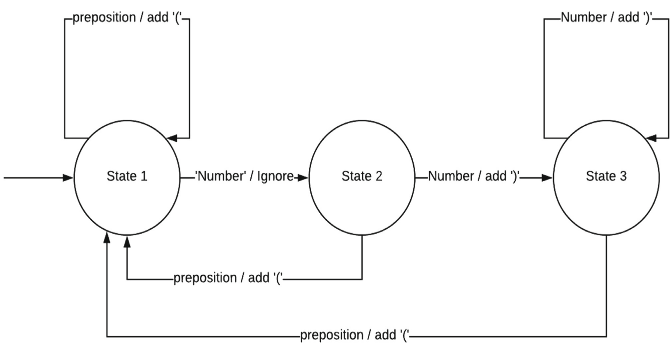

Fig. 3. State transition diagram for insertion of brackets.

Query" phase, the sentence is reduced to "*Can you tell me the + of 1 and 5?*". Then, the unnecessary words are removed to have the final meaningful form of the input "*+ of 1 and 5*" as **Filtered Query**. After the sentence simplification phase, we get a sentence that falls under either Category 0 or Category 1. Depending upon the categories, we proceed further to the next phase.

3.3 Modification of Category 1 Sentences

If the simplified sentence is a Category 1 sentence, then we need to modify the sentence further.

Insertion of Brackets in the Filtered Query. In this phase, we insert brackets (left as well as right) in their correct places in the preprocessed input sentence to generate an expression in prefix like notation. In this paper, the term "mathematical expression" is used to specifically refer to strings that have only numbers

```
Result:
Input: + of * of 3 and 5 and 6
Output: + of ( * of ( 3 and 5 ) and 6 )
Start;
STATE ← 1 ;
word_token ← first_word_token in the sentence;
while word_token not traversed last_word_token in the problem sentence do
    if word_token is a preposition ('of') then
        if STATE == 1 then
            insert (left bracket);
        end
        else if STATE ≠ 1 then
            STATE ← 1 ;
            insert (left bracket);
        end
    end
    else if word_token is a number then
        if STATE == 1 then
            STATE ← 2 ;
        end
        else if STATE ≠ 1 then
            STATE ← 3 ;
            insert (right bracket);
        end
    end
    word_token=next_word_token;
end
Exit;
```

Algorithm 1: APPLY_BRACKET – Applying brackets in the appropriate positions as per state diagram and mathematical operators. Our calculator turns an arbitrary sentence to its corresponding mathematical expression and finds the result. The brackets are inserted according to the state transition diagram in Fig. 3. As is evident from Fig. 3, whenever we see a preposition, we add a left parenthesis ('(') and go to state 1. Here, state 1 is the start node and the transition ends when we reach end of sentence. The first time we see a number, we just make a transition to state 2 and do nothing. At state 2, if we again see another number, then we go to state 3 and add a right parenthesis (')'). At state 3 we keep on adding ')' as long as we encounter numbers. Finally, we balance the brackets with right parenthesis, if required. E.g., after the step, the Filtered Query "*+ of * of 3 and 5 and 6*", is reduced to "*+ of (* of (3 and 5) and 6)*" (cf. Algorithm 1). The system also balances the end brackets if required (algorithm is not given).

Insertion of Operators. After inserting brackets in the appropriate places, we now map the operators to their corresponding operands. Therefore, we replace every 'and' by the correct operator. Additionally, we remove all the prepositions and the extra operators (operators present in incorrect places) so that our sentence finally reduces to a Category 0 sentence, i.e., a simple mathematical expression. This is done according to the **APPLY_OPERATOR** algorithm (cf. Algorithm 2). This algorithm maps the operators to their correct operands and removes all the prepositions and extra operators, if any. E.g., for the **Input** "*sum of one and five*", the corresponding **Filtered Query** is "*+ of 1 and 5*" and after applying the **APPLY_BRACKET** algorithm, this reduces to "*+ of (1 and 5)*". Here, if we carefully look, we can see the following things in the list given below.

– The correct place of the '+' operator should be between '1' and '5' and not at the beginning of the sentence.
– The extra '+' (present at the beginning of the sentence), 'of' and 'and' will no longer be required after the '+' operator has been put in its correct place by the **APPLY_OPERATOR** algorithm.

When we insert '+' in its correct place, the sentence becomes "*+ of (1 + and 5)*". Now, we remove the extra '+' in the beginning and the prepositions ('*of*' and '*and*'), the sentence reduces to "*(1 + 5)*".

In Algorithm 2, 'LB' keeps count of the left brackets and 'STATE' keeps track of the state of the system. 'STACK' used to store the operators in the sentence. 'LATEST_POP' stores the last element popped from 'STACK'.

After inserting the operators, the sentences are transformed to Category 0 sentences, e.g., the sentence "*+ of (* of (3 and 5) and 6)*" is turned into "*((3 * 5) + 6)*". It is to be noted that this sentence is an infix expression. We now create a list, called the **Final List**, corresponding to the infix expression. This list contains the tokens of the infix expression as its elements. For the previous example, the **Final List** will be – ['(', '(', '3', '*', '5', ')', '+', '6', ')']. In this step, we also check if there are appropriate number of operators and operands. This is can be done by simply checking whether "number of operands - number

```
Result: Equation expression in infix notation with appropriate operators
        generated from the prefix input form
Start;
STATE ← 1 ;
word_token ← first_word_token in the sentence;
while word_token not traversed last_word_token in the problem sentence do
    if word_token is an operator then
        push(word_token);
        if next_word_token is not operator or next_word_token is not 'by' or
        next_word_token is not a number then
            delete(word_token);
        end
    end
    else if word_token is a preposition then
        delete(word_token);
    end
    else if word_token is a left bracket then
        Increment LB by 1;
        if LB > 0 then
            STATE ← 1 ;
        end
    end
    else if word_token is a right bracket then
        Decrement LB by 1;
        if LB == 0 then
            STATE ← 2 ;
        end
    end
    else if word_token is a conjunction ('and_CC') then
        if STACK is not empty then
            LATEST_POP =POP();
            Replace 'and' by LATEST_POP ;
        end
        else if STATE == 1 then
            Replace 'and' by LATEST_POP;
        end
        else if STATE == 2 then
            Replace 'and' by '+' ;
        end
    end
    word_token=next_word_token;
end
Exit;
```

Algorithm 2: APPLY_OPERATOR–Placing the operator(s) in the appropriate position to create final equation.

of operators = 1". This condition is sufficient to check whether the number of operators and operands is appropriate since in case of binary operators like +, −, *, /, the number of operands and operators differ by exactly 1. If the difference is less than 1, then sentence is erroneous and not considered as a valid problem. This kind of problems are also not part of the dataset we prepared.

Final Expression Evaluation The **Final List** obtained in the previous stage is in the form of an infix expression. We convert the infix expression to the equivalent postfix expression [1] and then evaluate the postfix expression to get the final result. E.g. [‘(’, ‘(’, ‘3’, ‘*’, ‘5’, ‘)’, ‘+’, ‘6’, ‘)’] is converted to [‘3‘, ‘5’, ‘*’, ‘6’, ‘+’] which evaluates to 21.

4 Dataset and Evaluation

4.1 Dataset

We developed a dataset[3] that consists of a total of 430 diverse single-sentence arithmetic problems with addition, subtraction, multiplication and division operations. The dataset also contains the actual results and the final results generated by our system and the Google calculator. The dataset consists of problems of 3 types as given below.

1. Problem statements with numbers and operators only (Category 0) of varying complexities. There are 210 problems of this type.
2. Problem statements having at least one preposition and no Category 0 subpart (referred to as Category 1 expressions). There are 77 problems of this type.
3. Problem statements that are a combination of Category 0 and Category 1 type sentences and can also contain unnecessary words like 'what', 'find', 'can', etc. e.g., "*Subtract 2 from (4 * 3)*", "*Product of one and (two plus three)*", etc. There are 143 problems of this type.

4.2 Result Analysis

The proposed system was evaluated on the dataset we prepared and it gave correct results for 371 problems and incorrect results for 4 problems. The rest 55 problems in the dataset are ambiguous and have multiple interpretations and hence multiple results. Here, the correct result depends on the intention of the user. An example of ambiguous question is "*Sum of one and three into four*"[4]. This can either mean "*1 + (3 * 4)*" or "*(1 + 3) * 4*". In case of such sentences, only the user can decide which one of the two is intended. The results given by our system for such questions (present in our dataset) may or may not be correct

[3] Available at: https://drive.google.com/file/d/1eiYRWRhguH82BwiMkOsTfPBQxr4-ZpWG/view?usp=sharing.

[4] In India, the word ‘into’ is used to refer to the multiplication operation.

since our system solves the sentences on the basis of operator precedence but we consider them as incorrect answer. Some examples are given below, for which our system considers them as wrong answer.

1. "*Subtraction of three from nine divided by six*"–The sentence can be interpreted in two ways, as given below.
   - "((9 − 3) / 6) = 1"
   - "((9 / 6) − 3) = −1.5" (Our system will give the second result).
2. "*Sum of three and five into product of nine and six*"– The sentence can be interpreted in two ways-
   - "((3 + 5) * (9 * 6))) = 432"
   - "(3 + (5 * (9 * 6))) = 273" (Our system will give the second result).

To summarize, our system gives correct results for 86.28% of the total questions in the dataset and ambiguous/wrong results for 13.72% of the total questions.

4.3 Comparison with Google Calculator

In comparison with the Google Calculator on the same dataset and we found that the Google Calculator generates results for only 255 questions. Out of these 255 results, only 222 are correct, 7 are incorrect and 26 are ambiguous results. Therefore, Google Calculator gives correct results for only 51.63 % of the total questions. Moreover, Google Calculator exhibits limitations (some given below) that we recognized and tried to overcome in our system.

– In the Google Calculator, an operator can only be mapped to two operands. E.g. Consider the question-"*Product of one and two and three*". The Google Calculator solves it as "*1 * (2 + 3) = 5*". Here, '*' is mapped to "*1 and (2 + 3)*" whereas it should have been solved as "*((1 * 2) * 3) = 6*".
– Google Calculator is also limited in its ability in handling words or phrases expressing mathematical functions. It does not produce results for questions involving phrases like "added to", "division of", etc.

5 Conclusion

Our system can efficiently solve single-sentence problems involving basic operations like addition, subtraction, multiplication and division. In future, we would like extend the system by adding features to find solutions to problems containing operations like squares, cubes, square roots, percentages, mod, scientific operations etc. Another direction is to extend the system to solve multi-sentence word problems. Our end objective to develop a calculator based on natural language text inputs in English or other languages containing such mathematical numbers and operational clues with the display of the generated equations and the final answers.

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Elderly Care Monitoring System with IoT Application

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Abstract. Falls among elderly can pose serious consequences such as injury or even fatal ones. Therefore, it is essential that fall are detected early and a way to that is by using IoT platform. The authors have been developing a wearable device for elderly monitoring system utilizing accelerometer. The data from accelerometer is connected to an Internet-of-Things (IoT) platform called *ThingSpeak*^{TM}. Based on IoT platform, elderly patients can be remotely monitored as long as the care providers have good internet access. The paper presents the experimental results of determining the sensitivity and specificity of the accelerometer used in the proposed system. This is the first step for developing an accurate data acquisition for monitoring purposes. Based on the experimental results, the average percentage for sensitivity obtained for this device is 73.3%, while the average for specificity obtained is 89.3%. Both sensitivity and specificity tests shows promising results which indicates that the device only has a fail rate of 26.7% and error rate of 10.7%.

Keywords: Fall detector · Internet of Things · Accelerometer

1 Introduction

It may seem that for elders to stay at home is the safest possible place for them to avoid health hazards. But one of the few problems faced by elderly people is the tendency for them to trip or fall down even when they are at home. Falls are a serious problem faced by the elderly; it contributes to disability for elderly people. A fall is an event which results in a person coming to rest inadvertently on the ground or other lower level, not as a consequence of the following: sustaining a violent blow, loss of consciousness, sudden onset of paralysis, or an epileptic seizure [1].

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As we age, it brings about a number of physiological changes, not only does it affects our physical appearance but it also causes some health deterioration. The older we are the more frail our body becomes. This causes the body to experience health disorders such as eye sight problems, hearing problems, body joint problems, memory losses and so on [2]. There is no avoiding the grace of aging. We can only help those affected by relieving their symptoms and prevent further deterioration of the present conditions. Perhaps the best step is to constantly monitor their health status and take immediate action if a serious condition catches our attention. This is the key idea of this project which is to monitor the health condition of the elderly through the advancement of Internet-of-Things (IoT) application. Therefore, with the advancement of technology, a fall detection system can promote home-based rehabilitation, reduce costs for traveling to healthcare providers and definitely can provide early interventions and treatments that can save lives.

In general, development of elderly monitoring systems related to fall detection can be divided into two (2): automated surveillance system and wearable devices. In an automated surveillance system, the monitoring of elderly involves the implementation of various sensors within the environment where the subject is located. These sensors provide information such as audio, video and even pressure that can be transmitted wirelessly to a healthcare provider located in a remote location [3, 4]. On the other hand, wearable devices provide a cost effective solutions. Wearables technology consists of many nodes all equipped with sensors, networks and power supplies, where nodes within the same area can communicate with each other [5–7]. They can be effectively used to monitor and track the conditions of patients in both developed cities and inaccessible rural areas using an intranet network or internet and thereby greatly reducing the workload of healthcare providers, minimize medical errors, and increase the efficiency of working hospital staff, and in the long run reduce the healthcare cost and also improve patient comfort.

The authors have been developing a wearable device for elderly monitoring system utilizing accelerometer. The data from accelerometer is connected to an Internet-of-Things (IoT) platform called *ThingSpeak*^{TM}, where users can collect and store data from various sensors in the cloud for developing IoT application. This paper presents the hardware design of the proposed device. Furthermore, preliminary results of determining the sensitivity and specificity of the accelerometer used in the proposed system are described. This is the first step for developing an accurate data acquisition for monitoring purposes. Sensitivity and specificity tests are done to determine the sensitivity of the accelerometer for various types of falls, to verify overall specificity of the sensor in which it is able to differentiate a fall and a daily life activity (ADL).

2 Methodology

2.1 Prototype Hardware Design

Figure 1 shows the overview of the proposed elderly care monitoring system. In this work, An Arduino UNO microcontroller is used to process movement data from an ADXL345 accelerometer to detect fall based on gravitational acceleration (g's). A GPS module is used as a location tracker providing real time data of user's location. The data from accelerometer and GPS module are processed and transmitted to an IoT platform called *ThingSpeak*^{TM}. The data can be monitored and viewed through the internet with web interface or mobile applications.

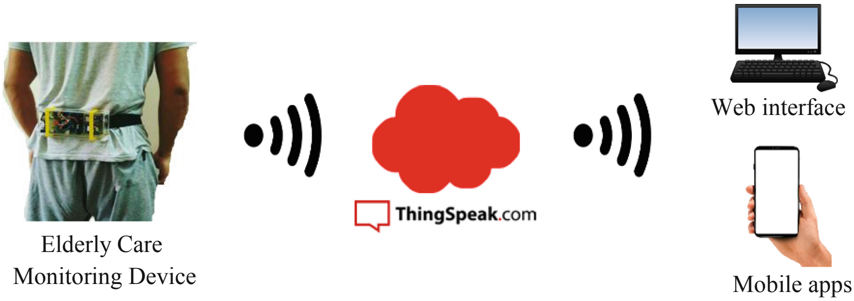


Fig. 1. PCB design of final circuit development

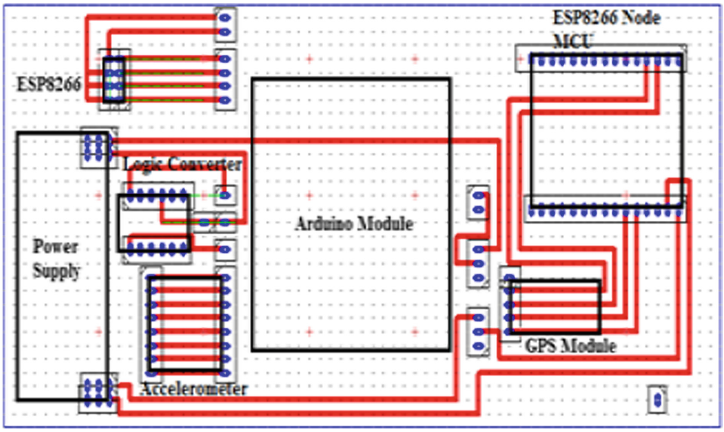


Fig. 2. PCB design of final circuit development

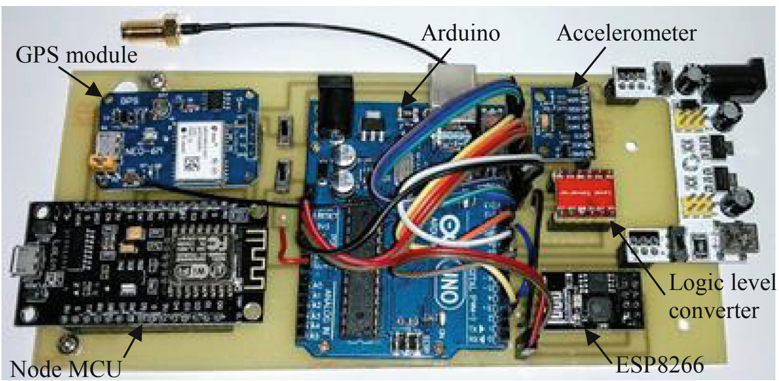


Fig. 3. Finalized actual circuit design

2.2 Prototype Hardware Design

Figure 2 shows the circuit designed using Proteus and PCB wizard software. It was designed with size and weight in mind, whereby the design is to be as compact as possible to minimize the size. Figure 3 shows the actual circuit design of the prototype. As shown in the figure, an Arduino UNO is used as the microcontroller. An ADXL345 accelerometer is used as a fall detector based on gravitational acceleration (g's) from three axis x, y and z of the accelerometer. A NEO-6M GPS module is used as a location tracker, providing real time data of the user's location. The prototype also uses an ESP8266 Wi-Fi transceiver module that allows information to be uploaded to the internet for IoT platform. The module is used to transfer the data from accelerometer to *ThingSpeak™* platform. Furthermore, an ESP8266 Node MCU is also used in this prototype. It performs similar with the ESP8266 Wi-Fi transceiver, but the Node MCU allows self-programming through its micro USB ports. This Node MCU is connected to the Arduino UNO board and the NEO-6M GPS module, where the data location data from GPS module is transmitted to *ThingSpeak™* platform via this Node MCU. The device is also consists of a logic level converter that converts signal voltage to a safe level without affecting the data. Here, the converter is used for ESP8266 Wi-Fi transceiver module as it operates on 3.3 V, and the Arduino communicates with its ports using 5 V power supply. Thus, a logic converter must be used between the two devices in order to prevent over voltage damages.

2.3 IoT Platform Using *ThingSpeak™*

This project focuses on using the Internet-of-Things (IoT) platform as a medium to monitor and store data and even provide notifications. *ThingSpeak™* platform is an online website that supports IoT development and it is free to use. In this work, *ThingSpeak™* platform is used to upload sensor data for both accelerometer and GPS module via the ESP8266 Wi-Fi transceiver and ESP8266 Node MCU. Therefore, users can view the uploaded data on a dedicated webpage using computer or even mobile phones. Another plus side for this website is that it supports notification based on our settings and even MATLAB development. Figure 4 shows the data that can be viewed for fall detection consists of plotted graph for x, y and z axes over time, and a graph that shows if there is a fall detected or not. This platform refreshes data every 15 s and has no limit of usage. Furthermore, notifications can be in the form of email alerts, tweeter feeds and so much more.

2.4 The Proposed Fall Detection System

Figure 5 shows the flow chart of the proposed fall detection system. The accelerometer will firstly detect the 3 axis values (x, y, and z) for the acceleration values in g force. It will constantly updates values to *ThingSpeak™* platform as a database for IoT. The system is designed to detect falls based on a certain threshold value and formula that will be explained later on the sections after this. If a fall is detected, *ThingSpeak™* will immediately send an alert to the care taker within 15 s. The alerts come in the form of emails. A mobile application is also able to monitor and provide alerts by acquiring data from the *ThingSpeak™* database and react accordingly.

Fig. 4. ThingSpeakTM platform showing accelerometer data

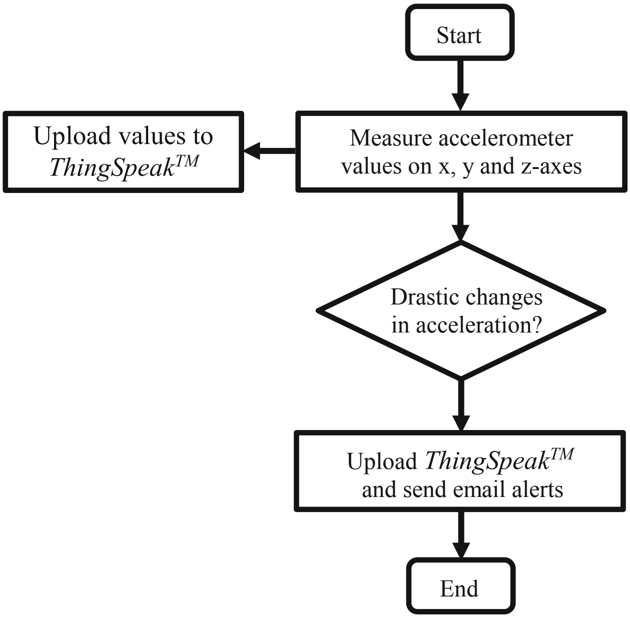


Fig. 5. Elderly care fall monitoring program flow chart.

2.5 Fall Prediction Method

Figure 6 shows a Cartesian coordinate system for the accelerometer utilized in the device. The accelerations along the X, Y and Z axes are defined as $\boldsymbol{a_x(t)}$, $\mathbf{a_y(t)}$ and $\boldsymbol{a_z(t)}$ respectively during time $\boldsymbol{t}$, where $\boldsymbol{a(t)} = \{\boldsymbol{a_x(t)}, \boldsymbol{a_y(t)}, \boldsymbol{a_z(t)}\}$. Therefore, the sum-vector of acceleration of all spatial components $\boldsymbol{a(t)}$, can be calculated using the following equation:

$$\boldsymbol{a(t)} = \sqrt{\boldsymbol{a_x(t)}^2 + \boldsymbol{a_y(t)}^2 + \boldsymbol{a_z(t)}^2} \tag{1}$$

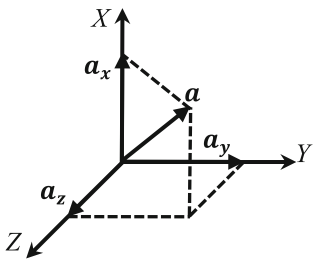


Fig. 6. Cartesian coordinate system for accelerometer

In order to obtain the fall prediction, the predicted number value of sum-vector threshold to effectively obtain a fall is deemed to be 0.75 g based on the work done by [7]. The combination of sum-vector $\boldsymbol{a(t)}$, must not be lesser than the indicated threshold value. Once the sum-vector is less than the indicated threshold value, the system will determine it as a fall and thus transmitting the alert status. The reason the sum-vector threshold value of 0.75 g is used is based on the theory that gravitational acceleration while standing still on a surface is 1 g on earth's gravity, during a free fall state the acceleration value will be 0 g this is indicating a free fall status. Through trial and error it was determined that the best threshold for this particular device is at the value of 0.75 g.

Based on the work done in [8], falls can be evaluated on statistical tests. As a single fall result cannot determine the overall effectiveness of the sensor, therefore, four (4) categories are used in this work to analyze and evaluate the fall detection. The categories are as follows,

- True positive (*TP*): A fall is present.
- False negative (*FN*): A fall occurs, but the device does not pick it up.
- True negative (*TN*): A normal activity is performed; the device did not declare a fall.
- False positive (*FP*): The device indicates a fall, but it did not occur at all.

Testing and data collection were done on multiple subjects to ensure a varied data collection, and the overall data are collected and plotted on a table. The calibration of the proposed system consists of sensitivity and specificity tests. The sensitivity test is to determine the sensitivity of the sensor for various types of falls. The falls that were tested are lateral falls left and right, forward falls and backward falls [9]. The falls were repeated several times and the results were recorded in the form of a table. The equation to calculate the sensitivity is as follows:

$$Sensitivity = TP/(TP + FN) \times 100\% \tag{2}$$

On the other hand, the specificity test is to determine the overall specificity of the sensor in which it is able to differentiate a fall and a daily life activity (ADL). Several tests have been carried out to test the specificity namely, sitting down on a chair, lying down on a bed, bending down to pick something up and coughing or sneezing. The results obtained are plotted on a table. Based on [9], the equation to determine the specificity is as follows:

$$Specificity = TN/(TN + FP) \times 100\% \tag{3}$$

3 Results and Discussions

3.1 Elderly Care Monitoring System with IoT

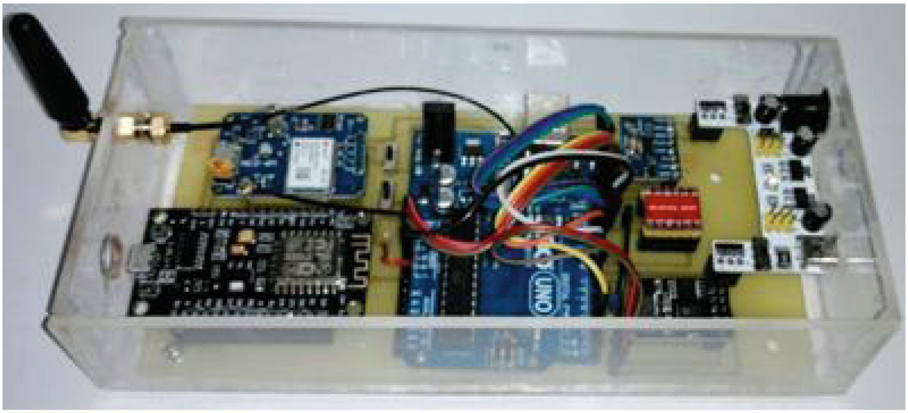
The assistive device for elderly care and monitoring was developed mainly in terms of fall detection and GPS locating. The developed monitoring system is consists of an Arduino UNO microcontroller, ADXL345 accelerometer for the fall detection, ESP8266 Wi-Fi transceiver to communicate from the Arduino to the internet, a logic level converter to maintain a safe input voltage for devices, a NEO-6M GPS module and ESP8266 Node MCU for GPS locating, and finally a power supply to power the device. The dimension of this device is 19 cm (length) × 8.5 cm (height) × 4 cm (width), and the case is made of acrylic hard plastic. The overall weight of this hardware is approximately 200 g.

Figure 7(a) shows the actual image of the developed elderly monitoring system with IoT application that is placed inside an acrylic case. Taking into consideration the ease-of-use, the device is attached on a belt as shown in Fig. 7(b). Therefore, the prototype device is developed as a wearable that can be worn on the waist as shown in Fig. 7(c).

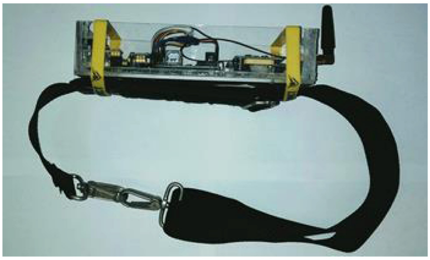
The device operates by switching ON the ON/OFF switch. Once the ON/OFF switch is turned ON the device begin sending accelerometer and location data to *ThingSpeak*^{TM} database for every 15 s. Furthermore, from the *ThingSpeak*^{TM} account, caregivers can monitor the status of a patient and it will provide alerts in the form of email if a fall is detected. Caregivers can also monitor patients through the internet via computer or any mobile devices such as tablet or smartphones.

3.2 Preliminary Experimental Results

For the purpose of testing and monitoring the results of the program, the microcontroller was connected to the computer via the USB serial connection. Since this project does not support a built in LCD, we have to monitor the data through the Arduino IDE program development environment to view the constant data streams. This section is divided into two (2) experiments: fall detection experiment and GPS tracking experiment.



(a)



(b)

(c)

Fig. 7. (a) The hardware device placed inside an acrylic case, (b) the device attached on a belt, (c) the device placement on the body

Fall Detection Experiment. First, the Arduino microcontroller in the device was connected to the USB serial port of the computer to test the accelerometer. Figure 7 shows a screen shot of Arduino's serial monitor that displays the accelerometer values on x, y and z axes, and also the fall status if a fall was detected. As shown in the figure, during a normal state a fall will not be detected and the values of accelerations (g's) can be seen. However, once a fall is detected the sum-vector of the acceleration $\boldsymbol{a}(\boldsymbol{t})$ calculated using Eq. (1), will be less than the threshold value, in this case 0.75 g. If this condition occurs, it will register as a fallen state and will send an email notification.

The data displayed on the serial monitor as shown in Fig. 8 is also sent to the *ThingSpeak*^{TM} platform via the ESP8266 Wi-Fi module. Figure 9 shows an example of x, y and z axes values plotted over time. If there is a fall detected it will also be registered in the graph labeled as "Fall". As shown in Fig. 9(d), "Fall" is recorded around 15:38PM, that correspond to the fluctuation of acceleration values on all axes of the accelerometer as shown in Fig. 9(a) to (c).

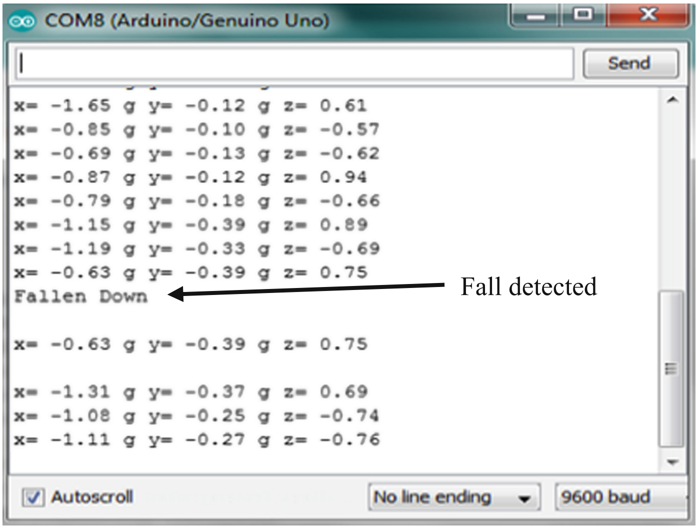


Fig. 8. Arduino's serial monitor showing sensor values

Fig. 9. *ThingSpeak*^{TM} platform showing accelerometer data: (a) x-axis, (b) y-axis, (c) z-axis, (d) Fall detection results based on sum-vector of the acceleration $\boldsymbol{a}(\boldsymbol{t})$

Once a fall has been detected, *ThingSpeak*^{TM} will send notifications in the form of email. This type of automatic notifications can be set in the form of "REACT" function available in *ThingSpeak*^{TM}. Figure 10 shows an example of an email notification received from the server, once a fall has occurred.

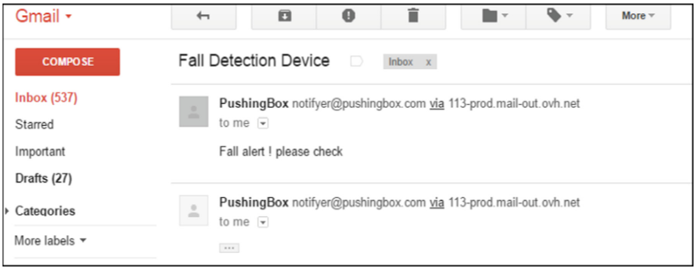


Fig. 10. Email notification

GPS Tracking Experiment. The second component of this device is the GPS tracker. It provides real-time location in terms of longitude and latitude of the device, made possible by a GPS module. The GPS module provides the coordinates of the device, and then transmits it to the *ThingSpeak*^{TM} database server.

In this experiment, the coordinates of four (4) locations derived from *ThingSpeak*^{TM} (these coordinates are originally sent from the developed GPS tracker) are compared with the coordinates obtained from Google Maps to verify the effectiveness of the developed GPS tracker. Figure 11 shows graphs of the longitude and latitude of four (4) locations derived from *ThingSpeak*^{TM} platform. Table 1 shows the detail longitude and latitude of the coordinates based on the graphs.

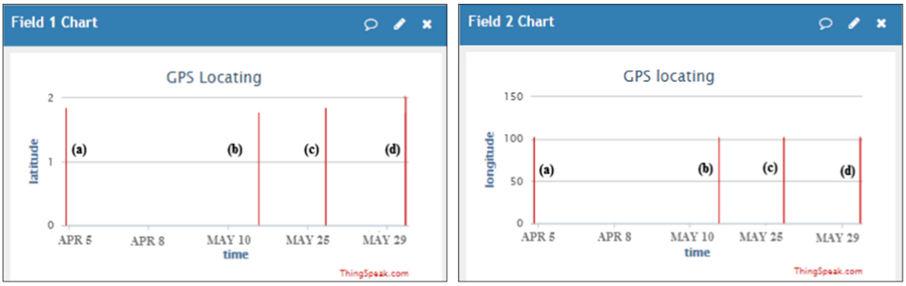
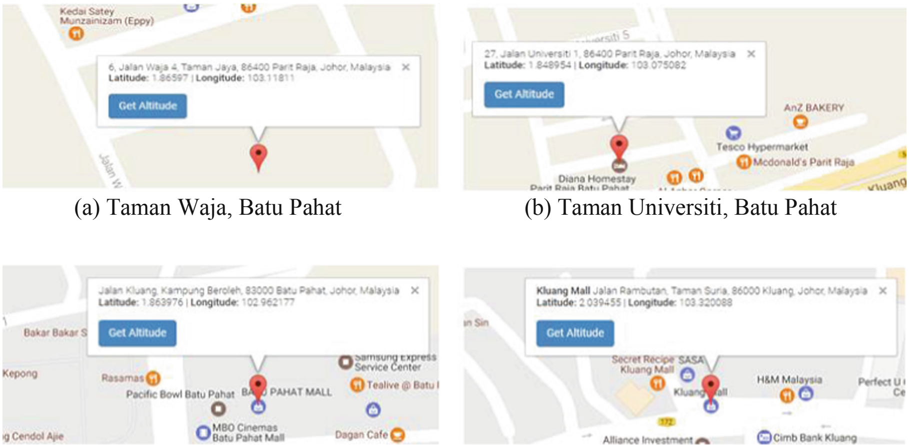


Fig. 11. GPS latitude (left) and longitude (right) coordinates for four (4) locations derived from *ThingSpeak*^{TM}: On both figures, (a)Taman Waja, Batu Pahat (b)Taman Universiti, Batu Pahat (c)Batu Pahat Mall, (d)Kluang Mall

Table 1. Coordinates for four (4) locations derived from *ThingSpeak*^{TM}

Location	Latitude	Longitude
Taman Waja	1.86597	103.1181
Taman Universiti	1.84895	103.0750
Batu Pahat Mall	1.86397	102.9621
Kluang Mall	2.03945	103.3200

On the other hand, Fig. 12 the longitude and latitude of the same four (4) locations obtained from Google Maps. Table 2 shows the detail coordinates based on Fig. 12. From Figs. 11 and 12, all coordinates obtained from the GPS tracker that are displayed on the *ThingSpeak*^{TM} platform are accurate when being compared with the coordinates obtained from Google Maps. Therefore, this experiment shows the effectiveness of the developed GPS tracker. However, for the current prototype, the coordinates have to be entered manually on the Google Maps application to find the location. For future improvements a software that automatically reads the coordinates and displays it on the maps without user intervention will be designed.



(a) Taman Waja, Batu Pahat (b) Taman Universiti, Batu Pahat

(c) Batu Pahat Mall (d) Kluang Mall

Fig. 12. Latitude and longitude coordinates for four (4) locations derived from Google Maps

Table 2. Coordinates for four (4) locations derived from Google Maps

Location	Latitude	Longitude
Taman Waja	1.86597	103.1181
Taman Universiti	1.84895	103.0750
Batu Pahat Mall	1.86397	102.9621
Kluang Mall	2.03945	103.3200

3.3 Sensitivity and Specificity Test

Sensitivity and specificity tests for fall detection have been carried out to determine the sensitivity of the device in determining fall/non-fall conditions when operating the proposed device. A total of eight (8) healthy subjects between the ages of 22 to 26 years old have participated in the tests. For experimental steps, in each activity, five (5) subjects were required to conduct each particular activity five (5) times per person. GPS tracker data collection does not require any sensitivity or specificity tests because the obtained data are based on GPS satellite positioning, and is usually accurate depending on the number of decimals, the longer the decimals the more pin-point the location is. The GPS signal will always receive the coordinates if it has an uninterrupted connection between the satellites, therefore is best if used outdoors.

Sensitivity Test. Table 1 shows the results for the sensitivity test. The results are recorded based on three (3) activities that were repeated 25 times by five (5) subjects for each activity. The activities were done by a healthy male subjects. The activities are forward falls, backward falls and side falls. The results are in the form of "Fall" for true fall detection, and "None" for negative fall detection. The sensitivity rate of the device can be calculated using Eq. (2). Based on the results shown in Table 3, the sensitivity for this device is acceptable for a prototype; the forward fall has a higher percentage (80%) than the others, while the side falls has the lowest percentage (62%). Several factors may have affected the readings of the accelerometer including tilt angle and fall acceleration. The tests were conducted under a controlled setting and the subjects' safety have been a priority to avoid injuries. As such, the results obtained might be much more accurate if tested on actual fall conditions.

Table 3. Sensitivity test results

	No. of trials	Forward falls	Backward falls	Side falls
Total	25	TP = 20 FN = 5	TP = 18 FN = 7	TP = 17 FN = 8
Percentage	100%	80%	62%	78%

Specificity Test. Table 4 shows the specificity tests in which experiments have been carried out to determine whether the device detects a fall under a 'Non-fall' condition. Three (3) types of activities were carried out namely, sitting down, standing up and walking. "Fall" indicates a false positive reading while "None" indicates a true negative reading. The data shows a good specificity percentage for the device with all the activities above 80%. Based on Eq. (3), sitting down has 84%, standing up has 88% and walking has 96% specificity. This means that the device has at most 10% error rate of giving false readings. The experiment shows a good specificity data. The whole activity was conducted under a controlled environment where each movement and action is deliberately carried out. If the activities were to occur naturally, the specificity percentage might be very different and can be lower.

Table 4. Specificity test results

	No. of trials	Sitting down	Standing up	Walking
Total	25	TP = 21 FN = 4	TP = 22 FN = 3	TP = 24 FN = 1
Percentage	100%	84%	88%	96%

4 Conclusions

The placement of the device around the lower abdomen or torso of the patient is found to be the best location since it is the center of gravity for the human body, and it is much more stable and not prone to incorrect prediction along the course of experiments. As a fall occurs, a human center of gravity is destabilized, and this is what causes the human to fall down. The average percentage for sensitivity of the device is 73.3% while the average for specificity is 89.3%. Therefore, both sensitivity and specificity tests show promising results for this device. This indicates that the device only has a fail rate of 26.7% and error rate of 10.7%.

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Multiple Criteria Analysis Based Robot Selection for Material Handling: A *De Novo* Approach

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Abstract. A novel Multiple Criteria Analysis (MCA) approach has been proposed for evaluation, selection and ranking of industrial robots, which is hitherto not applied in open literature for the purpose. The current investigation reveals the employability of engineering economy in selection of robots considering multiple conflicting attributes. Aggregate fiscal Terminal Value (TV) of normalized score of alternatives is determined to evaluate Specific Benefit (or Benefit-Cost Ratio) which is considered as the robot selection index. The proposed method has been illustrated with an example in order to demonstrate its aptness as a decision aid. The result so obtained has been compared with other methods and techniques available in papers already published in various journals.

Keywords: MCDM · Robot selection · Multiple criteria analysis · Engineering economy · Terminal value · Benefit-cost ratio

1 Introduction

Ever since the beginning of time, man has been faced with the problem of moving himself and the materials needed for his existence. Movement is the common denominator of all economic activities involving physical goods, and in much that supports intellectual activity. The steel mill, the supermarket, the automatic factory, the construction project, the library and the bank - all are engaged in moving things. Man himself frequently acts as a material handler throughout his daily life. Basically material handling is the art of implementing movement -by easier, faster, safer and economic way [1].

Modern manufacturing industries have been experiencing extremely tough competition in global market. To survive in the competition robots have been increasingly and extensively used in industries for performing repetitive, hard and hazardous job with precision, accuracy, and rapidness in material handling, spot welding, arc welding, mechanical assembly, electronic assembly, material removal, inspection and testing, water jet cutting, loading and unloading, spray painting and finishing etc. There is large numbers of robot manufacturers; the specifications of the robots are different in many cases and of course the attributes of them are not same, also the same performance

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attributes of manufacturers can not be expected. On the contrary, the materials to be handled are versatile in nature, e.g. powdered, sticky, fragile, bulky etc. So it is hard to select a suitable industrial robot as a material handling equipment for a particular material from a feasible set of different robots. One may easily be misled. End-users are faced with many options in both economical factors and technical factors in the evaluation of the industrial robots. The industrial robot selection problem with multiple conflicting attributes is complex. This complexity makes the selection procedure hard and vague. In the present paper an attempt has been made to mitigate this complexity at a great extent.

The rest of the paper is organized as follows. Section 2 furnishes a detailed description of literature review on robot selection. Section 3 establishes the proposed methodology. Section 4 illustrates the methodology with a numerical example. Section 5 envelops the framework with some essential conclusions.

2 Literature Review

In the past, several models have already been suggested for the robot selection. These models can be classified into five categories viz. (1) multi-criteria decision-making (MCDM) models, (2) production system performance optimization models, (3) computer assisted models, (4) statistical models, and (5) other approaches [2, 3].

An MCDM process always contains at least two criteria and two alternatives [4, 5]. There are two conditions in MCDM approach; one is the criterion must be more than one in number and the other is the criteria will be conflicting in nature [6–8]. MCDM models include multi-attribute decision-making (MADM) models, multi-objective decision-making (MODM) models and other similar approaches [9–12].

In MODM, the decision-maker's objective, such as optimal utilization of resources and improved quality remain explicit and are assigned weights reflecting their relative importance [13, 14]. In MADM, all objectives of decision makers are unified under a super function termed the decision-maker's utility, which depends on robot attributes. The main advantage of MCDM models is their ability to consider a large number of robot attributes. Using MCDM, the decision-maker can consider engineering, vendor-related, and cost attributes [15]. Optimization models related to performance of production system select a robot that optimizes some performance measures of the production system, such as quality or throughput, with robot attributes treated as decision variables. Shih [6] proposed an 11- step procedure to demonstrate robot selection. In the study, multiple criteria of robots were first recognized as two categories - benefits and costs. The performance of robots were evaluated on their incremental benefit-cost ratios and the robots were ranked by applying group TOPSIS. The incremental benefit-cost or the cut-off ratio is the key for selection of robot. Computer assisted models have been advocated by many researchers to deal with the large number of robot attributes [17].

Khouja and Offodile [18] reviewed the literature on industrial robots selection problem and provided directions for future research. Chu and Lin [19] developed a fuzzy TOPSIS method where the values of objective criteria were converted into dimensionless indices to ensure compatibility between the rating of linguistic variables of subjective criteria and the values of objective criteria. Through internal arithmetic of fuzzy numbers, the defuzzying of weighted rating into crisp value by the ranking method of mean of removals and determination of closeness coefficient, the robots were ranked.

Parkan and Wu [20] suggested a method that demonstrated the use and compare some of the current multi-attribute decision making (MADM) and performance measurement procedures through a robot selection problem. But this paper is not adequately robust and effective for simultaneously handling both tangible and intangible factors. Kaharaman et al. [21] proposed a fuzzy hierarchical TOPSIS model for selection of industrial robotic systems. An application was also presented with sensitivity analysis by changing the critical parameters. Kumar and Grag [22] presented a distance based approach for optimal selection of robots. They compared their method with some other two methods viz. TOPSIS and, diagraph and matrix.

Kamali et al. [23] considered the robot selection problem from the point of view of manual and automation. Nof and Lechtman [24] suggested robot time motion system for rating of human operator as an alternative of method time measurement system (MT). Zadeh [25] introduced linguistic variables for expressing the weights of criteria for evaluation of robot suitability versus subjective criteria.

Jones et al. [11] used the marginal value function for evaluation and ranking. Huang and Ghandforoush [26] presented a procedure for evaluation and selection of robots based on the investment and budget requirements. Dooner [27] simulated robot operation in work space and used the workspace as an aid to robot selection. Diagraph and matrix method for robot selection were used in some investigations [28, 29]. Khouja and Booth [30] applied fuzzy clustering procedure, Liao [31] used fuzzy multi criteria decision making method, and Goh and Haut [32] employed analytical hierarchy process to give a new direction for robot selection.

3 Proposed Methodology

In the current investigation multiple criteria analysis has been employed for the selection of robot from a set of finite number of feasible alternatives. Engineering economy/financial management has been exploited for the evaluation, ranking and selection procedure. The proposed methodology of robot selection is described below.

Step 1: *Formation of decision matrix:* A decision matrix consists of thee components namely alternatives, criteria, performance ratings. First, a decision making committee consisting of Engineers, Production managers, technicians, veteran, and any person responsible for decision is formed. The committee may be formed even with a single member. The decision making committee has the objective to select the best alternative or to rank the alternatives in order of relative benefit (from the best to the worst). The committee identifies the problem and criteria for robot selection. Selection criteria are attributes of robots under consideration on the basis of which performances of robots are measured. The criteria are divided into two groups- one is benefit criteria (higher value is better and desirable) and the other is non-benefit/cost criteria (lower value is better and desirable). In the other way selection criteria are divided into three categories by their nature, viz. objective (quantitative/measurable), subjective (qualitative/not measurable) and critical (those must be satisfied) criteria.

The committee makes a short list of alternative robots for further assessment on the basis of the selection criteria. Each alternative robot is given a score by the committee (or each member of the committee) with respect to each attribute; this score is termed as performance rating or simply rating. Performance ratings under objective criteria are expressed by crisp (specific) values but performance ratings under subjective criteria are expressed by linguistic variables due to vagueness, imprecision, indistinctness and ambiguity. The words or phrases like 'good', 'very good', 'medium', 'poor', 'very poor' are called linguistic variables which are measured by human perception, feelings, experience etc. A decision matrix with ***m*** number of alternative robots $(R_1, \ldots, R_j, \ldots, R_n)$, ***n*** number of criteria $(C_1, \ldots, C_j, \ldots, C_n)$ and ***m* x *n*** number of performance ratings $(x_{11}, \ldots, x_{ij,} \ldots, x_{mn})$ is formed as follows.

$$[D]_{m\times n} = \begin{array}{c} \\ R_1 \\ \vdots \\ R_i \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{ccc} C_1 \ldots & C_j \ldots & C_n \end{array} \\ \begin{bmatrix} x_{11} & \ldots & x_{1j} & \ldots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i1} & \ldots & x_{ij} & \ldots & x_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{m1} & \ldots & x_{mj} & \ldots & x_{mn} \end{bmatrix} \end{array} \tag{1}$$

Step 2: *Construction of weight matrix:* Decision making committee members may award different importance weights to the criteria depending upon requirement and personal view. In case of multiple decision makers, average (algebraic mean) of importance weights is used. It may be added that importance weights do not have any unit. Weight matrix is constructed as follows:

$$[W]_{1\times n} = \begin{bmatrix} w_1 \ldots w_j \ldots w_n \end{bmatrix} \tag{2}$$

Where, sum of importance weights of all criteria is one hundred percent i.e. $\sum_{j=1}^{n} w_j = 1$

Step 3: *Normalization:* The performance ratings along any particular column (with respect to a particular criterion) are expressed in same unit but along different columns may have different units. Hence elements of different columns can not be compared one another. By normalization, ratings are converted into unit less element to combine together or compare one another. The values of performance ratings can be normalized using the following equation.

$$x_{ij}^{N} = \frac{x_{ij}}{\max_i (x_{ij})} \tag{3}$$

Where x_{ij}^{N} is normalized value of performance rating of alternative i under criterion j.

Step 4: *Evaluation of Terminal Value:* In finance, the Terminal Value (TV) (continuing value or horizon value) of a security is the present value at a future point in time of all future cash flows when we expect stable growth rate forever. It is most often used in multi-stage discounted cash flow analysis, and allows for the limitation of cash flow projections to a several-year period. Forecasting results beyond such a period is impractical and exposes such projections to a variety of risks limiting their validity, primarily the great uncertainty involved in predicting industry and macroeconomic conditions beyond a few years.

Thus, the terminal value allows for the inclusion of the value of future cash flows occurring beyond a several-year projection period while satisfactorily mitigating many of the problems of valuing such cash flows. The terminal value is calculated in accordance with a stream of projected future free cash flows in discounted cash flow analysis.

In the proposed methodology, terminal value of normalized rating is suggested for the assessment of alternative robot with respect to each beneficial and non-beneficial criterion. Here performance weight substitutes both interest factor and period of time under consideration. The advantage of using importance weight in stead of both interest rate and period of time is that the calculated coefficient (that resembles future value factor) of normalized rating gives a modified importance weight which is a function of corresponding weight. Terminal values of benefit and non benefit criteria are calculated using following equations.

$$tv_{ij}^{B} = x_{ij}^{N}\left(1 + w_j\right)^{w_j} \tag{4}$$

$$tv_{ij}^{NB} = x_{ij}^{N}\left(1 + w_j\right)^{w_j} \tag{5}$$

tv_{ij}^{B} = terminal value of normalized rating of alternative i under benefit criterion j,
tv_{ij}^{NB} = terminal value of normalized rating of alternative i under non-benefit criterion j,
x_{ij}^{N} = corresponding normalized rating of alternative i under criterion j,
w_j = corresponding importance weight of criterion j.

Step 5: *Aggregate Terminal Value:* Terminal Values (TV) of ratings under benefit criteria and non-benefit criteria are separately added for calculating Aggregate Terminal Value (ATV). ATV under benefit criteria and non-benefit criteria reflect the assessment of total beneficial scores and non-beneficial scores respectively. Aggregate Terminal Values are calculated using the following simple equations.

$$TV_i^{B} = \sum_{j=1}^{p} x_{ij}^{N}\left(1 + w_j\right)^{w_j} \tag{6}$$

$$TV_i^{NB} = \sum_{j=1}^{q} x_{ij}^{N}\left(1 + w_j\right)^{w_j} \tag{7}$$

TV_i^{B} = aggregate terminal value of alternative i under all benefit criteria,
TV_i^{NB} = aggregate terminal value of alternative i under all non-benefit criteria, Where, $i = 1, 2, \ldots, m$.

Step 6: *Benefit-Cost Ratio (Benefit-Non Benefit Ratio or Specific Benefit):* Jules Dupuit, an engineer from France, first introduced the concept of benefit cost ratio in 1848. Alfred Marshall, a British economist further enhanced the formula that became the basis for benefit cost ratio. However, the formalized development of it did not occur until the Federal Navigation Act of 1936 was introduced. This act required that projects that were carried out by the U.S. Corps of Engineers have a higher benefit to the general public than the total investment in the projects. In its simplest form, benefit cost ratio is a figure that is used to define the value of a project versus the money that will be spent in doing the project in the overall assessment of a cost-benefit analysis.

A Benefit-Cost Ratio (BCR) is an indicator, used in the formal discipline of cost-benefit analysis, that attempts to summarize the overall value for money of a project or proposal. A BCR is the ratio of the benefits of a project or proposal, expressed in monetary terms, relative to its costs, also expressed in monetary terms. All benefits and costs should be expressed in discounted present values.

In the current paper, Benefit-Cost Ratio (Benefit-Non Benefit Ratio or Specific Benefit) is proposed to consider robot selection index. Benefit-Cost Ratio (or specific benefit) of an alternative is expressed by the ratio of aggregate terminal value of ratings under benefit criterion to that of the non-benefit criteria. The higher value of specific benefit is desirable. The following equation is used for calculating specific benefit (Benefit-Cost Ratio) of alternative ***i***.

$$SB_i = \frac{TV_i^B}{TV_i^{NB}} \tag{8}$$

The higher the specific benefit, the better the corresponding alternative.

Step 7: *Ranking and selection:* Arrange the robots in descending order of their specific benefits. Higher value of specific benefit is desirable and better. Ranking order of the robots is the same while they are arranged in descending order of their specific benefits. The best robot is associated with the highest specific benefit. The worst robot is associated with the lowest specific benefit, so on and so forth.

4 Illustrative Example

The following example is considered to demonstrate and validate the application of the proposed methodology in robot selection for particular type of material handling in industry. The objective of the demonstration is to develop a procedure for combining various attributes relevant to robot working so that a comprehensive ranking of the alternative robots could be obtained. The following example has been cited from Kumar and Grag [22] for an industrial application of robots. The following figure is not exhaustive. A lot of other criteria may be considered depending upon the requirement and decision makers' opinion. The demonstration is to test the applicability of the model and to explain its application. In the current example five alternative robots R1, R2, R3, R4 and R5 are short listed after preliminary screening, considering three benefit criteria-Load capacity (C1), Degree of freedom (C2), Velocity ratio (C3) and one non-benefit criterion- Repeatability (C4).

Average preference weights of criteria and performance ratings of alternatives are given in Table 1. Average preference weights of the four criteria have been given by the decision making committee. Since the criteria have different dimensions, they are normalized in order to convert into dimensionless quantity so as to compare one anther. Normalization of performance ratings is carried out by using Eq. (3) in such a manner so that all the elements become zero to one. The normalized ratings have been shown in Table 2.

Now terminal value of each normalized rating is calculated. For example terminal value of alternative R1 with respect to benefit criterion C1 (Load capacity) is calculated as $tv_{11}^{B} = 0.88(1 + 0.09633)^{0.09633} = 0.8878$ using Eq. (4). Similarly Eq. (5) is used for non-benefit criterion for the same purpose. Aggregate terminal value of an alternative is calculated using Eq. (6). For example, aggregate terminal value of robot R1 for benefit criteria (C1, C2 & C3) is evaluated as $TV_1^B = (tv_{11}^B + tv_{12}^B + tv_{13}^B) = (0.8878 + 0.8774 + 1.008) = 2.7740$ and that of for non-benefit criteria (C4) as $TV_1^{NB} = tv_{14}^{NB} = 0.5122$ using Eq. (7). Finally specific benefit (or benefit-cost ratio) is estimated by Eq. (8). For example specific benefit of robot R1 is calculated as $SB_1 = TV_1^B / TV_1^{NB}$. Terminal values and aggregate terminal values have been shown in Table 3. Specific benefits of robots are represented in Fig. 1 which clearly shows that the ranking of robots according to proposed method is R3 $\gg$ R2 $\gg$ R1 $\gg$ R5 $\gg$ R4. The best robot is R3 and the worst robot is R4.

Table 1. Attribute level of each alternate robot and Average Preference Weight (APW)

	Load capacity (LC)	Degree of freedom (DF)	Velocity ratio (VR)	Repeatability (R)
APW ⟶	0.09633	0.24949	0.09633	0.55786
R1	60	5	125	0.40
R2	60	6	125	0.40
R3	68	6	75	0.13
R4	50	6	100	1.00
R5	30	5	55	0.60

Source: Kumar and Garg [22]

Table 2. Normalization of attribute level of each alternate robot with Average Preference Weight (APW)

	Load Capacity (LC)	Degree of freedom (DF)	Velocity ratio (VR)	Repeatability (R)
APW ➔	0.09633	0.24949	0.09633	0.55786
R1	0.88	0.83	1.00	0.40
R2	0.88	1.00	1.00	0.40
R3	1.00	1.00	0.60	0.13
R4	0.73	1.00	0.80	1.00
R5	0.44	0.83	0.44	0.60

Table 3. Terminal value, aggregate terminal value, specific benefit and ranking of robots

Robots	tv_{i1}^{B}	tv_{i2}^{B}	tv_{i3}^{B}	tv_{i4}^{NB}	TV_{i}^{B}	TV_{i}^{NB}
R1	0.8878	0.8774	1.0088	0.5122	2.7740	0.5122
R2	0.8878	1.0571	1.0088	0.5122	2.9537	0.5122
R3	0.8878	1.0571	0.6053	0.1665	2.6713	0.1665
R4	0.7365	1.0571	0.8071	1.2806	2.6007	1.2806
R5	0.4439	0.8774	0.4439	0.7683	1.7652	0.7683

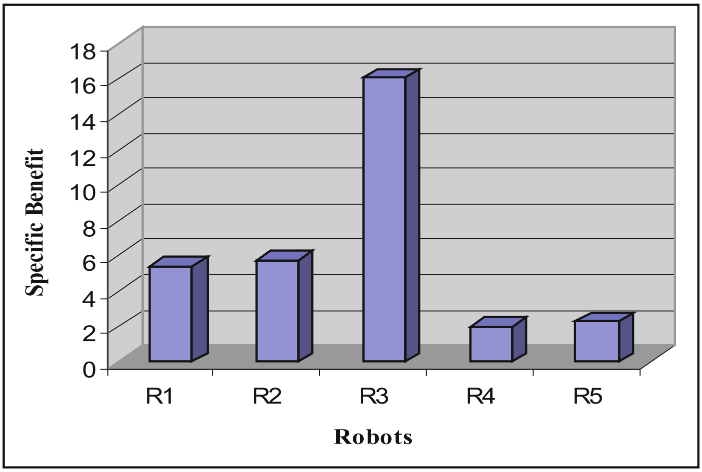

Fig. 1. Specific benefit of robots

A comparison of the proposed method with TOPSIS, Diagraph and Matrix method, and DBA (Distance Based Analysis) method is shown in Table 4 as well as in Fig. 2.

Table 4. Comparison of methods

Robot	Ranking of robots by applying			
	TOPSIS method*	Diagraph and Matrix method *	DBA method*	Proposed method
R1	3	3	3	3
R2	2	2	2	2
R3	1	1	1	1
R4	5	5	5	5
R5	4	4	4	4

*Source: Kumar and Garg [22]

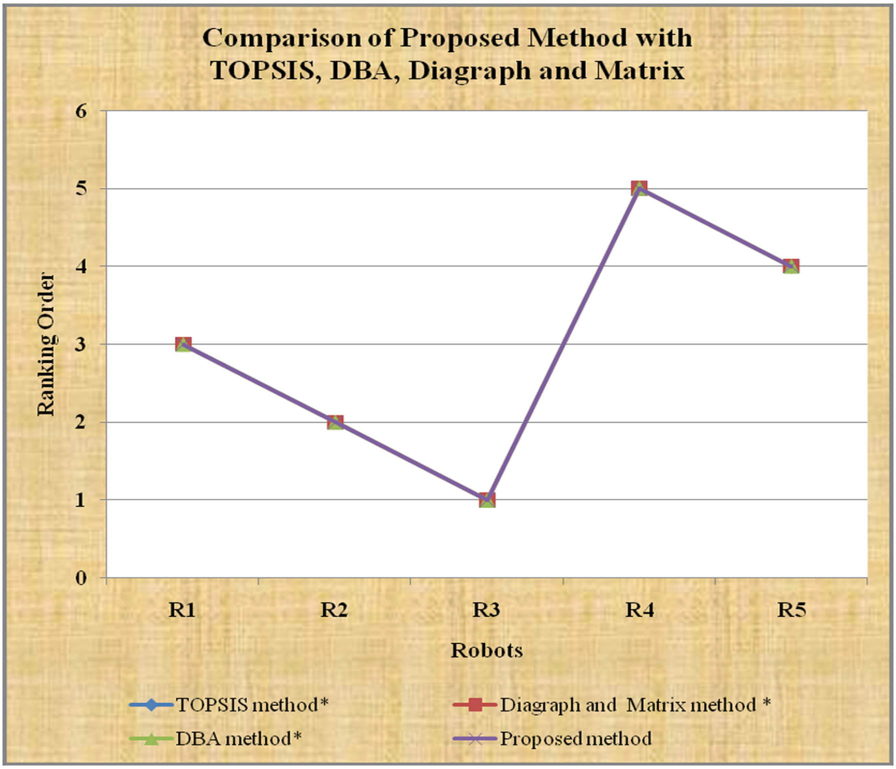

Fig. 2. Comparison of proposed method with TOPSIS, DBA, Diagraph and Matrix method.

The Spearman's rank correlation coefficient of the proposed with respect to all the three methods is 1. The comparison clearly shows that the ranking using proposed method completely matches with those of other methods, which validate the proposed approach of robot selection.

5 Conclusions

In this paper, we have reconsidered and proposed the Terminal Value (TV) of Engineering Economy for the assessment of alternatives with respect to beneficial and non-beneficial multiple conflicting criteria. Hitherto TV has not yet been reported or used in open literature for this purpose. Benefit to non-benefit ratio has been considered as the key selection parameter for robots. The importance weight of criteria plays a great role in the evaluation process as it simultaneously acts as the interest rate and number of periods of cash flows. The comparison shows that the obtained ranking of robots by the proposed method completely matches with the works published in different open journals. This validates and gives corroborate to the proposed methodology for applying in robot selection as a simple but effective strategy. One may easily conclude that this method of robots selection is straight forwarded, effective and one of the simplest methods those have ever been proposed in open literature.

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Green Manufacturing in a Decentralized Supply Chain

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Abstract. This study analyzes the influence of power structure on pricing, greening decisions, and profits of the channel members in a decentralized supply chain. In order to identify the characteristics of the optimal greening level and the profits of the supply chain members, three different models are developed and analyzed under manufacturer-Stackelberg, retailer-Stackelberg, and vertical Nash game settings. Results from the present study suggest that both the retailer and manufacturer prefers their respective leaderships. But the sales volume is higher in Nash game structure. However, consumer gets more benefits under the retailers' leadership.

Keywords: Green supply chain · Pricing · Game theory

1 Introduction

Today consumers are very much aware about the greening level of the purchased items, due to rising awareness about health and environmental damages caused by those products. Accordingly, eco-friendly products such as LED bulbs, plug-in electric cars, BPA-free products, recycled goods are receiving increasing attention among modern consumers. Moreover, in order to protect the environment, regulatory agencies such as BEE in India, EPA in USA, EEA in Europe are employing stricter regulations on the manufacturing industries. All this greening initiatives also provides an opportunity for the modern manufacturers to stimulate the market demand of their products, by increasing the investment in green product innovations. Global manufacturers such as Apple Inc. are using only recycled papers in packaging of their products. Moreover, modern car manufacturers are installing exhaust emission control systems into their products in order to reduce pollution. In the existing literature articles such as Liou et al.

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(2016), Kaur et al. (2017), Ma et al. (2018) and so on, are some studies where the issue of green supply chain is addressed.

In practice the optimal decisions are significantly effected by the decision making power of the supply chain participants. In the existing literature there are several articles where the power structure is treated as the key factor that largely influences the process of optimal decision making. In some of the articles the manufacturer is considered as the leader and the retailer is the follower (For example, Hing et al. (2007), Ma et al. (2013), Swami and Shah (2013), Dey and Saha (2018) etc.). However, powerful retailers, such as Costco, Wal-Mart, Kroger etc. have more dominating power over the other supply chain members (We and Jing (2018)). Chen et al.(2015) studied the impact of power structure in a mixed dual-channel and found that the pricing decision as well as the performance of the supply chain members varies with the power structure. Touboulic et al. (2014) concluded that the organizational responses in implementing sustainable initiatives is influenced by the power affect. Chen and Wang (2015) studied a smart phone supply chain channel selection under different power structures and concluded that the firm that has more power will receive more profits. The authors claimed that the profits of the firms and the supply chain members as well as the channel selection decision are significantly affected by the power of the supply chain members.

The present study considers three power structures, namely manufacturer-Stackelberg (MS), retailer-Stackelberg (RS) and Nash game in our study. In MS game the manufacturer plays the role of a Stackelberg leader and has the power to take the decision first. Whereas in RS game retailer works as a Stackelberg leader and the first decision is taken by the retailer and followed by the other members of the supply chain. However in Nash game both the members take their decisions simultaneously.

2 Model Description, Notation and Assumptions

This study considers a single period decentralized supply chain and analyzes the influence of power structure on the greening incentives of the manufacture in a two-level supply chain consisting of one manufacturer and one retailer where manufacturer sells a single type of product through an independent retailer in a 'green' sensitive consumer market. To formulate analytical models the following assumptions are made:

(i) The functional form of the demand function is as follows:

$$D = a - bp + c\theta, \tag{1}$$

Where a represents the market potential, b represents price elasticity and c is the consumer's sensitivity to greening level. p represents the retail price of the product and θ represents greening level. The above demand function is extensively adopted in the existing supply chain management literature (Ghosh and Shah (2012)).

(ii) The per unit greening level improvement cost varies with production volume and is considered as $\delta\theta^2$. Constant δ represents sensitivity for the green manufacturing investment(Zhu and He (2017)).
(iii) All the parameters related to market demand are common knowledge between the supply chain members (Ghosh and Shah (2012)).

To distinguish the outcomes under the MS, RS, and Nash games, the following list of notations are used:

Model	Notations
i	green supply chain models under MS, RS, and Nash games, $i = m, r, n$
p_i	per unit retail price of the product
w_i	per unit wholesale price of the product
m_i	per unit profit margin of the retailer
π_{mi}	manufacturer's profit
π_{ri}	retailer's profit
Q_i	sales volume in two consecutive selling periods
θ_i	greening level of product

3 Decentralized Models Under Different Power Structure

In this section, we present the profit structures and optimal solutions under the MS, RS, and Nash games.

Under MS game, the manufacturer as Stackelberg leader determines wholesale price of the product w_m and greening level of the product θ_m. Depending on manufacturer response, the retailer sets the retail price (p_m). The profit functions for the retailer and manufacturer under MS game are as follows:

$$\pi_{rm} = (p_m - w_m)(a - bp_m + c\theta_m) \tag{2}$$

$$\pi_{mm} = (w_m - \delta\theta_m^2)(a - bp_m + c\theta_m) \tag{3}$$

The retailers profit function is concave with respect to p_m as $\frac{\partial^2 \pi_{rm}}{\partial {p_m}^2} = -2b < 0$ and $\frac{\partial^2 \pi_{mm}}{\partial {w_m}^2} = -b < 0$ and $\Phi_1 = \frac{\partial^2 \pi_{mm}}{\partial {w_m}^2} \times \frac{\partial^2 \pi_{mm}}{\partial {\theta_m}^2} - \left(\frac{\partial^2 \pi_{mm}}{\partial w_m \partial \theta_m}\right)^2 = 2bc\delta\theta_m + b\delta(a - b(w_m + \delta{\theta_m}^2)) - c^2/4$.

Therefore, first order condition of Eq. (2) with respect to p_m yields $p_m = \frac{a+bw_m+c\theta_m}{2b}$. Substituting this value in Eq. (3) one can get the following equation:

$$\pi_{mm} = \frac{1}{2}(a - bw_m + c\theta_m)(w_m - \delta{\theta_m}^2) \tag{4}$$

Taking first order condition of Eq. (4) with respect to w_m and θ_m and solving them one can get the following set solutions:

$$(w_m, \theta_m) \equiv (w_1, \theta_1) = \left(\frac{3c^2+4ab\delta}{8b^2\delta}, \frac{c}{2b\delta}\right)$$
$$(w_m, \theta_m) \equiv (w_2, \theta_2) = \left(\frac{c^2+2ab\delta-c\sqrt{c^2+4ab\delta}}{2b^2\delta}, \frac{c-\sqrt{c^2+4ab\delta}}{2b\delta}\right)$$
$$(w_m, \theta_m) \equiv (w_3, \theta_3) = \left(\frac{2ab\delta+c(c+\sqrt{c^2+4ab\delta})}{2b^2\delta}, \frac{c+\sqrt{c^2+4ab\delta}}{2b\delta}\right)$$

Because, $\Phi_1|_{(w_1,\theta_1)} = \frac{1}{8}(c^2 + 4ab\delta) > 0$, therefore, the profit function is concave at first-pair of solution. One may verify that $\Phi_1|_{(w_2,\theta_2)} = \Phi_1|_{(w_3,\theta_3)} = -\frac{1}{4}(c^2 + 4ab\delta) < 0$, therefore, the second and third set of solution is not optimal. Substituting the optimal values one can get the following optimal solutions:

$$p_m = \frac{7c^2+12ab\delta}{16b^2\delta}; \pi_{rm} = \frac{(c^2+4ab\delta)^2}{256b^3\delta^2}; \pi_{rm} = \frac{(c^2+4ab\delta)^2}{128b^3\delta^2}; Q_m = \frac{4ab\delta+c^2}{16b\delta}.$$

Again, in RS game, the retailer plays the role of a stackelberg leader and maximizes her profit with respect to the manufacturer optimal responses. We use a transformation $m_r = p_r - w_r$, which is similar to that of Aust and Buscher (2012). The retailer sets profit margin (m_r)and consequently the manufacturer sets wholesale price w_r and GL θ_r. The profit functions for the retailer and manufacturer in are obtained as follows:

$$\pi_{rr} = m_r(a - b(m_r + w_r) + c\theta_r) \tag{5}$$

$$\pi_{mr} = (w_r - \delta(\theta_r)^2)(a - b(m_r + w_r) + c\theta_r) \tag{6}$$

Note that $\frac{\partial^2 \pi_{mr}}{\partial {w_r}^2} = -2b < 0$ and $\Phi_2 = \frac{\partial^2 \pi_{mr}}{\partial {w_r}^2} \times \frac{\partial^2 \pi_{mr}}{\partial {\theta_r}^2} - \left(\frac{\partial^2 \pi_{mr}}{\partial w_r \partial \theta_r}\right)^2 = -c^2 + 8bc\delta\theta - 4b\delta(-a + b(m + w + \delta\theta^2))$. Taking first order condition of Eq. (6) with respect to w_r and θ_r and solving them one can get the following set solutions:

$$(w_r, \theta_r) \equiv (w_4, \theta_4) = \left(\frac{3c^2+4b(a-bm_r)\delta}{8b^2\delta}, \frac{c}{2b\delta}\right)$$
$$(w_r, \theta_r) \equiv (w_5, \theta_5) = \left(\frac{c^2+2b(a-bm_r)\delta-c\sqrt{c^2+4b(a-bm_r)\delta}}{2b^2\delta}, \frac{c-\sqrt{c^2+4b(a-bm_r)\delta}}{2b\delta}\right)$$
$$(w_r, \theta_r) \equiv (w_6, \theta_6) = \left(\frac{2b(a-bm_r)\delta+c(c+\sqrt{c^2+4b(a-bm_r)\delta})}{2b^2\delta}, \frac{c+\sqrt{c^2+4b(a-bm_r)\delta}}{2b\delta}\right)$$

The profit function is concave at the first pair of values as $\Phi_2|_{(w_4,\theta_4)} = \frac{1}{2}(c^2 + 4b(a-bm)\delta) > 0$ and negative at the remaining set of values. Similarly, the profit function of the retailer is also concave with respect to p_r as $\frac{\partial^2 \pi_{rr}}{\partial {p_r}^2} = -b < 0$. Substituting the optimal responses of the manufacturer into Eq. (5) one can get the following equation:

$$\pi_{rr} = \frac{m(c^2 + 4b(a - bm)\delta)}{8b\delta} \tag{7}$$

Solving $\frac{d\pi_{rr}}{dm_r} = 0$ one may obtain $m_r = \frac{c^2+4ab\delta}{8b^2\delta}$. Using backward substitution the following solutions are obtain:

$$\theta_r = \frac{c}{2b\delta};\ \pi_{rr} = \frac{(c^2+4ab\delta)^2}{128b^3\delta^2};\ \pi_{mr} = \frac{(c^2+4ab\delta)^2}{256b^3\delta^2};\ Q_r = \frac{4ab\delta+c^2}{16b\delta}.$$

Finally, the profit functions in Nash game are same as in the RS game. But the difference is that, in Nash game both the retailer and manufacturer takes their decisions simultaneously. The simplified values of equilibrium outcomes in the Nash game are the followings:
$m_n = \frac{c^2+4ab\delta}{12b^2\delta}$; $w_n = \frac{c^2+ab\delta}{3b^2\delta}$; $\theta_n = \frac{c}{2b\delta}$; $\pi_{rn} = \frac{(c^2+4ab\delta)^2}{144b^3\delta^2}$; $\pi_{mn} = \frac{(c^2+4ab\delta)^2}{144b^3\delta^2}$; $Q_n = \frac{4ab\delta+c^2}{12b\delta}$.

Next we propose the following theorem:

Theorem 1.

(i) The wholesale price of the product under different power structures satisfy

$$w_m \geq w_n \geq w_r$$

(ii) The retail price of the product under different power structures satisfy

$$p_n < p_r = p_m.$$

Proof. The wholesale price of the product under different power structures satisfy the following inequalities:

$$w_n - w_r = \frac{c^2 + 4ab\delta}{48b^2\delta} > 0$$

$$w_m - w_n = \frac{c^2 + 4ab\delta}{24b^2\delta} > 0$$

The retail price of the product under different power structures satisfy:

$$p_m - p_r = 0$$

$$p_r - p_n = \frac{c^2 + 4ab\delta}{48b^2\delta} > 0$$

Hence the Theorem.

From Theorem 1 one can observe that the consumer get benefited under the retailers leadership. Next we propose the following theorem to compare optimal profits, sales volume, and greening level under different game structures.

Theorem 2.

(i) Retailers profit function under different power structure satisfy

$$\pi_{rr} \geq \pi_{rn} \geq \pi_{rm}$$

(ii) Manufacturers profit function under different power structure satisfy
$$\pi_{mm} \geq \pi_{mn} \geq \pi_{mr}$$

(iii) Optimal sales volume under different power structure satisfy
$$Q_n \geq Q_r = Q_m$$

(iv) Optimal greening level under different power structure does not change.

Proof. Retailers profit function under different power structure satisfy the following inequalities:
$$\pi_{rr} - \pi_{rn} = \frac{(c^2 + 4ab\delta)^2}{1152b^3\delta^2} > 0$$
$$\pi_{rn} - \pi_{rm} = \frac{7(c^2 + 4ab\delta)^2}{2304b^3\delta^2} > 0$$

Similarly, manufacturers profit function under different power structure satisfy the following inequalities:
$$\pi_{mn} - \pi_{mr} = \frac{7(c^2 + 4ab\delta)^2}{2304b^3\delta^2} > 0$$
$$\pi_{mm} - \pi_{mn} = \frac{(c^2 + 4ab\delta)^2}{1152b^3\delta^2} > 0$$

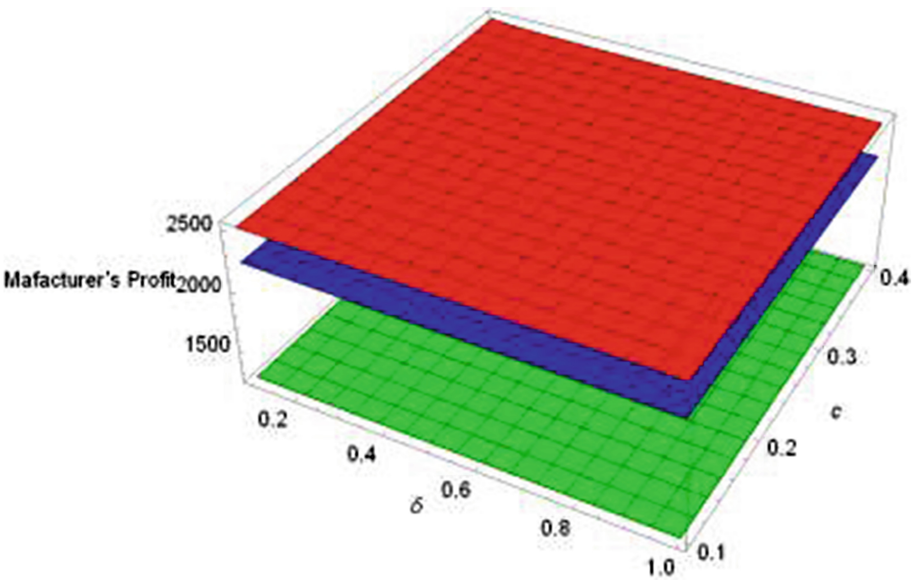

Fig. 1. The profit functions of the manufacturer π_{mm} (Red), π_{mr} (Green), and π_{mn} (Blue).

Finally, the optimal sales volume under different power structure satisfy:

$$Q_m - Q_r = 0$$

$$Q_n - Q_r = \frac{c^2 + 4ab\delta}{48b\delta} > 0$$

Above relations ensure the proof of Theorem 2.

From Theorem 2 one can observe that power structure within supply chain members does not create any perturbation to greening level of the product. Moreover, both the retailer and the manufacturer get higher profits under their individual leaderships. But the sales volume reaches its highest value under the Nash game structure. In order to support these analytical findings made in Theorem 2 we present numerical illustrations with the following values of the parameters: $a = 200$, $b = 0.5$, $c \in (0.01, 0.4)$, and $\delta \in (0.1, 1)$.

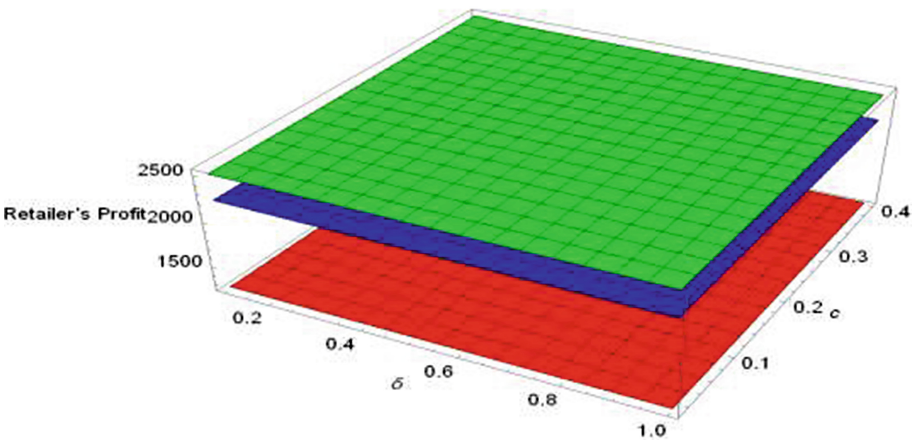

Fig. 2. The profit functions of the retailer π_{mm} (Red), π_{mr} (Green), and π_{mn} (Blue).

Above Figs. 1 and 2 justifies the claims made in Theorem 2.

4 Conclusion

In this study we analyzed the characteristics of pricing and greening level under different power structures in a decentralized single period supply chain environment. By analyzing the present model we have found that power structure do not make any impact on the optimal per unit greening investment of the manufacturer. Moreover, each supply chain member prefers their own leadership in order to increase the profitability. Sales volume is maximum under the Nash game and the retail price is minimum under the retailers leadership.

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Selection of Resource for Re-assignment of a Job Due to Break Down Failure Under Agent Based Holonic Manufacturing Environment

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Abstract. Selection of resource for re-assignment of a job due to break down failure under holonic manufacturing environment is very important activity especially while the decision is to make under conflicting subjective criteria. Due to the multiple conflicting criteria and their complexity, selection of proper resource can be considered as a multi criteria decision-making (MCDM) problem. This analysis aims to provide a rationally new approach for resource evaluation and selection that might be a suitable surrogate of the existing methodologies. This paper explores an evaluation algorithm capable of considering subjective criteria. The proposed methodology is illustrated with a suitable example. In the methodology, the subjective measures of performance rating and importance weight are integrated using the proposed logical technique for determination of respective Resource Selection Index (MSI). Resources are ranked in the descending order of their respective RSI values and the resource associated with the highest MSI value is selected as the best one. Finally, the proposed technique is verified with a conventional MCDM tool. This shows that the proposed technique is very simple but extensively effective and extremely applicable in selection of resource under holonic manufacturing environment.

Keywords: MCDM · Holonic manufacturing · Agent based systems · Cognitive · Resource Selection Index

1 Introduction

Ever-increasing market complexities together with the shop-floor disturbances have enhanced the business uncertainty by many times. To deal with such a tumultuous situation, distributed approach to problem solving (DPS) [1, 2] is the possible solution and agent based holonic manufacturing systems, a sub-set of DPS, has already made remarkable breakthrough in this regard. Although the holonic concept was originated for living and societal systems [3], it is successfully mapped to business organization. Holons are autonomous and co-operative building blocks of a manufacturing system that are intended to execute shop-floor functions. A holon consists of an information processing part and often a physical processing part. The benefits holonic behavior provides

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to the manufacturing business organization are same as it offers to the living organisms and societies i.e., stability in the face of disturbances, adaptability and flexibility in the face of change, and efficient use of available resources. The HMS concept combines the best features of hierarchical and heterarchical organization. It preserves the stability of a hierarchy while providing the dynamic flexibility of a heterarchy [3, 4]. This holonic behavior is accomplished by an agent. An agent is a computer system which is capable of acting autonomously in its environment in order to meet its design objectives [5]. From the objected-oriented programming view point, agents are a special type of software objects which have their own internal algorithms, use a common language for communication and in contrast with objects, have the possibility to reason, interpret incoming messages, and take decisions according to its specific beliefs and objectives. Colombo et al. [6] defined agent as a software entity situated in a production environment, with enough intelligence that is capable of autonomous control actions in this environment and of co-operation relationships by participating in associations' agreements with other entities in order to meet its design objectives. An agent should be able to act without the direct intervention of humans or other agents, and should have control over its own actions and internal state. In essence holonic behavior is captured and implemented by agent based systems. The details of holonic and agent based systems are available in literatures [7, 8].

However, holonic behavior can be enriched, if the agents are augmented with cognitive properties [9]. Decision making is one of the basic cognitive processes of human behaviors by which a preferred option or a course of actions is chosen from a set of alternatives based on certain criteria [10]. In a holonic manufacturing system, the successful implementation of the holonic behavior depends on how efficiently and effectively varieties of decisions are made to deal with disturbances. Disturbances refer to random occurrence of events that hinder pursue the goal and causes perturbations to the system such as rush order, cancellation/modifications of order quantity, machine partial and complete malfunctioning, change in delivery pattern and priority, addition/alteration of resources etc. [11]. It is expected that in the wake of such disruptions, agents would rise to the occasion and with the help of cognitive decision making ability, would provide solution. Among various disturbances, machine breakdown is perhaps one of the most critical to deal with. Automatic diagnostic and fault recovery management will try to repair the fault and put the machine under working condition at the earliest at the expense of time, production loss, effort, and cost. Under such condition it is also desirable that other remaining machines will try to shoulder extra work by virtue of cooperative property so as to minimize the production loss. It is the proactiveness of the agents that invigorates a sense of responsibility to shoulder extra work. Since the agents are intelligent, they consider a number of criteria with varying weight for decision making.

In the present work, a novel multi criteria decision making method (MCDM) is proposed to deal with machine malfunctioning. An MCDM problem must contain at least two alternatives and at least two conflicting criteria. MCDM problems are classified into diverse aspects and strategies. The criteria are classified in two ways; firstly, as subjective (qualitative/intangible), objective (quantitative/tangible) and critical criteria (that need to be satisfied before further processing) and secondly, as benefit criteria (higher value is desirable) and cost criteria (lower value is desirable). In an MCDM problem a finite set of alternatives/strategies can be evaluated considering multi-criteria. The MCDM techniques include TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), AHP (Analytical Hierarchy Process), MOORA (Multi Objective Optimization on the basis of Ratio Analysis), VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje i.e. Multi-criteria Optimization and Compromise Solution), GRA (Grey Relational Analysis), COPRAS-G (Complex Proportional Assessment method with the applications of the Grey systems theory), SAW (Simple Additive Weighting), ELECTRE (ELimination Et Choix Traduisant la REalité that means ELimination and Choice Expressing REality), PROMETHEE (Preference Ranking Organization METHod for Enrichment of Evaluations) etc. [12–14].

The rest of the paper is organized in the following manner. A problem in this respect is presented in Sect. 2 and proposed algorithm is presented in Sect. 3. The results and related discussions regarding problem, and illustration of the proposed method are furnished in Sect. 4. Finally Sect. 5 concludes the paper.

2 Problem Definition

A South Asian company newly introduces Holonic Manufacturing System in its organization. To decide how to reassign an already allocated job to available resources due to a sudden breakdown failure, the shop-floor control holon wants to make a judicious decision. There are five remaining resources to which incomplete job can be re-allocated considering six subjective criteria viz. Precision, Availability, Eagerness, Responsibility, Processing and Processing.

- Precision: Precision can be defined as the closeness of two or more measurements. It is a statistical variability that is random error.
- Availability: It is the characteristic of a resource which is useable, operable and committable to perform its expected function. It is the aggregate of the resources reliability, maintainability accessibility, secure-ability and serviceability.
- Eagerness: It is the manifestation of proactiveness of an agent by virtue of which it extends cooperation to other agents by shouldering extra responsibility for the fulfillment of the organizational goal. This property stems from the cognitive behavior of an agent to contribute from its own in the wake of any changing circumstances.
- Responsibility: It may be defined as the obligation and the duty to perform satisfactorily a task assigned by a by some holon that must be fulfilled.
- Processing Cost: Processing costs refers to the cost incurs for conversion of raw material or semi finished product to finished product. It involves labor costs, depreciation of resource etc.
- Processing Time: It refers to the period of time or interval required for conversion of raw material or semi finished product to finished product, through manufacturing process and operations.

3 Proposed Algorithm

Step 1: Formation of decision matrix:

$$D = \left(LV_{ij}\right)_{m \times n} \tag{1}$$

LV_{ij} is Linguistic variable for ith alternative with respective to jth criteria, where

$$LV_{ij} = \left(l_{ij}, m_{ij}, u_{ij}\right) \tag{2}$$

Step 2: Formation of weight matrix:

$$W = \left(w_j\right)_{1 \times n} \tag{3}$$

weight is awarded in linguistic variable and subsequently in triangular fuzzy number where $w_j = \left(\alpha_j, \beta_j, \gamma_j\right)$

Step 3: Defuzzification of linguistic variable is accomplished by using

$$x_{ij} = \left(\frac{l_{ij} + 2m_{ij} + u_{ij}}{4}\right) \tag{4}$$

Step 4: Standardization of defuzzified performance rating using the equations

$$r_{ij} = \frac{x_{ij}}{\max\limits_i\left(x_{ij}\right)} \quad j \in B \tag{5}$$

$$r_{ij} = \frac{\min\limits_i\left(x_{ij}\right)}{x_{ij}} \quad j \in NB \tag{6}$$

Step 5: Defuzzification of weight of criteria is calculated by the following equation

$$w_j = \left(\frac{\alpha_j + 2\beta_j + \gamma_j}{4}\right) \tag{7}$$

Step 6: Normalization of weight

$$\bar{w}_j = \frac{w_j^d}{\sum\limits_{j=1}^{n} w_j^d} \tag{8}$$

Step 7: Weighted normalized decision matrix

$$D_{wn} = \left(\bar{w}_j * r_{ij}\right)_{m \times n} = \left(v_{ij}\right)_{m \times n} \tag{9}$$

Step 8: Elementary Positive and negative contribution:

$$PC_{ij}^{+} = \left(e^{v_{ij}} - 1\right) = \left[e^{\left(\bar{w}_j * r_{ij}\right)} - 1\right] \tag{10}$$

Elementary Negative contribution is measured by

$$NC_{ij}^{-} = \left(e^{-v_{ij}} - 1\right) = \left[e^{-(\bar{w}_j * r_{ij})} - 1\right] \tag{11}$$

Step 9: Total Positive contribution is measured by

$$TPC_i^{+} = \left(\sum_{j=1}^{n} e^{v_{ij}} - n\right) = \left[\sum_{j=1}^{n} e^{(\bar{w}_j * r_{ij})} - n\right] \tag{12}$$

Total Negative contribution is measured by

$$TNC_i^{-} = \left(\sum_{j=1}^{n} e^{-v_{ij}} - n\right) = \left[\sum_{j=1}^{n} e^{-(\bar{w}_j * r_{ij})} - n\right] \tag{13}$$

Step 10: Determination of Resource Selection Index (RSI) by

$$RSI_i = \frac{p * \left(TPC_i^{+} - TNC_i^{-}\right)}{\left(TPC_i^{+}\right)_{\max} - \left(TNC_i^{-}\right)_{\min}} = \frac{p * \left(\sum_{j=1}^{n} e^{v_{ij}} - \sum_{j=1}^{n} e^{-v_{ij}}\right)}{\left(\sum_{j=i}^{n} e^{v_{ij}}\right)_{\max} - \left(\sum_{j=1}^{n} e^{-v_{ij}}\right)_{\min}}$$

$$= \frac{p * \left(\sum_{j=1}^{n} e^{(\bar{w}_j * r_{ij})} - \sum_{j=1}^{n} e^{-(\bar{w}_j * r_{ij})}\right)}{\left(\sum_{j=1}^{n} e^{\bar{w}_j * r_{ij}}\right)_{\max} - \left(\sum_{j=1}^{n} e^{-\bar{w}_j * r_{ij}}\right)_{\min}} \tag{14}$$

Here p is the desired point of scale. Select the best alternative with the highest closeness coefficient.

4 Calculation and Discussion

In the current analysis the decision maker (in this case an agent) considers the five alternative resources on the basis of six subjective criteria. These criteria are evaluated by seven degrees of linguistic variables. The linguistic variables with their abbreviation and the corresponding Triangular Fuzzy Numbers (TFN) are given in Table 1. Resource assessment matrix with respect to criteria in terms of linguistic variables is shown in Table 2. It also indicates that, of the six criteria, four viz. Precision, Availability, Eagerness and Responsibility are of benefit category whereas the remaining two, Processing Cost and Processing Time are of non-benefit category. The weight of the criteria as assessed by the decision makers are also furnished in the Table 2 against the respective criterion. Table 3 shows the resource assessment matrix in fuzzy triangular numbers by converting linguistic variables into corresponding TFN. Table 4 converts the TFN of resource assessment matrix into crisp numbers by using Eq. (4), whereas

Eq. (7) is used for weight for the same purpose. Table 5 shows the standardized value of resource assessment obtained by using Eqs. (5) and (6). It is noticed that each element in the matrix lies between 0 and 1. The sum of weights of all criteria is unity. It is also noticeable that the criterion "Availability" is associated with the highest value (0.2302**)** of all. It indicates that the decision makers give the highest importance to "Availability" in selection process. Table 6 depicts weighted standardized assessment matrix for individual criterion by using Eq. (9). Table 7 shows the Elementary Positive Contribution (EPC) and Elementary Negative Contribution (NPC) calculated by using Eqs. (10) and (11). Table 8 describes total positive and total negative contribution of various criteria by using Eqs. (12) and (13). Resource selection index (RSI) is measured by using Eq. (14) for the resources and is shown in Table 8. The same problem is considered and solved by applying two well-known existing methodologies viz. TOPSIS and MOORA. The obtained ranks are shown in the adjacent column of the same table. Figure 1 depicts resources selection Index and Fig. 2 shows ranking of resources graphically. This shows ranking of the resources by proposed method match absolutely with the ranking by MOORA and TOPSIS methodologies. This validates the proposed technique in selection of resource for re-assignment of a job for break down failure under agent based holonic manufacturing environment.

Table 1. Abbreviation, linguistic variables and Triangular Fuzzy Number (TFN)

Abbreviation	Linguistic variable	Triangular fuzzy number
EH	Extremely High	(7, 8, 9)
VH	Very High	(6, 7, 8)
H	High	(5, 6, 7)
M	Medium	(4, 5, 6)
A	Average	(3, 4, 5)
P	Poor	(2, 3, 4)
VP	Very Poor	(1, 2, 3)

Table 2. Resource assessment matrix with respect to criteria in terms of linguistic variables

	Precision (+)	Availability (+)	Eagerness (+)	Responsibility (+)	Processing cost (−)	Processing time (−)
Weight	VH	EH	VH	A	M	H
Resources	C1	C2	C3	C4	C5	C6
R1	VH	A	H	**G**	VH	H
R2	H	H	VH	F	H	H
R3	EH	VH	H	P	EH	F
R4	A	M	VH	M	P	H
R5	P	H	H	VH	M	VH

Table 3. Resource assessment matrix in fuzzy triangular numbers

	Precision (+)	Availability (+)	Eagerness (+)	Responsibility (+)	Processing cost (−)	Processing time (−)
Weight	(6, 7, 8)	(7, 8, 9)	(6, 7, 8)	(2, 3, 4)	(4, 5, 6)	(6, 7, 8)
Resources	C1	C2	C3	C4	C5	C6
R1	(6, 7, 8)	(3, 4, 5)	(1, 2, 3)	(5, 6, 7)	(5, 6, 7)	(5, 6, 7)
R2	(5, 6, 7)	(3, 4, 5)	(6, 7, 8)	(2, 3, 4)	(5, 6, 7)	(5, 6, 7)
R3	(1, 2, 3)	(6, 7, 8)	(5, 6, 7)	(1, 2, 3)	(7, 8, 9)	(3, 4, 5)
R4	(3, 4, 5)	(4, 5, 6)	(6, 7, 8)	(4, 5, 6)	(1, 2, 3)	(5, 6, 7)
R5	(6, 7, 8)	(5, 6, 7)	(5, 6, 7)	(6, 7, 8)	(4, 5, 6)	(6, 7, 8)

Table 4. Conversion of resource assessment matrix into crisp numbers

	Precision (+)	Availability (+)	Eagerness (+)	Responsibility (+)	Processing cost (−)	Processing time (−)
Weight	7	8	7	3	5	7
Resource	C1	C2	C3	C4	C5	C6
R1	7.00	4.00	2.00	6.00	7.00	6.00
R2	6.00	4.00	7.00	3.00	6.00	6.00
R3	2.00	7.00	6.00	2.00	8.00	4.00
R4	4.00	5.00	7.00	5.00	2.00	6.00
R5	7.0	6.0	6.0	7.0	5.0	7.0
MAX	7.0	7.0	7.0	7.0	8.0	7.0

Table 5. Standardization of resource assessment

	Precision (+)	Availability (+)	Eagerness (+)	Responsibility (+)	Processing cost (−)	Processing time (−)
Weight	0.1858	0.2302	0.1857	0.0796	0.1327	0.1858
Resources	C1	C2	C3	C4	C5	C6
R1	1.0000	0.5714	0.2857	0.8571	0.8750	0.8571
R2	0.8571	0.5714	1.0000	0.4286	0.7500	.8571
R3	0.2857	1.0000	0.8571	0.2857	1.0000	0.5714
R4	0.5714	0.7143	1.0000	0.7143	0.2500	0.8571
R5	1.0000	0.8571	0.8571	1.0000	0.6250	1.0000

Table 6. Weighted standardized assessment matrix

	Precision (+)	Availability (+)	Eagerness (+)	Responsibility (+)	Processing cost (−)	Processing time (−)
Resource	C1	C2	C3	C4	C5	C6
R1	0.2042	0.1406	0.0545	0.0706	0.1232	0.1727
R2	0.1727	0.1406	0.2042	0.0347	0.1047	0.1727
R3	0.0545	0.2588	0.1727	0.0230	0.1419	0.1120
R4	0.1120	0.1787	0.2042	0.0585	0.0337	0.1727
R5	0.2042	0.2181	0.1727	0.0829	0.0865	0.2042

Table 7. Elementary Positive Contribution (EPC) and Elementary Negative Contribution (NPC)

	Precision (+)	Availability (+)	Eagerness (+)	Responsibility (+)	Processing cost (−)	Processing time (−)
Resource	C1	C2	C3	C4	C5	C6
R1	0.2042	0.1406	0.0545	0.0706	0.1232	0.1727
R2	0.1727	0.1406	0.2042	0.0347	0.1047	0.1727
R3	0.0545	0.2588	0.1727	0.0230	0.1419	0.1120
R4	0.1120	0.1787	0.2042	0.0585	0.0337	0.1727
R5	0.2042	0.2181	0.1727	0.0829	0.0865	0.2042

Table 8. TPC, NPC and RSI measures for the resources and ranking of resources

Resources	TPC	TNC	TPC-TNC	RSI	Ranking by		
					Proposed method	TOPSIS	MOORA
R1	0.4699	0.2958	0.1741	4.50	5	5	5
R2	0.5522	0.2773	0.2748	7.10	3	3	3
R3	0.5090	0.2540	0.2550	6.59	4	4	4
R4	0.5535	0.2064	0.3471	8.96	2	2	2
R5*	0.6779	0.2907	0.3871	10.00*	1*	1*	1*

*RS5 is Best resources associated with the highest RSI value.

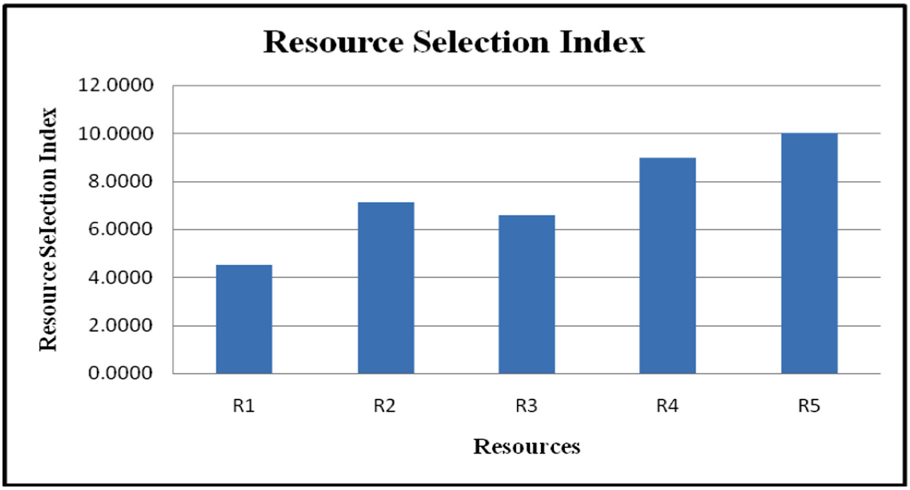

Fig. 1. Resources selection index

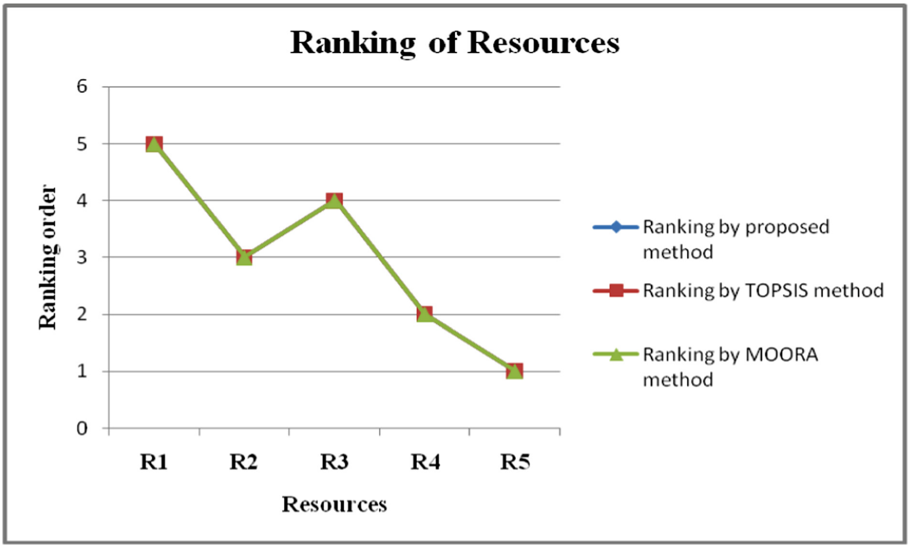

Fig. 2. Ranking of resources

5 Conclusion

In an agent based holonic manufacturing environment, sudden breakdown of a resource due to unpredictable and unassigned causes is a common phenomenon. In this circumstance, the reassignment of the concerned job to an existing resource is essential for completion of the job in scheduled time as far as practicable. So, application of Multi-Criteria Decision Making in re-assignment of the job to the most suitable resource is inevitable in an intelligent system. The illustration of the proposed methodology by solving the given re-assignment and evaluation problem shows that the proposed methodology is simple and useful technique in this domain. The Spearman's Rank Correlation coefficient (=1) of the proposed technique with those of the existing methodologies ensures the validity of the technique. Hence the proposed method can be considered a useful, effective and new technique in evaluation of resources and re-assignment of a job due to sudden break down failure in an agent based intelligent holonic manufacturing environment. Consideration of both tangible and intangible factors with heterogeneous decision makers may be an important direction of future research.

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Uncertain Demand Allocation with Insufficient Resource in Disaster by Using Facebook Disaster Map Under Limited Fund

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Abstract. Disasters are the unfortunate suddenly occurring events which collapse a living society within a moment. Facebook, the most trending social media have initiated an active role to response the crisis developing a safety check feature to know about the demand of victims in an area. Through this feature the Facebook authority can confirm the requirement of the affected people in a specific area. This research have introduced a mathematical model for multi-objective solid transportation problem to quick delivery of the relief products to the devastating society according to the most prior demand of an area acquiring the best information through safety check feature launched by Facebook with limited resources. The two main objectives of the model are to minimize the total cost and minimize the total time so that relief can be executed under a limited budget responding quickly to the affected area. In this paper, fuzzy goal programming is adopted to solve the multi-objective solid transportation problem. To contemplate the performance of the model, a numerical example is presented in this paper. The numerical example is solved with LINGO optimization solver.

Keywords: Disaster relief · Social media · Facebook disaster map · Intensity of demand · Fuzzy goal programming · LINGO

1 Introduction

The events of disaster are unforeseeable. After the strike, disaster devastates a living society within a moment. Lots of researches are ongoing to predict the occurrence of disaster, but no community is immune from the occurrence of disaster yet. Several disasters may be natural (e.g. earthquake, flood, drought, landslide, volcanic eruption etc.) or human caused (e.g. terrorist attack, nuclear exploitation, wrong laboratory experiments etc.) create not only trouble for living society but also degrade the environment also. Death of thousand numbers of

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people and destruction of million worth of properties have been reported every year. The role of social media has become more acceptable as it widely spreads information about the catastrophic environment worldwide [1,2]. In today's life social media is such a tool which passes every information from a single user to national and international community. When disaster strikes an area, the information about demolition and affected population are not bounded on its own community, it has extended over the local and state agencies, large organization such as Centers for Disease Control and Prevention (CDC), the Federal Emergency Management Agency (FEMA) and the Red Cross. Sometimes particular individual or organizations provide vague information to mislead the society which creates a major distraction of the response authority in disaster. But social media takes its strong step against the malicious information and let the people to know the real fact. But network unavailability in the disastrous environment is the main demerits of social media.

Facebook (FB) is the most active social media in the entire world[1]. FB has shown its active association with disaster relief launching a new site feature known as crisis response which acts as safety check tool through which FB users able to confirm his/her safety to family and friends during crisis environment like natural or human caused disaster, other life threatening incidents etc. It was first deployed in Nepal earthquake in 2015, providing the safety features to the users to mark them alive and well amidst the catastrophe. Traditional communicating social media are often disconnected in the time which is the most significant for response victims after the hitting of a disaster. In the critical time proper allocation of resources identifying major damages in the society is not an easy deal for response authority. But an area having lots of FB users, if suddenly goes off-line, then also high concentration is given by the FB management and inform govern authority to response those areas accordingly[2].

It is very important to recognize the location and movement of the affected people after disaster to covey the relief materials. FB offers 'Disaster Maps Data' which illustrate about the people's movement and concentration of FB users before and after the calamity. FB management has mentioned that this initialization of FB always response authority like disaster management authority, non-governmental organization (NGO), humanitarian aid agencies etc. to collect information about the effect of disaster and response the affected community with proper care. In June, 2017 in Peru, FB disaster map was introduced for the first time and uses information about user's aggregated location which is shared through FB. FB has commenced disaster map in November, 2017 in India[3]. It has partnered with National Disaster Management Authority (NDMA) and

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1 https://www.google.co.in/search?q=most+active+social+media&rlz=1C1EJFA_enIN782IN782&oq=most+active+social+media&aqs=chrome..69i57.12901j0j7&sourceid=chrome&ie=UTF-8.

2 https://www.thehindu.com/news/national/facebook-brings-in-disaster-response-tools-to-india/article20009679.ece.

3 https://indianexpress.com/article/technology/social/facebook-rolls-out-disaster-maps-in-india-4929557/.

Sustainable Environment and Ecological Development Society (SEEDS) to response the affected community and reduced the critical gap between the authority and victims in the purpose of conveying relief materials due to lack of proper information about the requirements in particular areas. FB provides multiple type of map during disaster viz. location density map explaining the location of the population before and after the calamity and movement map explaining the movement's patterns of population between neighborhood cities for some period. This pattern makes understand the response authority about the demand of relief resources.

As immediate response is very urgent for a society demolished by the unpredictable calamity, resources are kept in central response centers (CRCs) set up by response authority. After the strike of disaster CRCs are active to deliver relief product to the affected areas (AAs) through some conveyance. The relief products are stored in some local response center (LRCs) near to the AAs so that people can obtain relief products according to their needs. Through this community help feature of FB people can ask help posting their primary needs food, shelter, water, transport system, baby items, cloths etc. for survive in the calamity. On the basis of demand, FB management authority categorized it according to priority with proper verification of location and transforms the information to relief response authority to supply the relief product to its proper destination. Consequently, it is important to follow proper strategies and manage of resource allocation in disaster in a right direction efficiently through mathematical modeling [3–6]. For smooth dealing with the crisis providing relief products, a mathematical model for multi-objective solid transportation problem (MOSTP) is established in the paper. The objective of the MOSTP is to minimize the total cost and to minimize the total time of the disaster operation. This research has considered the CRCs with limited relief product are as source point. LRCs very near to AAs are the demand points which are arranged for storing sufficient amount of relief products. For delivering the products from CRCs to LRCs authority hires conveyance from commercial supply chain which has also some working schedule for every day. As the demand of LRCs is very high and CRCs have limited resources, therefore authority has confronted a big problem to fulfill LRCs in such anarchic situation. To overcome the situation this research investigates an intensity factor for measuring the demand of AAs in the mathematical model based on the FB information derived by safety check feature. According to it, the FB users confirm their demand from a particular area and disaster management authority measure the importance of the requirement and transfer the relief product to the LRCs. The intensity measure for demand basically helps the authority to smooth maintaining of relief materials in the catastrophe. The MOSTP is solved through fuzzy goal programming technique.

This paper has introduced basically a mathematical model for solid transportation problem (STP) in disaster relief operation. The basic concept of solid transportation problem (STP) for the first time in the literature in 1962, by Haley [7]. STP is a an extension of transportation problem (TP) considering an extra constraint for conveyance [8]. When STP has more than one objective function it is MOSTP. In the literature, we have seen an growing interest for STP and MOSTP from [9–11]. As emergency situation after disaster urges for quick response with relief items under well planned management system, therefore supply chain of resources is very important factor [12–14]. The survey of operation research/management science (OR/MS) to the literature is done by Atlay and Green in [15]. Analysis of some barrier in humanitarian supply chain that can hinder the disaster operation are discussed in [16]. Application of some multi-criteria optimization techniques are reviewed in the article [17]. Berkoune et al. define transportation problem in emergency situation [18]. There are some article determines the delivery schedule for vehicles and allocate resources based on supply, vehicle capacity and delivery restriction are reviewed in the article [19,20]. There has been seen growing interest of mathematical modeling for MOSTP for crisis management in the literature [21–23]. But probably no one has considered the FB safety check features for demand allocation by formulating s mathematical model. The concept of the mathematical model is described in the following section.

2 Concept of the Model

In this research, we have designed a mathematical model for solid transportation problem in the disastrous environment to allocate the relief material among the victims. When disaster strikes in areas, most of the FB users confirm their situation and requirements through safety check feature. Knowing the information about location and demand of the affected people forwarding from FB management, the authority responsible for disaster starts to convey relief products to the affected areas through some conveyances. There is a set of CRCs with limited relief products established by the disaster relief authority from where relief products are shipped to set of LRCs with sufficient amount of space to store relief materials in the locality of disaster prone areas. LRC are the storing center for relief materials where people can obtain easily their needs. But due to the limited amount of relief products, CRCs are not capable to fulfill the LRCs. CRCs transfer the relief materials to LRCs on the basis of intensity of demand of the AAs. The intensity of demand of the AAs demand is measured by the collection of information from FB. There different types of conveyance which have a limited capacity to load products. These conveyance work for a limited time period per day (Fig. 1).

Assumption for the Model

- The mathematical model is a MOSTP.
- CRCs have limited amount of relief products.
- LRCs are arranged to store sufficient amount of relief products.
- Relief products are transported to the LRCs according to the intensity of requirements for the AAs.
- Intensity factor is measured by the authority by the information derived from FB.

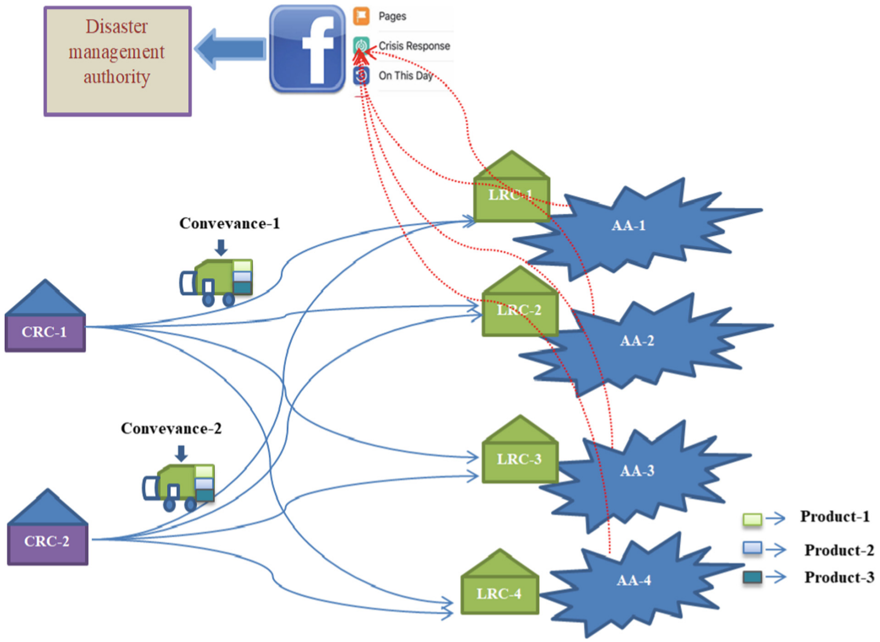

Fig. 1. Distribution process of the relief products

- Storage cost is specified for particular products.
- Conveyance are commercial so it has limited working time.

Indices

- I: Set of CRCs from where humanitarian goods are supplied, indexed by i.
- J: Set of LRCs in the AAs, indexed by j.
- J$'$: Set of AAs indexed j'.
- K: Set of conveyance, indexed by k.
- P: Set of type of relief products, indexed by p.

Parameter

c^p_{ijk}: Per unit transportation cost of p th relief products from i th CRC to j th LRCs to through k th conveyance.
s^p: Storage cost for the p th relief products.
t^p_{ijk}: Transportation time for supplying p-th relief product from i-th CRC to j-th LRCs through k-th conveyance.
a^p_i: Capacity of i-th CRC to store p-th relief product.
b^p_j: Storage capacity of p-th relief product at j-th LRCs.
v^p: Unit volume of the p-th relief product.
V_k: Maximum volume loading capacity for k-th conveyance.
L_k: Limited working time in a day for k-th conveyance.

B: Limited fund for the emergency management operation.
$\alpha^p_{j'}$: Intensity factor to measure the requirement of p-th relief material in a j'-th particular AAs.

Decision Variables

- x^p_{ijk} Unit amount of p-th relief product supplied to j-th LRCs from i-th CRCs through k-th type of conveyance.
- y^p_{ijk} is binary variable defined as follows:
$y^p_{ijk} = \begin{cases} 1 \text{ if } x^p_{ijk} > 0 \\ 0 \text{ otherwise} \end{cases}$

Mathematical Model. The mathematical model developed for a MOSTP in disaster aims to minimize the total cost and time of the disaster management operation. The mathematical formulation given as follows:

$$Min\ Z_1 = \sum_{p=1}^{P} \Big(\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} c^p_{ijk} x^p_{ijk} + s^p \Big) \tag{1}$$

$$Min\ Z_2 = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{p=1}^{P} t^p_{ijk} y^p_{ijk} \tag{2}$$

subject to

$$\sum_{j=1}^{J} \sum_{k=1}^{K} x^p_{ijk} \leq a^p_i \quad \forall i, p \tag{3}$$

$$\sum_{i=1}^{I} \sum_{k=1}^{K} x^p_{ijk} = \alpha^p_{j'} b^p_j \quad \forall j, j', p \ and\ j' = j \tag{4}$$

$$\sum_{i=1}^{I} \sum_{j=1}^{J} v^p x^p_{ijk} \leq V^p_k \quad \forall k, p \tag{5}$$

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{p=1}^{P} t^p_{ijk} y^p_{ijk} \leq L_k \tag{6}$$

$$\sum_{p=1}^{P} \Big(\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} c^p_{ijk} x^p_{ijk} + s^p \Big) \leq B \tag{7}$$

$$x^p_{ijk}, y^p_{ijk} \in R^+, \tag{8}$$

The objective function (1) minimizes the total emergency relief cost of the disaster management operation. The total cost includes transportation cost of the relief products with storage cost for the products. The objective function (2) minimizes the total transportation time to meet the demand of the affected people in the devastation scenario quickly. The constraint (3) guarantees that the total shipment of the relief items from CRCs to LRCs does not exceed the total capacity of product in of the CRCs. The constraint (4) attempts to fulfill the requirements of AAs according to the intensity of requirement of demand of relief products by delivering the product to LRCs. The constraint (5) is the volume loading capacity of the conveyance. The constraint (6) is the working time of conveyance attempt to complete their delivery of relief product as per day schedule. The constraint (7) is total utilize cost cannot cross the limited relief fund granted for the disaster management operation. The constraint (8) is non-negativity condition.

3 Solution Methodology

The mathematical model is a MOSTP define in the Sect. 2. To solve the model we have used fuzzy goal programming. Fuzzy goal programming is used in multi-objective transportation problem in [24]. According to this method, we have to solve each objective function (Z_r) separately and ignoring the others. Then we have found (L_r) the aspired level of achievement and (U_r) the highest acceptable level of achievement for the r-th objective function respectively. To solve the model with this method we have constructed the model as follows:

$$Min\ \phi \tag{9}$$

subject to

$$\left(\frac{U_r - Z_r}{U_r - L_r}\right) + d_r^+ - d_r^- = 1 \tag{10}$$

$$\phi \geq d_r^+, \tag{11}$$

$$r = 1, 2$$

$$d_r^+ d_r^- = 0, \tag{12}$$

$$\sum_{j=1}^{J}\sum_{k=1}^{K} x_{ijk}^p \leq a_i^p \quad \forall i, p \tag{13}$$

$$\sum_{i=1}^{I}\sum_{k=1}^{K} x_{ijk}^p = \alpha_{j'}^p b_j^p \quad \forall j, j', p\ and\ j' = j \tag{14}$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J} v^p x_{ijk}^p \leq V_k \quad \forall k, p \tag{15}$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{p=1}^{P} t_{ijk}^p y_{ijk}^p \leq L_k \quad \forall k \tag{16}$$

$$\sum_{p=1}^{P}\Big(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} c_{ijk}^p x_{ijk}^p + s^p\Big) \leq B \tag{17}$$

$$d_r^+ d_r^- \geq 0, \tag{18}$$

$$\phi \leq 1, \phi \geq 0, \tag{19}$$

$$x_{ijk}^p, y_{ijk}^p \in R^+, \tag{20}$$

The amount of underachieved and overachieved of the goal d_r^- and d_r^+ to achieve the objective of the model.

4 Computational Analysis of the Model

The model introduced for relief response authority is explained with a numerical data set. The numerical set up is constructed considering 2 CRCs, 4 LRCs for 4 AAs, 2 conveyance, 3 types of product (e.g. clothes, food, medicine etc.). As CRCs have limited resources and LRCs are arranged to store sufficient resources therefore it is not easy to fulfill the capacity of LRCs. The shipment from CRCs to LRCs takes place on the basis of the intensity factor defined the management authority for the requirement in a particular area defines through a constraint for the model.

4.1 Input for the Mathematical Model

The objective function of the MOSTP are minimization of total cost of the disaster operation and minimization of transportation time for delivering relief products from CRCs to LRCs. The objective function for cost minimization is consists of addition of total transportation cost (c_{ijk}^p) multiplied by the transporting amount of relief products from CRCs to LRCs (x_{ijk}^p) and storage cost of relief products (s^p). The objective function for time minimization is the total time (t_{ijk}^p) multiplied with the binary variable (y_{ijk}^p). The input for the objective functions and the constraints are given in the Table 1.

4.2 Result Analysis

The model is coded with LINGO optimization solver and run through an Intel core i5 processor. After 395 iterations, we get an optimal result of the model. The optimal value for the cost is 780.0039 and time is 26.8 h. The value of $d_1^- = 0.1416217$ and $d_2^- = 0.1717949$. The shipment of relief products from CRC to LRC are shown in the Table 2 and depicted in the Fig. 2.

The Fig. 2 shows that relief materials are fulfilled by the CRCs to LRCs according to the demand of AAs on intensity basis of requirements. The intensity of AA-1 for the product-1, product-2 and product-3 are 0.2, 0.3 and 0.5. From the result, we have seen that the allocation of the product-1, product-2 and product-3 from LRCs are 20%, 30% and 50% of the total storage of the LRCs. Based on the intensity of AA-2, AA-3 and AA-4 for the products the allocation of the product-1, product-2 and product-3 from LRCs are 40%, 20% and 50%; 50%, 70% and 80%; 40%, 50% and 40%; of the total storage of the LRCs. Since CRCs have limited capacity to deliver relief products, so it has transfer only most prior demand to the LRCs, although it has more capacity to store.

Table 1. Input of the objective function and constraint for the model

Transportation cost of relief products from CRCs to LRCs

	$k=1$						$k=2$					
	$i=1$			$i=2$			$i=1$			$i=2$		
j	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$
1	5	6	5.8	5.7	5.6	5.8	5.8	6	6.3	5.3	5	5.8
2	5.7	5.4	4.8	6	6.5	7	5.2	5.6	4.5	5.2	5	6
3	5	6	5	5.2	5.7	5.8	7.5	5.9	6	6.5	6.4	5.8
4	5.4	6.2	6.4	6.3	7	5.5	5.6	5.2	6.5	6	6.3	6.5

Transportation time for delivering relief products from CRCs to LRCs

	$k=1$						$k=2$					
	$i=1$			$i=2$			$i=1$			$i=2$		
j	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$
1	2	2.1	2	2	1.8	1.5	1.8	2.1	1.5	1.6	1.7	2.1
2	1.2	1.5	2.1	2.2	2.4	1.4	2.1	2.6	1.3	2.8	2.5	1.7
3	1.5	1.8	1.4	1.8	1.6	2.1	1.3	1.2	1.2	2.2	2.5	2.6
4	1.5	1.7	1.8	2.4	2.3	2.4	2	2.3	2.4	2.1	2.4	2.5

Capacity of CRCs

i	p = 1	p = 2	p = 3	i	p = 1	p = 2	p = 3
1	30	40	55	2	35	45	40

Capacity to store in LRCs

j	p = 1	p = 2	p = 3	j	p = 1	p = 2	p = 3
1	28	16	20	2	15	25	28
3	26	30	28	4	25	27	26

Intensity factor for demand

j	p = 1	p = 2	p = 3	j	p = 1	p = 2	p = 3
1	0.2	0.3	0.5	2	0.4	0.2	0.6
3	0.5	0.7	0.8	4	0.4	0.5	0.4

Storage cost of relief product			Unit volume of relief product		
p = 1	p = 2	p = 3	p = 1	p = 2	p = 3
10	12	15	0.5	0.3	0.4

Maximum volume loading capacity of conveyance		Limited working time of the conveyance (hours/day)	
k = 1	k = 2	k = 1	k = 2
100	150	12	18

Total budget: 10000

In table, index j indicates the number of CRCs, i for LRCs, k for conveyances used, p for relief products.

Table 2. Allocation of relief products from CRCs to LRCs based on intensity measure of demand

Shipment of relief products from CRCs to LRCs

	$k=1$						$k=2$					
	$i=1$			$i=2$			$i=1$			$i=2$		
j	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$
1	5.6	–	–	–	–	–	–	–	–	–	4.8	10.0
2	–	–	16.8	–	–	–	6.0	–	–	–	5.0	–
3	8.4	–	22.4	4.6	21.0	–	–	–	–	–	–	–
4	10.0	–	10.4	–	–	–	–	4.60555	–	–	8.894450	–

In table, index j indicates the number of CRCs, i for LRCs, k for conveyances used, p for relief products.

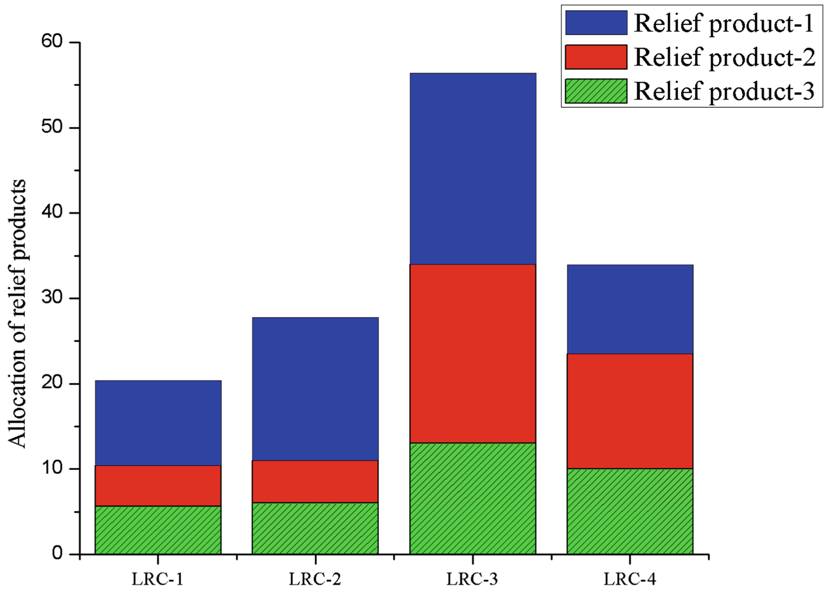

Fig. 2. Allocation of relief products from CRC to LRC

5 Conclusion and Future Scope

A mathematical model for STP with two objective functions minimizing the total cost and total transportation time for disaster management is presented in this paper. Through this model demand allocation is properly completed with intensity of requirement of relief products in the affected areas, as the limited resources in the source point from where it is delivered. The intensity factor is determined from best information acquiring from safety check feature of Facebook launched for response after disaster. The local response center are arranged to store more relief materials for large scale response in disaster operation. A numerical example is presented in this paper which shows the efficiency of smooth functioning of the model. The multi-objective solid transportation is solved fuzzy goal programming technique in LINGO optimization solver. In future, we can extend the mathematical model in uncertain environment using fuzzy logic, stochastic, genetic algorithm etc.

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Solution of a Bi-level Programming Problem with Inexact Parameters

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Abstract. In a bi-level programming, coefficients in the objective functions and the constraints may not to be fixed real numbers due to the presence of uncertainties in the domain of the model. This paper develops a methodology to solve these type bi-level programming problems whose parameters in the upper and lower level objective functions and constraints vary in intervals. A methodology is derived to find a compromising solution. The methodology is illustrated through numerical example.

Keywords: Nonlinear optimization · Uncertain optimization · Interval valued function · Interval inequality · Order relation

1 Introduction

A general bi-level programming problem, denoted by **BLP**, involves two optimization problems, **UP** as upper level problem and **LP** as lower level problem, where the feasible region of the first problem is implicitly determined by second problem.

BLP:

$$\mathbf{UP}: \quad \begin{cases} \min\limits_{y \in \mathbb{R}^{n_1}} & f_1(x, y) \\ \text{subject to} & g_{1i}(x, y) \leq b_{1i}, \; i = 1, 2, ..., m_1, \end{cases}$$

where x solves the lower level problem

$$\mathbf{LP}: \quad \begin{cases} \min\limits_{x \in \mathbb{R}^{n_2}} & f_2(x, y) \\ \text{subject to} & g_{2j}(x, y) \leq b_{2j}, \; j = 1, 2, ..., m_2, \end{cases}$$

$f_1, \; f_2, \; g_{1i}, \; g_{2j} : \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$, $b_{1i} \in \mathbb{R}^{m_1}, \; b_{2j} \in \mathbb{R}^{m_2}$.

For $y \in \mathbb{R}^{n_1}$, denote

$$\Psi(y) = \arg \min_{x \in \mathbb{R}^{n_2}} \{f_2(x, y) : g_{2j}(x, y) \leq b_{2j}, \; j = 1, 2, ..., m_2\}.$$

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$(x, y) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1}$ is said to be a feasible solution of **BLP** if $x \in \Psi(y)$ and $g_{1i}(x, y) \leq b_{1i},\ i = 1, 2, ..., m_1$. The lower level problem may or may not have unique solution. If the lower level optimal solution is $x(y)$ and uniquely determined then $\Psi(y) = x(y)$, corresponding to which, **BLP** becomes

$$\min_{y \in \mathbb{R}^{n_1}} f_1(x(y), y) \quad \text{subject to } g_{1i}(x(y), y) \leq b_{1i},\ i = 1, 2, ..., m_1.$$

In case the solution of **LP** is not uniquely determined, then the value of $f_1(x, y)$ cannot be predicted directly in general. There are two main approaches to overcome this situation, which consider the optimistic and the pessimistic reformulations of **BLP**.

There are many real life optimization problems related to Stackelberg games, environmental economics, biofuel production problems, supply chain management problem etc, which can be expressed as bi-level programming models. In general, the coefficients in the objective functions and the constraints of **BLP** are assumed to be real numbers. But in real life there are many situations where this assumption is not always true due to the presence of various types of uncertainties in the domain of **BLP** model. These uncertain parameters can be handled mathematically by estimating their lower and upper bounds from historical data. In other words these parameters may be considered as closed intervals. Consequently the real valued functions $f_1(x, y),\ f_2(x, y),\ g_{1i}(x, y)$ and $g_{2j}(x, y)$ with interval parameters become interval valued functions. In these circumstances we call **BLP** as interval bi-level programming problem and denote it by **IBP**. Here we provide an example of such a real life model in supply chain.

Example 1. **Production-distribution planning in a supply chain as an interval bi-level problem**

Consider a supply chain with two levels: a production company with production plants $(P_i,\ i = 1, 2, ..., p)$ at lower level, and a distribution company with distributers $(D_j,\ j = 1, 2, ..., d)$ at upper level. Production company has hegemonic power in the chain and distribution company as follower, should adopt the best decision with the regard of production company. The production company produces products $(PD_k,\ k = 1, 2, ..., m)$ and distributes them to the distributers who sale products to the retailers $(R_l,\ l = 1, 2, ..., r)$ in the market. Due to the presence of several inexact information in the system, which arise due to change in market price, market demand, quality of ingredients and instruments with respect to time, the parameters are not fixed. Lower and upper bound of these parameters can be estimated from the historical data. As a result of which, the parameters of the total system become intervals and hence the model can be converted to an interval bi level optimization model.

The distribution company which owns the distributers $(D_j,\ j = 1, 2, ..., d)$ aims to minimize the distribution cost. This consists of the transportation cost of shipping the products from each plant to each distributer, together with the transportation cost from each distributer to the customer. At the same time, the production company that controls the production plants wants to minimize the production costs at lower level. The following assumptions and notations are used to describe the model:

$p_{ik}^L(p_{ik}^R)$	lower (upper) bound of production cost of manufacturing one unit of product PD_k at plant P_i
$\alpha_{ijk}^L(\alpha_{ijk}^R)$	lower (upper) bound of transportation cost of one unit of product PD_k from plant P_i to distributer D_j
$\beta_{jkl}^L(\alpha_{jkl}^R)$	lower (upper) bound of transportation cost of one unit of product PD_k from distributer D_j to retailer R_l
w_{kl}	demand of product PD_k at retailer R_l
h_i	available production capacity at production plant P_i
v_{ik}	amount of production capacity of one unit of product PD_k at production plant P_i
c_{ij}	amount of capacity used by one unit of product P_i at distributer D_j
q_{ij}	available capacity to store product P_i at distributer D_j
x_{ijk}	amount of product PD_k sent from plant P_i to distributer D_j
y_{jkl}	amount of product PD_k sent from distributer D_j to retailer R_l

Objective Functions and Constraints

- Aim of the distribution company is to minimize the total transportation cost of the distributers. This consists of the transportation cost of shipping products from each production plant to each distributer, together with the transportation cost from each distributer to each retailer. Hence the total transportation cost is

$$\sum_{j=1}^{d}\left(\sum_{i=1}^{p}\sum_{k=1}^{m}\left[\alpha_{ijk}^L,\alpha_{ijk}^R\right]x_{ijk}\oplus\sum_{i=1}^{p}\sum_{l=1}^{r}\left[\beta_{jkl}^L,\alpha_{jkl}^R\right]y_{jkl}\right).$$

- The manufacturing company, which owns the production plant, aims to minimize the total production cost, which consists of the sum of the production cost at each plant. The production cost is

$$\sum_{i=1}^{p}\sum_{k=1}^{m}\sum_{j=1}^{d}\left[p_{ik}^L,p_{ik}^R\right]x_{ijk}.$$

- For each retailer, demand of each type of product is satisfied. Hence $\sum\limits_{j=1}^{d} y_{jkl}\geq w_{kl},\ \forall k\ l.$

- Available capacity to store each product at each distribution center satisfies $\sum\limits_{l=1}^{r} c_{ij}y_{jkl}\leq q_{ij},\forall i\ j.$

- At each plant, available capacity should not be exceeded. Hence $\sum\limits_{k=1}^{m}\sum\limits_{j=1}^{d} v_{ik}x_{ijk}\leq h_i,\ \forall i.$

- Each plant sends to each distributer at least what is to be delivered to retailers. Hence $\sum_{i=1}^{s} x_{ijk} \geq \sum_{l=1}^{r} y_{jkl},\ \forall j,\ k$.
- Non-negativity restrictions are $x_{ijk} \geq 0,\quad y_{jkl} \geq 0,\ \forall i, j, k, l$.

Model Formulation

At lower level, the manufacturing company that controls the production plants, wants to minimize the production cost. At upper level, the distribution company which owns distributers aims to minimize the distribution cost. Hence the model is:

$$
\begin{aligned}
\mathbf{UP}: \quad & \min_{y_{jkl}} \sum_{j=1}^{d} \left(\sum_{i=1}^{p} \sum_{k=1}^{m} \left[\alpha_{ijk}^{L}, \alpha_{ijk}^{R}\right] x_{ijk} \oplus \sum_{i=1}^{p} \sum_{l=1}^{r} \left[\beta_{jkl}^{L}, \alpha_{jkl}^{R}\right] y_{jkl} \right) \\
\text{subject to} \quad & \sum_{j=1}^{d} y_{jkl} \geq w_{kl},\ k = 1, 2, ..., m;\ l = 1, 2, ..., r \\
& \sum_{l=1}^{r} c_{ij} y_{jkl} \leq q_{ij},\ i = 1, 2, ..., s;\ j = 1, 2, ..., d \\
& y_{jkl} \geq 0,\ \forall j, k, l,
\end{aligned}
$$

where x_{ijk} solves:

$$
\begin{aligned}
\mathbf{LP}: \quad & \min_{x_{ijk}} \sum_{i=1}^{p} \sum_{k=1}^{m} \sum_{j=1}^{d} \left[p_{ik}^{L}, p_{ik}^{R}\right] x_{ijk} \\
\text{subject to} \quad & \sum_{k=1}^{m} \sum_{j=1}^{d} v_{ik} x_{ijk} \leq h_i,\ i = 1, 2, ..., s \\
& \sum_{i=1}^{s} x_{ijk} \geq \sum_{l=1}^{r} y_{jkl},\ j = 1, 2, ..., d,\ k = 1, 2, ..., m \\
& x_{ijk} \geq 0,\ \forall i, j, k,
\end{aligned}
$$

which is an interval bi-level programming problem. A general interval bi-level problem is properly proposed in Sect. 3.

Theory of bi-level programming with parameters as real numbers is studied by many researchers (see [1,4,5,11,16] etc.). Some researchers have discussed supply chain inventory model with fuzzy parameters, (see [3,6–8] etc.). [2,14,15] have discussed bi-level programming in which, only the coefficients of objective functions vary in closed intervals. They have focused on the determination of the optimal bounds of the objective function of a linear bi-level programming model with interval parameters. These papers do not address the theoretical justification of the existence of the solution of the model. More over, study on nonlinear bi level programming problem with interval parameters in the objective function as well as constraints remains an open area for research. Due to the presence of interval uncertainty, general interval nonlinear bi level models are more complex to handle. In the present work, we consider a particular bi-level nonlinear programming problem, where the parameters in the objective function as well as in the constraints are intervals. So we do not discuss the optimistic and pessimistic case of the solution of **IBP**. Both the lower and upper level problems are indirectly solved through a deterministic equivalent of **IBP**. The existence of an efficient solution of this problem is studied with respect to a partial order relation which is based on decision maker's satisfaction level.

Section 2 describes some prerequisites on interval analysis which are used in subsequent sections. The existence of the solution of a general interval bi-level programming problem, which addresses intervals in the objective function as well as in the constraints, is discussed in Sect. 3, and a numerical example is provided in Sect. 4 to support the theoretical developments of the paper. Section 5 includes some concluding remarks.

2 Prerequisites

Following notations are required to explain the methodology.

- $I(\mathbb{R})$: The set of closed intervals on $\mathbb{R}$. For $\hat{a} \in I(\mathbb{R})$, $\hat{a} = [a^L, a^R]$, $a^L,\ a^R \in \mathbb{R}$. $\hat{a}$ is said to be a degenerate interval if $a^L = a^R$ and is denoted by $\widehat{a}$.
- $\Lambda_k :\ \{1, 2, ..., k\}$.
- An algebraic operation $\circledast$ $(* \in \{+, -, \cdot, /\})$ in $I(\mathbb{R})$ is defined as follows. For $\hat{a} = [a^L, a^R]$ and $\hat{b} = [b^L, b^R]$ in $I(\mathbb{R})$, $\hat{a} \circledast \hat{b} = \{a * b : a \in \hat{a},\ b \in \hat{b}\}$. For $\hat{a} \oslash \hat{b},\ 0 \notin \hat{b}$.
- The spread of the interval $\hat{a}$, denoted by $\mu(\hat{a})$, is $\mu(\hat{a}) = a^R - a^L$.
- $I(\mathbb{R})^n :\ \{\hat{a}_v : \hat{a}_v = (\hat{a}_1\ \hat{a}_2\ ...\ \hat{a}_n)^T,\ \hat{a}_j \in I(\mathbb{R}),\ j \in \Lambda_n\}$.

2.1 χ Partial Ordering in $I(\mathbb{R})$ and $I(\mathbb{R})^n$ ([12])

$I(\mathbb{R})$ is not a totally ordered set. Two intervals may overlap, one interval may lie behind another interval or one interval may include another interval. Several partial orderings in $I(\mathbb{R})$ exist in literature (see [9,10,13]). Order relations between two intervals $\hat{a}$ and $\hat{b}$ can be explained in two ways; first one is an extension of $<$ on real line, that is, $\hat{a} < \hat{b}$ iff $a^R < b^L$, and the other is an extension of the concept of set inclusion, that is, $\hat{a} \subseteq \hat{b}$ iff $a^L \geq b^L$ and $a^R \leq b^R$. These order relations cannot explain ranking between two overlapping intervals. To meet this gap, [12] introduced a partial order relation (χ-partial order relation), which is based on the decision maker's satisfaction level for the comparison of the intervals. Mathematically, the decision maker's satisfaction associated by a function $\chi : I(\mathbb{R}) \times I(\mathbb{R}) \to [0, 1]$ as follows. For $\hat{a}$ and $\hat{b}$ in $I(\mathbb{R})$,

$$\chi(\hat{a}, \hat{b}) = \begin{cases} 1, & a^R \leq b^L \\ 0, & a^L \geq b^R \\ \frac{b^R - a^L}{(b^R - b^L) + (a^R - a^L)} \in (0, 1), & a^L < b^R \text{ and } a^R > b^L. \end{cases} \tag{1}$$

Definition 1. *For two intervals* $\hat{a},\ \hat{b} \in I(\mathbb{R})$,

$$\begin{aligned}
&\hat{a} \preceq_{\chi} \hat{b} \ \ \textit{iff}\ \mu(\hat{a}) \le \mu(\hat{b})\ \textit{and}\ \chi(\hat{a},\hat{b}) \in [1/2,1],\\
&\hat{a} \prec_{\chi} \hat{b} \ \ \textit{iff}\ \mu(\hat{a}) \le \mu(\hat{b})\ \textit{and}\ \ \chi(\hat{a},\hat{b}) = 1,\\
&\hat{a} = \hat{b} \ \ \textit{iff}\ \mu(\hat{a}) = \mu(\hat{b})\ \textit{and}\ \chi(\hat{a},\hat{b}) = 1/2.
\end{aligned}$$

Definition 2. *For* $\hat{a}_v = (\hat{a}_1\ \hat{a}_2\ ...\ \hat{a}_n)^T$ *and* $\hat{b}_v = (\hat{b}_1\ \hat{b}_2\ ...\ \hat{b}_n)^T$ *in* $I(\mathbb{R})^n$,

$$\hat{a}_v \preceq_{\chi}^{n} \hat{b}_v \ \ \textit{iff}\ \ \hat{a}_i \preceq_{\chi} \hat{b}_i,\ \forall i \in \Lambda_n.$$

Using the concept of degree of inferiority between two intervals, degree of inferiority between two interval vectors $\hat{a}_v$ and $\hat{b}_v$ of dimension n can be defined as

$$\chi(\hat{a}_v,\hat{b}_v) = \min_{i\in\Lambda_n} \{\chi(\hat{a}_i,\hat{b}_i)\}. \tag{2}$$

Throughout this paper we consider the partial ordered sets $(I(\mathbb{R}),\preceq_{\chi})$ and $(I(\mathbb{R})^n,\preceq_{\chi}^{n})$.

3 General Interval Bi-level Programming Problem

Interval valued function is defined by many authors in several ways (See [9,13] etc). In general, interval valued function is a mapping from one or more interval arguments onto an interval number. In this paper we consider an interval valued function $\hat{f} : D \subseteq \mathbb{R}^n \to I(\mathbb{R})$, as $\hat{f}(x) = [f^L(x), f^R(x)]$ such that $f^L(x) \le f^R(x)\ \forall x \in D$. In the proposed bi-level programming problem with bounded parameters, in short **IBP**, the constraints are interval inequalities and objective functions are interval valued functions.

Consider a general bi-level programming problem with bounded parameters as

$$\textbf{IBP}:\ \min_{y}\ \ \hat{f}_1(x,y)\ \ \text{subject to}\ \hat{g}_{1i}(x,y) \preceq_{\chi} \hat{b}_{1i},\ i \in \Lambda_{m_1},\ \ y \in \mathcal{D}_2 \subseteq \mathbb{R}^{n_1},$$

where x solves:

$$\min_{x}\ \ \hat{f}_2(x,y)\ \ \text{subject to}\ \hat{g}_{2j}(x,y) \preceq_{\chi} \hat{b}_{2j},\ j \in \Lambda_{m_2},\ \ x \in \mathcal{D}_1 \subseteq \mathbb{R}^{n_2}.$$

$\hat{f}_1, \hat{f}_2, \hat{g}_{1i}, \hat{g}_{2j} : \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \to I(\mathbb{R}),\ i \in \Lambda_{m_1},\ j \in \Lambda_{m_2}$ are interval valued functions. $\hat{f}_1(x,y) = [f_1^L(x,y), f_1^R(x,y)]$, $\hat{f}_2(x,y) = [f_2^L(x,y), f_2^R(x,y)]$, $\hat{g}_{1i}(x,y) = [g_{1i}^L(x,y), g_{1i}^R(x,y)]$, $\hat{g}_{2j}(x,y) = [g_{2j}^L(x,y), g_{2j}^R(x,y)]$, $\forall (x,y) \in \mathcal{D}_1 \times \mathcal{D}_2$, and $\hat{b}_{1i} = [b_{1i}^L, b_{1i}^R]$, $\hat{b}_{2j} = [b_{2j}^L, b_{2j}^R]$. $\hat{f}_1,\ \hat{f}_2,\ \hat{g}_{1i},\ \hat{g}_{2j}$ may be linear or nonlinear interval valued functions. One may observe that two types of uncertain factors are associated with **IBP**.

(i) Uncertainty in feasible region (uncertainty in feasible region is measured by the acceptable degree of feasibility of the decision).
(ii) Uncertainty in objective function (uncertainty in objective function is measured by the acceptable degree of flexibility of the objective functions towards their goals).

Example 2. An example of **IBP** is

$$\mathbf{IBP}: \quad \min_{0<y\leq 1} \; [1,2]x_1y^3 \oplus [\frac{1}{2},2]x_2$$

where x_1, x_2 solve the problem

$$\begin{aligned} &\min_{x_1,x_2} && [-2,-1]x_2 \\ &\text{subject to} && [1,2]x_1y \preceq_\chi [9,10],\;\; [1,1]x_1^2 \oplus [1,2]x_2y \preceq_\chi [1,2] \\ & && x_1, x_2 \geq 0. \end{aligned} \tag{3}$$

Here

$$\begin{aligned} &\hat{f}_1(x,y) = [1,2]x_1y^3 \oplus \left[\frac{1}{2},2\right]x_2 = [f_1^L(x,y), f_1^R(x,y)] = \left[x_1y^3 + \frac{1}{2}x_2, 2x_1y^3 + 2x_2\right] \\ &\hat{f}_2(x,y) = [f_2^L(x,y), f_2^R(x,y)] = [-2,-1]x_2 = [-2x_2, -x_2] \\ &\hat{g}_{11}(x,y) = [g_{11}^L(x,y), g_{11}^R(x,y)] = [y,y] \\ &\hat{g}_{21}(x,y) = [g_{21}^L(x,y), g_{21}^R(x,y)] = [1,2]x_1y = [x_1y, 2x_1y] \\ &\hat{g}_{22}(x,y) = [g_{22}^L(x,y), g_{22}^R(x,y)] = [1,1]x_1^2 \oplus [1,2]x_2y = [x_1^2 + x_2y, x_1^2 + 2x_2y] \\ &\hat{b}_{11} = [1,1],\;\; \hat{b}_{21} = [1,2]. \end{aligned}$$

3.1 Solution Methodology

To study the existence of the solution of **IBP**, we first consider two subproblems $\mathbf{P}_2(y)$ and $\mathbf{P}_1$ as follows, and derive the goals of the objective functions of these problems. These goals are used to derive the solution of **IBP**.

$$\begin{aligned} &\mathbf{P}_1: && \min_{(x,y)\in\mathcal{D}_1\times\mathcal{D}_2} \hat{f}_1(x,y), \quad \hat{g}_{1i}(x,y) \preceq_\chi \hat{b}_{1i},\; i \in \Lambda_{m_1} \\ &\mathbf{P}_2(y): && \min_{x\in\mathcal{D}_1} \hat{f}_2(x,y), \hat{g}_{2j}(x,y) \preceq_\chi \hat{b}_{2j},\; j \in \Lambda_{m_2}. \end{aligned}$$

Feasible regions of $\mathbf{P}_2(y)$ and $\mathbf{P}_1$ are described by interval inequalities. For fixed $y \in D_2$, the feasible region of $\mathbf{P}_2(y)$ is governed by m_2 number of interval inequalities $\hat{g}_{2j}(x,y) \preceq_\chi \hat{b}_{2j}, j \in \Lambda_{m_2}$, which can be described by the set

$$S^l(y) = \bigcap_{j\in\Lambda_{m_2}} \{x \in \mathcal{D}_1 : \hat{g}_{2j}(x,y) \preceq_\chi \hat{b}_{2j}\}.$$

The feasible region for $\mathbf{P}_1$ is

$$S^u = \bigcap_{i \in \Lambda_{m_1}} \{(x, y) \in \mathcal{D}_1 \times \mathcal{D}_2 : \hat{g}_{1i}(x, y) \preceq_\chi \hat{b}_{1i}\},$$

which is also governed by m_1 number of interval inequalities $\hat{g}_{1i}(x, y) \preceq_\chi \hat{b}_{1i},\ i \in \Lambda_{m_1}$.

Uncertainty in Feasible Region. For any x in $S^l(y)$, the interval $[g^L_{2j}(x, y), g^R_{2j}(x, y)]$ may lie before or overlap or exceed $[b^L_{2j}, b^R_{2j}]$ for every j. Accordingly, the feasibility of x satisfying interval inequality $[g^L_{2j}(x, y), g^R_{2j}(x, y)] \preceq_\chi [b^L_{2j}, b^R_{2j}]$ for $\mathbf{P}_2(y)$ is completely acceptable, or partially acceptable, or not acceptable. One may observe that

(i) $x \in S^l(y)$ is a fully acceptable feasible solution if

$$x \in \{x \in D_1 : g^R_{2j}(x, y) \leq b^L_{2j} \forall j\};$$

(ii) x is not at all an acceptable feasible point if x goes beyond the region $\{x \in D_1 : g^L_{2j}(x, y) \leq b^R_{2j} \forall j\}$, i.e.

$$x \in \{x \in D_1 : g^L_{2j}(x, y) > b^R_{2j} \forall j\};$$

(iii) x is a partially acceptable feasible solution if it lies in

$$\{x \in D_1 : g^L_{2j}(x, y) \leq b^R_{2j} \text{ and } g^R_{2j}(x, y) > b^L_{2j} \forall j\}.$$

Hence every point x in $S^l(y)$ is associated with certain degree of feasibility between the interval vectors

$$\begin{pmatrix} [g^L_{21}(x, y), g^R_{21}(x, y)] \\ [g^L_{22}(x, y), g^R_{22}(x, y)] \\ \vdots \\ [g^L_{2m_2}(x, y), g^R_{2m_2}(x, y)] \end{pmatrix} \quad and \quad \begin{pmatrix} [b^L_{21}, b^R_{21}] \\ [b^L_{22}, b^R_{22}] \\ \vdots \\ [b^L_{2m_2}, b^R_{2m_2}] \end{pmatrix}.$$

Using the concept of comparison between two interval vectors discussed in Subsect. 2.1, if $\chi^F_{2j} : I(\mathbb{R}) \times I(\mathbb{R}) \to [0, 1]$ given by,

$$\chi^F_{2j}(\hat{g}_{2j}(x, y), \hat{b}_{2j}) = \begin{cases} 1, & g^R_{2j}(x, y) \leq b^L_{2j} \\ 0, & g^L_{2j}(x, y) > b^R_{2j} \\ \frac{b^R_{2j} - g^L_{2j}(x,y)}{(b^R_{2j} - b^L_{2j}) + (g^R_{2j}(x,y) - g^L_{2j}(x,y))}, & \text{elsewhere,} \end{cases} \tag{4}$$

describes the degree of inferiority of $[g^L_{2j}(x, y), g^R_{2j}(x, y)]$ with $[b^L_{2j}, b^R_{2j}]$ for every j, then the acceptable degree of feasibility of x is maximum possible value of α satisfying $\alpha \leq \chi^F_{2j}(\hat{g}_{2j}(x, y), \hat{b}_{2j}),\ \forall j$. Define a set

$$S^l_f(y) = \left\{(x, \tau_2) : x \in S^l(y),\ \mu(\hat{g}_{2j}(x, y)) \leq \mu(\hat{b}_{2j}),\ \tau_2 = \min_{j \in \Lambda_{m_2}} \left\{\chi^F_{2j}(\hat{g}_{2j}(x, y), \hat{b}_{2j})\right\}\right\}.$$

This implies

$$\tau_2 \le \chi^F_{2j}(\hat{g}_{2j}(x,y), \hat{b}_{2j})$$
$$\Rightarrow \tau_2 \le \frac{b^R_{2j} - g^L_{2j}(x,y)}{(b^R_{2j} - b^L_{2j}) + (g^R_{2j}(x,y) - g^L_{2j}(x,y))}. \tag{5}$$

Definition 3. *$x \in \mathcal{D}_1$ is said to be an acceptable feasible point of $\mathbf{P}_2(y)$ with degree of feasibility τ_2 if for fixed y, $(x, \tau_2) \in S^l_f(y)$ with degree of feasibility $\tau_2 \in [\frac{1}{2}, 1]$.*

In similar way, feasibility of $\mathbf{P}_1$ can be described. Acceptable feasible region of $\mathbf{P}_1$ is given by

$$S^u_f = \left\{((x,y),\tau_1) : (x,y) \in S^u, \mu(\hat{g}_{1i}(x,y)) \le \mu(\hat{b}_{1i}),\ \tau_1 = \min_{i \in \Lambda_{m_1}} \left\{\chi^F_{1i}(\hat{g}_{1i}(x,y), \hat{b}_{1i})\right\}\right\},$$

where for each i, $\chi^F_{1i} : I(\mathbb{R}) \times I(\mathbb{R}) \to [0,1]$ is defined by

$$\chi^F_{1i}(\hat{g}_{1i}(x,y), \hat{b}_{1i}) = \begin{cases} 1, & g^R_{1i}(x,y) \le b^L_{1i} \\ 0, & g^L_{1i}(x,y) > b^R_{1i} \\ \frac{b^R_{1i} - g^L_{1i}(x,y)}{(b^R_{1i} - b^L_{1i}) + (g^R_{1i}(x,y) - g^L_{1i}(x,y))} & \text{elsewhere.} \end{cases}$$

Here $\tau_1 = \min\limits_{1 \le i \le m} \left\{\chi^F_{1i}(\hat{g}_{1i}(x,y), \hat{b}_{1i})\right\}$ implies

$$\tau_1 \le \frac{b^R_{1i} - g^L_{1i}(x,y)}{(b^R_{1i} - b^L_{1i}) + (g^R_{1i}(x,y) - g^L_{1i}(x,y))}. \tag{6}$$

Feasible region of **IBP** is defined in the following subsection.

Uncertainty in Objective Function. One may observe that, uncertainties present in objective functions are in the form of intervals. To find the χ-optimal solution of **IBP**, we use a variant of goal programming technique. Goals for the objective functions of $\mathbf{P}_2(y)$ and $\mathbf{P}_1$ can be found in several ways by the decision maker. In this section, we provide a procedure to determine the goals of the objective functions.

Methodology to Find the Goals of the Objective Functions of $\mathbf{P}_{2(y)}$ and $\mathbf{P}_1$. For fixed y, any x satisfying the interval inequalities $\hat{g}_{2j}(x,y) \preceq_\chi \hat{b}_{2j},\ \forall j$, belongs to the feasible set $S^l(y)$. Minimum and maximum feasible region of $\mathbf{P}_2(y)$, which may be denoted by $S^l_{min}(y)$ and $S^l_{max}(y)$ respectively, are

$$S^l_{min}(y) = \{x \in \mathcal{D}_1 : g^R_{2j}(x,y) \le b^L_{2j},\ \forall j \in \Lambda_{m_2}\}$$
$$S^l_{max}(y) = \{x \in \mathcal{D}_1 : g^L_{2j}(x,y) \le b^R_{2j},\ \forall j \in \Lambda_{m_2}\}$$
$$S^{l^c}_{min}(y) = \{x \in \mathcal{D}_1 : g^R_{2j}(x,y) \ge b^L_{2j},\ g^L_{2j}(x,y) \le b^R_{2j},\ \forall j \in \Lambda_{m_2}\}. \tag{7}$$

One may observe that $S^l(y) = S^l_{max}(y) \cup S^l_{min}(y) \cup S^{l^c}_{min}(y)$. Hence, $S^l_{min}(y) \subseteq S^l(y)$.

Suppose that

$$\Psi_l^R(y) = \arg \min_{x \in S^l_{min}(y)} f_2^R(x,y),\ \Psi_l^L(y) = \arg \min_{x \in S^l_{max}(y)} f_2^L(x,y).$$

We denote

$$\Phi_l^L(y) = f_2^L(\Psi_l^L(y), y);\ \Phi_l^R(y) = f_2^R(\Psi_l^R(y), y).$$

Using the similar concept, as discussed above, the minimum and maximum feasible region of the problem $\mathbf{P}_1$ can be described as

$$\begin{aligned} S^u_{min} &= \left\{(x,y) \in \mathcal{D}_1 \times \mathcal{D}_2 : g^R_{1i}(x,y) \le b^L_{1i}\right\} \\ S^u_{max} &= \left\{(x,y) \in \mathcal{D}_1 \times \mathcal{D}_2 : g^L_{1i}(x,y) \le b^R_{1i}\right\}. \end{aligned} \tag{8}$$

Denote

$$\begin{aligned} L^l &= \min\{\min_{(x,y)\in S^u_{max}} \{f_1^L(x,y),\ x \in \Psi_l^L(y)\}\} \\ L^r &= \min\{\min_{(x,y)\in S^u_{max}} \{f_1^L(x,y),\ x \in \Psi_l^R(y)\}\} \\ U^l &= \max\{\min_{(x,y)\in S^u_{min}} \{f_1^R(x,y),\ x \in \Psi_l^L(y)\}\} \\ U^r &= \max\{\min_{(x,y)\in S^u_{min}} \{f_1^R(x,y),\ x \in \Psi_l^R(y)\}\}. \end{aligned}$$

Denote $l = \min\{L^l, L^r\}$ and $u = \max\{U^l, U^r\}$.

Lemma 1. *For any* $(x,y) \in \mathcal{D}_1 \times \mathcal{D}_2$,

(a) $\Phi_l^L(y) \le \min_{x\in S_l(y)} f_2(x,y) \le \Phi_l^R(y)$,

(b) $l \le \min_{x\in \Psi_l^L(y) \bigcup \Psi_l^R(y),\ (x,y)\in S^u} f_1(x,y) \le u$,

where $f_i(x,y) = \lambda_i f_i^L(x,y) + (1-\lambda_i) f_i^R(x,y),\ i = 1,2,\ \lambda_i \in [0,1]$.

Proof. From the definition of minimum and maximum feasible region in Expression (7), it is true that $S^l_{min}(y) \subseteq S^l_{max}(y)$. Also $f_2^L(x,y) \le f_2(x,y) \le f_2^R(x,y)$. So for every fixed y, if $x \in S^l(y)$ then

$$\min_{x\in S^l_{max}(y)} f_2^L(x,y) \le \min_{x\in S_l(y)} f_2(x,y) \le \min_{x\in S^l_{min}(y)} f_2^R(x,y).$$

That is, $\Phi_l^L(y) \le \min_{x\in S_l(y)} f_2(x,y) \le \Phi_l^R(y)$. Hence the proof of (a) follows.

From the definition of S^u_{max} and S^u_{min} in Expression (8), one may observe that $S^u_{min} \subseteq S^u_{max}$. Also $f_1^L(x,y) \le f_1(x,y) \le f_1^R(x,y)$. So

$$x \in \Psi_l^L(y) \Rightarrow \min_{\substack{x\in\Psi_l^L(y)\\(x,y)\in S^u_{max}}} f_1^L(x,y) \le \min_{\substack{x\in\Psi_l^L(y)\\(x,y)\in S^u}} f_1(x,y) \le \min_{\substack{x\in\Psi_l^L(y)\\y\in S^u_{min}}} f_1^R(x,y) \tag{9}$$

$$x \in \Psi_l^R(y) \Rightarrow \min_{\substack{x\in\Psi_l^R(y)\\(x,y)\in S^u_{max}}} f_1^L(x,y) \le \min_{\substack{x\in\Psi_l^R(y)\\(x,y)\in S^u}} f_1(x,y) \le \min_{\substack{x\in\Psi_l^R(y)\\(x,y)\in S^u_{min}}} f_1^R(x,y). \tag{10}$$

The inequalities (9) and (10) are equivalent to

$$L^l \le \min_{\substack{x\in\Psi_l^L(y)\\(x,y)\in S^u}} f_1(x,y) \le U^l \quad\text{and}\quad L^r \le \min_{\substack{x\in\Psi_l^R(y)\\(x,y)\in S^u}} f_1(x,y) \le U^r \tag{11}$$

respectively.

Therefore $l \le \min_{\substack{x\in\Psi_l^L(y)\bigcup\Psi_l^R(y)\\(x,y)\in S^u}} f_1(x,y) \le u$, where $l = \min\{L^l, L^r\}$ and $u = \max\{U^l, U^r\}$. Hence the proof of (b) follows. ■

Note 1. Lemma 1 provides the goals of the objective functions of $\mathbf{P}_1$ and $\mathbf{P}_2(y)$. $\Phi_l^L(y)$ and $\Phi_l^R(y)$ are the lower and upper bound of the goal of $f_2^L(x,y)$ for fixed y respectively, and l and u are the lower and upper bound of the goal of $f_1(x,y)$ respectively.

$[\Phi_l^L(y), \Phi_l^R(y)]$ may be considered as the goal of $\hat{f}_2(x,y)$. Hence the decision maker has to take the decision so that

$$\hat{f}_2(x,y) \preceq_\chi [\Phi_l^L(y), \Phi_l^R(y)] \text{ for fixed } y.$$

These goals can also be provided by the decision maker directly.

To obtain the acceptable degree of flexibility of $\hat{f}_2(x,y)$ towards its goal $[\Phi_l^L(y), \Phi_l^R(y)]$, we associate a function $\chi_2^O\left(\hat{f}_2(x,y), [\Phi_l^L(y), \Phi_l^R(y)]\right)$, which can be defined by

$$\chi_2^O\left(\hat{f}_2(x,y), [\Phi_l^L(y), \Phi_l^R(y)]\right) = \begin{cases} 1, & \text{if } f_2^R(x,y) < \Phi_l^L(y) \\ 0, & \text{if } f_2^L(x,y) > \Phi_l^R(y) \\ \frac{\Phi_l^R(y) - f_2^L(x,y)}{(\Phi_l^R(y)-\Phi_l^L(y)) + \left(f_2^R(x,y) - f_2^L(x,y)\right)}, & \text{elsewhere,} \end{cases} \tag{12}$$

with $\mu(\hat{f}_2(x,y)) \le \mu([\Phi_l^L(y), \Phi_l^R(y)])$.

Since $[l, u]$ is considered as the desired goal of $\hat{f}_1(x,y)$, the decision maker has to take the decision so that

$$\hat{f}_1(x,y) \preceq_\chi [l, u].$$

Using the above concept, the acceptable degree of flexibility of $\hat{f}_1(x,y)$ towards its goal $[l,u]$ can be defined by

$$\chi_1^O\left(\hat{f}_1(x,y), [l,u]\right) = \begin{cases} 1, & \text{if } f_1^R(x,y) < l \\ 0, & \text{if } f_1^L(x,y) > u \\ \frac{u - f_1^L(x,y)}{(u-l) + \left(f_1^R(x,y) - f_1^L(x,y)\right)}, & \text{elsewhere,} \end{cases} \tag{13}$$

with $\mu(\hat{f}_1(x,y)) \le \mu([l,u])$.

3.2 Uncertainty in the Feasible Region and Objective Function Taken Together

In $\mathbf{P}_2(y)$, the objective function is characterized by the acceptable degree of flexibility $\chi_2^O\left(\hat{f}_2(x,y),[\Phi_l^L(y),\Phi_l^R(y)]\right)$ of $\hat{f}_2(x,y)$ towards its goal, and the constraints are characterized by the acceptable degree of feasibility $\chi_{2j}^F(\hat{g}_{2j}(x,y),\hat{b}_{2j})$ of $x\in S^l(y)$. So in this environment, for fixed y, a decision $x\in S^l(y)$ for $\mathbf{P}_2(y)$ is the selection of activities that simultaneously satisfies the objective function, and the constraints. Such a decision $x\in S^l(y)$ can be derived by solving the following optimization problem.

$$\begin{aligned}&\min\left\{\chi_2^O\left(\hat{f}_2(x,y),[\Phi_l^L(y),\Phi_l^R(y)]\right),\ (x,\tau_2)\in S_f^l(y)\right\}\\ &=\min_{(x,\tau_2)\in S_f^l(y)}\frac{\Phi_l^R(y)-f_2^L(x,y)}{(\Phi_l^R(y)-\Phi_l^L(y))+\left(f_2^R(x,y)-f_2^L(x,y)\right)}.\end{aligned}\tag{14}$$

From the definition of $S_f^l(y)$, $\tau_2\le\chi_{2j}^F(\hat{g}_{2j}(x,y),\hat{b}_{2j}),\ \forall j\in\Lambda_{m_2}$.

For fixed y, let $\theta_2=\theta_2(y)$ be given by

$$\theta_2=\min\left\{\chi_2^O\left(\hat{f}_2(x,y),[\Phi_l^L(y),\Phi_l^R(y)]\right),\chi_{2j}^F(\hat{g}_{2j}(x),\hat{b}_{2j}),\forall j\in\Lambda_{m_2}\right\}.$$

This means

$$\begin{aligned}&\theta_2\le\chi_2^O\left(\hat{f}_2(x,y),[\Phi_l^L(y),\Phi_l^R(y)]\right)\\&\theta_2\le\chi_{2j}^F(\hat{g}_{2j}(x),\hat{b}_{2j}),\forall j\in\Lambda_{m_2}.\end{aligned}$$

To get the highest satisfaction of the decision $x\in S^l(y)$, θ_2 has to be considered as the maximum possible value, satisfying these inequalities, which can be achieved by solving the following optimization problem

$$\begin{aligned}\max\ &\theta_2\\ \text{subject to}\ &\theta_2\le\chi_2^O\left(\hat{f}_2(x,y),[\Phi_l^L(y),\Phi_l^R(y)]\right)\\ &\theta_2\le\chi_{2j}^F(\hat{g}_{2j}(x),\hat{b}_{2j})\\ &\mu(\hat{f}_2(x,y))\le\mu([\Phi_l^L(y),\Phi_l^R(y)])\\ &\mu(\hat{g}_{2j}(x))\le\mu(\hat{b}_{2j})\\ &x\in\mathcal{D}_1,\ \frac{1}{2}\le\theta_2\le1,\ j\in\Lambda_{m_2}.\end{aligned}$$

After substituting the values of χ_2^O and χ_{2j}^F, the above problem reduces to

$$
\begin{aligned}
\mathbf{P'}_2(y): \quad & \max \;\; \theta_2 \\
\text{subject to } \; & \theta_2 \leq \frac{\Phi_l^R(y) - f_2^L(x,y)}{(\Phi_l^R(y) - \Phi_l^L(y)) + \left(f_2^R(x,y) - f_2^L(x,y)\right)} \\
& \theta_2 \leq \frac{b_{2j}^R - g_{2j}^L(x,y)}{(b_{2j}^R - b_{2j}^L) + (g_{2j}^R(x,y) - g_{2j}^L(x,y))} \\
& \mu(\hat{f}_2(x,y)) \leq \mu([\Phi_l^L(y), \Phi_l^R(y)]) \\
& \mu(\hat{g}_{2j}(x)) \leq \mu(\hat{b}_{2j}) \\
& x \in \mathcal{D}_1, \; \frac{1}{2} \leq \theta_2 \leq 1, \; j \in \Lambda_{m_2}.
\end{aligned}
$$

Let the solution of $\mathbf{P'}_2(y)$ is $\Psi(y)$ for fixed y.

Next, we define the acceptable feasible solution and the χ-optimal solution of **IBP** as follows.

Definition 4. *$(x,y) \in \mathcal{D}_1 \times \mathcal{D}_2$ is said to be an acceptable feasible solution of* **IBP** *with degree of feasibility τ, where $\tau = \min\{\tau_1, \tau_2\} \in [\frac{1}{2}, 1]$ if $x \in \Psi(y)$, and (x,y) is an acceptable feasible solution of $\mathbf{P}_1$.*

Definition 5. *An acceptable feasible solution (x^*, y^*) with acceptable degree of feasibility τ^*, is said to be a χ-optimal solution of* **IBP** *if there exists no acceptable feasible solution (x,y), with $(\tau > \tau^*)$, such that $\hat{f}_1(x,y) \prec_\chi \hat{f}_1(x^*,y^*)$.*

For $\mathbf{P}_1$, the decision (x,y) is the selection of activities that simultaneously satisfies the objective function and constraints, which is

$$
\min \left\{ \chi_1^O \left(\hat{f}_1(x,y), [l,u] \right), \; x \in \Psi(y), \; y \in S^u \right\}. \tag{15}
$$

Following the steps, discussed to formulate $\mathbf{P'}_2(y)$, we get the deterministic equivalent of (15) as

$$
\begin{aligned}
\mathbf{P}_1^{'}: \quad & \max \;\; \theta_1 \\
\text{subject to } \; & \theta_1 \leq \frac{b_{1i}^R - g_{1i}^L(x,y)}{(b_{1i}^R - b_{1i}^L) + (g_{1i}^R(x,y) - g_{1i}^L(x,y))} \\
& \theta_1 \leq \frac{u - f_1^L(x,y)}{(u-l) + \left(f_1^R(x,y) - f_1^L(x,y)\right)} \\
& \mu(\hat{g}_{1i}(x,y)) \leq \mu(\hat{b}_{1i}) \\
& \mu(\hat{f}_1(x,y)) \leq \mu([l,u]) \\
& x \in \Psi(y), \; y \in \mathcal{D}_2, \; \frac{1}{2} \leq \theta \leq 1, \; i \in \Lambda_{m_1}.
\end{aligned}
$$

Let $\theta = \min\{\theta_1, \theta_2\}$. Combining the $\mathbf{P'}_2(y)$ and $\mathbf{P}'_1$, we construct the following optimization problem

$$\begin{aligned}
\mathbf{IBP}': \qquad & \max_{x,y,\theta} \ \theta \\
\text{subject to} \ & \theta g^R_{1i}(x,y) + (1-\theta) g^L_{1i}(x,y) \leq \theta b^L_{1i} + (1-\theta) b^R_{1i} && (16)\\
& \theta g^R_{2j}(x,y) + (1-\theta) g^L_{2j}(x,y) \leq \theta b^L_{2j} + (1-\theta) b^R_{2j} && (17)\\
& \theta f^R_1(x,y) + (1-\theta) f^L_1(x,y) \leq l\theta + u(1-\theta) && (18)\\
& \theta f^R_2(x,y) + (1-\theta) f^L_2(x,y) \leq \theta \Phi^L_l(y) + (1-\theta)\Phi^R_l(y) && (19)\\
& \mu(\hat{f}_1(x,y)) \leq \mu([l,u]) && (20)\\
& \mu(\hat{g}_{1i}(x,y)) \leq \mu(\hat{b}_{1i}) && (21)\\
& \mu(\hat{f}_2(x,y)) \leq \mu\left([\Phi^L_l(y), \Phi^R_l(y)]\right) && (22)\\
& \mu(\hat{g}_{2j}(x,y)) \leq \mu(\hat{b}_{2j}) \\
& (x,y) \in \mathcal{D}_1 \times \mathcal{D}_2, \ \frac{1}{2} \leq \theta \leq 1, \ i \in \Lambda_{m_1}, \ j \in \Lambda_{m_2}. && (23)
\end{aligned}$$

$\mathbf{IBP}'$ is a classical optimization problem, which can be solved using nonlinear programming techniques. Let (x^*, y^*, θ^*) be a solution of the problem $\mathbf{IBP}'$. Accordingly χ-optimal solution of $\mathbf{IBP}$ can be defined as follows.

The relation between the solution of $\mathbf{IBP}$ and $\mathbf{IBP}'$ is established in the following theorem.

Theorem 1. *If $f^p_t(x,y), g^p_{ti}(x,y)$, and $g^p_{tj}(x,y)$ for all i,j, $t \in \{1,2\}$, $p \in \{L,R\}$ are convex, continuously differentiable functions, and (x^*, y^*, θ^*) is the optimal solution of* $\mathbf{IBP}'$, *then (x^*, y^*) is a χ-optimal solution of* $\mathbf{IBP}$ *with degree of acceptability θ^*.*

Proof. Suppose (x^*, y^*, θ^*) is an optimal solution of $\mathbf{IBP}'$.

- First, we need to prove that (x^*, y^*) is an acceptable feasible solution of $\mathbf{IBP}$.

Since x^* and θ^* satisfy the inequalities (17) and (23), so $(x^*, \tau^*_2) \in S^l_f(y^*)$ for fixed $y = y^*$, with acceptable degree of feasibility $\tau^*_2 = \min\limits_{j \in \Lambda_{m_2}} \chi^F_{2j}(\hat{g}_{2j}(x^*, y^*), \hat{b}_{2j})$. $\theta^* \leq \tau^*_2$

Since (x^*, y^*, θ^*) satisfies the inequalities (19) and (22), so

$$\theta^* \leq \chi^O_2\left(\hat{f}_2(x^*, y^*), [\Phi^L_l(y^*), \Phi^R_l(y^*)]\right).$$

Since $\theta^* \in \left[\frac{1}{2}, 1\right]$, so (x^*, y^*, θ^*) is a solution of $\mathbf{P'}_2(y)$. Hence $x^* \in \Psi(y^*)$.

Again, (x^*, y^*) and θ^* satisfy (16) and (21). So $((x^*, y^*), \tau^*_1) \in S^u_f$, with $\tau^*_1 = \min\limits_{j \in \Lambda_{m_1}} \chi^F_{1i}(\hat{g}_{1i}(x^*, y^*), \hat{b}_{1i})$. $\theta^* \leq \tau^*_1$.

From the above discussions it is clear that (x^*, y^*) is an acceptable feasible point of the problem $\mathbf{P}_1$ and $x^* \in \Psi(y^*)$. Hence following the Definition 4, we can say that (x^*, y^*) is an acceptable feasible solution of $\mathbf{IBP}$ with acceptable degree of feasibility $\tau^* = \{\tau^*_1, \tau^*_2\}$, and $\theta^* \leq \tau^*$.

- Next, we need to show that (x^*, y^*) is a χ-optimal solution of **IBP**.

Suppose that (x^*, y^*) is not a χ-optimal solution of **IBP**. So, following Definition 5, we can say that there exists an acceptable feasible solution $(x', y') \neq (x^*, y^*)$, with $\tau' \geq \tau^*$, such that

$$\hat{f}_1(x', y') \prec_\chi \hat{f}_2(x^*, y^*). \tag{24}$$

This interval inequality implies that

$$\mu(\hat{f}_1(x', y')) \leq \mu(\hat{f}_1(x^*, y^*)) \tag{25}$$
$$f_1^R(x', y') \leq f_1^L(x^*, y^*). \tag{26}$$

Since $[l, u]$ is a non degenerate interval, Inequality (25) leads to

$$\begin{aligned} & f_1^R(x', y') - f_1^R(x', y') \leq f_1^R(x^*, y^*) - f_1^R(x^*, y^*) \\ & \Rightarrow 0 < (u - l) + (f_1^R(x', y') - f_1^L(x', y')) \leq (u - l) + (f_1^R(x^*, y^*) - f_1^L(x^*, y^*)). \end{aligned} \tag{27}$$

From Inequality (26), we get

$$\begin{aligned} & f_1^L(x', y') \leq f_1^R(x', y') \leq f_1^L(x^*, y^*) \leq f_1^R(x^*, y^*) \\ & \Rightarrow u - f_1^L(x', y') \geq u - f_1^L(x^*, y^*). \end{aligned} \tag{28}$$

The inequalities (27) and (28) imply

$$\frac{u - f_1^L(x', y')}{(u - l) + (f_1^R(x', y') - f_1^R(x', y'))} \geq \frac{u - f_1^L(x^*, y^*)}{(u - l) + (f_1^R(x^*, y^*) - f_1^R(x^*, y^*))},$$

and we have $\tau' \geq \tau^*$.
Hence

$$\begin{aligned} & \min\left\{\frac{u - f_1^L(x^*, y^*)}{(u - l) + (f_1^R(x^*, y^*) - f_1^R(x^*, y^*))}, \tau^*\right\} \leq \min\left\{\frac{u - f_1^L(x', y')}{(u - l) + (f_1^R(x', y') - f_1^R(x', y'))}, \tau'\right\} \\ & \Rightarrow \theta_1^* \leq \theta_1'. \end{aligned}$$

So (x^*, y^*) is not an optimal solution of **IBP′**. Accordingly, (x^*, y^*) is not an χ-optimal solution of **IBP**. That is, our assumption "(x^*, y^*) is not a χ-optimal solution of **IBP**" is wrong. Hence the theorem follows. ■

The above theoretical development is numerically illustrated step by step in the following section.

4 Numerical Example

Example 3. Consider the Example 2, proposed in Sect. 3.
Using Lemma 1, we find the upper and lower bound of the goals of the objective functions of **IBP** and using the methodology of Sect. 3.1, we find the χ-optimal solution of **IBP**.

The two subproblems $\mathbf{P}_2(y)$ and $\mathbf{P}_1$ for this example are

$$\begin{aligned}\mathbf{P}_2(y):\quad &\min_{x_1,x_2}\ [-2,-1]x_2\\ \text{subject to}\quad &[1,2]x_1y \preceq_\chi [9,10],\ x_1^2+[1,2]x_2y \preceq_\chi [1,2]\\ &x_1,x_2\geq 0,\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}_1\ :\ &\min_{y}\ [1,2]x_1y^3 \oplus \left[\frac{1}{2},2\right]x_2\\ \text{subject to}\ &0\leq y\leq 1\\ &[1,2]x_1y \preceq_\chi [9,10],\ x_1^2+[1,2]x_2y \preceq_\chi [1,2]\\ &x_1,x_2\geq 0.\end{aligned}$$

The feasible region of the problem $\mathbf{P}_2(y)$ is associated with interval uncertainties. Here

$$S_f^l(y)=\left\{((x_1,x_2),\tau_2): x_1y\leq 1, x_2y\leq 1, \tau_2\leq \frac{10-x_1y}{x_1y+1},\ \tau_2\leq \frac{2-x_1^2-x_2y}{x_2y+1},\ x_1,\ x_2\geq 0\right\}.$$

But, $\mathbf{P}_1$ has deterministic feasible region

$$S^u=\{(x,y):0<y\leq 1\}.$$

The minimum and maximum feasible region for $\mathbf{P}_2(y)$ are

$$\begin{aligned}&S_{min}^l(y)=\{(x_1,x_2):2x_1y\leq 9,\ x_1^2+2x_2y\leq 1,\ x_1,x_2\geq 0\}\\ \text{and}\qquad &S_{max}^l(y)=\{(x_1,x_2):x_1y\leq 10,\ x_1^2+x_2y\leq 2,\ x_1,x_2\geq 0\}\end{aligned}$$

respectively.

To get the lower and upper bound of the lower level objective function, we need to solve the problems

$$\min_{(x_1,x_2)\in S_{max}^l(y)} f_2^L(x,y) \text{ and } \min_{(x_1,x_2)\in S_{min}^l(y)} f_2^R(x,y),$$

which are

$$\min_{(x_1,x_2)\in S_{max}^l(y)} -2x_2 \text{ and } \min_{(x_1,x_2)\in S_{min}^l(y)} -x_2$$

respectively. For fixed y, solutions of these problems are

$$\Psi_l^L(y)=x_l(y)=\begin{pmatrix}0\\ \frac{2}{y}\end{pmatrix},\ y>0,\ \text{and } \Psi_l^R(y)=x_r(y)=\begin{pmatrix}0\\ \frac{1}{2y}\end{pmatrix},\ y>0.$$

For fixed y, $\Psi_l^L(y)$ and $\Psi_l^R(y)$ are unique. $\Phi_l^L(y)=-\frac{4}{y}$ and $\Phi_l^R(y)=-\frac{1}{2y}$. Following the Lemma 1 and Note 1, the goal of $\hat{f}_2(x,y)$ becomes $[-\frac{4}{y},-\frac{1}{2y}]$.

Here

$$L^l = \min_y\{x_1y^3 + \frac{1}{2}x_2,\ x_1 = 0,\ x_2 = \frac{2}{y},\ 0 < y \le 1\} = 1$$

$$L^r = \min_y\{x_1y^3 + \frac{1}{2}x_2,\ x_1 = 0,\ x_2 = \frac{1}{2y},\ 0 < y \le 1\} = \frac{1}{4}.$$

The lower bound of the objective function of $\mathbf{P}_1$ is $\min\{L^l,\ L^r\} = \frac{1}{4}$, corresponding to the solution $(0, \frac{1}{2}, 1)$.

Also

$$U^l = \min_y\{2x_1y^3 + 2x_2,\ x_1 = 0,\ x_2 = \frac{2}{y},\ 0 < y \le 1\} = 4$$

$$U^r = \min_y\{2x_1y^3 + 2x_2,\ x_1 = 0,\ x_2 = \frac{1}{2y},\ 0 < y \le 1\} = 1.$$

The upper bound of the objective function of $\mathbf{P}_1$ is $\max\{U^l,\ U^r\} = 4$, corresponding to the solution $(0, 2, 1)$.

Following the Lemma 1 and Note 1, goal of $\hat{f}_1(x, y)$ becomes $[\frac{1}{4}, 4]$.

Hence

$$\hat{f}_2(x, y) \preceq_\chi \left[-\frac{4}{y}, -\frac{1}{2y}\right]$$
$$\hat{f}_1(x, y) \preceq_\chi \left[\frac{1}{4}, 4\right].$$

Acceptable degree of flexibility of the function of $\hat{f}_2(x, y)$ towards its goal $[-\frac{4}{y}, -\frac{1}{2y}]$ is

$$\chi_2^O\left(\hat{f}_2(x, y), \left[-\frac{4}{y}, -\frac{1}{2y}\right]\right) = \begin{cases} 1, & \text{if } f_2^R(x, y) < -\frac{4}{y} \\ 0, & \text{if } f_2^L(x, y) > -\frac{1}{2y} \\ \frac{-\frac{1}{2y}+2x_2}{\frac{7}{2y}+x_2}, & \text{elsewhere,} \end{cases}$$

with $\mu(\hat{f}_2(x, y)) \le \mu\left(\left[-\frac{4}{y}, -\frac{1}{2y}\right]\right)$, which is $x_2 \le \frac{7}{2y}$.
Acceptable degree of flexibility of $\hat{f}_1(x, y)$ towards its goal $\left[\frac{1}{4}, 4\right]$ is

$$\chi_1^O\left(\hat{f}_1(x, y), \left[\frac{1}{4}, 4\right]\right) = \begin{cases} 1, & \text{if } f_1^R(x, y) < \frac{1}{4} \\ 0, & \text{if } f_1^L(x, y) > 4 \\ \frac{4 - x_1y^3 - \frac{1}{2}x_2}{x_1y^3 + \frac{3}{2}x_2 + \frac{15}{4}}, & \text{elsewhere} \end{cases}$$

with $\mu(\hat{f}_1(x, y)) \le \mu\left(\left[\frac{1}{4}, 4\right]\right)$, which is $x_1y^3 + \frac{3}{2}x_2 \le \frac{15}{4}$.

Using χ_1^O and χ_2^O, and the methodology in Subsect. 3.1, the deterministic problem $\mathbf{IBP'}$ is formulated as

$$
\begin{aligned}
\mathbf{IBP'}: \qquad & \max_{x_1,x_2,y,\theta} \quad \theta \\
& \text{subject to} \quad \theta \leq \frac{10 - x_1 y}{x_1 y + 1} \\
& \qquad\qquad\quad \theta \leq \frac{2 - x_1^2 - x_2 y}{x_2 y + 1} \\
& \qquad\qquad\quad \theta \leq \frac{-\frac{1}{2y} + 2x_2}{x_2 + \frac{7}{2y}} \\
& \qquad\qquad\quad \theta \leq \frac{4 - x_1 y^3 - \frac{1}{2}x_2}{x_1 y^3 + \frac{3}{2}x_2 + \frac{15}{4}} \\
& \qquad\qquad\quad x_1 y \leq 1,\ x_2 y \leq 1 \\
& \qquad\qquad\quad x_2 \leq \frac{7}{2y},\ x_1 y^3 + \frac{3}{2}x_2 \leq \frac{15}{4} \\
& \qquad\qquad\quad x_1, x_2 \geq 0,\ 0 < y \leq 1,\ \frac{1}{2} \leq \theta \leq 1.
\end{aligned}
$$

Solution of $\mathbf{IBP'}$ is found using Mathematica 8.0 as $x_1^* = 0,\ x_2^* = 1.400, y^* = 0.663, \theta^* = 0.557$. That is, χ-optimal solution of **IBP** is $((x_1^*, x_2^*), y^*) = ((0, 1.400), 0.663)$ with 55% degree of satisfaction. The optimal value of **IBP** is $[0.700, 2.800]$.

5 Conclusion

Most of the existing literature on IBP focus on the derivation of bounds of linear bi-level programming problem only in which the objective functions are interval valued functions and the feasible region is governed by real valued functions. This paper studies the existence of solution of non linear bi-level problems where both the objective functions and constraints are interval valued linear/nonlinear functions. This method provides one compromising solution of the problem, which is feasible and optimal up to certain acceptable degree for the decision maker. Here we have not discussed the optimistic and pessimistic case of the solution of **IBP**. Both the lower and upper level problems are indirectly solved through the deterministic equivalent $\mathbf{IBP'}$. Due to the presence of interval uncertainty other general structures including multi level models are more complex to handel, which may be treated as the future scope of the present work.

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Dynamics of Effector -Tumor-Interleukin-2 Interactions with Monod-Haldane Immune Response and Treatments

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Abstract. In this paper, we modify and study a deterministic mathematical model governing interactions between tumor cell (TCs), effector-cells (ECs) and interleukin-2 with treatments. We investigate local stability at equilibria. The system exhibits transcritical bifurcation at tumor free equilibrium which is proved analytically by using Sotomayer's theorem and illustrated numerically. Our study demonstrates that tumor is being eradicated after administration of external treatments. We execute extensive computer simulations and analyze with biological relevance.

Keywords: Monod-Haldane immune response · Local stability · Transcritical bifurcation

1 Biological Background and Motivation

"Cancer or malignant tumor" is still a leading health problem and showing the cause of death of millions of people all over the world [1]. The mechanism of creation and destruction as well as prohibition and treatment are still relatively not known and in its infancy. In cancer, cells enhance and divide uncontrollably and also are invaded in the body. Kirschner and Panetta profounded a model with three compartments namely, activated immune cells (effector cells (ECs)) such as natural killer cells and cytotoxic T-cells which are cytotoxic to tumor cells (TCs), tumor cells and concentrated interleukin-2 (IL-2). They introduced Michaelis-Menton form of immune response to indicates the saturated effects of immune response between effector cells and interleukin-2 and also to identifies the self-limited production of IL-2.

Modern treatments are included with surgery, chemotherapy, and immunotherapy [2,3]. Immunotherapy mainly indicates to the treatment with a cytokine. Interleukin-2 controls T-helper 1 according to the autocrine manner and excites the growth of T-helper 2 [4]. Indeed, the tumor cells excites T-lymphocytes to conduct further growth [5,6].

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The immune system shows both stimulatory and inhibitory roles in tumorigenesis [7,8]. The immune response is taken as two different types based on their activities, (i) anti-tumor (IL-2, IL-12, IFN-γ) and pro-tumor (such as IL-4, IL-6, IL-10, TGF β).

Most studies have discussed the interaction between tumor-immune system [9,10,21] and few works have been investigated on the role of pro-tumor factors [10].

From the brief discussion, it can be noted that the immune system can be suppressed for different factors contained in cells [11,12]. At the primary stage, the immune system is strong to defend the tumor which could be considered as dormancy [13,14]. As the phenomena of tumor immune interaction is likely unpredictable [9]. A non-monotone function as the immune response can be taken in the mathematical model. In 1930, Haldane proposed response function in enzymology [15]. Sokol and Howell [16] simplified the Monod-Haldane response function. Influenced studies about cellular phenomena, we take modified Monod-Handane form of kinetics to describe non-monotonic immune interaction between effector cells and interleukin-2.

It is obvious that time delays has influence on the dynamics of physiological system [17]. It should be considered as time for proliferation, molecule production, differentiation and transportation of cells, etc. In 2008, Banerjee discussed tumor model of Kirschner-Panetta with time delay interacting between effector cells and interleukin-2 [18]. Tumor-immune dynamics with discrete time delay has been a interest of researchers for a long time [19].

Our main objective is to illustrate the dynamical behavior of effector cells (ECs), tumor cells (TCs) and interleukin-2 (IL-2) based on Monod-Haldane form of immune-response with time delay.

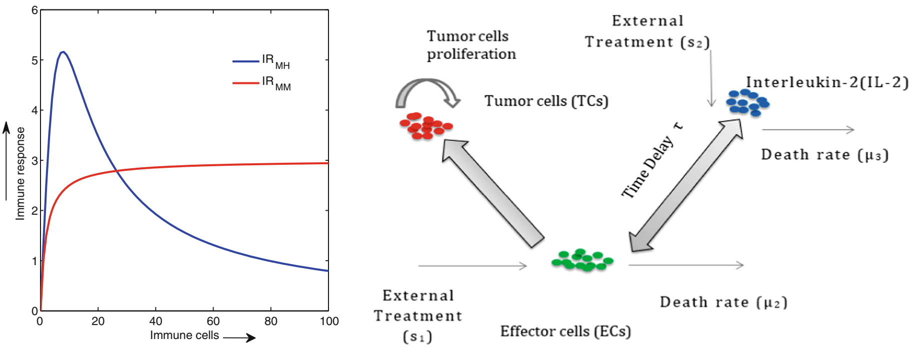
The subsequent part of this paper as follows. In (2) is the formulation of the mathematical model using delay differential equations have been considered which includes the interaction between ECs, TCs, and IL-2. In (3), the qualitative and numerical simulation of the system is performed with non-negative with initial conditions and positivity. Local stability is discussed at the meaningful biological equilibrium point. Transcritical bifurcation using **Sotomayer's theorem** has been studied. We carried out extensive computer simulations and results are analyzed with proper biological meanings. Finally, in (5), brief concluding remarks have been drawn up key findings of the problem.

2 Mathematical Model and Preliminaries

In tumor dynamics, an immune response shows that effector cells annihilate tumor cells. Kirschner-Panetta studied a model with Michaelis-Menten. Michaelis-Menten kinetics indicate the enzymatic reactions which are a non monotone in nature. But dynamics of tumor-immune interaction is likely unpredictable [10]. Figure 1a depicts the characteristic changes of non-monotone function. It can be noted that modified Monod-Haldane kinetics is able to describe the tumor-immune dynamics.

$$IR_{MM}(x) = \frac{\alpha x}{\beta + x}, \quad IR_{MH}(x) = \frac{\alpha x}{\beta + x^2}$$

where $\beta > 0, \alpha > 0$ represent half-saturation constant and growth rate of cells respectively. Other immune-response function is Which is modified Monod-Haldane kinetics [16].



(a) (b)

Fig. 1. (a) comparison between linear response function and Michalis-Menton response function, (b) schematic diagram of the interactions between tumor-cells, effector-cells and host cells.

In this article, we have modified the Kirschner-Panetta model [3] by introducing Monod-Haldane type immune response for interaction between effector cells and interleukin-2.

The modified system of equations is given by

$$\begin{aligned}\frac{dE}{dt} &= cT - \mu_2 E + \frac{p_1 E(t-\tau)I(t-\tau)}{g_1 + I^2(t-\tau)} + s_1\\ \frac{dT}{dt} &= r_2 T(1-bT) - \frac{aE(t)T(t)}{g_2 + T(t)}\\ \frac{dI(t)}{dt} &= \frac{p_2 E(t)T(t)}{g_3 + T(t)} - \mu_3 I + s_2\end{aligned} \tag{1}$$

with initial conditions $E(0) = E_0, T(0) = T_0, I(0) = I_0$. The first equation describes to the changing rate of effector cells. The effector cells increases due to tumor cells with antigenicity c.Effector cells are stimulated by interleukin-2 thrugh autocrine and paracrine signaling. A time delay is added for interaction between effector cells and interleukin-2. Treatment s_1 is added for the source of effector cells. The second equation corresponds the change of tumor cells. The death rare of tumor cells is a in the form of Michael-Menton kinetics due to tumor-immune interaction. The third equation represents the rate of change of density of IL-2. Michael-Menton kinetics is included for self-limiting production of IL-2. μ_3 describes the natural death rate of IL-2. Treatment s_2 is administered as an external source of IL-2 into the system.

Using scaling, $x = \frac{E}{E_0}, y = \frac{T}{T_0}, z = \frac{I_L}{I_{L_0}}$

$c' = \frac{cT_0}{E_0 t_s}, p_1' = \frac{p_1}{I_{L_0} t_s}, g_1' = \frac{g_1}{I_{L_0}^2}, t' = t_s t, \mu_2' = \frac{\mu_2}{t_s}, s_1' = \frac{s_1}{t_s}, r_2' = \frac{r_2}{t_s}, b' = bT_0, a' = \frac{aE_0}{t_s T_0}, g_2' = \frac{g_2}{T_0}, p_2' = \frac{P_2 E_0}{T_0 I_{L_0} t_s}, g_3' = \frac{g_3}{T_0}, s_2' = \frac{s_2}{t_s I_{L_0}}, \mu_3' = \frac{\mu_3}{t_s}, \tau' = nT_0\tau.$

For utility, '$\prime$' is omitted and for a proper choice of scaling to avoid numerical stiffness. We take $E_0 = T_0 = I_{L_0} = 1/b$ and $t_s = r_2$.
The non-dimensionalized system (1) is

$$\begin{aligned}
\frac{dx}{dt} &= cy - \mu_2 x + \frac{p_1 x(t-\tau) z(t-\tau)}{g_1 + z^2(t-\tau)} + s_1 \\
\frac{dy}{dt} &= r_2 y(1 - by) - \frac{axy}{g_2 + y} \\
\frac{dz}{dt} &= \frac{p_2 xy}{g_3 + y} - \mu_3 z + s_2
\end{aligned} \tag{2}$$

subject to the following initial condition $\eta = (\eta_1, \eta_2, \eta_3)$, defined in the space

$$\mathbb{C}_+ = \{\chi \in \mathbb{C}([-\tau, 0], \mathbb{R}_+^3) : x(\eta) = \chi_1(\eta), y(\eta) = \chi_2(\eta), z(\eta) = \chi_3(\eta)\}. \tag{3}$$

where $\chi_i(\eta) \geq 0, i = 1, 2, 3, \eta \in [-\tau, 0]$ and $\mathbb{C}$ is Banach space of continuous functions. $\chi : [-\tau, 0] \to \mathbb{R}_+^3$, with suitable sub-norm and $\mathbb{R}_+^3 = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$.

3 The Qualitative Analysis of the System

3.1 Positive Invariance

Theorem 1. *Each solution of the system (2) with initial condition χ_1, χ_2, χ_3 given in (3) defined on $[-\tau, 0]$ remains always positive for all finite time $t > 0$.*

Consider the model system (2) in the vector form $G = (x, y, z)^T \in \mathbb{R}_+^3$ and

$$G(Y) = \begin{pmatrix} G_1(Y) \\ G_2(Y) \\ G_3(Y) \end{pmatrix} = \begin{pmatrix} cy + \frac{p_1 x(t-\tau) z(t-\tau)}{g_1 + z^2(t-\tau)} + s_1 \\ r_2 y(1 - by) - \frac{axy}{g_2 + y} \\ \frac{p_2 xy}{g_3 + y} - \mu_3 z + s_2 \end{pmatrix}$$

here, $G : \mathbb{C}_+ \to \mathbb{R}_+^3$ and $G \in \mathbb{C}^\infty(\mathbb{R}_+^3)$, then the system can be expressed as

$$\dot{Y} = G(Y_t) \tag{4}$$

with initial condition $Y_t(\eta) = Y(t + \eta), \eta \in [-\tau, 0]$. Considering Eq. (4) and choosing $Y(\eta) \in \mathbb{C}_+$ such that $Y_i = 0$. We get $G_i(Y)|_{Y_i(t)} = G_i(0) \geq 0, \ \forall \ Y_t \in \mathbb{C}_+, \ i = 1, 2, 3$. Based on lemma from Das et al. [20], for any solution of $\dot{Y} = F(Y_t)$ with $Y_t(\xi) \in \mathbb{C}_+$ and $Y(t) = Y(t, Y(0))$, we get $Y(t) \in \mathbb{R}_+^4, \ \forall \ t \geq 0$, *i.e,* it remains always non-negative throughout region $\mathbb{R}_+^3$.

3.2 Boundedness

Theorem 2. *There exist positive constants N_x, N_y, N_z for positive solution $(x(t), y(t), z(t))$ of the system (2) with large t such that $x(t) \leq N_x, y(t) \leq N_y$, and $z(t) \leq N_z$.*

The model system (2) satisfies Lipschitz conditions function. Therefore, there exists a unique solution of (2) with initial condition (χ_1, χ_2, χ_3). Since solution exists, then $x(t-\tau) = \chi_1(t-\tau), y(t-\tau) = \chi_2(t-\tau), z(t-\tau) = \chi_3(t-\tau)$. From the first equation of (2), we have

$$x(t) = \chi(0)e^{-\mu_2 t} + e^{-\mu_2 t} \int_0^t \left(s_1 + \frac{p_1 \chi_1(t-\tau)\chi_3(t-\tau)}{g_1 + \chi_3^2(t-\tau)} \right) e^{\mu_2 \delta} d\delta$$

Since, $\frac{z}{g_1+z^2} < 1$ and $0 < e^{-\tau t} \leq 1$, by Gronwall's Lemma [22]

$$x(t) \leq \chi(0) + \frac{s_1}{\mu_1} e^{\mu_2 t} + \int_0^t p_1 \chi_1(t-\tau) e^{\mu_2 \delta} d\delta = N_x$$

Hence, $x(t) \leq N_x$(say), where N_x is uniformly bounded on $[-\tau, 0]$.

From second equation, we have

$$\frac{dy}{dt} \leq (r_2 + N_x)(1 - \frac{b}{r_2 + N_x} y)$$

By Kamke's comparison theorem

$$\lim_{t \to \infty} \sup y(t) \leq \frac{r_2 + N_x}{b}$$

and

$$y(t) \leq \max\{\chi_2(0), \frac{r_2 + N_x}{b}\} = N_y(say)$$

Similarly, the third equation

$$\frac{dz}{dt} \leq s_2 - \mu_3 z$$

Hence,

$$z(t)e^{\mu_3 t} = \textstyle\int_0^t s_2 e^{-\mu_2 t} dt = \frac{s_2}{\mu_3} + 1$$
$$z(t) \leq N_z(say).$$

Here, N_x, N_y, N_z are non-negative. From above, it is clear that $x(t), y(t), z(t)$ are bounded on $\in \mathbb{R}_+^3, \quad \forall \ t \geq 0$.

3.3 Equilibria

The model system (2) has following equilibrium points:

(i) Trivial equilibrium $E_0(x, y, z) = (0, 0, 0)$
(ii) Tumor-free equilibrium $E_1(x, 0, 0) = (\frac{s_1}{\mu_2}, 0, 0)$, E_1 exists if $r_2 < \frac{as_1}{\mu_2 g_1}$
(iii) Tumor-presence equilibrium $E^*(x^*, y^*, z^*) = \Big(\frac{1}{a} r_2(1 - by^*)(g_2 + y^*), \frac{p_2(1-by^*)(g_2+y^*)}{a\mu_3(g_3+y^*)} + \frac{\mu_3}{s_2}, \frac{1}{c}(\mu_2 x^* - s_1 - \frac{p_1 x^* z^*}{g_1+z^{*2}})\Big)$, E^* exists if $y^* < \frac{1}{b}$ and $\mu_2 > \frac{p_1 z^*}{g_1+z^{*2}}$.

3.4 Local Stability Analysis

Trivial-Equilibrium: The Jacobian matrix J_{E_0} of the model system (2) at trivial-equilibrium point $E_0(0, 0, 0)$ is given by

$$J_{E_0} = \begin{bmatrix} -\mu_2 & c & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -\mu_3 \end{bmatrix}$$

Thus, the eigenvalues of J_{E_0} are $\lambda_1 = -\mu_2, \lambda_2 = r_2$ and $\lambda_3 = -\mu_3$. Hence, we have the theorem.

Theorem 3. *The trivial-equilibrium E_0 is repeller.*

Tumor-Free Equilibrium: The Jacobian matrix J_{E_1} of the system (2) about tumor-free equilibrium point $E_1(x, 0, 0)$ is given by

$$J_{E_1} = \begin{bmatrix} -\mu_2 & c & \frac{p_1 x}{g_1} \\ 0 & r_2 - \frac{ax}{g_2} & 0 \\ 0 & \frac{p_2 x}{g_3} & -\mu_3 \end{bmatrix}$$

Thus, the eigenvalue of characteristics equation of J_{E_1} at $E_1(x, 0, 0) = (\frac{s_1}{\mu_2}, 0, 0)$ are $\lambda_1 = -\mu_2 < 0, \lambda_2 = r_2 - \frac{as_1}{\mu_2 g_2}$ and $\lambda_3 = -\mu_3 < 0$. Hence, we have the following theorem.

Theorem 4. *E_1 is stable equilibrium point if $r_2 < \frac{as_1}{\mu_2 g_2}$.*

To check the feasibility of our analytical findings, we present the numerical results of the model system using MATLAB to verify the effect of discrete time delay. The parameter values are given in Table 1 which have been used to find numerical simulations. The tumor-free equilibrium points $E^*(x, 0, 0) = (2.99401, 0, 0)$ for $s_1 = 0.00005, s_2 = 0.5$ are The tumor-presence equilibrium points $E^*(x^*, y^*, z^*)$ are obtained as $(0.000330959, 0.042687, 0.003855541)$, $(0.000646938, 0.996377, 0.000324677)$, and $(0.00736132, 0.955141, 0.00385348)$.

Table 1. Parameter values used for numerical simulations

Dimensional parameter	Value	Dimensionless value
c (antigenicity of tumor)	$0 \leq 0.05$	$0 \leq 0.278$
p_1 (growth rate of effector cells)	0.1245	0.69167
g_1 (half saturation constant)	2×10^7	0.02
μ_2 (natural death rate of effector cells)	0.03	0.1667
r_2 (growth rate of tumor cells)	0.18	1
b (carrying capacity of tumor cells)$^{-1}$	1.0×10^{-9}	1
a (natural death rate of tumor cells)	1	5.5556
g_2 (half saturation constant)	1×10^5	0.0001
μ_3 (natural death rate of IL-2)	10	55.556
p_2 (growth rate of IL-2)	5	27.778
g_3 (half saturation constant)	1×10^3	0.000001

Analysis of Transcritical Bifurcation

One of the eigenvalues of J_{E_1} is zero at $r^* = r_2 = \frac{as_1}{\mu_2 g_2}$, we apply **Sotomayer's theorem** to analyze bifurcation theory [23].

Suppose, J_{E_1} and its transpose have eigenvalue $\lambda = 0$ with corresponding eigenvectors $U = (u_1, u_2, u_3)^T$ and $V = (v_1, v_2, v_3)^T$ at bifurcating points r^*. We get $V = (c\mu_3 + \frac{p_1 p_2 s_1^2}{\mu_2 \mu_3 g_1 g_3}, \mu_2, \frac{p_2 s_1}{\mu_2 g_3})^T$ and $V = (0, 1, 0)^T$. Now, w_{r^*} denotes the vector function of partial derivatives of the system (2) with respect to r_2 and $Dw(E_1, r^*)U$ corresponds with directional derivatives of w in the direction of U at E_1. Now,

$$V^T.w_{r_2}(E_1, r^*) = (0,1,0).\begin{pmatrix} 0 \\ y(1-by) \\ 0 \end{pmatrix}_{E_1} = 0,$$

$$V^T.[Dw_{r_2}(E_1, r^*)U] = (0,1,0).\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-2by & 0 \\ 0 & 0 & 0 \end{pmatrix}_{E_1} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_2 = \mu_3 \neq 0$$

and

$$V^T.[D^2 w_{r_2}(E_1, r^*).(U,U)] = (0,1,0).D\begin{pmatrix} 0 \\ u_2(1-2by) \\ 0 \end{pmatrix}_{E_1} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= (0,1,0).\begin{pmatrix} 0 \\ -2bu_1u_2 \\ 0 \end{pmatrix}_{E_1} = -2bu_1u_2 \neq 0.$$

Hence, all three conditions for the existence of Transcritical-bifurcation satisfy. So we have the theorem.

Theorem 5. *The model system (2) exhibits a Transcritical-bifurcation with respect r_2 at tumor-free equilibrium point E_1 for $r_2 = r^* = \frac{as_1}{\mu_2 g_2}$.*

Tumor-Presence Equilibrium: The Jacobian matrix J_{E^*} of the model system (2) at tumor-presence equilibrium point $E^*(x^*, y^*, z^*)$ is given by

$$J_{E^*} = \begin{bmatrix} -\mu_2 + \frac{m_1 z^*}{g_1+z^{*2}} e^{-\lambda\tau} & c & p_1 x^* \frac{g_1 - z^{*2}}{(g_1+z^{*2})^2} e^{-\lambda\tau} \\ -\frac{ay^*}{g_2+y^*} & r_2 - 2by^* - \frac{ag_2 x^*}{(g_2+y^*)^2} & 0 \\ \frac{p_2 y^*}{g_3+y^*} & \frac{p_2 g_3 x^*}{(g_3+y^*)^2} & -\mu_3 \end{bmatrix}$$

and the characteristic equation is,

$$J_{E^*} = \lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0, \tag{5}$$

where λ is eigenvalue and

$$m_2(\lambda) = -\left\{ r_2 - 2by^* - \frac{ag_2x^*}{(g_2+y^*)^2} \right\} + (\mu_2 + \mu_3)$$

$$m_1(\lambda) = -(\mu_2 + \mu_3)\left\{ r_2 - 2by^* - \frac{ag_2x^*}{(g_2+y^*)^2} \right\} - \frac{cay^*}{g_2+y^*} + \mu_3\mu_2$$

$$m_0(\lambda) = -\mu_2\mu_3\left\{ r_2 - 2by^* - \frac{ag_2x^*}{(g_2+y^*)^2} \right\} - \frac{ca\mu_3 y^*}{g_2+y^*}$$

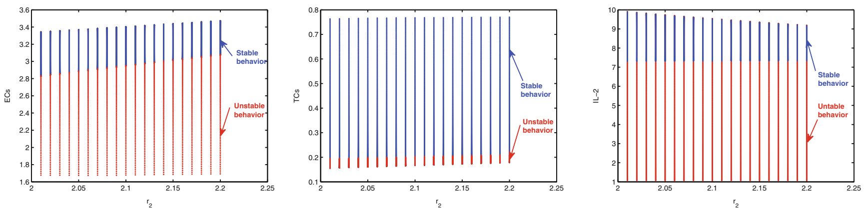
$$n_2(\lambda) = -\frac{p_1 z^*}{g_1 + z^{*2}}$$

$$n_1(\lambda) = \frac{p_1 z^*}{g_1 + z^{*2}} - \frac{p_1^2 x^* y^* (g_1 - z^{*2})}{(g_3+y^*)(g_1+z^{*2})^2} + \frac{p_1 z^*}{g_1 + z^{*2}}\left\{ r_2 - 2by^* - \frac{ag_2x^*}{(g_2+y^*)^2} \right\}$$

$$n_0(\lambda) = \frac{p_1\mu_3 z^*}{g_1 + z^{*2}}\left\{ r_2 - 2by^* - \frac{ag_2x^*}{(g_2+y^*)^2} \right\} - \frac{ap_1p_2g_3x^{*2}y^*(g_1 - z^{*2})}{(g_2+y^*)(g_3+y^*)^2(g_1+z^{*2})^2}.$$

4 Discussion with Computer Simulation

In this section, we validate our analytical results of local stability and Transcritical bifurcation. The parameters values are demonstrated in Table 1.



(a) **(b)** **(c)**

Fig. 2. The figures show the transcritical bifurcations with respect to parameter r_2 around E_1.

As unstable region is lesser than stable region in Fig. 2(b), it can be mentioned that effector cells are sufficiently strong to annihilate immature tumor cells.

Now, we study the system with different cases based on delay, adoptive cellular immunotheraphy (S_1) and concentrated interleukin-2 ((S_2)) as external treatments.

Case I: $\tau = 0, s_1 = 0, s_2 = 0$.
The characteristic equation becomes
$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 = 1, n_2\lambda^2 + n_1\lambda + n_0 = 0,$
$i.e,\ \lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0.$
The system is stable if the roots of equation have negative real parts. According to Routh-Hurwitz criterion, the system is stable for $m_2+n_2 > 0, m_1 + n_1 > 0$ and $(m_2 + n_2)(m_1 + n_1) - (m_0 + n_0) > 0$.

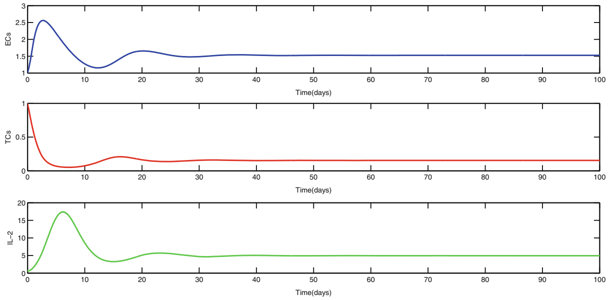


Fig. 3. The figure shows that tumor-presence equilibrium E^* is locally asymptotically stable for $\tau = 0$.

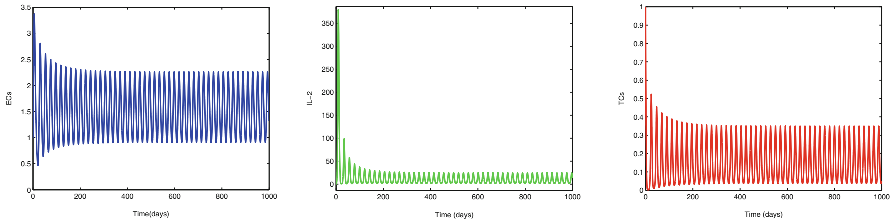
We have plotted Fig. 3 to verify Routh-Hurwitz criterion. The figure shows that the system is asymptotically stable with eigenvalues (−50.24, −21.75, −35.28) at E^* for time delay $\tau = 0$. It can be noted that effector cells and interleukin-2 together can destroy tumor cells instantaneous.

Now we investigate the system at tumor-presence equilibrium in presence of discrete time delay τ.

Case II: $\tau > 0, s_1 = 0, s_2 = 0$.

From Fig. 4, it is observed that the system shows unstable behavior. Here, we can conclude that the effector cells and interleukin-2 are unable to control the periodic oscillation of tumor cell in presence of delay. This implies that tumor cells are sufficiently strong and shows immunosuppressing nature as pro-tumor factors.

As the tumor exhibits unstable behavior as well as uncontrolled growth, we administer either adoptive cellular immunotherapy (ACI) or concentrated interleukin-2 (IL-2) as an external treatment.

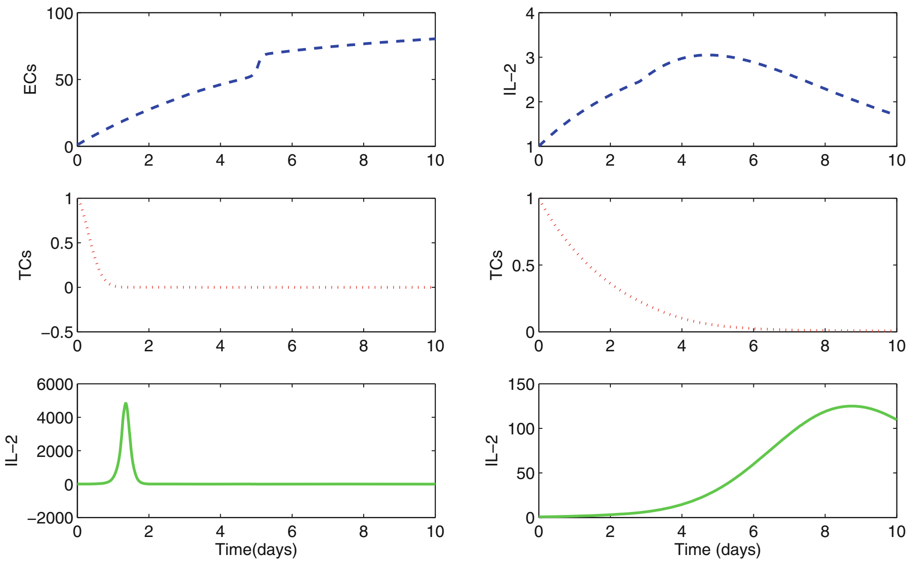


(a) **(b)** **(c)**

Fig. 4. (a–c) The figures show the unstable periodic solution of the system in presence of time delay $\tau > 0$.

Case III: $\tau > 0, s_1 > 0, s_2 = 0$.

After administration of adoptive cellular immunotheraphy (ACI) into the system, it is observed that tumor is eradicated as well as IL-2 is fully eliminated in Fig. 5(a). IL-2 reaches to highest position to incite effector cells. Moreover, excessive doses of adoptive cellular immunotheraphy increases density of effector cells in patients body.



(a) **(b)**

Fig. 5. Time trajectories for effector cells, tumor cells and IL-2, (a): $s_1 = 0.000000973, s_2 = 0, \tau = 0.1$ days $= 2.4$ h, (b): $s_1 = 0, s_2 = 0.005, \tau = 0.1$ day $= 2.4$ h.

Case IV: $\tau > 0, s_1 = 0, s_2 > 0$.

In similar way, after administration of concentrated interleukin-2 (IL-2) in absence of ACI, it is noticed that effector cells and IL-2 decrease after eradication of tumor in Fig. 5(b).

5 Concluding Remarks

In this paper, our main aim was to get a understanding of the three components of the tumor-immune system along with time delay between effector cells and interleukin-2. We have discussed the Monod-Haldane form of immune response based delayed mathematical model including external treatments in the human immunological system. We have showed the positivity of the solution, boundedness. The local stability of equilibria has been discussed. The transcritical bifurcation is seen at tumor-free equilibria.

The system shows that tumor is eradicated after inputs of Adoptive cellular immunotherapy (ACI) in absence of concentrated IL-2. Adoptive cellular immunotheraphy can defend the tumor growth but it should be kept in mind that much amount of ACI causes side-effects to patients. In other words. Moreover, the external input of concentrated IL-2 in absence of ACI is more effective whereas immune system decreases after eradication tumor.

The main difference between the presented work and recent literatures is the role of anti-tumor and pro-tumor factors on the dynamics of the tumor-immune system. Here, the influence of time delay is considerable for biologically meaningful values of parameters. Further studies are needed for understanding the role of pro-tumor and anti-tumor factors which is crucial for the tumor-immune process. We hope that our discussion will help the researches to gather appropriate knowledge about tumor-immune interaction.

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An Inventory Model of Time and Reliability Dependent Demand with Deterioration and Partial Backordering Under Fuzziness

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Abstract. This paper present a deterministic inventory model for durable product where both demand and deterioration are time and reliability dependent. The paper develop a cost effective ordering inventory model where the increase in reliability of the item leads to increase in demand and decreases the rate of deterioration of the item. In this paper we have considered an inventory model for deteriorating items under fuzziness. The shortages are allowed in this economic order quantity model which are partially lost in sales and partially backlogged. To evaluate the optimum solution, the inventory model is solved under fuzzy environment. While incorporating symmetric triangular fuzzy number we use total λ-integral value to defuzzify the solution. Numerical example along with graphical representation are provided to illustrate the proposed model and the significant features of the results are discussed. Finally, the effects of different parameters present through sensitivity analysis on the numerical example.

Keywords: Inventory · Deterioration · Shortage · Partial backlog · Reliability · Lost in sales · Triangular fuzzy number

1 Introduction

In economic order quantity (EOQ) model demand and deterioration has been always the important factors in the inventory system. Generally, various types of demand rate are considered such as constant demand, price dependent demand, time-dependent demand (Goswami and Chaudhuri 1991; Teng et al. 2002; Dye et al. 2005; Khanna et al. 2013; Banerjee and Agrawal 2017; Jain et al. 2018), Time-and-price dependent demand (Sarkar et al. 2014), linear trend in demand (Donaldson 1977; Dave 1989), fuzzy random demand (Bag et al. 2009), variable demand (Shaikh et al. 2018), stochastic demand (Pal and Mahapatra 2017; Tiwari et al. 2018), etc. Palanivel et al. (2016) discussed on two-warehouse inventory model with non-instantaneously deteriorating items, stock-dependent demand, shortages and inflation. Kundu et al. (2017) presented an imperfect

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EPQ model for deteriorating items with promotional exoort dependent demand. Adak and Mahapatra (2018) studied on an inventory model of flexible demand for price, stock and reliability with deterioration under inflation incorporating delay in payment. Bhunia et al. (2018) developed an application of genetic algorithm and PSO in an inventory model for single deteriorating item with variable demand dependent on marketing strategy and displayed stock level. In real life problem it would also be more realistic to consider the demand as time and reliability dependent. This paper consider such type of items whose reliability increases with time and so a major part of the customers tends to buy more products which leads to decrease in deteriorated items and hence the shortages arises. Most of the EOQ models were considered with the realistic assumption of various types of demand but in real life the assumption is not always true in practice. For example the demand sometimes not only depend on time only it may vary with time and reliability. Thus in this case we consider the demand is time and reliability dependent one.

Maintenance of inventories of deteriorating items is a problem of major concern in the supply chain of any business organizations due to undergo decay or deterioration over time. In most of the cases researchers considered the deteriorated items satisfying Weibull distribution. The researchers constructed inventory models with a comprehensive Weibull distribution deterioration (Covert and Philip 1973; Giri et al. 2003; Pal et al. 2014a, b; Chakraborty et al. 2018), inventory models with a time dependent deterioration rate (Goyal 1987; Abad 1996; Papachristos and Skouri 2000; Manna and Chaudhuri 2006), inventory model with cyclic deterioration items (Mandal and Pal 1998; Deng et al. 2007; Panda et al. 2008), models with random demand and deterioration (Chung et al. 2011) and fuzzy deterioration (Shabani et al. 2016). Roy et al. (2011) studied an optimal shipment strategy for imperfect items in a stock-out situation. In general it is observed that the items starts deteriorating as time increases and the product of low reliability. Sanni and Chukwu (2016) discussed an inventory model with three-parameter Weibull deterioration, quadratic demand rate and shortages. Mishra et al. (2017) introduced an inventory model under price and stock dependent demand for controllable deterioration rate with shortages and preservation technology investment. Tiwari et al. (2018) presented joint pricing and inventory model for deteriorating items with expiration dates and partial backlogging under two-level partial trade credits in supply chain. In this paper we have considered this realistic scenario and consider the deterioration as time and reliability dependent.

In most of the research papers, complete backlogging of unsatisfied demand is assumed. In practice, there are customers who are willing to wait and receive their orders at the end of shortage period, while others are not. In the last few years, considerable attention has been given to inventory modeling with partial backlog which consider a mixture of backorders and lost sales for non deteriorating products (Montgomery et al. 1973). Many researchers worked on models with partial backlogging (Hill 1995; Abad 1996; Wu 2001; Jose et al. 2006; Bhunia et al. 2014; Wee et al. 2014) by taking into account of the customers' behavior.

Sana and Goyal (2015) studied (Q, r, L) model for stochastic demand with lead-time dependent partial back-logging. Mahapatra et al. (2017) presented an inventory model for deteriorating items with time and reliability dependent demand and partial backorder. We consider the backlogging rate is any decreasing function of the waiting time up to the next replenishment.

The cost function becomes imprecise in nature due to the uncertainty of inventory parameters, here we have considered the cost function that is holding cost, deterioration cost and shortage cost is fuzzy in nature. Several researchers work on fuzzy inventory model due to impreciseness of the cost and other parameters. De and Goswami (2006) developed an EOQ model with fuzzy inflation and fuzzy deterioration rate. Mahapatra et al. (2011) developed a production inventory model with fuzzy coefficients using parametric geometric programming. Mahapatra et al. (2012) introduced an EPQ model with imprecise space constraint based on intuitionistic fuzzy optimization technique. Mahapatra et al. (2013) presented inventory model with EPQ model with fuzzy coefficient of objective and constraint via parametric geometric programming. Samal and Pratihar (2014) developed optimization of variable demand fuzzy economic order quantity inventory models without and with backordering. Pal et al. (2014a, b) developed an EPQ model for a ramp type demand with Weibull deterioration under inflation in finite time horizon in crisp and fuzzy environment. Jana et al. (2014) constructed multi-item partial backlogging inventory models over random planning horizon in random fuzzy environment. De and Sana (2015) developed a backlogging EOQ model for promotional effort and selling price sensitive demand-an intuitionistic fuzzy approach. Pal et al. (2015) introduced a production inventory model for deteriorating item with ramp type demand allowing inflation and shortages under fuzziness. Shekarian et al. (2016) developed an economic order quantity model considering dixoerent holding costs for imperfect quality items subject to fuzziness and learning. Bera and Jana (2017) studied Multi-item imperfect production inventory model in bi-fuzzy environments.

We consider an EOQ models with time and reliability dependent demand as well as time and reliability dependent deterioration. We also consider that the model undergoes shortages which are partially backlogged and partially lost in sales. Further this paper is designed as follows: In Sect. 2 some notations and assumptions are presented which are used in this paper. Section 3 presents the formulation of the inventory model and details analytical analysis of the model present in Sect. 4. In Sect. 5, we derive the condition for the total cost of our model to be convex and hence optimized our model. In Sect. 6, we have observed the effect of fuzziness on the proposed inventory model. Further a numerical example is given in Sect. 7 to test the model. The sensitivity analysis of various parameter along with its graphical explanation are observed, and the managerial impact are discussed in Sect. 8. Section 9 presents some conclusions based on the study of the proposed model with future scope of research.

2 Assumptions and Notation

Notation

T is the time horizon or the total replenishment time.
t_1 is the time when the inventory level finishes.
t_1^* is the optimal solution for t_1.
c_d is the cost of each deteriorated item.
c_h is the inventory holding cost per unit item.
c_s is the shortage cost per unit of time.
c_l is the penalty cost for lost sale per unit item.
r is the reliability parameter which always lies between $0 < r < 1$.
θ is the deterioration rate of the on-hand inventory over $[0, t_1]$.
$I(t)$ is the on-hand inventory level at time t over the ordering cycle $[0, T]$.
A is the ordering cost.
S is the maximum inventory level for each ordering cycle that means $S = I(0)$.

Assumption

(i) The demand rate $D(t)$ is assumed to be any non negative function, that is we assume $D(t) = at^r$ where a is the shape parameter and $a > 0$.
(ii) The deterioration function of items are also considered as time and reliability dependent function $\theta(t) = \frac{t}{r^\alpha}$ where α is the shape parameter.
(iii) Shortage is allowed. The demand during shortage is partially lost and partially backordered. The fraction which is backlogged follows the backlog function taken as $\beta(x) = e^{-\gamma x}$ with $\beta(0) = 1$ and $\beta(T) \geq 0$ where γ is a shape parameter.
(iv) Delivery lead time is zero.
(v) Single item is considered in the prescribed time cycle.
(vi) The model start with an on hand inventory stock.

3 Mathematical Modeling of Inventory System

Replenishment occurs at time $t = 0$ when the inventory level attains its maximum S. At $t = t_1$ the inventory level reduces to zero (i.e., the inventory finishes) due to demand and deterioration. Still if there is a demand that leads to shortages. Thus shortage is allowed to occur during the time interval (t_1, T). The demand during the shortage period (t_1, T) is partially backlogged and partially lost in sales. The backlogging rate is dependent on the waiting time of the customer where demand is fulfilled in the next replenishment cycle. The shortages which leads to lost in sales are due to those customers who are not interested to wait and they opt for other product available in the market (Fig. 1).

A typical behavior of the inventory in a cycle is depicted in following The inventory levels of the model are described and formulated by the following equations:

$$\frac{d}{dt}I(t) + \frac{t}{r^\alpha}I(t) = -at^r, 0 \leq t \leq t_1 \tag{1}$$

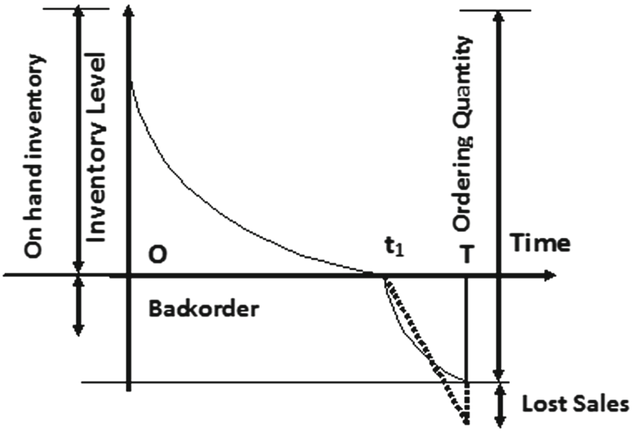

Fig. 1. Proposed inventory model with inventory vs time.

$$\frac{d}{dt}I(t) = -\beta(T-t)at^r, t_1 \leq t \leq T \tag{2}$$

with the boundary conditions $I(0) = S,\ I(t_1) = 0$.

4 Mathematical Analysis of the Proposed Inventory Model

We directly solve Eqs. (1) and (2) under the condition that $I(t_1) = 0$ to imply that

$$I(t) = ae^{-\frac{t^2}{2r^\alpha}}\left[\frac{t_1^{r+1}-t^{r+1}}{r+1} + \frac{t_1^{r+3}-t^{r+3}}{r+3}\right],\ 0 \leq t \leq t_1 \tag{3}$$

$$= a\left[\frac{(1-\gamma T)\left(t_1^{r+1}-t^{r+1}\right)}{r+1} + \frac{\gamma\left(t_1^{r+2}-t^{r+2}\right)}{r+2}\right],\ t_1 \leq t \leq T \tag{4}$$

Lemma 1. *The maximum inventory level S must satisfy the following relation in terms of shape parameter a, reliability r and time t_1 where the inventory level reduces to zero.* $S = at_1^{r+1}\left[\frac{1}{r+1} + \frac{t_1^2}{r+3}\right]$.

Proof. Using the boundary conditions $I(0) = S$ in 3, a relation is obtained as follows

$$I(0) = a\left[\frac{t_1^{r+1}}{r+1} + \frac{t_1^{r+3}}{r+3}\right]$$
$$S = at_1^{r+1}\left[\frac{1}{r+1} + \frac{t_1^2}{r+3}\right]$$

Hence we can say that, the inventory level S must satisfy the following relation in terms of shape parameter a, reliability r and time t_1 where the inventory level reduces to zero. Now, the proof is complete.

The amount of deteriorated items (DT) during $[0, t_1]$ is evaluated

$$I(0) - \int_0^{t_1} as^r ds = \int_0^{t_1} as^r (e^{\frac{s^2}{2r^\alpha}} - 1) ds = \frac{at_1^{r+3}}{2r^\alpha (r+3)} \tag{5}$$

The holding cost (HC) during $[0, t_1]$ is evaluated

$$c_h \int_0^{t_1} I(t)dt = ac_h \int_0^{t_1} e^{-\frac{t^2}{2r^\alpha}} \left[\frac{t_1^{r+1} - t^{r+1}}{r+1} + \frac{t_1^{r+3} - t^{r+3}}{r+3}\right] dt = ac_h t_1^{r+2} \left[\frac{1}{r+2} + \frac{2t_1^2}{6r^\alpha (r+4)} - \frac{t_1^4}{12r^{2\alpha}(r+6)}\right] \tag{6}$$

The shortage cost (SC) during $[t_1, T]$ is evaluated

$$c_s \int_{t_1}^{T} -I(t)dt = ac_s \int_{t_1}^{T} \left[\frac{(1-\gamma T)\left(t_1^{r+1} - t^{r+1}\right)}{r+1} + \frac{\gamma\left(t_1^{r+2} - t^{r+2}\right)}{r+2}\right] dt = ac_s \left[\frac{(1-2\gamma T)}{r+2} t_1^{r+2} + \frac{\gamma}{r+3}\left(t_1^{r+3} + \frac{T^{r+3}}{r+2}\right) + \frac{(1-\gamma T)\, T}{r+1}\left(\frac{T^{r+1}}{r+2} - t_1^{r+1}\right)\right] \tag{7}$$

The lost sales (LS) during $[t_1, T]$ is evaluated

$$ac_l \int_{t_1}^{T} \{1 - \beta(T-t)\}\, t^r dt = ac_l \gamma \left[\frac{T^{r+2}}{(r+1)(r+2)} + t_1^{r+1}\left(\frac{t_1}{r+2} - \frac{T}{r+1}\right)\right] \tag{8}$$

Ordering cost (OC) is

$$OC = A$$

Therefore the average total cost is expressed as:
$TC(t_1) = \frac{1}{T}[OC + DT + HC + SC + LS]$

$$\begin{aligned} TC(t_1) &= \frac{1}{T}[OC + DT + HC + SC + LS] \\ &= \frac{1}{T}\left[A + ac_h\left\{\frac{t_1^{r+2}}{r+2} + \frac{r^{-\alpha} t_1^{r+4}}{3(r+4)} - \frac{r^{-2\alpha} t_1^{r+6}}{12(r+6)}\right\} + ac_l\gamma\left\{\frac{T^{r+2}}{(r+1)(r+2)} + t_1^{r+1}\left(\frac{t_1}{r+2} - \frac{T}{r+1}\right)\right\}\right. \\ &\quad + \frac{ac_d t_1^3}{6r^\alpha}\left\{1 + r\log t_1 - \frac{r}{3}\right\} + ac_s\left\{\frac{(1-2\gamma T)}{(r+2)\, t_1^{-(r+2)}} + \frac{\gamma}{r+3}\left(t_1^{r+3} + \frac{T^{r+3}}{r+2}\right)\right. \\ &\quad \left.\left. + \frac{\left(T - \gamma T^2\right)}{r+1}\left(\frac{T^{r+1}}{r+2} - t_1^{r+1}\right)\right\}\right] \end{aligned} \tag{9}$$

5 Optimization of Proposed Model

In the management point of view the production/manufacturing cost should be optimum and hence minimizing the total cost per unit time. So optimizing the case of the proposed model by differentiation method is as follows.

From (14) it follows that $t_1 = 0$ and for that value the shortage will start at the beginning of the model, which is impossible in general case.

So we consider any arbitrary positive number ε (i.e. $\varepsilon > 0$), no matter how small such that

$$TC'(\varepsilon) < 0 \tag{10}$$

$$TC'(T) = a\left[\frac{c_d T^2}{2r^\alpha}(1 + r\log T) + 2c_s T^{r+1}\{(1-\gamma T)\} + c_h T^{r+1}\left(1 + \frac{T^2}{3r^\alpha} - \frac{T^4}{12r^{2\alpha}}\right)\right] > 0 \tag{11}$$

Hence $TC''(t_1^*) > 0$ to imply that $TC'(t_1)$ is increasing from $TC'(\varepsilon) < 0$ to $TC'(T) > 0$. So there is a unique point say t_1^* that satisfies $TC'(t_1^*) = 0$. Therefore it follows that

$$TC'(t_1) < 0,\ for\, \varepsilon < t_1 < t_1^* \tag{12}$$

and

$$TC'(t_1) > 0,\ for\, t_1^* < t_1 < T \tag{13}$$

According to (12), (13) we obtain $TC'(t_1) \le 0$ with $TC(t_1)$ non-increasing for $\varepsilon < t_1 < t_1^*$. On the other hand as $TC'(t_1) \ge 0$ with $TC(t_1)$ non decreasing for $t_1^* < t_1 < T$, t_1^* is the minimum point or the optimal solution.

Lemma 2. *The cost function TC is convex at* $t = t_1^*$ *if* t_1^* *satisfies the conditions* $c_h t_1\left(\frac{t_1^4}{12r^{2\alpha}} - \frac{t_1^2}{3r^\alpha} - 1\right) + c_s\left\{T(\gamma T - 1) + (2\gamma T - 1)t_1 - \gamma t_1^2\right\} + \gamma c_l(T - t_1) = \frac{c_d t_1^{2-r}}{2r^\alpha}(1 + r\log t_1)$ *and* $\frac{c_h t_1^3}{3r^\alpha}\left(2 - \frac{t_1^2}{r^\alpha}\right) + c_s\left\{\gamma t_1^2 - T(1-\gamma T)\right\} + c_l\gamma T > \frac{c_d t_1^{2-r}}{2r^{\alpha-1}}\left\{(1 - \log t_1 - t_1) - \frac{1}{r}\right\}$ *then* $TC(t_1^*)$ *is minimum.*

Proof. We have the total cost $TC(t_1^*)$ as follows

$$\begin{aligned}TC(t_1) = \frac{1}{T}\Bigg[&A + ac_h\left\{\frac{t_1^{r+2}}{r+2} + \frac{r^{-\alpha}t_1^{r+4}}{3(r+4)} - \frac{r^{-2\alpha}t_1^{r+6}}{12(r+6)}\right\} + ac_l\gamma\left\{\frac{T^{r+2}}{(r+1)(r+2)} + t_1^{r+1}\left(\frac{t_1}{r+2} - \frac{T}{r+1}\right)\right\}\\ &+ \frac{ac_d t_1^3}{6r^\alpha}\left\{1 + r\log t_1 - \frac{r}{3}\right\} + ac_s\left\{\frac{(1-2\gamma T)}{(r+2)\,t_1^{-(r+2)}}\right.\\ &\left.+ \frac{\gamma}{r+3}\left(t_1^{r+3} + \frac{T^{r+3}}{r+2}\right) + \frac{\left(T - \gamma T^2\right)}{r+1}\left(\frac{T^{r+1}}{r+2} - t_1^{r+1}\right)\right\}\Bigg]\end{aligned}$$

For minimization of the total cost function $TC'(t_1) = 0$ and $TC''(t_1) < 0$ must be satisfied. Hence,

$$\begin{aligned}TC'(t_1) = \frac{a}{T}\Bigg[&\frac{c_d t_1^2}{2r^\alpha}(1 + r\log t_1) + c_s t_1^r\left\{(1-2\gamma T)t_1 + \gamma t_1^2 + T(1-\gamma T)\right\}\\ &+ c_l\gamma t_1^r(t_1 - T) + + c_h t_1^{r+1}\left(1 + \frac{t_1^2}{3r^\alpha} - \frac{t_1^4}{12r^{2\alpha}}\right)\Bigg]\end{aligned} \tag{14}$$

and

$$\begin{aligned}TC''(t_1) = \frac{a}{T}\Bigg[&\frac{c_d t_1}{2r^\alpha}(2 + 2r\log t_1 + rt_1) + c_s t_1^{r-1}\left\{(1-2\gamma T)(r+1)t_1 + \gamma(r+2)t_1^2 + T(1-\gamma T)r\right\}\\ &+ \gamma c_l t_1^{r-1}\{(r+1)t_1 - rT\} + c_h t_1^r\left((r+1) + \frac{(r+3)\,t_1^2}{3r^\alpha} - \frac{(r+5)\,t_1^4}{12r^{2\alpha}}\right)\Bigg]\end{aligned}$$

For $TC'(t_1) = 0$, we have

$$c_h t_1 \left(\frac{t_1^4}{12r^{2\alpha}} - \frac{t_1^2}{3r^\alpha} - 1 \right) + c_s \left\{ T(\gamma T - 1) + (2\gamma T - 1) t_1 - \gamma t_1^2 \right\} + \gamma c_l (T - t_1) = \frac{c_d t_1^{2-r}}{2r^\alpha} (1 + r \log t_1)$$

and from $TC''(t_1) > 0$

$$\frac{c_h t_1^3}{3r^\alpha} \left(2 - \frac{t_1^2}{r^\alpha} \right) + c_s \left\{ \gamma t_1^2 - T(1 - \gamma T) \right\} + c_l \gamma T > \frac{c_d t_1^{2-r}}{2r^{\alpha-1}} \left\{ (1 - \log t_1 - t_1) - \frac{1}{r} \right\}$$

Hence, we prove the lemma.

6 Fuzzy Mathematical Modeling of Proposed Inventory Model

Zadeh (1965) designed the fuzzy set theory in real-life problems as a mathematical way to measure inherent fuzziness to deal with the uncertainty and haziness of human thinking.

Fuzzy Set: A fuzzy set $\tilde{A}$ in A is defined by a set or ordered pairs, a binary relation,

$$\tilde{A} = (x, \mu_{\tilde{A}}(x)) : x \in A$$

in which $\mu_{\tilde{A}}(x) \in [0, 1]$ is called the membership function, $\mu_{\tilde{A}}(x)$ specifies the grade or degree to which any element x in A belongs to the fuzzy set $\tilde{A}$.

Triangular Fuzzy Number: A fuzzy number $\tilde{A}$ on is defined to be a triangular fuzzy number, if its membership function $\mu_{\tilde{A}}(x)\ R \to [0, 1]$ is equal to

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} if\ a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2} if\ a_2 \le x \le a_3 \\ 0\ otherwise \end{cases} \tag{15}$$

A triangular fuzzy number, as expressed by (15) will be denoted as (a_1, a_2, a_3). A TFN $\tilde{A} = (a_1, a_2, a_3)$ is represented pictorially in (Fig. 2).

Graded Mean Value: Let λ be a pre-assigned parameter also called the degree of optimism and $\lambda \in [0, 1]$. The total λ integral value or the graded mean value (Mahapatra and Roy 2006) of $\tilde{A}$ is defined as $I_\lambda(\tilde{A}) = \lambda I_R(\tilde{A}) + (1-\lambda) I_L(\tilde{A})$ where the left and right interval values of $\tilde{A}$ are $I_L(\tilde{A})$ and $I_R(\tilde{A})$ respectively and it is defined as

$$I_L(\tilde{A}) = \int_0^1 A_L(\alpha) d\alpha \ and\ I_R(\tilde{A}) = \int_0^1 A_R(\alpha) d\alpha$$

Therefore the left and right integral values are $I_L(\tilde{A}) = \frac{a_1+a_2}{2}$ and $I_R(\tilde{A}) = \frac{a_2+a_3}{2}$. Hence total λ integral value or the graded mean value of $\tilde{A}$ is $I_\lambda(\tilde{A}) = \lambda \left(\frac{a_2+a_3}{2} \right) + (1-\lambda) \left(\frac{a_1+a_2}{2} \right) = \frac{1}{2} \left[(1-\lambda) a_1 + a_2 + \lambda a_3 \right].$

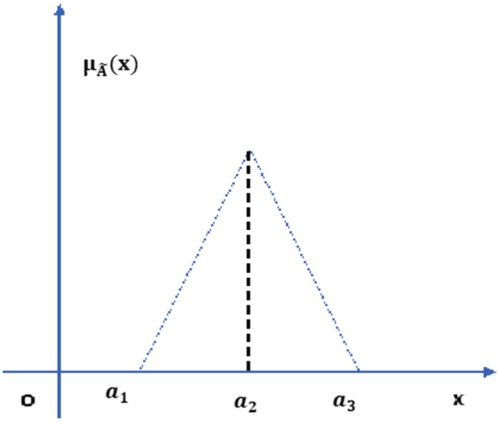

Fig. 2. Triangular fuzzy number

We introduce fuzzy for the cost parameters to modify the proposed inventory model to coincide with the real situation and check the effect on proposed inventory model.

The holding cost C_h is replaced by TFN $C_{\widetilde{h}} = (C_h - \delta_1, C_h, C_h + \delta_1)$ where $0 < \delta_1 < C_h$. Hence total λ integral value (or the graded mean value) of $C_{\widetilde{h}}$ is $I_\lambda(C_{\widetilde{h}}) = \frac{1}{2}\left[(1-\lambda)(C_h - \delta_1) + C_h + \lambda(C_h + \delta_1)\right] = C_h + \left(\lambda - \frac{1}{2}\right)\delta_1$.

The deterioration cost C_d is represent by TFN $C_{\widetilde{d}} = (C_d - \delta_2, C_d, C_d + \delta_2)$ where $0 < \delta_2 < C_d$ and in similar way we get $I_\lambda(C_{\widetilde{d}}) = C_d + \left(\lambda - \frac{1}{2}\right)\delta_2$.

The shortage cost C_s is represent by TFN $C_{\widetilde{s}} = (C_s - \delta_3, C_s, C_s + \delta_3)$ where $0 < \delta_3 < C_s$ and we get $I_\lambda(C_{\widetilde{s}}) = C_s + \left(\lambda - \frac{1}{2}\right)\delta_3$.

The fuzzy parameters $C_{\widetilde{h}}$, $C_{\widetilde{d}}$ and $C_{\widetilde{s}}$ can be used on the inventory model and hence the total costs become fuzzy in nature and this determines the fuzzy total expected cost $(\widetilde{TC})$ per unit time.

$$\begin{aligned}
\widetilde{TC}(t_1) &= \frac{1}{T}\left[A + a\left\{C_h + \left(\lambda - \frac{1}{2}\right)\delta_1\right\}\left\{\frac{t_1^{r+2}}{r+2} + \frac{r^{-\alpha}t_1^{r+4}}{3(r+4)} - \frac{r^{-2\alpha}t_1^{r+6}}{12(r+6)}\right\}\right. \\
&\quad + ac_l\gamma\left\{\frac{T^{r+2}}{(r+1)(r+2)} + t_1^{r+1}\left(\frac{t_1}{r+2} - \frac{T}{r+1}\right)\right\} + a\left\{C_s + \left(\lambda - \frac{1}{2}\right)\delta_3\right\} \\
&\quad \left\{\frac{(1-2\gamma T)}{(r+2)t_1^{-(r+2)}} + \frac{\gamma}{r+3}\left(t_1^{r+3} + \frac{T^{r+3}}{r+2}\right) + \frac{(T-\gamma T^2)}{r+1}\left(\frac{T^{r+1}}{r+2} - t_1^{r+1}\right)\right\} \\
&\quad \left. + \frac{at_1^3}{6r^\alpha}\left\{C_d + \left(\lambda - \frac{1}{2}\right)\delta_2\right\}\left\{1 + r\log t_1 - \frac{r}{3}\right\}\right] \qquad (16)
\end{aligned}$$

Solving (16) we get the optimal time and optimal value of the total inventory cost for the degree of optimism by changing the values of the parameter.

7 Numerical Illustration

Here we illustrate the results and present an example for the proposed model through numerical presentation under crisp and fuzzy environments.

To illustrate and validate the developed model, a hypothetical system with the following values of different parameters has been considered. In this section we have presented an example for numerical exposure of the presented inventory model. The input parameters are: $c_d = \$1.5$ per unit per year, $c_h = \$5$ per unit per year, $c_s = \$3.5$ per unit, $c_l = \$8$ per unit, $a = 4.7$, $A = 25$, $T = 1$ year, $r = 0.7$, $\alpha = 5$, $\beta(x) = e^{-\gamma x}$ and $\gamma = 1.3$. The above problem is solved by using Mathematica and MATLAB software.

Therefore the optimal value of t_1 is $t_1^* = 0.614$ and the minimum cost is $TC(t_1^*) = 31.329$.

Table 1 represents the numerical solution of fuzzy inventory model by considering the fuzzy holding cost $C_{\tilde{h}} = (C_h - \delta_1, C_h, C_h + \delta_1)$, deterioration cost $C_{\tilde{d}} = (C_d - \delta_2, C_d, C_d + \delta_2)$ and shortage cost $C_{\tilde{s}} = (C_s - \delta_3, C_s, C_s + \delta_3)$ where the values of $\delta_1 = 2.5$, $\delta_2 = 0.7$, $\delta_3 = 1.7$. To finding the optimal cost we consider the effect of different values of λ, such as 1 (Optimistic), 0.7 (About optimistic), 0.5 (Moderate), 0.2 (About pessimistic) and 0 (pessimistic).

Table 1. Optimal solution in fuzzy environment

λ	t_1^*	$TC(t_1^*)$
0 (Pessimistic)	0.695	29.705
0.2 (About pessimistic)	0.687	29.854
0.5 (Moderate)	0.677	30.071
0.7 (About optimistic)	0.670	30.213
1 (Optimistic)	0.661	30.420

From Table 1, it was seen that if we increases the values of λ the total cost $TC(t_1^*)$ gradually increases as well as the time t_1^* decreases. The model is realistic in fuzzy environment since in reality the holding cost, deterioration cost and shortage cost are varying and indeterministic in nature.

From Fig. 3 it is seen that the total expected cost of the proposed model is followed concavity property.

8 Sensitivity Analysis of Numerical Example

We discuss the effect of changes of different parameters c_d, c_h, c_s, c_l, α, γ, r, T, A in different label of percentage. For analysis we taking one parameter at a time and the other parameters are unchanged. On the basis of sensitivity analysis obtained in the above Table 2 it is observed that the model is highly sensitive to ordering cost (A) and the total replenishment time (T), moderately sensitive

Table 2. Optimal average cost for different parameter.

Parameter	Change (%)	t_1^*	Change $t_1^*(\%)$	$TC(t_1^*)$	Change $TC(t_1^*)$
C_h	−20	0.653	6.352	30.775	−1.768
	−10	0.633	3.094	31.064	−0.846
	10	0.596	−2.932	31.570	0.769
	20	0.580	−5.537	31.792	1.478
C_d	−20	0.621	1.140	31.270	−0.188
	−10	0.618	0.651	31.300	−0.093
	10	0.611	−0.489	31.357	0.089
	20	0.607	−1.140	31.384	0.176
C_s	−20	0.608	−0.977	31.197	−0.421
	−10	0.611	−0.489	31.263	−0.211
	10	0.617	0.489	31.393	0.204
	20	0.620	0.977	31.458	0.412
C_l	−20	0.572	−6.840	30.703	−1.998
	−10	0.594	−3.257	31.029	−0.958
	10	0.632	2.932	31.604	0.878
	20	0.648	5.537	31.858	1.689
α	−20	0.627	2.117	31.205	−0.396
	−10	0.621	1.140	31.266	−0.201
	10	0.607	−1.140	31.393	0.204
	20	0.601	−2.117	31.459	0.415
γ	−20	0.581	−5.375	30.783	−1.743
	−10	0.598	−2.606	31.065	−0.843
	10	0.629	2.443	31.576	0.788
	20	0.643	4.723	31.808	1.529
r	−15	0.562	−8.469	32.388	3.380
	−10	0.581	−5.375	31.993	2.119
	10	0.641	4.397	30.808	−1.663
	15	0.652	6.189	30.592	−2.352
T	−20	0.514	−16.287	35.328	12.765
	−10	0.565	−7.980	32.924	5.091
	10	0.661	7.655	30.346	−3.138
	20	0.707	15.147	29.843	−4.743
A	−20	0	0	26.329	−15.960
	−10	0	0	28.829	−7.980
	10	0	0	33.829	7.980
	20	0	0	36.329	15.960

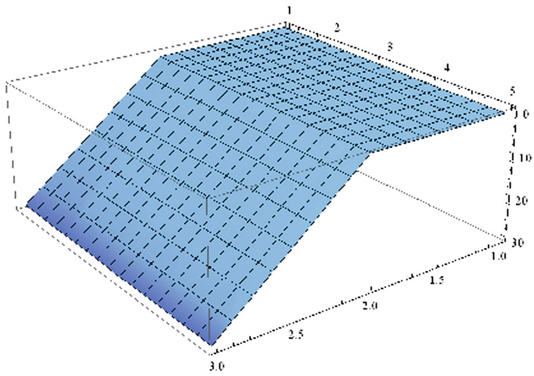

Fig. 3. Graph of total profit vs degree of optimism vs time

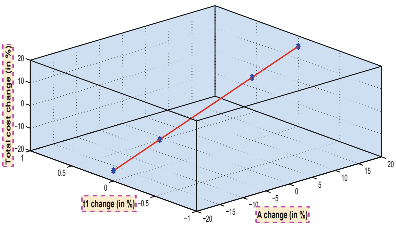

Fig. 4. Effect of optimal total cost for ordering cost 'A' parameter

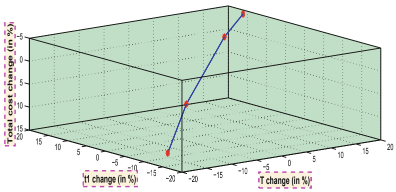

Fig. 5. Effect of optimal total cost for total replenishment time 'T'

to holding cost (c_h), lost in sales cost (c_l), reliability (r), backlogged parameter (γ), shape parameter (a) and less sensitive to deterioration cost (c_d), shortage cost (c_s) and shape parameter (α).

The model is highly sensitive to ordering cost (A) i.e., if there is increase ordering cost (A), the total cost increases, which means if we order items at higher cost obviously the total costing increases which is true in reality. We have noted that in our model that the change in the ordering cost does not affect the inventory time which is obvious. The model is also highly sensitive to the replenishment time (T) i.e., as the replenishment time increases the total cost decreases. Also as the replenishment time increases the inventory retaining time increases. This is realistic because if we have short replenishment time i.e., for small cycle, we have to order less items and hence we have to order new items frequently as the items will finish. Also we have to upgrade machine quality to produce better quality of items and hence the total cost also increases. Figure 4 gives a graphical representation of the percentage change in total cost with change of time and change of ordering cost parameters (A) and from Fig. 5 gives a graphical representation of the percentage change in total cost with change

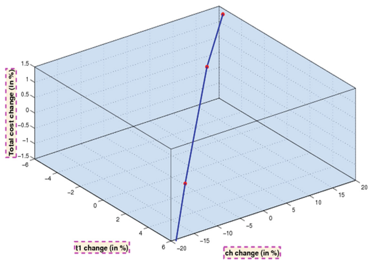

Fig. 6. Effect of optimal total cost for holding cost 'Ch' parameter

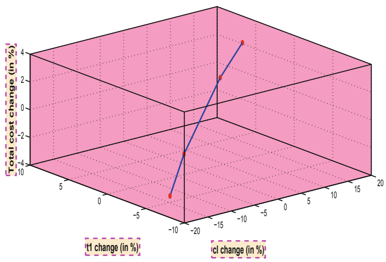

Fig. 7. Effect of optimal total cost for lost sale 'Cl' parameter

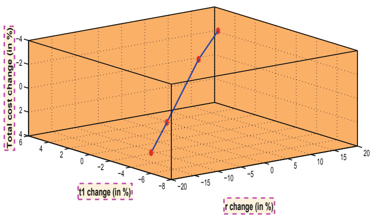

Fig. 8. Effect of optimal total cost for reliability 'r' parameter

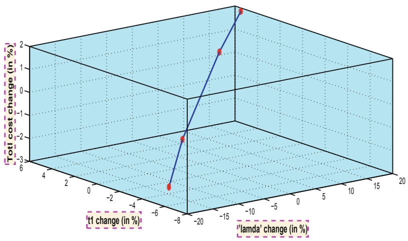

Fig. 9. Effect of optimal total cost for shape parameter 'γ'

of time and change of total replenishment time (T). Hence from Figs. 4 and 5 it is observed that ordering cost and replenishment time are highly sensitive parameters.

The model is moderately sensitive to holding cost (c_h) i.e., if there is increase the holding cost the total cost increases and t_1^* will decreases. It means if we hold any item for longer time with high value of holding cost then obviously the total cost will increase whereas our aim is to reduce the total cost. Thus with the increase in holding cost the replenishment time decreases i.e., we have to hold the item for less time, in order to obtain the optimum (i.e., minimum) total cost. The model is moderately sensitive to the penalty cost for lost in sales cost (c_l) and backlogged parameter (γ) . i.e. if there is increase the penalty cost and backlogged parameter, the total cost increases and t_1^* will also increases. If there is high penalty cost the retailer has to pay more for the lost in sale. Also the items which are backlogged are actually backordered and those demand has to be fulfilled in the next replenishment cycle thus the total cost increases. It is also observed that as the penalty cost and backlogged factor increases the optimal replenishment time increases. Also as reliability (r) increases the total cost decreases. This is obvious because as the reliability of the product increases, its demand in the market will also increases and hence the company will gain more profit hence the total cost decreases. Figure 6 gives a graphical representation of the percentage change in total cost with change of time and change of holding cost (c_h). Figure 7 gives a graphical representation of the percentage change in total cost with change of time and change of lost sale cost parameter (c_l). Figure 8 gives a graphical representation of the percentage change in total cost with change of time and change of reliability parameter (r) and Fig. 9 gives a graphical representation of the percentage change in total cost with change of time and change of total backlogged parameter (γ). Hence from the above Figs. 6, 7, 8, 9 it is observed that out of all the above parameters, reliability and penalty cost for the lost in sales are sensitive.

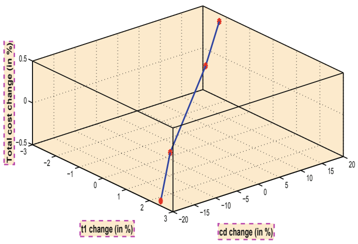

Fig. 10. Effect of optimal total cost for deterioration cost 'Cd' parameter

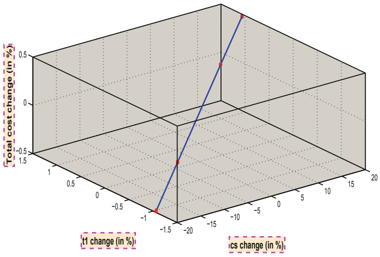

Fig. 11. Effect of optimal total cost for shortage cost 'Cs' parameter

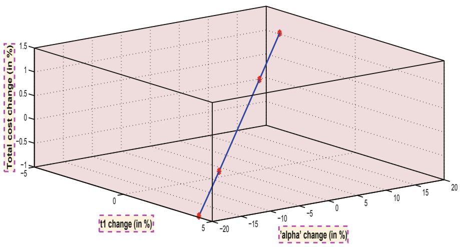

Fig. 12. Effect of optimal total cost for shape parameter 'α'

The model is less sensitive to deterioration cost (c_d), shortage cost (c_s) and shape parameter (α). As both the deterioration cost (c_d) and shortage cost (c_s) increases the total cost also increases which is obvious. Also if the shape parameter α increases then the deterioration function θ increases as well as the deterioration cost is increases and so the total cost increases. Figure 10 gives a graphical representation of the percentage change in total cost with change of time and change of deterioration cost parameters (c_d) while Fig. 11 observe the percentage change in total cost with change of time and change of shortage cost parameters (c_s) and lastly Fig. 12 observe the percentage change in total cost with change of time and change of shape parameter (α).

8.1 Managerial Impact

Managerial implication is an important part of any business organisation. It helps retailer to make decisions in different replenishment policies based on the minimum inventory total cost. A right decision strategy would benefit both the manufacturer and the retailer in the long run. The study is particularly useful for the inventory systems where the manufacturers and their retailers form a strategic alliance with a mutually beneficial objective. The proposed model can be used in any type of durable product such as photographic film, electronic components, radioactive materials, automobiles, furniture, jewelry, sporting goods, toy makers, small tool manufacturers etc. and semi durable product such as apparels, preserved eatables, foot wears, home furnishing etc.

9 Concluding Remarks

In this study, an inventory model has presented to determine the optimal replenishment time for a time and reliability dependent demand and deterioration along with shortages. This paper has presented a part of the shortages are backlogged and part of them are lost in sales and incorporates some realistic feature such as time and reliability dependent deterioration, which is a natural phenomenon of goods because as time pass the items will deteriorate but the deterioration rate of highly reliable items are less. We have considered two cases one when the parameters are crisp value and second when the parameters are fuzzy in nature. On calculating the above we have observed that the total cost in the crisp case is more than that of the fuzzy case. We have obtained a condition to optimize our model. Hence our model will provide a new managerial insight that helps a manufacturing system/industry to gain the profit at optimal level. A numerical example is provided in support of the analytical form of the proposed model and sensitivity analysis is also performed. In our model effect of reliability, cycle length and the ordering cost are highly sensitive on optimal total cost. A future study should further incorporate the proposed model into more realistic assumptions, such as probabilistic demand, fuzzy demand, inflation, the demand which depends on the advertisement of the items, availability of current stock, etc.

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Attribute Reduction of Incomplete Information Systems: An Intuitionistic Fuzzy Rough Set Approach

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Abstract. Nowadays, fast expansion of data processing tools leads to increase in databases in terms of objects as well as attributes in different fields like image processing, pattern recognition and risk prediction in management. Attribute reduction is a process of selecting those attributes that are mutually sufficient and individually necessary for retaining basic property of the given information system. In this paper, we introduce a novel approach for attribute reduction of an incomplete information system based on intuitionistic fuzzy rough set theory. We define an intuitionistic fuzzy tolerance relation between two objects and calculate rough approximations of an incomplete information space by using tolerance classes of each object. The degree of dependency method is used for calculating reduct set of an incomplete information system in order to handle noise and irrelevant data. An algorithm is presented for better understanding of the proposed approach and is applied to an incomplete information system. Finally, we compare proposed approach with an existing approach for attribute reduction of an incomplete information system through an example.

Keywords: Incomplete information system · Set-valued data · Attribute reduction · Tolerance relation · Degree of dependency

1 Introduction

Due to continuous generation of data from various fields like census information, bioinformatics, social media, etc., a large number of objects as well as attributes are stored in different databases. Knowledge extraction from such data deluge is the most difficult task due to the abundance of redundant and irrelevant data. In order to decrease processing time and to enhance the classification accuracy, attribute reduction or feature selection [1–5] plays a very important role by eliminating redundant and irrelevant attributes. Attribute reduction is the process of selecting those attributes of a given information system, which contains almost all information as the original information system to reduce the classification time, cost, complexity and overfitting.

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Pawlak's rough set approximations established a whole new path in the field of knowledge discovery [6, 7]. The key benefit of Rough set theory (RST) is that it uses only internal information, not prior model conventions. In rough sets, several discretization methods are required for handling real-valued information systems. Due to this discretization process, the problem of intrinsic error and hence information loss become a main issue while computing reduct of a real-valued information system. Hybridization of rough set with fuzzy set (proposed by Zadeh [8, 9]) by Dubois and Prade [10, 11] provide an important tool in order to tackle uncertainty available in real-valued datasets. Fuzzy rough set based concepts can easily handle uncertainty and vagueness available in the datasets.

Apart from all these benefits, fuzzy set theory also has some limitations, for example: voting problem. Suppose 15 voters are voting for an item, in which 8 voters are in "favour", 5 voters are "against" and rest 2 "abstain". Since fuzzy sets only considers membership grade, hence with fuzzy set one can only handle the first case, i.e., "favour". To handle other two scenarios, we should add a non-membership degree for "against" and hesitancy degree for "abstain". Hence, some generalizations of fuzzy sets are needed in order to tackle other uncertainties. Intuitionistic fuzzy set (proposed by Atanassov [12, 13]) is an extension of fuzzy set which is used to deal with uncertainty having membership, non-membership and hesitancy degrees simultaneously. Therefore, in comparison with fuzzy set based approaches, intuitionistic fuzzy set based approaches can handle uncertainty in a much efficient manner.

In many real-world problems, due to ambiguity and incompleteness, lots of missing data existed in the information system generated from various sources. An incomplete information system can be converted into a set-valued information system by replacing all missing values with the set of all possible values of each attribute [14, 15]. In set-valued information systems, the attribute set may vary with time as new information is arrived. It is used to represent the inexact and lost information in a given dataset. Wu proposed a concept for attribute reduction of incomplete decision systems based on evidence theory [16]. The concept of knowledge entropy and feature selection in incomplete decision systems are presented by Xu and Sun [17]. Kryszkiewicz defined a rough set approach for rule generations in incomplete information systems [18, 19]. For rule acquisition in incomplete information systems, the technique of maximal consistent block is given by Leung and Li [20]. Dai and Xu studied approximations as well as uncertainty measures in incomplete information systems [21]. A rough set approach based on a neighborhood system in incomplete information system is presented by Yang et al. [22]. Yang et al. proposed the concept of multi granulation rough sets in an incomplete information system [23]. Dai and Tian presented an approach for attribute reduction of an incomplete information system by converting it into a set-valued data using fuzzy rough set theory [24].

Liu et al. proposed an incremental approach based on rough set theory in dynamic incomplete information systems [25]. Focus on solving the problem of attribute reduction of incomplete systems with dynamically varying attribute set is carried out by Shu and Shen [26]. Zhao and Qin introduced a neighborhood-tolerance relation and applied it on attribute reduction of incomplete data with mixed categorical and numerical features [27]. Zeng et al. presented a new influence-guided group decision making model based on the assumptions that experts influence each other [28]. The mutual information criterion is presented in order to evaluate candidate features in an incomplete information system by Qian and Shu, which considers the redundancy between selected attributes [29]. Zhang et al. proposed three different parallel matrix-based approaches to process incomplete data for the computation of approximations [30].

All above mentioned approaches are based on statistical, rough set and fuzzy rough set techniques. None of the researchers have considered an intuitionistic fuzzy set assisted rough approximation based technique for attribute reduction of an incomplete information system using degree of dependency approach. Due to availability of an extra information in intuitionistic fuzzy set, i.e. non-membership grade, proposed approach handles uncertainty in a much efficient manner as compared to statistical, rough set and fuzzy rough set based approaches. In this paper, we introduce a novel approach for attribute selection in incomplete information system based on intuitionistic fuzzy rough set theory. We transform incomplete information system into set-valued information system and define an intuitionistic fuzzy tolerance relation between two objects. The similarity of two objects in transformed set-valued information system is considered up to a threshold value using two parameters in order to avoid misclassification and perturbation. For attribute selection of a given information system, the degree of dependency approach is used in order to find smallest reduct set.

The rest of the paper is structured as follows. In Sect. 2, basic definitions related to intuitionistic fuzzy set, incomplete information system and set-valued information system are given. The proposed concept of attribute reduction in an incomplete information system is presented and thoroughly studied in Sect. 3. In Sect. 4, an algorithm for proposed approach is presented for demonstration. Illustrative examples are given for better understanding of the proposed model in Sect. 5. In Sect. 6, a comparison between proposed model and existing model is performed. Section 7 concludes our work.

2 Preliminaries

In this section, we describe some basic definitions and examples of incomplete as well as set-valued information systems.

Definition 2.1 [12]: Let $\boldsymbol{U}$ be a finite non-empty set. A set $\boldsymbol{A}$ on the universe $\boldsymbol{U}$ of the form $\boldsymbol{A} = \{<x, \boldsymbol{\mu_A}(\boldsymbol{x}), \boldsymbol{\vartheta_A}(\boldsymbol{x})>|x \in U\}$ is said to be an intuitionistic fuzzy (IF) set, where $\boldsymbol{\mu_A} : \boldsymbol{U} \to [\mathbf{0}, \mathbf{1}]$ and $\boldsymbol{\vartheta_A} : \boldsymbol{U} \to [\mathbf{0}, \mathbf{1}]$ satisfy the condition $\mathbf{0} \leq \boldsymbol{\mu_A}(\boldsymbol{x}) + \boldsymbol{\vartheta_A}(\boldsymbol{x}) \leq 1$ for all $\boldsymbol{x}$ in $\boldsymbol{U}$. $\boldsymbol{\mu_A}(\boldsymbol{x})$ and $\boldsymbol{\vartheta_A}(\boldsymbol{x})$ are the membership degree and non-membership degree of the element $\boldsymbol{x}$ in $\boldsymbol{A}$, respectively and $\boldsymbol{\pi_A}(\boldsymbol{x}) = \mathbf{1} - \boldsymbol{\mu_A}(\boldsymbol{x}) - \boldsymbol{\vartheta_A}(\boldsymbol{x})$ is the degree of hesitancy (or non-determinacy) of the element $\boldsymbol{x}$ in IF set $\boldsymbol{A}$.

Any ordinary fuzzy set $\boldsymbol{A} = \{<x, \boldsymbol{\mu_A}(\boldsymbol{x})>|x \in U\}$ can be characterized by an IF set having the form $\{<x, \boldsymbol{\mu_A}(\boldsymbol{x}), \mathbf{1} - \boldsymbol{\mu_A}(\boldsymbol{x})>|\boldsymbol{x} \in \boldsymbol{U}\}$. Thus, every fuzzy set is an IF set.

Definition 2.2 [31]: An IF binary relation $<\mu_A(x_i, x_j), \vartheta_A(x_i, x_j)>$ between objects $x_i, x_j \in U$ is said to be an IF tolerance relation if it is reflexive (i.e. $\mu_A(x_i, x_i) = 1$ **and** $\vartheta_A(x_i, x_i) = 0, \forall x_i \in X$) and symmetric (i.e. $\mu_A(x_i, x_j) = \mu_A(x_j, x_i)$ **and** $\vartheta_A(x_i, x_j) = \vartheta_A(x_j, x_i)$, $\forall x_i, x_j \in X$).

Definition 2.3 [32]: A quadruple $IS = (U, AT, V, h)$ is called an Information System, where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects, called the universe of discourse, $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set of attributes. $V = \bigcup_{a \in AT} V_a$, where V_a is the set of attribute values associated with each attribute $a \in AT$ and $h : U \times AT \rightarrow V$ is an information function that assigns particular values to the objects against attribute set such that $\forall a \in AT, \forall u \in U, h(u, a) \in V_a$.

An information system $IS = (U, AT, V, h)$ is said to be a decision system if $AT = C \cup D$ where C is a non-empty finite set of conditional attributes and D is a non-empty collection of decision attributes with $C \cap D = \emptyset$.

Definition 2.4 [14]: In a decision system $IS = (U, C \cup D, V, h)$ if V_a encloses null values i.e. V_a has missing attribute values of an object for at least one attribute $a \in AT$, then IS is called an incomplete decision system. Table 1 is an example of an incomplete decision system, in which some attribute values of objects are missing.

Due to ambiguity and incompleteness, lots of missing data existed in the information systems associated with many real-world application problems. All missing values in any information system can be replaced by the set of all possible values for any attribute.

Definition 2.5 [14]: In a decision system, if each attribute has a single entity as attribute value, then it is called single-valued decision system, otherwise it is known as set-valued decision system. Set-valued decision system is a generalization of the single-valued decision system, in which an object can have more than one attribute values. The transformed incomplete decision system as given in Table 1 into a set-valued decision system is given in Table 2.

Table 1. Incomplete decision system

U	c_1	c_2	c_3	c_4	D
x_1	0	1	1.0	10	1
x_2	1	2	*	15	1
x_3	*	*	*	20	2
x_4	0	3	2.0	15	2
x_5	*	*	1.5	25	1
x_6	1	1	2.5	20	1

Table 2. Set-valued decision system obtained from Table 1

U	c_1	c_2	c_3	c_4	D
x_1	0	1	1.0	10	1
x_2	1	2	{1.0, 1.5, 2.0, 2.5}	15	1
x_3	{0, 1}	{1, 2, 3}	{1.0, 1.5, 2.0, 2.5}	20	2
x_4	0	3	2.0	15	2
x_5	{0, 1}	{1, 2, 3}	1.5	25	1
x_6	1	1	2.5	20	1

3 Proposed Work

In this section, we present an IF rough set model for attribute reduction of an incomplete information system by using degree of dependency approach. We divide this section into two subsections:

3.1 IF tolerance relation and associated rough set

In this subsection, we first define an IF relation between two objects of a set-valued information system. We define an IF tolerance relation using thresholds on membership and non-membership grades in order to avoid some misclassification and perturbation. Lower and upper approximations are defined and basic properties of rough set are investigated.

Definition 3.1: Let $\boldsymbol{IS} = (\boldsymbol{U}, \boldsymbol{AT}, \boldsymbol{V}, \boldsymbol{h})$ be a set-valued information system and $\boldsymbol{A} \subseteq \boldsymbol{AT}$, then we define an IF relation $<\boldsymbol{\mu_A}, \boldsymbol{\vartheta_A}>$ as:

$$\begin{cases} \mu_A\left(x_i, x_j\right) = min_{a \in A} S_a\left(x_i, x_j\right) \\ \vartheta_A\left(x_i, x_j\right) = 1 - max_{a \in A} S_a\left(x_i, x_j\right) \end{cases} \tag{1}$$

where, $x_i, x_j \in U, i, j \in [1, 2, \ldots, |U|]$ and $S_a\left(x_i, x_j\right)$ is the similarity between x_i, x_j is given by

$$S_a\left(x_i, x_j\right) = \frac{2\left|a(x_i) \cap a\left(x_j\right)\right|}{|a(x_i)| + \left|a\left(x_j\right)\right|} \tag{2}$$

Example 3.1: For $A = \{c_3\}$, $S_A(x_1, x_2) = \frac{2|\{1.0\} \cap \{1.0, 1.5, 2.0, 2.5\}|}{|\{1.0\}| + |\{1.0, 1.5, 2.0, 2.5\}|} = \frac{2}{5} = 0.4$
Hence, $\mu_A(x_1, x_2) = 0.4$, $\vartheta_A(x_1, x_2) = 0.6$
For $A = \{c_2, c_3\}$, $\mu_A(x_1, x_2) = \min\left\{S_{c_2}(x_1, x_2), S_{c_3}(x_1, x_2)\right\} = \min\{0, 0.4\} = 0$ and $\vartheta_A(x_1, x_2) = 1 - \max\left\{S_{c_2}(x_1, x_2), S_{c_3}(x_1, x_2)\right\} = 1 - \max\{0, 0.4\} = 0.6$.

For enlargement of IF positive region, we ignore some misclassification and perturbation by using a threshold value on membership grade and another on non-membership grade of IF relation. Thus, ability of model in knowledge representation becomes much stronger with respect to misclassification.

So we define a new kind of binary relation using two threshold values $\alpha, \beta \in (0, 1)$ as follows:

$$R_A^{\alpha,\beta} = \{(x_i, x_j) | \mu_A(x_i, x_j) \geq \alpha,\ \vartheta_A(x_i, x_j) \leq \beta\} \tag{3}$$

where, α, β are similarity thresholds and are user-oriented. For insertion of objects within tolerance classes, these parameters provide similarity upto a required level.

Lemma 3.1: $R_A^{\alpha,\beta}$ is an IF tolerance relation.

Proof

(i) **Reflexive**

$$\mu_A(x_i, x_i) = min_{a\in A}\frac{2|a(x_i) \cap a(x_i)|}{|a(x_i)| + |a(x_i)|} = 1 \geq \alpha$$

$\vartheta_A(x_i, x_i) = 1 - max_{a\in A}\frac{2|a(x_i)\cap a(x_i)|}{|a(x_i)|+|a(x_i)|} = 1 - 1 = 0 \leq \beta$, Since $0 \leq \alpha, \beta \leq 1$
Hence, $(x_i, x_i) \in R_A^{\alpha,\beta}$, $\forall x_i \in U$ implies that $R_A^{\alpha,\beta}$ is reflexive.

(ii) **Symmetric**

Let $(x_i, x_j) \in R_A^{\alpha,\beta} \Rightarrow \mu_A(x_i, x_j) \geq \alpha, \vartheta_A(x_i, x_j) \leq \beta$

$$min_{a\in A}S_a(x_i, x_j) \geq \alpha, 1 - max_{a\in A}S_a(x_i, x_j) \leq \beta$$

Now, $min_{a\in A}S_a(x_j, x_i) \geq \alpha, 1 - max_{a\in A}S_a(x_j, x_i) \leq \beta$

$$\Rightarrow \mu_A(x_j, x_i) \geq \alpha,\ \vartheta_A(x_j, x_i) \leq \beta$$

Hence, $(x_j, x_i) \in R_A^{\alpha,\beta}$ implies that $R_A^{\alpha,\beta}$ is symmetric.
Therefore, $R_A^{\alpha,\beta}$ is an IF tolerance relation.

Now, tolerance class of an object x_i with respect to attribute set $A \subseteq AT$ can be defined as follows:

$$\left[R_A^{\alpha,\beta}\right](x_i) = \{x_j \in U | x_i R_A^{\alpha,\beta} x_j\} \tag{4}$$

Then, we propose IF lower and upper approximations of any object set $X \subseteq U$ as:

$$\underline{R}_A^{\alpha,\beta}X = \{x_i \in U | \left[R_A^{\alpha,\beta}\right](x_i) \subset X\} \tag{5}$$

$$\overline{R}_A^{\alpha,\beta}X = \left\{x_i \in U | \left[R_A^{\alpha,\beta}\right](x_i) \cap X \neq \emptyset\right\} \tag{6}$$

The tuple $<\underline{R}_A^{\alpha,\beta}X,\ \overline{R}_A^{\alpha,\beta}X>$ is called an intuitionistic fuzzy set based rough set.

Theorem 3.1. Let $(U, C \cup D, V, h)$ be a decision system with $X \subseteq U$, $\alpha, \beta \in (0, 1)$. Let $A_1 \subseteq A_2 \subseteq C$, then

(i) $\underline{R}_{A_1}^{\alpha,\beta} X \subseteq \underline{R}_{A_2}^{\alpha,\beta} X$,
(ii) $\overline{R}_{A_2}^{\alpha,\beta} X \subseteq \overline{R}_{A_1}^{\alpha,\beta} X$

Proof

(i) Let $x \in \underline{R}_{A_1}^{\alpha,\beta} X \Rightarrow \left[R_{A_1}^{\alpha,\beta}\right](x) \subseteq X$

Since $A_1 \subseteq A_2 \Rightarrow \left[R_{A_2}^{\alpha,\beta}\right](x) \subseteq \left[R_{A_1}^{\alpha,\beta}\right](x) \Rightarrow \left[R_{A_2}^{\alpha,\beta}\right](x) \subseteq X$
Therefore, $x \in \underline{R}_{A_2}^{\alpha,\beta} X$. Hence, $\underline{R}_{A_1}^{\alpha,\beta} X \subseteq \underline{R}_{A_2}^{\alpha,\beta} X$.

(ii) Let $x \in \overline{R}_{A_2}^{\alpha,\beta} X \Rightarrow \left[R_{A_2}^{\alpha,\beta}\right](x) \cap X \neq \emptyset$

Since $A_1 \subseteq A_2 \Rightarrow \left[R_{A_2}^{\alpha,\beta}\right](x) \subseteq \left[R_{A_1}^{\alpha,\beta}\right](x) \Rightarrow \left[R_{A_1}^{\alpha,\beta}\right](x) \cap X \neq \emptyset$
Therefore, $x \in \overline{R}_{A_1}^{\alpha,\beta} X$, hence, $\overline{R}_{A_2}^{\alpha,\beta} X \subseteq \overline{R}_{A_1}^{\alpha,\beta} X$.

Theorem 3.2. Let $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, then

(i) $\underline{R}_{A}^{\alpha_1,\beta_1} X \subseteq \underline{R}_{A}^{\alpha_2,\beta_2} X$,
(ii) $\overline{R}_{A}^{\alpha_2,\beta_2} X \subseteq \overline{R}_{A}^{\alpha_1,\beta_1} X$

Proof

(i) Let $x \in \underline{R}_{A}^{\alpha_1,\beta_1} X \Rightarrow \left[R_{A}^{\alpha_1,\beta_1}\right](x) \subseteq X$

Since, $\left[R_{A}^{\alpha_2,\beta_2}\right](x) = \{y \in U | \mu_A(x, y) \geq \alpha_2,\ \vartheta_A(x, y) \leq \beta_2\}$
Also, $\{y \in U | \mu_A(x, y) \geq \alpha_1,\ \vartheta_A(x, y) \leq \beta_1\}$ (Since, $\alpha_1 \leq \alpha_2, \beta_1 \geq \beta_2$)

$$\Rightarrow \left[R_{A}^{\alpha_2,\beta_2}\right](x) \subseteq \left[R_{A}^{\alpha_1,\beta_1}\right](x) \subseteq \mathrm{X} \tag{7}$$

$\Rightarrow x \in \underline{R}_{A}^{\alpha_2,\beta_2} X$. Therefore, $\underline{R}_{A}^{\alpha_1,\beta_1} X \subseteq \underline{R}_{A}^{\alpha_2,\beta_2} X$.

(ii) Let $x \in \overline{R}_{A}^{\alpha_2,\beta_2} X \Rightarrow \left[R_{A}^{\alpha_2,\beta_2}\right](x) \cap X \neq \emptyset$.

Since $\left[R_{A}^{\alpha_2,\beta_2}\right](x) \subseteq \left[R_{A}^{\alpha_1,\beta_1}\right](x)$ (by Eq. 7)

$$\Rightarrow \left[R_{A}^{\alpha_1,\beta_1}\right](x) \cap X \neq \emptyset \Rightarrow x \in \overline{R}_{A}^{\alpha_1,\beta_1} X$$

Therefore, $\overline{R}_{A}^{\alpha_2,\beta_2} X \subseteq \overline{R}_{A}^{\alpha_1,\beta_1} X$.

3.2 IF rough set based attribute reduction for set-valued information systems

In this subsection, we propose a degree of dependency based approach for attribute reduction of a set-valued decision system based on rough approximations as defined in Sect. 3.1.

Based on IF rough set as defined in Eqs. (5) and (6), positive region of a set of decision attributes D over a set of conditional attributes A can be defined as:

$$POS_A^{\alpha,\beta}(D) = \bigcup_{X \in U/D} \underline{R}_A^{\alpha,\beta} X \tag{8}$$

where, $U/D =$ collection of classes having objects with same decision values.

Now, using the definition of IF positive region, we compute the degree of dependency of D over A as:

$$\Gamma_A(D) = \frac{\left|POS_A^{\alpha,\beta}(D)\right|}{|U|} \tag{9}$$

where, $|.| =$ cardinality of a set and $\Gamma_A(D) \in [0, 1]$.

Definition 3.2: A subset A of the conditional attribute set C is said to be a reduct of a decision system if

$$\begin{aligned} &\Gamma_A(D) = \Gamma_C(D) \\ &\Gamma_{A-\{c_i\}}(D) < \Gamma_A(D),\ \forall c_i \in A \end{aligned} \tag{10}$$

First calculate the degree of dependency of decision attribute over each conditional attribute. Then select that conditional attribute for which degree of dependency is highest. Now, add other conditional attributes one by one to the obtained conditional attribute and calculate corresponding degree of dependencies for such pairs. Repeat this process till there is no increase in the degree of dependency of updated subsets of conditional attributes than previous one. So obtained updated set will be the reduct set of given incomplete decision system.

4 An Algorithm for IF Rough Set Based Attribute Selection of an Incomplete Information System

In this section, an algorithm for attribute selection of incomplete information systems is presented by using degree of dependency approach. The proposed algorithm is given as follows:

Step 1. Take an incomplete information system.
Step 2. Convert it into set-valued information system by replacing all missing values with the set of all possible values for each conditional attribute.
Step 3. Calculate IF tolerance relation between objects with respect to the set of conditional attributes.
Step 4. With the help of IF tolerance classes, compute lower approximations of the set of objects having same decision value.

Step 5. Calculate positive region and finally, the degree of dependency of D over each conditional attribute.
Step 6. Select that conditional attribute for which degree of dependency is highest (In case of more than one conditional attributes having the highest and same degree of dependencies, any one can be chosen randomly). That will be the first candidate for reduct set.
Step 7. Add other conditional attributes to the newly obtained candidate of reduct set and further calculate the degree of dependencies of such coupled conditional attributes.
Step 8. Repeat Steps 6 and 7. If there is no increase in the degree of dependency of updated subsets of conditional attributes than previous one, the process will terminate. The updated set will be the reduct set.

The main advantage of our algorithm is that it provides a close-to-minimal reduct set of an incomplete decision system without checking every possible subsets of set of conditional attributes.

5 Illustrative Example

In this section, the proposed algorithm is applied to an example decision system to illustrate our approach.

Example 5.1. An incomplete decision system is given in Table 1 in which some conditional attribute values are missing. The system is converted into a set-valued decision system by replacing the missing values with the set of all possible attribute values for each conditional attribute, given in Table 2. In the decision system four conditional attributes c_1, c_2, c_3 and c_4 of the object set U along with a decision attribute D are present.

Now, we calculate $\Gamma_{\{c_1\}}(D)$ as follows:
Take $\alpha = 0.7, \beta = 0.2$.
In Table 3, similarity degree between objects with respect to each attribute is given by using Eq. (2). For a conditional attribute c_k, the membership part of relation is $\mu_{c_k}(x_i, x_j) = S_{c_k}(x_i, x_j)$ and the non-membership part is $\vartheta_{c_k}(x_i, x_j) = 1 - \mu_{c_k}(x_i, x_j)$ with $k = 1, 2, 3, 4$ and $i, j = 1, 2, \ldots, 6$.
Tolerance classes of each object is calculated by Eq. (4),

$$\left[R_{c_1}^{\alpha,\beta}\right](u_1) = \{u_1, u_4\}, \left[R_{c_1}^{\alpha,\beta}\right](u_2) = \{u_2, u_6\}, \left[R_{c_1}^{\alpha,\beta}\right](u_3) = \{u_3, u_5\},$$
$$\left[R_{c_1}^{\alpha,\beta}\right](u_4) = \{u_1, u_4\}, \left[R_{c_1}^{\alpha,\beta}\right](u_5) = \{u_3, u_5\}, \left[R_{c_1}^{\alpha,\beta}\right](u_6) = \{u_2, u_6\}$$

Here, $U/D = \{D_1, D_2\}$, $D_1 = \{u_1, u_2, u_5, u_6\}$, $D_2 = \{u_3, u_4\}$.
Lower approximations of D_1 and D_2 with respect to attribute c_1 are computed using Eq. (5),

$$\underline{R}_{c_1}^{\alpha,\beta} D_1 = \{u_2, u_6\}, \underline{R}_{c_1}^{\alpha,\beta} D_2 = \emptyset$$

Now, the positive region of decision attribute D over c_1 is calculated by Eq. (8),

$$POS_{c_1}^{\alpha,\beta}(D) = \bigcup_{X \in U/D} (\underline{R}_{c_1}^{\alpha,\beta} X) = \left(\underline{R}_{c_1}^{\alpha,\beta} D_1\right) \cup \left(\underline{R}_{c_1}^{\alpha,\beta} D_2\right) = \{u_2, u_6\} \cup \emptyset = \{u_2, u_6\}$$

Table 3. Table for $S_{c_k}(x_i, x_j)$ with $k = 1, 2, 3, 4$ and $i, j = 1, 2, \ldots, 6$.

$S_{c_1}(x_i, x_j)$						$S_{c_2}(x_i, x_j)$						$S_{c_3}(x_i, x_j)$						$S_{c_4}(x_i, x_j)$					
1	0	0.67	1	0.67	0	1	0	0.5	0	0.5	1	1	0.4	0.4	0	0	0	1	0	0	0	0	1
0	1	0.67	0	0.67	1	0	1	0.5	0	0.5	0	0.4	1	1	0.4	0.4	0.4	0	1	0	1	0	0
0.67	0.67	1	0.67	1	0.67	0.5	0.5	1	0.5	1	0.5	0.4	1	1	0.4	0.4	0.4	0	0	1	0	0	0
1	0	0.67	1	0.67	0	0	0	0.5	1	0.5	0	0	0.4	0.4	1	0	0	0	1	0	1	0	0
0.67	0.67	1	0.67	1	0.67	0.5	0.5	1	0.5	1	0.5	0	0.4	0.4	0	1	0	0	0	0	0	1	0
0	1	0.67	0	0.67	1	1	0	0.5	0	0.5	1	0	0.4	0.4	0	0	1	1	0	0	0	0	1

After getting the positive region, the degree of dependency of D over c_1 is evaluated using Eq. (9),

$$\Gamma_{\{c_1\}}(D) = \frac{\left|POS_{c_1}^{\alpha,\beta}(D)\right|}{|U|} = \frac{2}{6} = 0.33$$

In the same manner, the degree of dependencies of D over other conditional attributes are calculated,

$$\Gamma_{\{c_2\}}(D) = \frac{4}{6} = 0.67, \quad \Gamma_{\{c_3\}}(D) = \frac{4}{6} = 0.67, \quad \Gamma_{\{c_4\}}(D) = \frac{4}{6} = 0.67$$

Since, the degree of dependencies of c_2, c_3 and c_4 are equal and highest. We can choose any of them as a first candidate of the reduct set. Taking c_2 as reduct candidate. We add other attributes to c_2. The new IF relations is calculated using Eq. (1) and the process of finding the new degree of dependencies of the attributes is again started.

IF similarity relation between objects for attribute set $\{c_1, c_2\}$ is given as

$$\left\langle \mu_{\{\boldsymbol{c}_1,\boldsymbol{c}_2\}}(\boldsymbol{x}_i,\boldsymbol{x}_j), \boldsymbol{\vartheta}_{\{\boldsymbol{c}_1,\boldsymbol{c}_2\}}(\boldsymbol{x}_i,\boldsymbol{x}_j)\right\rangle$$

$$= \begin{pmatrix} U & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & \langle 1,0\rangle & \langle 0,1\rangle & \langle 0.5,0.33\rangle & \langle 0,0\rangle & \langle 0.5,0.33\rangle & \langle 0,0\rangle \\ x_2 & \langle 0,1\rangle & \langle 1,0\rangle & \langle 0.5,0.33\rangle & \langle 0,1\rangle & \langle 0.5,0.33\rangle & \langle 0,0\rangle \\ x_3 & \langle 0.5,0.33\rangle & \langle 0.5,0.33\rangle & \langle 1,0\rangle & \langle 0.5,0.33\rangle & \langle 1,0\rangle & \langle 0.5,0.33\rangle \\ x_4 & \langle 0,0\rangle & \langle 0,1\rangle & \langle 0.5,0.33\rangle & \langle 1,0\rangle & \langle 0.5,0.33\rangle & \langle 0,1\rangle \\ x_5 & \langle 0.5,0.33\rangle & \langle 0.5,0.33\rangle & \langle 1,0\rangle & \langle 0.5,0.33\rangle & \langle 1,0\rangle & \langle 0.5,0.33\rangle \\ x_6 & \langle 0,0\rangle & \langle 0,0\rangle & \langle 0.5,0.33\rangle & \langle 0,1\rangle & \langle 0.5,0.33\rangle & \langle 1,0\rangle \end{pmatrix}$$

$$\left[R_{\{c_1,c_2\}}^{\alpha,\beta}\right](u_1) = \{u_1\}, \left[R_{\{c_1,c_2\}}^{\alpha,\beta}\right](u_2) = \{u_2\}, \left[R_{\{c_1,c_2\}}^{\alpha,\beta}\right](u_3) = \{u_3, u_5\},$$

$$\left[R_{\{c_1,c_2\}}^{\alpha,\beta}\right](u_4) = \{u_4\}, \left[R_{\{c_1,c_2\}}^{\alpha,\beta}\right](u_5) = \{u_3, u_5\}, \left[R_{\{c_1,c_2\}}^{\alpha,\beta}\right](u_6) = \{u_6\}$$

Now, $\underline{R}_{\{c_1,c_2\}}^{\alpha,\beta} D_1 = \{u_1, u_2, u_6\}$, $\underline{R}_{\{c_1,c_2\}}^{\alpha,\beta} D_2 = \{u_4\}$
and $POS_{\{c_1,c_2\}}^{\alpha,\beta}(D) = \{u_1, u_2, u_6\} \cup \{u_4,\} = \{u_1, u_2, u_4, u_6\}$
Therefore, $\Gamma_{\{c_1,c_2\}}(D) = \frac{4}{6} = 0.67$
Similarly, $\Gamma_{\{c_2,c_3\}}(D) = \frac{6}{6} = 1$, $\Gamma_{\{c_2,c_4\}}(D) = \frac{4}{6} = 0.67$.
Since, the degree of dependency can not exceed 1, hence $\{c_2, c_3\}$ is the reduct set of the incomplete decision system as given in Table 1.

6 Comparative Study

In this section, we compare proposed approach with a fuzzy-rough set based approach (Dai et al. [24]) on an incomplete decision system.

Example 6.1. Consider a set-valued decision system in which some conditional attributes are missing, is given in Table 4. Four conditional attributes c_1, c_2, c_3 and c_4 and a decision attribute D is contained in the system.

Table 4. Incomplete decision system

U	c_1	c_2	c_3	c_4	D
u_1	{1, 2, 3, 4}	{0, 1}	{1, 2}	0	1
u_2	{2, 3}	{2, 3}	{1}	1	1
u_3	{1, 2, 3, 4}	{1, 2}	{1, 2}	0	2
u_4	{2, 3, 4}	*	{0, 1)	*	1
u_5	{2, 4}	{0, 1, 2}	{0, 1}	*	2

Table 5. Set-valued decision system obtained from Table 4

U	c_1	c_2	c_3	c_4	D
x_1	{1, 2, 3, 4}	{0, 1}	{1, 2}	0	1
x_2	{2, 3}	{2, 3}	{1}	1	1
x_3	{1, 2, 3, 4}	{1, 2}	{1, 2}	0	2
x_4	{2, 3, 4}	{0, 1, 2, 3}	{0, 1)	{0, 1}	1
x_5	{2, 4}	{0, 1, 2}	{0, 1}	{0, 1}	2

After replacing missing values with the set of all possible values for each attribute, we get Table 5.

IF tolerance relation between objects is calculated and given in Table 6.

For $c_i, \vartheta_{c_i}\left(x_i, x_j\right) = 1 - \mu_{c_i}\left(x_i, x_j\right)$

We calculate the degree of dependency of D over c_1 as follows:

On taking $\alpha = 0.7$ and $\beta = 0.2$, tolerance classes are

$$\left[R_{c_1}^{\alpha,\beta}\right](u_1) = \{u_1, u_3, u_4\}, \left[R_{c_1}^{\alpha,\beta}\right](u_2) = \{u_2, u_4\}, \left[R_{c_1}^{\alpha,\beta}\right](u_3) = \{u_1, u_3, u_4\},$$
$$\left[R_{c_1}^{\alpha,\beta}\right](u_4) = \{u_1, u_2, u_3, u_4\}, \left[R_{c_1}^{\alpha,\beta}\right](u_5) = \{u_4, u_5\}$$

Here, $U/D = \{D_1, D_2\}$, $D_1 = \{u_1, u_2, u_4\}$, $D_2 = \{u_3, u_5\}$

Lower approximations of $X \in U/D$ are evaluated and given as,

$$\underline{R}_{c_1}^{\alpha,\beta} D_1 = \{u_2\},\ \underline{R}_{c_1}^{\alpha,\beta} D_2 = \emptyset$$

Using above lower approximations, we get the positive region of decision attribute D as follows:

$$POS_{c_1}^{\alpha,\beta}(D) = \{u_2\} \cup \emptyset = \{u_2\}$$

The degree of dependency of D w.r. t. c_1 is given as:

$$\Gamma_{\{c_1\}}(D) = \frac{\left|POS_{c_1}^{\alpha,\beta}(D)\right|}{|U|} = \frac{1}{5} = 0.2$$

Table 6. IF similarity relation

$\mu_{c_1}(x_i, x_j)$					$\mu_{c_2}(x_i, x_j)$					$\mu_{c_3}(x_i, x_j)$					$\mu_{c_4}(x_i, x_j)$				
1	0.67	1	0.86	0.67	1	0	0.5	0.67	0.8	1	0.67	1	0.5	0.5	1	0	1	0.67	0.67
0.67	1	0.67	0.8	0.5	0	1	0.5	0.67	0.4	0.67	1	0.67	0.67	0.67	0	1	0	0.67	0.67
1	0.67	1	0.86	0.67	0.5	0.5	1	0.67	0.8	1	0.67	1	0.5	0.5	1	0	1	0.67	0.67
0.86	0.8	0.86	1	0.8	0.67	0.67	0.67	1	0.86	0.5	0.67	0.5	1	1	0.67	0.67	0.67	1	1
0.67	0.5	0.67	0.8	1	0.8	0.4	0.8	0.87	1	0.5	0.67	0.5	1	1	0.67	0.67	0.67	1	1

Similarly, the degree of dependencies of D over other conditional attributes are:

$$\Gamma_{\{c_2\}}(D) = \frac{2}{5} = 0.4, \quad \Gamma_{\{c_3\}}(D) = \frac{1}{5} = 0.2, \quad \Gamma_{\{c_4\}}(D) = \frac{1}{5} = 0.2$$

Since, conditional attribute c_2 has the highest degree of dependency. Therefore c_2 is the first member of reduct set. Proceeding in the same manner as in Example 5.1, the attributes are combined to c_2 one by one and the corresponding degree of dependencies are calculated,

$$\Gamma_{\{c_1,c_2\}}(D) = \frac{3}{5} = 0.6, \Gamma_{\{c_2,c_3\}}(D) = \frac{3}{5} = 0.6, \ \Gamma_{\{c_2,c_4\}}(D) = \frac{3}{5} = 0.6$$

Let $\{c_1, c_2\}$ is the newly obtained member of reduct set. After further addition of attributes to $\{c_1, c_2\}$, we get

$$\Gamma_{\{c_1,c_2,c_3\}}(D) = \frac{3}{5} = 0.6, \Gamma_{\{c_1,c_2,c_4\}}(D) = \frac{3}{5} = 0.6$$

Since there is no increment in degree of dependencies, hence, $\{c_1, c_2\}$ or $\{c_1, c_3\}$ or $\{c_1, c_4\}$ is the reduct set of the incomplete decision system, given in Table 5.

If we apply the method of Discernibility matrix given by Dai et al. [24], we get the reduct set $\{c_1, c_2, c_3, c_4\}$.

Therefore, after applying the fuzzy-rough set approach proposed by Dai et al., based on discernibility function, we get all conditional attributes $\{c_1, c_2, c_3, c_4\}$ as the reduct of the Table 4. In this example, we did not get reduced dataset which contains the same information as a whole while on applying our proposed IFRS approach based on the degree of dependency method, we get $\{c_1, c_2\}$ as the reduct set. Thus, the proposed approach is better than the method given by Dai et al. in order to find the smallest reduct set.

7 Conclusion and Future Work

In this paper, a novel approach has been proposed for attribute reduction of an incomplete information system based on IF rough set concept. We have defined an IF tolerance relation and presented a method for calculating lower and upper approximations of a set. A method has been proposed to calculate positive region and degree of dependency. Furthermore, we have presented a greedy algorithm and one illustrative example for clear understanding of the proposed approach. A comparison has been made between the proposed approach and an existing approach and observed that our model performs better in order to find close-to-minimal reduct set.

In the future, we will investigate some robust models for attribute reduction of an incomplete information system to avoid misclassification and noise. So far, we have taken missing values in conditional attribute set, but we wish to extend our concept for information systems with missing decision attribute values.

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Cellular Automata Based Solution for Detecting Hardware Trojan in CMPs

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Abstract. Nowadays, Hardware Trojan threats have become inevitable due to the growing complexities of Integrated Circuits (ICs) as well as the current trend of Intellectual Property (IP) based hardware designs. An adversary can insert a Hardware Trojan during any of its life cycle phases-during the design, fabrication or even manufacturing phase. Once a Trojan is inserted into a system, it can cause an unwanted modification to system functionality which may degrade system performance or sometimes Trojans are implanted with the target to leak secret information. Once Trojans are implanted, it is hard to detect and impossible to remove it from the system as it is already fabricated into the chip. In this paper, we propose a stealthy Trojan model which affect the coherence mechanism of a Chip Multiprocessors' (CMPs) cache system by arbitrarily modifying the cache block state which in turn may leave the cache lines states as incoherent. We have evaluated the payload of such modelled Trojan and proposed a cellular automata (CA) based solution for detection of such Trojan.

Keywords: Hardware Trojan · Chip Multiprocessors · Cellular automata

1 Introduction

Due to growing complexities of IC chip design and shorting of time to market, IC chips are brought from trusted or even untrusted third party. This makes IC chip vulnerable to malicious modification of its sequential/combination circuit design, which is known as Hardware Trojan [3]. Hardware Trojan can be inserted by an adversary during its design, manufacturing or fabrication phase. The infected ICs which are brought from the third party can degrade system performance or even leak secret information from the system.

In this work, we consider a Trojan model for a Chip Multiprocessors' (CMPs) cache system. Once the Trojan is activated, it arbitrarily modifies the state of a cache line which may lead to an inconsistency in the states cache line and in turn affect cache coherence mechanism. This poses serious threats in ensuring data consistency in CMPs and motivates researchers to explore high speed detection logic for Trojan without a commitment of major computation cost.

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Conventionally in CMPs, the top level caches (L_1s) are kept coherent with the help of cache coherence protocols. Each L_1 cache miss generates the coherence messages. The all other L_1s of the system are updated to the valid data (cache line) states in accordance with the coherence messages. The cache coherence controller (CC) (Fig. 1) has the responsibility to ensure the coherency of shared data [8] in L_1 caches.

The presence of a Trojan in the computation logic of CC can lead to a major data inconsistency in CMPs. For example, the modification of a cache line state as Shared (S) from Modified (M) by the Trojan within the CC, effectively denies the issuance of invalidation message that may cause a serious damage to the system performance as well as its reliability. On the other hand, modification to 'M' state from 'S' results in unwanted message (invalidation) delivery that leads to huge power loss.

Therefore, maintaining coherency of shared data in CMPs is of utmost necessity. The schemes ensuring coherency in CMPs through frequent communication along the global wires, are reported in [1]. This communication affects the system performance as well as the energy usage [8]. Reliability and dependability, the two key issue in CMPs, are addressed by the researchers in [4]. The schemes ensuring coherency in CMPs with large number of cores are reported in [1,6,8]. These deal with the interactions between the on-chip interconnection network and the cache coherence protocol. In [8], the researchers have proposed a verification logic that can dynamically detect wrong state computation in coherence controller (CC) logic in a snoop based system. However, a very few of the researchers have addressed the issue of coherence state obfuscation by a stealthy Trojan in a CMPs cache system [9]. The current work, addresses a Cellular Automata based solution for detecting the presence of such a Trojan.

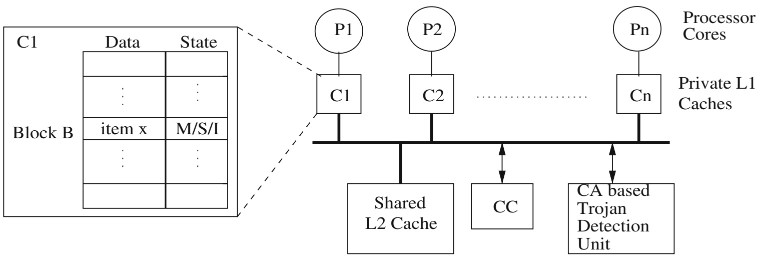


Fig. 1. CMPs with CC and Trojan detection unit.

2 CA Preliminaries

A Cellular Automaton (CA) can be viewed as an autonomous finite state machine (FSM). A CA cell is having two states - 0 or 1 and the next state (NS) of i^{th} CA cell is $S_i^{t+1} = f_i(S_{i-1}^t, S_i^t, S_{i+1}^t)$, where S_{i-1}^t, S_i^t and S_{i+1}^t are the present states (PS) of the left neighbor, self and right neighbor of the i^{th} cell at time t. f_i is the next state function. On the other hand, the states of cells $\mathcal{S}^t = (S_1^t, S_2^t, \cdots, S_n^t)$ at t is the present state of CA.

The f_i can be expressed in the form of a truth table (Table 1). The decimal equivalent of the 8 outputs is called 'Rule' R_i. In a 2-state 3-neighborhood CA, there can be 2^8 (256) rules. Five such rules 15, 14, 192, 207, and 240 are illustrated in Table 1. The first row lists the possible 2^3 (8) combinations of present states of $(i-1)^{th}$, i^{th} and $(i+1)^{th}$ cells at t.

A combination of the present states (1^{st} row of Table 1) can be considered as Min Term of a 3-variable $S_{i-1}^t, S_i^t, S_{i+1}^t$ switching function and is called RMT (rule min term). Column 011 of Table 1 is the 3^{rd} RMT. The next states corresponding to this RMT are 1 for Rule 15, 14 & 207, and 0 for Rule 192 & 240.

The set R = $<R_1, R_2, \cdots, R_i, \cdots, R_n>$ configures the cells of a CA. If all the R_is are same, the CA is a uniform CA; otherwise it is a non-uniform/hybrid CA. For the current work, we consider null boundary CA as in Fig. 2(a).

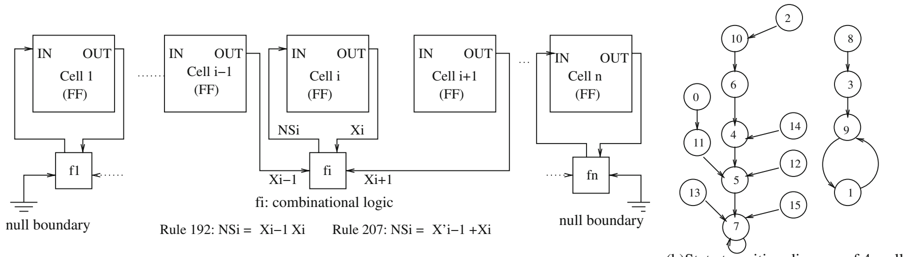
Table 1. RMTs of the CA $<15, 14, 192, 207, 240>$

PS	111	110	101	100	011	010	001	000	*Rule*
RMT	(7)	(6)	(5)	(4)	(3)	(2)	(1)	(0)	
NS	0	0	0	0	1	1	1	1	15
NS	0	0	0	0	1	1	1	0	14
NS	1	1	0	0	0	0	0	0	192
NS	1	1	0	0	1	1	1	1	207
NS	1	1	1	1	0	0	0	0	240

A CA is reversible if its states form only cycles in the state transition diagram; otherwise, the CA is irreversible (Fig. 2(b)). The set of states that forms cycle (7→7 and 9→1→9 of Fig. 2(b)) is referred to as the *attractor* [2]. The attractors of single length cycle, that is, 7→7 of Fig. 2(b) is of our current interest.

3 Modeled Trojan

A Trojan is usually categorized based on the trigger - that is, activation of the Trojan and its payload - that is, the effect of Trojan. Based on triggering conditions, a Trojan is classified in two ways - analog and digital Trojan. Analog Trojan is activated due to some analog conditions - like device aging effect, temperature and delay whereas a digital Trojan is activated due to the functioning of a sequential or combinational boolean logic function. After triggering of a Trojan, it can cause some functional failure or leak some secret information using power trace. In terms of payload, Trojan can also be categorized in two ways - analog and digital. Digital Trojan can cause incorrect logic value or it can modify the content of a memory location. Analog Trojan can affect the performance



(a) An n−cell CA

(b)State transition diagram of 4−cell irreversible CA<1,236,165,69>

Fig. 2. Null boundary CA

of the circuit as well as can change power and path delay. Trojan can also leak secret information or cause DoS attack.

Trojan can be detected in two ways - logic testing and side channel analysis. In logic testing, a test vector is generated to activate an unknown Trojan and propagate the effect to the output port and observe it. In side channel analysis, a golden circuit is required to observe the change in power consumption and path delay of the circuit.

Cache line state at jth cache

Cache line state at ith cache		O	E M	S	I	
M	X	X	X	X	√	
O			X	√	X √	X
E	X	X	X	X	√	
S	X	√	X	√	√	
I	√	√	√	√	√	

Fig. 3. Function table of MOESI protocol.

In this work, we have considered a Trojan model, which is activated by repeated access to two consecutive cache blocks by consecutive access by a particular processor. We consider the implantation of a Trojan within one 3PIP core which works hand-in-hand in a CMPs system and therefore shares the same LLC with all other cores. In CMPs' cache system following MOESI protocol, only one cache can hold the data in 'O' (own) state. For a given pair of caches (C_i of processor P_i and C_j of P_j), realizing MOESI protocol, the permitted states of cache lines for a data block B are noted in Fig. 3. Here, we consider a Trojan model which converts a cache line state from shared to modified state and considered processor P_1 as an infected core. Whenever processor P_1 performs read request, it changes its state to modified state instead of shared state. This affect the cache coherence protocol and leads to inconsistent state because two processors can not have a block in modified state at the same time. Total number of read and write miss also increases due to this Trojan. Next section briefly analyze payload of this Trojan model.

4 Payload Evaluation of the Modelled Trojan

In this section, we describe the payload evaluation of our modeled Trojan. For experimental evaluation, we have used the Multi2sim [7] simulation framework, a CPU-GPU heterogeneous simulation platform. We implemented the proposed model of Hardware Trojan in Multi2sim. The L_1 cache in each core is unified for instructions and data and L_2 is shared. The following test environment and parameters as described in Table 2, are considered for the experimentation. The benchmarks programs are run with input data sets as listed in Table 3.

- **Operating System:** Ubuntu 12.04LTS (64 bit)
- **Number of Cores used:** 16 and 32

Table 2. Parameters used in full system simulation

Parameter	Value
Processors	x86 cores, 1 GHz, single issue, in-order
L_1 cache	128 KB per core, unified, LRU replacement policy 8-way associative, 2-cycles latency, 64 byte line
L_2 cache	1 MB, LRU replacement policy, 8-way associative, 20-cycles latency, 64 byte line
Memory	1 GB, 200 cycles latency
Network	2D mesh topology, 2-cycle router, 1-cycle link latency, 36 bytes wide

Table 3. Benchmark and input data set

Benchmark	Problem size
FFT	64K points
Ocean	258×258
Radix	1M integers, radix 1024
LU	512×512 matrix, 16×16 blocks
Radiosity	room, -ae 5000.0 -en 0.050 -bf 0.10

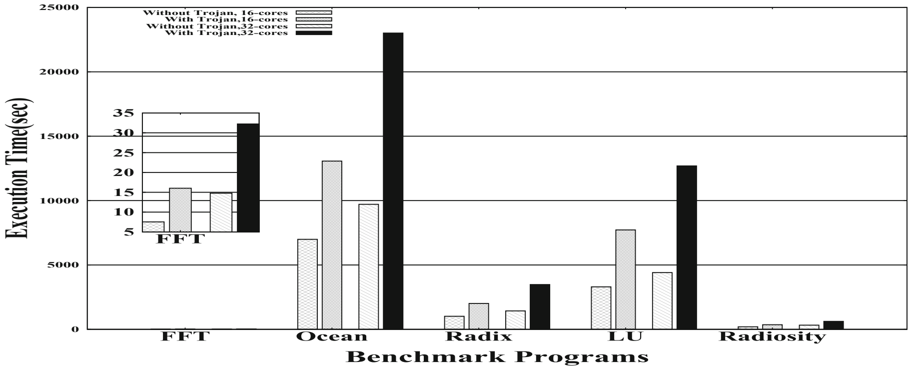


Fig. 4. Execution time of different benchmark program

After the successful implantation of the Trojans, we run the full system simulation with the programs of SPLASH II benchmark suite as the standard workload. From the simulation statistics, we try to estimate the Trojan payload. We consider the total execution time, total number of read misses, number of write misses, and runtime dynamic power consumption for payload assessment. We use McPAT [5], a power, area and timing modeling tool, for measuring the run-time dynamic power consumption.

We have shown the total execution time (sec) for different benchmark programs in Fig. 4. The execution time for two different models (with Trojan and without Trojan)-system with 16 cores and 32 cores with MOESI protocol have been reported. FFT and Radiosity take very less execution time for Trojan model as well as Trojan free model among all benchmark programs. From experimental result, we can see that there is a significant amount of increase in execution time in presence of the Trojan. In Fig. 5, total read misses for different benchmark programs have been shown. Total number of read misses increase in presence of Trojan except Radiosity. Total number of write misses for different benchmark programs have been shown in Fig. 6. From experimental result, we can see total number of write misses increase in presence of the Trojan for all the benchmark programs. Figure 7 shows the comparison of run time dynamic power for different benchmark programs in presence of Trojan and without Trojan. As total execution time, total number of read misses and total number write misses increases, it is expected that run time dynamic power will also increase in presence of Trojan. Figure 7 shows that for all the benchmark programs except Lu, total dynamic power increases significantly. Hence, run-time detection of such Trojan is very essential. In this work, we have proposed a Cellular Automata (CA) based logic to detect the presence of such a Trojan and is introduced in the next section.

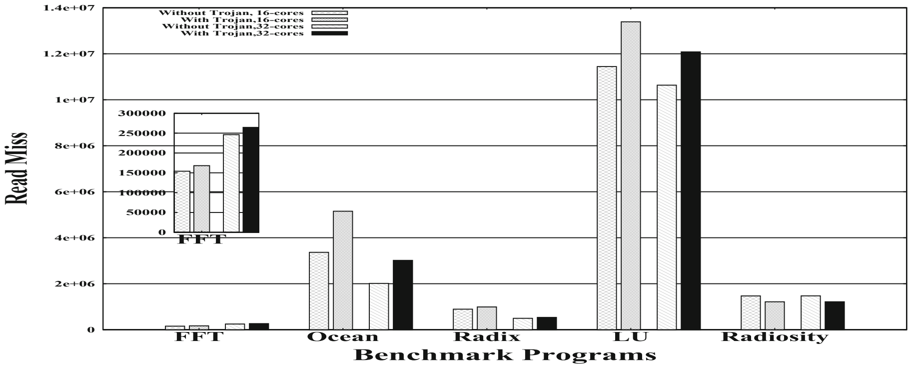


Fig. 5. Read misses of different benchmark program

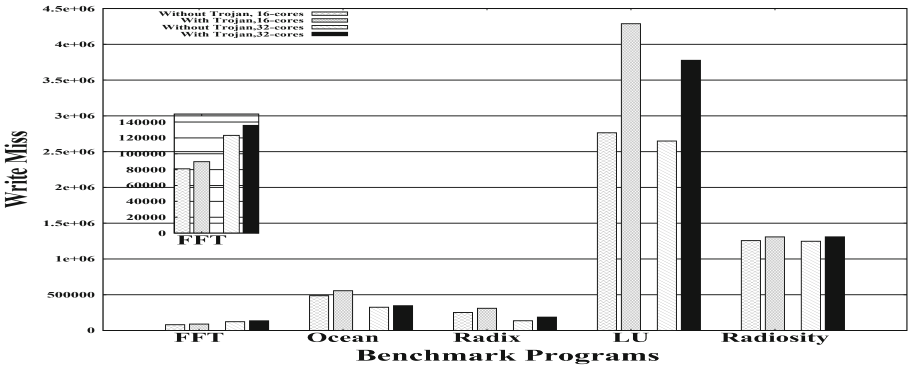


Fig. 6. Write misses of different benchmark program

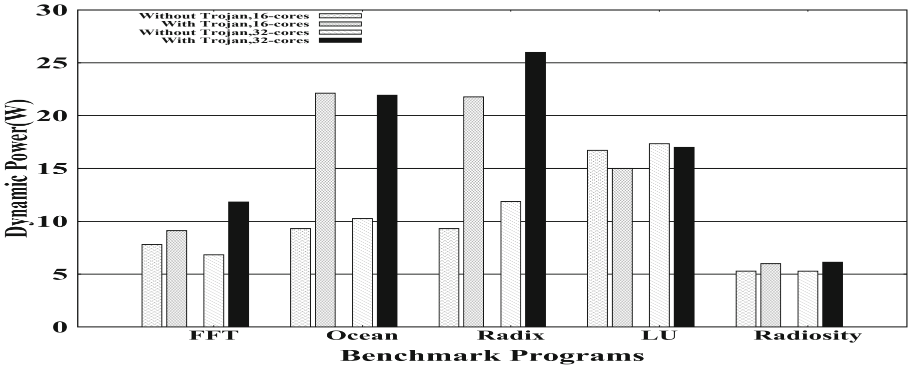


Fig. 7. Dynamic power of different benchmark program

5 Proposed Design for Detection of Modelled Trojan

Once the Trojan is activated, it can affect system performance or may leak secret information. As our modelled Trojan significantly affect the system performance by changing the cache block state arbitrarily, it is essential to detect the presence of such Trojan in the system (Fig. 8).

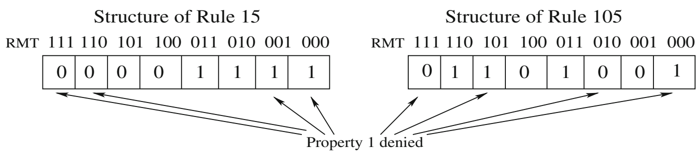


Fig. 8. Structure of rule 15 and 105.

Table 4. State transitions (MOESI protocol)

Current cache states (1)	Event (2)	Desired next states (3)	Resulting next states cause by Trojan (4)	Effect of Trojan (5)
All Cs I	Pi writes	Ci M and all others I	Ci M and all others I	CH
	Pi reads	Ci E and all others I	Ci M and all others I	CH
All Cs S	Pi writes	Ci M and all others I	Ci M and all others I	CH
	Pi reads	Ci S and all others S	Ci M and others S & I	ICH
Cs are I & S	Pi writes	Ci M and all others I	Ci M and others I	CH
	Pi reads	Ci S and all others S & I	Ci M and others S & I	ICH
Ci M, all others I	Pj writes	Cj M and all others I	Ci & Cj M and others I	ICH
	Pj reads	Ci O & Cj S others I	Cj S Ci M and others I	ICH
Cj E, all others I	Pi writes	Ci M and others I	Ci M and others I	CH
	Pi reads	Ci S & Cj O others I	Cj O Ci M and others I	ICH

Table 4 describes the possible combinations of states of cache lines for a block B and its possible state transitions (desirable as well as transitions resulted due to the Trojan payload). The possible current states of column 1 are the coherent states -that is, the Trojan detection unit responds as 'CH' (Coherent) in such cases. The event shown in column 2 causes transition of cache line states at different Cs (L_1 caches). The desired next state is noted in column 3. The next state due to incorrect recording of states are shown in column 4. The effect of Trojan is either 'CH' or 'ICH' (Incoherent) is shown in the last column. In summary, detection unit responds as 'CH', when the states of cache lines for B
Case 1: At all the caches (Cs) are I,
Case 2: At all the caches (Cs) are S,
Case 3: At some Cs are I and at others are S,

Case 4: At one cache is M and at all others are I,
Case 5: At one cache is E and at all others are I,
Case 6: At one cache is O and at all others are I,
Case 7: At one cache is O and at all others are S,
Case 8: At one cache is O and at least in one is S and at others are I.
On the other hand, for following incoherent states
Case 9: At one cache is M, at least in one is S and at others are I,
Case 10: At one cache is M, at one is O and at others are I,
Case 11: At two caches are M and at others are I,

the Trojan detection unit should respond ‘ICH’.

Therefore, the task of the detection unit is to respond correctly either ‘CH’ or ‘ICH’ following the above eleven cases. For a CMPs system with n L_1 caches (C_1, C_2, ..., C_n, where C_i is the L_1 of processor P_i), we employ an $(n+2)$ - cell CA (each of the two terminal cells are not representing a cache). If the cache line of block B at C_i has its state as ‘M’ then i^{th} cell of the CA is configured with rule R_M. Similarly, if the state of the cache line for B at C_i is ‘O’ or ‘E’ or ‘S’ or ‘I’, the corresponding CA cell is configured with rule R_O or R_E or R_S or R_I. At each transition from a current cache state to next cache state of Table 4, the detection unit forms the $(n+2)$ - cell CA through setting up of each CA cell rule R_i. Then the CA is run for ‘t’ $= (n+2)$ times steps with all 1s seed. It settles to either in attractor ‘CH’ for coherent state or ‘ICH’ for incoherent state to satisfy the eleven cases.

The design of the detection unit demands that the CA constructed from R_M, R_O, R_E, R_S and R_I should form single length cycle attractors, preferably, a single attractor. That is the design demands single length cycle single attractor CA (SACA) and the SACA should correctly distinguish CASE 1–8 and 9–11.

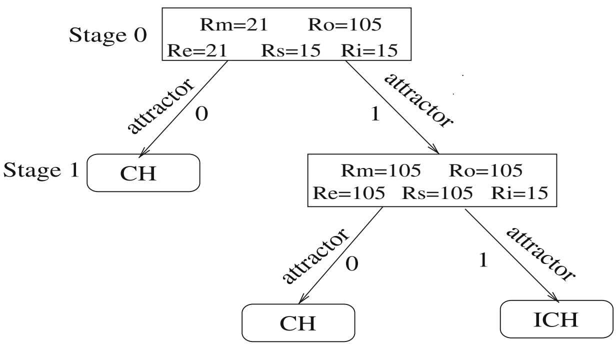


Fig. 9. *CA* based two-stage Trojan detection.

In Case 1 and Case 2, the *CA* formed are the uniform *CA* (say, SACA). Let, the attractors of these class of *CA* are X_1 and X_2. However, for Case 3 to 8, it can result in hybrid *CA*, say, with attractors Y_3, Y_4 ... Y_8.

Similarly, the CA formed in Case 9 to Case 11 are also hybrid. Let us assume these form attractors Z_9, Z_{10}, Z_{11} respectively. The best selection of R_M, R_O, R_E, R_S and R_I, therefore, can be such that

Cond 1: $A_1 = \{X_1, X_2, Y_3, Y_4, \ldots Y_8\}$ *belongs to CH and* $A_2 = \{Z_9, Z_{10}, Z_{11}\}$ *belongs to ICH and* A_1 *and* A_2 *are distinguishable.*

Table 4 describes that the CA for Case 11 is resulted from the uniform CA for Case 1 through hybridization of R_M at two cells. Further, CA formed for Case 4 (Case 5) are resulted from hybridization of SACA of Case 1 at single cell. The 2-stage (Stage 0 and Stage 1) solution to satisfy *Cond 1* assuming even number of L_1 caches in the system (Fig. 9) is described next.

Stage 0: In Stage 0, the selected rules are $\mathrm{R}_M = 21$, $\mathrm{R}_O = 105$, $\mathrm{R}_E = 21$, $\mathrm{R}_S = 14$ & $\mathrm{R}_I = 15$.

Observation 1. *A null-boundary n-cell (n is even) CA, configured with rule 15, if hybridized with rule 105 at a single cell, reaches a final state with LSB '0' when run for (n + 2) time steps from all 1s initial seed. If hybridized at more than one cell, it settles to a final state with LSB '1'.*

Lemma 1. *An n-cell null boundary even length uniform CA configured with rule 15 settles to an attractor with LSB as 0, for odd length CA the LSB of the attractor is 1.*

Observation 1 and *Lemma* 1 state that the combinations of such rules form two classes of even length $(n+2)$-CA (the two terminal cells are set to rule 15), while considering all 1s seed (initial state), for Case 1–11. The attractor's LSB '0' should signify Case 1–3 and Case 6–8 and '1' is for Case 4–5 and Case 9–11. That is, Case 4 and Case 5 are misclassified with Case 9 and Case 10 (*Cond 1*). The misclassification is corrected at Stage 1 of detection unit.

Stage 1: The rules selected in Stage 1 are $\mathrm{R}_M = 105$, $\mathrm{R}_O = 105$, $\mathrm{R}_E = 105$, $\mathrm{R}_S = 105$ and $\mathrm{R}_I = 15$. Now, the hybrid CA for Case 9 and Case 10 (hybridization with 105 at multiple cells) settle to the attractors that are different from that of the hybrid CA for Case 4 and Case 5 (hybridization with 105 at a single cell) at the LSB.

6 Hardware Realization of CA-based Design

For hardware realization (Fig. 10) of the Trojan detection unit, the states 'M', 'O', 'E', 'S' and 'I' are represented as the 000, 001, 010, 011 and 100 respectively (101, 110 and 111 are don't care). At Stage 0 (encoded as '0') the state of cache line for block B at C_i '000' implies that the rule 21 (R_M) is to be set for CA cell i, it is 105 (R_O) for state '001', for state '010' it is 21 (R_E), for state '011' the rule is 15 (R_S) and also 15 (R_I) is for '100'. To set the i^{th} CA cell rule at Stage 1 (encoded as '1'), the design accepts the cache line state (000/001/010/011/100) of B at C_i and the LSB ('0' or '1') of the *attractor* at Stage 0.

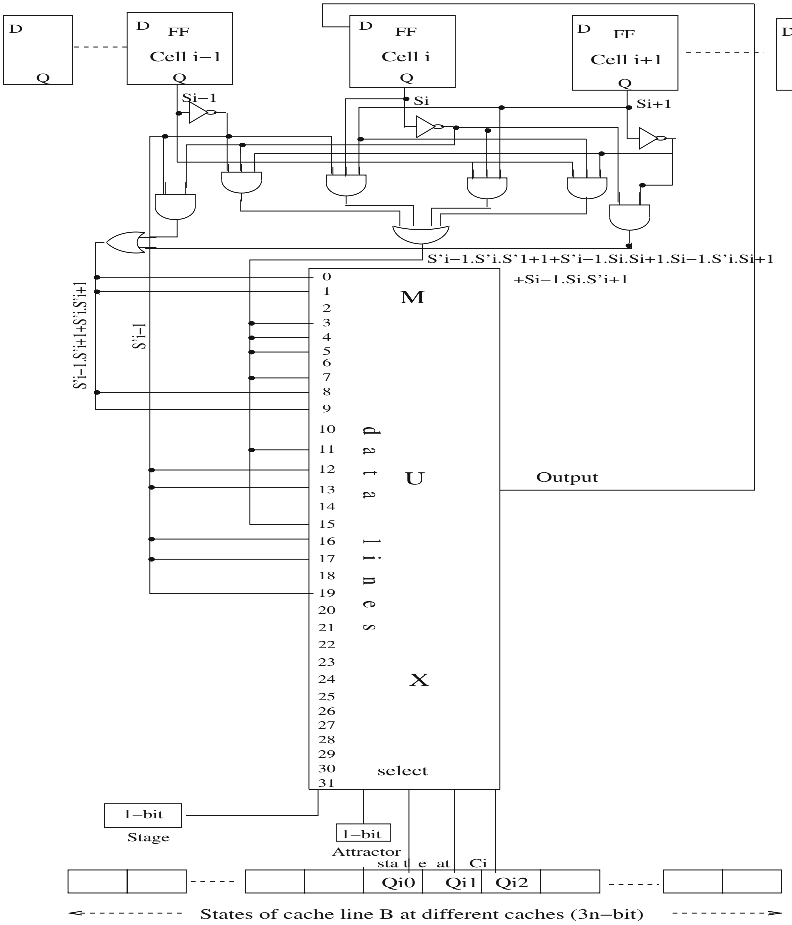


Fig. 10. Realization of CA based detection unit.

Table 5. Hardware requirements for the CA based Trojan detection unit

No. of cores (1)	Scheme [8]			CAVU$_{MOESI}$			Reduction in area (8)
	No. of FFs (2)	No. of NANDs (3)	Area (units) (4)	No. of FFs (5)	No. of NANDs (6)	Area (units) (7)	
16	12824	159	47824016	52	972	1353024	97.17%
32	21152	159	78737552	100	2032	2828544	96.40%
64	41512	159	154313872	196	3952	5501184	96.43%
128	80432	159	298784912	388	7792	10846464	96.36%
256	159288	159	591498384	772	15472	21537024	96.35%

Table 5 reports the gate counts (FFs and the 2-input NAND gates) and the area requirement of the designs for the CMPs with 16 to 256 processor cores. Column 1 notes the number of processor cores. The gate count and the area requirement for the scheme proposed in [8] for detection of incorrect recording of cache line state are reported in column 2, column 3 and column 4. The columns 5, 6, and 7 report the overhead (gate count and area) of detection unit. Column 8 notes the percentage reduction achieved in area.

7 Conclusion

This work establishes the application of SACA (single length cycle single attractor cellular automata) in designing a Trojan detection logic for CMPs. Based on the theoretical framework introduced in this work for irreversible cellular automata (CA), the SACA structure has been derived to realize detection logic for MOESI cache coherence protocol in a directory based framework. The solution targets CMPs with thousands of processor cores. It avoids rigorous computational and communication overhead while assuring highly robust and scalable design ensured by the inherent robust and scalable structure of CA.

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A Stochastic Programming Approach to Design Perishable Product Supply Chain Network Under Different Disruptions

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Abstract. Considering today's market competition, operational challenges, uncertain events, and demand volatility, industry practitioners face tremendous challenges as a decision makers. The purpose of this study is to identify, examine and suggest feasible solutions to Supply Chain (SC) practitioners under various disruptions. The operational level challenges are high for perishable product supply chains due to lack of proper infrastructure, inadequate cold storage, government interventions and improper transportation etc. compared to traditional manufacturing and automobile SC. The attempt is made in this study to gauge the effect of supplier and quality disruption on SC operations.

In order to achieve this, we have adopted a two stage stochastic programming approach to provide solutions related to ordering inventories, finding inefficient linkages in existing SC network and calculating respective SC cost. For this a SC configuration consists of two stages is considered along with two foreign suppliers and a local supplier with two distribution centers. The distribution centers receive supply of goods on regular basis from local suppliers as well as from foreign collaboration. However, in the face of uncertainty the distribution centers have to order emergency quantities from local supplier incurring extra cost depending on order size. The literature fails to consider partial order fulfillment even under cases of disruption. Our mathematical formulation has covered this missing gap and is solved using CPLEX (V.12) optimization package. Apart from this we have also performed an uncertainty analysis using @Risk. As the model is time-independent, we restrict our analysis without considering such parameters into proposed study. Moreover, incorporating such dimensions will always be a scope of future study.

Keywords: Supply chain · Stochastic programming · Disruptions · Scenario based planning

1 Introduction

Ideally the Supply Chains (SC) transfers the goods and services from point of origin to point of consumption. The information is transferred from top to bottom whereas the cash flow happens in reverse direction. However, in reality the SC gets shocks due to

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various natural and manmade disasters. For example, a wall street journal mentioned that a toy manufacturer has to sell its toys for the next Christmas selling season due to a catastrophic hit at west coast port, and the impact was seen for 10 consecutive days due to improper disaster management situations, Kahn (2002). DHL published report in 2015 on top 10 SC disruptions on SC operations mentions that the losses due to such events are immediate and huge (DHL 2015). 2 out of 10 disruptions mentioned were happened in India, one due to the Chennai floods and another due to the labor strike at Mumbai Nhava Sheva port.

SCs are vulnerable to risks, disruptions and uncertainties. Moreover, such unwanted events are categorized in various risk types, as proposed by Chopra and Sodhi (2004). A disruption majorly occurs due to the natural disasters, war, terrorism, and labor disputes. System inefficiencies results due the information infrastructure breakdown or power outages or any technical glitches (Chopra and Sodhi 2004). The severity of such impact could lead to permanent damages to the operations facilities. Yearly published report by Business Continuity Institute (BCI[1]) on supply chain resiliency captures the various causes and effects of different disruptions on supply chain performances. The empirical study considers interaction with SC experts across hundreds of SC managers from different geographical regions. These events severely damages the supply chains and leaves behind unattainable challenges such as loss of productivity, increased cost of working, revenue loss, and customer loss etc. It was also seen in the report that 48% times the recovery cost was approx. 50,000 euros which is equivalent to 42.30 Lakhs of Indian rupees. Such a huge repair losses would definitely make the SCs jeopardizes and takeaway large portion of human power, machineries and significant time, if ever assigned to generate extra revenue from other opportunities.

Impact of such a devastating event is huge for developing countries like India. The vulnerabilities for such events differ country-wise. Most of the Southeast Asian (Malaysia) countries face operational level challenges due to storms, heavy rain fall, tornadoes, and earth quakes. In India problems such as labor strike, governmental intervention, local level issues, transportation delays puts the SC in menace (Fig. 1).

Event like demonetization stalls the supply chain operations for longer period, resulting in the loss of heavy businesses. The one nation, one tax policy has also affected the supply chain operations and diminished the value chain. This change in policy can either boost the SC operations or impede the functioning badly. The occurrence of such events could be less in number but the severity could be overwhelming. Such impact is evidently high for the perishable SCs considering its vulnerability and poor infrastructure resources. Van der Vorst et al. (2009) categorized a Food Supply Chain Network (FSCN) into two types; (1) FSCN for fresh agricultural products, such as fresh vegetables and fruit; and (2) FSCN for processed food products, such as portioned meats, snacks, desserts. The author has also listed the design or redesign process-challenges for FSCN such as global sourcing requirements, biological variations, seasonality, preserve quality requirements, product shrinkages and stock out in retail, and special conditions for transportation and storage etc.

[1] The BCI has around 8,000 members in more than 100 countries worldwide. https://www.thebci.org/. https://www.thebci.org/resource/supply-chain-resilience-report-2017.html.

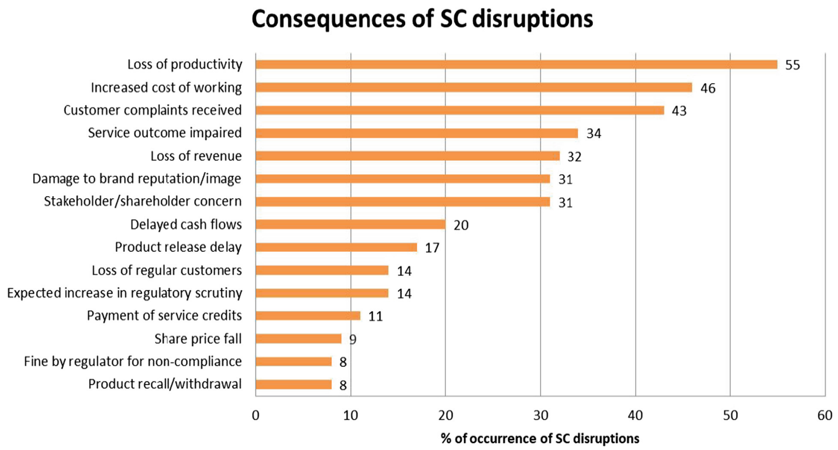

Fig. 1. Consequences of SC disruptions post occurrence (Source: [1]BCI resiliency report 2017)

Apart from this, problems such as improper packaging, quality and safety issues, financial issues, transportation problem, and linkages and integration issues, are seen prominent in case of perishable product SC networks Negi and Anand (2015). For fruits, the post-harvest losses are on an average 10% of the time it reaches the customers end (Post Harvest Losses in India, ICAR). In the proposed study, we intend to consider the effect of such challenges on operations of perishable products SCs. Further in the mathematical model, we have also included the effect of supplier unreliability with the help of supplier uncertainty along with the effect of quality loss due to perishable nature of the supply chain structure.

The supply chain structure consists of two Distribution Centers (DC), one local supplier, and two foreign suppliers. The candidate locations for the facilities are known in advance. However, the major task involved is to understand the effect of different disruptions when an order is placed at the DCs to other echelons in SC structure. A regular and minimum quantity of goods is received from local supplier at the DCs along with larger portion of order from foreign supplier. Since the cost incurred to purchase goods from local market is higher, it is cost advantageous to order larger portion of demand from foreign market. However, the foreign suppliers are subjected to different disruptions and delays whereas the local supplier fails in meeting the quality requirements. Therefore, decision making under such disruptive and uncertain environment becomes complex and requires optimization techniques. Thus we have used stochastic programming approach to capture the effect of disruption due to foreign supplier and reliability issue in case of local supplier in the study.

2 Relevant Literature

The supply chain planning decisions are categorized based on the time horizon into: strategic, tactical, and operational (Simchi-Levi et al. 2004). (Chopra and Sodhi 2004) prepared a classification for different risk and their causes, viz. forecast, inventory, procurement, and delays and disruption. Christopher tang (2006) proposed a work on SC disruption, the author proposed robust strategies in order to handle the disruption scenarios. The authors mentioned two properties for the strategies considered in order to fight disruptions. (a) Manage the inherent fluctuations efficiently regardless of major disruptions and (b) Make the SC more resilient. Basically these strategies are focused to improve the flexibilities of SC operations such as product, transportation, and supply flexibility etc. According to Daskin et al. (2005); Shen (2007) Supply Chain Network Design (SCND) is considered as a suitable application for facility location models. However, the review proposed by authors lacks detail explanation on multi-echelon SC models, limited scope w.r.t exact solution and over-emphasis on deterministic models. Melo et al. (2009) proposed review on SC models based on nature of SC parameters such as deterministic and stochastic models. The classification considers models with single or multiple commodities as well as for the single or multi-period environment. Moreover, the authors have encouraged performing more work on multi-layer stochastic SC network design. Uncertainty is a elementary aspect of any decision making process and thus author Min and Zhou (2002) proposed various solution alternatives for decision making under such situations. A location-inventory model for the situation where retailers' demands have normal distribution with known daily mean and variance, as proposed by Daskin et al. (2002), has been used in present to develop advance stochastic models. Stochastic programming the term introduced by Birge and Louveaux (2011), proposes various solutions approaches to handle such classes of optimization problem where data follows a certain distribution. Claypool et al. (2014) designed a supply chain network for new product as the risk is inherent in this process. The novelty on the research paper is combining the effect of risk due to product development challenges apart from inherent SC risk. Authors formulated the mixed integer programming (MIP) model while concurrently considering strategic exposure risk, time-to-market risk, and supplier reliability risk. The authors have also performed simulations in order to test the robustness of MIP. Govindan et al. (2017) proposed a literature review on SCND under uncertainty. The author classified all the papers in essentially three groups. The characteristics of each groups is presented as follow;

G1: Decision making (DM) environment with random parameters in which their probability distributions are known; **G2:** DM environment with random parameters in which DM has no information about the probability distribution. **G3:** Fuzzy decision making environment. Further, the author has proposed scenario dependent modelling approach where a probability for each discrete scenario is known and the solution approaches are classified under two categories depending on level of decision making; (1) Two stage stochastic programming and (2) multi-stage stochastic programming (Birge and Louveaux 2011). For any two two-stage stochastic SCND problems, second stage decisions are continuous and positive. And value of first stage decisions can be obtained through solving a linear program for each scenario. The author mentioned that decomposition algorithm is suitable tools to solve such type of problems. Sadghiani et al. (2015) developed retail SCN considering operational and disruption risks. The authors validated the model using illustrative case examples and a real case study in retail SC. FSCN distribution differs that from distribution of other products due to quality deterioration along SC. The current model considers the supplier unreliability due to its delay and unfulfillment of order placed from distribution centers. Data et al. (2003) published a paper considering the reliability and cost of supplier selection into its objective function. This paper has bridged the literature on newsvendor aspects and random yield perspective. The author suggest that the cost overcomes the reliability while selecting supplier whereas study proposed by Chopra et al. (2007) shows that the it is better to order from reliable supplier than disruption prone supplier, contradicting what has proposed in Data et al. (2003). However, what was missing in these papers that the researcher has not incorporated (1) effect of quality, safety, and sustainability on FSC; (2) Similar such effect in multi-echelon supply chain configurations and limit the research to dual or multiple sourcing situation. Akkerman et al. (2010); Zokaee et al. (2017) considered scenario based strategic and tactical SC design under uncertain demand, supply and various cost data. The authors has also tested the model for real industry data for bread supply chain in Iran, The authors lacks in considering the production planning and transportation routes in the SC operations. We in our study, consider supplier unreliability and effect of quality disruptions due to perishable nature of SC. Consider a two-echelon supply chain that involves foreign suppliers and a local supplier. The supply facilities are subjected to different disruptions and there is reliability issue in terms of order fulfillment from these foreign suppliers (Chopra and Sodhi 2004). At the same time there is contract established with a local supplier which supply the same products but at higher rates with substandard quality. It becomes cost advantage for the distribution centers to order from the outside/foreign suppliers in bulk but at cheaper rates. Due to the unwanted reasons the supply from these foreign facilities could not reach the distribution centers situated locally. The disruption is not limited to the supply but has an effect on the quality due to numerous factors. These factors are inefficient storage capacities and chilling facilities, lack of transport infrastructure and underutilization of vehicles, government interventions etc. Nagi and Anand (2015). Not only this but also the international price shocks and exchange rate volatility affects the SC operations, Halder and Pati (2011).

3 Model Formulation

Consider a two-echelon supply chain that involves foreign suppliers and a local supplier. The supply facilities are subjected to different disruptions and there is reliability issue in terms of order fulfillment from these foreign suppliers (Chopra and Sodhi 2004). At the same time there is contract established with a local supplier which supply the same products but at higher rate and with substandard quality. It becomes cost advantage for the distribution centers to order from the outside/foreign suppliers in bulk but at cheaper rates. Due to the unwanted reasons the supply from these foreign facilities could not reach the distribution centers situated in Mumbai (India). The disruption is not limited to the supply but has an effect on the quality due to numerous factors. These factors are inefficient storage capacities and chilling facilities, lack of transport infrastructure and underutilization of vehicles, government interventions etc. Not only this but also the international price shocks and exchange rate volatility affects the SC operations Halder and Pati (2011). Figure 2 depicts the schematic view of the SC network under consideration.

Complexity-wise it becomes difficult to solve an optimization model considering all the aspects of the real world issues. Moreover, following are the assumptions considered while formulating and solving the mathematical model;

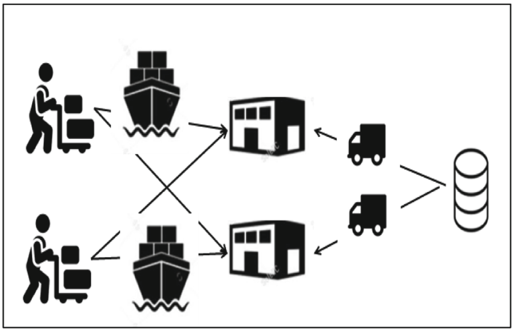

Fig. 2. Schematic view of two echelons SC with foreign supplier, DCs and a local supplier

1. The number of candidate facility locations is known and the fixed facility investment cost for each location is predetermined.
2. All the model parameters are deterministic except the products demand, disruption due to supplier denoted as α, transport disruption (β) and quality loss during whole process (γ).
3. The model does not consider time period and related factors like lead time or cycle time etc.
4. Though we have considered a distribution planning where foreign suppliers are involved, we are not considering any import duty, agricultural levies, excise duty and VAT etc. in the objective function or in the mathematical model.
5. The probability of occurrence of the each scenario is same for convenience.
6. The model holds true for the purchase cost of outside supplier lower than the purchase cost of local supplier.
7. The random demand considered in the model is mutually independent such that there is neither cross-nor autocorrelation (Helber et al. 2013).

a. Nomenclature

In order to facilitate the understanding of the mathematical modeling, the sets, parameters, and the decision variables are introduced in this section;

Sets

$f \in F$ → Set of foreign suppliers
$p \in P$ → Set of different product categories
$d \in D$ → Set of Distribution Centers (DC)
$s \in S$ → Set for scenarios

Parameters

FC_{fd}^{p} → Fixed cost for ordering p-type product from 'f' foreign supplier at 'd' DC
FL_{d}^{p} → Fixed cost for ordering p-type product from local supplier at 'd' DC

Cl_d^p → Unit ordering cost to local supplier from 'd' DC
C_{fd}^p → Unit ordering cost to 'f' foreign supplier from 'd' DC
CP_{ds}^p → Unit penalty cost for losing a customer due to quality disruption
EC_d^p → Unit additional emergency purchase cost for 'p' type product at 'd' DCs under scenario 's'
$Imax_{ds}^p$ → Maximum inventory for 'p' type product at 'd' DC under scenario 's'
D_{ds}^p → Demand at 'd' DC for 'p' type product under scenario 's'
η_{ds}^p → Supplier disruption factor for 'p' type product at 'd' DC from 'f' foreign supplier under scenario 's'
ξ_{ds}^p → Transport quality disruption factor of 'p' type product at 'd' DC from 'f' foreign supplier under scenario 's'
$\gamma_{\mathrm{dfs}}^{\mathrm{p}}$ → Parameter to decide proportion of quality waste of goods due to local supplier for 'p' type product received at 'd' DC under scenario 's'
v_{df}^p → Binary variable for selecting 'p' type product from 'f' foreign supplier at 'd' DC
vl_{df}^p → Binary variable for selecting 'p' type product from local supplier at 'd' DC
$Qlmin_d^p$ → Minimum quantities of 'p' type product ordered from local supplier at 'd' DC
Q_{fds}^p → Quantities of 'p' type product ordered from 'f' foreign supplier at 'd' DC under scenario 's'
Ql_{ds}^p → Total Quantities of 'p' type product ordered from local supplier at 'd' DC under scenario 's'
QE_{ds}^p → Emergency quantities of 'p' type product ordered from local supplier at 'd' DC under scenario 's'

The Total Supply Chain Cost (TSC) denoted as Ω. There are six terms in the objective function. Term 1 and 2 describes the fixed costs for selecting suppliers, either foreign or local for order placement. Term 3 in the objective function is for the ordering cost for the minimum quantities of goods ordered every time from the local supplier. All the subsequent terms thereafter are scenario dependent. Term 4 is for the quantities of goods received at the DCs post the supplier disruption and loss in quality during transportation. Considering the SC visibility, the quantities missed due to the foreign supplier disruption is ordered from the local supplier by paying an extra premium cost which is presented in the term 5. Term 6 in the objective function

$$\Omega = \sum_{f\in F}\sum_{p\in P} FC_{fd}^p.v_i^p + \sum_{p\epsilon P} FL_d^p.vl_f^p + \sum_{p\epsilon P}\sum_{l\epsilon L} Cl_d^p\left(Qlmin_d^p\right)+$$
$$P_s^*\left\{\left\{\sum_{f\epsilon F}\sum_{p\epsilon P}\sum_{s\epsilon S}\left(1-\eta_{ds}^p-\xi_{ds}^p+\eta_{ds}^p\xi_{ds}^p\right)C_{fds}^p.Q_{fds}^p\right\}+\right.$$
$$\sum_{p\epsilon P}\sum_{f\epsilon f}\sum_{s\epsilon S}(EC_d^p+Cl_d^p).QE_{ds}^p+\sum_{f\epsilon zF}\sum_{d\epsilon D}\sum_{p\epsilon P}\sum_{s\epsilon S}(\gamma_{dfs}^p((1-$$
$$\left.\eta_{dfs}^p)*\xi_{dfs}^p\,(Qlmin_d^p+QE_{ds}^p)).C_{ds}^p)\right\}$$

Subject to;

$$\sum_{f\epsilon F}\left(1-\eta_{ds}^{p}-\xi_{ds}^{p}+\eta_{ds}^{p}\xi_{ds}^{p}\right)Q_{fd}^{p}+Qlmin_{d}^{p}+QE_{ds}^{p}=INV_{ds}^{p}$$
$$\forall d\in D,\forall p\in P,\forall s\in S \quad (1)$$

$$\sum_{d\epsilon D}Ql_{d}^{p}\leq\sum_{d\epsilon D}Imax_{ds}^{p} \qquad \forall s\in S,\ \forall p\in P \quad (2)$$

$$\sum_{f\epsilon F}\eta_{ds}^{p}Q_{fd}^{p}=QE_{ds}^{p} \qquad \forall d\in D,\forall s\in S \quad (3)$$

$$\sum_{d\epsilon D}Qlmin_{d}^{p}+QE_{ds}^{p}=Ql_{d}^{p} \qquad \forall p\in P,\forall s\in S \quad (4)$$

$$\sum_{d\epsilon D}QE_{ds}^{p}\leq 2\sum_{d\epsilon D}Qlmin_{d}^{p} \qquad \forall p\in P,\forall l\in L,\forall s\in S \quad (5)$$

$$Z\leq\sum_{d\epsilon D}Ql_{d}^{p}/\sum_{d\epsilon D}\sum_{p\epsilon P}\sum_{s\epsilon S}D_{ds}^{p} \qquad \forall p\in P,\forall s\in S \quad (6)$$

$$Q_{fds}^{p},\ QE_{ds}^{p},\ Q_{lp}^{p}\geq 0 \qquad \forall i\in I,\forall p\in P,\forall l\in L \quad (7)$$

Equation 1 is to calculate the inventory at DCs under different scenarios. Equation 2 is putting the restriction on capacity of DCs for the given SC configuration. The emergency quantities ordered from the local supplier are calculated as in Eq. 3. And Eq. 4 calculates total quantities of goods ordered from local supplier. Equation 5 this equation is to define the capacity of local supplier. Following to term 3 in objective function, as we know there is certain minimum quantities are ordered from the local supplier. During emergency, DCs cannot order more than twice the regular quantities ordered from the local supplier. and any order beyond this capacity is considered lost sales. A service level constraint is also added in the mathematical formulation using Eq. 6. Finally, in Eq. (7) the variables are considered non-negative.

4 Results and Discussion

The mathematical model presented above is more generalized structure for multiple foreign suppliers, multiple DCs and a single hub acting as local supplier. As discussed in the above, the foreign suppliers are prone to disruptions and therefore the fails to meet the demand at DCs, however, the local supplier which act as more reliable supplier helps the DCs during the regular supplies as well as during emergency ordering but at advent of extra cost. Furthermore, it was also observed that the goods received from the foreign supplier has better quality and comparatively lesser cost. Table 1 outline the cost data under consideration. First two rows presents the cost for the quantities of goods ordered from the foreign suppliers based on the product type. Whereas the row 3rd and 4th presents the per kg cost incurred whenever an order is placed to the local supplier as well as the extra premium the DCs has to pay during the emergency order. The values for η and ξ lies in between 0 and 1. The severity due to these disruptions increases as the values for η and ξ tends to 1. As mentioned earlier there is aggregate demand that arises at both the DCs. For the simplicity, we have considered the demand with a mean

Table 1. Cost parameters for the problem under considerations

Cost Data	DC 1 (Product-1)	DC 1 (Product-1)	DC 2 (Product-2)	DC 2 (Product-2)
Supplier 1	650	650	900	900
Supplier 2	700	700	750	750
Local supplier	900	900	1050	1050
Local premium	50	50	50	50

of 3000 and Standard Deviation of 500 for type-1 product category and the demand with mean of 2000 and SD of 500 for product 2 categories. The minimum quantities ordered for both the product categories are 300 and 200 respectively.

In the Table 2, we have obtained the result for the single period model subjected to a demand distribution mentioned above. Apart from this, we used pert distribution for supplier (η) and quality disruption (ξ). Pert distribution, also called as beta distribution is widely used in the project management (PMBOK 2001). The inputs to the distribution are minimum, maximum and most likely value. The beta distribution is chosen after understanding the pattern for the occurrences of such disruptive events in real life. The decision variable for the above setting is shown in Table 2.

Table 2. Decision variable for quantities of product shipped between suppliers and DCs

Quantity shipped	DC 1 (Product-1)	DC 1 (Product-2)	DC 2 (Product-1)	DC 2 (Product-2)
Supplier 1	100	2301	100	100
Supplier 2	100	111	1356	100
Local supplier	300	300	200	200
Local emergency	100	150	350	300

It can be seen from the Table 2, the amount of goods ordered from the DC 1 from foreign supplier 1 is higher compared to other combinations. Similarly, foreign supplier 2 sends maximum lot of goods to DC 2 as compared to the other linkages in the SC configuration. In order to understand the effect of disruption due to all the SC members, we allow both the suppliers to distribute the quantities of goods to both the DCs under disrupted supplier condition. Therefore, the quantities obtained in the table are continuous values and such results are obtained for the different values of demands and supplier and quality disruptions. We have performed an uncertainty analysis for the above configuration by setting all the values in the excel sheet and used Palisade tool called @Risk optimization solver for analysis. We have also captured the penalty cost which goes for losing the customers due to low quality goods received at the DCs and is also accounted in the objective function (term 6). The values and the graph for same are presented in the Table 3 and Fig. 4 respectively. It is evident that the penalty cost due to the losses in the supply chain accounts approximately for 20% as compared to overall supply chain cost required to run the business.

Table 3. TSC and Penalty cost for the SC under consideration

TSC (in millions)	2.5	2.6	2.6	2.5	2.6	2.6	2.7	2.6	2.6	2.6
Penalty cost (in Lakhs)	5.0	5.1	5.1	5.1	5.2	5.2	5.2	5.2	5.3	5.3
TSC (in million)	2.5	2.5	2.6	2.6	2.6	2.5	2.5	2.7	2.5	2.7
Penalty cost (in Lakhs)	5.3	5.3	5.3	5.3	5.3	5.3	5.3	5.3	5.3	5.3

Since the model is a single period stochastic program, during different scenarios the penalty cost is measured and is plotted against the expected TSC as mentioned in the Fig. 4 below. Following to this analysis, we have also found some insightful conclusions for managerial application where the impact of which disruption or uncertainty such as demand has high or low impact on total supply chain cost and overall performance of supply chain in subsequent paragraph (Fig. 3).

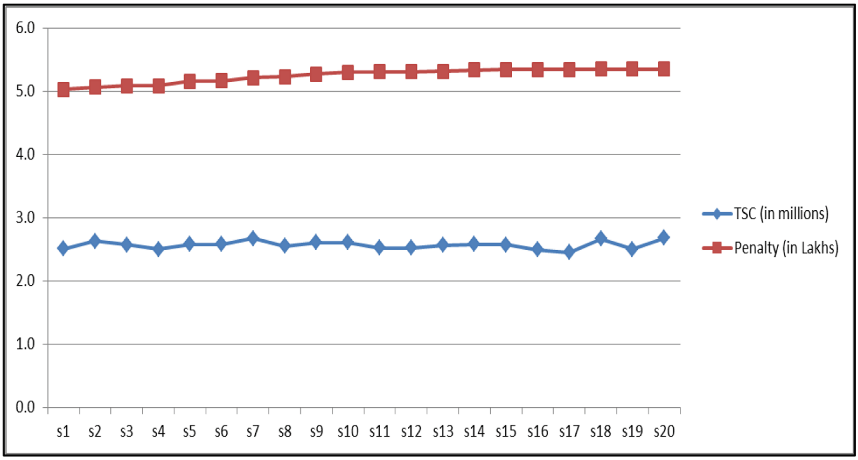

Fig. 3. Penalty cost w.r.t TSC during different scenarios

The expected total SC cost is 26,72,721.71. Also with a confidence interval of 90%, the TSC lies in the range of 2.537 million to 2.823 million. Figure 4 shows the expected value graph for the TSC under various distributions plotted using @Risk modeling software[2]. For the SC setting under study the minimum cost that a decision maker has to invest is 2.45 million while the maximum is till 3 million.

[2] http://www.palisade.com/risk/default.asp

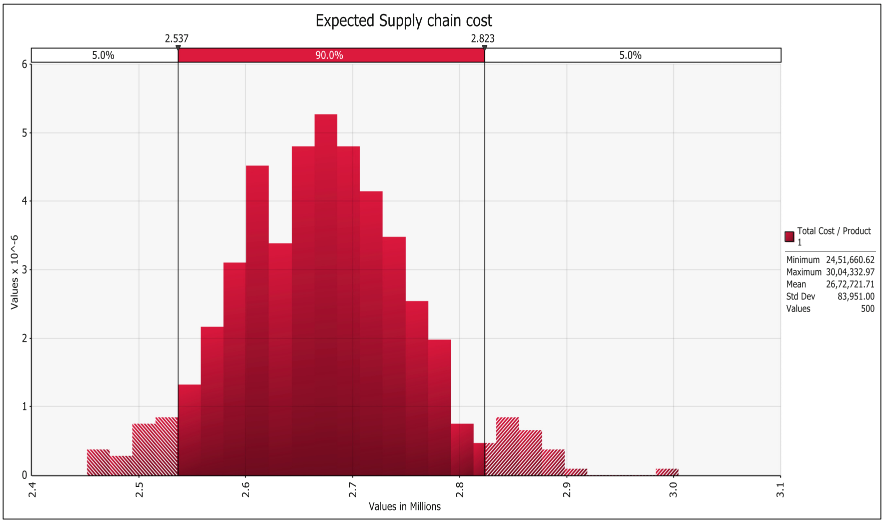

Fig. 4. Expected total supply chain cost

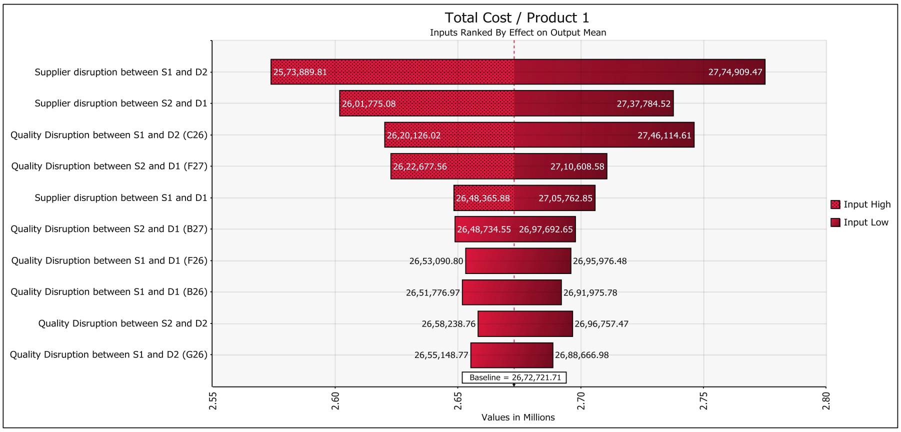

Fig. 5. Tornado graph for TSC against different scenario

Tornado is the special type of bar chart in which the values are arranged in the descending order. Such type of graph is effective in SC to identify the affected links or the connection between two points which under or over perform depending on values. We used the tornado graph for finding the most affected links in the SC configuration under consideration. Such type of graphs also shows the variation in the TSC due to respective change in the values of factor affecting the link. For example in Fig. 5 above the top 10 factor which contribute to the variation in the TSC are listed using Tornado graph. The most affected link i.e. supplier 1 and distribution center 2 shows a minimum cost of 2.57 million to the maximum of 2.77 million. The probable reason for the variation in TSC due to this link could be many. It may be because the supplier has failed to supply the quantities due to supplier failure as well as the disruption during transportation. Apart from this, Fig. 6 shows the variation in the TSC due to other factors. It is evident from the Fig. 6 that the second most reason for the significant variation in the SC cost is due to the quality of goods disrupted during shipment from S1 and DC2 followed by linkage between supplier 2 and DC1. The Figs. 5 and 6 are for Product category 1 and similar such result can be obtained for the product category 2. There are other reasons for the variation in the SC cost such as demand volatility which is not been captured in this analysis. Moreover, whenever a demand is considered in the modeling, challenges of approximation arises. Furthermore, we proposed a future direction for the study and conclusion.

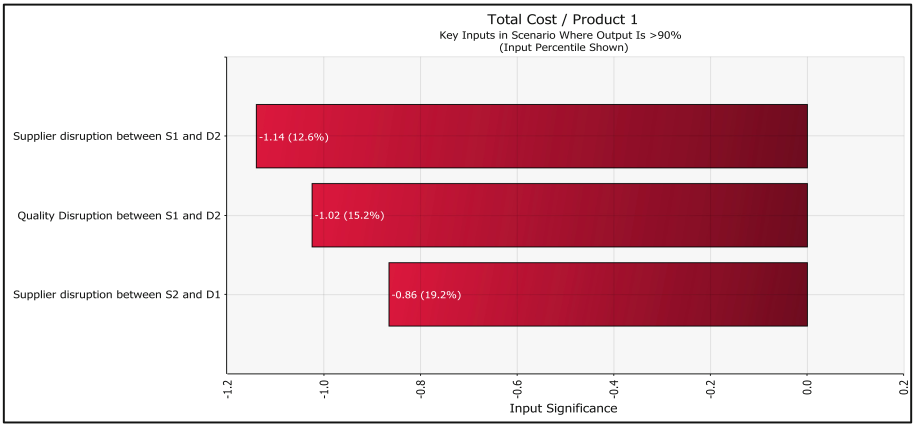

Fig. 6. Top 3 major SC affected links in SC configuration

5 Conclusion

Modeling various SC disruptions in a SC network has been a growing field of research for academicians and industry practitioners. The proposed study majorly looks into calculating the expected SC cost under various disruptions such as supplier and quality losses due to transportation. The MILP is solved using CPLEX (V.12.7.1) optimization solver package and uncertainty analysis for the given SC configuration is performed using Palisade tool (@Risk). Moreover we enlist few of the important characteristics of the above study as follow;

- We have measured the impact of disruptions due to foreign supplier along with the losses incurred during transportation on two-stage, multi-product SC network.
- We have only considered two types of disruptions i.e. disruption due to suppliers failure (full or partial) and quality loss during transportation.
- Identifying the low performing linkages in the SC configuration using tornado graph.
- Sensitivity analysis is performed on the given model by varying various parameters considered in the mathematical model such as η and ξ.

However, we have limitations to our study by not considering the planning horizon into modeling. The opportunity for selling low quality goods received after quality disruption is not considered in the model and can be a promising future research direction. The model majorly differs from the one proposed in the literature in terms of methodology and design. The proposed model is robust for the small and medium scale data and might requires a consideration of advanced optimization techniques such as meta-heuristics etc. We also see prospect in considering multiple objective in the same SC configuration such as profit or reliability maximization etc. Future work can be considered for incorporating the decay in product quality due to time and perishable nature, for more refer to Vorst et al. (2009). Flexibility into the model can be incorporated using robust optimization technique but that has kept out of the scope of this study.

We have only considered the stochastic nature in the model. Moreover, the model can be applied considering imprecise (fuzzy) and rough (approximation) environment, for directions on this refer to work developed by Garai et al. (2017) etc.

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Factors Affecting Quality of Schools in India
An Interpretive Structural Modeling Approach

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Abstract. This paper investigates and models the factors that determine the quality dimension in appraising school performance in India. It also aims to identify the factors that affect the quality of school education and throw light on their nature, significance and mutual influences, using interpretive structural modeling (ISM). The interdependency and correlation of identified factors are done through Factor analysis. Bartlett's test and Kaiser-Meyer-Olkin approach are used to measure sampling adequacy and reliability analysis is done by using Cronbach's Alpha index to measure the internal consistency. ISM demonstrates the interrelationships and the hierarchy of the factors. The results suggest that teacher motivation is the most influential factor influencing both teacher's experience and class size, which in turn, are related to teacher qualification, transport facilities, curriculum, and teacher's salary. MICMAC analysis is conducted to understand the interrelationships among the explored factors and classify them into autonomous, linkage, independent and dependent factors.

Keywords: Primary and secondary education · School · ISM · Interpretive structural modeling · Quality of education · Quality of school

1 Introduction

Primary education is a global concern. The world conference on 'Education For All' held in March, 1990 in Jomtien, Thailand, had called upon all member countries to initiate steps to improve and achieve Elementary Education for All by the year 2000. The ultimate goal affirmed by the world declaration on 'Education For All' was to meet the basic needs of children, youth, and adults, globally. India was also party to the declaration.

While India has witnessed a rise in schools imparting primary and secondary education, the growth has not really been steady. Post-independence, the government had set up a number of schools to impart education; however, it was observed that the quality of education was far from satisfactory.

The government has also played an active role to set up new schools in remote areas to achieve this objective. However, there is a difference between the private players and the government. While the government is keen to provide basic education to all, free of cost, the private sector players seek to provide higher quality education motivated by profit. So, their motives seem to be in conflict with each other.

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A literature review was conducted to identify the factors that could be used to evaluate the performance of schools in India. A number of researchers, including Lezotte et al. (2002), have found a common factor governing all good schools which is that they all have unique characteristics and processes, to impart quality education. Many studies have, therefore, been carried out to determine the factors for improving the quality of education.

Existing literature showcases the various factors that affect quality of primary and secondary education in India. However, these factors have been studied in isolation and their inter-relations have not been studied. However, studying the inter-relations is necessary because they all influence each other. These inter-relations can be categorized into input, environmental or output variables which come together to create complex educational systems (Biggs 1993). Our paper contributes to extant literature by studying these inter-relations using interpretive structural modeling (ISM). ISM is a qualitative and interpretive tool based on mathematical modeling. We have also integrated expert opinion in our framework.

The aim of this study is to:

- To identify the factors that impact quality of education through literature review; and
- To explain the nature, significance and interactions of these factors using ISM.

The paper studies the Indian context but the factors affecting education are relevant to both the Indian and the global context. In the first section, we account for the literature based on which we have identified our factors. In the next section, we describe the methodology and then in the final section, we follow up with results, discussions, implications, limitations and directions for further research.

2 Quality Concern in School Education

The aim of education is basically to fulfill the needs and aspirations of the students to further their careers, play a constructive role in society, participate in nation-building and further the cause of humanity. All this is possible only when we provide quality education. According to Dewney et al. (1994) quality means "meeting, exceeding and delighting customer's needs and expectations with the recognition that these needs and desires will change over time".

Cliff et al. (1987), on the other hand, describe quality as "somewhat problematic: like beauty, it lies in the eyes – or rather in the mind of the beholder".

However, quality should not be a privilege for those who can afford – in any participatory democracy such as India, it has to be accessible to all, irrespective of caste, gender, economic status or any other social division. J.P. Naik finds 'equality, quality and quantity' as essential to Indian education. Providing these three qualities requires deep understanding and implementation, both of which seem to be lacking in the education system of the country.

A global monitoring report by the United Nations Educational, Scientific and Cultural Organization (UNESCO) finds that systematic standards should be the focal point of the quality debate (Global Monitoring Report 2006 – Literacy for Life, UNESCO, 2006). What this implies is that the child's performance should be seen as an indicator of systematic quality.

A school providing quality education should base the teaching experience, including preparing of curriculum and training teachers on the accepted principles of child pedagogy. In the context of parents, quality is something that raises "the levels of academic performance usually measured in the test scores in the various subjects which form part of school curriculum". In the context of education, quality is all the more abstract, hard to define and equally hard to measure in terms of tangible outcomes. With so many conflicting views it is difficult for educators, policy makers and other stakeholders to achieve common ground. For parents, aspirations to provide quality education are sacrificed at the altar of poverty and economic and social inequality.

3 Factors Affecting Quality of School Education

Teacher Motivation
Naseer et al. (2012) feel that teacher motivation is integral to providing quality education. A teacher whose basic necessities are fulfilled and who gets the required support and recognition for efforts is more likely to deliver quality output. Motivation directs behavior toward particular goals. So, teacher motivation is an important determinant of whether the person adopts an enthusiastic or lackluster approach to teaching.

Teacher Qualification
Chang and Yie (2006) have indicated Teacher's Qualification to be an important factor for indicating quality. It has been observed that parents give adequate attention to teacher qualification while shortlisting schools for their children. Yet, studies show that high qualification is not a guarantee to quality education, especially if the person is not motivated. On the other hand, a less qualified teacher may be more enthusiastic about her job.

Teacher Student Ratio
Duflo et al. (2015) have indicated that this factor is a very important determinant of quality. They have defined ratio to be the number of students in a school divided by the number of teachers it has. For a quality school, the ratio should be less so that teachers can provide individual attention to each student.

Geographical Location
Funkhouser and Cowden (1994) find location be an important determinant in selecting a school. Most parents prefer schools that are nearby with the aim to reduce money, resources and time to pick up and drop children to school.

Staff Behavior
General attitude of staff is another factor affecting school and quality. A pleasant and constructive interaction bodes well for the students, teachers and parents, leading to a better output in studies.

Curriculum
Barneston (1997) finds that parents tend to rely on word-of-mouth information when deciding on a school for children. He also feels that schools should advertise their advantages to reach potential parents as well as other influencers.

Pedagogy
Wu et al. (2014) feel that school quality is directly related to the pedagogical practices it follows. Pedagogy can be defined as "a conscious activity by a person designed to adapt or inculcate the learning of another". The method defines each child to be unique and shows that one universal system of education will not hold good for the entire population of students across all divides.

Learning Material
Wu et al. (2014) indicate the equipment at the school directly impact teaching. Excellent, state-of-the-art equipment and facilities can facilitate multiple teaching methods. Learning materials are directly proportional to the education learning system. Students prefer practical or lab-oriented classes rather than theoretical monotonous classes.

Teacher Salary
Britton and Propper (2016) highlighted the relationship between a teacher's salary and school quality. Teacher's salary impacts their motivation. Well-paid teachers will be more dedicated to their work. Money is always a motivated factor.

Class Size
Akerhielm (1995) showed that class size was directly related to school quality. If the class size is less, teacher can provide personalized attention to each student.

4 Methodology

We conducted primary and secondary research to meet the objectives of this research. The process included:

- Identifying appropriate factors from literature;
- Validating them through Factor Analysis; and
- Modeling and classification using ISM.

4.1 Identification of Factors

Based on the review of existing literature, we identified the factors influencing the performance of secondary and primary schools. These factors are presented in Table 1. We found literature to be scarce on factors affecting quality of secondary education in the Indian context. The interrelations among those research papers on performance indicators have also not been studied.

Table 1. Factors and literature support

Sub factor	Support in existing literature
Student attendance, Family background of the student	Mushtaq and Khan (2012)
Geographical location, Transport facility Physical resource	Makworo et al. (2014), Bulala et al (2014), Greenfield et al. (2016)
Teacher's qualification, Teachers salary Teachers motivation to teach, Teachers experience, Teacher student ratio, Behavior of the non-teaching staff, Leadership School governing body	Betts (1995), Duflo (2015), Nyagosia et al. (2013), Komba et al. (2013)
Learning material, Pedagogy, Curriculum	Mushtaq and Khan (2012)
Teaching learning environment, Safety, Class size	Betts (1995), Nuri and Dağlı (2014)

Source: Prepared by the author.

4.2 Factor Analyzed Finding

We have already identified factors affecting quality education in our previous paper. From those, we identified 12 factors for this study based on feedback from various government and private schools from across the country. The respondents were asked to rate the factors on a scale of 1 to 5 (Likert 1932) in the increasing order of importance. In our previous paper, we received 135 responses in a questionnaire format. We then conducted reliability analysis and found internal consistency among the factors: Rotated factor matrix for the survey. Because of cross-loading, we conducted the varimax rotation and found loading of more than 0.5 to be reliable for the study.

The total variance was explained at the five stages for the factors affecting performance of secondary and higher secondary schools in the country. We shortlisted five factors whose eigen values were greater than 1 to explain more than 76% of the variances.

Bartlett's test of sphericity and the Kaiser-Meyer-Olkin measure of sampling adequacy: Our results showed that Bartlett's test of sphericity is significant ($p < 0.001$, $p = 0.000$) while the Kaiser-Meyer-Olkin measure is 0.684, which is greater than 0.6.

Reliability analysis was conducted with the data and the result suggests a Cronbach's Alpha value of 0.836 for the study. Usually, the reliability coefficient, Cronbach's Alpha value of above 0.7 is considered good in the Indian context. In this study, the reliability analysis result showed a good amount of internal consistency.

4.3 ISM Technique and Model Development

Once the factors were identified, we took inputs from 38 experts (Table 2) to prepare Excel pair-wise compare matrix. We did not find any consensus amongst researchers or from the experts we had consulted. However, extant literature shows that this number usually varies from eight (Janes 1988) to 42 (Prasad and Suri 2011) in the different studies carried out using similar methods.

Table 2. Experts classification

Age group	Experience (years)	Gender	Number
40–45	>15	Male	4
		Female	2
45–50	>20	Male	12
		Female	5
55–60	>30	Male	11
		Female	4

Source: Prepared by the author.

4.3.1 Conceptual Framework

ISM is a qualitative and interpretive method proposed by Warfield (1974). The method seeks to evaluate complex socioeconomic systems. It is also extensively used to perform structural mapping of interrelations of elements (Malone 1975). Once the mapping is done, unclear mental models are then translated into useful, well-defined models (Ahuja et al. 2009) to arrive at an appropriate outcome. Unlike many traditional modeling approaches, ISM can use words, graphs and mathematics to retain the qualitative information (Janes 1988).

This approach is also interpretive in nature because it can also incorporate expert opinions. It is also structural given that it gives an overall structure of the relationships between all factors under study. Finally, it is a modeling one because it presents a visual representation of the system.

4.3.2 Steps in ISM

ISM follows a sequence (see Fig. 1). The advantages of this process find adequate mention in literature. It has been put to use to study various subjects such as information technology, knowledge management and education (Mandal and Deshmukh 1994; Sahney et al. 2010).

The steps of ISM followed in this study are:

(1) Establishing contextual relationship between factors to identify the words that would relate them together. For our study, we used the words "leads to" to relate the factors and determine a structure.
(2) Constructing structural self-interaction matrix (SSIM) using pair wise comparison (see Table 3). We used one of the following four symbols to denote the nature of the relationship between factors I and j (I refers to the factor in the row of the matrix while j refers to a factor in the column):

A – Is related from i to j only;
B – Is related from j to i only;
C – Is related both from i to j and j to i; and
D – Is not related between factors i and j.

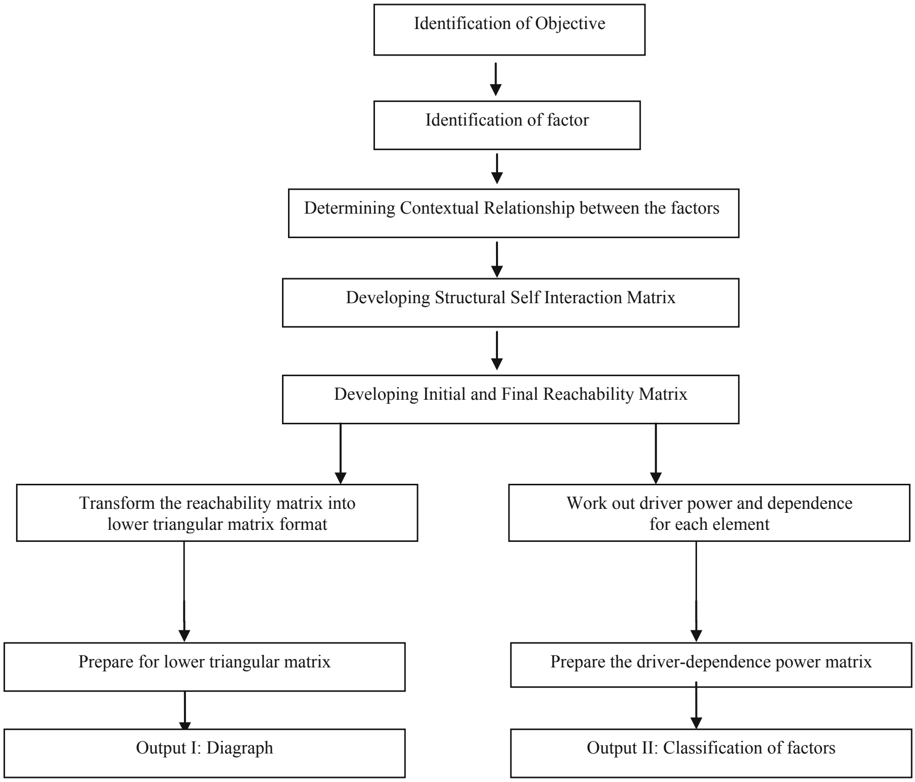

Fig. 1. ISM methodology (Source: Prepared by the author).

(3) Determine Reachability Matrix – It can be formed by square binary matrix M represended by a set S is if and only if

$$M + I = M = M^2$$

where I is the identity matrix. There must be unique reachability matrix M to any square binary matrix B there corresponds, these two are correlated by the Boolean equation:

$$(B + I)\text{fl n } (B + I)\text{fn} = (B + J)n + 1 = M$$

where the exponent n is a positive integer (< set elements of S) and The matrix M is the reachability matrix of B (Tables 6, 7, 8, 9 and 10).

(4) Generating a reachability matrix from SSIM and checking for transitivity: In the next step, SSIM was rehabilitated into reachability matrix (see Table 4) and Verified transitivity to develop the final reachability matrix (see Table 5).
(5) Level partitioning the reachability matrix: The final reachability matrix was partitioned on the basis of reachability and antecedent sets for each of the factors. Factors were grouped into six levels by a series of iterations (see Tables 4 and 11).

Table 3. Structural self-interaction matrix

Factor (Enter into the cells A, B, C or D as reflected in the drop down list)	Teacher motivation	Teacher qualification	Teacher student ratio	Geographical location	Transport facilities	Behavior non teaching staff	Curricular	Pedagogy	Learning materials	Teacher salary	Teacher experience	Class size
1	X	B	D	D	D	D	A	A	B	A	D	D
2		X	D	D	D	D	C	C	B	C	D	D
3			X	D	D	D	D	B	D	D	D	C
4				X	B	D	B	D	A	D	D	D
5					X	D	D	D	D	D	D	D
6						X	D	D	D	B	D	D
7							X	B	B	B	D	D
8								X	C	B	B	B
9									X	B	D	D
10										X	C	D
11											X	D
12												X

Source: Prepared by the author.

Table 4. Initial reachability matrix

Factor (Enter into the cells A, B, C or D as reflected in the drop down list)	Teacher motivation	Teacher qualification	Teacher student ratio	Geographical location	Transport facilities	Behavior non teaching staff	Curricular	Pedagogy	Learning materials	Teacher salary	Teacher experience	Class size
1	1	0	0	0	0	0	1	1	0	1	0	0
2	1	1	0	0	0	0	1	1	0	1	0	0
3	0	0	1	0	0	0	0	0	0	0	0	1
4	0	0	0	1	0	0	0	0	1	0	0	0
5	0	0	0	1	1	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0	0
7	0	1	0	1	0	0	1	1	1	1	0	0
8	0	1	1	0	0	0	1	1	1	0	0	0
9	1	1	0	0	0	0	1	1	1	0	0	0
10	0	1	0	0	0	0	1	1	1	1	1	0
11	0	0	0	0	0	0	0	1	0	1	1	0
12	0	0	1	0	0	0	0	1	0	0	0	1

Source: Prepared by the author.

Table 5. Final reachability matrix, Note: *Indicate changes made due to transitivity check

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1*	1*	1*	0	1*	1	1	1*	1	1*	0
2	1	1	1*	1*	0	1*	1	1	1*	1	1*	0
3	0	0	1	0	0	0	0	1*	0	0	0	1
4	0	0	0	1	0	0	0	0	1	0	0	0
5	0	0	0	1	1	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0	0
7	1*	1	0	1	0	0	1	1	1	1	0	0
8	1*	1	1	0	0	0	1	1	1	1*	0	1*
9	1	1	1*	1*	0	1*	1	1	1	1*	0	0
10	1*	1	1*	1*	0	1*	1	1	1	1	1	0
11	0	1*	1*	0	0	1*	1*	1	1*	1	1	0
12	0	1*	1*	0	0	1*	0	1	1*	0	0	1

Source: Prepared by the author.

Table 6. Partition of reachability matrix first iteration

Components	Reachability set	Antecedent set	Intersection set	Level
1	1, 2, 3, 4, 6, 7, 8, 9, 10, 11	1, 2, 7, 8, 9, 10	1, 2, 7, 8, 9, 10	
2	1, 2, 3, 4, 6, 7, 8, 9, 10, 11	1, 2, 7, 8, 9, 10, 11, 12	1, 2, 7, 8, 9, 10, 11	
3	3, 8, 12	1, 2, 3, 8, 10, 11, 12	3, 8, 12	I
4	4, 9	1, 2, 4, 5, 7, 10	4	
5	4, 5	5	5	
6	6	1, 2, 6, 9, 10, 11	6	I
7	1, 2, 4, 7, 8, 9, 10	1, 2, 7, 8, 9, 10, 11	1, 2, 7, 8, 9, 10	
8	1, 2, 3, 7, 8, 9, 10, 12	1, 2, 3, 7, 8, 9, 10, 11, 12	1, 2, 3, 7, 8, 9, 10, 12	I
9	1, 2, 3, 4, 6, 7, 8, 9, 10	1, 2, 4, 7, 8, 9, 10, 11, 12	1, 2, 3, 4, 7, 8, 9, 10	
10	1, 2, 3, 4, 6, 7, 8, 9, 10, 11	1, 2, 7, 8, 9, 10, 11	1, 2, 7, 8, 9, 10, 11	
11	2, 3, 6, 7, 8, 9, 10, 11	1, 2, 10, 11	2, 10, 11	
12	2, 3, 6, 8, 9, 12	3, 8, 12	3, 8, 12	

Source: Prepared by the author.

Table 7. Partition of reachability matrix - 2^{nd} iteration

Components	Reachability set	Antecedent set	Intersection set	Level
1	1, 2, 4, 7, 10,, 11	1, 2, 7, 10	1, 2, 7, 10	
2	1, 2, 4, 7, 10, 11	1, 2, 7, 10, 11, 12	1, 2, 7, 10, 11	
4	4	1, 2, 4, 5, 7, 10	4	II
5	4, 5	5	5	
7	1, 2, 4, 7, 10	1, 2, 7, 10, 11	1, 2, 7, 10	
9	1, 2, 4, 7, 9, 10	1, 2, 4, 7, 9, 10, 11, 12	1, 2, 4, 7, 9, 10	II
10	1, 2, 4, 7, 10, 11	1, 2, 7, 10, 11	1, 2, 7, 10, 11	
11	2, 7, 10, 11	1, 2, 10, 11	2, 10, 11	
12	2, 12	12	12	

Source: Prepared by the author.

Table 8. Partition of reachability matrix - 3rd iteration

Components	Reachability set	Antecedent set	Intersection set	Level
1	1, 2, 7, 10, 11	1, 2, 7, 10	1, 2, 7, 10	
2	1, 2, 7, 10, 11	1, 2, 7, 10, 11, 12	1, 2, 7, 10, 11	III
5	5	5	5	III
7	1, 2, 7, 10	1, 2, 7, 10, 11	1, 2, 7, 10	III
10	1, 2, 7, 10, 11	1, 2, 7, 10, 11	1, 2, 7, 10, 11	III
11	2, 7, 10, 11	1, 2, 10, 11	2, 10, 11	
12	2, 12	12	12	

Source: Prepared by the author.

(6) Draw ISM-based model.

To classify the factors, MICMAC analysis was done (see Fig. 3). MICMAC refers to Matrice d'Impacts Croisés Multiplication Appliquée à un Classement. We then plotted the factors on a graph based on their dependence and driving power, as calculated from the final reachability matrix. The four quadrants indicate the four groups with the following characteristics:

Quadrant I: autonomous factors which neither have high dependence nor high driving power.
Quadrant II: dependent factors with low driving power and high dependence.
Quadrant III: linkage factors with high driving power and high dependence.
Quadrant IV contained drivers with high driving power and low dependence.

Table 9. Partition of reachability matrix - 4th iteration

Components	Reachability set	Antecedent set	Intersection set	Level
1	1, 11	1	1	
11	11	1, 11	11	IV
12	12	12	12	IV

Source: Prepared by the author.

Table 10. Partition of reachability matrix - 5th iteration

Components	Reachability set	Antecedent set	Intersection set	Level
1	1	1	1	V

Source: Prepared by the author.

5 Results and Discussion

5.1 ISM-Based Model

Figure 2 shows the ISM model. It demonstrates the interrelationships and the hierarchy of the factors. The contextual relation or wording we have used is "leads to", accordingly, each arrow in the model represents this. We divide the 12 factors in five levels, with the fifth level indicating the highest level of importance.

From the model, it becomes clear that teacher motivation is the most important factor, influencing both teacher experience and class size, which in turn, are related to teacher qualification, transport facilities, curriculum, and teacher salary. At the third level, the model leads to geographical location and learning materials. The fourth level leads to the remaining three factors of the first level, i.e., behavior of non-teaching staff, pedagogy and teacher student ratio.

5.2 Driving Power - Dependence Graph

Figure 3 represents driving power dependence graph classifying these factors. These include behavior of the non-teaching staff, teacher-student ratio and geographical location. Linkage factors include curriculum, teacher salary, teacher qualification, pedagogy and learning factors. Dependent factors include teacher experience, teacher motivation and class size. Of these, teacher motivation has the highest dependent power which implies that it should be given top priority to improve the quality of education. There is only one autonomous factor, teacher experience, which indicates that all the factors are important and an integral part of the system.

Another interesting highlight is that five out of twelve factors under study are interdependent in that they influence others, and are in turn, influenced by them. Any change in any one of them, will have a ripple effect on other factors as well.

Table 11. Partition of reachability matrix - 6th iteration

Components	Reachability set	Antecedent set	Intersection set	Level
3	3, 8, 12	1, 2, 3, 8, 10, 11, 12	3, 8, 12	I
6	6	1, 2, 6, 9, 10, 11	6	I
8	1, 2, 3, 7, 8, 9, 10, 12	1, 2, 3, 7, 8, 9, 10, 11, 12	1, 2, 3, 7, 8, 9, 10, 12	I
4	4	1, 2, 4, 5, 7, 10	4	II
9	1, 2, 4, 7, 9, 10	1, 2, 4, 7, 9, 10, 11, 12	1, 2, 4, 7, 9, 10	II
2	1, 2, 7, 10, 11	1, 2, 7, 10, 11, 12	1, 2, 7, 10, 11	III
5	5	5	5	III
7	1, 2, 7, 10	1, 2, 7, 10, 11	1, 2, 7, 10	III
10	1, 2, 7, 10, 11	1, 2, 7, 10, 11	1, 2, 7, 10, 11	III
11	11	1, 11	11	IV
12	12	12	12	IV
1	1	1	1	V

Source: Prepared by the author.

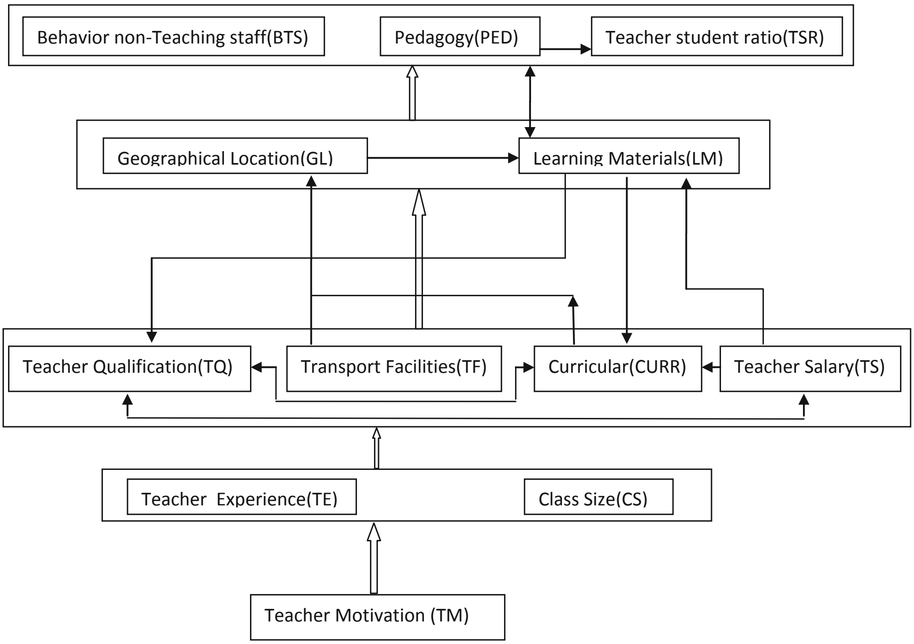

Fig. 2. ISM model (Source: Prepared by the author).

Table 12. Generate driving power and dependence weight

	1	2	3	4	5	6	7	8	9	10	11	12	
Factor (Enter into the cells A, B, C or D as reflected in the drop down list)	Teacher motivation	Teacher qualification	Teacher student ratio	Geographical Location	Transport facilities	Behavior non-teaching staff	Curricular	Pedagogy	Learning materials	Teacher salary	Teacher experience	Class size	Driving power
1	1	1	1	1	0	1	1	1	1	1	1	0	10
2	1	1	1	1	0	1	1	1	1	1	1	0	10
3	0	0	1	0	0	0	0	1	0	0	0	1	3
4	0	0	0	1	0	0	0	0	1	0	0	0	2
5	0	0	0	1	1	0	0	0	0	0	0	0	2
6	0	0	0	0	0	1	0	0	0	0	0	0	1
7	1	1	0	1	0	0	1	1	1	1	0	0	7
8	1	1	1	0	0	0	1	1	1	1	0	1	8
9	1	1	1	1	0	1	1	1	1	1	0	0	9
10	1	1	1	1	0	1	1	1	1	1	1	0	10
11	0	1	1	0	0	1	1	1	1	1	1	0	8
12	0	1	1	0	0	1	0	1	1	0	0	1	6
Dependence	6	8	8	7	1	7	7	9	9	7	4	3	

Source: Prepared by the author.

	Driving power	Dependence
TM	10	6
TQ	10	8
TSR	3	8
GL	2	7
TF	2	1
BTS	1	7
CURRI	7	7
PED	8	9
LM	9	9
TSAL	10	7
TE	8	4
CZ	6	3
	6	6.36

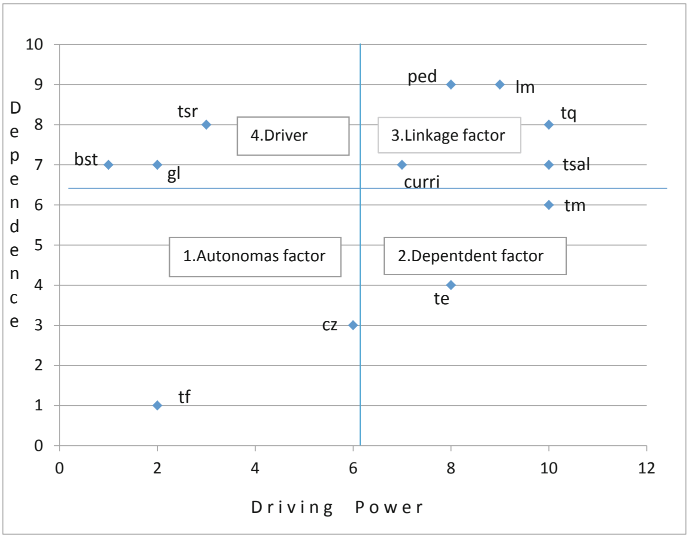

Fig. 3. Driving power dependence graph (Source: Prepared by the author).

6 Conclusion, Limitations and Future Research

This study identifies and models factors affecting the quality of primary and secondary education in India. We find all the identified factors to be important. Of the 12 factors studied, we find teacher motivation to be the most important factor (Table 12). Significant interrelations are also found which are sometimes not exposed by mere observation.

Many school lack focus and clarity in devising and implementing methods to impart and improve quality of the education they provide. Consequently, they fail to make the necessary impact. The framework proposed in this paper can be utilized by schools to develop practical and effective strategies to improve quality of education.

According to Watson (1978), ISM is slightly inflexible in nature “because of the difficulty of adding, deleting, combining or redefining elements in the course of an ISM session, the inability to easily incorporate minority viewpoints, and the inability to review the structure as it evolves.” Also, the contextual relations used in the study are based on the feedback of the experts. While these inter-relations provide deep insights, their practical implications may be different in real time settings. Therefore, it would be better if the insights are used in individual school settings, however, it is important to keep in mind that this could translate into money and time constraints.

After the factors and sub-factors are identified through factor analysis, we will need to define the hierarchy of factors using a graph. The next step would be to construct a mathematical model of those factors using the fuzzy embedded analytical hierarchy process. The findings would have to be validated through some schools selected from different parts of the country. As suggested by Kannan et al. (2008), the ISM-based model is hierarchical in nature and does not showcase the relative weights associated with each factor. To do this, we can use the analytic network process. Further research could focus on calibrating the importance of those major factors and sub-factors using the Fuzzy Analytical Hierarchy Process (FAHP) to determine the ranking of the secondary and higher secondary schools in India. Also, researchers can use structure equations modeling to test for the various schools on certain inputs and validate the findings with the existing ranking.

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Improved User Adaptable Human Fall Detection and Verification Using Statistical Analysis

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Abstract. Human fall detection systems are an important part of human monitoring systems especially for elderlies. Different studies were conducted using varieties of sensor to develop systems to accurately classify unintentional human falls from other activities of daily life. The major issues with the current studies using depth maps were the use of single threshold based algorithms and highly complex machine learning to detect falls. Therefore, the available systems cannot cater for the detection of fall events from people with different physical capabilities and differing environments they are living. This study proposes a user adaptable fall detection system with a statistical analysis based fall verification to overcome the issues of related works.

Keywords: Falls · Human fall · Depth maps · Assistive living · Daily activities · Fall risk levels

1 Introduction

Fall detection systems play an important role in providing better living and allowing elderly people to live in their own home independently as long as possible without having to change their life style. Fall detection systems are vitally important since statistics [1, 2] has shown that fall is the main reason of injury related death for seniors aged 79 [3, 4] or above and it is the second common cause of injury related (unintentional) death for all ages [5, 6]. Furthermore, fall is the biggest threat among all other incidents to elderly and those people who are in need of support [3, 7–16]. Accordingly for elderly people fall can have severe consequences, especially if not attended in a short period of time [17]. Human falls also represents the main source of morbidity and mortality among elderly [18].

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Several approaches with different sensors are used to develop such systems including wearable sensors, non-wearable sensors and vision based sensors. Among them wearable based devices such as belts and other non-wearable devices such as floor vibration sensor are very cheap and easy to setup. On the other hand, such sensors are prone to generate high false alarms and therefore are not reliable. As far as vision based devices are concerned, they are very accurate in classifying falls from other activities of daily life. Even though vision based devices are expensive and difficult to setup, they are reliable and generates less false alarms. But, such systems using normal cameras are subject to privacy issues and therefore are often rejected by users. Vision based systems using depth sensors could eliminate this issue with the benefit of no colour videos being captured. But the related works using depth sensor, basically uses joint measurements or employs complex algorithms including machine learning to detect fall events from other daily life activities. They basically used a single algorithm to detect falls without considering the differences in nature of falls from different people from different communities and the environment they are living. It is also to be noted that the nature of a fall event and the impact of it will differ from person to person and use of a single algorithm can easy miss-interpret any other similar fall like activity as a fall event. Therefore, this paper proposes a reliable human fall detection using depth sensor which is adaptive to the user's physical condition and uses statistical analysis to verify fall events if the position of the joints fail to classify fall from other daily life activity.

2 Related Works

The related works either used joint measurement from depth maps or employed some algorithms with features from machine learning. Some of the studies that employed joint measurement uses velocity or the speed and the changes in height pattern for fall detection. The method proposed by Nghiem et al., used the vertical speed and the distance from the ground to head and the centroid for fall detection. This algorithm first detects possible head position and then based on the head position, the subject is recognized by detecting head and shoulder. Then calculates its vertical speed and checks if the falling conditions is satisfied which is speed threshold of 2 m/s and height threshold of 0.5 m [19]. In another study, a fall is detected by thresholding the distance between these joint to the floor, if the floor is visible or detected. If the floor plane is not detected than a second algorithm is used which depends on the skeleton coordinate system. The second algorithm detects a fall if the y-coordinates of the joints mentioned are less than a given threshold [20]. A new fall detection algorithm method proposed by Yang et al. uses the distance of the centroids and angles of an ellipse for fall detection. The algorithm detects a fall event when the distance and angle between the ellipses and floor plane are lower than some threshold [21]. While Bian et al. presented a method for fall detection based on distance between human skeleton joints to the floor, and the joint velocity. The velocity of the joint hitting the floor is used to distinguish the fall accident from sitting or lying down on the floor. Fall is detected if the distance between the head and floor is lower than a recover threshold for certain period of time. The distance threshold used for the three joints and the recover threshold used for head is adaptive to the height of the person [22].

In another study presented by Planinc and Kampel, uses a bounding box based approach for fall detection. A fall is detected, if the major orientation of the person is parallel to floor and the height of the spine is near the floor. The developed algorithm using Kinect depth sensor was evaluated against State-of-the-Art approaches using 2D sensors or microphones. The results after improving the tracking of the skeleton when the person leaves the frame, shows an accuracy of 87.5% with 100% precision and 77.5% recall [23]. Another study, presented an unobtrusive fall detection system, running in real time with a novel algorithm which is made-up of several steps. A fall from sitting or standing is confirmed if the body motion gets involved in Y or Z coordinate. A sudden fall is also monitored by the time scale of the fall from the human towards the respective axis [24].

Another method proposed by Ma et al., uses a combination two computer vision approach, fall characterization using shape based approach and a machine learning classifier to identify human fall from other activities. At first human silhouette is extracted from depth images. Adaptive Gaussian Mixture Model (GMM) is used for human segmentation from background. The second step involves finding of the features of the detected subject. Kinect sensor was placed at a height of 1.5 m and features were extracted using c++ for experimental dataset. Experimental results show that human silhouettes are extracted even with no light conditions and the proposed approach is able to gain an accuracy of 86.83% using single camera [25].

3 Method

The proposed fall detection uses the changes in velocity and height which is applied in different order depending on the fall risk levels of the subject. If fall risk level of the user is high, then the algorithm will switch to detect fall by considering the fall risk level and possibly fall is verified using statistical analysis rather than using joint positions for fall confirmation.

The proposed fall detection methodology primarily consists of two stages for fall detection. The first stage will identify any potential fall event and the second stage will confirm or verify the fall event. These two stages of fall detection are designed into six processes in the proposed fall detection algorithm. The first process generates skeleton data from the depth map if a person is detected in the scene. The next process will compute the initial height, velocity and fall risk factors. This process can generate a potential fall alert if the initial velocity is high and pass to Process 5, for fall confirmation. Whereas, Process 3, which starts if the fall risk is high, will detect such an alert (potential fall) using another new velocity and height. On the other hand, Process 4, which starts if initial velocity is low, will use height, activity and acceleration to identify a potential fall event. These four processes belong to Stage 1, of fall detection and the remaining two processes (Process 5 and Process 6) will do the fall confirmation or verification of Stage 2 as shown in Fig. 1.

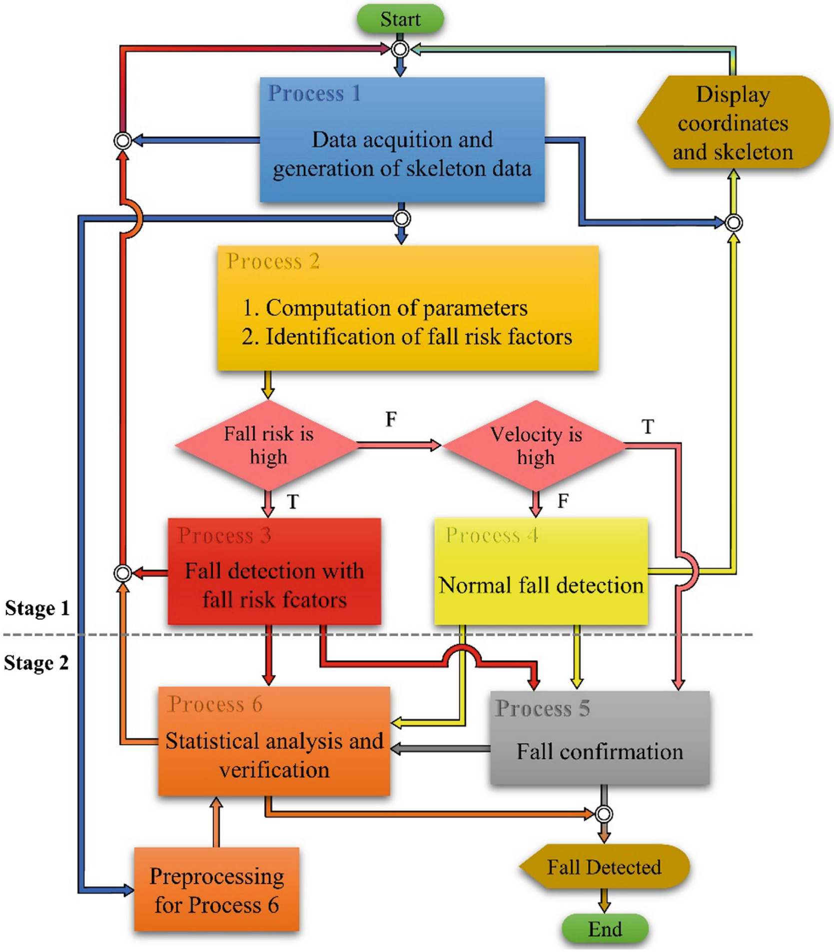


Fig. 1. Proposed fall detection algorithm.

The Process 5, is dedicated for fall confirmation if the velocity from Process 2 or Process 3, is flagged as high and no activity or high acceleration is flagged from Process 4. In case, if the Process 5, couldn't confirm a fall, then the Process 6, will be executed which will play the role of fall verification. The Process 6, will also be executed from Process 3 and Process 4 and it is primarily designed to detect fall using statistical analysis. The small block below the Process 6, block is dedicated to do all pre-processing required for fall detection using statistical analysis. With the proposed algorithm in Fig. 1, a fall is normally confirmed by Process 5, or verified through Process 6, or directly detected from Process 6. Once a fall event is confirmed, the third stage of the proposed algorithm which is responsible for fall injury estimation will be executed.

The parameters used in this proposed algorithm are the height and speed of the joints for basic activity classification. The parameters used for the identification of the fall risk level and verification of fall event are step_symmetry, trunk_sway, spread_arm, the total head drop observed, standard deviation of the person's height with respect to floor at any given time and fraction of number of frames where a drop of person's height with respect to previous frame is observed over the total number of frames.

The height of any joint is computed by applying the joint coordinates and floor plane equation can be applied to the following Eq. (1).

$$\text{Height (H)} = \frac{|\mathrm{Ax} \; + \mathrm{By} \; + \mathrm{Cz} \; + \mathrm{D}|}{\sqrt{\left(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2\right)}} \tag{1}$$

Where: x, y and z are the coordinates of the joint.

The Speed (Magnitude component of velocity) is calculated by dividing the distance travelled over the time taken as shown in Eq. (2). The time taken for the movement is 1/15 s, because the sensor generates 30 frames per second and the joint position is taken after skipping one frame (time for two frames).

$$\text{Speed (Magnitude component of velocity)} = \frac{\mathrm{D_c} - \mathrm{D_p}}{\mathrm{t_c} - \mathrm{t_p}} \,\text{Meter/second} \tag{2}$$

Where D_c is the Current Distance (current joint coordinate), D_p is the Previous Distance (previous joint coordinate), t_c is the current time in second and t_p is the previous time in second.

If the direction is vertically (irregular) to any side (any axis), the distance travelled, cannot be simple calculated by subtracting the position between two frames on any axis, because the changes is not on the axis and so if the changes is considered as on the axis, then the distance will be less than the actual distance travelled. Once distance for any irregular movement is calculated the magnitude part of the velocity can be calculated by using the Eq. (3).

$$\text{D (distance for irregular movements)} = \sqrt{\left((\mathrm{y} - \mathrm{y}')^2 + (\mathrm{x} - \mathrm{x}')^2\right)} \tag{3}$$

For statistical analysis the total head drop is calculated by taking the difference of height from the beginning of the fall to the end of the fall event. In other words, the difference of head position from the beginning to the end of a fall event. This parameter is calculated by subtracting the observed head height at the start and at the end of the pre-processing block (beginning of Process 6) in Fig. 1.

Standard deviation is a measure of the amount of variation or how much the data is spread out. If the data values are maintained at a smaller interval than the standard deviation value will be small and if the data are spread out than the standard deviation value will be higher. This can be used to classify unintentional fall and lying on floor from other activities of daily life. Because for unintentional fall or lying on floor from standing, the height decreases to more than 75% of the height while standing, thus the spread will be higher and therefore standard deviation will have noticeable difference with respect to the standard deviation from other activities. Standard deviation of the head height is computed by taking the head height of all the frames. The head height of each frame is computed and stored by the pre-processing block for Process 6 in Fig. 1.

Standard deviation (σ) is computed using the following formula [26] in Eq. (4).

$$\sigma = \sqrt{\frac{1}{N}\sum\nolimits_{i=1}^{N}(x_i - \mu)^2} \tag{4}$$

Where $\sum$ means "sum of", μ is mean of the data sets, x is a value in data set, N is the number of data elements and i is the i^{th} data value.

The fraction of number of frames where the person's height is decreased with respect to previous frame over the total number of frames is calculated through a loop. The loop will calculate the subject's height at the current frame and compare it with the previous frame. If the height in the current frame is below the previous frame than the loop will increment the frame count. At the end of the Process (pre-processing for Process 6 of fall detection algorithm), the frame count will be divided with the total number of frames to get the fraction of frames where height drop has observed. It is than multiplied by 100 to convert to percentage.

Fall risk factors are basically used to identify any abnormality from the movements and identify any existence of physical weakness that can easily cause an unintentional fall. The step_symmetry, is the estimate of the step inequality which can be realized by measuring the left and the right step lengths. Step length is the distance between left and right step, which can be measured using x-axis or z-axis coordinates depending on the direction of the movement. If the direction of the movement is on x-axis then the following Eq. (5), is used to compute the Step_symmetry and if the direction of the movement is on z-axis then simple z-values are used instead of x-values in the Eq. (5). Left and right step length and trunk sway is illustrated in Fig. 2.

$$\text{Step\_symmetry} = (R\_foot_x - L\_foot_x)_{PF} - (R\_foot_x - L\_foot_x)_{CF} \tag{5}$$

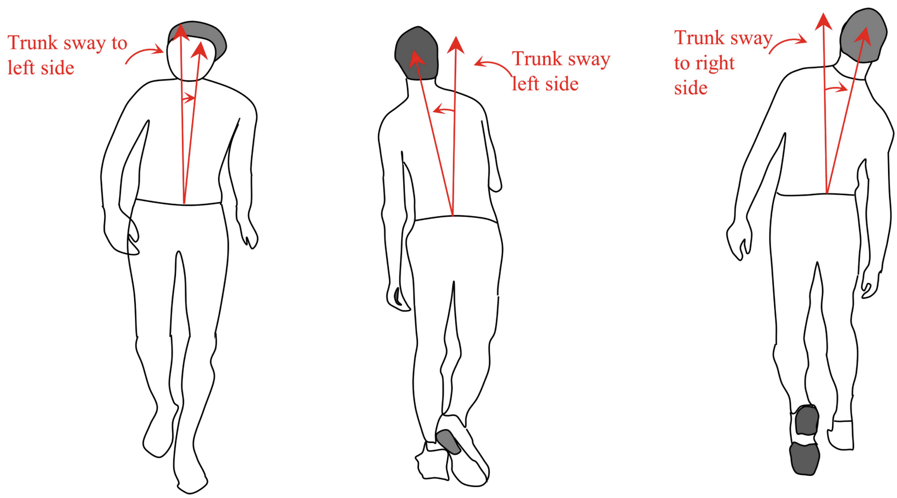
Where, *R_foot* is the right foot, *L_foot* is the left foot, x is the x-value or x-axis coordinate value, *PF* is the Previous Frame and *CP* is Current Frame.

Trunk_sway is a measure of how far the subject is bend side to side from trunk and it calculated by taking the changes of torso position with respect to the hip position. The amount of bend or the Trunk_sway value is simply an average of the difference of torso and hip position between frames. This variation can be calculated by taking x-axis values, if the direction of the movement is on z-axis as shown in the following Eq. (6) and using z-axis values instead of x-values if the direction of movement is on x-axis.

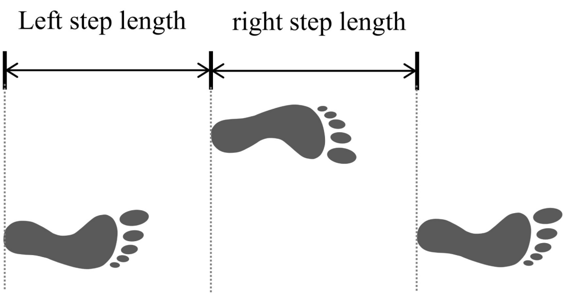
$$\text{Trunk\_sway} = \frac{\left(Torso_x - \left(\frac{L\_hip_x + R\_hip_x}{2}\right)\right)_{PF} + \left(Torso_x - \left(\frac{L\_hip_x + R\_hip_x}{2}\right)\right)_{CF}}{2} \tag{6}$$

Where, *L_hip* is the left hip position and *R_hip* is the right hip position.

Spread_arm is a measure of how much the two arms are spread. This parameter is computed by taking the difference of torso position and the two (left and right) arms. Similar like Trunk_sway, this value is also calculated from x-axis if the direction of the movement is on z-axis as shown in Eq. (7) and using z-axis values instead of x-values if the direction is on the x-axis. The average of the distance of the two arms to the torso



(a)Trunk sway



(b)Step length

Fig. 2. Illustration of trunk sway and step length

are threshold between frames to identify any action where the subject is spreading the arms to balance the body or trying to hold something to control the body.

$$\text{Spread\_arm} = \left(\frac{(Torso_x - R\_arm_x) + (Torso_x - L\_arm_x)}{2}\right)_{CP} - \left(\frac{(Torso_x - R\_arm_x) + (Torso_x - L\_arm_x)}{2}\right)_{PF} \tag{7}$$

Where, R_arm is right arm and L_arm is the left arm.

4 Experimental Results and Discussion

The experimental results showed that the proposed system can classify human fall from other activities of daily life using height changes and velocity of the subject together with the position of the subject after a fall event. The order of velocity and height of the subject used in the proposed algorithm greatly helped in eliminating human activities that are closely analogous to falls, before going to the final stage for fall confirmation. Thus, the algorithm proved to reduce the error rate since those activities that are mostly misinterpreted as fall (such as lying on floor) is classified out here before fall confirmation. This is not taken care in previous studies such as in [19, 20, 22] which uses distance and velocity for fall detection. The same goes to the distance and angled based approach used in [21] and the approach used in [23] which uses orientation and height of the subject for fall detection. The results of some of the critical events are illustrated in this section.

One of a transitional activity which can lead to a fall event is sitting on floor from standing. Fall while trying to sit on floor is worse than a fall event while trying to sit on chair. This is because in this case, the subject may not have anything (a chair or any other object) to hold for avoiding the fall or minimizing the injury. The following Fig. 3, shows the observed changes in height of the subject and the corresponding instant velocities for a fall event while trying to sit on floor from standing posture. The series of action include standing posture to lowing the upper body with a bend over torso and once the body gets balanced with the hand and limbs, the hip is placed on floor and the balance transferred. All the series of actions went smoothly until the body is almost balanced with the hand which took 6.9 s. After that the subject tried to sit on floor and transfer the body weight to hips in 1.43 s, but before the body gets balanced, a fall event was induced and in 1.23 s the subject was on floor. In the sample of data shown in Fig. 3, the height of the subject was not on floor for about 1.8 s (from 9.56 s to 11.366 s) even after the fall event. The reason is the fact that the impact of the fall event was not severe, and the subject completely rested on floor at 11.366 s. Therefore, the actual duration of the fall event was 3.03 s.

Another such event illustrated in the following Fig. 4, shows the changes in height pattern and the instantaneous velocity for a fall event while trying to sit on a chair. The total duration of the whole event (from standing posture to completely rest the body on floor) was 8.4 s. But the actual fall event started at 5.7 s when the subject lost control of the body. Therefore, the time taken to fall was 2.7 s. At the beginning, the subject was standing and then started to sit on a chair. At 5.7 s from the beginning of the event, the subject lost control and induced the fall event.

One of the observation was that the changes in height pattern itself cannot classify all the activities. The major issue arises when lying on floor and fall from standing are concerned. The two events were very similar in terms of height change pattern. But the two events were easily classified using instant speed and the time taken for the action. The unique flow of the proposed algorithm could eliminate this issue, since the algorithm was designed by considering all those matters and it consists of many flows to cover such different scenarios.

It was also found that the computed parameters from the depth images were subject to an average error of ± 4.66 cm. Even though, the parameters generated from running

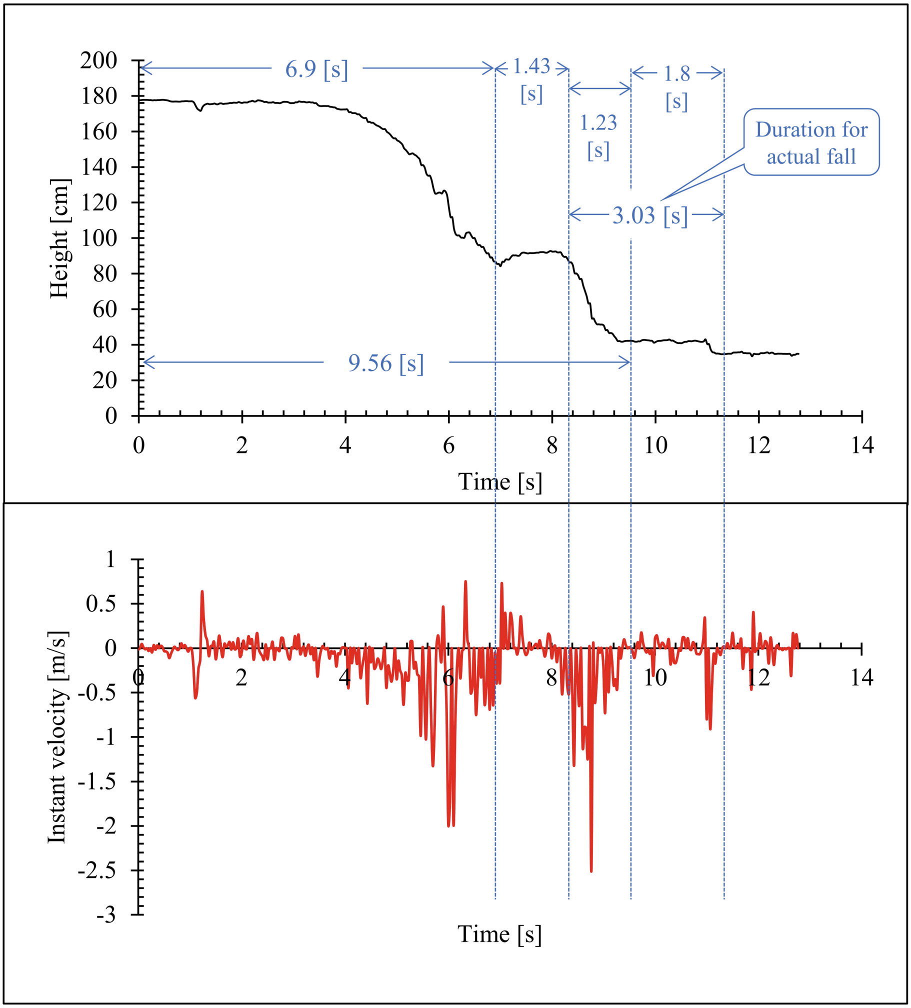


Fig. 3. Observation from a fall event while trying to sit on floor

was found to be with a high error rate, the computed parameters for the other activities including walking was accurate enough for the application. The proposed algorithm was designed by considering this error ranges. In addition, it was found that the use of more than one joints greatly helped to minimize these errors and most importantly the issue of obstacles blocking the view of the subject to the sensor. Furthermore, the issue of loss of information between frames was properly dealt by the proposed statistical analysis.

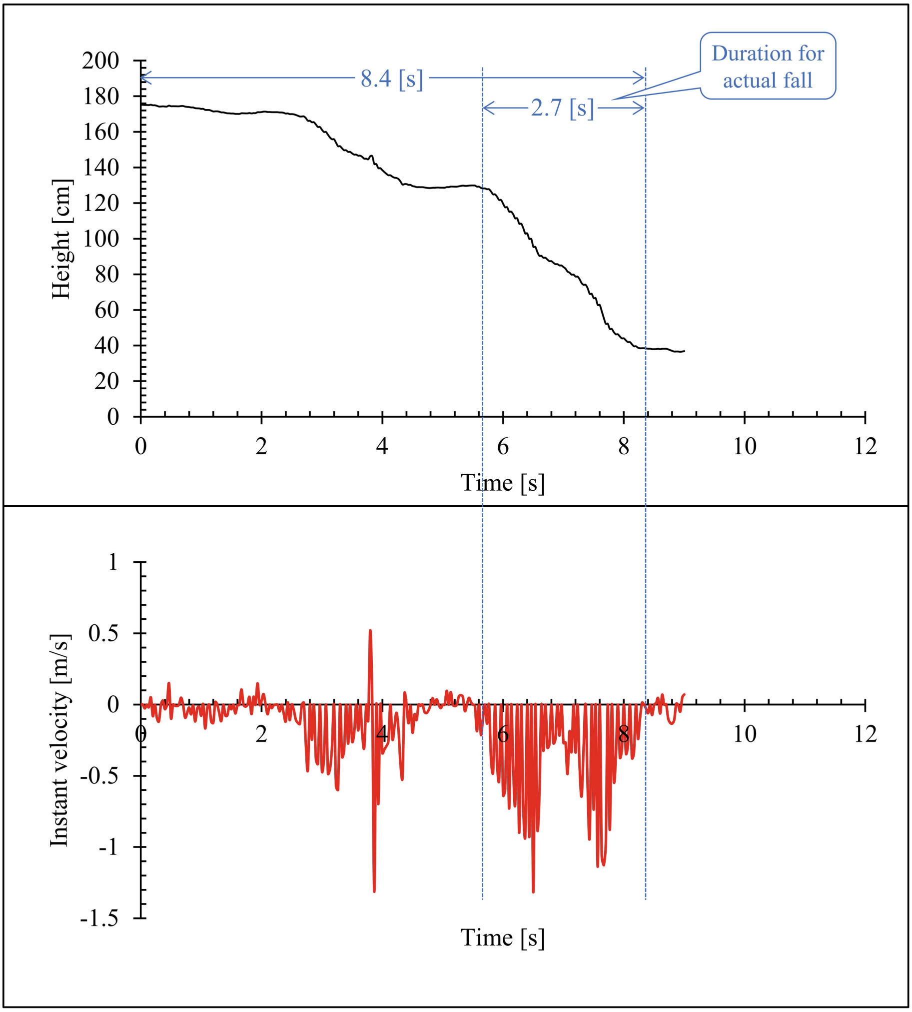


Fig. 4. Observations from a fall event while trying to sit on a chair

5 Conclusion

This paper proposed a fall detection algorithm which is adaptable to the user based on the physical strength or the fall risk level at any given time. The proposed algorithm for fall detection, checks the fall risk level at present time before proceeding with fall detection. The algorithm performs different approach for fall detection depending on the fall risk level of the user. A fall event is confirmed using position of joint and statistical analysis after a potential fall activity. The use of statistical analysis greatly helped to minimize false alarms because the parameters used for this analysis is computed just before the potential fall event and the process employed can handle loss of information from few frames. Furthermore the identification of fall risk level of the user reduced the complexities of the classification of fall events from other fall like activities by adjusting the algorithm and allowing the process to flow into the correct fall confirmation procedure.

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Convergence of Soft Filter and Soft Compactification in Redefined Soft Topological Spaces

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Abstract. In this paper we introduce a notion of soft filter and its convergence in redefined soft topological spaces. The relation of convergence between soft nets and soft filters is studied. Finally analogue of one point compactification is dealt with.

Keywords: Soft set · Soft element · Soft compactness · Weak soft compactness · Soft cluster element · Weak soft cluster element · Soft filter · Soft subfilter · Soft compactification

1 Introduction

With the passage of time, the uncertainty problems in different fields of real life like engineering, medical science, economics became a headache for the mathematicians where the traditional crisp set theory failed to answer. The fuzzy set theory of Zadeh [27] served in this purpose for times. On the virtue of some difficulties of the parameterization process of fuzzy set theory Molodtsov [16] in 1999 introduced the soft set theory for modeling uncertainty in a parametric manner. He also shown several applications of soft sets in different field. Then Maji et al. in [14] defined some basic operations on soft sets. After that researches has developed different mathematical structures such as algebraic, topological etc. on soft sets (for references please see [1–5,7,9,11,12,15,18–21,24]). Shabir and Naz [25] in 2011 for the first time tried their hands over soft topological spaces. In 2015, Shi et al. in [26] commented that the soft topology due to Shabir and Naz can be interpreted as a crisp topology. Recently a new soft topology was introduced by Chiney and Samanta in [4] which is different from the soft topology of Shabir and Naz. In their redefined soft topology Chiney and Samanta used the notions of elementary union and elementary intersection which is non distributive and the elementary complement do not obey the law of excluded middle. Recently Roy et al. [22] studied the notions of compactness and connectedness and Dutta

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et al. [10] introduced the notions of convergence of soft nets in this setting. The focus of this paper is to introduce the notions of soft filter and soft compactification and also to investigate the relation between the soft compactness and soft filters.

Straightforward proofs are omitted.

2 Preliminaries

Soft set is introduced by Molodtsov [16] in 1999. Later Maji et al. [14], Ma et al. [13], Nazmul and Samanta [17] worked with this. In this paper we follow the definition considered by Nazmul and Samanta [17] which is as follows:

Definition 1 *[17]. Let X be a universal set and A be a set of parameters. Let $P(X)$ denotes the power set of X. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of α approximate elements of the soft set (F, A).*

Operations on soft sets such as union, intersection, complement and absolute soft, set null soft set etc. are taken from [17].

Definition 2 *[6] . Let X be a non-empty set and A be a non-empty parameter set. Then a function, $\tilde{x} : A \rightarrow X$ is said to be a soft element of X. A soft element $\tilde{x}$ of X is said to belong to a soft set (F, A) over X, which is denoted by $\tilde{x}\tilde{\in}(F, A)$, if $\tilde{x}(\lambda) \in F(\lambda)$, $\forall \lambda \in A$.*

Let X be an initial universal set and A be a non-empty parameter set. Throughout the paper we consider the null soft set $(\tilde{\Phi}, A)$ and those soft sets (F, A) over X for which $F(\alpha) \neq \phi$, $\forall \alpha \in A$. We denote this collection by $S(\tilde{X})$ and $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft elements such that $\bar{r}(\lambda) = r$, $\forall \lambda \in A$.

Definition 3 *[8]. For any two soft sets $(F, A), (G, A) \in S(\tilde{X})$,*

(i) elementary union of (F, A) and (G, A) is denoted by $(F, A) \Cup (G, A)$ and is defined by $(F, A) \Cup (G, A) = SS(\mathscr{B})$, where, $\mathscr{B} = \{\tilde{x}\tilde{\in}(\tilde{X}, A) : \tilde{x}\tilde{\in}(F, A)$ or $\tilde{x}\tilde{\in}(G, A)\}$; i.e. $(F, A) \Cup (G, A) = SS(SE(F, A) \cup SE(G, A))$.

(ii) elementary intersection of (F, A) and (G, A) is denoted by $(F, A) \Cap (G, A)$ and is defined by $(F, A) \Cap (G, A) = SS(\mathscr{B})$, where, $\mathscr{B} = \{\tilde{x}\tilde{\in}(\tilde{X}, A) : \tilde{x}\tilde{\in}(F, A)$ and $\tilde{x}\tilde{\in}(G, A)\}$ i.e. $(F, A) \Cap (G, A) = SS(SE(F, A) \cap SE(G, A))$.

Definition 4 *[8]. For any soft set $(F, A) \in S(\tilde{X})$, the elementary complement of (F, A) is denoted by $(F, A)^{\mathbb{C}}$ and is defined by $(F, A)^{\mathbb{C}} = SS(\mathscr{B})$, where, $\mathscr{B} = \{\tilde{x}\tilde{\in}(\tilde{X}, A) : \tilde{x}\tilde{\in}(F, A)^C\}$ and $(F, A)^C$ is the complement of (F, A).*

Proposition 1 *[8]. For any collection of soft sets $(F_i, A) \in S(\tilde{X})$.*

(i) $\underset{i \in \Delta}{\Cup}(F_i, A) = \underset{i \in \Delta}{\tilde{\cup}}(F_i, A)$.

(ii) $\underset{i\in\Delta}{\Cap}(F_i, A) = \underset{i\in\Delta}{\tilde{\cap}}(F_i, A)$ *if* $\underset{i\in\Delta}{\Cap}(F_i, A) \neq (\tilde{\Phi}, A)$.

Definition 5 *[4]. Let τ be a collection of soft sets of $S(\tilde{X})$. Then τ is said to be a soft topology on $(\tilde{X}, A)$ if*

(i) $(\tilde{\Phi}, A)$ and $(\tilde{X}, A)$ belong to τ.
(ii) the elementary intersection of any two soft sets of τ belong to τ.
(iii) the elementary union of any number of soft sets of τ belong to τ.

The triplet $(\tilde{X}, \tau, A)$ is called a soft topological space.

Definition 6 *[4]. In a soft topological space $(\tilde{X}, \tau, A)$, the members of τ are called soft open sets in $(\tilde{X}, \tau, A)$ and a soft set $(F, A) \in S(\tilde{X})$ is called a soft closed set in $(\tilde{X}, \tau, A)$ if its relative complement $(F, A)^C \in S(\tilde{X})$ and $(F, A)^{\mathbb{C}} \in \tau$.*

Definition 7 *[4]. Let $(\tilde{X}, \tau, A)$ be a soft topological space and $(F, A) \in S(\tilde{X})$. Then the soft closure of (F, A) is denoted by $\overline{(F, A)}$ and is defined as the elementary intersections of all soft closed super sets of (F, A).*

Definition 8 *[4]. Let $(\tilde{X}, \tau, A)$ be a soft topological space. Then a subcollection $\mathscr{B}$ of τ, consisting $(\tilde{\Phi}, A)$, is said to be a soft base for τ if $\forall \tilde{x}\tilde{\in}(\tilde{X}, A)$ and for any soft open set (F, A) consisting the soft element $\tilde{x}$, there exists $(G, A) \in \mathscr{B}$ such that $\tilde{x}\tilde{\in}(G, A)\tilde{\subseteq}(F, A)$.*

Definition 9 *[4]. Let $(\tilde{X}, \tau, A)$ be a soft topological space. Then $(F, A)[\neq (\tilde{\Phi}, A)] \in S(\tilde{X})$ is said to be a soft neighbourhood (soft nbd) of a soft element $\tilde{x}$ if there exists a soft open set (G, A) such that $\tilde{x}\tilde{\in}(G, A)\tilde{\subseteq}(F, A)$. The soft nbd system at a soft element $\tilde{x}$, denoted by $\aleph_\tau(\tilde{x})$, is the family of all its soft nbds.*

Definition 10 *[4]. Let $(\tilde{X}, \tau, A)$ and $(\tilde{Y}, \nu, A)$ be two soft topological spaces and $f : SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft function associated with the family of functions $\{f_\lambda : X \to Y, \lambda \in A\}$. Now $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu, A)$ is said to be soft continuous at $\tilde{x_\circ}\tilde{\in}(\tilde{X}, A)$, if for every $(V, A) \in \nu$ such that $f(\tilde{x_\circ})\tilde{\in}(V, A)$, there exist $(U, A) \in \tau$ such that $\tilde{x_\circ}\tilde{\in}(U, A)$ and $f[(U, A)]\tilde{\subseteq}(V, A)$. f is said to be soft continuous on $(\tilde{X}, \tau, A)$ if it is soft continuous at each soft element $\tilde{x}\tilde{\in}(\tilde{X}, A)$.*

Definition 11 *[4]. A soft topological space $(\tilde{X}, \tau, A)$ is said to be a soft T_2 space if for $\tilde{x}, \tilde{y} \in SE(\tilde{X})$ with $\tilde{x}(\lambda) \neq \tilde{y}(\lambda)$, $\forall \lambda \in A$, there exists $(F, A), (G, A) \in \tau$ such that $\tilde{x}\tilde{\in}(F, A)$ and $\tilde{y}\tilde{\in}(G, A)$ and $(F, A)\tilde{\cap}(G, A) = (\tilde{\Phi}, A)$.*

Proposition 2 *[23]. Let $(\tilde{X}, \tau, A)$ be a soft topological space and $Y \subset X$. Let $\tau_{\tilde{Y}} = \{(\tilde{Y}, A) \Cap (G, A); (G, A) \in \tau\}$. Then $(\tilde{Y}, \tau_{\tilde{Y}}, A)$ is a soft subspace of $(\tilde{X}, \tau, A)$.*

Definition 12 *[23]. Let $(\tilde{X}, \tau, A)$ be a soft topological space. A soft set $(F, A) \in S(\tilde{X})$ is said to be soft compact in $(\tilde{X}, \tau, A)$ if any soft open cover of (F, A) has a finite soft subcover.*

Proposition 3 *[23]. Soft continuous image of a soft compact set is soft compact.*

Proposition 4 *[23]. Let $(\tilde{X}, \tau, A)$ be a soft T_2 space and (F, A) be a soft compact set such that $(F, A)^{\mathbb{C}} \neq (\tilde{\Phi}, A)$, then (F, A) is soft closed in $(\tilde{X}, \tau, A)$.*

Proposition 5 *[22]. A soft closed set in a soft compact topological space is soft compact.*

Proposition 6 *[22]. Let $(\tilde{X}, \tau, A)$ be a soft compact topological space. Then any family of soft closed set having finite intersection property (elementary intersection) has non-empty intersection.*

Definition 13 *[22]. A soft topological space $(\tilde{X}, \tau, A)$ is said to be a strong soft regular space if for all $\tilde{x} \in SE(\tilde{X})$ and for all soft open set (U, A) such that $\tilde{x}\tilde{\in}(U, A), \exists(V, A) \in \tau$ such that $\tilde{x}\tilde{\in}(V, A)\tilde{\subseteq}\overline{(V, A)}\tilde{\subseteq}(U, A)$.*

Proposition 7 *[22]. Soft closure of a soft compact set in a strong soft regular space is soft compact.*

Definition 14 *[22]. A soft topological space $(\tilde{X}, \tau, A)$ is said to be locally soft compact if for each $\tilde{x} \in SE(\tilde{X})$ there exists a soft compact nbd of $\tilde{x}$.*

Proposition 8 *[22]. Let $(\tilde{X}, \tau, A)$ be a strong soft regular and locally soft compact topological space. Then each $\tilde{x} \in SE(\tilde{X})$ has a soft compact nbd base.*

Definition 15 *[22]. Let $(\tilde{X}, \tau, A)$ be a soft topological space. A soft set $(F, A)[\neq (\tilde{\Phi}, A)] \in S(\tilde{X})$ is said to be soft dense in $(\tilde{X}, A)$, if $\overline{(F, A)} = (\tilde{X}, A)$.*

Definition 16 *[10]. A soft topological space $(\tilde{X}, \tau, A)$ is said to be a soft p-T_2 space if for any $\tilde{x}, \tilde{y} \in SE(\tilde{X})$ with $\tilde{x} \neq \tilde{y}$, there exists $(F, A), (G, A) \in \tau$ such that $\tilde{x}\tilde{\in}(F, A)$, $\tilde{y}\tilde{\in}(G, A)$ and $(F, A) \Cap (G, A) = (\tilde{\Phi}, A)$.*

Definition 17 *[10]. Let D be a directed set with order relation $\geq$ and $(\tilde{X}, \tau, A)$ be a soft topological space. Then a function $\tilde{S} : D \rightarrow SE(\tilde{X})$ is said to be a soft net.*

Definition 18 *[10]. If $\tilde{S}$ is a soft net in a soft topological space $(\tilde{X}, \tau, A)$ and $\tilde{x}$ is a soft element of $(\tilde{X}, A)$, we say that the soft net $\tilde{S}$ converges to $\tilde{x}$ if for any soft nbd (G, A) of $\tilde{x}$ there exist $n \in D$ such that for all $m \in D$, $\tilde{S}_m\tilde{\in}(G, A)$, $\forall\ m \geq n$.*

3 Soft Filter

Definition 19. *Let X be a non-empty set and A be a parameter set. A non-empty collection $\tilde{\mathscr{F}}$ of soft sets of $S(\tilde{X})$ is said to be a soft filter in $(\tilde{X}, A)$ if*

(i) $(\tilde{\Phi}, A)\tilde{\notin}\tilde{\mathscr{F}}$.

(ii) *For* $(F_i, A)\tilde{\in}\tilde{\mathscr{F}},\ i = 1, 2, ..., n,\ \ \Cap_{i=1}^{n} (F_i, A)\tilde{\in}\tilde{\mathscr{F}}$.

(iii) *For any* $(G, A)\tilde{\supset}(F, A)\tilde{\in}\tilde{\mathscr{F}} \Rightarrow (G, A)\tilde{\in}\tilde{\mathscr{F}}$.

Example 1. Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{x} \in SE(\tilde{X})$. Consider the soft nbd system $\aleph_{\tilde{x}}$. Clearly $(\tilde{\Phi}, A)\tilde{\notin}\aleph_{\tilde{x}}$. For $(F_i, A)\tilde{\in}\aleph_{\tilde{x}}$ for $i = 1, 2, ...n$, $\underset{i=1}{\overset{n}{\Cap}}(F_i, A)\tilde{\in}\aleph_{\tilde{x}}$. Also for any $(F, A)\tilde{\in}\aleph_{\tilde{x}}$ and for any $(G, A)\tilde{\supseteq}(F, A)$ $(G, A)\tilde{\in}\aleph_{\tilde{x}}$. Hence $\aleph_{\tilde{x}}$ is a soft filter called the soft nbd filter of $\tilde{x}$.

Definition 20. *Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. $\tilde{\mathscr{B}} \subseteq \tilde{\mathscr{F}}$ is said to be a a soft filter base of $\tilde{\mathscr{F}}$ if for each $(F, A)\tilde{\in}\tilde{\mathscr{F}}$ $\exists(G, A)\tilde{\in}\tilde{\mathscr{B}}$ such that $(G, A)\tilde{\subseteq}(F, A)$.*

Proposition 9. *Let X be a non-empty set and A be a parameter set. A non-empty collection $\tilde{\mathscr{B}}$ of soft sets of $S(\tilde{X})$ is a soft base for some soft filter in $(\tilde{X}, A)$ iff for any $(F_1, A), (F_2, A)\tilde{\in}\tilde{\mathscr{B}}$ $\exists$ $(F_3, A)\tilde{\in}\tilde{\mathscr{B}}$ such that $(F_3, A)\tilde{\subseteq}(F_1, A) \Cap (F_2, A)$.*

Proof. Let $\tilde{\mathscr{B}}$ be a soft base for some soft filter $\tilde{\mathscr{F}}$. Let $(F_1, A), (F_2, A)\tilde{\in}\tilde{\mathscr{B}}$. Then $(F_1, A), (F_2, A)\tilde{\in}\tilde{\mathscr{F}}$ (since $\tilde{\mathscr{B}} \subseteq \tilde{\mathscr{F}}$). Then $(F_1, A) \Cap (F_2, A)\tilde{\in}\tilde{\mathscr{F}}$. Since $\tilde{\mathscr{B}}$ is a soft base of $\tilde{\mathscr{F}}$ $\exists$ $(F_3, A)\tilde{\in}\tilde{\mathscr{B}}$ such that $(F_3, A)\tilde{\subseteq}(F_1, A) \Cap (F_2, A)$.

Conversely suppose the conditions hold. Let $\tilde{\mathscr{F}}$ be the collection of all soft supersets of members of $\tilde{\mathscr{B}}$. Then $\tilde{\mathscr{B}} \subseteq \tilde{\mathscr{F}}$. Now we show that $\tilde{\mathscr{F}}$ is a soft filter in $(\tilde{X}, A)$. Since $(\tilde{\Phi}, A)\tilde{\notin}\tilde{\mathscr{B}}$, so $(\tilde{\Phi}, A)\tilde{\notin}\tilde{\mathscr{F}}$. Let $(F_1, A), (F_2, A)\tilde{\in}\tilde{\mathscr{F}}$. Then $\exists$ $(B_1, A), (B_2, A)\tilde{\in}\tilde{\mathscr{B}}$ such that $(B_1, A)\tilde{\subset}(F_1, A)$ and $(B_2, A)\tilde{\subset}(F_2, A)$. Then $(B_1, A) \Cap (B_2, A)\tilde{\subset}(F_1, A) \Cap (F_2, A)$. By the given condition $\exists$ $(B_3, A)\tilde{\in}\tilde{\mathscr{B}}$ such that $(B_3, A)\tilde{\subset}(B_1, A)\Cap(B_2, A)\tilde{\subset}(F_1, A)\Cap(F_2, A)$. Therefore $(F_1, A)\Cap(F_2, A)\tilde{\in}\tilde{\mathscr{F}}$. Let $(G, A)\tilde{\supset}(F, A)\tilde{\in}\tilde{\mathscr{F}}$. Then $\exists$ $(B, A)\tilde{\in}\tilde{\mathscr{B}}$ such that $(B, A)\tilde{\subset}(F, A)\tilde{\subset}(G, A)$. So $(G, A)\tilde{\in}\tilde{\mathscr{F}}$. Hence $\tilde{\mathscr{F}}$ is a soft filter.

Definition 21. *Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. $\mathbb{S} \subseteq \tilde{\mathscr{F}}$ is said to be a a soft sub-base of $\tilde{\mathscr{F}}$ if the family of all finite elementary intersections of members of $\mathbb{S}$ is a soft base of $\tilde{\mathscr{F}}$.*

Proposition 10. *Let X be a non-empty set and A be a parameter set. $\mathbb{S}(\neq \Phi) \subseteq S(\tilde{X})$ is a soft sub-base for some soft filter in $(\tilde{X}, A)$ iff the elementary intersections of any finite number of members of $\mathbb{S}$ is not equal to $(\tilde{\Phi}, A)$.*

Definition 22. *Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. Then $\tilde{\mathscr{F}}$ is said to be soft convergent if there is $\tilde{x}\tilde{\in}SE(\tilde{X})$ such that for any soft nbd (V, A) of $\tilde{x}$ $\exists$ $(F, A)\tilde{\in}\tilde{\mathscr{F}}$ such that $(F, A)\tilde{\subseteq}(V, A)$.*

Definition 23. *Let $(\tilde{X}, \tau, A)$ be a soft topological space a and $\tilde{S} : (D, \geq) \to SE(\tilde{X})$ be a soft net. A soft element $\tilde{x}\tilde{\in}SE(\tilde{X})$ is said to be a soft cluster element of $\tilde{S}$ if for any soft nbd (V, A) of $\tilde{x}$ and for any $n \in D$, $\exists$ $p \geq n$ such that $\tilde{S}_p\tilde{\in}(V, A)$.*

Definition 24. *Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. A soft element $\tilde{x}\tilde{\in}SE(\tilde{X})$ is said to be a soft cluster element of $\tilde{\mathscr{F}}$ if for any soft nbd (V, A) of $\tilde{x}$ and for any $(F, A)\tilde{\in}\tilde{\mathscr{F}}$, $(V, A) \Cap (F, A) \neq (\tilde{\Phi}, A)$.*

Definition 25. *Let $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}$ be two soft filters in $(\tilde{X}, A)$. Then $\tilde{\mathscr{G}}$ is said to be a soft sub-filter of $\tilde{\mathscr{F}}$ if $\tilde{\mathscr{F}}\tilde{\subseteq}\tilde{\mathscr{G}}$.*

Proposition 11. *A soft topological space $(\tilde{X}, \tau, A)$ is a soft $p{-}T_2$ space iff every soft filter in $(\tilde{X}, A)$ converges to at most one soft element.*

Proof. Let $(\tilde{X}, \tau, A)$ be a soft $p - T_2$ space. If possible let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$ converge to two distinct soft elements $\tilde{x}$ and $\tilde{y}$. Since $(\tilde{X}, \tau, A)$ is a soft $p - T_2$ space $\exists$ a soft nbd (U, A) of $\tilde{x}$ and (V, A) of $\tilde{y}$ such that $(U, A) \Cap (V, A) = (\tilde{\Phi}, A)$. Again since $\tilde{\mathscr{F}}$ converges to $\tilde{x}$ and $\tilde{y}$ so $\exists$ $(F_1, A), (F_2, A) \tilde{\in} \tilde{\mathscr{F}}$ such that $(F_1, A) \tilde{\subset} (U, A)$ and $(F_2, A) \tilde{\subset} (V, A)$. This implies $(F_1, A) \Cap (F_2, A) = (\tilde{\Phi}, A) \tilde{\in} \tilde{\mathscr{F}}$ which is a contradiction. Hence every soft filter converge to at most one soft element.

Conversely let the condition hold. If possible let $(\tilde{X}, \tau, A)$ is not a soft $p - T_2$ space. Then $\exists$ $\tilde{x} \neq \tilde{y} \tilde{\in} SE(\tilde{X})$ such that $(U, A) \Cap (V, A) \neq (\tilde{\Phi}, A)$ $\forall (U, A) \tilde{\in} \aleph_{\tilde{x}}$ and $\forall (V, A) \tilde{\in} \aleph_{\tilde{y}}$. Then by Proposition 10 there is a soft filter $\tilde{\mathscr{F}} \tilde{\supset} \aleph_{\tilde{x}} \Cup \aleph_{\tilde{y}}$. Then $\tilde{\mathscr{F}}$ converges to both $\tilde{x}$ and $\tilde{y}$ which is a contradiction. Hence $(\tilde{X}, \tau, A)$ is a soft $p - T_2$ space.

Proposition 12. *Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. A soft element $\tilde{x} \tilde{\in} SE(\tilde{X})$ is a soft cluster element of $\tilde{\mathscr{F}}$ iff $\exists$ a soft sub-filter $\tilde{\mathscr{G}}$ of $\tilde{\mathscr{F}}$ converging to $\tilde{x}$.*

Proof. Let $\tilde{x}$ be a soft cluster element of $\tilde{\mathscr{F}}$. Consider the soft filter $\tilde{\mathscr{F}}$ and the soft nbd filter $\aleph_{\tilde{x}}$ of $\tilde{x}$. Since $\tilde{x}$ is a soft cluster element of $\tilde{\mathscr{F}}$, $(F, A) \Cap (V, A) \neq (\tilde{\Phi}, A)$ $\forall (V, A) \tilde{\in} \aleph_{\tilde{x}}$ and $\forall (F, A) \tilde{\in} \tilde{\mathscr{F}}$. Hence there is a soft filter $\tilde{\mathscr{G}} \tilde{\supset} \tilde{\mathscr{F}} \Cup \aleph_{\tilde{x}}$. Then $\tilde{\mathscr{G}}$ is a soft sub-filter of $\tilde{\mathscr{F}}$ converging to $\tilde{x}$.

Conversely let $\tilde{\mathscr{G}}$ be a soft sub-filter of $\tilde{\mathscr{F}}$ converging to $\tilde{x}$. Then $\tilde{\mathscr{G}} \tilde{\supset} \tilde{\mathscr{F}}$. Since $\tilde{\mathscr{G}}$ converges to $\tilde{x}$ so $\tilde{\mathscr{G}} \tilde{\supset} \aleph_{\tilde{x}}$. So $\tilde{\mathscr{G}} \tilde{\supset} \tilde{\mathscr{F}} \Cup \aleph_{\tilde{x}}$. Then $\forall (F, A) \tilde{\in} \tilde{\mathscr{F}}$ and $\forall (V, A) \tilde{\in} \aleph_{\tilde{x}}, (F, A), (V, A) \tilde{\in} \tilde{\mathscr{G}}$. Since $\tilde{\mathscr{G}}$ is a soft filter $(F, A) \Cap (V, A) \neq (\tilde{\Phi}, A)$. Hence $\tilde{x}$ is a soft cluster element of $\tilde{\mathscr{F}}$.

Proposition 13. *Let X and Y be two non-empty sets and A be a parameter set. Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$ and $\tilde{f} : SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft function. Then $\tilde{f}(\tilde{\mathscr{F}}) = \{\tilde{f}(F, A) \ : (F, A) \tilde{\in} \tilde{\mathscr{F}}\}$ is a soft base for some soft filter in $(\tilde{Y}, A)$.*

Remark 1. The soft filter in $(\tilde{Y}, A)$ obtained from $\tilde{f}(\tilde{\mathscr{F}})$ as a soft base is denoted by $\tilde{f}_*(\tilde{\mathscr{F}})$.

Proposition 14. *Let $(\tilde{X}, \tau, A)$ and $(\tilde{Y}, \nu, A)$ be two soft topological space and $\tilde{f} : SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft function. Then $\tilde{f}$ is soft continuous at a soft element $\tilde{x} \tilde{\in} SE(\tilde{X})$ iff for any soft filter $\tilde{\mathscr{F}}$ in $(\tilde{X}, A)$ converging to $\tilde{x}$, $\tilde{f}_*(\tilde{\mathscr{F}})$ converges to $\tilde{f}(\tilde{x})$.*

Proof. Suppose $\tilde{f}$ is soft continuous at a soft element $\tilde{x} \tilde{\in} SE(\tilde{X})$. Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$ converging to $\tilde{x}$. Let (G, A) be any soft nbd of $\tilde{f}(\tilde{x})$. Since $\tilde{f}$ is soft continuous at $\tilde{x}$, $\exists$ a soft nbd (V, A) of $\tilde{x}$ such that $\tilde{f}(V, A) \tilde{\subset} (G, A)$. Since $\tilde{\mathscr{F}}$ converges to $\tilde{x}$, $\exists (F, A) \tilde{\in} \tilde{\mathscr{F}}$ such that $(F, A) \tilde{\subset} (V, A)$. Then $\tilde{f}(F, A) \tilde{\subset} \tilde{f}(V, A) \tilde{\subset} (G, A)$. Now $\tilde{f}(F, A) \tilde{\in} \tilde{f}_*(\tilde{\mathscr{F}})$. Hence $\tilde{f}_*(\tilde{\mathscr{F}})$ converges to $\tilde{f}(\tilde{x})$.

Conversely let for any soft filter $\tilde{\mathscr{F}}$ converging to $\tilde{x}$, $\tilde{f}_*(\tilde{\mathscr{F}})$ converges to $\tilde{f}(\tilde{x})$. Let (V, A) be any soft nbd of $\tilde{f}(\tilde{x})$. Now $\aleph_{\tilde{x}}$ is a soft filter in $(\tilde{X}, A)$ converging to $\tilde{x}$. Then by the given condition $\tilde{f}_*(\aleph_{\tilde{x}})$ converges to $\tilde{f}(\tilde{x})$. So $\exists$ $(G, A)\tilde{\in}\tilde{f}_*(\aleph_{\tilde{x}})$ such that $(G, A)\tilde{\subset}(V, A)$. So $\exists$ $(U, A)\tilde{\in}\aleph_{\tilde{x}}$ such that $\tilde{f}(U, A)\tilde{\subset}(G, A)\tilde{\subset}(V, A)$. Hence $\tilde{f}$ is soft continuous at $\tilde{x}$.

4 Associated Soft Nets and Associated Soft Filters

Proposition 15. *Let $\tilde{S} : (D, \geq) \rightarrow SE(\tilde{X})$ be a soft net in a soft topological space $(\tilde{X}, \tau, A)$. For any $n \in D$, let $B_n = \{\tilde{S}_m \ ; \ m \geq n\}$. Let $\tilde{\mathscr{F}} = \{(F, A) \in S(\tilde{X}) \ ; \ SS(B_n)\tilde{\subset}(F, A)$ for some $n \in D\}$. Then $\tilde{\mathscr{F}}$ is a soft filter in $(\tilde{X}, A)$.*

Proof. Clearly $(\tilde{\Phi}, A)\tilde{\notin}\tilde{\mathscr{F}}$. Let $(F_1, A), (F_2, A)\tilde{\in}\tilde{\mathscr{F}}$. Then $\exists$ $n_1, n_2 \in D$ such that $SS(B_{n_1})\tilde{\subseteq}(F_1, A)$ and $SS(B_{n_2})\tilde{\subseteq}(F_2, A)$. Since D is a directed set $\exists$ $n \in D$ such that $n \geq n_1, \ n \geq n_2$. Then $SS(B_n)\tilde{\subseteq}(F_1, A) \Cap (F_2, A)$. Hence $(F_1, A) \Cap (F_2, A)\tilde{\in}\tilde{\mathscr{F}}$. Let $(F, A)\tilde{\in}\tilde{\mathscr{F}}$ and $(G, A)\tilde{\supset}(F, A)$. Then $(G, A)\tilde{\supset}(F, A)\tilde{\supset}SS(B_n)$ for some $n \in D$. Hence $(G, A)\tilde{\in}\tilde{\mathscr{F}}$. Hence $\tilde{\mathscr{F}}$ is a soft filter in $(\tilde{X}, A)$.

Definition 26. *The soft filter $\tilde{\mathscr{F}}$ as obtained in Proposition 15 is called the soft filter associated to the soft net $\tilde{S}$ and is denoted by $\tilde{\mathscr{F}}_{\tilde{S}}$.*

Proposition 16. *Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{S} : (D, \geq) \rightarrow SE(\tilde{X})$ be a soft net. Then*

(i) $\tilde{S}$ converges to a soft element $\tilde{x}\tilde{\in}SE(\tilde{X})$ iff the associated soft filter $\tilde{\mathscr{F}}_{\tilde{S}}$ converges to $\tilde{x}$.
(ii) A soft element $\tilde{\xi}$ is a soft cluster element of $\tilde{S}$ iff $\tilde{\xi}$ is a soft cluster element of $\tilde{\mathscr{F}}_{\tilde{S}}$.

Proof

(i) Let $\tilde{S} : (D, \geq) \rightarrow SE(\tilde{X})$ converges to $\tilde{x}$. Let $(V_{\tilde{x}}, A)$ be a soft nbd of $\tilde{x}$. Then $\exists$ $n \in D$ such that $\tilde{S}_m\tilde{\in}(V_{\tilde{x}}, A)$ $\forall m \geq n$. Let $B_n = \{\tilde{S}_m \ ; \ m \geq n\}$. Thus $SS(B_n)\tilde{\in}\tilde{\mathscr{F}}_{\tilde{S}}$ and $SS(B_n)\tilde{\subset}(V_{\tilde{x}}, A)$. So $(V_{\tilde{x}}, A)\tilde{\in}\tilde{\mathscr{F}}_{\tilde{S}}$. Since $(V_{\tilde{x}}, A)$ is arbitrary so $\aleph_{\tilde{x}}\tilde{\subset}\tilde{\mathscr{F}}_{\tilde{S}}$. Hence $\tilde{\mathscr{F}}_{\tilde{S}}$ converges to $\tilde{x}$.
Conversely suppose $\tilde{\mathscr{F}}_{\tilde{S}}$ converges to $\tilde{x}$. Let $(V_{\tilde{x}}, A)$ be a soft nbd of $\tilde{x}$. Then $(V_{\tilde{x}}, A)\tilde{\in}\tilde{\mathscr{F}}_{\tilde{S}}$. So $\exists$ $n \in D$ such that $(V_{\tilde{x}}, A)\tilde{\supset}SS\{\tilde{S}_m \ ; \ m \geq n\}$. Therefore $\tilde{S}_m\tilde{\in}(V_{\tilde{x}}, A)$ $\forall m \geq n$. Hence $\tilde{S}$ converges to $\tilde{x}$.

(ii) Let $\tilde{\xi}$ be a soft cluster element of $\tilde{S}$. Let (V, A) be any soft nbd of $\tilde{\xi}$ and (F, A) be any member of $\tilde{\mathscr{F}}_{\tilde{S}}$. Since $(F, A)\tilde{\in}\tilde{\mathscr{F}}_{\tilde{S}}$ $\exists$ $n \in D$ such that $SS\{\tilde{S}_m \ ; \ m \geq n\}\tilde{\subset}(F, A)$. Since $\tilde{\xi}$ is a soft cluster element of $\tilde{S}$ so $\tilde{S}_m\tilde{\in}(V, A)$. So $(V, A) \Cap SS\{\tilde{S}_m \ ; \ m \geq n\} \neq (\tilde{\Phi}, A) \Rightarrow (V, A) \Cap (F, A) \neq (\tilde{\Phi}, A)$. Hence $\tilde{\xi}$ is a soft cluster element of $\tilde{\mathscr{F}}_{\tilde{S}}$.
Conversely let $\tilde{\xi}$ be a soft cluster element of $\tilde{\mathscr{F}}_{\tilde{S}}$. Let (V, A) be any soft nbd of $\tilde{\xi}$ and $n \in D$. Since $\tilde{\xi}$ is a soft cluster element of $\tilde{\mathscr{F}}_{\tilde{S}}$, so $SS\{\tilde{S}_m \ ; \ m \geq n\} \Cap (V, A) \neq (\tilde{\Phi}, A)$. So $\exists$ $m \geq n$ such that $\tilde{S}_m\tilde{\in}(V, A)$. So $\tilde{\xi}$ is a soft cluster element of $\tilde{S}$.

Definition 27. *Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. Let $D = \{(\tilde{x}, (F, A)) \in SE(\tilde{X}) \times \tilde{\mathscr{F}}\ ; \tilde{x}\tilde{\in}(F, A)\}$. Define $\geq$ on D by $(\tilde{x}, (F, A)) \geq (\tilde{y}, (G, A))$ if $(F, A)\tilde{\subseteq}(G, A)$. Then $(D, \geq)$ is a directed set. Define $\tilde{S} : (D, \geq) \to SE(\tilde{X})$ by $\tilde{S}(\tilde{x}, (F, A)) = \tilde{x}$. Then $\tilde{S}$ is a soft net called the associated soft net of the soft filter $\tilde{\mathscr{F}}$ and is denoted by $\tilde{S}_{\tilde{\mathscr{F}}}$.*

Proposition 17. *Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. Then*

(i) $\tilde{\mathscr{F}}$ converges to a soft element $\tilde{x}\tilde{\in}SE(\tilde{X})$ iff the associated soft net $\tilde{S}_{\tilde{\mathscr{F}}}$ converges to $\tilde{x}$.
(ii) $\tilde{\xi}$ is a soft cluster element of $\tilde{\mathscr{F}}$ iff $\tilde{\xi}$ is a soft cluster element of $\tilde{S}_{\tilde{\mathscr{F}}}$.

Proof

(i) Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$ converging to $\tilde{x}$. Let (V, A) be any soft nbd of $\tilde{x}$. Then $\exists (F, A)\tilde{\in}\tilde{\mathscr{F}}$ such that $(F, A)\tilde{\subseteq}(V, A)$.Then $(\tilde{x}, (F, A)) \in D$. Then for any $(\tilde{y}, (W, A)) \in D$ with $(\tilde{y}, (W, A)) \geq (\tilde{x}, (F, A))$, $\tilde{S}(\tilde{y}, (W, A)) = \tilde{y}\tilde{\in}(W, A)\tilde{\subset}(F, A)\tilde{\subseteq}(V, A)$. Hence $\tilde{S}_{\tilde{\mathscr{F}}}$ converges to $\tilde{x}$.
Conversely suppose $\tilde{S}_{\tilde{\mathscr{F}}}$ converges to $\tilde{x}$. Let (V, A) be any soft nbd of $\tilde{x}$. Since $\tilde{S}_{\tilde{\mathscr{F}}}$ converges to $\tilde{x}$, $\exists\ (\tilde{y}, (F, A)) \in D$ such that $\tilde{S}(\tilde{z}, (W, A))\tilde{\in}(V, A)$ $\forall (\tilde{z}, (W, A)) \geq (\tilde{y}, (F, A))$. Then $(W, A)\tilde{\subset}(V, A)$. So $(V, A)\tilde{\in}\tilde{\mathscr{F}}$. Since (V, A) is arbitrary $\aleph_{\tilde{x}}\tilde{\subset}\tilde{\mathscr{F}}$. Hence $\tilde{\mathscr{F}}$ converges to $\tilde{x}$.

(ii) Let $\tilde{\xi}$ be a soft cluster element of $\tilde{\mathscr{F}}$. Let (V, A) be any soft nbd of $\tilde{\xi}$ and $(\tilde{y}, (F, A)) \in D$. Then $(F, A)\tilde{\in}\tilde{\mathscr{F}}$ and $(V, A) \Cap (F, A) \neq (\tilde{\Phi}, A)$. Let $\tilde{x}\tilde{\in}(V, A) \Cap (F, A)$. Then $(\tilde{x}, (F, A)) \in D$ and $(\tilde{x}, (F, A)) \geq (\tilde{y}, (F, A))$. Now $\tilde{S}(\tilde{x}, (F, A)) = \tilde{x}\tilde{\in}(V, A)$. Hence $\tilde{\xi}$ is a soft cluster element of $\tilde{S}_{\tilde{\mathscr{F}}}$.
Conversely let $\tilde{\xi}$ be a soft cluster element of $\tilde{S}_{\tilde{\mathscr{F}}}$. Let (V, A) be any soft nbd of $\tilde{\xi}$ and (F, A) be any member of $\tilde{\mathscr{F}}$. Let $\tilde{y}\tilde{\in}(F, A)$. Then $(\tilde{y}, (F, A)) \in D$. Since $\tilde{\xi}$ is a soft cluster element of $\tilde{S}_{\tilde{\mathscr{F}}}$ there is $(\tilde{x}, (G, A)) \geq (\tilde{y}, (F, A))$ such that $\tilde{S}(\tilde{x}, (G, A))\tilde{\in}(V, A)$. Now $\tilde{S}(\tilde{x}, (G, A)) = \tilde{x}\tilde{\in}(G, A)\tilde{\subset}(F, A)$. So $(V, A) \Cap (F, A) \neq (\tilde{\Phi}, A)$. Hence $\tilde{\xi}$ is a soft cluster element of $\tilde{\mathscr{F}}$.

5 Soft Compactness and Soft Filter

Definition 28. *A soft topological space $(\tilde{X}, \tau, A)$ is said to be a weak soft compact space if any family of soft closed set having elementary finite intersection property has a non empty intersection (elementary intersection).*

Remark 2. A soft compact topological space is weak soft compact but not conversely.

Example 2. Let $\mathbb{N}$ be the set of natural numbers and $A = \alpha_i,\ i \in \mathbb{N}$ be the parameter set. Define a mapping for each $i \in \mathbb{N},\ F_i : A \to P(\mathbb{N})$ by

$$F_i(\alpha_j) = \begin{cases} \mathbb{N} & if\ i = j \\ i & if\ i \neq j. \end{cases}$$

Let $\tau = \{\underset{i\in\mathbb{J}}{\cup}(F_i, A),\ \mathbb{J} \subseteq \mathbb{N}\} \cup \{(\tilde{\Phi}, A)\}$. Then $(\tilde{\mathbb{N}}, \tau, A)$ is a soft topological space. Let $\mathscr{B} = \{(F_i, A),\ i \in \mathbb{N}\}$. Then $\mathscr{B}$ is a soft open cover of $(\tilde{\mathbb{N}}, A)$ having no finite soft subcover. Hence $(\tilde{\mathbb{N}}, \tau, A)$ is not soft compact. Since there is only one non empty soft closed set $(\tilde{\mathbb{N}}, A)$, so $(\tilde{\mathbb{N}}, \tau, A)$ is weak soft compact.

Definition 29. *Let $(\tilde{X}, \tau, A)$ be a soft topological space and $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. A soft element $\tilde{x}\tilde{\in}SE(\tilde{X})$ is said to be a weak soft cluster element of $\tilde{\mathscr{F}}$ if $\tilde{x}\tilde{\in}\overline{(F, A)}$ $\forall (F, A)\tilde{\in}\tilde{\mathscr{F}}$.*

Remark 3. A soft cluster element and a weak soft cluster element are different from each other. This is shown in the following two Examples.

Example 3. Let $X = \{x, y, z\}$ and $A = \{\alpha, \beta\}$. Let $\tau = \{(\tilde{\Phi}, A), (\tilde{X}, A), (G_1, A), (G_2, A), (G_3, A), (G_4, A), (G_5, A)\}$, where, $G_1(\alpha) = \{x, y\}, G_1(\beta) = \{x, y, z\}$; $G_2(\alpha) = \{y\}, G_2(\beta) = \{x, y\}; G_3(\alpha) = \{y\}, G_3(\beta) = \{x\}; G_4(\alpha) = \{y, z\}, G_4(\beta) = \{x, y\}; G_5(\alpha) = \{x\}, G_5(\beta) = \{y, z\}$. Then $(\tilde{X}, \tau, A)$ is a soft topological space. Let $(H, A) \in S(\tilde{X})$, where, $H(\alpha) = \{x\}, H(\beta) = \{z\}$. Let $\tilde{\mathscr{F}} = \{(F, A) \in S(\tilde{X})\ ; (F, A)\tilde{\supseteq}(H, A)\}$. Then $\tilde{\mathscr{F}}$ is a soft filter in $(\tilde{X}, A)$. Now $(G, A)\tilde{\in}\tilde{\mathscr{F}}$, where, $G(\alpha) = \{x, z\}; G(\beta) = \{z\}$. Consider the soft element $\bar{x}$. Then $\bar{x}$ is a soft cluster element of $\tilde{\mathscr{F}}$ but $\bar{x}\tilde{\notin}\overline{(G, A)}$. Hence $\bar{x}$ is not a weak soft cluster element of $\tilde{\mathscr{F}}$.

Example 4. Let $X = \{x, y, z\}$ and $A = \{\alpha, \beta\}$. Let $\tau = \{(\tilde{\Phi}, A), (\tilde{X}, A), (G_1, A), (G_2, A), (G_3, A)\}$, where, $G_1(\alpha) = \{z\}, G_1(\beta) = \{y, z\}; G_2(\alpha) = \{y\}, G_2(\beta) = \{x\}; G_3(\alpha) = \{y, z\}, G_3(\beta) = \{x, y, z\}$. Then $(\tilde{X}, \tau, A)$ is a soft topological space. Let $(H, A) \in S(\tilde{X})$, where, $H(\alpha) = \{x\}, H(\beta) = \{y\}$. Let $\tilde{\mathscr{F}} = \{(F, A) \in S(\tilde{X})\ ; (F, A)\tilde{\supseteq}(H, A)\}$. Consider the soft element $\bar{z}$. Then (G_1, A) is a soft nbd of $\bar{z}$ and $(G_1, A) \Cap (H, A) = (\tilde{\Phi}, A)$. So, $\bar{z}$ is not a soft cluster element of $\tilde{\mathscr{F}}$. Now $\bar{z}\tilde{\in}\overline{(G, A)},\ \forall (G, A)\tilde{\in}\tilde{\mathscr{F}}$. Then $\bar{z}$ is a weak soft cluster element of $\tilde{\mathscr{F}}$.

Proposition 18. *A soft topological space $(\tilde{X}, \tau, A)$ is a weak soft compact space iff every soft filter in $(\tilde{X}, A)$ has a weak soft cluster element.*

Proof. Let $(\tilde{X}, \tau, A)$ be a weak soft compact topological space. Let $\tilde{\mathscr{F}}$ be a soft filter in $(\tilde{X}, A)$. Consider the collection $\mathscr{B} = \{\overline{(F, A)}\ ;\ (F, A)\tilde{\in}\tilde{\mathscr{F}}\}$. Then $\mathscr{B}$ is a family of soft closed set having elementary F.I.P. Since $(\tilde{X}, \tau, A)$ is a weak soft compact topological space so $\Cap\{\overline{(F, A)} :\ (F, A)\tilde{\in}\tilde{\mathscr{F}}\} \neq (\tilde{\Phi}, A)$. Hence $\exists\ \tilde{x}\tilde{\in}SE(\tilde{X})$ such that $\tilde{x}\tilde{\in}\overline{(F, A)}$ $\forall (F, A)\tilde{\in}\tilde{\mathscr{F}}$. Then $\tilde{x}$ is a weak soft cluster point of $\tilde{\mathscr{F}}$. Since $\tilde{\mathscr{F}}$ is arbitrary every soft filter in $(\tilde{X}, A)$ has a weak soft cluster element.

Conversely let every soft filter in $(\tilde{X}, A)$ has a weak soft cluster element. Let $\tilde{\mathscr{F}}$ be a family of soft closed set having elementary F.I.P. Then there is a soft filter $\tilde{\mathscr{G}}$ of which $\tilde{\mathscr{F}}$ is a soft sub-base i.e $\tilde{\mathscr{F}} \subseteq \tilde{\mathscr{G}}$. Then by assumption $\tilde{\mathscr{G}}$ has a weak soft cluster element. Hence $\Cap\{\overline{(F, A)}\ :\ (F, A)\tilde{\in}\tilde{\mathscr{G}}\} \neq (\tilde{\Phi}, A)$. So $\Cap\{\overline{(F, A)} :\ (F, A)\tilde{\in}\tilde{\mathscr{F}}\} \neq (\tilde{\Phi}, A)$. Hence $(\tilde{X}, \tau, A)$ is a weak soft compact topological space.

Remark 4. In a soft compact topological space every soft filter has a weak soft cluster element. But the converse is not true which is shown in the following Example.

Example 5. Consider the soft topology of Example 2. Then every soft filter in $(\tilde{\mathbb{N}}, A)$ has a weak soft cluster element though $(\tilde{\mathbb{N}}, \tau, A)$ is not soft compact.

6 Soft Compactification

Let X and X^+ be two non-empty set such that $X \subset X^+$ and A be a non-empty set of parameters. Let $(F, A) \in S(\tilde{X}) \subset S(\tilde{X}^+)$. Then the elementary complement of (F, A) in $(\tilde{X}, A)$ will be denoted by $(F, A)^{\mathbb{C}|\tilde{X}}$ and the elementary complement of (F, A) in $(\tilde{X}^+, A)$ will be denoted by $(F, A)^{\mathbb{C}|\tilde{X}^+}$.

Lemma 1. *Let X be an initial universal set A be an arbitrary parameter set. Let $\infty \notin X$. Let $X^+ = X \cup \{\infty\}$. Let $(\tilde{X}, \tau, A)$ be a soft topological space. Let $\tau_1 = \{(G, A) \in S(\tilde{X}^+) : \bar{\infty}\tilde{\in}(G, A)$ and $(G, A)^{\mathbb{C}|\tilde{X}^+}$ is soft closed and soft compact in $(\tilde{X}, \tau, A)\}$. Let $\tau^+ = \tau \cup \tau_1$. Then, for any $(G, A)[\neq (\tilde{\Phi}, A)] \in \tau_1,\ \exists\ (H, A) \in \tau$ such that $(G, A) = (H, A) \Cup (\bar{\infty}, A)$.*

Proof. Let $(G, A) \in \tau_1$, then $\bar{\infty}\tilde{\in}(G, A)$ and $(G, A)^{\mathbb{C}|\tilde{X}^+}$ is soft closed and soft compact in $(\tilde{X}, \tau, A)$. Let $(F, A) = (G, A)^{\mathbb{C}|\tilde{X}^+}$ and $(H, A) = (F, A)^{\mathbb{C}|\tilde{X}}$. Then $(H, A) \in \tau$ and $(G, A) = (H, A) \Cup (\bar{\infty}, A)$.

Proposition 19. *Let X be an initial universal set A be an arbitrary parameter set. Let $\infty \notin X$. Let $X^+ = X \cup \{\infty\}$. Let $(\tilde{X}, \tau, A)$ be a soft topological space. Let $\tau_1 = \{(G, A) \in S(\tilde{X}^+) : \bar{\infty}\tilde{\in}(G, A)$ and $(G, A)^{\mathbb{C}|\tilde{X}^+}$ is soft closed and soft compact in $(\tilde{X}, \tau, A)\}$. Let $\tau^+ = \tau \cup \tau_1$. Then,*

(i) $(\tilde{X}^+, \tau^+, A)$ *is a soft topological space.*
(ii) $(\tilde{X}, \tau, A)$ *is a soft subspace of* $(\tilde{X}^+, \tau^+, A)$.
(iii) $(\tilde{X}^+, \tau^+, A)$ *is soft compact.*
(iv) $(\tilde{X}, A)$ *is soft dense in* $(\tilde{X}^+, \tau^+, A)$ *iff* $(\tilde{X}, A)$ *is not soft compact.*

Proof

(i) $(\tilde{\Phi}, A) \in \tau \Rightarrow (\tilde{\Phi}, A) \in \tau^+$.
$(\tilde{X}^+, A)^{\mathbb{C}|\tilde{X}^+} = (\tilde{\Phi}, A)$ which is soft closed and soft compact in $(\tilde{X}, \tau, A)$. So, $(\tilde{X}^+, A) \in \tau_1$. Hence $(\tilde{X}^+, A) \in \tau^+$.
Let $(F_1, A), (F_2, A) \in \tau^+$. Then the following cases may arise
Case I : Let $(F_1, A), (F_2, A) \in \tau$. Then $(F_1, A) \Cap (F_2, A) \in \tau$. Hence $(F_1, A) \Cap (F_2, A) \in \tau^+$.
Case II : Let $(F_1, A) \in \tau$ and $(F_2, A) \in \tau_1$. Then by Lemma 1 there is $(F_2', A) \in \tau$ such that $(F_2, A) = (F_2', A) \Cup (\bar{\infty}, A)$. Then $(F_1, A) \Cap (F_2, A) = (F_1, A) \Cap (F_2', A) \in \tau$ (since $\bar{\infty}\tilde{\notin}(F_1, A)$). Hence $(F_1, A) \Cap (F_2, A) \in \tau^+$.

Case III : Let $(F_1, A), (F_2, A) \in \tau_1$. Then $\bar{\infty}\tilde{\in}(F_1, A), \bar{\infty}\tilde{\in}(F_2, A)$ and $(F_1, A)^{\mathbb{C}|\tilde{X}^+}, (F_2, A)^{\mathbb{C}|\tilde{X}^+}$ are soft closed and soft compact in $(\tilde{X}, \tau, A)$. Again by Lemma 1 $\exists$ $(G_1, A), (G_2, A) \in \tau$ such that $(F_1, A) = (G_1, A) \Cup (\bar{\infty}, A)$ and $(F_2, A) = (G_2, A) \Cup (\bar{\infty}, A)$. Now $(F_1, A)^{\mathbb{C}|\tilde{X}^+} = [(G_1, A) \Cup (\bar{\infty}, A)]^{\mathbb{C}|\tilde{X}^+} = (G_1, A)^{\mathbb{C}|\tilde{X}^+} \Cap (\tilde{X}, A) = (G_1, A)^{\mathbb{C}|\tilde{X}}$. Similarly, $(F_2, A)^{\mathbb{C}|\tilde{X}^+} = (G_2, A)^{\mathbb{C}|\tilde{X}}$. Again, $[(F_1, A) \Cap (F_2, A)]^{\mathbb{C}|\tilde{X}^+} = [\{(G_1, A) \Cap (G_2, A)\} \Cup (\bar{\infty}, A)]^{\mathbb{C}|\tilde{X}^+} = [(G_1, A) \Cap (G_2, A)]^{\mathbb{C}|\tilde{X}^+} \Cap (\tilde{X}, A) = (G_1, A)^{\mathbb{C}|\tilde{X}} \Cup (G_2, A)^{\mathbb{C}|\tilde{X}}$ which is soft compact in $(\tilde{X}, \tau, A)$. We now show that $(G_1, A)^{\mathbb{C}|\tilde{X}} \Cup (G_2, A)^{\mathbb{C}|\tilde{X}}$ is soft closed in $(\tilde{X}, \tau, A)$.

We claim that $(G_1, A) \Cap (G_2, A) \neq (\tilde{\Phi}, A)$. For if $(G_1, A) \Cap (G_2, A) = (\tilde{\Phi}, A)$, then $[(G_1, A) \Cap (G_2, A)]^{\mathbb{C}|\tilde{X}} = (\tilde{X}, A) \Rightarrow (G_1, A)^{\mathbb{C}|\tilde{X}} \Cup (G_2, A)^{\mathbb{C}|\tilde{X}} = (\tilde{X}, A)$ which implies $(\tilde{X}, A)$ is soft compact, so $(G_1, A) \Cap (G_2, A) \neq (\tilde{\Phi}, A)$. Now $[(G_1, A)^{\mathbb{C}|\tilde{X}}]^{\mathbb{C}|\tilde{X}} \Cap [(G_2, A)^{\mathbb{C}|\tilde{X}}]^{\mathbb{C}|\tilde{X}} = (G_1, A) \Cap (G_2, A) \neq (\tilde{\Phi}, A)$. So, $(G_1, A)^{\mathbb{C}|\tilde{X}} \Cup (G_2, A)^{\mathbb{C}|\tilde{X}}$ is soft closed in $(\tilde{X}, \tau, A)$. Hence $[(F_1, A) \Cap (F_2, A)]^{\mathbb{C}|\tilde{X}^+}$ is soft closed in $(\tilde{X}, \tau, A)$. Also $\bar{\infty}\tilde{\in}(F_1, A) \Cap (F_2, A)$. Therefore $(F_1, A) \Cap (F_2, A) \in \tau_1 \Rightarrow (F_1, A) \Cap (F_2, A) \in \tau^+$.

So in all cases τ^+ is closed under finite intersection.

Let $(F_i, A) \in \tau^+$ $\forall i \in \Delta$. Then if $(F_i, A) \in \tau$ $\forall i \in \Delta$, then $\underset{i\in\Delta}{\Cup}(F_i, A) \in \tau \Rightarrow \underset{i\in\Delta}{\Cup}(F_i, A) \in \tau^+$. Otherwise there is $\alpha \in \Delta$ such that $(F_\alpha, A) \in \tau_1$. Then $\bar{\infty}\tilde{\in}(F_\alpha, A)$ and $(F_\alpha, A)^{\mathbb{C}|\tilde{X}^+}$ is soft closed and soft compact in $(\tilde{X}, \tau, A)$. We show that $\underset{i\in\Delta}{\Cup}(F_i, A) \in \tau_1$ i.e we show that $[\underset{i\in\Delta}{\Cup}(F_i, A)]^{\mathbb{C}|\tilde{X}^+}$ is soft closed and soft compact in $(\tilde{X}, \tau, A)$. Clearly $\bar{\infty}\tilde{\in}\underset{i\in\Delta}{\Cup}(F_i, A)$. Now $[\underset{i\in\Delta}{\Cup}(F_i, A)]^{\mathbb{C}|\tilde{X}^+} = \underset{i\in\Delta}{\Cap}(F_i, A)^{\mathbb{C}|\tilde{X}^+} \tilde{\subset} (F_\alpha, A)^{\mathbb{C}|\tilde{X}^+}$. Consider the following two cases

Case I : Suppose $(F_i, A)^{\mathbb{C}|\tilde{X}^+} = (\tilde{\Phi}, A)$ for some $i \in \Delta$. Then $\underset{i\in\Delta}{\Cap}(F_i, A)^{\mathbb{C}|\tilde{X}^+} = (\tilde{\Phi}, A)$ which is soft closed and soft compact in $(\tilde{X}, \tau, A)$. Hence $\underset{i\in\Delta}{\Cup}(F_i, A) \in \tau_1 \Rightarrow \underset{i\in\Delta}{\Cup}(F_i, A) \in \tau^+$.

Case II : Suppose $(F_i, A)^{\mathbb{C}|\tilde{X}^+} \neq (\tilde{\Phi}, A)$ $\forall i \in \Delta$. Now $\underset{i\in\Delta}{\Cap}(F_i, A)^{\mathbb{C}|\tilde{X}^+}$ is soft closed in $(\tilde{X}, \tau, A)$. Since $(F_\alpha, A)^{\mathbb{C}|\tilde{X}^+}$ is soft compact and the soft closed set $\underset{i\in\Delta}{\Cap}(F_i, A)^{\mathbb{C}|\tilde{X}^+} \tilde{\subset} (F_\alpha, A)^{\mathbb{C}|\tilde{X}^+}$ so $\underset{i\in\Delta}{\Cap}(F_i, A)^{\mathbb{C}|\tilde{X}^+}$ is soft compact in $(\tilde{X}, \tau, A)$. Hence $\underset{i\in\Delta}{\Cup}(F_i, A) \in \tau_1 \Rightarrow \underset{i\in\Delta}{\Cup}(F_i, A) \in \tau^+$. Therefore τ^+ is closed under arbitrary union. Hence $(\tilde{X}^+, \tau^+, A)$ is a soft topological space.

(ii) Let $(F, A) \in \tau^+$. If $(F, A) \in \tau$ then $(F, A) \Cap (\tilde{X}, A) = (F, A) \in \tau$. If $(F, A) \in \tau_1$ then by Lemma 1, there is $(F', A) \in \tau$ such that $(F, A) = (F', A) \Cup (\bar{\infty}, A)$. Then $(F, A) \Cap (\tilde{X}, A) = (F', A) \Cap (\tilde{X}, A)$ [since $\bar{\infty}\tilde{\notin}(\tilde{X}, A)$] $= (F', A) \in \tau$. Therefore $(\tilde{X}, \tau, A)$ is a soft subspace of $(\tilde{X}^+, \tau^+, A)$.

(iii) Let $\mathscr{B}$ be a soft open cover of $(\tilde{X}^+, A)$. Then there is $(G, A) \in \mathscr{B}$ such that $\bar{\infty}\tilde{\in}(G, A)$ and $(G, A) \in \tau_1$. Then $(G, A)^{\mathbb{C}|\tilde{X}^+}$ is soft closed and compact in $(\tilde{X}, \tau, A)$. Let $\mathscr{B}' = \{(F, A) \Cap (\tilde{X}, A) \ : \ (F, A) \in \mathscr{B}\}$. Then $\mathscr{B}'$ is a soft open cover of $(G, A)^{\mathbb{C}|\tilde{X}^+}$. Since $(G, A)^{\mathbb{C}|\tilde{X}^+}$ is soft compact in $(\tilde{X}, \tau, A)$ there is a finite soft subcover $\{(F_i, A) \Cap (\tilde{X}, A) \ ; \ i = 1, 2, ..., n\}$ of $\mathscr{B}'$. Then $\{(F_i, A) \ ; i = 1, 2, ..., n, (G, A)\}$ is a finite soft subcover of $(\tilde{X}^+, A)$. Hence $(\tilde{X}^+, \tau^+, A)$ is soft compact.

(iv) Let $(\tilde{X}, A)$ be soft dense in $(\tilde{X}^+, \tau^+, A)$. Then $(\bar{\infty}, A) \notin \tau_1$ (since $(\tilde{X}, A)\tilde{\cap}(\bar{\infty}, A) = (\tilde{\Phi}, A)$). Then $(\bar{\infty}, A)^{\mathbb{C}|\tilde{X}^+}$ is not soft closed and soft compact in $(\tilde{X}, \tau, A)$ i.e $(\tilde{X}, A)$ is not soft compact.

Conversely suppose $(\tilde{X}, \tau, A)$ is not soft compact. Then $(\bar{\infty}, A) \notin \tau_1$ i.e $(\bar{\infty}, A)^{\mathbb{C}|\tilde{X}^+} = (\tilde{X}, A)$ is not soft closed in $(\tilde{X}^+, \tau^+, A)$. Hence $\overline{(\tilde{X}, A)} = (\tilde{X}^+, A)$. Hence$(\tilde{X}, A)$ is soft dense in $(\tilde{X}^+, \tau^+, A)$.

Remark 5. Let $(\tilde{X}, \tau, A)$ be a non compact soft topological space. Then the soft topological space $(\tilde{X}^+, \tau^+, A)$ defined as above is called the soft compactification of $(\tilde{X}, \tau, A)$.

Proposition 20. *Let $(\tilde{X}, \tau, A)$ be a locally soft compact, soft T_2 and strong soft regular topological space and let $(\tilde{X}^+, \tau^+, A)$ be the soft compactification of $(\tilde{X}, \tau, A)$. Then for the soft element $\bar{\infty}$ and for any $\tilde{x} \in SE(\tilde{X})$ $\exists$ $(U, A), (V, A) \in \tau^+$ such that $\tilde{x}\tilde{\in}(U, A)$ and $\bar{\infty}\tilde{\in}(V, A)$ and $(U, A)\tilde{\cap}(V, A) = (\tilde{\Phi}, A)$.*

Proof. Let $\tilde{x} \in SE(\tilde{X})$ be arbitrary. Since $(\tilde{X}, \tau, A)$ is locally soft compact $\exists$ a soft compact nbd (U, A) of $\tilde{x}$. Since $(\tilde{X}, \tau, A)$ is a strong soft regular topological space so $\overline{(U, A)}$ is also soft compact in $(\tilde{X}, \tau, A)$. Then if $(U, A)^{\mathbb{C}|\tilde{X}} = (\tilde{\Phi}, A)$ then $\overline{(U, A)} = (\tilde{X}, A)$ which imply $(\tilde{X}, A)$ is soft compact which is a contradiction. So $(U, A)^{\mathbb{C}|\tilde{X}} \neq (\tilde{\Phi}, A)$. Since $(\tilde{X}, \tau, A)$ is soft T_2, and (U, A) is a soft compact set such that $(U, A)^{\mathbb{C}|\tilde{X}} \neq (\tilde{\Phi}, A)$, so (U, A) is soft closed in $(\tilde{X}, \tau, A)$. Let $(V, A) = (U, A)^{\mathbb{C}|\tilde{X}} \Cup (\bar{\infty}, A)$. Then $(V, A)^{\mathbb{C}|\tilde{X}^+} = (U, A)$ which is soft closed and soft compact in $(\tilde{X}, \tau, A)$. Hence $(V, A) \in \tau_1$. Clearly $(U, A)\tilde{\cap}(V, A) = (\tilde{\Phi}, A)$.

7 Conclusion

In this paper we have studied the notions like convergence of soft filters, relation between soft filter and soft compactness and the soft compactification in process. We hope that the results in this article will help and motivate researchers to study this topological structure on soft sets. Next to solidify the importance of this structure we look forward to introduce the notions like soft bitopological spaces and soft topological groups etc. in this settings.

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Decision Making for Medical Diagnosis Through Credibility Theory

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Abstract. The area of medical diagnosis becomes more important and interesting for application of fuzzy variables due to imprecise, vague, uncertain character of medical information and documentation as well. Although numerous studies have been encountered in recent decades, however most of the studies lead to counterintuitive output more often. Keeping this in mind, this article presents an effort to carry out medical diagnosis using credibility distribution and for this purpose an algorithm has been formulated. It is observed that the present approach provides realistic and analytically correct result which also tallies with human intuition.

Keywords: Uncertainty · Fuzzy variable · Credibility distribution · Medical diagnosis

1 Introduction

Problems of real world are always found to be tainted with uncertainty. In 1965 Zadeh [1] in his seminal paper ‘ Fuzzy Sets’ introduced the theory of fuzzy sets to deal with uncertainty; soon this theory was extended in different ways such as interval valued fuzzy set (IVFS) by Sambuc [2] in 1975; intuitionistic fuzzy set (IFS) by Atanassov [3] in 1986. Another representation of uncertainty is the credibility measure developed by Liu [4] in 2004 which is based on the average of the necessity measure and possibility measure and later redefined by Liu [5]. These theories were then used to deal with different real world problems; medical diagnosis being a prominent one.

1.1 Existing Related Works for Medical Diagnosis

After that various direct/indirect extensions of fuzzy set (FST) those have been successfully applied in most of the problems of real world situation including medical diagnosis. Initial applications of FST in the meadow of medical diagnosis (MD) were initiated by Zadeh [6] and after that, Sanchez [7, 8] exclusively prompted the idea under uncertain domain using fuzzy matrices. Some other recent application of FST in MD can be witnessed in Yao and Yao [9], Dash et al. [10], Dagar et al. [11], Çelik and Yamak [12], Elizabeth and Sujatha [13], Porchelvi et al. [14], Medeiros et al. [15], Farhadinia [16], Dutta and Limboo [17], Dutta and Dash [18].

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Again application of IVFSs in MD are encountered in Chetia and Das [19], Ahn et al. [20], Meenakshi and Kaliraja [21], Elizabeth and Sujatha [22], Li et al. [23] and Dutta [24].

Yet again application of IFSs in MD are observed in De et al. [25], Szmidt and Kacprzyk [26], Own [27], Choi et al. [28], Samuel and Balamurugan [29–31], Hung and Tuan [32], Chang and Hung [33], Maheshwari and Srivastava [34], Davvaz and Sadrabadi [35], Jemal et al. [36], Samuel and Rajakumar [37].

Nevertheless, credibility theory that has been evolved from the possibility theory has not much been recognized for application in MD. Only one unique application of credibility theory in MD is observed viz., Dutta [38] through the distance measure.

1.2 Shortcomings of Existing Literature

Even though various applications of IFS and its extensions are encountered in literature, however, most of these applications lead to illogical or counterintuitive output. For instance, the effort of De et al. [25] was rebuted by Szmidt and Kacprzyk [26] but their results for the same IFS data set were different and counterintuitive. Similarly, dissimilar to analytical and human intuition results are also observed in (Valchos and Sergiadis [39]; Own, [27]; Ye [40]; Boran and Akay [41]; Song et al. [42], Davvaz and Sadrabadi [35]). Again, steady and scrupulous investigation demonstrates that the studies made by Çelik and Yamak [12], Elizabeth and Sujatha [13] and Porchelvi et al. [14] also endow with ridiculous yields where fuzzy sets are defined on continuous universe. Çelik and Yamakand [12] approach gives that patient-2 suffers from disease-2 which is dubious but from the meticulous investigation point of view it should be disease-1. Likewise, in the approach of Elizabeth and Sujatha [13] suffering of patient-1 from disease-2 is debatable even as it should also be disease-1. In the similar way, Porchelvi et al. [14] approach shows that patient-1 suffers from disease-2 which is again doubtful and it should be again disease-1.

1.3 Major Contribution

The foremost contribution of this manuscript is to formulate a medical investigation process through the concept of sampling of credibility distributions. Furthermore, medical investigation has also been achieved by constructing an effective algorithm and according medical decision making problem has been fitted. Here Çelik and Yamakand [12] medical diagnosis problem which produced illogical results has been resolved and it is found that the present approach has the capability to solve diagnosis problem in a specific and effective manner and obtained results tally with analytical and logical results along with human intuition.

2 Preliminaries

Here some essential notions of credibility theory are depicted that will be required in this swot up.

Definition: Let Ω be a non-null set, and ρ be the power set of Ω, and $Pos : \rho \to \mathbb{R}$ which is called possibility measure if it satisfies the subsequent three conditions (Liu [43])

1. $Pos(\Omega) = 1$
2. $Pos(\phi) = 0$
3. $Pos(\cup_i \xi_i) = \sup_i Pos\{\xi_i\}$ for any events $\{\xi_i\}$.

Furthermore, the triplet (Ω, ρ, Pos) is called possibility space.

Definition: Let (Ω, ρ, Pos) be the possibility space and ξ be a set in ρ. Then the necessity measure Nec, of ξ is $Nec\{\xi\} = 1 - Pos\{\xi^c\}$, where ξ^c is the compliment of ξ.

Definition: If Ω is a non null set, and ρ is the power set of Ω and $\xi \in \rho$ then the credibility measure is (Liu and Liu [44])

$$Cr\{\xi\} = \frac{1}{2}(Pos\{\xi\} + Nec\{\xi\})$$

Furthermore, for any $\xi \in \rho$, $Pos\{\xi\} = \min(2Cr, 1)$

The triplet (Ω, ρ, Cr) is known as credibility space if Cr, called the credibility measure, a non –negative set function satisfies the following axioms (Liu and Liu, [44])

$$Cr\{\Omega\} = 1$$

$Cr\{\xi_1\} \leq Cr\{\xi_2\}$ for whenever $\xi_1 \subset \xi_2$,
$Cr\{\xi\} + Cr\{\xi^c\} = 1$ for any ξ.
$Cr\{\underset{i}{\cup} \xi_i\} = \underset{i}{\sup} Cr\{\xi_i\}$ for any events $\{\xi_i\}$ with $\sup_i Cr\{\xi_i\} < 0.5$.

Definition: A fuzzy variable is a function from a credibility space (Ω, ρ, Cr) to the set of real numbers (Liu [45]).

Suppose ζ is a fuzzy variable defined on the credibility space (Ω, ρ, Cr) whose membership function defined from the credibility measure is

$$\mu_\zeta(x) = \min(2Cr : \zeta = x\}, 1); x \in \mathbb{R}$$

Definition: A credibility distribution is defined as (Liu [5])

$\varphi_\zeta : \mathbb{R} \to [0, 1]$ of a fuzzy variable ζ as

$$\varphi_\zeta(x) = Cr\{\theta \in \Theta : \zeta(\theta) \leq x\}$$

which gives that the credibility of the fuzzy variable (ζ) takes a value less than or equal to x. If the fuzzy variable ζ is expressed by a membership function μ, then its credibility distribution is given by

$$\mu_\zeta(x) = \frac{1}{2}\{\sup_{y \leq x} \mu(y) + \sup_{y < x} \mu(y)\}\}, x, y \in \mathbb{R}.$$

Definition: The credibility density function defined is as $\varphi_\zeta : \mathbb{R} \to [0, \infty)$ of any fuzzy variable ζ is a function such that (Liu [5])

$$\varphi_\zeta(x) = \int_{-\infty}^{x} \varphi(y)dy, \forall x \in \mathbb{R}.$$

If ζ be a fuzzy variable denoted as $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ with $\zeta_1 < \zeta_2 < \zeta_3$; then its the membership function is defined as

$$\mu_\zeta(x) = \begin{cases} \frac{x-\zeta_1}{\zeta_2-\zeta_1}, \; if\; \zeta_1 \le x \le \zeta_2 \\ \frac{\zeta_3-x}{\zeta_3-\zeta_2}, \; if\; \zeta_2 \le x \le \zeta_3 \\ 0, \; otherwise \end{cases}$$

The credibility distribution function of fuzzy number ζ is

$$\Phi_\zeta(x) = \begin{cases} 0, & if\; x < \zeta_1 \\ \frac{x-\zeta_1}{2(\zeta_2-\zeta_1)} & if\; \zeta_1 \le x \le \zeta_2 \\ \frac{x+\zeta_3-2\zeta_2}{2(\zeta_3-\zeta_2)} & if\; \zeta_2 \le x \le \zeta_3 \\ 1 & if\; x \ge \zeta_3 \end{cases}$$

The credibility density function of fuzzy number ζ is obtained as

$$\phi_\zeta(x) = \begin{cases} \frac{1}{2(\zeta_2-\zeta_1)}, & if\; \zeta_1 \le x \le \zeta_2 \\ \frac{1}{2(\zeta_3-\zeta_2)} & if\; \zeta_2 \le x \le \zeta_3 \\ 0, & otherwise \end{cases}$$

In the same way, the credibility distribution function and credibility density function of a trapezoidal fuzzy variable $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ are given by

$$\Phi_\zeta(x) = \begin{cases} 0 & if\; x \le \zeta_1 \\ \frac{x-\zeta_1}{2(\zeta_2-\zeta_1)} & if\; \zeta_1 \le x \le \zeta_2 \\ \frac{1}{2} & if\; \zeta_2 \le x \le \zeta_3 \\ \frac{x+\zeta_4-2\zeta_3}{2(\zeta_4-\zeta_3)} & if\; \zeta_3 \le x \le \zeta_4 \\ 1 & if\; x \ge \zeta_4 \end{cases}$$

and

$$\phi_\zeta(x) = \begin{cases} \frac{1}{2(\zeta_2-\zeta_1)} & if\; \zeta_1 \le x \le \zeta_2 \\ \frac{1}{2(\zeta_4-\zeta_3)} & if\; \zeta_3 \le x \le \zeta_4 \\ 0 & otherwise \end{cases}$$

3 Sampling Technique for Credibility Distribution

Sampling technique to produce random numbers usually used in probabilistic technique can also be used for credibility distribution of fuzzy variable ζ (Dutta [38]). For this purpose, uniformly distributed random numbers from [0, 1] are generated first and after that random variables are generated by equating these number to credibility distribution function. One number (say, z_c) is generated in this process, corresponding to the credibility distribution function.

Suppose w is a uniformly distributed random number then the uncertain variable z_c is obtained as

$$z_c = \Phi_\zeta^{-1}(w)$$

For example, if $\zeta = (4, 6, 9)$ is fuzzy variable whose corresponding credibility distribution is depicted in Fig. 1, then, for the uniformly distributed random number 0.90, the sampling technique produces 8.4 is the values of the uncertain variable corresponding to credibility distribution.

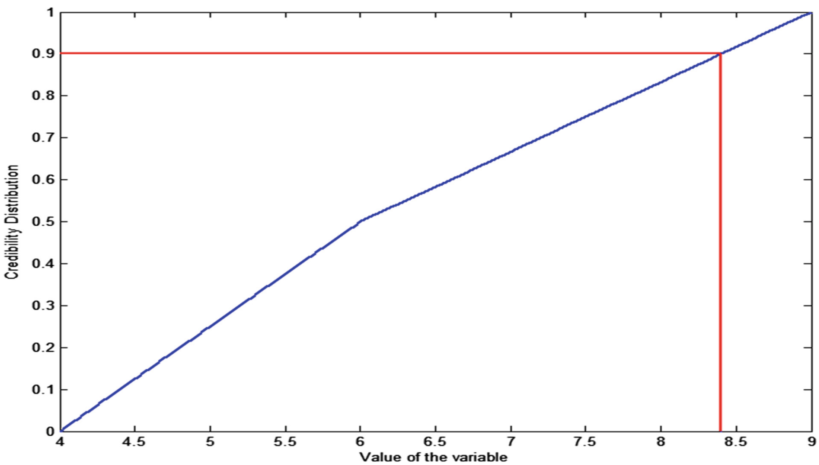

Fig. 1. Credibility distribution of the fuzzy variable $\zeta = (4, 6, 9)$

4 Max-Min Composition of Credibility Vector

Let $\xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n)$ be two credibility vectors where ξ_i and $\zeta_i (i = 1, 2, 3, \ldots, n)$ are fuzzy variables. Suppose $\Phi_{\xi_i}^{-1}$ and $\Phi_{\zeta_i}^{-1}$ are the corresponding credibility distribution of ξ_i and ζ_i respectively. Let $w \in [0, 1]$ be a uniformly distributed random number and $\xi_i^{-1}(w)$ and $\zeta_i^{-1}(w)$ are the sampling values of the credibility vectors where ξ_i and ζ_i respectively. Then, the max-min composition of the credibility vectors where ξ_i and ζ_i is defined as

$$\Phi_\xi^{-1} \circ \Phi_\zeta^{-1}(w) = \underset{i}{Max}\Big[Min\Big\{\Phi_{\xi_i}^{-1}(w), \Phi_{\zeta_i}^{-1}(w)\Big\}\Big], w \in [0, 1].$$

This composition rule will be explored to obtain patients-diseases direct relationship from patients-symptoms and symptoms-diseases relationships.

5 Medical Diagnosis

The Decision making and Medicine are the two most fertile and fascinating regions where FST finds applications. Real world problems are usually ill defined and also because of the vague uncertain and imprecise character of data/information, decision making requires the use of fuzzy variables. In this part, an attempt has been made to perform medical investigation using credibility distribution. An algorithm is developed here whose stages are presented below and to implement the algorithm computer program has been formulated via Matlab M-file.

5.1 Algorithm to for Medical Diagnosis

Suppose $\Psi = \{\Psi_1, \Psi_2, \ldots, \Psi_p\}, \Upsilon = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_q\}$ and $\Gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_r\}$ are the collections of patients, symptoms and diseases respectively. To establish the patients and – diseases relationship the following steps are adopted.

- Determine patients and symptoms first.
- Construction of medical knowledge based on fuzzy relations i.e., to establish patient-symptoms fuzzy relation ($\Im$) and symptoms-diseases fuzzy relation ($\Re$).
- Conversion of all the components of symptoms patient-symptoms fuzzy relation ($\Im$) and symptoms-diseases fuzzy relations ($\Re$) into credibility distributions.
- Evaluation of diagnosis value using the max-min composition of patient-symptoms ($\Im$) and symptoms-diseases ($\Re$) relations to obtain patients and – diseases relation ($\mho$)

$$\text{i.e., } \Phi^{-1}_{\mho(\Psi_j,\Gamma_k)}(w) = \underset{l}{Max}\left[Min\left\{\Phi^{-1}_{\Im(\Psi_j,\Upsilon_l)}(w), \Phi^{-1}_{\Re(\Upsilon_l,\Gamma_k)}(w)\right\}\right]$$

  where $j = 1, 2, \ldots, p$; $l = 1, 2, \ldots, q$ and $k = 1, 2, \ldots, r$.
- Highest diagnosis value in every row will indicate that the patient Ψ_j is expected to have the respective disease Γ_k.
- If in any row two/more highest diagnosis values are acquired then it is difficult take proper decision. In such situations for those particular patients-diseases relation only the following composition operation will be adopted to calculate diagnosis value in fresh.

$$\Phi^{-1}_{\mho(\Psi_j,\Gamma_k)}(w) = \sum_{l}\left[Min\left\{\Phi^{-1}_{\Im(\Psi_j,\Upsilon_l)}(w), \Phi^{-1}_{\Re(\Upsilon_l,\Gamma_k)}(w)\right\}\right]$$

- Then here too, the highest diagnosis value in those rows will indicate that the patient Ψ_j is expected to have the respective disease Γ_k.

5.2 Drawbacks of Existing Work and Its Apposite Solution

In this section, the validity, applicability and superiority of the present approach has been showcased by exposing the shortcomings of Celik and Yamak [12] approach first and also by overcoming the drawbacks.

Suppose there are three patients *Dhurba*, *Debasish* and *Ankur* in a hospital of Dibrugarh district of Assam. The symptoms of the patients' are *temperature*, *headache*, *cough* and *stomach problem*. The patient-symptoms fuzzy relation ($\Im$) and symptoms-diseases fuzzy relations ($\Re$) are presented in Tables 1 and 2 respectively in tabular form which are adopted from Celik and Yamak [12]. Execution of Celik and Yamak [12] method gives that *Dhurba* is suffering from *viral fever* whereas *Debasish* and *Ankur* suffer from *Typhoid*. On the other hand, meticulous inv analysis of the data set leads to establish that *Debasish* also suffers from *viral fever*. It can be observed that for *Debasish* the maximum value among the symptoms crop up for the symptom *temperature* while for the symptom *temperature* the maximum value among the diseases crops up for *viral fever* and hence the extensive observation is that *Debasish* suffers from *viral fever*. Similar way, it can be opined that *Dhruba* is also suffering from *viral fever* while *Ankur* suffers from *Typhoid*.

Table 1. Patients-symptoms fuzzy relation

Q	Temperature	Headache	Cough	Stomach problem
Dhruba	(6,7,8)	(2,3,4)	(4,5,6)	(1,2,3)
Debasish	(5,6,7)	(1,2,3)	(2,3,4)	(4,5,6)
Ankur	(2,3,4)	(4,5,6)	(2,3,4)	(5,6,7)

Table 2. Symptom-disease

R	Viral fever	Typhoid	Malaria
Temperature	(8,9,10)	(4,5,6)	(0,1,2)
Headache	(2,3,4)	(4,5,6)	(4,5,6)
Cough	(4,5,6)	(1,2,3)	(4,5,6)
Stomach problem	(1,2,3)	(7,8,9)	(7,8,9)

To apply the proposed approach it is necessary to convert all the fuzzy variables into credibility distributions and sampling technique has to be performed. For example, if Fig. 2 represent a credibility distribution then at value 0.9 the generated number by the sapling is 132.36.

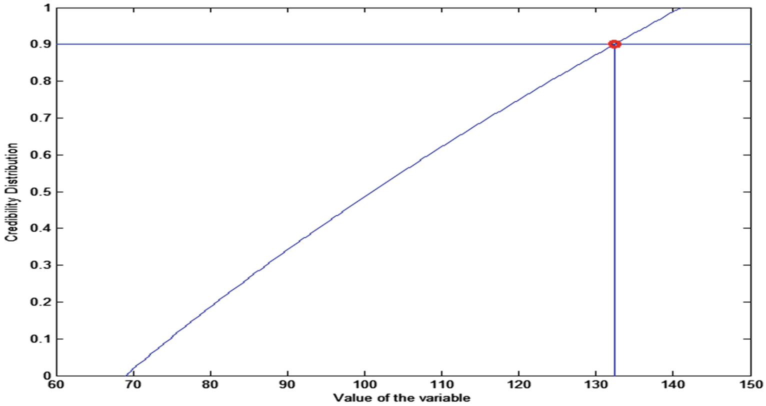

Fig. 2. The sampling value at 0.9 is 132.36

For simplicity, the three values 0, 0.5 and 1 are considered here for sampling purpose and accordingly the medical diagnosis is performed. The diagnosis values of the patient-disease relation obtained by the present approach at 0, 0.5 and 1 are presented in Tables 3, 4 and 5 respectively.

Table 3. The diagnosis values of the patients-diseases relation at 0

	Viral fever	Typhoid	Malaria
Dhruba	**6**	4	4
Debasish	**5**	4	4
Ankur	2	**5**	**5**

Table 4. The diagnosis values of the patients-diseases relation at 0.5

	Viral fever	Typhoid	Malaria
Dhruba	**7**	5	5
Debasish	**6**	5	5
Ankur	3	**6**	**6**

Table 5. The diagnosis values of the patients-diseases relation at 1.

	Viral fever	Typhoid	Malaria
Dhruba	**8**	6	6
Debasish	**7**	6	6
Ankur	4	**7**	**7**

In Table 3, the highest value in the first row is obtained to be 6 which acquaintances *Dhruba* with the disese *viral fever* and Table 4 and 5 also corroborate the same result. Similarly, in the second row the maximum value is 5 which also acquaintances *Debasish* with *viral fever*. In this case also Tables 4 and 5 also substantiate the same result. But from Tables 3 and 4, it can be observed that in the last row two maximum values are obtained which makes difficulties in exact diagnosis. For this case, second composition rule i.e.,

$$\Phi^{-1}_{\mho(\Psi_j,\Gamma_k)}(w) = \sum_l \left[Min\left\{ \Phi^{-1}_{\Im(\Psi_j,\Upsilon_l)}(w),\ \Phi^{-1}_{\Re(\Upsilon_l,\Gamma_k)}(w) \right\} \right]$$

has been adopted and the acquired results are presented in Tables 6, 7 and 8 respectively.

Table 6. The diagnosis values of the patients-diseases relation at 0

	Viral fever	Typhoid	Malaria
Dhruba	**6**	4	4
Debasish	**5**	4	4
Ankur	2	**12**	11

Table 7. The diagnosis values of the patients-diseases relation at 0.5

	Viral fever	Typhoid	Malaria
Dhruba	**7**	5	5
Debasish	**6**	5	5
Ankur	3	**16**	15

Table 8. The diagnosis values of the patients-diseases relation at 1.

	Viral fever	Typhoid	Malaria
Dhruba	**8**	6	6
Debasish	**7**	6	6
Ankur	4	**20**	19

In Table 6, is found that in third row the maximum value is 12 and which connects patient *Ankur* with the disease *Typhoid.* Tables 7 and 8 also confirm the same result. Finally, it is found that *Dhruba and Debasish* both are suffering from viral fever *Ankur* alone suffers from *Typhoid* which is logically and analytically true and even it corroborates human intuitions, however the restriction of this method is that it not proficient to tackle the diagnosis problems with one composition rule.

6 Conclusion

It is pragmatic that most of the medical decision making problems engage treating with uncertainties and even it is utmost important to conscious about all the uncertainties and must try to incorporate all the information into analysis. Most frequently, fuzzy variables are used to represent uncertainty and accordingly medical diagnosis problems are performed under fuzzy environment. However, some studies provide counter intuitive results. Hence an effort has been made to bring credibility theory into application to medical diagnosis. Here, medical diagnosis is carried out by applying our proposed approach in Celik and Yamak [12] diagnosis problem. In this approach, first we determined patients & symptoms and formulate medical knowledge based on fuzzy relations. Then, we converted all the entries of the fuzzy variables into credibility distributions and using credibility sampling patient-disease relation was obtained. It is observed that the patient-disease relationship Tables 3, 4 and 5 easily show that *Dhurba* and *Debasish* both are suffering from *viral fever*. But confusions arises for patient *Ankur* whether he is suffering from *viral fever* or *typhoid*. Then, second composition rule has been considered and it rectifies the confusions. It is found that *Ankur* suffers from *typhoid*. Furthermore, though the results obtained by this present approach completely tally with analytical results and human intuition, but the limitation of this approach is that it is not capable to address the diagnosis problems with a single composition rule. Future work done can be made to devise a straightforward composition rule for better diagnosis.

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Klein-Gordon Equation with Double Ring Shaped Coulomb Potential via AIM

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Abstract. In this article Klein-Gordon equation is studied for Double Ring Shaped Coulomb Potential by Asymptotic Iteration Method. The bound state solution is obtained for inverse square potential from Radial part in terms of confluent hypergeometric function. Energy eigen value for isotropic harmonic oscillator and ring shaped oscillator with its solution in terms of Gauss hypergeometric function are also obtained from the angular part.

Keywords: Klein-Gordon (KG) equation · Double ring shaped Coulomb Potential · Asymptotic Iteration Method (AIM)

1 Introduction

It is well known that the relativistic part of the Schrödinger equation describing free particles is the Klein-Gordan (KG) equation. In relativistic quantum mechanics solution of KG-equation plays an important role for some physical potential. It has been created a line of great interest to solve KG-equation by equating scalar and vector potential for some typical central and non-central potential such as Hulthen [1], Morse [2], Ring shaped Hartmann [3], Kratzer Fuez [4], Pösch-Teller [5], Coulomb [6], Harmonic Oscillator [7] etc.

Particularly the Coulombic ring shaped Potential is introduced by Hartmann et al. [8] in Quantum Chemistry. The double ring shaped Coulomb potential is a 3-dimensional Coulomb potential surrounded by a double ring shaped inverse square potential. Mainly the double ring shaped Coulomb potential is a non-central potential in spherical coordinate and can be written as:-

$$V(r,\theta) = -\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta} \tag{1}$$

where, $A = \eta\sigma^2 e^2$, $B = \frac{\hbar^2\eta^2\sigma^2}{2\mu}$, $C = \frac{\hbar^2}{2\mu}a$ where, $a \geq 0$ μ is the mass of the particle, η and σ are positive real parameters which range from 1 to 10 and r and Θ are the spherical coordinates. In Eq. (1) when we put $B = 0$ and $C = 0$ then V(r) reduces to coulomb potential and when $B = 0$ then V(r) reduces to ring shaped Hartmann potential. Specially the ring shaped non central potentials

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are used to describe the molecular structure [9] and the interaction between the deformed nuclie [10] and specially the molecular structure of benzene [11].

The methods considered to solve KG-equation are Super Symmetric Approximation Method (SUSY) [12], Nikiforov Uvarov Method (NU) [13], WKB Approximation Method [14], Variational, Functional Analysis etc. The Asymptotic Iteration Method (AIM) [15], proposed by Ciffce et al. gives more accurate and efficient results for the spectrum of many particles in relativistic and non-relativistic Quantum Mechanics compared to the other techniques.

In this paper we solve the 3-dimensional KG-equation for the double ring shaped Coulomb potential considering equal scalar and vector potential by using AIM. This paper is as follows: In Sect. 2, we give a brief description about AIM method. Section 3 gives the stationary radial and angle dependent part of the KG equation. In Sect. 4 we found the solution of the Radial part of the KG equation. Section 5 gives the solution of the Angular part of the Kg equation. Section 6 is left for the conclusion.

2 Overview of Asymptotic Iteration Method

The AIM method is based on solving a second order differential equation of the form:

$$f_n''(x) = \lambda_0(x) f_n'(x) + s_0(x) f_n(x) \tag{2}$$

Where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x. The variables, $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To find a general solution to this equation, we differentiate (1) with respect to x and find

$$f_n'''(x) = \lambda_1(x) f_n'(x) + s_1(x) f_n(x) \tag{3}$$

Where $\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x)$,

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \tag{4}$$

Similarly, the second derivative of (1) yields

$$f_n^4(x) = \lambda_2(x) f_n'(x) + s_2(x) f_n(x), \tag{5}$$

Where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{6}$$

Equation (1) can be easily iterated up to $(k+1)th$ and $(k+2)th$ derivatives, $k = 1, 2, 3, ...$ Therefore, we have the recurrence relations

$$\begin{aligned} f_n^{(k+1)}(x) &= \lambda_{k-1}(x) f_n'(x) + s_{k-1}(x) f_n(x), \\ f_n^{(k+2)}(x) &= \lambda_k(x) f_n'(x) + s_k(x) f_n(x), \end{aligned} \tag{7}$$

Where

$$\lambda_k(x) = \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x),$$
$$s_k(x) = s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x). \tag{8}$$

From the ratio of the (k + 2)th and (k + 1)th derivatives, we have

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{f_n^{(k+2)}(x)}{f_n^{(k+1)}(x)} = \frac{\lambda_k(x)[f'_n(x) + \frac{s_k(x)}{\lambda_k(x)} f_n(x)]}{\lambda_{k-1}(x)[f'_n(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} f_n(x)]}. \tag{9}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x) \tag{10}$$

which is the "asymptotic" aspect of the method, then, (8) reduces to

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{11}$$

which yields

$$f_n^{(k+1)}(x) = C_1 exp(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)} dx) = C_1 \lambda_{k-1}(x) exp(\int [\alpha(x) + \lambda_0(x)]dx), \tag{12}$$

where C_1 is the integration constant and the right hand side of (11) is obtained by using (9) and (10). By inserting (11) into (6), the first-order differential equation is obtained as

$$f'_n(x) + \alpha(x) f_n(x) = C_1 exp(\int [\alpha(x) + \lambda_0(x)]dx). \tag{13}$$

This first-order differential equation can easily be solved and the general solution of (1) can be obtained as:

$$f_n(x) = exp(-\int^x \alpha(x_1)dx_1)[C_2 + C_1 \int^x exp(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)]dx_2)dx_1] \tag{14}$$

For a given potential, the radial Klein-Gordon equation is converted to the form of (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (7). The termination condition of the method in (9) can be arranged as

$$\Delta_k(x) = \lambda_k(x) s_{k-1}(x) - \lambda_{k-1}(x) s_k(x) = 0, \tag{15}$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of (15) and the radial quantum number n is equal to the iteration number k for this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential and the approximate energy eigenvalues are obtained from the roots of (15) for sufficiently great values of k with iteration for which k is always greater than n in these numerical solutions.

The general solution of (1) is given by (13). The first part of (13) gives us the polynomial solutions that are convergent and physical, whereas the second part of (13) gives us non-physical solutions that are divergent. Although (13) is the general solution of (1), we take the coefficient of the second part (C_1) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials:

$$f_n(x) = C_2 exp(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)} dx_1), \tag{16}$$

where n represents the principal quantum number.

3 Stationary Radial and Angle-Dependent Klein-Gordon Equation with Equal Scalar and Vector Potential

The stationary 3D K-G equation with the coupling of a vector potential V(r) and a scalar potential S(r) for a particle of rest mass m_0 in the natural units $\hbar$ = c = 1 can be expressed as

$$\nabla^2 \Psi(r,\theta,\phi) + [(E - V(r,\theta,\phi))^2 - (m_0 + S(r,\theta,\phi)^2)]\Psi(r,\theta,\phi) = 0 \tag{17}$$

where, E, V(r) and S(r) are the relativistic energy of the particle, vector and scaler potentials, respectively. Assuming V(r) = S(r) we get from Eq. (17),

$$\nabla^2 \Psi(r,\theta,\phi) + [(E^2 - {m_0}^2) - 2(E + m_0)V(r,\theta)]\Psi(r,\theta,\phi) = 0 \tag{18}$$

Now considering the double ring shaped Coulomb potential the KG-equation reduces to,

$$\nabla^2 \Psi(r,\theta,\phi) + [(E^2 - {m_0}^2) - 2(E + m_0)\{-\frac{A}{r} + \frac{B}{r^2 sin^2\Theta} + \frac{C}{r^2 cos^2\Theta}\}]\Psi(r,\theta,\phi) = 0 \tag{19}$$

To separate the variables for the stationary wave function we assume,

$$\Psi(r,\theta,\phi) = \frac{R(r)}{r}\frac{\Theta(\theta)}{sin^{\frac{1}{2}}\theta}\Phi(\phi) \tag{20}$$

By following the standard procedure of separation of variables we get the component equations as follows:-

$$\frac{d^2 R}{dr^2} + (\delta^2 + \frac{\lambda A}{r} - \frac{\alpha^2}{r^2})R(r) = 0 \tag{21}$$

$$\frac{d^2\Theta}{d\theta^2} - [\frac{\lambda B}{sin^2\theta} + \frac{\lambda C}{cos^2\theta} - \frac{1}{4} - \frac{1}{4sin^2\theta} - \alpha^2 + \frac{\beta^2}{sin^2\theta}]\Theta(\theta) = 0 \tag{22}$$

$$\frac{d^2\Phi}{d\phi^2} = -\beta^2\Phi(\phi) \tag{23}$$

where, $\lambda = 2(E + m_0)$, $\delta^2 = (E^2 - {m_0}^2)$, represents the relativistic energy of a particle and α^2 and β^2 are separation constants. Putting $\alpha^2 = l(l+1)$, which we often encounter in various Schrödinger quantum systems, with the orbital angular momentum $l = 0, 1, 2, ...$ and the magnetic quantum number $\beta = 0, \pm 1, \pm 2, ...$.

The solution of Eq. (23) is the azimuthal angle solution and it is,

$$\Phi(\phi) = De^{i\beta\phi} \tag{24}$$

The Eq. (21) is radial equation and Eq. (22) is angle-dependent equation for the KG equation. For these two equations we use AIM in our next parts.

4 Solution for Radial Part of KG Equation

To solve the Eq. (21) with AIM for $l \neq 0$, we should transform Eq. (21) to the form of Eq. (2). For bound state solution of the Eq. (21) we consider $R(0) = 0$ and $R(\infty) = 0$. Therefore for the physically acceptable radial solution we consider the radial wave function as follows:

$$R(r) = r^{(l+1)}e^{-i\delta}f(r) \tag{25}$$

Thus by substituting $y = -2i\delta r$ and taking $R(r)$ as in Eq. (21) the wave function reduces to,

$$\frac{d^2f}{dy^2} - 2(\frac{1}{2} - \frac{l+1}{y})\frac{df}{dy} - (\frac{l+1}{y} - \frac{i\xi A}{y\delta})f(y) = 0 \tag{26}$$

where, $\xi = \frac{\lambda}{2}$ and $\lambda_0(y) = 2(\frac{1}{2} - \frac{l+1}{y})$ and $S_0(y) = \frac{l+1}{y} - \frac{i\xi A}{y\delta}$. After calculating $\lambda_n(y)$ and $S_n(y)$ we get,

$$\begin{aligned}
\lambda_0(y) &= 2(\frac{1}{2} - \frac{l+1}{y}) \\
S_0(y) &= \frac{l+1}{y} - \frac{i\xi A}{y\delta} \\
\lambda_1(y) &= \frac{2(l+1)}{y^2} + \frac{(l+1)-a}{y} - \frac{4(l+1)}{y} + \frac{4(l+1)^2}{y^2} \\
S_1(y) &= \frac{(l+1-a)(y-2l-3)}{y^2} \\
\lambda_2(y) = 1 - \frac{2(a-2l+2)}{y} &+ \frac{(a+3l+3) - 2(l+1)(l+a)}{y^2} - \frac{4(l+1)(2l^2+9l+8)}{y^3} \\
S_2(y) &= \frac{1}{y^3}[(l+1-a)\{y^2 - y(a+3l+4) + (4l^2+16l+15)\}]...etc.
\end{aligned} \tag{27}$$

where, $a = \frac{i\xi A}{\delta}$ combining these results with the condition given in Eq. (10) yields:

$$\frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n=0} = (l+1) \tag{28}$$

$$\frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n=1} = 2(l+1) \tag{29}$$

...

$$\frac{S_n}{\lambda_n} = \frac{S_{(n+1)}}{\lambda_{(n+1)}} \Rightarrow \left[\frac{i\xi A}{\delta}\right]_{n} = (n+1)(l+1) \tag{30}$$

Thus the generalised term in Eq. (30) gives the energy spectrum of the KG-equation with double ring shaped Coulomb potential, where n is radial quantum number $(n = 0, 1, 2, ..)$.

For $A = 0$ in Eq. (21) we get a singular solution which corresponds to the inverse square potential which gives bound state only if the separation constant is negative specially less than $-\frac{1}{4}$. The bound states are determined by potential well type and hence the quantum number n is limited by the study of potential well.

Equation (26) satisfies the confluent hypergeometric function and the solution of the differential equation given in (26) can be written as,

$$f(r) = 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r) \tag{31}$$

Thus the radial part of the wave function can be written as,

$$R(r) = N_1 r^{(l+1)} e^{-i\delta} 1F_1(l+1-\frac{i\xi A}{\delta}, 2l+2; -2i\delta r) \tag{32}$$

where N_1 is the normalizing constant.

5 Solution for Angular Part of KG Equation

The angular part of the KG equation for double ring shaped coulomb potential is given by Eq. (22). We consider $\tilde{l} = l + \frac{1}{2}$ and $l(l+1) = \tilde{l}^2 - \frac{1}{4}$ and the Eq. (22) becomes,

$$\frac{d^2\Theta}{d\theta^2} - \left[\frac{\lambda C}{cos^2\theta} + \frac{\lambda B - \frac{1}{4} + \beta^2}{sin^2\theta}\right]\Theta = -\tilde{l}^2\Theta(\theta) \tag{33}$$

Defining P and Q as follows,

$$P = -\frac{1}{2} \pm \sqrt{\lambda B + \beta^2}\, and \tag{34}$$

$$Q = -\frac{1}{2} \pm \sqrt{\lambda C + \frac{1}{4}} \tag{35}$$

Equation (33) reduces to,

$$\frac{d^2\Theta}{d\theta^2} - \left[\frac{Q(Q+1)}{cos^2\theta} + \frac{P(P+1)}{sin^2\theta}\right]\Theta = -\tilde{l}^2\Theta(\theta) \tag{36}$$

To solve this equation by AIM with boundary conditions i.e. $\theta(0)$ and $\theta(\pi)$ are finite we consider the following wave function,

$$\Theta(\theta) = sin^{P+1}\theta cos^{Q+1}\theta f(\theta) \tag{37}$$

By using this wave function in Eq. (36) we get the second order homogeneous differential equation as,

$$\frac{d^2 f}{d\theta^2} = 2\{(Q+1)tan\theta - (P+1)cot\theta\}\frac{df}{d\theta} + [(P+Q+2)^2 - (l+\frac{1}{2})^2]f \tag{38}$$

Now by using AIM method we have,

$$s_0 = [(P+Q+2)^2 - (l+\frac{1}{2})^2] \tag{39}$$

$$\lambda_0 = 2\{(Q+1)tan\theta - (P+1)cot\theta\} \tag{40}$$

$$s_1 = 2[(P+Q+2)^2 - (l+\frac{1}{2})^2]\{(Q+1)tan\theta - (P+1)cot\theta\} \tag{41}$$

$$\lambda_1 = 2\{(Q+1)sec^2\theta + (P+1)cosec^2\theta\} + [(P+Q+2)^2 - (l+\frac{1}{2})^2] + 4[\{(Q+1)tan\theta - (P+1)cot\theta\}]^2 \tag{42}$$

and so on. Thus combining these results with the condition given by Eq. (10) we have,

$$\frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \Rightarrow \quad \tilde{l}^2 = (P+Q+2)^2 \tag{43}$$

$$\frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} \Rightarrow \quad \tilde{l}^2 = (P+Q+4)^2 \tag{44}$$

$$....etc. \tag{45}$$

and finally we get the generalised form as,

$$\tilde{l}^2 = (P+Q+2n+2)^2 \quad for \quad n = 0,1,2,3,... \tag{46}$$

By putting the value of P , Q and $\tilde{l}^2$ into the Eq. (46) we get value of l,

$$l = \sqrt{\lambda B + \beta^2} + \sqrt{\lambda C + \frac{1}{4}} + 2n + \frac{1}{2} \tag{47}$$

now by inserting the value of l into the generalised form of energy eigen value for the radial part of the KG-equation with double ring shaped coulomb potential given by Eq. (30), we get the relativistic energy spectrum for a bound electron from the following equation,

$$\left[\frac{i\xi A}{\delta}\right]_n = (n+1)(\sqrt{\lambda B + \beta^2} + \sqrt{\lambda C + \frac{1}{4}} + 2n + \frac{1}{2} + 1) \tag{48}$$

By putting $B = C = 0$ into the above equation we get the energy eigen value for isotropic harmonic oscillator[] and by putting $B = 0$ and $C \neq 0$ into the above equation we get the eigen values for the ring shaped oscillator.

For the eigen function of the angular part we substitute $cos^2(\theta) = x$ in the Eq. (38) and it reduces to the form:

$$x(1-x)\frac{d^2 f}{dx^2} + [(Q + \frac{3}{2}) - (P + Q + 3)x]\frac{df}{dx} + \frac{1}{4}[(P + Q + 2)^2 - (l + \frac{1}{2})^2]f = 0 \tag{49}$$

The solution of the above differential equation is of the form of Gauss hypergeometric function given by,

$$f(\theta) = 2F_1(-n, P + Q + 2 + n, Q + \frac{3}{2}; cos^2(\theta) \tag{50}$$

Therefore, using the value of $f(\theta)$ in the Eq. (37) and we get the eigen function for the angular part as follows:

$$\Theta(\theta) = N_2 sin^{P+1}\theta cos^{Q+1}\theta 2F_1(-n, P + Q + 2 + n, Q + \frac{3}{2}; cos^2(\theta)) \tag{51}$$

where N_2 is the normalizing constant.

6 Conclusion

In this paper we have solved the KG-equation for the Double Ring Shaped coulomb Potential via Asymptotic Iteration Method. Using this method we get the general expression of the energy spectrums and the corresponding wave function in terms of confluent hypergeometric function multi-dimensional space. From the solution of the radial part we get the bound state solution of the inverse square potential by equating $A = 0$ in Eq. (21) and we also get a solution of the bound state of potential well type. From the solution of angular part we get the energy eigen values for isotropic harmonic oscillator and for the ring shaped oscillator by equating the coefficients in Eq. (48) with appropriate values. The Double Ring Shaped Potentials have many applications in the field of nuclear physics and quantum chemistry which are mainly used to describe the interaction between the deformed pair of nuclei in Physics and to describe the molecular structure of benzene in chemistry. Thus the non-central potential named double Ring Shaped Coulomb Potential is used to find the quantum information in chemical and Molecular Physics.

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Bound State Solutions of the Klein-Gordon Equation for Rosen-Morse Potential in Spin and Pseudo-Spin Symmetry

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Abstract. In this article, the bound state solutions of the Klein-Gordon equation for Rosen-Morse potential with mixed scalar-vector potentials are investigated within the framework of spin and pseudo-spin symmetric limits. It is also explained that the bound state energy spectrum exists only for the spin symmetric limit, but not for the pseudo-spin symmetric limit. To pursue the investigation the asymptotic iteration method is used. My results are in excellent agreement with previous work.

Keywords: Klein-Gordon equation · Rosen-Morse potential · Spin and pseudo-spin symmetry · Asymptotic iteration method

1 Introduction

In recent studies, spin and pseudo-spin symmetry created a line of interest of its applications in nuclear physics and other related fields. In such applications, the scalar and vector potentials of the relativistic problem are related in a simple way. This concepts have been used to establish an effective nuclear shell-model scheme. Traditionally, for spin and pseudo-spin symmetry, the attractive Lorentz scalar potential and the repulsive Lorentz vector potential of a physical problem are related by $V_s = \pm V_v + \eta$, where η is a constant. In finite nuclei, the constant η is zero.

In 1969, Hecht and Adler introduced the idea about spin symmetry and pseudo-spin symmetry with the nuclear shell model [1]. Recently, a large number of researcher have shown their interest in obtaining the solutions of the Dirac equations for some typical potentials within spin and pseudo-spin symmetric limit but a little number of study done under the framework of Klein-Gordon formulation. This idea have been used in the investigation of certain aspects of deformed and exotic nuclei in nuclear physics and relevant fields.

Spin symmetry is relevant to meson with one heavy quark, which is being used to explain the absence of quark spin orbit splitting (spin doublets) observed

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in heavy-light quark mesons. On the other hand, pseudo-spin symmetry concept has been successfully used to explain different phenomena in nuclear structure including deformation, super-deformation, identical bands, exotic nuclei and degeneracies of some shell model orbital in nuclei (pseudo-spin doublets). Ginocchio et al. [2] deduced that a Dirac Hamiltonian with scalar V_s and vector V_v harmonic oscillator potentials when $V_v = V_s$ possesses a spin symmetry, whereas for $V_v + V_s = 0$ or $V_v = -V_s$ it possesses a pseudo-spin symmetry.

Recent studies under spin and pseudo-spin symmetric limit include the potentials: pseudo harmonic oscillator potential [3], Kratzer-Fues potential [4], Hulthén potential [5,6], modified Eckart and modified deformed Hylleraas potential [7], Pöschl-Teller potential [8], the Rosen-Morse potential [9,10], harmonic oscillator potential [11], Manning-Rosen potential [12], Wood-Saxon potential [13], Kratzer potential with angle dependent potential [14], Scarf potential [15], the Hua potential [16]. And the methods include: Asymptotic Iteration Method (AIM) [17–22], the Nikiforov-Uvarov method (N-U) [23], super-symmetric and shape invariance method [24]. Here I adopt the AIM to cultivate the bound state energy eigenvalues and the corresponding eigenfunctions for Rosen-Morse potential with the vector potential V_v and scalar potential V_s for the Klein-Gordon equation.

In this article, the bound state energy eigenvalues and the corresponding eigenfunctions of Rosen-Morse potential with spin and pseudo-spin symmetric limit for the s-wave in the one-dimensional Klein-Gordon equation with the help of the asymptotic iteration method and also shown that the bound state energy spectrum exists only for the spin symmetric limit, but not for the pseudo-spin symmetric limit.

The manuscript is organized as follows: To make it self-contained, a brief review of AIM is presented in the next section. In Sect. 3, the Klein-Gordon Equation is expressed for Rosen-Morse potential with spin and pseudo-spin symmetric limit. In Sect. 4, the Spin and pseudo-spin symmetric solutions obtained for any k state and also given a remark on non-existence of bound state solution for the pseudo-spin symmetric limit. Finally, the conclusions are given in Sect. 5.

2 Overview of AIM Method

The AIM method is based on solving a second order differential equation of the form:

$$f_n''(x) = \lambda_0(x) f_n'(x) + s_0(x) f_n(x) \tag{1}$$

where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to x. The variables, $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To find a general solution to this equation, we differentiate (1) with respect to x and find

$$f_n'''(x) = \lambda_1(x) f_n'(x) + s_1(x) f_n(x) \tag{2}$$

where $\lambda_1(x) = \lambda_0'(x) + s_0(x) + \lambda_0^2(x)$,

$$s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \tag{3}$$

Similarly, the second derivative of (1) yields

$$f_n^4(x) = \lambda_2(x) f_n'(x) + s_2(x) f_n(x), \tag{4}$$

where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{5}$$

Equation (1) can be easily iterated up to $(k + 1)th$ and $(k + 2)th$ derivatives, $k = 1, 2, 3, ...$ Therefore, we have the reccurence relations

$$\begin{aligned} f_n^{(k+1)}(x) &= \lambda_{k-1}(x) f_n'(x) + s_{k-1}(x) f_n(x), \\ f_n^{(k+2)}(x) &= \lambda_k(x) f_n'(x) + s_k(x) f_n(x), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x). \end{aligned} \tag{7}$$

From the ratio of the (k + 2)th and (k + 1)th derivatives, we have

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{f_n^{(k+2)}(x)}{f_n^{(k+1)}(x)} = \frac{\lambda_k(x)[f_n'(x) + \frac{s_k(x)}{\lambda_k(x)} f_n(x)]}{\lambda_{k-1}(x)[f_n'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} f_n(x)]}. \tag{8}$$

For sufficiently large k, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x) \tag{9}$$

which is the "asymptotic" aspect of the method, then, (8) reduces to

$$\frac{d}{dx}\ln[f_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{10}$$

which yields

$$f_n^{(k+1)}(x) = C_1 exp(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)} dx) = C_1 \lambda_{k-1}(x) exp(\int [\alpha(x) + \lambda_0(x)] dx), \tag{11}$$

where C_1 is the integration constant and the right hand side of (11) is obtained by using (9) and (10). By inserting (11) into (6), the first-order differential equation is obtained as

$$f_n'(x) + \alpha(x) f_n(x) = C_1 exp(\int [\alpha(x) + \lambda_0(x)] dx). \tag{12}$$

This first-order differential equation can easily be solved and the general solution of (1) can be obtained as:

$$f_n(x) = exp(-\int^x \alpha(x_1) dx_1)[C_2 + C_1 \int^x exp(\int^{x_1} [\lambda_0(x_2) + 2\alpha(x_2)] dx_2) dx_1] \tag{13}$$

For a given potential, the radial Klein-Gordon equation is converted to the form of (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (7). The termination condition of the method in (9) can be arranged as

$$\Delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \tag{14}$$

where k shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of (14) and the radial quantum number n is equal to the iteration number k for this case. For nontrivial potentials that have no exact solutions, for a specific n principal quantum number, every one choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential and the approximate energy eigenvalues are obtained from the roots of (14) for sufficiently great values of k with iteration for which k is always greater than n in these numerical solutions.

The general solution of (1) is given by (13). The first part of (13) gives the polynomial solutions that are convergent and physical, whereas the second part gives non-physical solutions that are divergent. Although (13) is the general solution of (1), one take the coefficient of the second part (C_1) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials:

$$f_n(x) = C_2 exp(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)}dx_1), \tag{15}$$

where n represents the principal quantum number.

3 The Klein-Gordon Equation

In the conventional relativistic units $\hbar = c = 1$ the one-dimensional K-G equation for a particle of rest mass m can be expressed as

$$\Psi''(x) + [(E_n - V_v(x))^2 - (m + V_s(x))^2]\Psi(x) = 0 \tag{16}$$

where n, E_n, $V_v(x)$ and $V_s(x)$ are the quantum number, relativistic energy of the particle, vector and scaler potentials respectively.

Assuming $V_s(x) = \pm V_v(x) + \eta$, we get from (16)

$$\Psi''(x) + [(E_n^2 - m^2 - \eta^2 - 2m\eta) - 2(E_n \pm m \pm \eta)V_v(x)]\Psi(x) = 0 \tag{17}$$

4 Bound State Solutions Within Spin and Pseudo-Spin Symmetric Limit

Considering the Rosen-Morse potential [25] as

$$V_v(x) = -V_1 sech^2\alpha x + V_2 \tanh \alpha x \tag{18}$$

where α is the screening parameter, determining the range for the Rosen-Morse potential. The Rosen-Morse potential can be written in the exponential form:

$$V_v(x) = -4V_1\frac{e^{-2\alpha x}}{(1+e^{-2\alpha x})^2} + V_2\frac{1-e^{-2\alpha x}}{1+e^{-2\alpha x}} \tag{19}$$

Introducing a new variable $s = e^{-2\alpha x}$ it is straight forward to show that (17) takes the form:

$$\Psi''(s) + \frac{1}{s}\Psi'(s) + [-\frac{\epsilon_n^2}{s^2} + \frac{\gamma(\gamma-1)}{s(1+s)^2} - \frac{\beta^2(1-s)}{s^2(1+s)}]\Psi(s) = 0 \tag{20}$$

where we have used the notations

$$\epsilon_n^2 = \frac{m^2+\eta^2+2m\eta-E_n^2}{4\alpha^2}, \gamma(\gamma-1) = \frac{2(E_n \pm m \pm \eta)V_1}{\alpha^2}, \beta^2 = \frac{(E_n \pm m \pm \eta)V_2}{2\alpha^2} \tag{21}$$

The wave function should satisfy the boundary conditions, i.e. $\Psi(s) = 0$ at $s = 0$ and $\Psi(s) = 0$ at $s = -1$. Therefore the reasonable physical wave function we consider is

$$\Psi(s) = s^{\sqrt{\epsilon_n^2+\beta^2}}(1+s)^{\gamma}f_n(s), \tag{22}$$

and $\Psi'(s)$ and $\Psi''(s)$ becomes

$$\begin{aligned}
\Psi'(s) = {} & [f_n'(s) + (\frac{\sqrt{\epsilon^2+\beta^2}}{s} + \frac{\gamma}{1+s})f_n(s)]s^{\sqrt{\epsilon_n^2+\beta^2}}(1+s)^{\gamma} \\
\Psi''(s) = {} & [f_n''(s) + 2(\frac{\sqrt{\epsilon^2+\beta^2}}{s} + \frac{\gamma}{1+s})f_n'(s) \\
& + [(\frac{\sqrt{\epsilon^2+\beta^2}}{s} + \frac{\gamma}{1+s})^2 - \frac{\sqrt{\epsilon^2+\beta^2}}{s^2} - \frac{\gamma}{(1+s)^2}]f_n(s)]s^{\sqrt{\epsilon_n^2+\beta^2}}(1+s)^{\gamma}
\end{aligned}$$

Now inserting this wave function (22) into Eq. (20), we obtain the linear second-order homogeneous differential equation in the form:

$$\begin{aligned}
f_n''(s) & = -[\frac{(1+2\sqrt{\epsilon_n^2+\beta^2}) + (1+2\sqrt{\epsilon_n^2+\beta^2}+2\gamma)s}{s(1+s)}]f_n'(s) \\
& \quad -[\frac{\gamma^2+2\gamma\sqrt{\epsilon_n^2+\beta^2}+2\beta^2}{s(1+s)}]f_n(s)
\end{aligned} \tag{23}$$

which is amenable to an AIM solution. By comparing Eq. (23) with Eq. (1), we get the values of $\lambda_0(s)$ and $s_0(s)$ and we may calculate $\lambda_k(s)$ and $s_k(s)$ by using the recursion relation (7). This gives

$$\lambda_0(s) = -[\tfrac{(1+2\sqrt{\epsilon_n^2+\beta^2})+(1+2\sqrt{\epsilon_n^2+\beta^2}+2\gamma)s}{s(1+s)}]$$

$$s_0(s) = -[\tfrac{\gamma^2+2\gamma\sqrt{\epsilon_n^2+\beta^2}+2\beta^2}{s(1+s)}]$$

$$\lambda_1(s) = \tfrac{[4+6(1+\gamma)\sqrt{\epsilon_n^2+\beta^2}+3\gamma^2+6\gamma+4\epsilon_n^2+2\beta^2]s^2+[4+6(2+\gamma)\sqrt{\epsilon_n^2+\beta^2}+8\epsilon_n^2+6\beta^2-\gamma^2+4\gamma]s}{s^2(1+s)^2}$$

$$\tfrac{+(2+6\sqrt{\epsilon_n^2+\beta^2}+4\epsilon_n^2+4\beta^2)}{s^2(1+s)^2}$$

$$s_1 = \tfrac{(\gamma^2+2\gamma\sqrt{\epsilon_n^2+\beta^2}+2\beta^2)[2+3s+2(1+s)\sqrt{\epsilon_n^2+\beta^2}+2\gamma s]}{s^2(1+s)^2} ...etc. \tag{24}$$

Inserting these equations into termination condition, $\triangle_1(s) = s_0(s)\lambda_1(s) - s_1(s)\lambda_0(s)$, we obtain the first $\triangle_1(s)$ value as

$$\triangle_1(s) = -\frac{(\gamma^2 + 2\gamma\sqrt{\epsilon_n^2 + \beta^2} + 2\beta^2)[\gamma^2 + 2\gamma + 1 + 2\beta^2 + 2(1+\gamma)\sqrt{\epsilon_n^2 + \beta^2}]}{s^2(1+s)^2} \tag{25}$$

The root of Eq. (25) gives us the first value of ϵ_n^2 as $\epsilon_0^2 = \frac{\gamma^4+4\beta^2}{4\gamma^2}$. In similar way, using the quantization condition given by Eq. (14), we get other $\triangle_n(s)$ and ϵ_n^2 values. For a better understanding the expressions of $\triangle_2(s)$ and $\triangle_3(s)$ along with corresponding ϵ_2^2 and ϵ_3^2 are given bellow:

$$\triangle_2(s) = s_1(s)\lambda_2(s) - s_2(s)\lambda_1(s) = 0 \Rightarrow \epsilon_1^2 = \frac{(\gamma+1)^4 + 4\beta^2}{4(\gamma+1)^2},$$

$$\triangle_3(s) = s_2(s)\lambda_3(s) - s_3(s)\lambda_2(s) = 0 \Rightarrow \epsilon_2^2 = \frac{(\gamma+2)^4 + 4\beta^2}{4(\gamma+2)^2}, \tag{26}$$

Proceeding in this way we can generalize the above expressions by mathematical induction, the eigenvalues to be of the form

$$\epsilon_n^2 = \frac{(\gamma+n)^4 + 4\beta^2}{4(\gamma+n)^2}; n = 0, 1, 2, 3, ... \tag{27}$$

It is important to note that the term $\sqrt{\epsilon_0^2 + \beta^2}$ has two values, which are $\pm\frac{\gamma^2+2\beta}{2\gamma}$ but only the negative value satisfies the Eq. (25). The other value does not satisfy Eq. (25), which is also true for $\epsilon_1, \epsilon_2, \epsilon_3, ...$
From Eqs. (21) and (27), we obtain the energy eigenvalues E_n for the Rosen-Morse potential as

$$E_n^2 = (m+\eta)^2 - \alpha^2[(\gamma+n)^2 + \frac{4\beta^2}{(\gamma+n)^2}] \tag{28}$$

where, $\gamma = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{8(E_n \pm m \pm \eta)V_1}{\alpha^2}}$. This result is same as the s-state solution of the K-G equation for the Rosen-Morse potential with spin symmetry in (11).

It should be noted that, putting $s = -z$ the Eq. (23) transforms to the differential equation of the form

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0, \tag{29}$$

which is satisfied by the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$. Here, $a+b = 2(\sqrt{\epsilon_n^2 + \beta^2}+\gamma)$, $ab = \gamma^2+2\gamma\sqrt{\epsilon_n^2 + \beta^2}+2\beta^2$,$c = 1+2\sqrt{\epsilon_n^2 + \beta^2}$. Therefore the corresponding unnormalized eigenfunctions by using the wave function generator given by (15) could be obtained as

$$f_n(s) = (-1)^n C_2 \frac{\Gamma(n + 2\sqrt{\epsilon_n^2 + \beta^2} + 1)}{\Gamma(2\sqrt{\epsilon_n^2 + \beta^2} + 1)} {}_2F_1(-n, 2(\sqrt{\epsilon_n^2 + \beta^2} + \gamma) + n; 1 + 2\sqrt{\epsilon_n^2 + \beta^2}; -s) \quad (30)$$

where Γ and ${}_2F_1$ are the gamma function and Gauss hypergeometric function respectively [26,27]. Therefore, we can write the total radial wave function by using Eqs. (22) and (30) as

$$\Psi(s) = N s^{\sqrt{\epsilon_n^2+\beta^2}} (1 + s)^{\gamma} {}_2F_1(-n, 2(\sqrt{\epsilon_n^2 + \beta^2} + \gamma) + n; 1 + 2\sqrt{\epsilon_n^2 + \beta^2}; -s) \quad (31)$$

where N is normalization constant.

4.1 Bound State Conditions

Let us construct two positively defined wave number k_1 and k_2 related to following parameters

$$\sqrt{\epsilon_n^2 + \beta^2} = \frac{k_1}{\alpha} \quad (32)$$

$$\gamma = \frac{ik_2}{\alpha} \quad (33)$$

as

$$k_1 \equiv \frac{1}{2}\sqrt{-(E_n - m - \eta - V_2(1 \mp 1))(E_n + m + \eta - V_2(1 \pm 1))} \quad (34)$$

$$k_2 \equiv -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{8(E_n \pm m \pm \eta)V_1}{\alpha^2}} \quad (35)$$

Due to confinement of the KG particles, the wave functions should exponentially decay outside the potential well and should accompany to the particles within the well. The wave numbers defined in Eqs. (34) and (35) should be real to satisfy the confinement conditions

$$-(E_n - m - \eta - V_2(1 \mp 1))(E_n + m + \eta - V_2(1 \pm 1)) > 0 \quad (36)$$

$$\alpha^2 + 8(E_n \pm m \pm \eta)V_1 \geq 0 \quad (37)$$

in addition to the conditions $V_1 > 0$ and $V_2 > 0$. These conditions are used to comprehend the wave functions within spin and pseudo-spin symmetric limit.

4.2 Within Spin Symmetric Limit

The bound state conditions given in Eqs. (36) and (37) is visualized in Fig. 1 within spin symmetric limit. The intersection of the required conditions is shown by the shaded region (see Fig. 1). The first condition (given by Eq. (36)), restricts the energy eigenvalues in an interval whereas the other condition (given by Eq. (37)), indicates the minimum value of energy spectrum. If a KG particle with a rest mass m is confined in a Rosen-Morse potential well that has the depth parameter less than m, only positive bound states can be obtained. Negative energy eigenvalues start to appear with constriction, in Rosen-Morse potential wells. In a Rosen-Morse potential well, the whole range of energy spectrum from $-m - \eta + 2V_2$ to $m + \eta$ is obtained.

4.3 Within Pseudo-Spin Symmetric Limit

For the pseudo-spin symmetric limit, the analysis of the required conditions ends up with a surprising result. The inequalities given with Eqs. (36) and (37) are satisfied in the shaded area plotted in (see Fig. 2). On the other hand, a Rosen-Morse potential well occurs only for a positive potential depth parameter V_1. Therefore, the energy eigenvalue solution of bound state in the pseudo-spin symmetric limit is an empty set and consequently, a KG particle cannot be confined within a Rosen-Morse potential well.

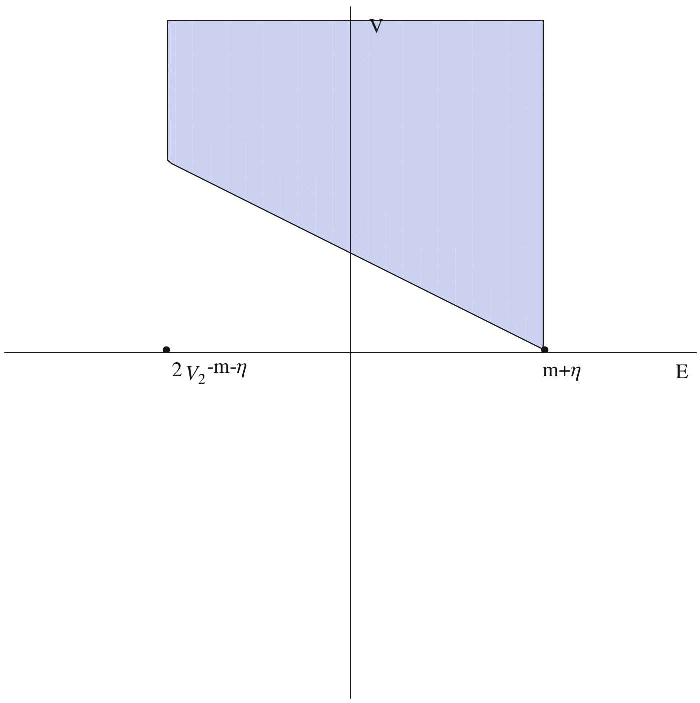

Fig. 1. Possible energy spectrum region for a confined particle within spin symmetric limit.

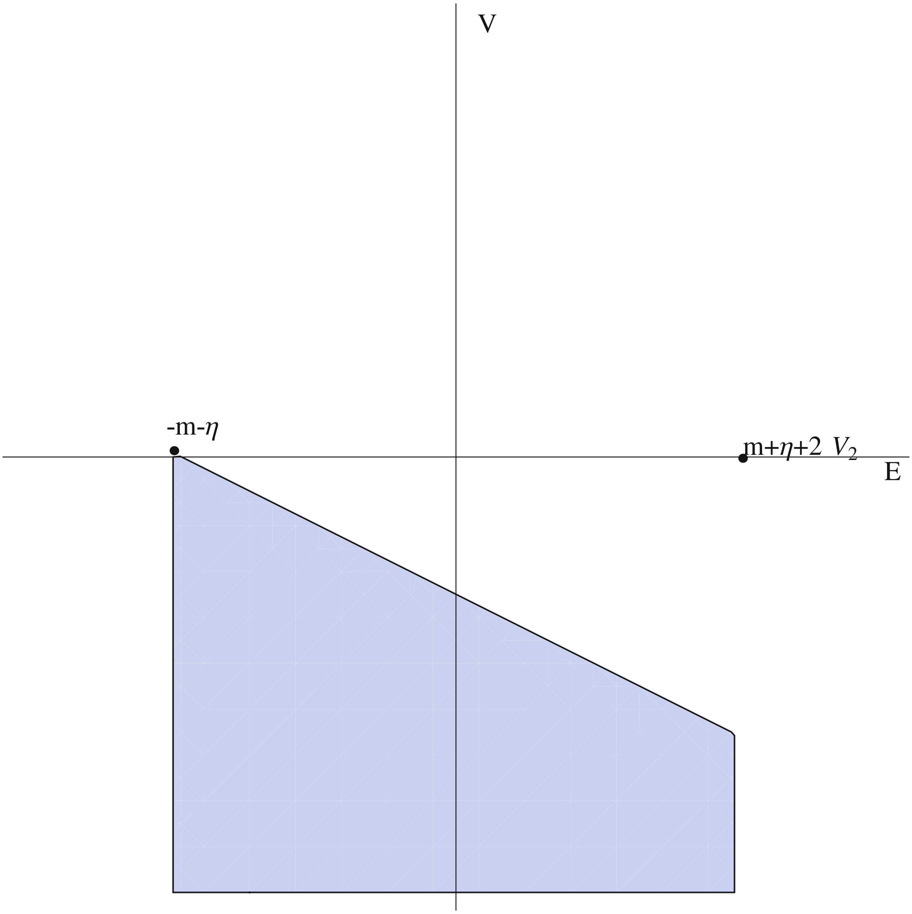

Fig. 2. Possible energy spectrum region for a confined particle within pseudo-spin symmetric limit. Since potential depth parameter defined positively, a particle cannot be confined in the pseudo-spin symmetric limit.

5 Conclusions

In this article, the one-dimensional K-G equation for Rosen-Morse potential is studied with spin and pseudo-spin symmetric limit within the framework of the AIM that gives the eigenvalues directly by transforming the second-order differential equation into a form $f_n''(x) = \lambda_0(x)f_n'(x) + s_0(x)f_n(x)$ along with the derivation of wave functions through iteration of $s_0(x)$ and $\lambda_0(x)$. The exact analytical expression of the energy eigenvalues and the corresponding normalized eigenfunctions has been found in terms of the gamma functions and Gauss hypergeometric functions.

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On g^*-Closed Sets in Fuzzy Topological Spaces

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Abstract. In this treatise, we propose a new type of closed set in fuzzy topological spaces called g^* fuzzy closed set, which is lying in between the fuzzy closed set and the generalized fuzzy closed set. Also, we study another class of fuzzy sets called θ-g^* fuzzy closed sets which is weaker than θ- fuzzy closed sets but stronger than θ-g fuzzy closed sets and an interrelationships among these newly defined fuzzy closed sets along with the existing generalized fuzzy closed sets are established. Furthermore, the idea of fuzzy g^*-connectedness is introduced in the light of g^*-fuzzy closed sets. Finally, we define $T_{1/2}{}^*$-space, ${}^*T_{1/2}$-space, ${}_\theta T_{1/2}{}^*$-space and ${}^*{}_\theta T_{1/2}$-space and some applications of these newly defined spaces are discussed.

Keywords: g^* fuzzy closed sets · θ-g^* fuzzy closed sets · fuzzy g^*-connectedness · fuzzy $T_{1/2}{}^*$-space · fuzzy ${}^*T_{1/2}$-spaces · fuzzy ${}_\theta T_{1/2}{}^*$-spaces · fuzzy ${}^*{}_\theta T_{1/2}$-spaces

1 Introduction and Preliminaries

The notion of fuzzy sets and their operations were first introduced by Zadeh [8] in 1965 while the concept of fuzzy topological space was first defined by Chang [3] which plays a significant role in the whole study of fuzzy set theory. In 1970, Levine [7] introduced the notion of generalized closed sets in topological space. Later on Balasubramanian et al. [1] defined the generalization of closed sets in fuzzy topological space. Dontchev and Maki [9] have introduced θ-generalized closed sets in topological spaces. After that El-shafei et al. [5] studied θ-generalized closed sets in fuzzy sense. Veerakumar [4] proposed and investigated a new type of closed set in topological space termed as g^* closed set which settled in between the class of closed set and the class of generalized closed set. Our aim is to explore two types of fuzzy closed sets namely, g^* fuzzy closed (in short, g^*fc) set in the direction of Balasubramaniam et al. [1] and θ-g^* fuzzy closed (in short, θ-g^*fc) set by following El-shafei et al. [5] Also we establish the interrelationship among these newly defined sets, g fuzzy closed sets, fuzzy closed set, θ-closed sets, θ-g closed sets and verified each non relationships along with suitable counter examples. In particular, we have shown

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that g^* fuzzy closed set is stronger than g fuzzy closed set due to Balasubramaniam et al. [1] and θ-g^* fuzzy closed set is weaker than θ-closed set due to El-shafei et al. [5] Moreover, using these newly defined fuzzy closed sets, we have proposed some different types of fuzzy spaces namely fuzzy ${T_{1/2}}^*$-space, fuzzy ${}^*T_{1/2}$-space, fuzzy ${}_\theta {T_{1/2}}^*$-space, fuzzy ${}^*{}_\theta T_{1/2}$-space by extending the notion of $T_{1/2}$ connectedness in fuzzy topological space. In this treatise, we have studied fuzzy g^*-connectedness and also investigated the interrelationships among fuzzy g^*-connectedness, fuzzy g-connectedness and fuzzy-connectedness.

Troughout this paper (X, τ) represent fuzzy topological space (in short, fts) with respect to Chang [3]. A fuzzy subset is denoted by λ, μ, η and fuzzy topology by τ and general topology by T. In this paper, 1_X and 0_X denote the null set and X itself respectively. Moreover, closure of a fuzzy set λ is denoted by $cl(\lambda)$ and g fuzzy closure of λ is denoted as $cl^*(\lambda)$. Furthermore, $1_X - \lambda$ represent the complement of the fuzzy set λ in fts (X, τ).

Now, we recalled some important definitions below as ready references for our research work.

1.1 Definition [4]

A subset A of a topological space (X, T) is called g^* closed set if $cl(A) \subset U$, whenever $A \subset U$ and U is g open in (X, T).

1.2 Definition

A fuzzy set λ of a fuzzy topological space (X, τ) is called:

(i) [1] Generalized fuzzy closed (in short, gfc) set iff $cl(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and $\mu \in \tau$. The complement of gf closed set is called gf open and the set of all gf closed sets are defined as $gfC(X)$.
(ii) [6] θ -closed iff $\lambda = cl_\theta(\lambda)$, where $cl_\theta(\lambda) = \wedge\{cl(\mu) : \lambda \leq \mu, \mu \in \tau\}$ and $cl(\lambda) \leq cl_\theta(\lambda)$.
(iii) [1] $FT_{1/2}$-space iff every gf closed set is fuzzy closed.

1.3 Definition [5]

A fuzzy set λ of a fuzzy topological space (X, τ) is called θ-generalized closed if $cl_\theta(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is open in (X, τ).

1.4 Definition [4]

A topological space (X, T) is called:

(i) ${T_{1/2}}^*$-space iff every g^* closed set in it is closed.
(ii) ${}^*T_{1/2}$-space iff every g closed set is g^* closed.

2 Some Properties of g^*-fuzzy Closed Sets

In this section, we introduced g^*-fuzzy closed sets and some of their properties.

2.1 Definition

Let (X, τ) be a fuzzy topological space. A fuzzy set λ in X is called g^*-fuzzy closed (in short, g^*fc) iff $cl(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is gf-open in (X, τ). The complement of g^*-fuzzy closed set is called g^*-fuzzy open (in short, g^*fo) set in (X, τ). The class of all g^*fc subsets of X is denoted by $g^*fC(X)$.

2.2 Theorem

Every fuzzy closed set is a g^*fc set in (X, τ).

Proof: Let λ be any fuzzy closed set in fts (X, τ) and U be any generalized open set where, $\lambda \leq U$. For any fuzzy closed set, $\lambda = cl\lambda \leq U$. Hence, λ is a g^*fc set.

2.3 Remark

The following example showing that the converse of the above theorem is not necessarily true in general.

2.4 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.2}, b_{0.3}\}\}$. Then $FC(X) = \{0_X, 1_X, \{a_{0.8}, b_{0.7}\}\}$. In this case, the fuzzy set $\{a_{0.9}, b_{0.9}\}$ is g^*fc but it is not fuzzy closed.

2.5 Theorem

Every g^*fc set is a gf closed set in (X, τ).

Proof: Let us suppose that λ be any g^*f closed set and μ be any fuzzy open set such that, $\lambda \leq \mu$. Now we know that every fuzzy open set is a generalized fuzzy open set, this implies that μ is gf open. From the hypothesis, we can write $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$. Therefore, λ is gf closed set.

2.6 Remark

The converse of the above theorem may not be true in general, which is shown by the following example.

2.7 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.1}, b_{0.9}\}\}$. Then $FC(X) = \{0_X, 1_X, \{a_{0.9}, b_{0.1}\}\}$. Now, the fuzzy set $\{a_{0.7}, b_{0.1}\}$ is gf closed but it is not g^*f closed set.

2.8 Theorem

Every θ-closed set is a g^*f closed set in (X, τ).

Proof: Let λ be any fuzzy set which is θ-closed and μ is any gf open set such that, $\lambda \leq \mu$. We know that $cl(\lambda) \leq cl_\theta(\lambda)$, this implies from our assumption, $\lambda = cl(\lambda) = cl_\theta(\lambda) \leq \mu$. Hence, λ is a g^*f closed set.

2.9 Remark

The converse of the above theorem may not be true in general. This is verified in the following Example.

2.10 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A, B\}$, where $A = \{a_{0.8}, b_{0.1}\}, B = \{a_{0.2}, b_0\}$. Then $FC(X) = \{0_X, 1_X, \{a_{0.2}, b_{0.9}\}, \{a_{0.8}, b_1\}\}$. In this case, we observe that, the fuzzy set $\{a_{0.2}, b_{0.9}\}$ is g^*f closed but it is not θ-closed.

2.11 Remark

θ-g fuzzy closed set and g^* fuzzy closed set are independent of each other which is shown by the following two examples.

2.12 Example

Let (X, τ) be a fts with $X = \{a\}$ and $\tau = \{0_X, 1_X, \{a_{2/3}\}, \{a_{3/4}\}\}$. Here the fuzzy set $\lambda = \{a_{1/3}\}$ is fuzzy closed, this implies that λ is a g^* closed set but it is not a θ-g closed set.

2.13 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.2}, b_{0.8}\}\}$. Here, the fuzzy set $\lambda = \{a_{0.3}, b_{0.2}\}$ is a θ-g closed set but it is not a g^* closed set.

2.14 Theorem

If λ and μ are g^*f closed sets, then $\lambda \vee \mu$ is also a g^*f closed set.

Proof: It follows from the fact that $cl(\lambda \vee \mu) = cl(\lambda) \vee cl(\mu)$.

2.15 Remark

The intersection of two g^*fc sets is not g^*fc. This is shown in the following example.

2.16 Example

Let $X = \{a, b\}$ and fuzzy sets of X are defined as follows:

$A = \{a_{0.4}, b_{0.7}\}$, $B = \{a_{0.1}, b_{0.8}\}$, $C = \{a_{0.4}, b_{0.8}\}, D = \{a_{0.1}, b_{0.7}\}$, $E = \{a_{0.9}, b_{0.2}\}$, $F = \{a_{0.7}, b_{0.3}\}$.

Let $\tau = \{0_X, 1_X, A, B, C, D\}$ be a fts on X. Here we observe that, the fuzzy sets E and F are g^*fc but $E \wedge F = \{a_{0.7}, b_{0.2}\}$ is not g^*fc set.

2.17 Theorem

If λ is a g^*f closed set of fts (X, τ) such that, $\lambda \leq \mu \leq cl(\lambda)$, then μ is also a g^*f closed set of (X, τ).

Proof: Let U be any gf open set of fts (X, τ) where, $\mu \leq U$. Since λ is g^*f closed, $cl(\lambda) \leq U$. We have, $\mu \leq cl(\lambda) \Rightarrow cl(\mu) \leq cl(cl(\lambda)) = cl(\lambda) \leq U$. Therefore, μ is also g^*f closed set in (X, τ).

2.18 Theorem

If λ is a g^*f open set in fts (X, τ) such that, $int(\lambda) \leq \mu \leq \lambda$. Then μ is g^*f open in (X, τ).

Proof: We have $int(\lambda) \leq \mu \leq \lambda \Rightarrow (1_X - \lambda) \leq (1_X - \mu) \leq (1_X - int(\lambda)) = cl(1_X - \lambda)$. Since, λ is g^*f open set which implies that, $(1_X - \lambda)$ is g^*f closed set. From Theorem 2.17 $(1_X - \mu)$ is g^*f closed set $\Rightarrow \mu$ is g^*f open set in (X, τ).

2.19 Theorem

A fuzzy set λ is said to be g^*fo iff $\mu \leq int(\lambda)$, whenever $\mu \leq \lambda$ and μ is gfc set in (X, τ).

Proof: Suppose, λ be a g^*f open set and μ is any gf closed set in X where, $\mu \leq \lambda$. So, $(1_X - \lambda)$ is g^*fc, this implies that, $cl(1_X - \lambda) \leq (1_X - \mu)$. So $(1_X - cl(1_X - \lambda)) \geq (1_X - (1_X - \mu)) = \mu$. Now, we know that $1_X - cl(1_X - \lambda) = int(\lambda)$. Hence, $\mu \leq int(\lambda)$.

Conversely, let λ is a fuzzy set such that $\mu \leq int(\lambda)$ whenever μ is any gf closed set and $\mu \leq \lambda$. Suppose $(1_X - \lambda) \leq \mu$ where μ is any gf open set. Our aim is to show that $(1_X - \lambda)$ is g^*fc. Here, $(1_X - \lambda) \leq \mu \Rightarrow (1_X - \mu) \leq \lambda$. Now by assumption, $(1_X - \mu) \leq int(\lambda) \Rightarrow (1_X - int(\lambda)) \leq \mu$. We know that $(1_X - int(\lambda)) = cl(1_X - \lambda)$. Hence, $cl(1_X - \lambda) \leq \mu$. This shows that $(1_X - \lambda)$ is g^*fc.

2.20 Definition

Let λ be any fuzzy set in fts (X, τ). Define interior and closure of g^* fuzzy set λ denoted as $Int^{**}(\lambda)$ and $cl^{**}(\lambda)$ respectively as follows:

$$Int^{**}(\lambda) = \vee\{\mu : \mu \text{ is } g^*fo \text{ and } \mu \leq \lambda\}.$$
$$cl^{**}(\lambda) = \wedge\{\mu : \mu \text{ is } g^*fc \text{ and } \lambda \leq \mu\}.$$

2.21 Theorem

Let λ be any fuzzy set of a fts (X, τ). Then, $cl^{**}(1_X - \lambda) = 1_X - int^{**}(\lambda)$ and $int^{**}(1_X - \lambda) = 1_X - cl^{**}(\lambda)$.

Proof: Let μ_1 be any g^*fo set such that, $\mu_1 \leq \lambda$. Then, for any g^*fc set μ_2, $\mu_2 = 1_X - \mu_1 \geq 1_X - \lambda$. Then

$$\begin{aligned} int^{**}(\lambda) &= \vee\{1_X - \mu_2 : \mu_2 \text{ is } g^*fc \text{ and } \mu_2 \geq (1_X - \lambda)\}. \\ &= 1_X - \wedge\{\mu_2 : \mu_2 \text{ is } g^*fc \text{ and } \mu_2 \geq (1_X - \lambda)\}. \\ &= 1_X - cl^{**}(1_X - \lambda). \\ cl^{**}(1_X - \lambda) &= 1_X - int^{**}(\lambda). \end{aligned}$$

Now, let μ_2 be any g^*fc-set such that, $\mu_2 \geq \lambda$, $\mu_1 = 1_X - \mu_2 \leq 1_X - \lambda$. Then,

$$\begin{aligned} cl^{**}(\lambda) &= \wedge\{1_X - \mu_1 : \mu_1 \text{ is } g^*fo \text{ and } \mu_1 \leq 1_X - \lambda\}. \\ &= 1_X - \vee\{\mu_1 : \mu_1 \text{ is } g^*fo \text{ and } \mu_1 \leq 1_X - \lambda\}. \\ &= 1_X - int^{**}(1_X - \lambda). \\ int^{**}(1_X - \lambda) &= 1_X - cl^{**}(\lambda). \end{aligned}$$

2.22 Definition

A fts (X, τ) is called a fuzzy ${T_{1/2}}^*$ space if every g^*f closed set is fuzzy closed.

2.23 Example

Let $X = \{a, b\}$ and λ, μ are two fuzzy sets defined as follows:

$\lambda = \{a_{0.8}, b_{0.9}\}$, $\mu = \{a_{0.2}, b_{0.1}\}$.

Let $\tau = \{0_X, 1_X, \lambda\}$ be any fts of X. Here, $g^*fC(X) = \{0_X, 1_X, \mu\} = FC(X)$. Hence, the given space is fuzzy ${T_{1/2}}^*$ space.

2.24 Theorem

Every fuzzy $T_{1/2}$ space is fuzzy ${T_{1/2}}^*$ space.

Proof: Let (X, τ) be fuzzy $T_{1/2}$ space and λ is any g^*f closed set in X. We know that every g^*f closed set is a gf closed set, this implies that, λ is gf closed set. Since, (X, τ) is $T_{1/2}$ space, this means that λ is fuzzy closed set. Hence, (X, τ) is fuzzy ${T_{1/2}}^*$ space.

2.25 Remark

A fuzzy ${T_{1/2}}^*$ space may not be a fuzzy $T_{1/2}$ space as it shown by the following counter example.

2.26 Example

If we take the above Example 2.23, in this case $g^*fC(X) = FC(X)$ but here we observe that, $gfC(X) \neq FC(X)$. Therefore the space is not fuzzy $T_{1/2}$ space.

2.27 Definition

A fts (X, τ) is called a fuzzy ${}^*T_{1/2}$-space if every gf closed set of (X, τ) is a g^*f closed set.

2.28 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0\leq\alpha\leq 0.4}, b_{\beta\geq 0.6}\}, \{a_{0\leq\alpha\leq 0.6}, b_{\beta\geq 0.4}\}\}$. Then $FC(X) = \{0_X, 1_X, \{a_{\alpha>0.6}, b_{\beta<0.4}\}, \{a_{\alpha>0.4}, b_{\beta<0.6}\}\}$. Here we observe that, $gfC(X) = \{0_X, 1_X, \{a_{\alpha>0.6}, b_{\beta<0.4}\}, \{a_{\alpha>0.4}, b_{\beta<0.6}\}, \{a_{\alpha>0.6}, b_{0\leq\beta\leq 1}\}\}$. Therefore, $gfO(X) = \{0_X, 1_X, \{a_{0\leq\alpha\leq 0.4}, b_{\beta\geq 0.6}\}, \{a_{0\leq\alpha\leq 0.6}, b_{\beta\geq 0.4}\}, \{a_{0\leq\alpha\leq 0.4}, b_{0\leq\beta\leq 1}\}\}$. Now, from the above we can conclude that, $g^*fC(X) = gfC(X)$. Hence the given space is fuzzy ${}^*T_{1/2}$-space.

2.29 Theorem

Every fuzzy $T_{1/2}$-space is a fuzzy ${}^*T_{1/2}$-space.

Proof: Suppose the fts (X, τ) be a fuzzy $T_{1/2}$-space and let λ be any gf closed set of (X, τ). Now by the hypothesis, λ is closed. Since, Every fuzzy closed $\Rightarrow g^*fc$ in (X, τ). So, the fts (X, τ) is a fuzzy ${}^*T_{1/2}$-space.

2.30 Remark

The following counter example shows that, converse of the above statement may not be true.

2.31 Example

If we take the above Example 2.28, in this case, $gfC(X) \neq FC(X)$. Thus this space is fuzzy ${}^*T_{1/2}$-space but it is not fuzzy $T_{1/2}$-space.

2.32 Theorem

A fts (X, τ) is a fuzzy $T_{1/2}$-space if and only if it is fuzzy ${}^*T_{1/2}$ and fuzzy ${T_{1/2}}^*$.

Proof: Necessity -which is follows from the theorems and Sufficiency - Let (X, τ) is both ${}^*T_{1/2}$ and ${T_{1/2}}^*$-space and λ is a gf closed set in (X, τ). Then by assumption, λ is both fuzzy closed and g^*fc of (X, τ). Hence, the fts (X, τ) is a fuzzy $T_{1/2}$-space.

2.33 Remark

Both fuzzy ${T_{1/2}}^*$ness and fuzzy ${}^*T_{1/2}$ness are independent of each other as we see in the next two examples.

2.34 Example

Let us take the Example 2.23, in which, the fts (X, τ) is not ${}^*T_{1/2}$-space because the set $\{a_{0.9}, b_{0.1}\}$ is gfc but it is not g^*fc but here the space is fuzzy ${T_{1/2}}^*$ space.

2.35 Example

Let us take the Example 2.28, in which, the fts (X, τ) is a ${}^*T_{1/2}$-space but it is not ${T_{1/2}}^*$-space because the set of all g^* fuzzy closed sets of X, $g^*fc(X) \neq FC(X)$.

3 Results on Fuzzy g^*-connectedness

Fatteh et al. [2] and Balasubramaniam et al. [1] studied the concept of fuzzy connectedness and fuzzy generalized (in short, fg) connectedness respectively as follows:

3.1 Definition [2]

An fts (X, τ) is said to be fuzzy connected if it has no proper fuzzy clopen (closed and open) set. [A fuzzy set λ in (X, τ) is proper if $\lambda \neq 0$ and $\lambda \neq 1$].

3.2 Definition [1]

A fuzzy topological space (X, τ) is said to be fg-connected iff the only fuzzy sets which are both gf open and gf closed are 0_X and 1_X.

3.3 Definition

A fts (X, τ) is said to be fg^*-connected iff the only fuzzy sets which are both g^*f open and g^*f closed are 0_X and 1_X.

3.4 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.3}, b_{0.5}\}$. Here, the sets which are both g^*fc and g^*fo are 1_X and 0_X. Hence, above given space is fg^*-connected.

3.5 Theorem

Every fg-connected space is fg^*-connected space.

Proof: Let us suppose that X is not fg^*-connected but it is fg-connected. This means that there exist a proper g^*f set $\lambda(\lambda \neq 0_X$ and $\lambda \neq 1_X)$ which is both fg^*o and fg^*c. We have, $g^*fo \Rightarrow gfo$, this means that X is not fg-connected, which is a contradiction to our assumption. Hence the above statement is proved.

3.6 Remark

The converse of the above theorem may not be true. This is shown by the following example.

3.7 Example

Let $X = \{a, b\}$ and λ, μ are some fuzzy sets defined as follows:
$\lambda = \{a_{0.4}, b_{0.6}\}$, $\mu = \{a_{0.4}, b_{0.7}\}$.

Let $\tau = \{0_X, 1_X, \lambda\}$ be a fts on X. Here, the set μ is both gfo and gfc but there exist only 0_X and 1_X which are both g^*fo and g^*fc. Hence the given space is not fg-connected but it is fg^*-connected.

3.8 Theorem

Let (X, τ) be a fuzzy ${T_{1/2}}^*$ space. Then X is fuzzy connected iff X is fg^*-connected.

Proof: If possible let X is not fg^*-connected but X is fuzzy ${T_{1/2}}^*$ space and also fuzzy connected. Now, by assumption, there exists a proper fuzzy set λ such that, λ is both g^*fo and g^*fc. Here, X is fuzzy ${T_{1/2}}^*$ space $\Rightarrow \lambda$ is both fuzzy open and fuzzy closed $\Rightarrow X$ is not fuzzy connected, which is a contradiction to our assumption.

Conversely, suppose that X be a fg^*-connected space and fuzzy ${T_{1/2}}^*$ space but it is not fuzzy connected. This means that there exists a proper fuzzy set λ $(\neq 0_X, 1_X)$, such that λ is both f-open and f-closed. We know that, f open $\Rightarrow$ g^*fo, which follows that X is not fg^*-connected; a contradiction.

3.9 Theorem

Let (X, τ) be a fuzzy $^*T_{1/2}$ space. Then X is fg^*-connected iff X is fg-connected.

Proof: If possible let X is not fg-connected but X is fuzzy $^*T_{1/2}$ space and also fg^* connected. Now, by assumption, there exists a proper fuzzy set λ such that, λ is both gfo and gfc. Here, X is fuzzy $^*T_{1/2}$ space $\Rightarrow \lambda$ is both g^*fo and $g^*fc \Rightarrow X$ is not fg^* connected, which is a contradiction to our assumption.

Conversely, suppose that X be a fg-connected space and fuzzy $^*T_{1/2}$ space but it is not fg^*-connected. This means that there exists a proper fuzzy set λ ($\neq 0_X, 1_X$), such that λ is both g^*f open and g^*f closed. We know that, g^*f open $\Rightarrow$ gfo, which follows that X is not fg-connected; a contradiction.

3.10 Corollary

A fts (X, τ) is said to be fg^*-connected iff it has no non- zero gf sets λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$, $cl(\lambda_1) + \lambda_2 = \lambda_1 + cl(\lambda_2) = 1$.

3.11 Definition

A fts (X, τ) is said to be g^*f strongly connected if it has no non-zero gf closed set λ_1 and λ_2 such that $\lambda_1 + \lambda_2 \leq 1$. Here X is called g^*f-weakly connected if it is not g^*f strongly connected.

3.12 Definition

A g^*f closed set is called regular g^*f closed if $\lambda = cl^{**}(Int^{**}\lambda)$. The fuzzy complement of regular g^*f closed set is called regular g^*f open.

4 θ-g^* Closed Sets in Fuzzy Topological Space

4.1 Definition

A fuzzy subset λ of a fts (X, τ) is called θ-g^* fuzzy closed (in short, θ-g^*fc) if $cl_\theta(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is gf open in (X, τ). The class of all θ-g^* fuzzy closed in X is denoted as θ-$g^*C(X)$. The complement of a θ-g^* fuzzy closed set is called θ-g^* fuzzy open set.

4.2 Proposition

Every θ-closed fuzzy set in (X, τ) is θ-g^*f closed.

Proof: Proof is straightforward. Hence, omitted.

4.3 Remark

The reverse of the above theorem may not be true which is verified by the following counter example.

4.4 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.6}, b_{0.2}\}\}$. In this case, we observe that the fuzzy set $\lambda = \{a_{0.6}, b_{0.9}\}$ is a fuzzy θ-g^*-closed set but it is not a θ-closed set.

4.5 Proposition

Every θ-g^*-closed set is a θ-gf closed set in (X, τ).

Proof: From the definition of θ-g^*f closed set and θ-gf closed set, it is obvious.

4.6 Remark

The following counter example shows that the reverse of the above theorem may not be true.

4.7 Example

If we take the above Example 4.4, the fuzzy set $\mu = \{a_{0.4}, b_{0.8}\}$ is a θ-gf closed but it is not a θ-g^*f closed set.

4.8 Proposition

Every θ-g^* fuzzy closed set is gf closed in (X, τ).

Proof: The proof is obvious.

4.9 Remark

The converse part of the above theorem may not be true. This can be shown by the following counter example.

4.10 Example

From the above Example 4.4, we observe that, the fuzzy set $\mu = \{a_{0.4}, b_{0.9}\}$ is a gf closed set but it is not a θ-g^* fuzzy closed set.

4.11 Theorem

Every θ-g^* fuzzy closed set is g^* fuzzy closed.

Proof: Proof is straightforward. Therefore, omitted.

4.12 Remark

The converse of the above theorem may not be true which is shown by the following counter example.

4.13 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.4}, b_{0.7}\}, \{a_{0.1}, b_{0.8}\}, \{a_{0.4}, b_{0.8}\}, \{a_{0.1}, b_{0.7}\}\}$. Here, we observe that the fuzzy set $\lambda = \{a_{0.9}, b_{0.2}\}$ is a g^* fuzzy closed but it is not a θ-g^*f closed set.

4.14 Definition

A fts (X, τ) is called a fuzzy ${}_\theta T_{1/2}{}^*$ space if every θ-g^*f closed set is θ-closed.

4.15 Example

Let (X, τ) be a fts with $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \{a_{0.5}, b_{0.5}\}\}$. Then θ-$g^*C(X) = \{0_X, 1_X, \{a_{0.5}, b_{0.5}\}\}$. Here we observe that, all θ-$g^*c(X)$ is θ-closed. Hence, the given space is ${}_\theta T_{1/2}{}^*$ space.

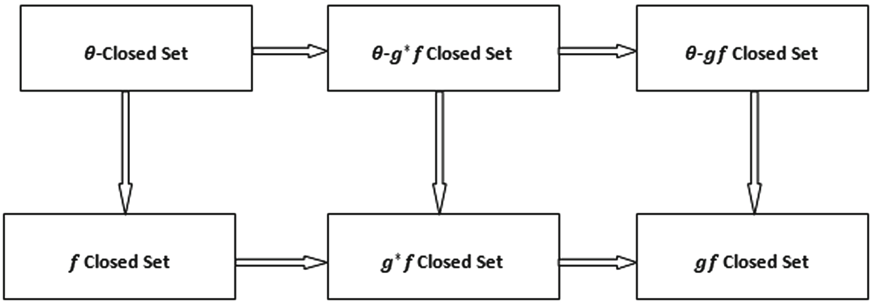
4.16 Definition

A fts (X, τ) is called a fuzzy ${}^*{}_\theta T_{1/2}$ space if every θ-gf closed set is θ-g^*f closed.

4.17 Example

If we take the Example 2.28, in this case, we observe that, θ-$gC(X) = gfC(X) = \theta$-$g^*C(X)$. Hence, the given space is fuzzy ${}^*{}_\theta T_{1/2}$ space.

We conclude this paper by depicting the following diagram of our whole study where $\Rightarrow$ means one sided implication but not the reverse one:



5 Conclusion

It is well accepted that g^* closed sets are lying between closed sets and g closed sets in general topology. Balasubramaniam et al. [1] and El-Shafai et al. [5] defined generalized closed sets and θ-generalized closed sets in fuzzy topological spaces respectively. In this treatise, the existence of g^* fuzzy closed sets in between fuzzy closed sets and generalized fuzzy closed sets are developed in the context of fts. Moreover, we have established interrelationships among θ-closed sets, θ-g^* fuzzy closed set, θ-g fuzzy closed sets, fuzzy closed sets, g^*f closed sets and gf closed sets. Also, in this paper, we introduced four different kinds of spaces namely fuzzy ${T_{1/2}}^*$-space, fuzzy $^*T_{1/2}$-spaces, fuzzy $_\theta {T_{1/2}}^*$-spaces, fuzzy $^*_\theta T_{1/2}$-spaces.

The detailed study on these sets may be extended in the direction of separation axioms and hyperconnectedness.

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Design and Development of Electronics Pest Repellent Using PIR Sensor and 8051 Micro-Controller

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Abstract. Rodents and pests are a huge problem in our daily lives and in the industry and getting relieve from them seems almost inevitable. They cause large scale damage to our crops, commodities and other useful things. Pest repellents already exist, but due to their toxicity or high price, they aren't feasible to use. In this paper, we have been able to design an Electronic Pest Repellent circuit using PIR sensor and 8051 micro-controller module, which prove to be lower in cost than the pre-existing electronic repellents. The frequency generated by this setup makes the environment unpleasant for the rodents and pests hence they evade the premises. This device can be beneficial to farmers, storage godowns, general public to repel these harmful rodents and pests.

Keywords: Pyroelectric Infra-red (PIR) sensor · 8051 micro-controller · Pest repellent · PCB design

1 Introduction

Nowadays, the pesticides are used to control pest that is very common method. Generally we use pesticides in and around homes which are poisonous. So, it is important to know the effectiveness of these chemicals [1]. Electronic Pest Repellent is an emerging technology which is cheapest, eco-friendly and produces no risk to the human [2,3]. In this paper, we have designed and developed an electronics pest repellent circuit that is very essential for domestic and as well as the industrial purposes. An electronics pest repellent is a weapon that one can utilize to keep away pest like rats, insects etc. Using poisonous pest repellent can be hazardous in a family. So, the proposed scheme is described to get rid of these hazardous problems. Pyroelectric Infra-red (PIR) Sensor and 8051 micro-controller module are the main components of this electronics pest repellent circuit. A PIR Sensor is a pyroelectric device that can sense infra-red (IR) radiation changes within its viewing range.

In other words, these sensors are sensitive to moving objects radiation IR light. Fresnel lens focuses irradiation on PIR sensor measures the changes in the

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IR rate. This is also cost efficient and simple than other micro-controllers. These pest repellent circuit senses infra-red radiation changes with viewing range and these devices broadcast sound waves that scare the pests away from your home.

This paper is organized for remainder portion as follows. Section 2 provides the related work. Section 3 presents the proposed design. Section 4 provides working of electronics pest repellent circuit. Section 5 presents the PCB design. Section 6 contains experimental result analysis. Section 7 provides the application. Section 8 presents the conclusion.

1.1 Features of Proposed Electronics Pest Repellent

The proposed electronics pest repellent device has many advantages. Some of the salient features are as follows:

1. **Power Efficient:** The device consumes less power and there is no need to modify the circuit for extra separate power supply.
2. **Cost-effective:** The cost of the electronic device is less low so that every individual and family can buy and use it.
3. **Compactness:** The device uses few electronic components compare to other existing systems.
4. **Indication of power:** A led on the device shows the power indication which identify whether the device is working or not.
5. **Simple Circuit Design:** The circuit can be simply made so that mass production can be easily done.
6. **No Harm and No Toxicity:** It doesn't produce any smoke, gases and radiation. It is harmless and produces no toxicity.

2 Related Works

The main aim to replace the chemical method and use of bio pesticide method with ultrasonic pest repelling system which is eco-friendly. The audible frequency range of pests is from 1 Hz to 100 KHz. Pests are forced to leave the particular area due to intense auditory stress because of higher frequency. So as an alternative, ultrasonic device is used which contains of the power unit and the pulse generator unit [1]. Due to compactness, cost effectiveness and pollution free source compared to other chemical repellents, this device is highly effective. The effectiveness of the device was increased by repeating and continuous varying of frequency of oscillation in order to prevent the pests from being habitual. Sound gets affected due to atmospheric factors like humidity, air velocity, temperature etc [2]. An ultrasonic pest repellent system is a very useful device to counter the various problems due to pests. The developed device can emit ultrasonic energy of varied frequencies. These frequencies do not affect the hearing ability of human but affect the auditory senses of pests like rodents, insects by making them uncomfortable in their abode. Astable Multivibrator (AMV), timer NE555 is used to generate the required ultrasonic frequency and varied the frequency automatically by pulse generating IC (CA3130) and a counter (CD4017). A D-type flip-flop IC (CD4013) was used to obtain a signal of symmetrical output [3]. The development of electronics pest repellent system is considered as useful and best design instead of pesticides and herbicides. This kind of design will better adapt to the environment of developing countries. This constructed device for proper performance is required evaluation [4,5].

The electronics pest control device is essential for developing countries in the field of agricultural now and in the future [6]. It plays an adverse effect on public opinion. As a result reputation, prosecution is loosed and poor staff relations are noticed.

Bhadriraju et al. designed ultrasound pest control devices which produced ultrasonic sound that has a marginal effect in repelling pests. This level of repellence may not be concerned in commercial purpose [7,8].

These repellent devices are very important for crop production and other day to day life issues [9].

Kolte et al. described the several diseases of mustard plants from pests, insects and diagnosis process of these diseases [11].

The main objective of the electronics pest control device should be to prevent these activities. Electronics pest control devices have yet not obtained wide popularity and publicity. Public are still dependent on chemical methods [10]. So, these devices need much publicity and can be used as the best alternative to chemical pesticides.

3 Proposed Design of Electronics Pest Repellent Circuit

Electronics pest repellent circuit is designed by using PIR sensor and 8051 micro-controller. This proposed system is capable of producing sound measurement, the changes in IR rate and it creates an electric potential difference corresponding to the variation in IR radiation and detects the presence of pests like rodents, insects etc. PIR Sensor and 8051 micro-controller module are the heart of this electronics pest repellent circuit. The block diagram of this proposed electronics pest repellent circuit is described in Fig. 1. Atmega16 micro-controller has been used to produce the different patterns of frequencies and an assembly consisting of audio power amplifier, buzzer and LCD for this purpose has been required. This proposed design is power efficient, low cost, compact and simple.

3.1 8051 Micro-controller

The main device in pest repellent circuit is 'AT89S52' which is a typical low power, high performance 8051 micro-controller. It is manufactured by Atmel^{TM}. Operating range is 4.0 V to 5.5 V. This micro-controller provides a versatile and profitable solution to many embedded applications. Internal RAM is 256 * 8 bit and there are 16-bit counter in this micro-controller.

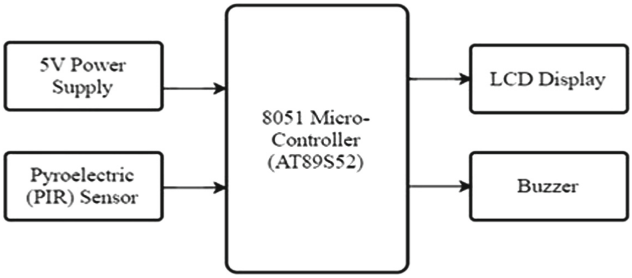

Fig. 1. Basic block diagram of electronics pest repellent

3.2 Pyroelectric (PIR) Sensor

Passive Infra-red sensor (PIR) is a sensor which detects the presence of human being, animals, and insects. There are two important parts in this sensor, one is Pyro-electric crystal which detects the heat from living organism like rats, insects etc and another is Fresnel lens which focuses IR radiation on PIR sensor.

3.3 Buzzer

It is an audio signaling device. It may be mechanical, electromechanical or piezoelectric. PIR sensor confirms the presence of pests, there after a message will be sent to the 8051 micro-controller module and then the buzzer will be buzzed.

4 Working of Electronics Pest Repellent Circuit

In Fig. 2 represents the mechanism of the proposed design. The detection of movements has always been important. This device has become very easy to detect pests like rats, insect's movements. In this paper, 8051 micro-controller and PIR sensor are interfaced. The buzzer is used as output to detect the motion of pests. The Power supply is given to the PIR sensor using 5Volts rail of 8051 micro-controller. The output pin of PIR Sensor is connected to P3.5 pin of 8051 micro-controller board and this pin will be the input pin of micro-controller. The VCC and ground pin are connected respectively. The PIR sensors consist of two slots. When a pest passes by, then it intercepts the first slot of the PIR sensor. When pest leaves the sensing area, the sensor generates a negative differential change between the two bisects. The 5 V power supply module consists of the 9 V batteries through the IC7805 voltage regulator. 16 * 2 LCD module is very common type of LCD module. It consists of 16 rows and 2 columns of 5 * 8 LCD matrices. In this work, the Enable (EN), Register-Select (RS), Read/Write(R/W) pins of LCD Display are connected to the 8051 micro-controller pin P2.7, P2.6, P2.5 respectively and data pins of LCD (DB0-DB7) are connected to the P0.0 to P0.7 of 8051 micro-controller board. Pin P2.0 is connected to the transistor BC547 through resistor. The negative side of the buzzer is connected to the collector of BC547 transistor and the positive side is connected to the VCC. The HEX file of written assembly program is generated and "AT89S52" micro-controller IC is burned. On other hand, the PCB layout is developed by PCB wizard and this PCB layout is designed and implemented by using PCB fabrication steps.

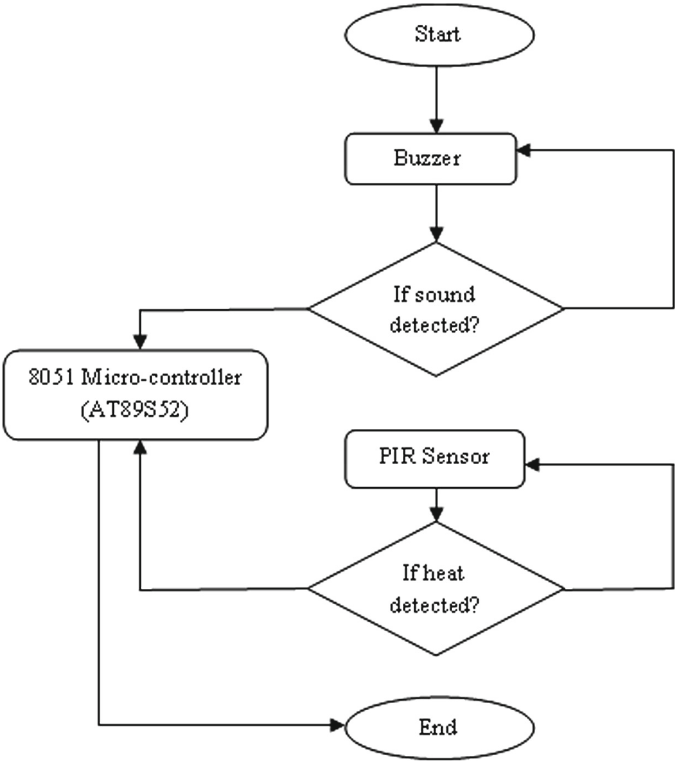

Fig. 2. Mechanism of proposed system

5 PCB Designing of Proposed Scheme in PCB Wizard

There are many types of software for the PCB design like the PCB wiz, express PCB, Eagle PCB, Proteus, Circuit wizard. PCB Wizard is used to design of proposed electronics pest repellent circuit's PCB. It is very easy to use and understand this proposed design. At first components are taken from PCB component gallery which is required for this circuit. Then the proposed circuit is designed in the PCB Wizard. Therefore, by clicking tools and convert, finally the PCB solder side artwork of proposed circuit is designed. Figure 3 shows the layout of PCB design of proposed circuit. This PCB design is developed using PCB fabrication steps.

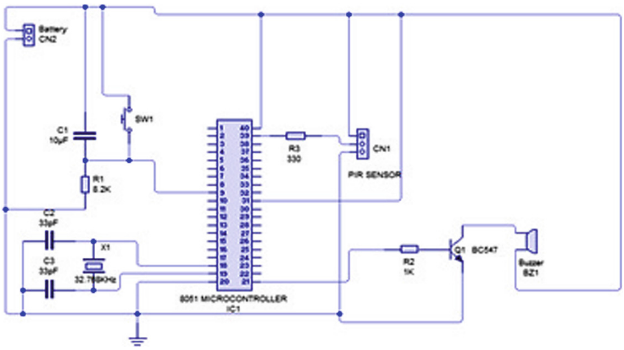

Fig. 3. Electronics pest repellent circuit using PCB Wizard

6 Experimental Results Analysis

This system has tested manually. This design is very much useful for domestically, industrially and farms. This device will generate sound within a range after certain time intervals, when heat is generated by pests. Implementation of proposed design is compares using PIR sensor and Ultrasonic sensor. The result is shown in the following Fig. 4.

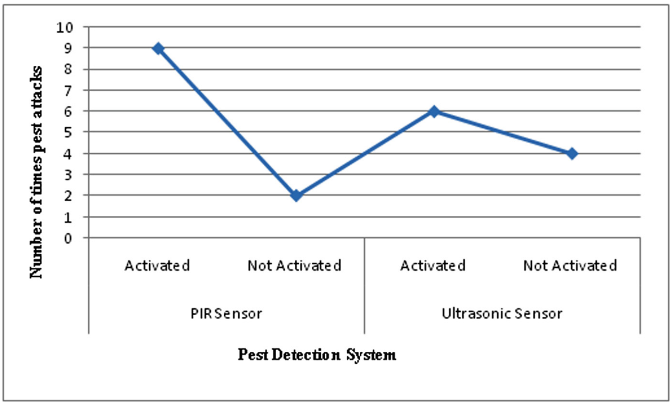

Fig. 4. Pest detection system graph

In Fig. 5 shows the graphical representation of the range of PIR sensor in terms of voltage with respect to distance. The voltage from PIR decreases according to increase the distance.

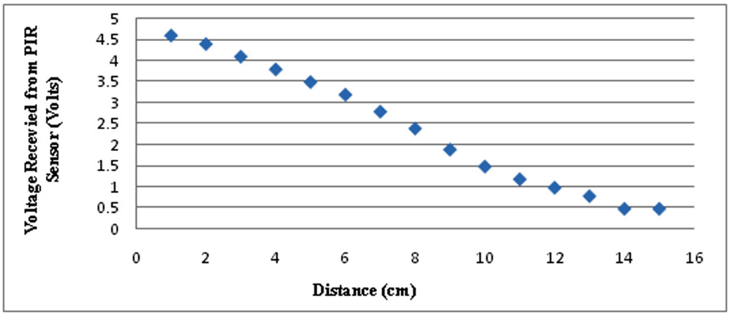

Fig. 5. The voltage received from PIR sensor versus distance graph

7 Application

The proposed electronics pest repellent devices are applicable in following cases:

1. **Household Applications:**
   The proposed device mostly used in household applications like kitchen chimney, store room etc.
2. **Agricultural Applications:**
   To protect the oil-seeds, crops etc from pests, this developed device is very much useful. By using this device the economic losses should be controlled.
3. **Industrial Applications:**
   This device is more suitable for industrial purpose. It can be used to protect the valuable documents, flies, machines from pests like rats, cockroaches, insects etc.
4. **Educational Applications:**
   To secure the important documents, papers, files etc of school, college, this device is very useful. This proposed device is also suitable for the laboratory and library.
5. **Others Applications:**
   Electronics pest repellent device is also applicable for hospitals, banks, offices etc.

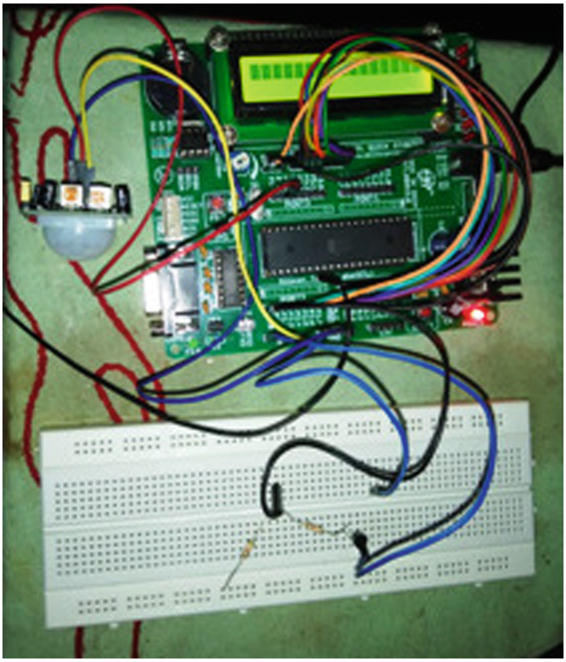

Fig. 6. Implemented electronics pest repellent system

8 Conclusion

The electronics pest repellent is designed to repel pests, usually rodents or insects. The performance of this proposed design could be improved with modification. This work is to increase the productivity of industry. The publicity of this design is very much needed. This device can be used in agricultural purpose to repel the pests.

By reducing the chemical pest repellents from the industry, from the home and farms, we can minimize the adverse effects and approach the problems electronically which will be beneficial for the future generations (Fig. 6).

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A Study on Weighted Doubly Truncated Renyi Divergence

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Abstract. In 1961, Renyi introduced the concept of parametric family of entropies as a generalization of the Shannon entropy. A significant application of this measure can be found in ecology, statistics and various other areas. Later, a divergence measure based on the Renyi entropy was introduced. In this study, we propose a weighted (shift-dependent) version of the Renyi divergence for two doubly truncated nonnegative random variables. Several bounds are obtained. The effect of monotone transformations on the proposed measure is discussed. Finally, a numerical study is performed to provide estimate of the proposed measure.

Keywords: Renyi divergence · Truncated random variable · Monotone transformation · Stochastic order · Maximum likelihood estimation

1 Introduction

In 1951, Kullback and Leibler introduced a divergence information measure between two probability density functions (PDFs). It is known as the Kullback-Leibler (KL) divergence measure. The KL divergence measure between two nonnegative absolutely continuous random variables X and Y with respective PDFs $f(x)$ and $g(x)$ is defined as (*see* Kullback and Leibler (1951))

$$\mathcal{KL}(X||Y) = \int_0^\infty f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx. \tag{1}$$

It measures the similarity of two PDFs. It is nonnegative. It equals to zero if and only if $f(x) = g(x)$ almost everywhere. The smaller value of (1) implies that the distributions corresponding to the random variables X and Y are more similar. Note that (1) is not a distance since it is neither symmetric nor satisfies the triangle inequality. There is an extensive literature regarding criteria to evaluate the best statistical model. One of them is KL divergence measure. For some results on divergence measure, we refer to Wang et al. (2011), Kayal (2015), Varma (1966), Misagh and Yari (2012), and Park and Shin (2014). Based on the Renyi entropy (see Renyi (1961)), the generalized divergence measure is given by

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$$\mathcal{R}_\alpha(X||Y) = \frac{1}{\alpha-1} \ln \int_0^\infty f(x) \left(\frac{g(x)}{f(x)}\right)^{1-\alpha} dx, \quad \alpha(\neq 1) > 0. \tag{2}$$

It is dubbed as the Renyi divergence measure of order α. Note that as $\alpha \to 1$, (2) reduces to (1). There are situations in real-life where the uncertainty contained in different populations are same in terms of the Renyi divergence measure given by (2). To overcome such problem, in this study, we consider weighted/shift-dependent version of the Renyi divergence measure. Henceforth, we call it as weighted Renyi divergence measure of order α. The weighted Renyi divergence of order α is defined as

$$\mathcal{R}^w_\alpha(X||Y) = \frac{1}{\alpha-1} \ln \int_0^\infty x f(x) \left(\frac{g(x)}{f(x)}\right)^{1-\alpha} dx, \quad \alpha(\neq 1) > 0. \tag{3}$$

In the following example we show the importance of the weighted Renyi divergence of order α.

Example 1. Suppose a nonnegative random variable X follows uniform distribution in the interval $(0, 1)$. Let Y and Z be two other nonnegative random variables with PDFs

$$g(x) = \begin{cases} 2x, \text{ if } 0 < x < 1, \\ 0, \quad \text{otherwise}, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 2(1-x), \text{ if } 0 < x < 1, \\ 0, \qquad\qquad \text{otherwise}, \end{cases}$$

respectively. Then, it is easy to obtain that

$$\mathcal{R}_\alpha(X||Y) = \frac{1}{\alpha-1} \ln \left(\frac{2^{1-\alpha}}{2-\alpha}\right) = \mathcal{R}_\alpha(X||Z),$$

that is, the Renyi divergence measure of order α between X and Y, and that between X and Z are equal. Hence, we can not compare the divergence contained in X and Y, and X and Z. But,

$$R^w_\alpha(X||Y) = \frac{1}{\alpha-1} \ln \left(\frac{2^{1-\alpha}}{3-\alpha}\right) \quad \text{and} \quad R^w_\alpha(X||Z) = \frac{1}{\alpha-1} \ln \left(\frac{2^{1-\alpha}}{(2-\alpha)(3-\alpha)}\right).$$

Now, it is not difficult to observe that the weighted Renyi divergence measure of order α between X and Y, and X and Z are not equal.

In reliability and life testing studies, there are some experiments where the current age of a system needs to be incorporated. Also, somebody may be interested to study uncertainty which relies in the past lifetime of a component. Hence the measures (1), (2) and (3) are not appropriate to handle these situations. To overcome such difficulty, researchers considered divergence measures between two residual and past lifetime distributions and studied their properties. Let X and Y be two absolutely continuous nonnegative random variables with cumulative distribution functions (CDFs) $F(x)$ and $G(x)$ and survival functions (SFs)

$\bar{F}(x)$ and $\bar{G}(x)$, respectively. The weighted residual and past Renyi divergence measures between two nonnegative random variables X and Y are given by

$$\mathcal{R}_{\alpha}^{w}(X||Y;t) = \frac{1}{\alpha-1}\ln\int_{t}^{\infty} x\frac{f(x)}{\bar{F}(t)}\left(\frac{g(x)/\bar{G}(t)}{f(x)/\bar{F}(t)}\right)^{1-\alpha} dx, \tag{4}$$

$$\text{and}\ \ \widetilde{\mathcal{R}}_{\alpha}^{w}(X||Y;t) = \frac{1}{\alpha-1}\ln\int_{0}^{t} x\frac{f(x)}{F(t)}\left(\frac{g(x)/G(t)}{f(x)/F(t)}\right)^{1-\alpha} dx, \tag{5}$$

respectively, where $\alpha(\neq 1) > 0$. The measures (4) and (5) respectively are known as the relative entropy of residual lifetimes $[X|X > t]$ and $[Y|Y > t]$; and past lifetimes $[X|X < t]$ and $[Y|Y < t]$.

Recently, there have been growing interests in analyzing doubly truncated data in the statistical analysis of survival data as well as in other fields like astronomy or economy. Doubly truncated failure time arises if an individual is observed and its failure time falls into a certain finite interval. Doubly truncated random variable appears in quasar survey, where an investigator assumes that the apparent magnitude is doubly truncated. Also, the times to progression for patients with certain disease who received chemotherapy, experienced tumor progression and subsequently died are doubly truncated. For various results on doubly truncated random variable, we refer to Sankaran and Sunoj (2004) and Misagh and Yari (2012). Analogous to (4) and (5), the weighted dynamic Renyi divergence measure of order α between two doubly truncated random variables $[X|t_1 < X < t_2]$ and $[Y|t_1 < Y < t_2]$ is defined as

$$\mathcal{R}_{\alpha}^{w}(X||Y;t_1,t_2) = \frac{1}{\alpha-1}\ln\int_{t_1}^{t_2} x\ \frac{f(x)}{\Lambda F}\left(\frac{g(x)/\Lambda G}{f(x)/\Lambda F}\right)^{1-\alpha} dx, \tag{6}$$

where $\alpha(\neq 1) > 0$, $\Lambda F = F(t_2) - F(t_1)$, $\Lambda G = G(t_2) - G(t_1)$ and $0 < t_1 < t_2$. The uncertainty measure given in (6) is also called as the weighted doubly truncated Renyi divergence measure of order α (WDRD$_\alpha$). In the following example, we show the importance of the WDRD$_\alpha$.

Example 2. Consider random lifetimes X and Y of two systems with joint PDF

$$f(x,y) = \begin{cases} \frac{x}{2}e^{-y}, \text{ if } 0 < x < 2, y > 0, \\ \quad 0, \quad \text{elsewhere.} \end{cases}$$

The marginal PDFs of X and Y are $f(x) = x/2, 0 < x < 2$ and $g(y) = \exp\{-y\}, y > 0$, respectively. We cannot compare these two random variables in terms of the uncertainty measure based on the tools given in (3), (4) and (5) since X and Y have different supports. In this case, the functionals given in (6) is useful. These enable us to compare uncertainty measures in a given interval. Below, we present some numerical values of the WDRD$_\alpha$. $\mathcal{R}_{0.5}^{w}(X||Y; 0.1, 1) = 1.218900; \mathcal{R}_{0.5}^{w}(X||Y; 0.5, 1.99) = -0.263013$ and $\mathcal{R}_{1.5}^{w}(X||Y; 0.1, 1) = -0.232800$; $\mathcal{R}_{1.5}^{w}(X||Y; 0.5, 1.99) = 1.270670$.

Remark 1. Respectively, the WDRD_α for the doubly truncated random variables reduces to the residual WDRD_α, past WDRD_α and weighted Renyi divergence as follows: (i) $\lim_{t_2\to\infty} \mathcal{R}^w_\alpha(X||Y;t_1,t_2) = \mathcal{R}^w_\alpha(X||Y;t_1)$, (ii) $\lim_{t_1\to 0} \mathcal{R}^w_\alpha(X||Y;t_1,t_2) = \widetilde{\mathcal{R}}^w_\alpha(X||Y;t_2)$ and (iii) $\lim_{\substack{t_1\to 0\\ t_2\to\infty}} \mathcal{R}^w_\alpha(X||Y;t_1,t_2) = \mathcal{R}^w_\alpha(X||Y)$.

Next, we present some important notations and results in terms of the definitions which are useful to obtain some of our main results.

Definition 1. *Let X be a nonnegative absolutely continuous random variable with PDF $f(x)$ and cumulative distribution function $F(x)$. The generalized failure rate functions of $[X|t_1 < X < t_2]$ are defined as (see Navarro and Ruiz, 1996)*

$$h_1^X(t_1,t_2) = \frac{f(t_1)}{F(t_2)-F(t_1)} \quad \text{and} \quad h_2^X(t_1,t_2) = \frac{f(t_2)}{F(t_2)-F(t_1)}$$

for all $(t_1,t_2) \in D = \{(x,y) : F(x) < F(y)\}$.

Definition 2. *The generalized conditional mean of a doubly truncated random variable $[X|t_1 < X < t_2]$ is defined by*

$$\mu_X(t_1,t_2) = E(X|t_1 < X < t_2) = \int_{t_1}^{t_2} \frac{xf(x)}{F(t_2)-F(t_1)}dx.$$

Definition 3. *Let X and Y be two nonnegative random variables with distribution functions $F(x)$ and $G(x)$, survival functions $\bar{F}(x)$ and $\bar{G}(x)$, PDFs $f(x)$ and $g(x)$, respectively. Then, X is said to be smaller than Y in the*

(i) likelihood ratio order, denoted by $X \leq^{LR} Y$, if $g(x)/f(x)$ is increasing in x,

(ii) usual stochastic order, denoted by $X \leq^{ST} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

It is well known that $X \leq^{ST} Y$ if $X \leq^{LR} Y$. For further reading on the usual stochastic ordering, one may refer to Shaked and Shanthikumar (2007). Throughout the paper, we assume that the random variables are nonnegative and absolutely continuous.

The paper is arranged as follows. In Sect. 2, we present main results on the WDRD_α. Specifically, we obtain some bounds of the mathematical measure (6). We study monotonic behavior of the WDRD_α and the effect of the monotone transformations on (6). Further, a simulation study is carried out to estimate the WDRD_α for two exponential populations in Sect. 3. A real data set is considered. Finally, some concluding remarks have been added in Sect. 4.

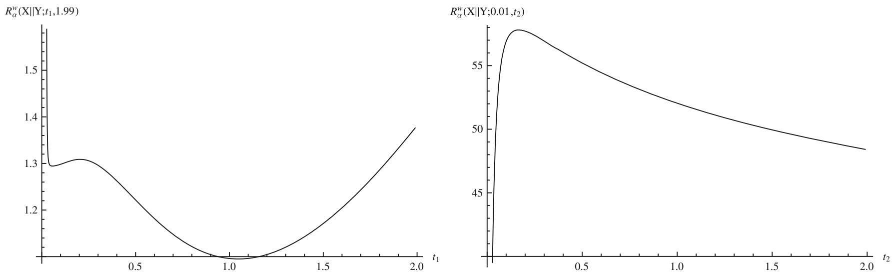
2 Properties of the WDRD_α

In this section, we obtain some bounds of the WDRD_α in terms of the generalized hazard rates and doubly truncated mean. The bounds have been obtained based on the monotonicity property of the proposed measure. Note that the measure given in (6) is not monotone always. The following example provides evidence in this direction.

Example 3. Let X and Y have the CDFs

$$F(x) = \frac{x^2}{4}, 0 < x < 2 \text{ and } G(x) = \exp\left\{\frac{1}{2} - \frac{1}{x}\right\}, 0 < x < 2,$$

respectively. The WDRD$_\alpha$ between X and Y is plotted in Fig. 1. This shows that the WDRD$_\alpha$ is not monotone.



(a) (b)

Fig. 1. Figures (a) and (b) represent the plots of $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$ for $\alpha = 1.5$. In Figure (a), we consider $t_2 = 1.99$ and $t_1 \in (0, t_2)$. In Figure (b), we consider $t_1 = 0.01$ and $t_2 \in (t_1, 2)$.

2.1 Bounds

First we obtain lower and upper bounds of (6) by using the increasing and decreasing behavior of WDRD$_\alpha$.

Proposition 1. *Let X and Y be have PDFs $f(x)$ and $g(x)$, respectively.*

(i) If $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$ is increasing (decreasing) in t_1, then for $\alpha > 1\,[\alpha < 1]$, we have $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$

$$\geq (\leq)\,[\leq (\geq)]\, \frac{1}{1-\alpha} \ln\left[\frac{\alpha}{t_1}\left(\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)}\right)^{1-\alpha} - \frac{\alpha-1}{t_1}\left(\frac{h_1^Y(t_1,t_2)}{h_1^X(t_1,t_2)}\right)^{\alpha}\right].$$

(ii) If $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$ is increasing (decreasing) in t_2 then for $\alpha > 1\,[\alpha < 1]$, we have $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$

$$\leq (\geq)\,[\geq (\leq)]\, \frac{1}{1-\alpha} \ln\left[\frac{\alpha}{t_2}\left(\frac{h_2^X(t_1,t_2)}{h_2^Y(t_1,t_2)}\right)^{1-\alpha} - \frac{\alpha-1}{t_2}\left(\frac{h_2^Y(t_1,t_2)}{h_2^X(t_1,t_2)}\right)^{\alpha}\right].$$

Proof. Rewriting the expression of $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$ given by (6), we get

$$\exp\{(\alpha-1)\mathcal{R}_\alpha^w(X||Y; t_1, t_2)\} = \int_{t_1}^{t_2} x \frac{f(x)}{\Lambda F}\left(\frac{g(x)/\Lambda G}{f(x)/\Lambda F}\right)^{1-\alpha} dx. \tag{7}$$

Differentiating (7) with respect to t_1 and t_2 we get

$$\frac{\partial}{\partial t_1}\mathcal{R}_\alpha^w(X||Y;t_1,t_2) = \frac{1}{\alpha-1}\Big[\alpha h_1^X(t_1,t_2)-(\alpha-1)h_1^Y(t_1,t_2) - \frac{t_1(h_1^X(t_1,t_2))^\alpha(h_1^Y(t_1,t_2))^{1-\alpha}}{\exp\{(\alpha-1)\mathcal{R}_\alpha^w(X||Y;t_1,t_2)\}}\Big] \tag{8}$$

$$\text{and } \frac{\partial}{\partial t_2}\mathcal{R}_\alpha^w(X||Y;t_1,t_2) = \frac{1}{\alpha-1}\Big[(\alpha-1)h_2^Y(t_1,t_2)-\alpha h_2^X(t_1,t_2) - \frac{t_2(h_2^X(t_1,t_2))^\alpha(h_2^Y(t_1,t_2))^{1-\alpha}}{\exp\{(\alpha-1)\mathcal{R}_\alpha^w(X||Y;t_1,t_2)\}}\Big], \tag{9}$$

respectively. Now, using $\frac{\partial}{\partial t_1}\mathcal{R}_\alpha^w(X||Y;t_1,t_2)\geq(\leq)0$, (8) and the provided condition on α, after some simplification we get the result as mentioned in (i). In similar way the other part can be proved. Thus, the detailed proof of (ii) is omitted. This completes the proof of the theorem.

In the following theorems we obtain the bounds of (6) by using the concept of the likelihood ratio order.

Theorem 1. *Consider two absolutely continuous nonnegative random variables X and Y such that $X\leq^{LR}Y$. Then, for $\alpha(\neq 1)>0$ we have*

$$\xi_2(t_1,t_2)\leq\mathcal{R}_\alpha^w(X||Y;t_1,t_2)\leq\xi_1(t_1,t_2),$$

where $\xi_1(t_1,t_2) = \frac{\ln\mu_X(t_1,t_2)}{\alpha-1}+\ln\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)}$ *and* $\xi_2(t_1,t_2) = \frac{\ln\mu_X(t_1,t_2)}{\alpha-1}+\ln\frac{h_2^X(t_1,t_2)}{h_2^Y(t_1,t_2)}$.

Proof. Given $X\leq^{LR}Y$. So, from definition of the likelihood ratio order, $t_1<x$ and $\alpha>1$, we have $f(x)^{\alpha-1}g(x)^{1-\alpha}\leq f(t_1)^{\alpha-1}g(t_1)^{1-\alpha}$. Hence, from (6) we get

$$\begin{aligned}\mathcal{R}_\alpha^w(X||Y;t_1,t_2) &\leq \frac{1}{\alpha-1}\ln\int_{t_1}^{t_2}x\frac{f(x)}{\varLambda F}\left(\frac{f(t_1)/\varLambda F}{g(t_1)/\varLambda G}\right)^{(\alpha-1)}dx\\ &=\frac{1}{\alpha-1}\ln\mu_X(t_1,t_2)+\ln\left(\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)}\right).\end{aligned} \tag{10}$$

Again, using the definition of likelihood ratio order, $x<t_2$ and $\alpha>1$ we get $f(x)^{\alpha-1}g(x)^{1-\alpha}\geq f(t_2)^{\alpha-1}g(t_2)^{1-\alpha}$. Thus, from (6) we obtain

$$\begin{aligned}\mathcal{R}_\alpha^w(X||Y;t_1,t_2) &\geq \frac{1}{\alpha-1}\ln\int_{t_1}^{t_2}x\frac{f(x)}{\varLambda F}\left(\frac{f(t_2)/\varLambda F}{g(t_2)/\varLambda G}\right)^{(\alpha-1)}dx\\ &=\frac{1}{\alpha-1}\ln\mu_X(t_1,t_2)+\ln\left(\frac{h_2^X(t_1,t_2)}{h_2^Y(t_1,t_2)}\right).\end{aligned} \tag{11}$$

Combining (10) and (11), we get the result for $\alpha>1$. Similarly, it can be proved for $\alpha<1$ as per the above steps.

In order to show the validity of Theorem 1 we consider the following example.

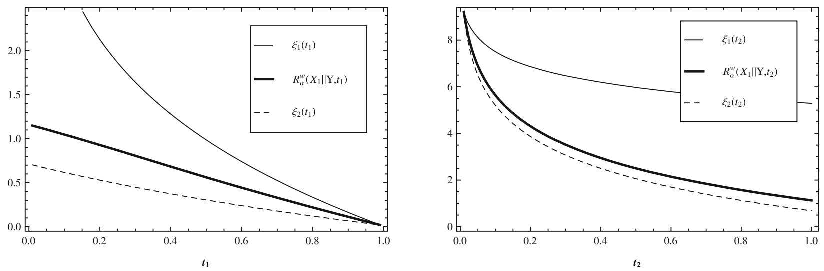
Example 4. Let X_1 and X_2 be the lifetimes of two components of a parallel system. Further, assume that these components are independently working. Then, $Y = \max\{X_1, X_2\}$ denotes the lifetime of this system. Let the components of the systems are uniformly distributed on $(0, 1)$. Therefore, the distribution function of Y is $G(x) = x^2, 0 < x < 1$. Clearly, $X_i \leq^{LR} Y,\ i = 1, 2$. For $0 < t_1 < t_2 < 1$, we obtain

$$\mathcal{R}^w_\alpha(X_i||Y; t_1, t_2) = \frac{1}{\alpha - 1} \ln\left[\frac{2^{1-\alpha}(t_2^2 - t_1^2)^{\alpha-1}(t_2^{3-\alpha} - t_1^{3-\alpha})}{(3-\alpha)(t_2 - t_1)^\alpha}\right], \tag{12}$$

$$\xi_1(t_1, t_2) = \frac{1}{\alpha - 1} \ln\left(\frac{t_1 + t_2}{2}\right) + \ln\left(\frac{t_2 + t_1}{2t_1}\right) \tag{13}$$

$$\text{and} \quad \xi_2(t_1, t_2) = \frac{1}{\alpha - 1} \ln\left(\frac{t_1 + t_2}{2}\right) + \ln\left(\frac{t_2 + t_1}{2t_2}\right). \tag{14}$$

In Fig. 2, we present graphs of the lower bounds, upper bounds and WDRD_α for $\alpha = 0.5$.



(a) (b)

Fig. 2. Figure (a) and (b) represent the graphs of (12), (13) and (14) for $\alpha = 0.5$. In Figure (a), we consider $t_2 = 0.99$ and $t_1 \in (0, t_2)$ and in Figure (b) we consider $t_1 = 0.01$ and $t_2 \in (t_1, 1)$.

Theorem 2. *Consider three nonnegative continuous random variables X_1, X_2 and X_3 with PDFs $f_1(x), f_2(x)$ and $f_3(x)$, respectively. If $X_1 \leq^{LR} X_2$, then for $\alpha > 1 (\alpha < 1)$ we have*

$$\frac{\alpha}{\alpha - 1} \ln\left(\frac{h_2^{X_1}(t_1, t_2)}{h_2^{X_2}(t_1, t_2)}\right) \leq (\geq)\ \mathcal{R}^w_\alpha(X_1||X_3; t_1, t_2) - \mathcal{R}^w_\alpha(X_2||X_3; t_1, t_2)$$
$$\leq (\geq)\ \frac{\alpha}{\alpha - 1} \ln\left(\frac{h_1^{X_1}(t_1, t_2)}{h_1^{X_2}(t_1, t_2)}\right).$$

Proof. The proof can be done by using the similar arguments as in Theorem 1. Hence omitted.

2.2 Monotone Transformations

In this section, we obtain the effect of the strictly monotone transformations on the weighted doubly truncated Renyi divergence of order α.

Theorem 3. *Let X and Y be two absolutely continuous nonnegative random variables with PDFs $f(x)$ and $g(x)$, and CDFs $F(x)$ and $G(x)$, respectively. Consider a bijective function ψ, which is strictly monotone and differentiable. Then, for all $0 \le t_1 < t_2 < +\infty$ and $\alpha(\neq 1) > 0$, we have*

$$\mathcal{R}_\alpha^w(\psi(X)||\psi(Y); t_1, t_2) = \begin{cases} \mathcal{R}_\alpha^{w,\psi}(X||Y; \psi^{-1}(t_1), \psi^{-1}(t_2)), \\ \qquad\qquad\qquad \textit{if } \psi \textit{ is strictly increasing}, \\ \mathcal{R}_\alpha^{w,\psi}(X||Y; \psi^{-1}(t_2), \psi^{-1}(t_1)), \\ \qquad\qquad\qquad \textit{if } \psi \textit{ is strictly decreasing}, \end{cases}$$

where $\mathcal{R}_\alpha^{w,\psi}(X||Y; t_1, t_2) = \frac{1}{\alpha-1} \ln \int_{t_1}^{t_2} \psi(x) \frac{f(x)}{\Lambda F} \left(\frac{g(x)/\Lambda G}{f(x)/\Lambda F} \right)^{1-\alpha} dx.$

Proof. If ψ is strictly increasing, then the PDF and CDF of $\psi(X)$ are given by

$$f_\psi(x) = \frac{f(\psi^{-1}(x))}{\psi'(\psi^{-1}(x))} \quad \text{and} \quad F_\psi(x) = F(\psi^{-1}(x)), \tag{15}$$

respectively and that of $\psi(Y)$ are respectively given by

$$g_\psi(x) = \frac{g(\psi^{-1}(x))}{\psi'(\psi^{-1}(x))} \quad \text{and} \quad G_\psi(x) = G(\psi^{-1}(x)). \tag{16}$$

Now, from (6), (15) and (16) we have

$$\begin{aligned} \mathcal{R}_\alpha^w(\psi(X)||\psi(Y); t_1, t_2) &= \frac{1}{\alpha-1} \ln \int_{t_1}^{t_2} x \frac{f_\psi(x)}{\Lambda F_\psi} \Big(\frac{g_\psi(x)/\Lambda G_\psi}{f_\psi(x)/\Lambda F_\psi} \Big)^{1-\alpha} dx \\ &= \frac{1}{\alpha-1} \ln \int_{t_1}^{t_2} \Big(\frac{g(\psi^{-1}(x))/\Lambda G_\psi}{f(\psi^{-1}(x))/\Lambda F_\psi} \Big)^{1-\alpha} \frac{x f(\psi^{-1}(x)) dx}{\psi'(\psi^{-1}(x)) \Lambda F_\psi}, \end{aligned} \tag{17}$$

where $\Lambda F_\psi = F(\psi^{-1}(t_2)) - F(\psi^{-1}(t_1))$ and $\Lambda G_\psi = G(\psi^{-1}(t_2)) - G(\psi^{-1}(t_1))$. Using the transformation $z = \psi^{-1}(x)$ in (17) we obtain

$$\begin{aligned} \mathcal{R}_\alpha^w(\psi(X)||\psi(Y); t_1, t_2) &= \frac{1}{\alpha-1} \ln \int_{\psi^{-1}(t_1)}^{\psi^{-1}(t_2)} \psi(z) \frac{f(z)}{\Lambda F_\psi} \Big(\frac{g(z)/\Lambda G_\psi}{f(z)/\Lambda F_\psi} \Big)^{1-\alpha} dz. \\ &= \mathcal{R}_\alpha^{w,\psi}(X||Y; \psi^{-1}(t_1), \psi^{-1}(t_2)). \end{aligned}$$

This proves of the first part of the theorem. In similar way the proof of the second part can be done when ψ is strictly decreasing function. This completes the proof of the theorem.

Remark 2. Let $\psi_1(x) = F(x)$ and $\psi_2(x) = \bar{F}(x)$, where $F(x)$ and $\bar{F}(x)$ are the CDF and survival function of X, respectively. Clearly, $\psi_1(x)$ and $\psi_2(x)$ satisfy all the assumptions given in Theorem 3. Therefore, as an application of Theorem 3 we have

$$\mathcal{R}_\alpha^w(F(X)||F(Y);t_1,t_2) = \mathcal{R}_\alpha^{w,F}(X||Y;F^{-1}(t_1),F^{-1}(t_2)),$$
$$\text{and}\ \ \mathcal{R}_\alpha^w(\bar{F}(X)||\bar{F}(Y);t_1,t_2) = \mathcal{R}_\alpha^{w,\bar{F}}(X||Y;\bar{F}^{-1}(t_1),\bar{F}^{-1}(t_2)).$$

Proposition 2. *Consider X and Y as in Theorem 3. If $\psi(x) = ax$, where $a > 0$, then from Theorem 3 we have*

$$\mathcal{R}_\alpha^w(aX||aY;t_1,t_2) = \frac{\ln a}{\alpha - 1} + \mathcal{R}_\alpha^w\left(X||Y;\frac{t_1}{a},\frac{t_2}{a}\right).$$

Proposition 3. *Consider X and Y as in Theorem 3. If $\psi(x) = x + b$, where $0 < b < min\{t_1, t_2\}$, then from Theorem 3 we have*

$$\mathcal{R}_\alpha^w(X + b||Y + b;t_1,t_2) = \mathcal{R}_\alpha^{w,\psi}\left(X||Y;t_1 - b,t_2 - b\right).$$

Remark 3. From Proposition 3, it is clear that the measure defined in (6) is shift-dependent. In some pragmatic circumstances of reliability and neurobiology, a shift-dependent measure is desirable.

3 An Estimate of $\mathbf{WDRD}_\alpha$

In this section, we present a simulation study to obtain estimates of the weighted doubly truncated Renyi divergence measure between two exponential populations with means $1/\theta_1$ and $1/\theta_2$, where $\theta_1, \theta_2 > 0$. Let X and Y follow exponential distributions with CDFs $F(x) = 1-e^{-\theta_1 x}, x > 0$ and $G(y) = 1-e^{-\theta_2 y}, y > 0$, respectively. Then, the WDRD_α between X and Y is given by

$$\mathcal{R}_\alpha^w(X||Y;t_1,t_2) = \frac{1}{\alpha - 1}\ln\left[\frac{\eta_1(\theta_1,\theta_2)}{(\eta_2(\theta_1,\theta_2))^\alpha \times (\eta_3(\theta_1,\theta_2))^{(1-\alpha)}}\right], \tag{18}$$

where $\alpha(\neq 1) > 0$, $\eta_1(\theta_1,\theta_2) = \frac{\theta_1^\alpha \theta_2^{1-\alpha}}{\theta_1\alpha+\theta_2(1-\alpha)}\Big[(1 + t_1)e^{-(\theta_1\alpha+\theta_2(1-\alpha))t_1} - (1 + t_2)e^{-(\theta_1\alpha+\theta_2(1-\alpha))t_2}\Big]$, $\eta_2(\theta_1,\theta_2) = e^{-\theta_1 t_1} - e^{-\theta_1 t_2}$ and $\eta_3(\theta_1,\theta_2) = e^{-\theta_2 t_1} - e^{-\theta_2 t_2}$. Now, using the method of maximum likelihood we estimate $\mathcal{R}_\alpha^w(X||Y;t_1,t_2)$ given by (18). First we estimate the unknown parameters θ_1 and θ_2 based on the doubly truncated exponential data. Consider $x_1, x_2, \ldots, x_n$ are the realizations of identically independently distributed (iid) random observations from exponential population with mean $1/\theta_1$ provided $x_i \in [t_1, t_2]$, $i = 1, 2, \ldots, n$. Then, the truncated density of X_i, subject to $X_i \in [t_1, t_2]$ is

$$f(x_i|\theta_1) = \begin{cases} \dfrac{\theta_1 e^{-\theta_1 x_i}}{e^{-\theta_1 t_1} - e^{-\theta_1 t_2}}, & \text{if } x_i \in [t_1, t_2], \\ 0, & \text{if } x_i \notin [t_1, t_2]. \end{cases} \tag{19}$$

Thus, the likelihood function for the data $(x_1, x_2, \ldots, x_n)$ is

$$\ell(\theta_1|x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{\theta_1 e^{-\theta_1 x_i}}{e^{-\theta_1 t_1} - e^{-\theta_1 t_2}}. \tag{20}$$

Similarly, the likelihood function for the iid random observations $y_1, y_2, \ldots, y_n$ from exponential population with mean $1/\theta_2$ provided $y_i \in [t_1, t_2]$, $i = 1, 2, \ldots, n$, can be obtained as

$$\tau(\theta_2|y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} \frac{\theta_2 e^{-\theta_2 y_i}}{e^{-\theta_2 t_1} - e^{-\theta_2 t_2}}. \tag{21}$$

Thus, the maximum likelihood estimate of θ_1, denoted by $\widehat{\theta}_1$ and the maximum likelihood estimate of θ_2, denoted by $\widehat{\theta}_2$ can be obtained after solving the differential equations $\frac{d \ln \ell}{d\theta_1} = 0$ in θ_1 and $\frac{d \ln \tau}{d\theta_2} = 0$ in θ_2, respectively. The solution of these differential equations can not be obtained explicitly. Hence, to obtain $\widehat{\theta}_1$ and $\widehat{\theta}_2$ we use Newton-Raphson method. Finally, we plug the value of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ in (3.1) to get the maximum likelihood estimate of $\mathcal{R}_\alpha^w(X||Y; t_1, t_2)$, denoted by $\widehat{\mathcal{R}}_\alpha^w(X||Y; t_1, t_2)$ is given by

$$\widehat{\mathcal{R}}_\alpha^w(X||Y; t_1, t_2) = \frac{1}{\alpha - 1} \ln \left[\frac{\eta_1(\widehat{\theta}_1, \widehat{\theta}_2)}{(\eta_2(\widehat{\theta}_1, \widehat{\theta}_2))^\alpha \times (\eta_3(\widehat{\theta}_1, \widehat{\theta}_2))^{(1-\alpha)}} \right], \tag{22}$$

where $\alpha(\neq 1) > 0$. To get $\widehat{\theta}_1$ and $\widehat{\theta}_2$, we generate data from two exponential populations using Monte Carlo simulation. The estimated values are computed based on 1000 samples with sample size 100 for different truncation limits and parameter values. In Table 1 we present few of them.

Real Data: In the following, two real data sets related to the failure times of the air conditioning system of two planes are considered. This data set is given by Bain and Engelhardt (1991).

Data Set-I (*Plane 7911*): 33, 47, 55, 56, 104, 176, 182, 220, 239, 246, 320.

Data Set-II (*Plane 7912*): 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 90, 95, 120, 120, 225, 246, 261.

It is easy to check that the above data sets can be fitted with exponential distributions. Assume that due to some causes, the observations in $[20, 250]$ are observed. The unknown parameters can be estimated by the method of maximum likelihood. Here, $\hat{\theta}_1 = 0.006567$ and $\hat{\theta}_2 = 0.016772$. From (22), we get the plot of $\widehat{\mathcal{R}}_\alpha^w(X||Y; 50, 100)$ for different values of α as shown in Fig. 3.

Table 1. Estimates of $\mathcal{R}_\alpha^w(X||Y;t_1,t_2)$ when $\alpha = 5$.

(θ_1,θ_2)	(t_1,t_2)	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\mathcal{R}_\alpha^w(X\|\|Y;t_1,t_2)$	$\widehat{\mathcal{R}}_\alpha^w(X\|\|Y;t_1,t_2)$
(0.1, 0.5)	(1, 5)	0.148284	0.490207	0.902910	0.797935
	(1, 10)	0.083241	0.465678	2.329150	2.203440
	(2, 10)	0.054765	0.510440	2.040310	2.312920
	(2, 15)	0.084277	0.545610	3.677920	4.296780
(0.5, 0.1)	(1, 5)	0.521316	0.029537	0.495919	0.624768
	(1, 10)	0.517224	0.135190	0.916082	0.832051
	(2, 10)	0.547211	0.199272	0.951809	0.757343
	(2, 15)	0.518270	0.114320	1.209020	1.177240
(1, 2)	(1, 5)	1.094420	1.485250	2.502910	0.823429
	(1, 10)	1.044000	1.508860	6.38183	2.127760
	(2, 10)	1.013800	1.861040	5.632090	4.529900
	(2, 15)	1.016250	1.861050	9.475350	7.556770
(2, 1)	(1, 5)	2.027590	1.020880	0.573715	0.569369
	(1, 10)	2.032550	0.940094	0.591658	0.657296
	(2, 10)	2.839850	0.742625	0.692812	1.269650
	(2, 15)	2.839850	0.754032	0.693145	1.258010

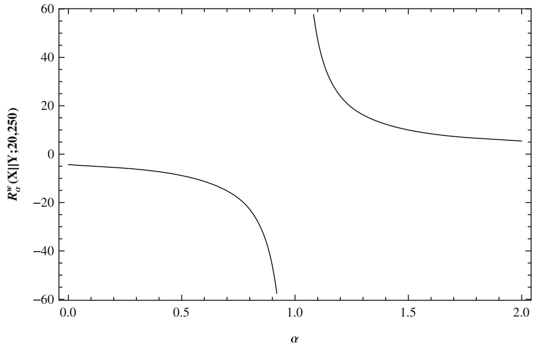


Fig. 3. It represents the plots of $\widehat{\mathcal{R}}_\alpha^w(X||Y;20,250)$ for different values of α.

4 Conclusion

In this communication, we have proposed a weighted version of the Renyi divergence measure between two doubly truncated random variables and showed its importance. Bounds are derived for the proposed measure based on the likelihood ratio order. Some examples are given to illustrate the results. The results obtained in this paper reduce to the cases of residual life time and past life time by assuming $t_2 \to \infty$ and $t_1 \to 0$, respectively. The effect of the monotone transformations has been studied on the proposed measure. Finally, a numerical study is carried out to estimate the WDRD_α for exponential populations.

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Tauberian Theorems for Statistical Cesàro Summability of Function of Two Variables over a Locally Convex Space

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Abstract. The notion of statistical convergence is more general than the classical convergence. Tauberian theorems via different ordinary summability means have been established by many researchers. In the present paper, we have established two new Tauberian theorems via statistical Cesàro summability mean of continuous function of two variables by using oscillating behavior and De la vallée Poussin means of double integral over a locally convex space. Finally, some concluding remarks and corollaries are provided here to support our theorems and demonstrated that our results are the non trivial extension of some existing results.

Keywords: Locally convex space · Double Cesàro summability · Slow oscillation; Improper integral · Tauberian theorem; $(C, 1, 1)$-summability · (C, k, r)-summability

1 Introduction

It makes no sense to speak of the sum of a divergent infinite series. Nevertheless, a series that is not "too badly divergent" can be assigned a generalized sum in a variety of natural ways. We are familiar with the notion of summability in connection with the theorems of Abel and Cesàro, which asserts that every convergent series is Abel or Cesàro summable to its ordinary sum. More generally, an Abelian theorem or Cesàro theorem work to the effect that, a method of summability assigns to each convergent series its ordinary sum. A Tauberian theorem goes in the opposite direction and asserts that every summable series, which is not too badly divergent is actually convergent.

Tauber [23] gave the first Tauberian theorems for single sequences, that an Abel summable sequence is convergent with some suitable conditions. A number of authors such as Landau [12], Hardy and Littlewood [6], and Schmidt [18] obtained some classical Tauberian theorems for Cesàro and Abel summability methods of single sequences. Recently, Çanak and Totur [2], and Jena et al. [7]

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investigated and studied several Tauberian theorems for single sequences. Knopp [11] obtained some classical type Tauberian theorems for Abel and $(C, 1, 1)$ summability methods of double sequences and proved that Abel and $(C, 1, 1)$ summability methods hold for the set of bounded sequences. Móricz [14] proved some Tauberian theorems for Cesàro summable double sequences and deduced Tauberian theorems of Landau [13] and Hardy [5] type. Recently, Totur [24] has extended some classical type Tauberian theorems for single and double sequences in connection with one-sided Tauberian theorems.

Very recently, Çanak and Totur [1] has been proved a Tauberian theorem for Cesàro summability of single integrals and also the alternative proofs of some classical type Tauberian theorems for the Cesàro summability of single integrals. Moreover, in the year 2017, Jena et al. [9] proved the Tauberian theorem for Harmonic summability of double integrable real-valued function over $\mathbb{R}^2$ and also, established the inclusion relation between statistical convergence and classical convergence.

The aim of this paper is to prove, statistical versions of Littlewood Tauberian theorems via $(C, 1, 1)$-summability method for double integrable functions over a locally convex space under slow oscillation by using the De la Vallée Poussin mean of the double integral. In fact, we extend a Tauberian theorem due to Çanak and Totur [1].

2 Preliminaries

The notion of statistical convergence was introduced by Fast [3] and Steinhaus [22]. Also, in this connection Fridy [4] has shown that $k(x_k - x_{k+1}) = o(1)$ is a Tauberian condition for the statistical convergence of (x_k). Subsequently, many researchers worked in this area to several settings. For more recent works in this direction, one may refer [10,16,17,19,20] and [21]. Existing works in this field based on statistical convergence appears to have been restricted to real or complex sequences; however, in the present paper, we extend the idea for a locally convex Hausdorff topological linear space.

Let $I = [0, \infty) \subseteq \mathbb{R}$ provided with the Lebesgue measure. Let $X(I)$ be the space of all real valued measurable functions on I. We say that, a functional $\omega : X(I) \to [0, \infty)$ be the modulus on $X(I)$ provided that the following conditions hold:

(i) $\omega(f) = 0$ if and only if $f = 0$, $\forall\ f \in [0, \infty)$,
(ii) $\omega(f + g) = \omega(f) + \omega(g)$, $\forall\ f, g \in [0, \infty)$ and $f, g \geqq 0$,
(iii) ω is increasing function and
(iv) ω is continuous on $[0, \infty)$.

Further suppose that, X denote a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous semi-norms q.

Let $f(x, y)$ be a function in X and let the partial sum of $f(x, y)$ is given by

$$s(x, y) = \int_0^x \int_0^y f(\zeta, \eta) d\zeta d\eta, \quad (0 < x, y < \infty).$$

The $(C,1,1)$ mean of $f(x,y)$ is,

$$\sigma(s(x,y)) = \sigma^{(1,1)}(s(x,y)) = \frac{1}{xy}\int_0^x \int_0^y s(\zeta,\eta)d\zeta d\eta \tag{1}$$

see [15].

The integral $\int_0^x \int_0^y s(\zeta,\eta)d\zeta d\eta$ is $(C,1,1)$-summable to a finite number $\ell \in X$ if and only if, for all $q \in Q$ and all $\epsilon > 0$, such that

$$\lim_{x,y\to\infty} q(\sigma(x,y) - \ell) \to 0.$$

In this case, we write

$$\sigma(x,y) \to \ell \;\; \text{over} \;\; X.$$

Furthermore, the $(C,1,0)$ and $(C,0,1)$ means of $f(x,y) \in X$ are defined by

$$\sigma^{(1,0)}(s(x,y)) = \frac{1}{x}\int_0^x s(\zeta,y)d\zeta \;\; \text{and} \;\; \sigma^{(0,1)}(s(x,y)) = \frac{1}{y}\int_0^y s(x,\eta)d\eta$$

respectively (see [15]).

The integral $\int_0^x \int_0^y s(\zeta,\eta)d\zeta d\eta$ is statistically $(C,1,1)$-summable to a finite number $\ell \in X$ if and only if, for all $q \in Q$ and all $\epsilon > 0$

$$\lim_{u,v\to\infty} \frac{1}{uv}|\{0 < x,y \leqq u,v \;\; \text{and} \;\; q(\sigma(x,y) - \ell) \geqq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} \lim_{x,y\to\infty} \sigma^{(1,1)}(s(x,y)) = \text{stat} \int_0^x \int_0^y (1-\zeta/x)(1-\eta/y)f(\zeta,\eta)d\zeta d\eta = \ell \in X. \tag{2}$$

Next, if the integral

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = \ell \in X \tag{3}$$

exists, then limit of (1) also exists; however, in general the converse is not true.

In order, to prove the sufficient part, we have to use the oscillatory behavior and De la vallée Poussin mean of the above double integral over X. For each non-negative integers k, r we define

$$\sigma^{(k,r)}(s(x,y)) = \begin{cases} \frac{1}{xy}\int_0^x \int_0^y \sigma^{(k-1,r-1)}s(\zeta,\eta)d\zeta d\eta, & \text{for } k,r \geq 1 \\ \int_0^x \int_0^y s(\zeta,\eta)d\zeta d\eta, & \text{for } k,r = 0 \end{cases}$$

Note that, $\sigma^{(1,1)}(s(x,y)) = \sigma(s(x,y))$.

A double integral $\int_0^\infty \int_0^\infty f(x,y)dxdy$ is said to be statistically (C,k,r)-summable to $\ell \in X$, if and only if $\sigma^{(k,r)}(s(x,y))$ is summable to $\ell \in X$.

Remark 1. If $k = 1$ and $r = 1$, then (C, k, r)-summabllity mean reduces to $(C, 1, 1)$-summability mean. Again if $k \neq 0$ and $r = 0$ then (C, k, r)-summabllity mean reduces to $(C, k, 0)$-summability mean. Further, if $k = 0$ and $r \neq 0$ then (C, k, r)-summabllity mean reduces to $(C, 0, r)$-summability mean.

We have the partial sum of function,

$$s(x,y) = \int_0^x \int_0^y f(\zeta,\eta) d\zeta d\eta.$$

Now

$$s(x,y) - \sigma(s(x,y)) = v(f(x,y)), \tag{4}$$

where

$$v(f(x,y)) = v^{(1,1)}(f(x,y)) = \frac{1}{xy}\int_0^x \int_0^y \zeta\eta f(\zeta,\eta) d\zeta d\eta.$$

Notice that,

$$\sigma'(s(x,y)) = \frac{v(f(x,y))}{xy}.$$

Let us define for each non negative integers k and r,

$$v^{(k,r)}(f(x,y)) = \begin{cases} \frac{1}{xy}\displaystyle\int_0^x \int_0^y v^{(k-1,r-1)}\zeta\eta f(\zeta,\eta) d\zeta d\eta, \text{ for} & k,r \geq 1 \\ \displaystyle\int_0^x \int_0^y \zeta\eta f(\zeta,\eta) d\zeta d\eta, & \text{for} \quad k,r = 0. \end{cases}$$

Moreover, a double integral $\int_0^\infty \int_0^\infty xyf(x,y)dxdy$ is statistically (C,k,r)-summable to $\ell \in X$, if and only if $v^{(k,r)}(f(x,y))$ is summable to $\ell \in X$.

The De la Vallée Poussin - mean of the double integral $\int_0^x \int_0^y f(\zeta,\eta) d\zeta d\eta$ is defined by

$$\tau(s(x,y)) = \frac{1}{(\lambda x - x)(\lambda y - y)}\int_x^{\lambda x} \int_y^{\lambda y} s(\zeta,\eta) d\zeta d\eta, \quad \lambda \in (1,\infty) \quad \text{and}$$

$$\tau(s(x,y)) = \frac{1}{(x - \lambda x)(y - \lambda y)}\int_{\lambda x}^{x} \int_{\lambda y}^{y} s(\zeta,\eta) d\zeta d\eta, \quad \lambda \in (0,1).$$

A double integral $\int_0^x \int_0^y f(x,y)dxdy$ belonging to X, is oscillating slowly [15], if

$$\lim_{\lambda\to 1^+} \limsup_{x,y\to\infty} \max_{x,y\leq\zeta,\eta\leq\lambda x,\lambda y} |s(\zeta,\eta) - s(x,y)| = 0. \tag{5}$$

Equivalently, Eq. (5) can be reformulated as

$$\lim_{\lambda\to 1^-} \limsup_{x,y\to\infty} \max_{\lambda x,\lambda y\leq\zeta,\eta\leq x,y} |q(s(\zeta,\eta)) - s(x,y)| = 0. \tag{6}$$

In the year 2016, Jena et al. proved some Tauberian theorems for Cesàro summability of single and double real sequences (see [7,8]). Again, Jena *et al.* [9] also proved a Tauberian theorem for Harmonic summability of double integrable real-valued function over $\mathbb{R}^2$. Motivated essentially by the above mentioned works, here in this paper, we have introduced the notion of statistical Cesàro summability over a locally convex space proved a generalized Littlewood Tauberian theorem for Cesàro summability of double integral.

3 Main Results

Theorem 1. *If $s(x, y)$ is statistically $(C, 1, 1)$-summable to ℓ in a locally convex space X and $s(x, y)$ is oscillating slowly, then $s(x, y) \to \ell \in X$, as $x, y \to \infty$.*

For proving the above theorem, we need the following lemmas.

Lemma 1. *The sequence of partial sum $s(x, y)$ of double integrable functions $f(x, y)$ over a locally convex space X is oscillating slowly if and only if $v(f(x, y)) \in X$ is bounded and oscillating slowly.*

Proof. Let $s(x, y)$ is oscillating slowly. Initially let us show that $v(f(x, y)) = O(1)$ as $x, y \to \infty$.

We have,

$$\int_0^x \int_0^y wzf(w,z)dwdz = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \int_{x/2^{i+1}}^{x/2^i} \int_{y/2^{j+1}}^{y/2^j} wzf(w,z)dwdz. \tag{7}$$

It follows from the identity,

$$\begin{aligned}
&\int_\alpha^\beta \int_\gamma^\delta wzf(w,z)dwdz = \int_\alpha^\beta \int_\gamma^\delta wzs'(w,z)dwdz. \\
&= \int_\alpha^\beta z \left(\int_\gamma^\delta ws'(w,z)dw \right) dz \\
&= \int_\alpha^\beta z \left[w(s(w,z))_\gamma^\delta - \int_\gamma^\delta s(w,z)dw \right] dz \\
&= \int_\alpha^\beta \int_\gamma^\delta zs(w,z)dwdz + \delta \int_\alpha^\beta zs(\delta,z)dz - \gamma \int_\alpha^\beta zs(\gamma,z)dz \\
&\qquad\qquad -\gamma \int_\alpha^\beta zs(\delta,z)dz + \gamma \int_\alpha^\beta zs(\delta,z)dz \\
&= -\int_\alpha^\beta \int_\gamma^\delta zs(w,z)dwdz + (\delta-\gamma) \int_\alpha^\beta zs(\delta,z)dz \\
&\qquad\qquad +\gamma \left(\int_\alpha^\beta zs(\delta,z)dz - \int_\alpha^\beta zs(\gamma,z)dz \right)
\end{aligned}$$

$$= \int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\left[z(s(w,z)-s(\delta,z))\right]dzdw + \gamma\left(\int_{\alpha}^{\beta} zs(\delta,z)dz - \int_{\alpha}^{\beta} zs(\gamma,z)dz\right)$$

$$= (\beta-\alpha)(\delta-\gamma)\max_{\alpha,\gamma\le x,y\le\beta,\delta}|s(x,y)-s(\beta,\delta)| + \gamma\left|\int_{\alpha}^{\beta} z(s(\delta,z)-s(\gamma,z))dz\right|.$$

If we choose $\beta = x/2^i$, $\beta/\alpha \le 2$ and $\delta = y/2^j$, $\delta/\gamma \le 2$, we obtain

$$\left|\int_0^x\int_0^y wzf(w,z)dwdz\right| \le A\sum_{i=0,j=0}^{\infty,\infty}\frac{xy}{2^{i+j}} = O(xy),\ \ as\ x,y\to\infty.$$

Now we have to show that, $\sigma(s(x,y))$ is oscillating slowly. Since $\sigma'(s(x,y)) = \frac{v(f(x,y))}{x}$, we get

$$\begin{aligned}|\sigma(s(\zeta,\eta))-\sigma(s(x,y))| &= \left|\int_x^{\zeta}\int_y^{\eta}\sigma'(s(w,z))dwdz\right|\\ &= \left|\int_x^{\zeta}\int_y^{\eta} f(w,z)dwdz\right|\\ &\le C\int_x^{\zeta}\left(\int_y^{\eta}\frac{dw}{w}\right)\frac{dz}{z}\\ &= C\log(\eta/y)\log(\zeta/x),\ \text{for}\ \ \text{any}\ x,y\le\zeta,\eta\le\lambda x,\lambda y.\end{aligned}$$

Clearly, we have

$$\max_{x,y\le\zeta,\eta\le\lambda x,\lambda y}|\sigma(s(\zeta,\eta))-\sigma(s(x,y))| \le C\log(\lambda)\log(\lambda).$$

Taking the limit sup to both sides as $\lambda\to 1^+$, we get

$$\lim_{\lambda\to 1^+}\lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le\lambda x,\lambda y}|\sigma(s(\zeta,\eta)-\sigma(s(x,y))| = 0.$$

That implies $v(f(x,y))$ is oscillating slowly by Kronecker identity (4).

To prove the converse part, suppose that $v(f(x,y))$ is bounded and oscillating slowly. The boundedness of $v(f(x,y))$ implies that $\sigma(s(x,y))$ is oscillating slowly. Since $v(f(x,y))$ is oscillating slowly, so $s(x,y)$ is oscillating slowly by Kronecker identity (4). Which establish the Lemma 1.

Lemma 2. *(i) For* $\lambda > 1$,

$$s(x,y)-\sigma(s(\lambda x,\lambda y)) = \frac{1}{(\lambda-1)^2}\left(\sigma(s(\lambda x,\lambda y))-\sigma(s(x,y))\right)$$
$$+\frac{2}{(\lambda-1)}\sigma(s(\lambda x,\lambda y)) - \frac{1}{(\lambda x-x)(\lambda y-y)}\int_x^{\lambda x}\int_y^{\lambda y}(s(\zeta,\eta)-s(x,y))d\zeta d\eta.$$

(ii) For $0 < \lambda < 1$,

$$s(x,y) - \sigma(s(\lambda x, \lambda y)) = \frac{1}{(1-\lambda)^2}(\sigma(s(x,y)) - s(\lambda x, \lambda y))$$
$$+\frac{2}{(1-\lambda)}\sigma(s(\lambda x, \lambda y)) - \frac{1}{(x-\lambda x)(y-\lambda y)}\int_{\lambda x}^{x}\int_{\lambda y}^{y}(s(x,y) - s(\zeta,\eta))d\zeta d\eta.$$

Proof. (i) We have by De la Vallée Poussin mean of $s(x,y)$,

$$\tau(s(x,y)) = \frac{1}{(\lambda x - x)(\lambda y - y)}\int_{x}^{\lambda x}\int_{y}^{\lambda y} s(\zeta,\eta)d\zeta d\eta$$
$$= \frac{1}{x(\lambda-1)y(\lambda-1)}\left(\int_0^{\lambda x}\int_0^{\lambda y} s(\zeta,\eta)d\zeta d\eta - \int_0^{x}\int_0^{y} s(\zeta,\eta)d\zeta d\eta\right), \ (\lambda > 1).$$

Again, since

$$\sigma(s(\lambda x, \lambda y)) = \frac{1}{\lambda x \lambda y}\int_0^{\lambda x}\int_0^{\lambda y} s(\zeta,\eta)d\zeta d\eta,$$

and

$$\sigma(s(x,y)) = \frac{1}{xy}\int_0^{x}\int_0^{y} s(\zeta,\eta)d\zeta d\eta,$$

$$\text{so} \quad \tau(s(x,y)) = \frac{\lambda^2}{(\lambda-1)^2}\sigma(s(\lambda x, \lambda y)) - \frac{1}{(\lambda-1)^2}\sigma(s(x,y))$$
$$= \left(1 + \frac{1}{(\lambda-1)}\right)^2 \sigma(s(\lambda x, \lambda y)) - \frac{1}{(\lambda-1)^2}\sigma(s(x,y)).$$

Now

$$\tau(s(x,y)) - \sigma(s(\lambda x, \lambda y)) = \frac{1}{(\lambda-1)^2}\sigma(s(\lambda x, \lambda y)) + \frac{2}{(\lambda-1)}\sigma(s(\lambda x, \lambda y))$$
$$-\frac{1}{(\lambda-1)^2}\sigma(s(x,y)). \tag{8}$$

Subtracting $\sigma(s(\lambda x, \lambda y))$ from the identity,

$$\text{Also,} \ s(x,y) = \tau(s(x,y)) - \frac{1}{(\lambda x - x)(\lambda y - y)}\int_{x}^{\lambda x}\int_{y}^{\lambda y}(s(\zeta,\eta) - s(x,y))d\zeta d\eta,$$

we have

$$s(x,y) - \sigma(s(\lambda x, \lambda y)) = \tau(s(x,y)) - \sigma(s(\lambda x, \lambda y))$$
$$-\frac{1}{(\lambda x - x)(\lambda y - y)}\int_{x}^{\lambda x}\int_{y}^{\lambda y}(s(\zeta,\eta) - s(x,y))d\zeta d\eta. \tag{9}$$

From Eqs. (8) and (9), we get

$$s(x,y)-\sigma(s(\lambda x,\lambda y))=\frac{1}{(\lambda-1)^2}\left(\sigma(s(\lambda x,\lambda y))-\sigma(s(x,y))\right)$$
$$+\frac{2}{(\lambda-1)}\sigma(s(\lambda x,\lambda y))-\frac{1}{(\lambda x-x)(\lambda y-y)}\int_x^{\lambda x}\int_y^{\lambda y}(s(\zeta,\eta)-s(x,y))d\zeta d\eta.$$

Which establish (i).

Next, the proof of (ii) is similar to (i).

Proof of Theorem 1.

Proof. Let $s(x,y)$ is statistically $(C,1,1)$-summable to $\ell\in X$, this implies $\sigma(s(x,y))$ is $(C,1,1)$-summable to $\ell\in X$. Now from Eq. (4), we have $v(f(x,y))$ is statistically $(C,1,1)$-summable to zero. Thus by Lemma 1, $v(f(x,y))$ is oscillating slowly. Again by Lemma 2(i), we get

$$\begin{aligned}v(f(x,y))-\sigma(v(f(\lambda x,\lambda y)))&=\frac{1}{(\lambda-1)^2}((\sigma(v(f(\lambda x,\lambda y))))-\sigma(v(f(x,y))))\\&+\frac{2}{(\lambda-1)}\sigma(v(f(\lambda x,\lambda y)))-\frac{1}{(\lambda x-x)(\lambda y-y)}\\&\cdot\int_x^{\lambda x}\int_y^{\lambda y}(v(f(\zeta,\eta))-v(f(x,y)))d\zeta d\eta.\end{aligned}\tag{10}$$

Next, by (10)

$$\begin{aligned}&|v(f(x,y))-\sigma(v(f(x,y)))|\le\frac{1}{(\lambda-1)^2}|(\sigma(v(f(\lambda x,\lambda y))))-\sigma(v(f(x,y)))|\\&+\frac{2}{(\lambda-1)}|\sigma(v(f(\lambda x,\lambda y)))|+\max_{x,y\le\zeta,\eta\le\lambda x,\lambda y}|v(f(\zeta,\eta))-v(f(x,y))|.\end{aligned}\tag{11}$$

Now taking lim sup to both sides of Eq. (11) as $x,y\to\infty$, we obtain

$$\begin{aligned}&\lim\sup_{x,y\to\infty}|v(f(x,y))-\sigma(v(f(x,y)))|\\&\le\lim\sup_{x,y\to\infty}\frac{1}{(\lambda-1)^2}|(\sigma(v(f(\lambda x,\lambda y))))-\sigma(v(f(x,y)))|\\&+\lim\sup_{x,y\to\infty}\frac{2}{(\lambda-1)}|\sigma(v(f(\lambda x,\lambda y)))|\\&+\lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le\lambda x,\lambda y}|v(f(\zeta,\eta))-v(f(x,y))|.\end{aligned}\tag{12}$$

Furthermore, as $\sigma(v(f(\lambda x,\lambda y)))\in X$ converges; so first and second terms in the right hand side of Eq. (12), must vanish.

This implies,

$$\begin{aligned}&\lim\sup_{x,y\to\infty}|v(f(x,y))-\sigma(v(f(x,y)))|\\&\le\lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le\lambda x,\lambda y}|v(f(\zeta,\eta))-v(f(x,y))|.\end{aligned}\tag{13}$$

As $\lambda \to 1^+$ in (13), so we get

$$\limsup_{x,y\to\infty} |v(f(x,y)) - \sigma(v(f(x,y)))| \le 0.$$

It implies that, $v(f(x,y)) = o(1)$ as $x, y \to \infty$. Since $s(x,y)$ is statistically summable to $\ell \in X$ by Cesàro mean and $v(f(x,y)) = o(1)$ as $x, y \to \infty$, so $\lim\limits_{x,y\to\infty} s(x,y) = \ell \in X$.

Corollary 1. *If $s(x,y)$ is statistically (C,k,r)-summable to ℓ in a locally convex space X and $s(x,y)$ is oscillating slowly, then $s(x,y) \to \ell \in X$ as $x, y \to \infty$.*

Proof. By Lemma 1, $s(x,y)$ is oscillating slowly and also $\sigma^{(k,r)}(s(x,y))$ is oscillating slowly. Furthermore, from Theorem 1, $s(x,y)$ being statistically (C,k,r)-summable to $\ell \in X$, so

$$\operatorname{stat}\lim_{x,y\to\infty} \sigma^{(k,r)}(s(x,y)) = \ell \in X. \tag{14}$$

Next, from the definition

$$\sigma^{(k,r)}(s(x,y)) = \sigma^{(1,1)}(s(x,y))(\sigma^{(k-1,r-1)}(s(x,y))). \tag{15}$$

Clearly, Eqs. (14) and (15) implies that $s(x,y)$ is statistically $(C,k-1,r-1)$-summable to $\ell \in X$. Again by Lemma 1, $\sigma^{(k-1,r-1)}(s(x,y))$ is also oscillating slowly.

Thus, Theorem 1 implies

$$\lim_{x,y\to\infty} \sigma^{(k-1,r-1)}(s(x,y)) = \ell \in X.$$

Continuing in this way, we get $\lim\limits_{x,y\to\infty}(s(x,y)) = \ell \in X$.

Theorem 2. *If $s(x,y)$ is statistically $(C,1,1)$-summable to ℓ over a locally convex space X and $v(f(x,y))$ is oscillating slowly, then $s(x,y) \to \ell \in X$ as $x, y \to \infty$.*

Proof. As $s(x,y)$ is statistically $(C,1,1)$-summable to $\ell \in X$, so $\sigma^{(1,1)}(s(x,y))$ is also statistically Cesàro summable to $\ell \in X$. Therefore, $v(f(x,y))$ is statistically Cesàro summable to zero by Eq. (4). Using identity (4) to $v(f(x,y))$, we get $v(v(f(x,y)))$ is statistically Cesàro summable to zero. So that $v(v(f(x,y)))$ is oscillating slowly by Lemma 1. Now by Lemma 2(i),

$$\begin{aligned} v(v(f(x,y))) &- \sigma(v(v(f(\lambda x,\lambda y)))) \\ &= \frac{1}{(\lambda-1)^2}[\sigma(v(v(f(\lambda x,\lambda y)))) - \sigma(v(v(f(x,y))))] \\ &\quad + \frac{2}{(\lambda-1)}(\sigma(v(v(f(\lambda x,\lambda y))))) \\ &- \frac{1}{(\lambda x - x)(\lambda y - y)} \int_x^{\lambda x}\int_y^{\lambda y} (v(v(f(\zeta,\eta))) - v(v(f(x,y))))d\zeta d\eta. \end{aligned} \tag{16}$$

Next,

$$\begin{aligned}
|v(v(f(x,y))) - \sigma(v(v(f(x,y))))| \\
&\leq \frac{1}{(\lambda-1)^2}|\sigma(v(v(f(\lambda x,\lambda y)))) - \sigma(v(v(f(x,y))))| \\
&\quad + \frac{2}{(\lambda-1)}|\sigma(v(v(f(\lambda x,\lambda y))))| \\
&\quad + \max_{x,y\leq\zeta,\eta\leq\lambda x,\lambda y}|(v(v(f(\zeta,\eta))) - v(v(f(x,y))))|. \qquad (17)
\end{aligned}$$

Now taking lim sup to both sides of Eq. (17) as $x, y \to \infty$, we have

$$\begin{aligned}
\lim\sup_{x,y\to\infty}|v(v(f(x,y))) - \sigma(v(v(f(x,y))))| \\
&\leq \lim\sup_{x,y\to\infty}\frac{1}{(\lambda-1)^2}|\sigma(v(v(f(\lambda x,\lambda y)))) - \sigma(v(v(f(x,y))))| \\
&\quad + \lim\sup_{x,y\to\infty}\frac{2}{(\lambda-1)}|\sigma(v(v(f(\lambda x,\lambda y))))| \\
&\quad + \lim\sup_{x,y\to\infty}\max_{x,y\leq\zeta,\eta\leq\lambda x,\lambda y}|v(v(f(\zeta,\eta))) - v(v(f(x,y)))|. \qquad (18)
\end{aligned}$$

Since $\sigma(v(v(f(\lambda x,\lambda y)))) \in X$ converges, then first and second terms in right hand side of Eq. (18), must be zero and then

$$\begin{aligned}
\lim\sup_{x,y\to\infty}|v(v(f(x,y))) - \sigma(v(v(f(x,y))))| \\
&\leq \lim\sup_{x,y\to\infty}\max_{x,y\leq\zeta,\eta\leq\lambda x,\lambda y}|v(v(f(\zeta,\eta))) - v(v(f(x,y)))|. \qquad (19)
\end{aligned}$$

Next, as $\lambda \to 1^+$ Eq. (19), implies

$$\lim\sup_{x,y\to\infty}|v(v(f(x,y))) - \sigma(v(v(f(x,y))))| \leq 0.$$

Clearly, $v(v(f(x,y))) = o(1)$ as $x, y \to \infty$. Furthermore, $s(x,y)$ is statistically summable to $\ell \in X$ by Cesàro mean and $v(v(f(x,y))) = o(1)$ as $x, y \to \infty$, so $s(x,y) \to \ell \in X$ as $x, y \to \infty$.

Corollary 2. *If $s(x,y)$ is statistically (C,k,r)-summable to ℓ over a a locally convex space X and $v(f(x,y))$ is oscillating slowly, then $s(x,y) \to \ell \in X$ as $x, y \to \infty$.*

Proof. As $v(f(x,y))$ is oscillating slowly, setting $v(f(x,y))$ in place of $s(x,y)$; $\sigma^{(k,r)}(v(f(x,y)))$ is oscillating slowly by Lemma 1. Again as $v(f(x,y))$ is statistically (C,k,r) summable to $\ell \in X$, so Theorem 2 implies

$$\text{stat}\lim_{x,y\to\infty}\sigma^{(k,r)}(v(f(x,y))) = \ell \in X. \qquad (20)$$

Next by definition,

$$\lim_{x,y\to\infty}\sigma^{(k,r)}(v(f(x,y))) = \sigma^{(1,1)}(v(f(x,y)))\sigma^{(k-1,r-1)}(v(f(x,y))). \qquad (21)$$

From (20) and (21), we have $v(f(x, y))$ is statistically $(C, k-1, r-1)$-summable to $\ell \in X$. Again by Lemma 1, since $\sigma^{k-1,r-1}(v(f(x, y)))$ is oscillating slowly, so Theorem 2 implies $\lim_{x,y\to\infty} \sigma^{k-1,r-1}(v(f(x, y))) = \ell \in X$. Continuing in this way, we get $\lim_{x,y\to\infty} v(f(x, y)) = \ell \in X$.

4 Conclusion

Tauberian theorems for single sequences as well as for functions of single variable have been achieved a high degree of development; however, it is still in its infancy for double sequences and function of two or more variables. The result established in this paper for a function of two variables generalizes some earlier existing results for the function of a single variable. Further, it will be encouraging, if one can extend the result for function of several variables.

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On Approximation of Signals in the Weighted Zygmund Class via $(E, 1)(\overline{N}, p_n)$ Summability Means of Conjugate Fourier Series

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Abstract. Approximation of signals in Lipschitz and Zygmund classes via different summability means have been demonstrated by many researchers. In the proposed paper, we have studied an estimation of the order of convergence of conjugate Fourier series in the weighted Zygmund class $W(Z_r^{(\omega)})$ by using $(E, 1)(\overline{N}, p_n)$-product summability means and accordingly established some new results. Also, the results obtained here generalizes some known results.

Keywords: Degree of approximation · Conjugate Fourier series · $(E, 1)$ - summability · $(\overline{N}, p_n)$ - summability · $(E, 1)(\overline{N}, p_n)$ - summability · Weighted Zygmund class

1 Introduction

Signal analysis is concerned with the reliable estimation, detection and classification of signals (functions) which are subject to random fluctuations and it has its roots in probability theory, mathematical statistics and, more recently, approximation theory and communications theory. These approximation analysis of signals have great importance in the field of science and engineering. Moreover, it has given a new dimensions because of their vast applications in signal analysis, radar system, telecommunications and image processing system. The error estimation of functions in different function spaces such as Hölder, Lipschitz, Zygmund and Besov spaces etc. by using different summability techniques of trigonometric Fourier series has been received a growing interest of several researchers in last decades. Functions in L_r $(r \geq 1)$-spaces assumed to be most practicable in signal analysis. Particularly, L_1, L_2 and L_∞ spaces are used by engineers for designing digital filters and matrix summability or

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matrix transformation plays a vital role in this context. Also, matrix summability generalizes different summability methods like Cesàro summability, Nörlund summability, Riesz summability etc. (see [1,3,6,13,14,17,19]). Recently, different Lipschitz classes of functions are considered for approximation results under various summability means. For more details, see the current works [8–11,18]. Subsequently, the generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ is an extension of $Z_{(\alpha)}, Z_{(\alpha),r}$ and $Z^{(\omega)}$-classes. The generalized Zygmund class $Z_r^{(\omega)}$ $(r \geq 1)$ is studied by Leindler [12], Moricz [15], Moricz and Nemeth [16]. Recently, Singh *et al.* [21] used Hausdörff mean to establish some approximation results for the functions in generalized Zygmund class. Lal and Shireen [10] considered matrix-Euler product summability mean of Fourier series for approximation of functions of generalized Zygmund class. Very recently, some results on statistical approximation and associated Korovkin-type theorems has been established by different researchers (see [2,5,7,20,22–24]). To get best approximation by product summability means (Ordinary versions), in the proposed paper, we have used $(E,1)(\overline{N},p_n)$ summability mean of conjugate Fourier series to estimate the degree of approximation of a function of $W(Z_r^{(\omega)})$ $(r \geq 1)$ class (weighted Zygmund class).

Let $\sum u_n$ be a given infinite series with its sequence of partial sum $\{s_n\}$. Let $\{p_k\}$ for $k = 0, 1, 2, ...$ be a sequence of nonnegative integers such that $p_0 > 0$ and

$$P_n = \sum_{k=0}^{n} p_k \to \infty \text{ as } n \to \infty \quad (p_{-k} = P_{-k} = 0,\ k \geq 1). \tag{1}$$

Let the sequence-to-sequence transformation,

$$\tau_n^{\overline{N}} = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k, \quad n = 0, 1, 2, \cdots \tag{2}$$

defines $(\overline{N}, p_n)$ mean of $\{s_n\}$ generated by the sequence $\{p_k\}$. The series $\sum u_n$ is known to be summable to s by $(\overline{N}, p_n)$ method, if $\lim_{n\to\infty} \tau_n^{\overline{N}} \to s$ as $n \to \infty$. Also, this $(\overline{N}, p_n)$ method is regular (see [4]).
The sequence to sequence transformation,

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{k}{v} s_k \tag{3}$$

defines the $(E, 1)$ transform of the sequence $\{s_n\}$. The series $\sum u_n$ is summable to s with respect to $(E, 1)$ summability, if $E_n^1 \to s$ as $n \to \infty$. Also, $(E, 1)$ method is regular (see [4]).

Now we define here a new composite transform by using the product $(E,1)(\overline{N},p_n)$ transform. As $(\overline{N},p_n)$ and $(E,1)$ summability methods are regular, the product $(E,1)(\overline{N},p_n)$ method is also regular.
Let

$$\tau_n^{E\overline{N}} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v s_v \right\} \tag{4}$$

defines the $(E,1)(\overline{N},p_n)$ transform of the sequence $\{s_n\}$. We say here that $\sum u_n$ is summable to s by product $(E,1)(\overline{N},p_n)$ transform, if $\tau_n^{E\overline{N}} \to s$ as $n \to \infty$.

Let f be a 2π periodic function in $L^r[0,2\pi]$, $r \geq 1$ with the partial sum $s_n(f)$, then

$$s_n(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{5}$$

The conjugate series of the Fourier series (5) is given by

$$\tilde{s_n}(f) = \sum_{k=1}^{\infty}(a_n \cos nx - b_n \sin nx). \tag{6}$$

Let f be a 2π-periodic integrable function belonging to $[0,2\pi]$ and let $\tilde{f}$, conjugate to f be a 2π periodic function. We have,

$$L_r[0,2\pi] = \{\tilde{f} : [0,2\pi] \to \mathbb{R}; \int_0^{2\pi} |f(x)|^r dx < \infty\}.$$

The L_r norm of a function f is defined by

$$\|f\|_r = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, & 1 \leq r < \infty \\ ess\sup_{0<x\leq 2\pi} |f(x)|, & r = \infty. \end{cases}$$

The Zygmund modulus of continuity of f is defined by

$$\omega(f,h) = \sup_{o\leq h, x\in\mathbb{R}} |f(x+t) + f(x-t)|.$$

Let $C_{2\pi}$ be the Banach space of all 2π-periodic continuous functions defined on $[0,2\pi]$ under the supremum norm. For $0 < \alpha \leq 1$, the function space

$$Z_{(\alpha)} = \{f \in \mathbb{C}_{2\pi} : |f(x+t) + f(x-t)| = O(|t|^\alpha)\}$$

is also a Banach space with the norm $\|\cdot\|_{(\alpha)}$, given by

$$\|f\|_{(\alpha)} = \sup_{0\leq x\leq 2\pi} |f(x)| + \sup_{x,t\neq 0} \frac{|f(x+t) + f(x-t)|}{|t|^\alpha}.$$

For $f \in L_r[0,2\pi], r \geq 1$, the integral Zygmund modulus of continuity is given by

$$\omega_r(f,h) = \sup_{0<t\leq h} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+t) + f(x-t)|^r dx\right\}^{\frac{1}{r}}.$$

For $f \in \mathbb{C}_{2\pi}$ and $r = \infty$,

$$\omega_\infty(f,h) = \sup_{0<t\leq h} \max_x |f(x+t) + f(x-t)|.$$

Also, it is known that $\omega_r(f,h) \to 0$ as $r \to 0$.
Now define,

$$Z_{(\alpha),r} = \left\{ f \in L_r[0,2\pi] : \left(\int_0^{2\pi} |f(x+t) + f(x-t)|^r dx \right)^{\frac{1}{r}} = O(|t|^\alpha) \right\}$$

and also the space $Z_{(\alpha),r}, r \geq 1, 0 < \alpha \leq 1$ is a Banach space with the norm $\|\cdot\|_{(\alpha),r}$ of the form

$$\|f\|_{(\alpha),r} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(\cdot+t) + f(\cdot-t)\|_r}{|t|^\alpha}.$$

The $Z^{(\omega)}$ class of function is defined as

$$Z^{(\omega)} = \{f \in \mathbb{C}_{2\pi} : |f(x+t) + f(x-t)| = O(\omega(t))\},$$

where ω is a Zygmund modulus of continuity, that is, ω is continuous, positive and non-decreasing function with the sub linearity property

(i) $\omega(0) = 0$ and
(ii) $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$.

Let $\omega : [0, 2\pi] \to \mathbb{R}$ be a function with $\omega(t) > 0$ for $0 \leq t < 2\pi$ and let $\lim_{t\to 0^+} \omega(t) = \omega(0) = 0$ and define

$$Z_r^{(\omega)} = \left\{ \tilde{f} \in L_r : 1 \leq r \leq \infty, \sup_{t\neq 0} \frac{\|f(\cdot+t) + f(\cdot-t)\|_r}{\omega(t)} < \infty \right\},$$

where

$$\|\tilde{f}\|_r^{(\omega)} = \|f\|_r + \sup_{t\neq 0} \frac{\|f(\cdot+t) + f(\cdot-t)\|_r}{\omega(t)}, r \geq 1.$$

Clearly, $\|\cdot\|_r^{(\omega)}$ is a norm on $Z_r^{(\omega)}$. As we know L_r $(r \geq 1)$ is complete, the space $Z_r^{(\omega)}$ is also complete. Hence, we can say $Z_r^{(\omega)}$ is a Banach space under the norm $\|\cdot\|_r^{(\omega)}$.
Now we define the weighted Zygmund class as

$$W(Z_r^{(\omega)}) = \left\{ f \in W(Z_r^{(\omega)}) : \sup_{t\neq 0} \frac{\|(f(\cdot+t) + f(\cdot-t)) \sin^\beta(\cdot)\|_r}{\omega(t)} \leq \infty \right\} \tag{7}$$

where

$$\|f\|_r^{(\omega)} = \|f\|_r + \sup_{t\neq 0} \frac{\|(f(\cdot+t) + f(\cdot-t)) \sin^\beta(\cdot)\|_r}{\omega(t)}, \quad r \geq 1. \tag{8}$$

Clearly, $\|\cdot\|_r^{*(\omega)}$ is a norm of $Z_r^{(\omega)}$. The space $Z_r^{(\omega)}$ is complete because $L_r,\ \ r \geq 1$ is complete. Hence we can say $W(Z_r^{(\omega)})$ is complete.

As $Z_r^{(\omega)}$ is a Banach space under $\|\cdot\|_r^{(\omega)}$. Hence $W(Z_r^{(\omega)})$ is also a Banach space under $\|\cdot\|_r^{(\omega)}$ norm. Here $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that $\left(\frac{\omega(t)}{v(t)}\right)$ be positive, non-decreasing, then

$$\|f\|_r^{(v)} \leq \max\left(1, \frac{\omega(2\pi)}{v(2\pi)}\right)\|f\|_r^{(\omega)} \leq \infty.$$

Thus, we have

$$Z_r^{(\omega)} \subseteq Z_r^{(v)} \subseteq L_r \quad (r \geq 1).$$

Hence,

$$W(Z_r^{(\omega)}) \subseteq W(Z_r^{(v)}) \subseteq W(L_r, \omega(t)).$$

We use the following notations through out this paper:

$$\psi(x,t) = f(x+t) + f(x-t);$$

$$\widetilde{K}_n^{E\overline{N}}(t) = \frac{1}{\pi 2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin(\frac{t}{2})} \right\}.$$

2 Main Theorem

Theorem 1. *Let $\tilde{f}$ be conjugate to a 2π periodic function f, Lebesgue integrable in $[0, 2\pi]$ and belonging to weighted Zygmund class $W(Z_r^{(\omega)})$ $(r \geq 1)$. Then the degree of approximation of signal (function) $\tilde{f}$, using product $(E, 1)(\overline{N}, p_n)$ mean of conjugate Fourier series (6) is given by*

$$E_n(\tilde{f}) = \inf_{\tilde{\tau}_n^{E\overline{N}}} \|\tilde{\tau}_n^{E\overline{N}} - \tilde{f}\|_r^v = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt\right) \tag{9}$$

where $\omega(t)$ and $v(t)$ denotes the Zygmund modulus of continuity such that $\frac{\omega(t)}{v(t)}$ is positive and increasing.

Theorem 2. *Let $\tilde{f}$ be conjugate to a 2π periodic function f, Lebesgue integrable in $[0, 2\pi]$ and belonging to weighted Zygmund class $W(Z_r^{(\omega)})$ $(r \geq 1)$. Then the degree of approximation of signal (function) $\tilde{f}$, using product $(E, 1)(\overline{N}, p_n)$ mean of conjugate Fourier series (6) is given by*

$$E_n(\tilde{f}) = \inf_{\tilde{\tau}_n^{E\overline{N}}} \|\tilde{\tau}_n^{E\overline{N}} - \tilde{f}\|_r^v = O\left((n+1)^{-\beta} \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \tag{10}$$

where $\omega(t)$ and $v(t)$ denotes the Zygmund moduli of continuity such that $\frac{\omega(t)}{tv(t)}$ is positive and decreasing.

To prove the theorems we need the following lemmas.

Lemma 1. $|\widetilde{K_n}(t)| = O(n)$, *for* $0 \leq t \leq \frac{1}{n+1}$.

Proof. For $0 \le t \le \frac{1}{n+1}$, we have $\sin\ nt \le n \sin\ t$.

$$
\begin{aligned}
|\widetilde{K_n}(t)| &= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}} \right\} \right| \\
&= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\frac{\cos\frac{t}{2} - \cos vt.\cos\frac{t}{2} + \sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}} \right) \right\} \right| \\
&= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\frac{\cos\frac{t}{2}(2\sin^2 v\frac{t}{2})}{\sin\frac{t}{2}} + \sin vt \right) \right\} \right| \\
&\le \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(O(2\sin v\frac{t}{2}\sin v\frac{t}{2}) \right) + v\sin t \right\} \right| \\
&\le \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(O(v) + O(v) \right) \right\} \right| \\
&= \frac{1}{\pi 2^{n}} \left| \sum_{k=0}^{n} \binom{n}{k} \frac{O(k)}{P_k} \sum_{v=0}^{k} p_v \right| \\
&= O(n).
\end{aligned}
$$

Lemma 2. $|\widetilde{K_n}(t)| = O(\frac{1}{t})$, *for* $\frac{1}{n+1} < t \le \pi$.

Proof. For $\frac{1}{n+1} < t \le \pi$ and by using Jordan's lemma, $\sin\frac{t}{2} \ge \frac{t}{\pi}$ and $\sin nt \le 1$.

$$
\begin{aligned}
|\widetilde{K_n}(t)| &= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin\frac{t}{2}} \right\} \right| \\
&= \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\frac{\cos\frac{t}{2} - \cos vt.\cos\frac{t}{2} + \sin vt.\sin\frac{t}{2}}{\sin\frac{t}{2}} \right) \right\} \right| \\
&\le \frac{1}{\pi 2^{n+1}} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left(\cos\frac{t}{2}(2\sin^2 v\frac{t}{2}) + \sin vt \right) \right\} \right| \\
&\le \frac{1}{2^{n+1}t} \left| \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \right\} \right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}
$$

Lemma 3. *Let* $f \in Z_r^{(\omega)}$, *then for* $0 < t \le \pi$,

(i) $\|\psi(\cdot,t)\|_r = O(\omega(t))$ *and*

(ii) $\|\psi(\cdot+y,t)+\psi(\cdot-y,t)\|_r = \begin{cases} O(\omega(t)) \\ O(\omega(y)). \end{cases}$

If $\omega(t)$ *and* $v(t)$ *defined as in Theorem 1, then*
$\|\psi(\cdot+y,t)+\psi(\cdot-y,t)\|_r = O\left(v(y)\frac{\omega(t)}{v(t)}\right)$,
where

$$\psi(x,t) = f(x+t) + f(x-t).$$

Proof. This Lemma can be proved easily by following [10].

Lemma 4. $\|(\phi(\cdot+y,t)+\phi(\cdot-y,t)-2\phi(\cdot,t))\sin^{\beta}(\cdot)\|_r = O\left(t^{\beta}v(y)\left(\frac{\omega(t)}{v(t)}\right)\right)$

Proof. Following Lemma 2, $|\sin^{\beta} t| \leq t^{\beta}$ and for v is positive, nondecreasing, $t \leq y$, we obtain

$$\begin{aligned}\|(\phi(\cdot+y,t)+\phi(\cdot-y,t)-2\phi(\cdot,t))\sin^{\beta}(\cdot)\|_r &= O(t^{\beta}\omega(t)) \\ &= O\left(t^{\beta}v(t)\left(\frac{\omega(t)}{v(t)}\right)\right) \\ &\leq O\left(t^{\beta}v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).\end{aligned}$$

Since $\frac{\omega(t)}{v(t)}$ is positive, non-decreasing, if $t \geq y$, then $\frac{\omega(t)}{v(t)} \geq \frac{\omega(y)}{v(y)}$, so that

$$\begin{aligned}\|(\phi(\cdot+y,t)+\phi(\cdot-y,t)-2\phi(\cdot,t))\sin^{\beta}(\cdot)\|_r &= O(t^{\beta}\omega(y)) \\ &= O\left(t^{\beta}v(y)\left(\frac{\omega(t)}{v(t)}\right)\right).\end{aligned}$$

3 Proof of Main Results

3.1 Proof of Theorem 1

Let $\tilde{s}_k(f;x)$ denotes the k^{th} partial sum of the series (6) and following [25], we have

$$\tilde{s}_k(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(x;t)\frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin(\frac{t}{2})}dt. \tag{11}$$

Therefore using (2), the $(\overline{N}, p_n)$ transform of $\tilde{s}_k(f;x)$ is given by

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k(\tilde{s}_k(\tilde{f};x) - \tilde{f}(x)) = \frac{1}{2\pi}\int_0^{\pi} \phi(x;t)\frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{\cos\frac{t}{2} - \cos(v+\frac{1}{2})t}{\sin(\frac{t}{2})}dt. \quad (12)$$

Now considering the $(E,1)(\overline{N}, p_n)$ transform of $\tilde{s}_k(f;x)$ by $\tilde{\tau}_n^{E\overline{N}}$, we write

$$\tilde{\tau}_n^{E\overline{N}} - \tilde{f}(x) = \frac{1}{\pi 2^{n+1}}\sum_{k=0}^{n}\int_0^{\pi} \frac{\psi(x;t)}{\sin(t/2)}\frac{1}{P_k}\sum_{k=0}^{v} p_v(\cos\frac{t}{2} - \cos(v+\frac{1}{2})t)dt. \quad (13)$$

Let

$$\tilde{\mathcal{L}}_n(x) = \tilde{\tau}_n^{E\overline{N}} - \tilde{f}(x) = \int_0^{\pi} \psi(x;t)\tilde{K}_n^{E\overline{N}}(t)dt \quad (14)$$

then

$$\tilde{\mathcal{L}}_n(x+y) + \tilde{\mathcal{L}}_n(x-y) = \int_0^{\pi} \left[\psi(x+y;t) + \psi(x-y;t)\right]\tilde{K}_n^{E\overline{N}}(t)dt. \quad (15)$$

Now,

$$\begin{aligned}&\left(\tilde{\mathcal{L}}_n(\cdot+y) + \tilde{\mathcal{L}}_n(\cdot-y)\right)\sin^{\beta}(\cdot)\\ &\qquad = \int_0^{\pi}\left((\psi(\cdot+y;t) + \psi(\cdot-y;t))\sin^{\beta}(\cdot)\right)\tilde{K}_n^{E\overline{N}}(t)dt. \quad (16)\end{aligned}$$

Clearly, we have

$$\begin{aligned}&\|(\tilde{\mathcal{L}}_n(\cdot+y) + \tilde{\mathcal{L}}_n(\cdot-y)-)\sin^{\beta}(\cdot)\|_r\\ &= \int_0^{\pi} \|(\psi(\cdot+y;t) + \psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{K}_n^{E\overline{N}}(t)dt\\ &= \int_0^{\frac{1}{n+1}} \|(\psi(\cdot+y;t) + \psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{K}_n^{E\overline{N}}(t)dt\\ &\quad + \int_{\frac{1}{n+1}}^{\pi} \|(\psi(\cdot+y;t) + \psi(\cdot-y;t))\sin^{\beta}(\cdot)\|_r\tilde{K}_n^{E\overline{N}}(t)dt\\ &= I_1 + I_2 \quad \text{(say)}. \quad (17)\end{aligned}$$

Further the function $f \in W(Z_r^{(\omega)})$ implies $\phi \in W(Z_r^{(\omega)})$ and applying Lemmas 1 and 4 and monotonicity of $\frac{\omega(t)}{v(t)}$ with respect to t, we have

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{n+1}} \|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^\beta(\cdot)\|_r \tilde{K}_n^{E\overline{N}}(t)dt \\
&= O\left(\int_0^{\frac{1}{n+1}} v(y)\frac{t^\beta\omega(t)}{v(t)}\,(n)\,dt\right) \\
&= O\left(n\ v(y)\int_0^{\frac{1}{n+1}} \frac{t^\beta\omega(t)}{v(t)}dt\right) \\
&= O\left(n\ v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\int_0^{\frac{1}{n+1}} t^\beta dt\right) \\
&= O\left(n\ v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\left(\frac{t^{\beta+1}}{\beta+1}\right)_0^{\frac{1}{n+1}}\right) \\
&= O\left(n\ v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\frac{1}{(n+1)^{\beta+1}}\right) \\
&= O\left(\frac{n}{n+1}(n+1)^{-\beta}v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \\
&= O\left((n+1)^{-\beta}v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \quad \left(\because \frac{n}{n+1}=O(1)\right).
\end{aligned} \tag{18}$$

Next, using Lemmas 2 and 4, we get

$$\begin{aligned}
I_2 &= \int_{\frac{1}{n+1}}^{\pi} \|(\psi(\cdot+y;t)+\psi(\cdot-y;t))\sin^\beta(\cdot)\|_r \tilde{K}_n^{E\overline{N}}(t)dt \\
&= O\left(\int_{\frac{1}{n+1}}^{\pi} v(y)\frac{t^\beta\omega(t)}{v(t)}\left(\frac{1}{t}\right)dt\right) \\
&= O\left(v(y)\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)}dt\right).
\end{aligned} \tag{19}$$

Thus using (17)–(19), we have

$$\begin{aligned}
&\|(\tilde{\mathcal{L}}_n(\cdot+y)+\tilde{\mathcal{L}}_n(\cdot-y)-)\sin^\beta(\cdot)\|_r \\
&\quad = O\left((n+1)^{-\beta}v(y)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right)+O\left(v(y)\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-1}\omega(t)}{v(t)}dt\right).
\end{aligned} \tag{20}$$

Therefore, we have

$$\sup_{y\neq 0}\frac{\|\tilde{\mathcal{L}}_n(\cdot+y)+\tilde{\mathcal{L}}_n(\cdot-y)\|_r}{v(|y|)}$$
$$=O\Bigg((n+1)^{-\beta}\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\Bigg)+O\Bigg(\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-1}\omega(t)}{v(t)}dt\Bigg). \tag{21}$$

Also, we have

$$\|\psi(x;t)\|_r=\|f(x+t)+f(x-t)\|_r. \tag{22}$$

Now using Lemmas 1, 2 and 3, we have

$$\begin{aligned}\|(\tilde{\mathcal{L}}_n(\cdot))\sin^{\beta}(\cdot)\|_r &\leq \left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\|(\psi(\cdot,t))\sin^{\beta}(\cdot)\|_r|K_n^{E\overline{N}}(t)|dt\\ &=O\left((n)\int_0^{\frac{1}{n+1}}t^{\beta}\omega(t)dt\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}t^{\beta-1}\omega(t)dt\right)\\ &=O\left((n)\omega(\frac{1}{n+1})\int_0^{\frac{1}{n+1}}t^{\beta}dt\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{1-\beta}}dt\right)\\ &=O\left((n+1)^{-\beta}w(\frac{1}{n+1})\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{1-\beta}}dt\right). \end{aligned}\tag{23}$$

Now from (22) and (23), we have

$$\begin{aligned}\|(\tilde{\mathcal{L}}_n(\cdot))\sin^{\beta}(\cdot)\|_r^v &=\|(\tilde{\mathcal{L}}_n(\cdot))\sin^{\beta}(\cdot)\|_r+\sup_{y\neq 0}\frac{(\|\tilde{\mathcal{L}}_n(\cdot+y)+\tilde{\mathcal{L}}_n(\cdot-y))\sin^{\beta}(\cdot)\|_r}{v(y)}\\ &=O\left((n+1)^{-\beta}w(\frac{1}{n+1})\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{1-\beta}}dt\right)\\ &\quad+O\Bigg((n+1)^{-\beta}\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\Bigg)+O\Bigg(\int_{\frac{1}{n+1}}^{\pi}\frac{t^{\beta-1}\omega(t)}{v(t)}dt\Bigg)\\ &=\sum_{i=1}^{4}O(J_i)\qquad\text{(say)}.\end{aligned}\tag{24}$$

Now we write J_1 in terms of J_3 and further J_2, J_3 in terms of J_4.

For the shake of monotonicity of $v(t)$ for $0<t\leq\pi$, we have

$$\omega(t)=\frac{\omega(t)}{v(t)}.v(t)\leq v(\pi)\frac{\omega(t)}{v(t)}.v(t)=O\left(\frac{\omega(t)}{v(t)}\right)\ \text{ for }\ <t\leq\pi.$$

Therefore, we can write for $t=(n+1)^{-\beta}$

$$J_1=O(J_3). \tag{25}$$

Again by using monotonicity of $v(t)$,

$$\begin{aligned} J_2 = \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} v(t)dt &\leq (n+1)^{-1} v(\pi) \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt \\ &\leq \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt \\ &= O(J_4). \end{aligned} \tag{26}$$

Now using $\left(\frac{\omega(t)}{v(t)}\right)$ is positive and non-decreasing, we have

$$\begin{aligned} J_4 = \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt &\geq \left(\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \int_{\frac{1}{n+1}}^{\pi} t^{\beta-1} dt \\ &= \left(\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right) \frac{1}{(n+1)^{\beta}} \\ &= (n+1)^{-\beta} \frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}. \end{aligned} \tag{27}$$

Therefore

$$J_3 = O(J_4). \tag{28}$$

Now combining (24) and (28), we get

$$\|(\tilde{\mathcal{L}}_n(\cdot)) \sin^{\beta}(\cdot)\|_r = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt\right). \tag{29}$$

Hence,

$$E_n(\tilde{f}) = \inf_n \|(\tilde{\mathcal{L}}_n(\cdot)) \sin^{\beta}(\cdot)\|_r^{(v)} = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt\right). \tag{30}$$

This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

Following the proof of Theorem 1, we have

$$E_n(\tilde{f}) = O\left(\int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-1}\omega(t)}{v(t)} dt\right). \tag{31}$$

From our assumption that $\left(\frac{\omega(t)}{tv(t)}\right)$ is positive and non-increasing with t, we have

$$
\begin{aligned}
E_n(\tilde{f}) &= O\left((n+1)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\int_{\frac{1}{n+1}}^{\pi} t^{\beta}dt\right) \\
&= O\left((n+1)\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\left(\frac{t^{\beta+1}}{\beta+1}\right)_{\frac{1}{n+1}}^{\pi}\right) \\
&= O\left((n+1)^{-\beta}\frac{\omega(\frac{1}{n+1})}{v(\frac{1}{n+1})}\right). \qquad (32)
\end{aligned}
$$

This completes the proof of Theorem 2.

4 Concluding Remarks

If we put $\beta = 0$ in $W(Z_r^{(\omega)})$ class, then it reduces to $Z_r^{(\omega)}$ class; next, taking $r \to \infty$, the $Z_r^{(\omega)}$ class reduces to $Z^{(\omega)}$ class. Further, putting $\omega(t) = t^{\alpha}$ in $Z_r^{(\omega)}$ class, it reduces to $Z_{(\alpha),r}$ class. Finally, if we put $\omega(t) = t^{\alpha}$, then $Z^{(\omega)}$ class reduces to $Z_{(\alpha)}$ class. Thus, the results established here generalizes several known results.

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A Fuzzy Inventory Model of Defective Items Under the Effect of Inflation with Trade Credit Financing

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Abstract. The most common problem arises in inventory system is the defective items, which may arise due to improper inspection or due to mishandling during the transportation from supplier to retailer. This paper develops a fuzzy inventory model with defective items under the effect of inflation and the time value of money, where the demand is a function of selling price and advertisement cost. Moreover, to keep the faith of the customers, the retailer either reworks the defective items before they sold or refund the amount with penalties if they reached the customers. Also, the model is considered here with finite replenishment rate under trade credit financing scheme. The objective of the model is to obtain minimum total inventory cost by finding the optimal cycle length and the optimal order quantity. Furthermore, as a particular case, the results of the perfect system (i.e., the system without defective items) are obtained. Finally, the optimal solution of the model is illustrated with the help of numerical example. The effect of parameters are also considered and presented in the form of tables.

Keywords: Inventory model · Finite replenishment · Inflation · Trade credit financing · Graded mean integration method

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1 Introduction

Almost in all business sectors the quality of items affects their demand and indirectly affects the profit, earn, the growth of their share value, business expansion. Earlier most of the researchers gave their attention towards inventory models concerned for deteriorating items or perfect items. But it is an unrealistic

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assumption for inventory of most of the products available in the market which are longer life span than business cycle or imperfect quality or no deterioration but may contain a fraction of defectiveness. For example, fashionable items, decorative items, luxury items, electronic items etc. The defective items or imperfect quality items may arises in the inventory system due to several reasons, in the production inventory, it occurs due to deficiency in number of labors, different type of machinery problems, availability of quality raw materials, etc., in other inventories it occurs due to improper inspection at the time of purchase, poor quality of packaging, mishandling in transportation or inadequacy of facilities at inventory house etc.

Initially in 1986, Rosenblatt and Lee [15] considered the imperfectness in the production cycle and defective items are repaired or reworked instantly with extra cost if produces. In the same year, Porteus [14] discussed the occurrence of defective items and solution to improve the quality production in possible ways. Later on, several researchers contributed their work towards inventory models having defective items or imperfect quality by considering more inventory parameters.

In the context of defective or imperfect items, recently many researchers extended their work, some of them are Mandal et al. [11], Sarkar and Moon [17], Sarkar et al. [18], Sarkar et al. [16], Taheri-Tolgari et al. [22], Soni and Patel [20], Mahata and Goswami [10], Uthayakumar and Palanivel [24], Jagi et al. [8], Das et al. [3].

Inflation plays a very interesting and significant role in demand and price rate of particular items, then it affects the optimal ordering policy. Due to high inflation rate, many developing countries financial situation has changed. As inflation increases, the value of money goes down and erodes the future worth of saving and forces one for more current spending. Usually, these spending is on peripherals and luxury items that give rise to the demand of those items. We may mark that, inflation mostly effects in the petroleum products. As a result, we can't ignore its importance when considering inventory of such type of goods.

The effect of inflation was first introduced by Buzacott [1] in his EOQ model for inventory system in the year 1975. Later, many researchers developed inventory models by considering the effect of inflation. In recent years the following researchers considered the effect of inflation in their model. They are Sarkar and Moon [17], Sarkar et al. [16], Mishra et al. [12] and [13], Taheri-Tolgari [22], Yang and Chang [25], Uthayakumar and Palanivel [24], Hossen et al. [6], Tiwari et al. [23], Singh and Kumar [19], Chakraborty et al. [2].

In the present global scenario, traders facing heavy competition, to overcome this they are applying many strategies. Among all allowing trade credit financing or delay in payment plays an important role, that is, by facilitating trade credit to retailers, traders can increase their sales, attract new customers, reduces on-hand stock level and earn interest if the retailer failed to clear the due amount within the permissible delay period. For retailers, trade credit financing allows to purchase more items, acts as an alternative, discount, investment capital becomes less, it is an alternative to getting finance from banking and finance companies.

Also retailer can earn some interest during the credit period, so it reduces the inventory cost. There are several types of trade credit financing facilities offered by the supplier to retailer and retailer to customer. The supplier can offer the trade credit depends on order quantity by retailer. Some suppliers offer partial trade credit, i.e. at the time of purchase retailer can pay partial amount and remaining at the end of the offered period. Some other suppliers offer two level trade credit, i.e. within the first delay period, no interest is charged, after the first delay period to second delay period some minimal interest is charged, after second delay period heavy interest is charged by the supplier.

Haley and Higgins [5] for the first time introduce the concept of trade credit financing in their inventory model, then Goyal [4] developed an EOQ model by incorporating trade credit policy. In subsequent years, many researchers considered the concept of delay in payment. Recently, the following authors Sarkar et al. [18], Hardik et al. [20], Uthayakumar and Palanivel [24], Sujatha and Parvathi [21], Tiwari et al. [23], Das et al. [3], Singh and Kumar [19], and Dipankar et al. [2] are considered trade credit financing in their models.

The costs associated with inventory are considered constant in generally by most of the researchers. But in a practical scenario, they are having vagueness and imprecision in nature. Such as the selling price, advertisement cost, holding cost, interest rates and more. For example, selling price may vary depending upon the number of items by each customer. To deal with this type of situation, Fuzzy set theory is practically more significant rather than probability theory. Because Fuzzy theory deals with vagueness, imprecision whereas probability theory is best suitable for uncertainty occurs due to randomness. Zadeh [26] introduce the concept of fuzzy set theory. Initially, Lee and Yao [9] incorporated the fuzziness in their model. Following Lee and Yao, many researchers extended their work in the fuzzy environment. Recently Mandal et al. [11], Mahata and Goswami [10], Sujata and Parvati [21], Hossen et al. [6], Indrajitsingha et al. [7] obtained optimal results for their inventory problems in fuzzy environment.

In this article, considering the importance of fuzziness, inflation, trade credit financing, we developed both Crisp and fuzzy inventory models intending for defective items under the trade credit financing facility. Also the models incorporated the demand depends on selling price and advertisement cost, the effect of inflation and constant rate of supply with finite replenishment, two different scenarios. The model is intended to obtain a minimal total cost, optimal cycle length of the inventory system. So, we formulated a mathematical solution. Numerical examples are provided to validate the proposed problem and its solution. Finally the effect of change in value of parameters is summarized in the sensitivity analysis section.

2 Notations and Assumptions

2.1 Notations

Following Notations are considered for developing the model.

τ = Duration of the constant rate of supply provided to inventory.
T = Time at which inventory cycle ends.
$I_1(t)$ = Function of inventory level at time t, $0 \leq t \leq \tau$.
$I_2(t)$ = Function of inventory level at time t, $\tau \leq t \leq T$.
N = The constant supply rate of finished goods by the supplier.
N_1 = Total number of defective items.
ϵ = The scaling parameter for defective items, where defective items per unit time = ϵN^{σ}, $\epsilon > 0$ and $0 < \sigma \leq 1$.
W = The order size per cycle for both the cases.
$\mathcal{C}$ = Ordering cost per cycle.
$\mathcal{H}$ = Holding cost per unit item per unit time.
$\mathcal{P}$ = Purchasing cost per unit.
$\mathcal{C}_D$ = Cost per unit defective item against rework or refund as the case $(i.e,\ \mathcal{C}_D = \mathcal{C}_{rw}\ or\ \mathcal{C}_D = \mathcal{C}_{rf})$.
$\mathcal{C}_{rw}$ = Rework cost of unit defective item.
$\mathcal{C}_{rf}$ = Refund cost of unit defective item including penalty.
$\mathcal{M}_1$ = The maximum trade credit period with no interest being charged.
$\mathcal{M}_2$ = Which is greater than $\mathcal{M}_1$ and is maximum trade credit period with less interest rate is applicable.
θ_1 = Interest rate chargeable on stock in inventory by supplier during $[\mathcal{M}_1, \mathcal{M}_2]$.
θ_2 = Interest rate chargeable on stock in inventory by supplier during $[\mathcal{M}_2, T]$.
ϑ = Interest rate earned on amount accumulated by selling the items during the trade credit period.
ξ = The inflation rate.
d = Discount rate on items. Which represents the time value of money.
R = Net discount rate of inflation = d-ξ
$Z(T)$ = Total cost of the inventory system.
$\mathbb{D}$ = Demand function.
s = Selling price unit item.
A = Advertisement Cost.
$\tilde{s}$ = Fuzzy selling price per unit item.
$\tilde{A}$ = Fuzzy advertisement cost.
$\tilde{\mathbb{D}}$ = Fuzzy demand.
$\tilde{\theta_1}$ = Fuzzy interest rate chargeable on stock in inventory by supplier during $[\mathcal{M}_1, \mathcal{M}_2]$
$\tilde{\theta_2}$ = Fuzzy interest rate chargeable on stock in inventory by supplier during $[\mathcal{M}_2, T]$.
$\tilde{\vartheta}$ = Fuzzy interest rate earned on amount accumulated by selling the items during the trade credit period.

2.2 Assumptions

The following assumptions are made to develop the mathematical model.

(i) Single item is considered over an infinite planing horizon.
(ii) $\mathbb{D} = A^{\zeta}\alpha s^{-\beta}, \alpha > 0, \beta > 1, 0 \leq \zeta < 1$. Here α is a scaling parameter, β is the index of price elasticity, and ζ is the shape parameter.
(iii) The replenishment takes place at finite rate.
(iv) Supplier offers the trade credit financing to retailer.
(v) No shortages are arises during the cycle.
(vi) The effects of inflation and time value of money are considered.
(vii) The lead time is zero.
(viii) The defective items are considered.

3 Mathematical Model

In practical scenario, the goodwill of customers depends on timely service and quality of items provided by retailer. But, sometimes the damaged (defective) items reached to customers unknowingly. In such situation, on report of defectiveness of items by customers, retailer either rework and return those or refund an amount with penalty to the customer. So, here we consider an inventory model for defective items by considering these two scenarios.

In both the scenarios, we assumed that W units are inserted into the inventory during the cycle. These W units includes N_1 defective items. All these items are sold to the customer at a rate of $\mathbb{D}$ units per unit time as good units. Later on, customers report the defectiveness of items to the retailer. So,

Scenario 1. In scenario 1, retailer refund an amount $\mathcal{C}_{rf}$ per unit item including penalty to the customer against defective item.

Scenario 2. In scenario 2, retailer rework the defective items with a cost $\mathcal{C}_{rw}$ per unit and return them to customer.

For both the scenarios, entire cycle $[0, T]$ is considered as two intervals $[0, \tau]$, $[\tau, T]$ and the inventory starts with zero items. In the interval $[0, \tau]$, the supplier supplies the items with a constant rate N units per unit time and the demand rate of items is $\mathbb{D}$. So, in the interval $[0, \tau]$, the inventory level rises with a rate N units and is depletes $\mathbb{D}$ units per unit time simultaneously (see Fig. 1). Thus, the inventory level at any time in the interval $[0, \tau]$ is

$$\begin{aligned}
&I_1(t + \Delta t) = I_1(t) + N\Delta t - \mathbb{D}\Delta t \\
\Rightarrow\ &I_1(t + \Delta t) - I_1(t) = N\Delta t - \mathbb{D}\Delta t \\
\Rightarrow\ &\frac{I_1(t + \Delta t) - I_1(t)}{\Delta t} = N - \mathbb{D} \\
\Rightarrow\ &\lim_{\Delta t \to 0} \frac{I_1(t + \Delta t) - I_1(t)}{\Delta t} = N - \mathbb{D}
\end{aligned}$$

$$\Rightarrow \frac{dI_1(t)}{dt} = N - \mathbb{D}, \quad 0 \leq t \leq \tau, \quad (I_1(0) = 0). \tag{1}$$

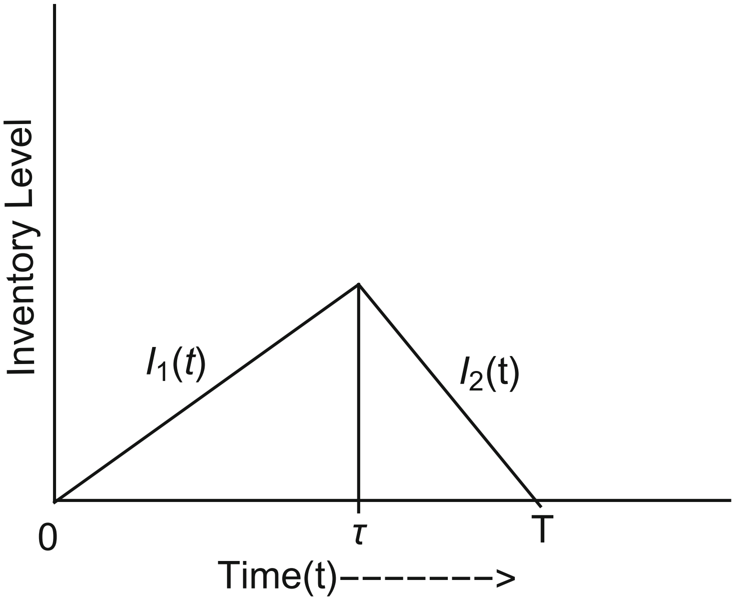

Fig. 1. Inventory level during the cycle

Solving Eq. (1), we have

$$I_1(t) = (N - \mathbb{D})t, \quad 0 \leq t \leq \tau. \tag{2}$$

Similarly, in the interval $[\tau, T]$, there is no supply of items by suppliers. But the demand rate of items is $\mathbb{D}$. So, the inventory level only depletes due to demand in the interval $[\tau, T]$, which finally falls to zero at time T (see Fig. 1) and the inventory level at any time t during the interval $[\tau, T]$ is

$$\frac{dI_2(t)}{dt} = -\mathbb{D}, \quad \tau \leq t \leq T, \quad (I_2(T) = 0). \tag{3}$$

It's solution is

$$I_2(t) = \mathbb{D}(T - t), \quad \tau \leq t \leq T. \tag{4}$$

At $t = \tau$, both $I_1(t)$ and $I_2(t)$ represents the same inventory level. So, using $t = \tau$ in Eqs. (2), (4) and solving for τ we get,

$$\tau = \frac{\mathbb{D}T}{N}. \tag{5}$$

As N is the constant rate of supply per unit time to the inventory during the interval $[0, \tau]$, the total order size can be placed at beginning of the cycle is

$$W = \int_0^{\tau} N \, dt = N\tau$$

Also the inventory is added with defective items with a rate ϵN^σ per unit time, then the total number of defective items may arises for both the scenarios is

$$N_1 = \int_0^\tau \epsilon N^\sigma \, dt = \epsilon N^\sigma \tau$$

Now the different inventory costs under the effect of inflation for both the scenarios are:

(i) Ordering Cost: Total ordering cost for both the scenarios-1 and scenario-2 is given by $OC = \mathcal{C}$

(ii) Purchase Cost: Total purchase cost for both the scenarios-1 and scenario-2 is given by

$$PC = \int_0^\tau \mathcal{P} N e^{-Rt} \, dt = \frac{\mathcal{P}N}{R}[1 - e^{-Rt}]$$

(iii) Holding Cost: Total Inventory Holding cost for both scenarios-1 and scenario-2 is given by

$$\begin{aligned} HC &= \mathcal{H}[\int_0^\tau I_1(t)e^{-Rt} \, dt + \int_\tau^T I_2(t)e^{-Rt} \, dt] \\ &= \frac{\mathcal{H}}{R^2}[\mathbb{D}e^{-R(T+\tau)}(e^{R\tau} + e^{RT}(R(T-\tau) - 1)) \\ &- e^{-R\tau}(\mathbb{D} - N)(e^{R\tau} - R\tau - 1)] \end{aligned}$$

(iv) Defective Cost: Total cost due to defectiveness of items.

$$DC = \mathcal{C}_D \int_0^\tau \epsilon N^\sigma e^{-Rt} \, dt = \frac{\mathcal{C}_D \epsilon N^\sigma}{R}[1 - e^{-R\tau}]$$

Here $\mathcal{C}_D = \mathcal{C}_{rf}$ for scenario-1 and $\mathcal{C}_D = \mathcal{C}_{rw}$ for scenario-2.

Trade Credit Financing Facility. In this model, we consider that the supplier offers two trade credit periods $\mathcal{M}_1, \mathcal{M}_2\ (> \mathcal{M}_1)$ to retailer for both the scenarios and retailer can pay the amount to supplier depending upon sale of the stock. If any stock is left in the inventory for sale, retailer pay the remaining amount with an interest rate as fixed by the supplier at the end of cycle T.

Case 1. If the retailer opt to pay the purchasing amount at the end of $\mathcal{M}_2\ (\leq T)$, then the supplier charge an interest rate θ_2 on the unpaid amount.

Case 2. If the retailer opt to pay the purchasing amount during the interval $[\mathcal{M}_1, T]$ (where $\mathcal{M}_1 \leq T \leq \mathcal{M}_2$), then the supplier charge an interest rate θ_1 on the unpaid amount.

Case 3. If the retailer opt to pay the purchasing amount at the end of the cycle T (where $T \leq \mathcal{M}_1$), then the retailer is not chargeable by any interest.

(v) Interest payable or earned by retailer:
Depending upon the choice of the retailer, the total interest payable or earned;

Case 1. If the retailer choose to pay the purchasing amount at the end of $\mathcal{M}_2(\leq T)$, then

$$
\begin{aligned}
I_{p1} &= \mathcal{P}\theta_2 \int_{\mathcal{M}_2}^{T} I_2(t)e^{-Rt}\,dt \\
&= \frac{\mathcal{P}\theta_2\mathbb{D}}{R^2} e^{-R(\mathcal{M}_2+T)}(e^{\mathcal{M}_2 T} + e^{RT}(RT - \mathcal{M}_2 R - 1)) \\
I_{e1} &= s\vartheta[\int_0^{\tau} (\tau - t)\mathbb{D}e^{-Rt}\,dt + \int_{\tau}^{\mathcal{M}_2} (T - t)\mathbb{D}e^{-Rt}\,dt] \\
&= \frac{s\vartheta\mathbb{D}}{R^2}\{e^{-R(\mathcal{M}_2+\tau)}e^{R\tau}(\mathcal{M}_2 R - RT + 1) + e^{\mathcal{M}_2 R}(R(T-\tau) - 1) \\
&\quad - \left(R\tau + e^{-R\tau} - 1\right)\}
\end{aligned}
$$

Case 2. If the retailer choose to pay the purchasing amount during the interval $[\mathcal{M}_1, T]$ (where $\mathcal{M}_1 \leq T \leq \mathcal{M}_2$), then

$$
\begin{aligned}
I_{p2} &= \mathcal{P}\theta_1 \int_{\mathcal{M}_1}^{T} I_2(t)e^{-Rt}\,dt \\
&= \frac{\mathcal{P}\theta_1\mathbb{D}}{R^2} e^{-R(\mathcal{M}_1+T)}(e^{\mathcal{M}_1 T} + e^{RT}(RT - \mathcal{M}_1 R - 1)) \\
I_{e2} &= s\vartheta[\int_0^{\tau} (\tau - t)\mathbb{D}e^{-Rt}\,dt + \int_{\tau}^{T} (T - t)\mathbb{D}e^{-Rt}\,dt] \\
&= \frac{s\vartheta\mathbb{D}}{R^2}\{e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}(R(T-\tau) - 1)\right) + (R\tau + e^{-R\tau} - 1)\}
\end{aligned}
$$

Case 3. If the retailer choose to pay the purchasing amount at the end of the cycle T (where $T \leq \mathcal{M}_1$), then

$$
\begin{aligned}
I_{p3} &= 0 \\
I_{e3} &= s\vartheta[\int_0^{\tau} (\tau - t)\mathbb{D}e^{-Rt}\,dt + \int_{\tau}^{T} (T - t)\mathbb{D}e^{-Rt}\,dt + \int_{T}^{\mathcal{M}_1} N\tau e^{-Rt}\,dt] \\
&= \frac{s\vartheta\mathbb{D}}{R^2}\{e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}(R(T-\tau) - 1)\right) + (R\tau + e^{-R\tau} - 1)\} \\
&\quad + \frac{s\vartheta}{R}N\tau\left(e^{-RT} - e^{-\mathcal{M}_1 R}\right).
\end{aligned}
$$

Now the total average cost per unit time $= \frac{1}{T}$ (Ordering Cost + Purchase Cost + Holding Cost + Defective Cost + Interest Payable-Interest Earned).

$$
\text{i.e.} \qquad Z_i(T) = \begin{cases} Z_{i1}(T), & \mathcal{M}_2(\leq T) \\ Z_{i2}(T), & \mathcal{M}_1 \leq T \leq \mathcal{M}_2 \\ Z_{i3}(T), & T \leq \mathcal{M}_1 \end{cases}
$$

Here, $Z_i(T)$ be the total average cost of i^{th}-scenario, where $i = 1, 2$. Also, defective cost(DC) is the refund cost ($\mathcal{C}_{rf}$) for Scenario-1 and rework cost ($\mathcal{C}_{rw}$) for scenario-2 for each case. Now the total average costs of different cases under each scenario-i ($i = 1, 2$) are as follows:

Case 1.

$$\begin{aligned}
Z_{i1}(T) &= \frac{1}{T}(OC + PC + HC + DC + I_{p1} - I_{e1})\\
&= \frac{1}{T}\Bigg\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Big[\mathbb{D}e^{-R(T+\tau)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau) - 1\big)\Big)\\
&- e^{-R\tau}(\mathbb{D} - N)(e^{R\tau} - R\tau - 1)\Big] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]\\
&+ \frac{\mathcal{P}\theta_2\mathbb{D}}{R^2}\Big[e^{-R(\mathcal{M}_2+T)}\big(e^{\mathcal{M}_2T} + e^{RT}(RT - \mathcal{M}_2R - 1)\big)\Big]\\
&- \frac{s\vartheta\mathbb{D}}{R^2}\Big[e^{-R(\mathcal{M}_2+\tau)}e^{R\tau}(\mathcal{M}_2R - RT + 1) + e^{\mathcal{M}_2R}\big(R(T-\tau) - 1\big)\\
&- \big(R\tau + e^{-R\tau} - 1\big)\Big]\Bigg\}
\end{aligned}$$

Case 2.

$$\begin{aligned}
Z_{i2}(T) &= \frac{1}{T}(OC + PC + HC + DC + I_{p2} - I_{e2})\\
&= \frac{1}{T}\Bigg\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Big[\mathbb{D}e^{-R(T+\tau)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau) - 1\big)\Big)\\
&- e^{-R\tau}(\mathbb{D} - N)(e^{R\tau} - R\tau - 1)\Big] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]\\
&+ \frac{\mathcal{P}\theta_1\mathbb{D}}{R^2}\Big[e^{-R(\mathcal{M}_1+T)}\big(e^{\mathcal{M}_1T} + e^{RT}(RT - \mathcal{M}_1R - 1)\big)\Big]\\
&- \frac{s\vartheta\mathbb{D}}{R^2}\Big[e^{-R(\tau+T)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau) - 1\big)\Big) + (R\tau + e^{-R\tau} - 1)\Big]\Bigg\}.
\end{aligned}$$

Case 3.

$$\begin{aligned}
Z_{i3}(T) &= \frac{1}{T}(OC + PC + HC + DC + I_{p3} - I_{e3})\\
&= \frac{1}{T}\Bigg\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Big[\mathbb{D}e^{-R(T+\tau)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau) - 1\big)\Big)\\
&- e^{-R\tau}(\mathbb{D} - N)(e^{R\tau} - R\tau - 1)\Big] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]\\
&- \frac{s\vartheta\mathbb{D}}{R^2}\Big[e^{-R(\tau+T)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau) - 1\big)\Big)\\
&+ (R\tau + e^{-R\tau} - 1)\Big] - \frac{s\vartheta}{R}N\tau\left(e^{-RT} - e^{-\mathcal{M}_1R}\right)\Bigg\}.
\end{aligned}$$

For $i = 1$, we take $\mathcal{C}_D = \mathcal{C}_{rf}$ and the corresponding costs for scenario-1. Similarly, for $i = 2$, we take $\mathcal{C}_D = \mathcal{C}_{rw}$ and the corresponding costs for scenario-2.

4 Solution Procedure

The best minimal total average cost can be obtained in both the scenarios by obtaining the optimal cycle length. This can be done as follow.

Step 1. Find $\frac{dZ_{ij}(T)}{dT}$ and $\frac{d^2Z_{ij}(T)}{dT^2}$ for $i = 1, 2$ & $j = 1, 2, 3$.

Step 2. The necessary and sufficient conditions for $Z_{ij}(T)$ is minimum are $\frac{dZ_{ij}(T)}{dT} = 0$ and $\frac{d^2Z_{ij}(T)}{dT^2} > 0$. So find T_{ij} such that $\frac{dZ_{ij}(T_{ij})}{dT} = 0$ and $\frac{d^2Z_{ij}(T_{ij})}{dT^2} > 0$ for each $i = 1, 2$ & $j = 1, 2, 3$.

Step 3. Now $Z_i^*(T) = min\{Z_{ij}(T_{ij}) : \mathcal{M}_2 \le T_{i1}, \mathcal{M}_1 \le T_{i2}, T_{i3} \le \mathcal{M}_1\}$ is the best minimum total cost for scenario-i ($i = 1, 2$). The corresponding T_{ij} is the best optimal cycle length for scenario-i ($i = 1, 2$).

5 Fuzzy Model

In real market situations, most of the costs are vagueness or imprecision in nature. So here we consider (i) selling price s as fuzzy; because some times it varies depending on the number of units purchased by customer (ii) advertisement cost A as fuzzy; because it varies depends upon the competition in the market and (iii) interest costs θ_1, θ_2 and ϑ are also fuzzy in nature.

Then, considering $\tilde{s}$, $\tilde{A}$ $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\vartheta}$ as the fuzzy parameters (remaining parameters are same as in crisp model). Taking all these parameters and proceeding in the similar lines of crisp model, we get the fuzzy demand as $\tilde{\mathbb{D}} = \tilde{A}^{\zeta}\alpha\tilde{s}^{-\beta}$, and the total fuzzy cost becomes

$$\text{i.e.} \qquad \widetilde{Z_i}(T) = \begin{cases} \widetilde{Z_{i1}}(T), & \mathcal{M}_2(\le T) \\ \widetilde{Z_{i2}}(T), & \mathcal{M}_1 \le T \le \mathcal{M}_2 \\ \widetilde{Z_{i3}}(T), & T \le \mathcal{M}_1 \end{cases}$$

Here, $\widetilde{Z_i}(T)$ represents the total fuzzy average cost of i^{th}-scenario. Now the total fuzzy average costs of different cases under each scenario-i ($i = 1, 2$) are as follow.

Case 1.

$$\begin{aligned}
\widetilde{Z_{i1}}(T) &= \frac{1}{T}(OC + PC + HC + DC + I_{p1} - I_{e1}) \\
&= \frac{1}{T}\Bigg\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Bigg[\tilde{\mathbb{D}}e^{-R(T+\tau)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau) - 1\big)\Big) \\
&\quad - e^{-R\tau}(\tilde{\mathbb{D}} - N)(e^{R\tau} - R\tau - 1)\Bigg] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^{\sigma}}{R}[1 - e^{-R\tau}] \\
&\quad + \frac{\mathcal{P}\tilde{\theta}_2\tilde{\mathbb{D}}}{R^2}\Big[e^{-R(\mathcal{M}_2+T)}\big(e^{\mathcal{M}_2T} + e^{RT}(RT - \mathcal{M}_2R - 1)\big)\Big] \\
&\quad - \frac{\tilde{s}\tilde{\vartheta}\tilde{\mathbb{D}}}{R^2}\Big[e^{-R(\mathcal{M}_2+\tau)}e^{R\tau}(\mathcal{M}_2R - RT + 1) + e^{\mathcal{M}_2R}\big(R(T-\tau) - 1\big) \\
&\quad - \big(R\tau + e^{-R\tau} - 1\big)\Big]\Bigg\}
\end{aligned}$$

Case 2.

$$\begin{aligned}
\widetilde{Z_{i2}}(T) &= \frac{1}{T}(OC + PC + HC + DC + I_{p2} - I_{e2}) \\
&= \frac{1}{T}\Bigg\{ \frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Bigg[\tilde{\mathbb{D}}e^{-R(T+\tau)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau)-1\big)\Big) \\
&\quad - e^{-R\tau}(\tilde{\mathbb{D}} - N)(e^{R\tau} - R\tau - 1)\Bigg] + \mathcal{C} + \frac{\mathcal{C}_D \epsilon N^{\sigma}}{R}[1 - e^{-R\tau}] \\
&\quad + \frac{\mathcal{P}\tilde{\theta}_1\tilde{\mathbb{D}}}{R^2}\Big[e^{-R(\mathcal{M}_1+T)}\big(e^{\mathcal{M}_1 T} + e^{RT}(RT - \mathcal{M}_1 R - 1)\big)\Big] \\
&\quad - \frac{\tilde{s}\tilde{\vartheta}\tilde{\mathbb{D}}}{R^2}\Bigg[e^{-R(\tau+T)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau)-1\big)\Big) + (R\tau + e^{-R\tau} - 1)\Bigg]\Bigg\}
\end{aligned}$$

Case 3.

$$\begin{aligned}
\widetilde{Z_{i3}}(T) &= \frac{1}{T}(OC + PC + HC + DC + I_{p3} - I_{e3}) \\
&= \frac{1}{T}\Bigg\{ \frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Bigg[\tilde{\mathbb{D}}e^{-R(T+\tau)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau)-1\big)\Big) \\
&\quad - e^{-R\tau}(\tilde{\mathbb{D}} - N)(e^{R\tau} - R\tau - 1)\Bigg] + \mathcal{C} + \frac{\mathcal{C}_D \epsilon N^{\sigma}}{R}[1 - e^{-R\tau}] \\
&\quad - \frac{\tilde{s}\tilde{\vartheta}\tilde{\mathbb{D}}}{R^2}\Bigg[e^{-R(\tau+T)}\Big(e^{R\tau} + e^{RT}\big(R(T-\tau)-1\big)\Big) \\
&\quad + (R\tau + e^{-R\tau} - 1)\Bigg] - \frac{\tilde{s}\tilde{\vartheta}}{R}N\tau\left(e^{-RT} - e^{-\mathcal{M}_1 R}\right)\Bigg\}.
\end{aligned}$$

For $i = 1$, we take $\mathcal{C}_D = \mathcal{C}_{rf}$ and the corresponding costs for scenario-1. Similarly for $i = 2$, we take $\mathcal{C}_D = \mathcal{C}_{rw}$ and the corresponding costs for scenario-2.

Now considering the fuzzy parameters as fuzzy triangular numbers, that is, $\widetilde{s} = (s1, s2, s3)$, $\widetilde{A} = (A1, A2, A3)$, $\widetilde{\theta_1} = (\theta_1 1, \theta_1 2, \theta_1 3)$, $\widetilde{\theta_2} = (\theta_2 1, \theta_2 2, \theta_2 3)$, $\widetilde{\vartheta} = (\vartheta 1, \vartheta 2, \vartheta 3)$ and defuzzified by *Graded Mean Integration Representation (GMIR) Method*, we obtain the total average costs.

$$GZ_{ij}(T) = \frac{1}{6}[\widetilde{Z^1_{ij}}(T) + 4\widetilde{Z^2_{ij}}(T) + \widetilde{Z^3_{ij}}(T)] \quad \text{for each } i = 1, 2 \;\;\&\;\; j = 1, 2, 3.$$

Here $GZ_{ij}(T)$ is the defuzzified total average cost of i^{th}-scenario and j^{th}-case ($i = 1, 2$ & $j = 1, 2, 3$). The $\widetilde{Z^k_{ij}}(T)$ is obtained by replacing the fuzzy parameters in $\widetilde{Z_{ij}}(T)$ by k^{th} fuzzy number of the corresponding triangular fuzzy number. So, the defuzzified total average costs of scenario-i ($i = 1, 2$) as

$$\text{i.e.} \qquad GZ_i(T) = \begin{cases} GZ_{i1}(T), & \mathcal{M}_2 (\leq T) \\ GZ_{i2}(T), & \mathcal{M}_1 \leq T \leq \mathcal{M}_2 \\ GZ_{i3}(T), & T \leq \mathcal{M}_1 \end{cases}$$

For each scenario, the defuzzified total average costs are

Case 1.

$$
\begin{aligned}
GZ_{i1}(T) = \frac{1}{6T}\Bigg\{\Bigg\{ & \frac{\mathcal{P}N}{R}[1-e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Big[\mathbb{D}1e^{-R(T+\tau)}\Big(e^{R\tau}+e^{RT}\big(R(T-\tau)-1\big)\Big) \\
& - e^{-R\tau}(\mathbb{D}1-N)(e^{R\tau}-R\tau-1)\Big] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^{\sigma}}{R}[1-e^{-R\tau}] \\
& + \frac{\mathcal{P}\theta_2 1\mathbb{D}1}{R^2}\Big[e^{-R(\mathcal{M}_2+T)}\big(e^{\mathcal{M}_2T}+e^{RT}(RT-\mathcal{M}_2R-1)\big)\Big] \\
& - \frac{s1\vartheta1\mathbb{D}1}{R^2}\Big[e^{-R(\mathcal{M}_2+\tau)}e^{R\tau}(\mathcal{M}_2R-RT+1)+e^{\mathcal{M}_2R}\big(R(T-\tau)-1\big) \\
& - \Big(R\tau+e^{-R\tau}-1\Big)\Big]\Bigg\} + 4\Bigg\{\frac{\mathcal{H}}{R^2}\Big[\mathbb{D}2e^{-R(T+\tau)}\Big(e^{R\tau}+e^{RT}\big(R(T-\tau)-1\big)\Big) \\
& - e^{-R\tau}(\mathbb{D}2-N)(e^{R\tau}-R\tau-1)\Big] + \frac{\mathcal{P}N}{R}[1-e^{-Rt}] + \frac{\mathcal{C}_D\epsilon N^{\sigma}}{R}[1-e^{-R\tau}] \\
& + \mathcal{C} + \frac{\mathcal{P}\theta_2 2\mathbb{D}2}{R^2}\Big[e^{-R(\mathcal{M}_2+T)}\big(e^{\mathcal{M}_2T}+e^{RT}(RT-\mathcal{M}_2R-1)\big)\Big] \\
& - \frac{s2\vartheta2\mathbb{D}2}{R^2}\Big[e^{-R(\mathcal{M}_2+\tau)}e^{R\tau}(\mathcal{M}_2R-RT+1)+e^{\mathcal{M}_2R}\big(R(T-\tau)-1\big) \\
& - \Big(R\tau+e^{-R\tau}-1\Big)\Big]\Bigg\} + \Bigg\{\frac{\mathcal{H}}{R^2}\Big[\mathbb{D}3e^{-R(T+\tau)}\Big(e^{R\tau}+e^{RT}\big(R(T-\tau)-1\big)\Big) \\
& - e^{-R\tau}(\mathbb{D}3-N)(e^{R\tau}-R\tau-1)\Big] + \frac{\mathcal{P}N}{R}[1-e^{-Rt}] + \frac{\mathcal{C}_D\epsilon N^{\sigma}}{R}[1-e^{-R\tau}] \\
& + \mathcal{C} + \frac{\mathcal{P}\theta_2 3\mathbb{D}3}{R^2}\Big[e^{-R(\mathcal{M}_2+T)}\big(e^{\mathcal{M}_2T}+e^{RT}(RT-\mathcal{M}_2R-1)\big)\Big] \\
& - \frac{s3\vartheta3\mathbb{D}3}{R^2}\Big[e^{-R(\mathcal{M}_2+\tau)}e^{R\tau}(\mathcal{M}_2R-RT+1)+e^{\mathcal{M}_2R}\big(R(T-\tau)-1\big) \\
& - \Big(R\tau+e^{-R\tau}-1\Big)\Big]\Bigg\}\Bigg\}
\end{aligned}
$$

Case 2.

$$
\begin{aligned}
GZ_{i2}(T) = \frac{1}{6T}\Bigg\{\Bigg\{ & \frac{\mathcal{P}N}{R}[1-e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Big[\mathbb{D}1e^{-R(T+\tau)}\Big(e^{R\tau}+e^{RT}\big(R(T-\tau)-1\big)\Big) \\
& - e^{-R\tau}(\mathbb{D}1-N)(e^{R\tau}-R\tau-1)\Big] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^{\sigma}}{R}[1-e^{-R\tau}] \\
& + \frac{\mathcal{P}\theta_1 1\mathbb{D}1}{R^2}\Big[e^{-R(\mathcal{M}_1+T)}\big(e^{\mathcal{M}_1T}+e^{RT}(RT-\mathcal{M}_1R-1)\big)\Big] \\
& - \frac{s1\vartheta1\mathbb{D}1}{R^2}\Big[e^{-R(\tau+T)}\Big(e^{R\tau}+e^{RT}\big(R(T-\tau)-1\big)\Big)+(R\tau+e^{-R\tau}-1)\Big]\Bigg\} \\
& + 4\Bigg\{\frac{\mathcal{P}N}{R}[1-e^{-Rt}] + \frac{\mathcal{H}}{R^2}\Big[\mathbb{D}2e^{-R(T+\tau)}\Big(e^{R\tau}+e^{RT}\big(R(T-\tau)-1\big)\Big) \\
& - e^{-R\tau}(\mathbb{D}2-N)(e^{R\tau}-R\tau-1)\Big] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^{\sigma}}{R}[1-e^{-R\tau}]
\end{aligned}
$$

$$+ \frac{\mathcal{P}\theta_1 2\mathbb{D}2}{R^2}\left[e^{-R(\mathcal{M}_1+T)}\left(e^{\mathcal{M}_1 T} + e^{RT}(RT - \mathcal{M}_1 R - 1)\right)\right]$$
$$- \frac{s2\vartheta 2\mathbb{D}2}{R^2}\left[e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right) + (R\tau + e^{-R\tau} - 1)\right]\Bigg\}$$
$$+ \left\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\left[\mathbb{D}3e^{-R(T+\tau)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.\right.$$
$$\left. - e^{-R\tau}(\mathbb{D}3 - N)(e^{R\tau} - R\tau - 1)\right] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]$$
$$+ \frac{\mathcal{P}\theta_1 3\mathbb{D}3}{R^2}\left[e^{-R(\mathcal{M}_1+T)}\left(e^{\mathcal{M}_1 T} + e^{RT}(RT - \mathcal{M}_1 R - 1)\right)\right]$$
$$\left.\left. - \frac{s3\vartheta 3\mathbb{D}3}{R^2}\left[e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right) + (R\tau + e^{-R\tau} - 1)\right]\right\}\right\}$$

Case 3.

$$GZ_{i3}(T) = \frac{1}{6T}\left\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\left[\mathbb{D}1e^{-R(T+\tau)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.\right.$$
$$\left. - e^{-R\tau}(\mathbb{D}1 - N)(e^{R\tau} - R\tau - 1)\right] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]$$
$$- \frac{s1\vartheta 1\mathbb{D}1}{R^2}\left[e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.$$
$$\left.\left. + (R\tau + e^{-R\tau} - 1)\right] - \frac{s1\vartheta 1}{R}N\tau\left(e^{-RT} - e^{-\mathcal{M}_1 R}\right)\right\}$$
$$+ 4\left\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\left[\mathbb{D}2e^{-R(T+\tau)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.\right.$$
$$\left. - e^{-R\tau}(\mathbb{D}2 - N)(e^{R\tau} - R\tau - 1)\right] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]$$
$$- \frac{s2\vartheta 2\mathbb{D}2}{R^2}\left[e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.$$
$$\left.\left. + (R\tau + e^{-R\tau} - 1)\right] - \frac{s2\vartheta 2}{R}N\tau\left(e^{-RT} - e^{-\mathcal{M}_1 R}\right)\right\}$$
$$+ \left\{\frac{\mathcal{P}N}{R}[1 - e^{-Rt}] + \frac{\mathcal{H}}{R^2}\left[\mathbb{D}3e^{-R(T+\tau)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.\right.$$
$$\left. - e^{-R\tau}(\mathbb{D}3 - N)(e^{R\tau} - R\tau - 1)\right] + \mathcal{C} + \frac{\mathcal{C}_D\epsilon N^\sigma}{R}[1 - e^{-R\tau}]$$
$$- \frac{s3\vartheta 3\mathbb{D}3}{R^2}\left[e^{-R(\tau+T)}\left(e^{R\tau} + e^{RT}\left(R(T-\tau)-1\right)\right)\right.$$
$$\left.\left. + (R\tau + e^{-R\tau} - 1)\right] - \frac{s3\vartheta 3}{R}N\tau\left(e^{-RT} - e^{-\mathcal{M}_1 R}\right)\right\}$$

For $i = 1$, we take $\mathcal{C}_D = \mathcal{C}_{rf}$ and the corresponding costs for scenario-1. Similarly for $i = 2$, we take $\mathcal{C}_D = \mathcal{C}_{rw}$ and the corresponding costs for scenario-2.

Remark 1. The solution procedure for fuzzy model is same as crisp model.

Remark 2. In both crisp and fuzzy model, if the value of ϵ is zero, then it leads to zero defective items. Thus, the resulting model is suitable for non-defective items. Also, the case of non-defective items are illustrated in the following examples.

6 Numerical Examples

Example 1. Crisp Model. The values of parameters in the proposed model can be taken as follow: $A = 50$, $\alpha = 40000$, $\beta = 2.5908$, $s = 15$, $\zeta = 0.4$, $\mathcal{P} = 10$, $\mathcal{C}_{rw} = 5$, $\mathcal{C}_{rf} = 20$, $\mathcal{C} = 100$, $\mathcal{H} = 3$, $\vartheta = 0.15$, $\theta_1 = 0.18$, $\theta_2 = 0.2$, $R = 1$, $\mathcal{M}_1 = 1.142$, $\mathcal{M}_2 = 1.753$, $\epsilon = 0.08$, $\sigma = 0.8$, $N = 300$. *Solution.* Using the solution procedure of the model, the optimal solutions in both the scenarios are obtained as follow:

Scenario-1. In this scenario, defective cost $(\mathcal{C}_D)$ per unit item is taken as the refund amount $(\mathcal{C}_{rf})$. So the optimal solution is

$$\tau = 0.317767 \quad T = 0.555391 \quad W = 95.33 \quad N_1 = 2.43722 \quad Z_1^* = 1753.58$$

Scenario-2. In this scenario, defective cost$(\mathcal{C}_D)$ per unit item is taken as the rework cost $(\mathcal{C}_{rw})$. So the optimal solution is

$$\tau = 0.316836 \quad T = 0.553765 \quad W = 95.0509 \quad N_1 = 2.43009 \quad Z_2^* = 1688.79$$

Fuzzy Model. Here, we taken fuzzy parameters as triangular fuzzy numbers and remain parameters are same as crisp. I.e. $\alpha = 40000$, $\beta = 2.5908$, $\zeta = 0.4$, $\mathcal{P} = 10$, $\mathcal{C}_{rw} = 5$, $\mathcal{C}_{rf} = 20$, $\mathcal{C} = 100$, $\mathcal{H} = 3$, $R = 1$, $\mathcal{M}_1 = 1.142$, $\mathcal{M}_2 = 1.753$, $\epsilon = 0.08$, $\sigma = 0.8$, $N = 300$. $\widetilde{A} = (40, 50, 60)$,$\widetilde{s} = (12, 15, 18)$, $\widetilde{\vartheta} = (0.13, 0.15, 0.17)$, $\widetilde{\theta_1} = (0.16, 0.18, 0.2)$, $\widetilde{\theta_2} = (0.18, 0.2, 0.22)$

Solution. Applying Graded Mean Integration Representation Method to defuzzify total average cost and using the solution procedure, we obtain the optimal results in both the scenarios as follow.

Scenario-1. In this scenario, defective cost$(\mathcal{C}_D)$ per unit item is taken as the refund amount $(\mathcal{C}_{rf})$. So the optimal solution is

$$\tau = 1.27618 \quad T = 2.1239 \quad W = 382.854 \quad N_1 = 9.78812 \quad GZ_1^* = 1798.54$$

Scenario-2. In this scenario, defective cost$(\mathcal{C}_D)$ per unit item is taken as the rework cost $(\mathcal{C}_{rw})$. So the optimal solution is

$$\tau = 1.27053 \quad T = 2.11448 \quad W = 381.158 \quad N_1 = 9.74475 \quad GZ_2^* = 1733.92$$

Example 2. Taking the values of all parameters same as in Example-1 except that of ϵ and let $\epsilon = 0$ in both scenarios. Then $N1 = 0$, i.e. no defective items, no defective cost $(DC = 0)$. Hence both the scenarios are identical and the resulting problem is for perfect(non-defective) items.

Solution

Crisp Model. The optimal solution is

$$\tau = 0.316528 \quad T = 0.553226 \quad W = 94.9584 \quad N1 = 0. \quad Z_1 = 1667.19$$

Fuzzy Model. The optimal solution is

$$\tau = 1.26866 \quad T = 2.11138 \quad W = 380.597 \quad N_1 = 0. \quad GZ_1 = 1712.37$$

Remark 3. In crisp model, the results obtained are inherent with vagueness. So, practically these results are not suitable. But the result obtained in fuzzy model are practically suitable as the vagueness of different parameters considered while developing the model.

7 Sensitivity Analysis

To study the sensitivity of different parameters involved in the model, we took the Example 1 and the results are summarized as follows (see Table 1):

Table 1. Sensitivity of different parameters involved in the inventory

Parameter	$Value$	T	GZ_1	Parameter	$Value$	T	GZ_1
α	25000	0.650639	1234.74		3.0908	1.90861	583.584
	30000	0.614119	1437.63	β	3.5908	2.54468	189.572
	35000	0.594013	1633.29		4.0908	4.36786	71.9633
ζ	0.1	1.87578	680.207		15	2.12075	1777
	0.2	1.84378	960.776	$\mathcal{C}_{rf}$	20	2.1239	1798.54
	0.3	1.89109	1342.28		25	2.12706	1820.08
A	125	2.14973	1810.24		3	2.1239	1798.54
	150	2.17534	1821.8	$\mathcal{H}$	4	2.01296	1857.94
	175	2.20073	1833.23		5	1.91697	1914.68

(i) The change in the value of parameters results the change in optimal cycle length and optimal cost of the Inventory system
(ii) Increase in the value of α results the increase in optimal cost, but decrease in optimal cycle length
(iii) Increase in the value of ζ results the increase in both optimal cycle length and optimal cost

(iv) Increase in the value of A results the increase in both optimal cycle length and optimal cost
(v) Increase in the value of β results the increase in optimal cycle length, but decrease in optimal cost
(vi) Increase in the value of $\mathcal{C}_{rf}$ results the increase in both optimal cycle length and optimal cost
(vii) Increase in the value of $\mathcal{H}$ results the increase in optimal cost, but decrease in optimal cycle length
(viii) Among all the parameters, a small increase in ζ results the high increase in total optimal cost. Similarly, a small increase in β results the a high increase in optimal cycle time, but rapid decrease in total optimal cost. But, a small increase in holding cost results the increase in optimal total cost and decrease in optimal cycle length, i.e, the increase in the value of parameters ζ, β and $\mathcal{H}$ effects the corresponding optimal cycle length or/and optimal total average cost significantly among all other parameters.

Remark 4. Here, we carried out the sensitivity of parameters in 1st scenario. If we consider scenario-2 for sensitivity of parameters, the result will be similar except for the parameter $\mathcal{C}_{rf}$. As it doesn't included in the scenario-2.

8 Conclusion

In the present article, we have developed a fuzzy inventory model for items that may contain a fraction of defectiveness under the effect of inflation with trade credit facility. The supply of items to the inventory model have constant rate per unit time within an interval during the cycle and the demand of the items are depends on selling price and advertisement cost throughout the cycle. Further, the model is considered under two scenarios. In scenario-1, retailer refunds an amount to the customer with a penalty against defective items. In scenario-2, retailer reworks the defective items and return to the customer. The main objective of the model is to overcome the vagueness, imprecision of different inventory parameters and minimize the total inventory cost of obtaining an optimal cycle length. So, we have considered fuzzy parameters, GMIR defuzzification technique, standard mathematical procedures and finally achieved the objective of the model. The detailed work is explained in the form of different sections. Also, the numerical examples are presented to evident the proposed model. Finally, the effect of change in values of different parameters are studied and presented in the form of a table. As fuzzy inventory model deals with vagueness and imprecision, we conclude that the fuzzy inventory model is more realistic than crisp model. Further, we hope that this article is helpful for inventory models considering in the context of defective items and it can be extended by considering shortages, backlogs, different type of demands etc.

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A Weak Contractive Condition and Some Fixed Point Theorems

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Abstract. Fixed point theorems for weak- (ψ, α, β)- contractive mappings have been introduced and investigated for different kinds of metric spaces. The paper first discusses a particular condition and then a weak contractive condition generalizing existing such conditions is defined. Our primary aim is to investigate the Banach, Kannan and Chatterjea's fixed point theorems in complete metric spaces satisfying the new weak contractive condition and their applications. In the sequel, several auxiliary results are investigated and also incorporated sufficient number of examples in suitable places to justify certain claims.

Keywords: Fixed point · Kannan fixed point theorem · Chatterjea fixed point theorem

1 Introduction and Preliminaries

Investigations for fixed point theorems in different settings as well as their generalizations have been going on intensively to meet several requirements both theoretical and applicable in nature, and in many cases, to fulfill several shortcomings as far as applications are concerned. These practices, in turn, contributing in the theory of fixed points to become one of the most explored topics of interest in pure and applied mathematics research.

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A mapping $T : X \to X$, where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y). \tag{1}$$

In [1], Banach proved that a contraction mapping having a unique fixed point in complete metric spaces. Also, inequality (1) implies the continuity of T. A natural question is that whether we can find contractive conditions which will imply the existence of fixed point in a complete metric space but not the continuity. Kannan [2] established the following result in which the above question has been answered in the affirmative.

Theorem 1. *([2]) If a mapping $T : X \to X$, where (X, d) is a complete metric space, satisfies the inequality*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \tag{2}$$

where $a \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point. The mappings satisfying (2) are called Kannan type mappings.

A similar contractive condition has been introduced by Chatterjea [3] as follows:

Theorem 2. *([3]) If a mapping $T : X \to X$, where (X, d) is a complete metric space, satisfies the inequality*

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)] \tag{3}$$

such that $b \in \left[0, \frac{1}{2}\right)$ and $x, y \in X$, then T has a unique fixed point. The mappings satisfying (3) are called Chatterjea type mapping.

In [4], Dorić (Theorem 2.1 of [4]), established a generalized contractive inequality to obtain fixed point for two maps. Also, Dorić introduced the following fixed point theorem as a corollary.

Theorem 3. *(Theorem 2.2 of [4]) Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mappings satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \beta(M(x, y))$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$$

and ψ, β defined as Theorem 2.1 of [4].

Afterward, Eslamian and Abkar [6] tried to give a generalization of Theorem 2.2 of [4] without $M(x, y)$.

Theorem 4. *(Eslamian and Abkar [6]) Let (X,d) be a complete metric space and $T:X\to X$ be a mapping satisfying*

$$\psi\left(d\left(Tx,Ty\right)\right)\leq\alpha\left(d\left(x,y\right)\right)-\beta\left(d\left(x,y\right)\right)$$

for all $x,y\in X$, where $\psi,\alpha,\beta:[0,\infty)\to[0,\infty)$ are such that ψ is altering distance function, α is continuous, β is lower semi-continuous,

$$\alpha\left(0\right)=\beta\left(0\right)=0 \text{ and } \psi\left(t\right)-\alpha\left(t\right)+\beta\left(t\right)>0 \text{ for all } t>0 \tag{4}$$

Then, T has a unique fixed point.

In Theorem 4, the restriction (4) seems not to be required. However, many other authors seem to use this condition (4) (see for instance [9–12]). The Theorem 4 is equivalent to the following:

Theorem 5. *(Modified Eslamian and Abkar Theorem) Let (X,d) be a complete metric space and $T:X\to X$ be a mapping satisfying*

$$\psi\left(d\left(Tx,Ty\right)\right)\leq\psi\left(d\left(x,y\right)\right)-\beta\left(d\left(x,y\right)\right)$$

for all $x,y\in X$, where $\psi,\beta:[0,\infty)\to[0,\infty)$ are such that ψ is altering distance function, β is lower semi-continuous. Then, T has a unique fixed point.

2 Fixed Point Theorems Depended on Another Function

In 2011, Moradi and Davood [5] introduced a new extension of Kannan type contractive mapping depending on another function T, which is continuous, one-to-one and subsequentially convergent.

Definition 1. *([5], [20], [21]) Let (X,d) be a metric space.*

(SSC) A mapping $T:X\to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent.

(SC) A mapping $T:X\to X$ is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

Theorem 6. *([5], [20], [21]) Let (X,d) be a complete metric space and $T,S:X\to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\lambda\in\left[0,\frac{1}{2}\right)$ and $x,y\in X$, S satisfying*

$$d\left(TSx,TSy\right)\leq\lambda\left[d\left(Tx,TSx\right)+d\left(Ty,TSy\right)\right] \tag{5}$$

then, S has a unique fixed point. Also if T is sequentially convergent, then for every $x_0\in X$ the sequence of iterates $\{S^nx_0\}$ converges to this fixed point.

Some of the subsequent improvements made can be summarized as follows:

P1 In 2013, Razani and Parvaneh [8], generalizing result of Moradi and Davood [5], gave fixed point theorems for weakly T-Chatterjea and weakly T-Kannan-contractive mappings in complete metric spaces.

P2 Very recently, Kir and Kiziltunc [9], generalizing result of [8] investigated generalized weak (ψ, α, β) contractions in order to establish extensions of Banach, Kannan and Chatterjea's fixed point theorems in complete metric spaces (see also [10, 20, 21]).

3 A New Generalization of Weak Contractive Conditions

In this section, a new generalization of weak contractive condition has been investigated for Banach, Kannan and Chatterjea's fixed point theorems without using conditions such as (4).

Now, we give some notations that we need to complete our results.

1. Let F be the set of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the condition $\xi(t) = 0$ if and only if $t = 0$,
2. We denote by Ψ the set of functions $\psi \in F$ such that ψ is continuous and nondecreasing,
3. We denote by Γ_1 the set of functions $\beta \in F$ such that β is lower semi-continuous.
4. We denote by Γ_2 the set of functions $\beta : [0, \infty)^2 \to [0, \infty)$ such that β is (lower semi) continuous in both argument and $\beta(a, b) = 0$ if and only $a = b = 0$.

Also, we denote by $SSC(X)$ the set of all mappings $T : X \to X$ such that T is one to one, continuous and subsequentially convergent, by $SC(X)$ the set of all mappings $T : X \to X$ such that T is one-to-one, continuous and sequentially convergent.

Definition 2. *We say that $F : [0, \infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

(1) $F(s, t) \le s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$;

for all $s, t \in [0, \infty)$.

Note that for such F function we have $F(0, 0) = 0$. We denote the set of C -class functions by $\mathcal{C}$.

Lemma 1. *([15, 22]) Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with*

$m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;
(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$;
(iv) $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon$.

Theorem 7. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC(X)$, F is C-class function and$f$ satisfying the inequality*

$$\psi\left(d\left(Tfx, Tfy\right)\right) \leq F(\psi\left(d\left(Tx, Ty\right)\right), \beta\left(d\left(Tx, Ty\right)\right)) \tag{6}$$

where $\psi \in \Psi, \beta \in \Gamma_2$. Then, f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = fx_n$ (equivelently, $x_n = f^n x_0$), $n = 1, 2, ...$ From (6), we have

$$\begin{aligned}\psi\left(d\left(Tx_n, Tx_{n+1}\right)\right) &= \psi\left(d\left(Tfx_{n-1}, Tfx_n\right)\right)\\ &\leq F(\psi\left(d\left(Tx_{n-1}, Tx_n\right)\right), \beta\left(d(Tx_{n-1}, Tx_n)\right))\\ &\leq \psi\left(d\left(Tx_{n-1}, Tx_n\right)\right).\end{aligned} \tag{7}$$

The inequality (7) implies that $\{d\left(Tx_n, Tx_{n+1}\right)\}$ is a nonincreasing sequence of non-negative real numbers and consequently, there exists $r \geq 0$ such that

$$d\left(Tx_n, Tx_{n+1}\right) \to r \ \textit{as}\ n \to \infty.$$

Letting $n \to \infty$ in (7), we obtain that

$$\psi\left(r\right) \leq F(\psi\left(r\right), \beta\left(r\right)) \leq \psi\left(r\right).$$

This last inequality implies that

$$F(\psi\left(r\right), \beta\left(r\right)) = \psi\left(r\right)$$

From definition of F class, we have $\psi\left(r\right) = 0$ or $\beta\left(r\right) = 0$. Thus $r = 0$. Now, we prove that $\{Tx_n\}$ is a Cauchy sequence. If possible, let $\{Tx_n\}$ be not a Cauchy sequence. Then, by Lemma 1 there exists $\epsilon > 0$ for which we can find subsequences $\left\{Tx_{m(k)}\right\}$ and $\left\{Tx_{n(k)}\right\}$ of $\{Tx_n\}$ with $n\left(k\right) > m\left(k\right) > k$ such that

$$\lim_{k\to\infty} d\left(Tx_{m(k)}, Tx_{n(k)}\right) = \lim_{k\to\infty} d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right) = \epsilon. \tag{8}$$

Now, consider the Eqs. (8) with (6)

$$\begin{gathered}\psi\left(d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \leq\\ F(\psi\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right), \beta\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right))\end{gathered} \tag{9}$$

and letting $k \to \infty$ in (9), we have

$$\psi\left(\epsilon\right) \leq F(\psi\left(\epsilon\right), \beta\left(\epsilon\right)) \leq \psi\left(\epsilon\right).$$

Again, thanks to definition of F class and of ψ function, we get $\epsilon = 0$ which is a contradiction. Thus, $\{Tx_n\}$ is a Cauchy sequence in complete metric space X. Hence, there is $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v. \tag{10}$$

Note that T is a subsequentially convergent, $\{x_n\}$ has a convergent subsequence. Thus, there is $u \in X$ and a subsequence $\left\{x_{n(k)}\right\}$ such that

$$\lim_{k\to\infty} x_{n(k)} = u. \tag{11}$$

Also, T is continous and $x_{n(k)} \to u$, therefore

$$\lim_{n\to\infty} Tx_{n(k)} = Tu. \tag{12}$$

Note that $\left\{Tx_{n(k)}\right\}$ is a subsequence of $\{Tx_n\}$, *so* $Tu = v$. Now, we show that $u \in X$ is fixed point of f.

$$\begin{aligned}\psi\left(d\left(Tx_{n(k)+1}, Tfu\right)\right) &= \psi\left(d\left(Tfx_{n(k)}, Tfu\right)\right)\\ &\leq F(\psi\left(d\left(Tx_{n(k)}, Tu\right)\right), \beta\left(d\left(Tx_{n(k)}, Tu\right)\right))\\ &\leq \psi\left(d\left(Tx_{n(k)}, Tu\right)\right)\end{aligned} \tag{13}$$

letting $k \to \infty$ in (13)

$$\psi\left(d\left(Tu, Tfu\right)\right) \leq F\left(0, 0\right) \leq \psi\left(0\right). \tag{14}$$

This implies that $Tu = Tfu$. Since T is one to one so $fu = u$. This shows $u \in X$ is a fixed of f.

To prove the uniqueness of the fixed point, if possible let u and u' be two fixed points of f. Thus, we have $fu' = u'$ and

$$\begin{aligned}\psi\left(d\left(Tu, Tu'\right)\right) &= \psi\left(d\left(Tfu, Tfu'\right)\right)\\ &\leq F(\psi\left(d\left(Tu, Tu'\right)\right), \beta\left(d\left(Tu, Tu'\right)\right))\\ &\leq \psi\left(d\left(Tu, Tu'\right)\right).\end{aligned} \tag{15}$$

Again, thanks to definition F class function and of ψ function, unless $d\left(Tu, Tu'\right) = 0$. This implies that $Tu = Tu'$. Since T is one-to-one $u = u'$. Thus, the fixed point is unique.

Remark 1. In Theorem 7, if T is sequentially convergent, that is, $T \in SC\left(X\right)$, by replacing $\{n\}$ with $\{n\left(k\right)\}$ we obtain that

$$\lim_{n\to\infty} x_n = u.$$

This implies that $\{x_n\}$ converges to the fixed point of f.

Now, we introduce the same concept for Kannan fixed point theorem.

Theorem 8. *Let (X,d) be a complete metric space and $f:X\to X$ be a mapping. Let $T\in SSC(X)$, F is C-class function and f satisfying the inequality*

$$\psi\left(d\left(Tfx,Tfy\right)\right)\leq$$
$$F(\psi\left(\frac{1}{2}\left[d\left(Tx,Tfx\right)+d\left(Ty,Tfy\right)\right]\right),\beta\left(d\left(Tx,Tfx\right),d\left(Ty,Tfy\right)\right)) \tag{16}$$

where $\psi\in\Psi,\beta\in\Gamma_2$. Then f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1}=fx_n$ (equivelently, $x_n=f^nx_0$), $n=1,2,\ldots$. Using the (16), we have

$$\begin{aligned}
&\psi d\left(Tx_n,Tx_{n+1}\right)\\
&=\psi\left(d\left(Tfx_{n-1},Tfx_n\right)\right)\\
&\leq F\begin{pmatrix}\psi\left(\frac{1}{2}\left[d\left(Tx_{n-1},Tfx_{n-1}\right)+d\left(Tx_n,Tfx_n\right)\right]\right),\\ \beta\left(d\left(Tx_{n-1},Tfx_{n-1}\right),d\left(Tx_n,Tfx_n\right)\right)\end{pmatrix}\\
&\leq\psi\left(\frac{1}{2}\left[d\left(Tx_{n-1},Tx_n\right)+d\left(Tx_n,Tx_{n+1}\right)\right]\right)
\end{aligned} \tag{17}$$

We see that $\{d\left(Tx_n,Tx_{n+1}\right)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there is $r\geq 0$ such that

$$\lim_{n\to\infty}d\left(Tx_n,Tx_{n+1}\right)=r\ . \tag{18}$$

Letting $n\to\infty$ in (18), then we have

$$\psi\left(r\right)\leq F(\psi\left(r\right),\beta\left(r,r\right))\leq\psi\left(r\right)$$

From definition of F class function and ψ,β, we get that $r=0$. Next, we prove that $\{Tx_n\}$ is a Cauchy sequence. If possible, let $\{Tx_n\}$ be not a Cahucy sequence. Then there exists $\epsilon>0$ for which we can find subsequences $\left\{Tx_{m(k)}\right\}$ and $\left\{Tx_{n(k)}\right\}$ of $\{Tx_n\}$ with $n\left(k\right)>m\left(k\right)>k$ such that

$$d\left(Tx_{m(k)},Tx_{n(k)}\right)\geq\epsilon. \tag{19}$$

Taking advantage of (16), we have $\psi\left(\epsilon\right)\leq F(\psi\left(0\right),\beta\left(0,0\right))\leq\psi\left(\epsilon\right)$. But this case is in conflict with $\epsilon>0$. Thus, $\{Tx_n\}$ is a Cauchy sequence in complete metric space X. Hence, there is $u\in X$ such that $x_{n(k)}\to u$, as $k\to\infty$ and $\lim\limits_{n\to\infty}Tx_{n(k)}=Tu$. Also, we have

$$\begin{aligned}
\psi\left(d\left(Tfu,Tfx_{n(k)-1}\right)\right)&\leq F\begin{pmatrix}\psi\left(\frac{1}{2}\left[d\left(Tu,Tfu\right)+d\left(Tx_{n(k)-1},Tx_{n(k)}\right)\right]\right),\\ \beta\left(d\left(Tfu,Tu\right),d\left(Tx_{n(k)-1},Tx_{n(k)}\right)\right)\end{pmatrix}\\
&\leq\psi\left(\frac{1}{2}\left[d\left(Tu,Tfu\right)+d\left(Tx_{n(k)-1},Tx_{n(k)}\right)\right]\right).
\end{aligned}$$

Letting $k\to\infty$ in the last inequality, we have

$$\psi\left(d\left(Tfu,Tu\right)\right)\leq$$

$$F\left(\psi\left(\frac{1}{2}d\left(Tu,Tfu\right)\right),\beta\left(d\left(Tfu,Tu\right),0\right)\right)\leq\psi\left(\frac{1}{2}d\left(Tu,Tfu\right)\right).$$

We obtain $d\left(Tu,Tfu\right)=0$ and as T is one to we get $u=fu$. It is easy to see the uniqueness of the fixed point.

Now, we introduce the same concept for Chatterjea fixed point theorem.

Theorem 9. *Let (X,d) be a complete metric space and $f:X\to X$ be a mapping. Let $T\in SSC\left(X\right)$, F is C-class function and f satisfying the inequality*

$$\psi\left(d\left(Tfx,Tfy\right)\right)\leq F\begin{pmatrix}\psi\left(\frac{1}{2}\left[d\left(Tx,Tfx\right)+d\left(Ty,Tfy\right)\right]\right),\\ \beta\left(d\left(Tx,Tfx\right),d\left(Ty,Tfy\right)\right),\end{pmatrix}$$

where $\psi\in\Psi,\beta\in\Gamma_2$. Then f has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1}=fx_n$ (equivelently, $x_n=f^nx_0$), $n=1,2,....$

$$\begin{aligned}\psi\left(d\left(Tx_n,Tx_{n+1}\right)\right)&=\psi\left(d\left(Tfx_{n-1},Tfx_n\right)\right)\\&\leq F\begin{pmatrix}\psi\left(\frac{1}{2}\left[d\left(Tx_{n-1},Tx_n\right)+d\left(Tx_n,Tx_{n+1}\right)\right]\right),\\ \beta\left(d\left(Tx_{n-1},Tx_n\right),d\left(Tx_n,Tx_{n+1}\right)\right)\end{pmatrix}\\&\leq\psi\left(\frac{1}{2}\left[d\left(Tx_{n-1},Tx_n\right)+d\left(Tx_n,Tx_{n+1}\right)\right]\right).\end{aligned}\tag{20}$$

We obtain that $\{d\left(Tx_n,Tx_{n+1}\right)\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there is $r\geq 0$ such that as $n\to\infty$ $d\left(Tx_n,Tx_{n+1}\right)\to r$ and $\psi\left(r\right)\leq F(\psi\left(r\right),\beta\left(r,0\right))\leq\psi\left(r\right).$ We obtain $r=0$. As in the proof of the Theorem 7, we get $\{Tx_n\}$ is a Cauchy sequence. To complete the proof the similar process in Theorems 7 and 8 would be used.

4 Some More Results and Applications

Example 1. The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathcal{C}$.

(1) $F(s,t)=ks$, $0<k<1$, $F(s,t)=s\Rightarrow s=0$;
(2) $F(s,t)=s-t$, $F(s,t)=s\Rightarrow t=0$;
(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,t)=s\Rightarrow s=0$;
(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1,r\in(0,\infty)$, $F(s,t)=s\Rightarrow t=0$;
(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$.
(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$.
(9) $F(s,t)=\sqrt[n]{\ln(1+s^n)}$, $F(s,t)=s\Rightarrow s=0$.

Now, we give some results based on our main theorems by picking out F-function.

4.1 Some Results and Applications for Generalized Banach Fixed Point Theorem

In Theorem 7, by picking out $F(s,t) = s - t$ we obtain the following result given by Kir and Kiziltunc [9].

Corollary 1. *(Kir and Kiziltunc [9]) Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx,Tfy)) \leq \alpha(d(Tx,Ty)) - \beta(d(Tx,Ty))$$

where $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma_1$ we have

$$\psi(t_1) \leq \alpha(t_2), \text{ implies } t_1 \leq t_2 \tag{21}$$

and for all

$$t > 0, \ \psi(t) - \alpha(t) + \beta(t) > 0. \tag{22}$$

Then, f has a unique fixed point.

Remark 2. Judging by origin, in Corollary 1, authors need not the conditions (21) and (22). The following Corollary is equivalent to Corollary 1.

Corollary 2. *(Modified Kir and Kiziltunc's Theorem) Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx,Tfy)) \leq \psi(d(Tx,Ty)) - \beta(d(Tx,Ty))$$

where $\psi \in \Psi, \beta \in \Gamma_1$. Then, f has a unique fixed point.

In Theorem 7, if we take $F(s,t) = ks$, then we obtain the following fixed point theorem given by Moradi and Beiranvand in [5].

Corollary 3. *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. If for $k \in [0,1)$ and for all $x,y \in X$,*

$$\psi(d(Tfx,Tfy)) \leq k\psi(d(Tx,Ty))$$

where

(1) $\psi : [0,\infty) \to [0,\infty)$, ψ is nondecreasing continuous from the right and $\psi^{-1}(0) = \{0\}$.
(2) T is one to one and graph closed (or subsequentially convergent and continuous).

Then, f has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Remark 3. Note that in Theorem 7, if we take $T = x$ and $F(s,t) = s - t$, we obtain Modified Eslamian and Abkar Theorem 5.

Remark 4. Note that Geraghty fixed point theorem [17] is a specific result of our Theorem 7.

Corollary 4. *(Geraghty, [17]) Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping satisfying the inequality*

$$d(fx, fy) \leq \varphi(d(x, y))\, d(x, y)$$

such that $\varphi : [0, \infty) \to [0, 1)$ and

$$\varphi(t_n) \to 1 \textit{ implies } t_n \to 0 \ .$$

Then f has a unique fixed point.

Remark 5. In Theorem 7, if we take $F(s, t) = ks, k \in (0, 1), \psi(s) = \int_0^s \varphi(t)\, dt$, then we obtain the following result that is more general than the result of Branciari [18]. Furthermore, taking T as identity mapping there, integral type contraction condition given by Branciari [18] holds.

Corollary 5. *Let (X, d) be a complete metric spaces. Let $f : X \to X$ be a mapping and $T \in SSC$ be such that for each $x, y \in X, k \in (0, 1)$*

$$\int_0^{d(Tfx, Tfy)} \varphi(t)\, dt \leq k \int_0^{d(Tx, Ty)} \varphi(t)\, dt$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)\, dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \to \infty} f^n x = a$.

The following example neither satisfies Banach contraction principle nor Corollary 3, but satisfies Theorem 7.

Example 2. Let $X = \left[0, \frac{1}{4}\right]$ endowed with Euclidean metric. Also, let $S : X \to X$ be given as $Sx = x - x^3$. Then, S is a not contraction mapping. If we take $Tx = \frac{x}{3}$ such that T is continuous, one to one subsequentially convergent then we obtain that

$$\begin{aligned} d(TSx, TSy) &= \left| \frac{x - x^3}{3} - \frac{y - y^3}{3} \right| \\ &\leq \left| \frac{x}{3} - \frac{y}{3} \right| + \left| \frac{x^3}{3} - \frac{y^3}{3} \right| \\ &\nleqq d(Tx, Ty)\,. \end{aligned} \tag{23}$$

Thus, (23) shows that S neither satisfies the condition of Banach Contraction Principle nor the condition of Corollary 3. Thus, we can not guarantee the existence of the fixed point of S. Now, we take $F(s,t) = \frac{s}{1+t}$, $\psi(t) = t$, $\beta(t) = \frac{9t^3}{1-9t^2}$. Assume that $x > y$, and we have

$$\begin{aligned} d(TSx, TSy) &= \left| \frac{x-x^3}{3} - \frac{y-y^3}{3} \right| \\ &= \frac{1}{3}\left[(x-y) - \left(x^3 - y^3\right)\right] \\ &\leq \frac{1}{3}\left[(x-y) - (x-y)^3\right] \\ &= \frac{\frac{x}{3} - \frac{y}{3}}{1 + \frac{9\left(\frac{x}{3}-\frac{y}{3}\right)^3}{1-9\left(\frac{x}{3}-\frac{y}{3}\right)^2}} \\ &= \frac{d(Tx, Ty)}{1+\beta(d(Tx, Ty))}. \end{aligned} \tag{24}$$

Thus, S satisfies the inequality (6) with T and β. Hence, (according to the Theorem 7) S has a unique fixed point in X. In fact $p = 0 \in X$ is unique fixed point of S.

Also, the following corollary is a specific result of our Theorem 7.

Corollary 6. *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(fx, fy)) \leq F(\psi(d(x,y)), \phi(d(x,y)))$$

where F is C-class function, $\psi : [0,\infty) \to [0,\infty)$ is continuous and monotone nondecreasing functions with $\psi(t) = 0$ if and only if $t = 0$, and $\phi : [0,\infty) \to [0,\infty)$ is continuous and monotone nondecreasing functions with $\psi(t) > 0, t > 0$ and $\psi(0) \geq 0$. Then f has a unique fixed point.

4.2 Some Results and Applications for Generalized Kannan and Chatterjea's Fixed Point Theorems

In Theorem 9, by picking out $F(s,t) = s - t$, we obtain Theorem 2.1 of [8].

Corollary 7. *(Weak $T-C$ Contraction Mapping Theorem) Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx, Tfy)) \leq$$
$$\alpha\left(\frac{1}{2}\left[d(Tx, Tfx) + d(Ty, Tfy)\right]\right) - \beta(d(Tx, Tfx), d(Ty, Tfy))$$

where $\psi \in \Psi, \beta \in \Gamma_2$. Then, f has a unique fixed point.

In Theorem 8, by picking out $F(s,t) = s - t$, we obtain Theorem 2.3 of [8].

Corollary 8. *(Weak $T-K$ Contraction Mapping Theorem) Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx,Tfy)) \leq$$

$$\psi\left(\frac{1}{2}\left[d(Tx,Tfx)+d(Ty,Tfy)\right]\right)-\beta(d(Tx,Tfx),d(Ty,Tfy))$$

where $\psi \in \Psi, \beta \in \Gamma_2$. Then, f has a unique fixed point.

Example 3. Let $X=[0,1]$ endowed with $d(x,y)=|x-y|$. Let $fx=\frac{x}{4}$ and $Tx=x^3$. If we consider $x=0$ and $y=1$, then f doesn't satisfies the condition of Kannan fixed point theorem. On the other hand if we take $\psi(t)=t$ and $F(s,t)=\frac{1}{5}s$, then for all $x,y\in X$, we have

$$d(Tfx,Tfy) \leq \frac{1}{10}\left[d(Tx,Tfx))+d(Ty,Tfy)\right].$$

Thus, f has a unique fixed point in X. Really, $p=0\in X$ is unique fixed point of f.

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A Large Class of Non-weakly Compact Subsets in a Renorming of c_0 with FPP

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Abstract. In 1979, Goebel and Kuczumow showed that very large class of non-weak* compact, closed, bounded and convex subsets of ℓ^1 has the fixed point property (FPP) for nonexpansive mappings. Later, in 2008, Lin proved that ℓ^1 can be renormed to have FPP for nonexpansive mappings. c_0-analogue of Lin's result is still open. However, renorming c_0, we prove that Goebel and Kuczumow analogy can be proved under affinity condition. That is, we prove that there exist a renorming of c_0 and a very large class of non-weakly compact, closed, bounded and convex subsets of c_0 with FPP for affine nonexpansive mappings whereas Dowling, Lennard and Turett proved in 2004 that weak compactness is equivalent to FPP for nonexpansive mappings when c_0 is considered with its usual norm instead of any equivalent norm.

Keywords: Nonexpansive mapping · Affine mapping · Fixed point property · Renorming · Asymptotically isometric c_0-summing basic sequence · Closed bounded convex set

1 Introduction

First by Goebel and Kuczumow [8] and recently in Everest's PhD thesis [6], under supervision of Chris Lennard, it was shown that very large classes of closed, bounded, convex, non-weak*-compact subsets K of l^1 have the fixed point property for nonexpansive mappings FPP(n.e.).

But as an intriguing result, in 2008, Lin [11] showed that there exists an equivalent norm for $(\ell^1, \|\cdot\|_1)$, denoted by $|||\cdot|||^*$ and given by

$$|||x|||^* = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|, \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^1,$$

such that $(\ell^1, |||\cdot|||^*)$ has FPP(n.e.).

Then, researchers have wondered $(c_0, \|\cdot\|_\infty)$ analogue of P.K. Lin's work. That is, it is still an open question whether or not c_0 can be renormed to have

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the fixed point property for nonexpansive mappings. Since this question could not be solved yet, we can say that researchers would be interested in the tools that may open ways to answer this famous question and so it would not be incorrect to say that fixed point theorists would be interested in a first stage of works for getting answer for the question of $(c_0, \|\cdot\|_\infty)$ analogue of Lin's work. Thus, we worked on c_0 analogue of Goebel and Kuczumow's theory (with an equivalent norm of course) which has also great importance by considering that getting very satisfied results for c_0 analogue of Goebel and Kuczumow's theory (with an equivalent norm) would be the first step to find a candidate equivalent norm to work on c_0 analogue of Lin's work.

We need to note that Dowling, Lennard and Turett [3] showed that any closed infinite dimensional subspace of $(c_0, \|\cdot\|_\infty)$ also fails FPP(n.e.). Hence, to think about c_0 analogue of Goebel and Kuczumow's work, firstly, we have to consider it with an equivalent norm for c_0. That is, we can work on a question "do there exist any renorming of c_0 and a nonempty closed, bounded and convex subset C so that every nonexpansive mapping has fixed point property?". Nezir and Dutta recently gave positive answer for this question when the mapping is also affine in their recent submitted work. However, it can be said that the class of sets having FPP for affine nonexpansive mappings with an equivalent norm was not large enough.

In order to get larger classes of sets with FPP for affine nonexpansive mappings, in this paper, we provide another equivalent norm and show that there is very large class of sets in c_0 with FPP for affine nonexpansive mappings. To introduce this class of sets, we need to recall Nezir's joint work with Lennard [10] but first we need to give some more literacy to reveal the connection between weak compactness in c_0 and fixed point property for affine nonexpansive mappings since in this work we also show that there exists a renorming of c_0 such that with that equivalent norm weak compactness in c_0 is not equivalent to fixed point property for affine nonexpansive mappings while there is a well-known fact by Dowling, Lennard and Turett [5]: weak compactness in c_0 is equivalent to fixed point property for nonexpansive mappings when the usual norm is considered. Now, let us recall these works motivating Lennard and Nezir and us to work on some nonweakly compact subsets of c_0 such that Lennard and Nezir observe that they fail fixed point property for affine nonexpansive mappings respect to the usual norm of c_0 whereas Nezir and Dutta observed that they have fixed point property for affine nonexpansive mappings respect to an equivalent norm that he constructed.

Consider the Banach space c_0, consisting of all scalar sequences converging to zero. In 1981 Maurey [12] proved that every weakly compact, convex subset C of $(c_0, \|\cdot\|_\infty)$ is such that every nonexpansive mapping $T : C \longrightarrow C$ has a fixed point; i.e., $(c_0, \|\cdot\|_\infty)$ has w-FPP (n.e.). In contrast to this result, in 2004 Dowling, Lennard and Turett [5] showed that every non-weakly compact, closed, bounded, convex (c.b.c.) subset K of $(c_0, \|\cdot\|_\infty)$ fails FPP for $\|\cdot\|_\infty$-nonexpansive mappings. This provided a converse to the important theorem of Maurey [12] that for all weakly compact convex subsets C of $(c_0, \|\cdot\|)_\infty$, every nonexpansive map $T: C \to C$ has a fixed point. (Note that in general Banach spaces the analogue of Maurey's result may fail. For example, $X = (L^1[0,1], \|\cdot\|_1)$, $C := \{f \in L^1[0,1] : 0 \le f \le 1\}$ and take $T: C \to C$ as Alspach's mapping [1]; i.e., fix an arbitrary $f \in C$ and define for all $t \in [0, \frac{1}{2})$, $(Tf)(t):\ = \min\{2f(2t), 1\}$ and for all $t \in [\frac{1}{2}, 1]$, $(Tf)(t):\ = \max\{2f(2t-1), 1\} - 1$. We remark that with respect to the norm $\|\cdot\|_1$, T is nonexpansive. It is seen that T is fixed point free. Thus, Alspach provided the first example to show weak compactness is not sufficient for a set to have FPP).

Recently, in 2015, Gallagher, Lennard and Popescu [7] show that weak compactness is not equivalent to FPP (n.e.) in c and they give an example of non-weakly compact, closed, bounded, convex subset W of the Banach space of convergent sequences $(c, \|\cdot\|_\infty)$ such that every nonexpansive mapping $T: W \to W$ has a fixed point. In their work, they define an equivalent norm $\|\cdot\|^\sim$ on c_0 for which there exist two non-weakly compact c.b.c. subsets that have the fixed point property (FPP) for $\|\cdot\|^\sim$-nonexpansive mappings since $(c, \|\cdot\|_\infty)$ is isomorphic to $(c_0, \|\cdot\|_\infty)$ and mentioned two subsets are isomorphic to two hyperconvex subsets of c with fixed point property. This is the first example of a non-weakly compact, closed, bounded, convex subset W of a Banach space X isomorphic to c_0, for which W has the fixed point property for nonexpansive mappings.

In contrast to this result, in 2004 Dowling, Lennard and Turett [5] showed that weak compactness is equivalent to fixed point property in c_0 for nonexpansive mappings and verified this conjecture by showing every non-weakly compact, closed, bounded, convex (c.b.c.) subset K of $(c_0, \|\cdot\|_\infty)$ is such that there exists a $\|\cdot\|_\infty$-nonexpansive mapping T on K that is fixed point free. This mapping T is generally not affine. It is an open question as to whether or not on every non-weakly compact, c.b.c. subset K of $(c_0, \|\cdot\|_\infty)$ there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping S that is fixed point free.

Motivated by this question, in 2011, Lennard and Nezir [10] prove that if a Banach space contains an asymptotically isometric (ai) c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$, then the closed convex hull of $(x_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$, fails the fixed point property for affine nonexpansive mappings.

In their paper, first of all, they work on some specific ai c_0-summing basic sequences in c_0.

For example, they fix $b \in (0,1)$ and define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := b\, e_1$, $f_2 := b\, e_2$, and $f_n := e_n$ for every $n \ge 3$ where $(e_n)_{n\in\mathbb{N}}$ is defined to be 1 in its n^{th} coordinate, and 0 in all other coordinates such that for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$, the sequence $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis.

Next, they define the closed, bounded, convex subset $E = E_b$ of c_0 by

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \ge t_2 \ge \cdots \ge t_n \downarrow_n 0 \right\} .$$

Then, they define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way: $\eta_1 := f_1$ and $\eta_n := f_1 + \cdots + f_n$ for every $n \geq 2$. Note that

$$E := \left\{ \sum_{n=1}^{\infty} \alpha_n \, \eta_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\} .$$

Thus, the following theorem is given in their work:

Theorem 1. *Assume $b \in (0,1)$. Then, the closed convex hull the sequence $(\eta_n)_{n\in\mathbb{N}}$, $E = \overline{\mathrm{co}}(\{\eta_n : n \in \mathbb{N}\})$ is such that there exists a fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$.*

Note that, it can be easily seen that the sequence $(\eta_n)_{n\in\mathbb{N}}$ is an ai c_0-summing basic sequence.

Next, they generalize their result more and prove the following theorems.

Theorem 2. *Assume $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ is any increasing sequence in $(0,1]$ with $b_n \uparrow_n 1$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in c_0 by the following way: $f_n := b_n \, e_n$, for every $n \in \mathbb{N}$. Next, take $E = E_{\overrightarrow{b}}$ of c_0 such that*

$$E := \left\{ \sum_{n=1}^{\infty} t_n \, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\}.$$

Then, there exists a fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$.

Theorem 3. *Assume $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ is any sequence in $(0,\infty)$ converging to some $\kappa > 0$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 given by $f_n := b_n \, e_n$ for every $n \in \mathbb{N}$. Next, using this sequence define a subset $E = E_{\overrightarrow{b}}$ of c_0 by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n \, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\}.$$

Then, there exists a fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$.

Theorem 4. *Assume $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ is any bounded sequence in $(0,\infty)$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 given by $f_n := b_n \, e_n$, for every $n \in \mathbb{N}$. Next, using this sequence define a subset $E = E_{\overrightarrow{b}}$ of c_0 by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n \, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\}.$$

Then, there exists a fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \longrightarrow E$.

More importantly, they give their main result as below:

Theorem 5. *Let $L \in (0, \infty)$. Banach space containing an L-scaled asymptotically isometric c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$ fails the fixed point property for affine nonexpansive mappings since when the closed convex hull of the sequence $(x_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$ is taken, then there exists a fixed point free affine contractive mapping $U : E \longrightarrow E$.*

In the recent work of Nezir and Güven, they defined the following equivalent norm $\|\cdot\|$ on c_0 depending on a scalar α satisfying some conditions and showed that there exist $0 < b < 1$ and $\alpha > 1$ such that for $E := \overline{\text{co}}(\{\sum\limits_{k=1}^{n} f_k \ : \ f_1 := b\, e_1,\ f_2 := b\, e_2\ ,\ f_n := e_n\ ,\ \forall n \geq 3\})$, similarly to the set E in Theorem 1 above, then for all affine $\|\cdot\|$-nonexpansive mappings (for all affine $|||\cdot|||$-nonexpansive mappings respectively) $T\colon E \longrightarrow E$, T has a fixed point in E.

Let $\alpha \in \Re$. For $x = (\xi_k)_k \in c_0$, define

$$\|x\| := \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha\xi_j\right|$$

$$\text{where } \sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0 \text{ and } \ Q_k > Q_{k+1},\ \forall k \in \mathbb{N}.$$

In this paper, firstly we show that for the equivalent norm we provide c_0 does not contain an asymptotically isometric c_0-copy. Next, we worked on a characterization about some c_0-summing basic sequences given in the first author's PhD thesis under supervision of Lennard [13]. There, they give the following results:

Assume $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $(0, \infty)$ for which there exists $\varGamma > 0$ with $[\varGamma \leq \gamma_N$, for all $N \in \mathbb{N}]$ and $\sigma := \sum\limits_{n=2}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$; and suppose $(b_n)_{n\in\mathbb{N}}$ is a sequence in $(0, \infty)$ converging to some $\lambda \in (0, \infty)$. Define the sequence by $\eta_n := \gamma_n(b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + ... + b_n e_n)$, for all $n \in \mathbb{N}$. Also suppose that $(\eta_n)_{n\in\mathbb{N}}$ satisfies a lower c_0-summing estimate. That is, suppose $\exists\, K \in (0, \infty)$ s.t. $\forall \alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$, $K \sup\limits_{n\geq 1} \left|\sum\limits_{j=n}^{\infty} \alpha_j\right| \leq \left\|\sum\limits_{j=1}^{\infty} \alpha_j\, \eta_j\right\|$. Then, $(\eta_n)_{n\in\mathbb{N}}$ is an L-scaled asymptotically isometric c_0-summing basic sequence. Moreover, on the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$, there exists a fixed point free affine $\|\cdot\|_\infty$-contractive mapping $U : E \longrightarrow E$.

After acknowledging all these sets E's given in the theorems above in previous works, we can conclude our introduction section with our results: on c_0 consider the equivalent norm $|||\cdot|||$ given below such that those sets E mentioned above have FPP for affine $|||\cdot|||$-nonexpansive mappings.

$$\begin{aligned}|||x||| := &\frac{1}{\gamma_1}\lim_{p\to\infty}\sup_{k\in\mathbb{N}}\gamma_k\left(\sum_{j=k}^{\infty}\frac{|\xi_j|^p}{2^j}\right)^{\frac{1}{p}}\\ &+\gamma_1\sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}Q_k\left|{\xi^\star}_k-\alpha{\xi^\star}_j\right|+\gamma_1\sqrt{\sup_{j\in\mathbb{N}}\sum_{k=1}^{\infty}{Q_k}^2|\xi_k-\alpha\xi_j|^2}\end{aligned}$$

where $\gamma_k \uparrow_k 1,\ \gamma_{k+2} > \gamma_{k+1}, \forall k \in \mathbb{N},\ \gamma_2 = \gamma_1,\ x^\star := ({\xi^\star}_j)_{j\in\mathbb{N}}$ is the decreasing rearrangement of x, $\sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0$ and $Q_k > Q_{k+1},\ \forall k \in \mathbb{N}$

such that from the definition of decreasing rearrangement, $\exists$ a 1-1 mapping $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ and $\exists(\varepsilon_j)_{j\in\mathbb{N}}$ s.t. each $\varepsilon_{\pi(j)} \in \{-1, 1\}$ and then $(\xi^\star)_k = |\xi_{\pi(k)}| = \varepsilon_{\pi(k)}\,\xi_{\pi(k)}, \forall k \in \mathbb{N}$.

Finally, we give our most generalized results below.

Theorem 6. *Let $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ be any sequence in $(0,\infty)$. We define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_n := b_n\, e_n$, for all $n \in \mathbb{N}$. Next, define the closed, bounded, convex subset $E = E_{\overrightarrow{b}}$ of c_0 by*

$$E := \left\{\sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \ge t_2 \ge \cdots \ge t_n \downarrow_n 0\right\}\ .$$

Then, the set E has the fixed point property for $|||.|||$-nonexpansive affine mappings.

Corollary 1. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ and let $(b_n)_{n\in\mathbb{N}}$ be any sequence in $(0,\infty)$. Define the sequence $(\eta_n)_{n\in\mathbb{N}}$ by setting $\eta_n := \mu_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n)$, for all $n \in \mathbb{N}$.*

Let E be the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E := \overline{\mathrm{co}}(\{\eta_n : n \in \mathbb{N}\})$.

Then, the set E has the fixed point property for $|||\cdot|||$-nonexpansive affine mappings.

We need to note that this paper forms M.Sc. thesis of Serap Oran [16] and some parts in the first two sections of this work come from the study by Nezir, Mustafa and Dutta [14].

2 Preliminaries

In this section, we give preliminaries to recall our equivalent norm we use and to give our main results.

Definition 1. *Let E be a non-empty closed, bounded, convex subset of a Banach space $(X, \|\cdot\|)$.*
Let $T : E \longrightarrow E$ be a mapping.

1. *we say T is affine if for all $\lambda \in [0,1]$, for all $x, y \in E$, $T\big((1-\lambda)\,x + \lambda\,y\big) = (1-\lambda)\,T(x) + \lambda\,T(y)$.*
2. *we say T is nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$, for all $x, y \in E$.*
   Also, we say that E has the fixed point property for nonexpansive mappings [FPP(n.e.)] if for all nonexpansive mappings $T : E \longrightarrow E$, there exists $z \in E$ with $T(z) = z$.

Let $(X, \|\cdot\|)$ be a Banach space and $E \subseteq X$. We will denote the closed, convex hull of E by $\overline{\text{co}}(E)$. As usual, $(c_0, \|\cdot\|_\infty)$ is given by

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \lim_{n\longrightarrow\infty} x_n = 0 \right\}.$$

Further, $\|x\|_\infty := \sup\limits_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$; and $(\ell^1, \|\cdot\|_1)$ is defined by

$$\ell^1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

We denote by c_{00} the vector space of all scalar sequences that have only finitely many non-zero terms.

We also recall the definition of an *asymptotically isometric c_0-summing basic sequence* in a Banach space $(X, \|\cdot\|)$ from Lennard and Nezir's paper [10] as below.

Definition 2. *Let $(X, \|\cdot\|)$ be a Banach space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X satisfying the following condition; then, we say $(x_n)_{n\in\mathbb{N}}$ is an asymptotically isometric (ai) c_0-summing basic sequence in $(X, \|\cdot\|)$: There exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ such that for every $(t_n)_{n\in\mathbb{N}} \in c_{00}$,*

$$\sup_{n\geq 1} \left(\frac{1}{1+\varepsilon_n}\right) \left|\sum_{j=n}^{\infty} t_j\right| \leq \left\|\sum_{j=1}^{\infty} t_j\, x_j\right\| \leq \sup_{n\geq 1}\,(1+\varepsilon_n) \left|\sum_{j=n}^{\infty} t_j\right|.$$

Note that here we can replace c_{00} by ℓ^1. Furthermore, if $L > 0$ and the sequence $(z_n/L)_{n\in\mathbb{N}}$ is an *asymptotically isometric c_0-summing basic sequence*, we will call the sequence $(z_n)_{n\in\mathbb{N}}$ an *L-scaled asymptotically isometric c_0-summing basic sequence* in $(X, \|\cdot\|)$.

Definition 3. [4] *We call a Banach space X contains an asymptotically isometric (ai) copy of c_0 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that*

$$\sup_n (1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sup_n |a_n|,$$

for all $(a_n)_n \in c_0$.

Theorem 7. [4] *If a Banach space X contains an ai copy of c_0, then X fails FPP(n.e).*

Definition 4.

$$\|\|x\|\| := \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{|\xi_j|^p}{2^j} \right)^{\frac{1}{p}}$$

$$+\gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|{\xi^\star}_k - \alpha {\xi^\star}_j\right| + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 |\xi_k - \alpha\xi_j|^2}$$

where $\gamma_k \uparrow_k 1,\ \gamma_{k+2} > \gamma_{k+1}, \forall k \in \mathbb{N},\ \gamma_2 = \gamma_1,\ x^\star := ({\xi^\star}_j)_{j\in\mathbb{N}}$ *is the decreasing rearrangement of* x, $\sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0$ *and* $Q_k > Q_{k+1},\ \forall k \in \mathbb{N}$

such that from the definition of decreasing rearrangement, $\exists$ a 1-1 mapping $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ and $\exists (\varepsilon_j)_{j\in\mathbb{N}}$ s.t. each $\varepsilon_{\pi(j)} \in \{-1, 1\}$ and then $(\xi^\star)_k = |\xi_{\pi(k)}| = \varepsilon_{\pi(k)}\, \xi_{\pi(k)}, \forall k \in \mathbb{N}$.

We can understand easily that the above expression is indeed an equivalent norm on c_0 due to the following facts:

Let $x = (\xi_i)_{i\in\mathbb{N}} \in c_0$. We will consider $x \neq (0, 0, \cdots)$ otherwise proof of the claim is clear.

Then,

$$\exists N \in \mathbb{N} \ni \|x\|_\infty = \sup_{k\in\mathbb{N}} |\xi_k| = \max_{k\in\mathbb{N}} |\xi_k| = |\xi_N| .$$

Due to power mean inequalities formula (see e.g. [9]) (using the one given with weighted power means),

$$\begin{aligned} \|x\|_\infty &= \max_{k\le N} |\xi_k| = \max\{|\xi_1|, |\xi_2|, \cdots, |\xi_N|\} \\ &= \lim_{p\to\infty} \left(\frac{|\xi_1|^p + |\xi_2|^p + \cdots + |\xi_N|^p}{2^N} \right)^{\frac{1}{p}} \\ &= \lim_{p\to\infty} \left(\sum_{k=1}^{N} \frac{|\xi_k|^p}{2^N} \right)^{\frac{1}{p}} . \end{aligned}$$

Thus,

$$\|x\|_\infty = \lim_{p\to\infty} \left(\sum_{k=1}^{\infty} \frac{|\xi_k|^p}{2^k} \right)^{\frac{1}{p}}$$

Then, it is easy to see that our norm is equivalent to the usual norm on c_0.

Lemma 1. [8] *If $\{x_n\}$ is a sequence in l^1 converging to x in weak-star topology, then for any $y \in l^1$,*

$$r(y) = r(x) + \|y - x\|_1 \text{ where } r(y) = \limsup_n \|x_n - y\|_1 .$$

In a recent paper of Nezir and Güven, they obtain the following partial analogue result to Lemma 1.

Lemma 2. *Let $(x_n)_n$ be a bounded sequence in a Banach space $(X, \|\cdot\|)$. Consider a function $s : X \to [0, \infty)$ given by*

$$s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_k - y \right\| , \ \forall y \in X.$$

Then, if X has weak Banach-Saks property and $x \in X$ is the weak limit of the sequence $(x_n)_n$, then there exists a subsequence $(x_{n_k})_k$ whose norm limit is x such that if s is redefined via this subsequence, we have $s(x) = 0$ and $s(y) = \|y - x\|$, $\forall y \in X$ and for any equivalent norm $\|\cdot\|$ on X.

Thus, since c_0 has weak Banach-Saks property [15], the above can be applied.

3 c_0 with Our Equivalent Norm Does Not Contain an Asypmtotically Isometric Copy of c_0

Theorem 8. *If $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$, then $(c_0, \||\cdot\||)$ does not contain an asymptotically isometric copy of c_0 where the norm $\||\cdot\||$ is given as in Definition 4.*

Proof. By contradiction, assume $(c_0, \||\cdot\||)$ does contain an asymptotically isometric copy of c_0. That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and a sequence $(x_n)_n$ in c_0 such that

$$\heartsuit \left[\begin{array}{l} \text{for every } n \in \mathbb{N} \text{ and every choice of scalars } t_1, t_2, \ldots, t_n, \\ \text{it follows that } \max\limits_{1 \le k \le n} (1 - \varepsilon_k) |t_k| \le \left\| \left| \sum\limits_{k=1}^{n} t_k x_k \right| \right\| \le \max\limits_{1 \le k \le n} |t_k| . \end{array} \right]$$

Let $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$, then $Q_1 > \frac{2|\alpha|}{1+2|\alpha|}$.

Hence, $\frac{1}{\gamma_1}2|\alpha| > 2 > 1$, $|2\alpha - 1| > |\alpha|$ and we can assume for the following equivalent norm $|\!|\!|.|\!|\!|^{\sim}$, $(c_0,\ |\!|\!|.|\!|\!|^{\sim})$ contains an asymptotically isometric copy of c_0:

$$|\!|\!|x|\!|\!|^{\sim} = \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{|\xi_j|^p}{2^j} \right)^{\frac{1}{p}} + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left| {\xi^\star}_k - \frac{2\alpha}{\gamma_1} {\xi^\star}_j \right|$$
$$+ \gamma_1 \sqrt{ \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left| \xi_k - \frac{2\alpha}{\gamma_1} \xi_j \right|^2 }$$

where $\gamma_k \uparrow_k 1$, $\gamma_{k+2} > \gamma_{k+1}, \forall k \in \mathbb{N}$, $\gamma_2 = \gamma_1$, $x^\star := ({\xi^\star}_j)_{j\in\mathbb{N}}$ is the decreasing rearrangement of x, $\sum_{k=1}^{\infty} Q_k = 1$, $Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k \in \mathbb{N}$

Without loss of generality we can assume that the sequence $(x_n)_n$ converges pointwise to 0.

For each $n \in \mathbb{N}$, let $x_n = (\xi_j^n)_j$.

Note that, for every $x \in c_0$, there exists $L > 1$ such that $\|x\|_\infty \geq |\!|\!|\frac{x}{L}|\!|\!|^{\sim}$. Now, without loss of generality, by passing to a subsequence if necessary, we may assume there exists $s \in \mathbb{N}$ such that $\|x_s\|_\infty > \frac{1}{|2\alpha-\gamma_1|}$. We can do this since for $L > 1$, the sequence $(x_n)_n$ can be replaced with $(\frac{x_n}{L})_n$ so that the condition $\heartsuit$ respect to newly defined norm is satisfied for null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and so there exists $s \in \mathbb{N}$ such that $\varepsilon_s < 1 - \frac{1}{|2\alpha-\gamma_1|}$ and $\|x_s\|_\infty \geq |\!|\!|\frac{x_s}{L}|\!|\!|^{\sim} > 1 - \varepsilon_s > \frac{1}{|2\alpha-\gamma_1|}$.

Now, there exists $r \in \mathbb{N}$ s.t. $\xi_r^s \neq 0$ and, as previously, since $x_s \in c_0$, there exists $N^{(s)} \in \mathbb{N}$ such that $\|x_s\|_\infty = |\xi^s_{N^{(s)}}| \geq |\xi_r^s|$. Hence, take $p = \min\{r \mid |\xi_r^s| = |\xi_{N^{(s)}}|\}$.

Now, let $\delta = \left(Q_1 - \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}\right) \frac{8|\alpha|(1+4|\alpha|)}{16|\alpha|^3+\left(68+\frac{4}{\gamma_1}\right)|\alpha|^2+\left(16+\frac{9}{\gamma_1}+8\gamma_1\right)|\alpha|+\frac{2}{\gamma_1}}$.

Now, choose $N_1 \geq p$ so that $\sum_{k=1+N_1}^{\infty} Q_k < (\frac{4}{\gamma_1}+4|\alpha|)\frac{\delta}{2}$. Choose $N_2 \in \mathbb{N}$ so that $\varepsilon_n < \min\left\{1 - \frac{1}{|2\alpha-\gamma_1|},\ \delta\right\}$ for all $n \geq \max\{s, N_2\}$. Choose $M \geq \max\{s, N_2\}$ so that $|2\alpha - \gamma_1|\left|\xi_j^n\right| < \frac{\left(\frac{1}{\gamma_1}+4\alpha\right)\delta}{8}$ and $\left|\xi_j^n\right| < \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{8|\alpha|}$ for $j = 1, 2, \ldots, N_1$ and for all $n \geq M$. Note that $1 \geq \|x_s\|^{\sim}$ and $1 \geq \|x_n\|^{\sim}$ and so $1 \geq \left|\xi_j^s\right|$ and $1 \geq \left|\xi_j^n\right|$ for all $j \in \mathbb{N}$.

Therefore, for each $n \geq M$,

$$\begin{aligned}
|||x_n|||^{\sim} &\leq \frac{1}{\gamma_1} \|x_n\|_\infty + \gamma_1 \sum_{k=1}^{\infty} Q_k \left|{\xi^\star}_k^n\right| + 2\left|\alpha\right| \sum_{k=1}^{\infty} Q_k \sup_{j\in\mathbb{N}} \left|{\xi^\star}_j^n\right| \\
&\quad + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left(|\xi_k^n| + \frac{2\alpha}{\gamma_1} |\xi_j^n| \right)^2} \\
&\leq \left(\frac{1}{\gamma_1} + 4|\alpha| \right) \|x_n\|_\infty + 2\gamma_1 \sum_{k=1}^{\infty} Q_k \left|\xi_k^n\right| \\
&\leq \left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right) \|x_n\|_\infty + 2\gamma_1 \sum_{k=1}^{N_1} Q_k \left|\xi_k^n\right| + 2\gamma_1 \sum_{k=1+N_1}^{\infty} Q_k \left|\xi_k^n\right| \\
&< \left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right) \|x_n\|_\infty + \frac{\frac{1}{\gamma_1}\left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right)\delta}{4|\alpha|} \sum_{k=1}^{N_1} Q_k + 2 \sum_{k=1+N_1}^{\infty} Q_k \\
&< \left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right) \|x_n\|_\infty + \frac{\frac{1}{\gamma_1}\left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right)\delta}{4|\alpha|} + \left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right)\delta \\
&< \left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right) \|x_n\|_\infty + \frac{\left(\frac{1}{\gamma_1} + 4\left|\alpha\right| \right)^2 \delta}{4|\alpha|}.
\end{aligned}$$

By the triangle inequality $\|x_n\|_\infty \leq \frac{1}{2}\|x_n + x_s\|_\infty + \frac{1}{2}\|x_n - x_s\|_\infty$ and so either $\|x_n + x_s\|_\infty \geq \|x_n\|_\infty$ or $\|x_n - x_s\|_\infty \geq \|x_n\|_\infty$.

If $\|x_n + x_s\|_\infty \geq \|x_n\|_\infty$ then we have

$$\begin{aligned}
1 &\geq |||x_s + x_n|||^{\sim} \\
&\geq \|x_s + x_n\|_\infty + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left| (\xi_k^s + \xi_k^n)^\star - \frac{2\alpha}{\gamma_1} \left(\xi_j^s + \xi_j^n\right)^\star \right| \\
&\quad + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left| \xi_k^s + \xi_k^n - \frac{2\alpha}{\gamma_1} \left(\xi_j^s + \xi_j^n\right) \right|^2} \\
&\geq \|x_s + x_n\|_\infty + \gamma_1 \sum_{k=1}^{\infty} Q_k \left| (\xi_k^s + \xi_k^n)^\star - \frac{2\alpha}{\gamma_1} (\xi_1^s + \xi_1^n)^\star \right| \\
&\geq \|x_s + x_n\|_\infty + Q_1 \left|2\alpha - \gamma_1\right| (\xi_1^s + \xi_1^n)^\star \\
&\geq \|x_n\|_\infty + Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^s + \xi_p^n\right| \\
&\geq \|x_n\|_\infty + Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^s\right| - Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^n\right| \\
&> \frac{\gamma_1}{1 + 4\gamma_1|\alpha|} |||x_n|||^{\sim} - \frac{\left(\frac{1}{\gamma_1} + 4|\alpha|\right)\delta}{4|\alpha|} + Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^s\right| - Q_1 \left|2\alpha - \gamma_1\right| \left|\xi_p^n\right|.
\end{aligned}$$

Hence,

$$
\begin{aligned}
1 &> \frac{\gamma_1}{1+4|\alpha|} |||x_n|||^{\sim} - \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{4|\alpha|} + Q_1 - |2\alpha-\gamma_1| \left|\xi_p^n\right| \\
&> \frac{\gamma_1}{1+4|\alpha|}(1-\varepsilon_n) - \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{4|\alpha|} + Q_1 - \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{8} \\
&> \frac{\gamma_1}{1+4|\alpha|}(1-\delta) - \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{4|\alpha|} + Q_1 - \frac{\left(\frac{1}{\gamma_1}+4|\alpha|\right)\delta}{8} \\
&> 1 + Q_1 - \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|} \\
&\quad -\delta\left(\frac{16|\alpha|^3 + \left(36+\frac{4}{\gamma_1}\right)|\alpha|^2 + \left(8+\frac{9}{\gamma_1}+8\gamma_1\right)|\alpha| + \frac{2}{\gamma_1}}{8|\alpha|(1+4|\alpha|)}\right) \\
&> 1+\delta
\end{aligned}
$$

which is not possible (contradiction).

Similarly we arrive at a contradiction if we assume that $\|x_n - x_s\|_\infty \geq \|x_n\|_\infty$.

4 Main Result: Many Sets in c_0 Having FPP (for n.e. and Affine Mappings) for an Equivalent Norm

Example 1. We will be considering the closed convex hull of summing basis. That is, we define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_n := e_n$, for every $n \geq 1$. Next, define the closed, bounded, convex subset $E = E_b$ of c_0 by

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\}.$$

Let us define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way. Let $\eta_1 := f_1$ and $\eta_n := f_1 + \cdots + f_n$, for every $n \geq 2$. It is straightforward to check that

$$E := \left\{ \sum_{n=1}^{\infty} \alpha_n\, \eta_n : \text{ each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

Then, it is well known that E is the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$ such that right shift mapping is fixed point free affine $\|\cdot\|_\infty$-nonexpansive mapping.

Theorem 9. *The set E defined as in the example above has the fixed point property for $|||\cdot|||$-nonexpansive affine mappings where the norm $|||\cdot|||$ is given as in Definition 4 if $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$.*

Proof. Note that we should understand the condition

$$\left[|\alpha| > 1 \text{ and } Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}\right]$$

is required to eliminate the possibility for inclusion of an asymptotically isometric copy of c_0 although we will not need to use it through the later stages of the proof.

Let $T : E \to E$ be an affine nonexpansive mapping. Then, there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}} \in E$ such that $|||Tx^{(n)} - x^{(n)}||| \underset{n}{\longrightarrow} 0$ and so $\left\|Tx^{(n)} - x^{(n)}\right\|_\infty \underset{n}{\longrightarrow} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $z \in c_0$ such that $x^{(n)}$ converges to z in weak topology. Then, by Lemma 2, we can define a function $s : c_0 \to [0, \infty)$ by

$$s\left(y\right) = \limsup_m \left|\left|\left|\frac{1}{m}\sum_{k=1}^{m} x^{(k)} - y\right|\right|\right| \text{ , } \forall y \in c_0.$$

Then,

$$s\left(y\right) = |||y - z|||,\ \forall y \in c_0.$$

Define

$$W := \overline{E}^{\sigma(l^\infty,\, l^1)} = \left\{\sum_{n=1}^{\infty} \alpha_n\, \eta_n :\ \text{each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n \leq 1\right\}$$

Case 1: $z \in E$.

Then, we have $s(Tz) = |||Tz - z|||$.

Also,

$$s\left(Tz\right) \leq \limsup_m \left|\left|\left|Tz - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\right|\right| \\ + \limsup_m \left|\left|\left|\frac{1}{m}\sum_{k=1}^{m} x^{(k)} - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\right|\right| .$$

But since T is affine,

$$\begin{aligned} s\left(Tz\right) &\leq \limsup_m \left|\left|\left|Tz - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\right|\right| + \limsup_m \left|\left|\left|\frac{1}{m}\sum_{k=1}^{m} x^{(k)} - \frac{1}{m}\sum_{k=1}^{m} Tx^{(k)}\right|\right|\right| \\ &\leq \limsup_m \left|\left|\left|z - \frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right|\right|\right| \\ &= s(z). \end{aligned}$$

Therefore, $|||z - Tz||| \leq 0$ and so $Tz = z$.

Case 2: $z \in W \setminus E$.

Then, z is of the form $\sum\limits_{n=1}^{\infty} \sigma_n \eta_n$ such that $\sum\limits_{n=1}^{\infty} \sigma_n \ < 1$.

Define

$$\delta := 1 - \sum_{n=1}^{\infty} \sigma_n$$

and define

$$h_\lambda := (\sigma_1 + \lambda\delta)\eta_1 + (\sigma_2 + (1-\lambda)\delta)\eta_2 + \sum_{n=3}^{\infty} \sigma_n \eta_n.$$

We want h_λ to be in E, so we restrict values of λ to be in $\left[-\frac{\sigma_1}{\delta},\ \frac{\sigma_2}{\delta} + 1\right]$, then

$$\begin{aligned} |||h_\lambda - z||| &= |||\lambda\delta\eta_1 + (1-\lambda)\delta\eta_2||| \\ &= |||\lambda\delta e_1 + (1-\lambda)\delta(e_1 + e_2)||| \\ &\leq \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] \lim_{p\to\infty} \sup \left\{ \gamma_1 \left[\frac{\delta^p}{2} + \frac{|1-\lambda|^p \delta^p}{4}\right]^{\frac{1}{p}}, \frac{\gamma_2|1-\lambda|\delta}{4} \right\} \\ &= \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] \max\left\{ \gamma_1|1-\lambda|\delta, \frac{\gamma_2|1-\lambda|\delta}{4} \right\}. \end{aligned}$$

Define

$$\Gamma := \min_{\lambda \in \left[-\frac{\gamma_1}{\delta},\ \frac{\gamma_2}{\delta}+1\right]} |||h_\lambda - z|||.$$

Hence, $\Gamma = 0$ and so there exists unique h_{λ_0} with $\lambda_0 \in \left[-\frac{\gamma_1}{\delta},\ \frac{\gamma_2}{\delta} + 1\right]$ such that $|||h_{\lambda_0} - z|||$ is minimizer of Γ. Now, define a subset in our set by $\Lambda := \{y : |||y - z||| \leq \Gamma\}$. Note that $\Lambda \subseteq E$ is a nonempty compact convex subset such that for any $h \in \Lambda$,

$$\begin{aligned} s\,(Th) \leq \limsup_m & \left|\left|\left| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\right|\right| \\ & + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\right|\right| . \end{aligned}$$

and since T is affine, the last inequality says that

$$
\begin{aligned}
s\left(Th\right) &\leq \limsup_m \left|\!\left|\!\left| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right)\right|\!\right|\!\right| \\
&\quad + \limsup_m \left|\!\left|\!\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - \frac{1}{m}\sum_{k=1}^{m} Tx^{(k)} \right|\!\right|\!\right| \\
&\leq \limsup_m \left|\!\left|\!\left| h - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right|\!\right|\!\right| \\
&= s(h).
\end{aligned}
$$

Also, $s(Th) = |||z - Th|||$ and $s(h) = |||z - h|||$.
Hence,

$$
|||z - Th||| \leq |||z - h||| \Longrightarrow |||z - Th||| = |||z - h||| \Longrightarrow Th \in \varLambda.
$$

Therefore, $T(\varLambda) \subseteq \varLambda$ and since T is continuous, Brouwer's Fixed Point Theorem [2] tells us that T has a fixed point such that $h = h_{\lambda_0}$ is the unique minimizer of $|||y - z|||\ :\ y \in E$ and $Th = h$.

Hence, E has FPP (n.e.) as desired.

Theorem 10. *Fix $b \in (0,1)$ and define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by the following way: $f_1 := b\,e_1$, $f_2 := b\,e_2$, and $f_n := e_n$, for every $n \geq 3$ where $(e_n)_{n\in\mathbb{N}}$ is defined to be 1 in its nth coordinate, and 0 in all other coordinates such that for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$, the sequence $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis.*

Next, define the closed, bounded, convex subset $E = E_b$ of c_0 by

$$
E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} .
$$

Then, define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way: $\eta_1 := f_1$ and $\eta_n := f_1 + \cdots + f_n$, for every $n \geq 2$. Note that E is the closed convex hull the sequence $(\eta_n)_{n\in\mathbb{N}}$. Then, the set E has the fixed point property for $|||\cdot|||$-nonexpansive affine mappings where the norm $|||\cdot|||$ is given as in Definition 4.

Proof. Again, the condition

$$
\left[|\alpha| > 1 \text{ and } Q_1 > \frac{1 - \gamma_1 + 4|\alpha|}{1 + 4|\alpha|} \right]
$$

is required only to eliminate the possibility for inclusion of an asymptotically isometric copy of c_0 although we will not need to use it through the later stages of the proof.

We will use exactly same method as the proof of Theorem 9 but just consider the following statements for the case 2 in the proof of Theorem 9.

$$\begin{aligned}\|\!|h_\lambda - z\|\!| &= \|\!|\lambda\delta\eta_1 + (1-\lambda)\delta\eta_2\|\!| \\ &= \|\!|\lambda\delta b e_1 + (1-\lambda)\delta(be_1 + be_2)\|\!| \\ &\le \frac{2\gamma_1(1+|\alpha|)+1}{\gamma_1} b \lim_{p\to\infty} \sup\left\{\gamma_1\left[\frac{\delta^p}{2} + \frac{|1-\lambda|^p\delta^p}{4}\right]^{\frac{1}{p}}, \frac{\gamma_2|1-\lambda|\delta}{4}\right\} \\ &= \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] b \max\left\{\gamma_1|1-\lambda|\delta, \frac{\gamma_2|1-\lambda|\delta}{4}\right\}.\end{aligned}$$

Define

$$\Gamma := \min_{\lambda\in\left[-\frac{\gamma_1}{\delta}, \frac{\gamma_2}{\delta}+1\right]} \|\!|h_\lambda - z\|\!|.$$

Hence, $\Gamma = 0$.

Now, we generalize our result more and give the following theorems.

Theorem 11. *Assume $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ is any increasing sequence in $(0,1]$ with $b_n \uparrow_n 1$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by the following way: $f_n := b_n\, e_n$, for every $n \in \mathbb{N}$. Next, take $E = E_{\overrightarrow{b}}$ of c_0 such that*

$$E := \left\{\sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \ge t_2 \ge \cdots \ge t_n \downarrow_n 0\right\} .$$

Then, the set E has the fixed point property for $\|\!|\cdot\|\!|$-nonexpansive affine mappings if $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$ where the norm $\|\!|\cdot\|\!|$ is given as in Definition 4.

Proof. Again, the condition

$$\left[|\alpha| > 1 \text{ and } Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}\right]$$

is required only to eliminate the possibility for inclusion of an asymptotically isometric copy of c_0 although we will not need to use it through the later stages of the proof.

We will use exactly same method as the proof of Theorem 9 but just consider the following statements for the case 2 in the proof of Theorem 9.

$$\begin{aligned}\|\!|h_\lambda - z\|\!| &= \|\!|\lambda\delta\eta_1 + (1-\lambda)\delta\eta_2\|\!| \\ &= \|\!|\lambda\delta b_1 e_1 + (1-\lambda)\delta(b_1 e_1 + b_2 e_2)\|\!| \\ &\le \frac{2\gamma_1(1+|\alpha|)+1}{\gamma_1} \lim_{p\to\infty} \sup\left\{\gamma_1\left[\begin{array}{l}\frac{{b_1}^p\delta^p}{2} \\ +\frac{{b_2}^p|1-\lambda|^p\delta^p}{4}\end{array}\right]^{\frac{1}{p}}, \frac{\gamma_2 b_2|1-\lambda|\delta}{4}\right\} \\ &= \frac{2\gamma_1(1+|\alpha|)+1}{\gamma_1} b_2 \lim_{p\to\infty} \sup\left\{\gamma_1\left[\begin{array}{l}\left(\frac{b_1}{b_2}\right)^p\frac{\delta^p}{2} \\ +\frac{|1-\lambda|^p\delta^p}{4}\end{array}\right]^{\frac{1}{p}}, \frac{\gamma_2|1-\lambda|\delta}{4}\right\} \\ &\le \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] b_2 \max\left\{\gamma_1|1-\lambda|\delta, \frac{\gamma_2|1-\lambda|\delta}{4}\right\}.\end{aligned}$$

Define $\varGamma := \min\limits_{\lambda\in\left[-\frac{\gamma_1}{\delta},\, \frac{\gamma_2}{\delta}+1\right]} |||h_\lambda - z|||$.

Hence, $\varGamma = 0$ since

$$\varGamma \leq \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] \min_{\lambda\in\left[-\frac{\gamma_1}{\delta},\, \frac{\gamma_2}{\delta}+1\right]} b_2 \max\left\{\gamma_1|1-\lambda|\delta, \frac{\gamma_2|1-\lambda|\delta}{4}\right\} = 0.$$

Theorem 12. *Assume $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ is any sequence in $(0,\infty)$ converging to some $\kappa > 0$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by the following way: $f_n := b_n\, e_n$, for every $n\in\mathbb{N}$. Next, take $E = E_{\overrightarrow{b}}$ of c_0 such that*

$$E := \left\{\sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0\right\}.$$

Then, the set E has the fixed point property for $|||\cdot|||$-nonexpansive affine mappings if $|\alpha| > 1$ and $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$ where the norm $|||\cdot|||$ is given as in Definition 4.

Proof. Again, the condition

$$\left[|\alpha| > 1 \text{ and } Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}\right]$$

is required only to eliminate the possibility for inclusion of an asymptotically isometric copy of c_0 although we will not need to use it through the later stages of the proof.

We will use exactly same method as the proof of Theorem 9 but just consider the following statements for the case 2 in the proof of Theorem 9.

$$\begin{aligned}
|||h_\lambda - z||| &= |||\lambda\delta\eta_1 + (1-\lambda)\delta\eta_2||| \\
&= |||\lambda\delta b_1 e_1 + (1-\lambda)\delta(b_1 e_1 + b_2 e_2)||| \\
&\leq \frac{2\gamma_1(1+|\alpha|)+1}{\gamma_1} \limsup_{p\to\infty} \left\{\gamma_1 \left[\begin{array}{c} \frac{{b_1}^p\delta^p}{2} \\ + \frac{{b_2}^p|1-\lambda|^p\delta^p}{4} \end{array}\right]^{\frac{1}{p}}, \frac{\gamma_2 b_2|1-\lambda|\delta}{4}\right\} \\
&\leq \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] \max\{b_1, b_2\} \max\left\{\gamma_1|1-\lambda|\delta, \frac{\gamma_2|1-\lambda|\delta}{4}\right\}.
\end{aligned}$$

Define $\varGamma := \min\limits_{\lambda\in\left[-\frac{\gamma_1}{\delta},\, \frac{\gamma_2}{\delta}+1\right]} |||h_\lambda - z|||$.

Hence, $\varGamma = 0$ since

$$\begin{aligned}
\varGamma &\leq \left[2(1+|\alpha|) + \frac{1}{\gamma_1}\right] \min_{\lambda\in\left[-\frac{\gamma_1}{\delta},\, \frac{\gamma_2}{\delta}+1\right]} \max\{b_1, b_2\} \max\left\{\gamma_1|1-\lambda|\delta, \frac{\gamma_2|1-\lambda|\delta}{4}\right\} \\
&= 0.
\end{aligned}$$

Then, the following corollaries are immediate by the proof of Theorem 12.

Corollary 2. *Assume* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *is any bounded sequence in* $(0,\infty)$*. Define the sequence* $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by the following way:* $f_n := b_n\, e_n$*, for every* $n \in \mathbb{N}$*. Next, take* $E = E_{\overrightarrow{b}}$ *of* c_0 *such that*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\} .$$

Then, the set E *has the fixed point property for* $|||\cdot|||$*-nonexpansive affine mappings if* $|\alpha| > 1$ *and* $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$ *where the norm* $|||\cdot|||$ *is given as in Definition 4.*

Then, we give our most generalized result by the proof of Theorem 12 and following corollaries are straightforward.

Theorem 13. *Let* $\overrightarrow{b} = (b_n)_{n\in\mathbb{N}}$ *be any sequence in* $(0,\infty)$*. We define the sequence* $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by setting* $f_n := b_n\, e_n$*, for all* $n \in \mathbb{N}$*. Next, define the closed, bounded, convex subset* $E = E_{\overrightarrow{b}}$ *of* c_0 *by* $E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : 1 = t_1 \geq t_2 \geq \cdots \geq t_n \downarrow_n 0 \right\}$*. Then, the set* E *has the fixed point property for* $|||\cdot|||$*-nonexpansive affine mappings if* $|\alpha| > 1$ *and* $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$ *where the norm* $|||\cdot|||$ *is given as in Definition 4.*

Corollary 3. *Assume* $(\mu_n)_{n\in\mathbb{N}}$*is a sequence in* $(0,\infty)$ *for which there exists* $\Gamma > 0$ *with [*$\Gamma \leq \mu_N$*, for every* $N \in \mathbb{N}$*] and* $\sigma := \sum_{n=2}^{\infty} |\mu_n - \mu_{n-1}| < \infty$ *; and assume* $(b_n)_{n\in\mathbb{N}}$ *is a sequence in* $(0,\infty)$ *converging to some* $\lambda \in (0,\infty)$*. Define the sequence* $(\eta_n)_{n\in\mathbb{N}}$ *by setting* $\eta_n := \mu_n(b_1e_1+b_2e_2+b_3e_3+b_4e_4+....+b_ne_n)$*, for all* $n \in \mathbb{N}$*. Also suppose that* $(\eta_n)_{n\in\mathbb{N}}$ *satisfies a lower* c_0*-summing estimate. That is, suppose* $\exists\, K \in (0,\infty)$ *s.t.* $\forall \alpha = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$*,* $K \sup_{n\geq 1} \left| \sum_{j=n}^{\infty} \alpha_j \right| \leq \left\| \sum_{j=1}^{\infty} \alpha_j\, \eta_j \right\|_{\infty}$*. Consider the closed convex hull of* $(\eta_n)_{n\in\mathbb{N}}$*,* $E := \overline{\mathrm{co}}(\{\eta_n : n \in \mathbb{N}\})$*.*

Then, the set E *has the fixed point property for* $|||\cdot|||$*-nonexpansive affine mappings if* $|\alpha| > 1$ *and* $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$ *where the norm* $|||\cdot|||$ *is given as in Definition 4.*

Corollary 4. *Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence in* $(0,\infty)$ *and let* $(b_n)_{n\in\mathbb{N}}$ *be any sequence in* $(0,\infty)$*. Define the sequence* $(\eta_n)_{n\in\mathbb{N}}$ *by setting* $\eta_n := \mu_n(b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + ... + b_ne_n)$*, for all* $n \in \mathbb{N}$*.*

Let E *be the closed convex hull of* $(\eta_n)_{n\in\mathbb{N}}$*,* $E := \overline{\mathrm{co}}(\{\eta_n : n \in \mathbb{N}\})$*.*

Then, the set E *has the fixed point property for* $|||\cdot|||$*-nonexpansive affine mappings if* $|\alpha| > 1$ *and* $Q_1 > \frac{1-\gamma_1+4|\alpha|}{1+4|\alpha|}$ *where the norm* $|||\cdot|||$ *is given as in Definition 4.*

5 Conclusion and Remarks

In 2004, Dowling, Lennard and Turett proved that weak compactness in c_0 is equivalent to the fixed point property for nonexpansive mappings [5]. Their result has been investigated for affine mappings but this equivalence has not been proved for affine nonexpansive mappings. However, in 2011, Lennard and Nezir proved that there exists very large class of nonweakly compact, closed, bounded and convex subsets of c_0 that fails FPP for affine nonexpansive mappings. Conversely, in 1979, Goebel and Kuczumow showed that there exists very large class of nonweakly compact, closed, bounded and convex subsets of ℓ^1 that has FPP for nonexpansive mappings.

Using the usual norm of c_0, Dowling, Lennard and Turett's result about equivalence of FPP for nonexpansive mappings and weak compactness shows that Goebel and Kuczumow's analogy for c_0 does not hold. That is why, it has been interesting question for researchers to think about the analogy of Goebel and Kuczumow's result using an equivalent norm. In fact, Lin proved that the whole Banach space ℓ^1 can be renormed to have FPP for nonexpansive mappings but c_0 analogy of Lin's result is still open. But our paper gives positive answer for Goebel and Kuczumow's analogy with an equivalent norm under affinity condition. We prove that there exist a renorming of c_0 and a very large class of non-weakly compact, closed, bounded and convex subsets of c_0 with FPP for affine nonexpansive mappings.

Our constructions use the sets and their generalizations in the paper by Lennard and Nezir [10]. In their study, they proved that for all bounded sequences $(b_n)_{n\in\mathbb{N}}$ in $(0,\infty)$, by defining $f_n := b_n\, e_n$ and $\eta_n := \sum\limits_{k=1}^{n} f_n$ for each $n \in \mathbb{N}$, the closed convex hull of $(\eta_n)_{n\in\mathbb{N}}$, $E = \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$ fails the fixed point property (FPP) for affine $\|\cdot\|_\infty$-nonexpansive mappings. Similarly, in the Ph.D. thesis of Nezir, they observed some c_0-summing basic sequences in $(c_0, \|\cdot\|_\infty)$ defined by $\eta_n := \gamma_n(b_1e_1 + b_2e_2 + b_3e_3 + ... + b_ne_n)$ for all $n \in \mathbb{N}$, they showed that whenever $0 < b_n$ converges to 1 and $0 < \gamma_n$ converges to 1 and $(\gamma_n)_{n\in\mathbb{N}}$ does not "oscillate too wildly", then $E = \overline{\text{co}}(\{\eta_n : n \in \mathbb{N}\})$ also fails FPP for affine $\|\cdot\|_\infty$-nonexpansive mappings.

In this paper, we construct an equivalent norm $|||\cdot|||$ on c_0 and show that all these sets mentioned above have FPP for affine $|||\cdot|||$-nonexpansive mappings. Moreover, we generalize our results for the closed convex hull of the sequence $\eta_n := \gamma_n(b_1e_1+b_2e_2+...+b_ne_n)$ when $0 < b_n$ and $0 < \gamma_n$ are arbitrarily choosen.

Working on finding a renorming and a class of non-weakly compact, closed bounded and convex subsets of c_0 with FPP for nonexpansive mappings that eliminates our required assumption of affinity will lead researchers to study a big open question, c_0-analogue of Lin's result. Hence, our paper might provide methods and candidates of renormings to work on further research to answer whether or not there exists a renorming of c_0 with FPP for nonexpansive mappings. But it would be an interesting study to check if one can eliminate the required affinity assumption on our class of sets to have FPP for nonexpansive mappings at first.

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4-Dimensional Transportation Problem for Substitute and Complementary Items Under Rough Environment

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Abstract. In this paper, an innovative 4-dimensional multifarious breakable items transportation problem (4DMBITP) has been proposed. Here, per unit selling expenses, per unit purchasing prices, per unit transportation expenditures, fixed charge, availability of the sources, demands of the destination, conveyances capacities and total available budget are expressed by rough intervals. The transported items are substitutable and complementary in nature. The demand of the items at the destination are directly related to the substitutability and complementary nature of the products and own selling price. The suggested model is converted into a deterministic one using lower and upper approximation intervals following Hamzehee et al. [1] as well as Expected Value Technique. The converted model is optimized through Generalized Reduced Gradient (GRG) techniques using LINGO 14 software. Finally, numerical examples are presented to illustrate the preciseness of the proposed model.

Keywords: 4-dimensional TP · Rough interval · Substitutable and complimentary items · Fixed charge · Budget constraint

1 Introduction

Transportation problems (TP) are one of the most common special type linear programming problems involving constraints (source constraint and demand constraint). TP is one of the optimization problems most widely used by private and public sectors. In genuine circumstance, we handle with constraints

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over and above, that is product constraints or/and transportation status constraints. In this way, the conventional 2D-transportation problem converts into a 3D-transportation problem i.e. STP (Solid Transportation Problem). As an assumption for customary TP, the introduction of STP, made by Haley [11]. As of late, the STP acquired much attraction and numerous algorithms have been developed. The problem of fixed charge (FC) was introduced by Hirsch and Dantzig [7]. Until now its application has been wide in the field of decision making and optimization problems.

Many researchers are working in this area such as Kennington and Unger [6] and others. Most of the practical concepts are vague in nature. In a transportation problem, impreciseness is observed due to deficiency of fixed data that is availability of the sources, demands of the destination, conveyances capacities and transportation expenditure etc. So investigation of transportation problem under uncertain environment is an important research issue.

In the last century where technology and industries govern the civilization, the need for sophisticated and easy techniques in the industry for optimizing products and profit is of immense importance rather than cost minimization. To further proceed and bring new field of transportation, we investigate four-dimensional transportation issues for multifarious item with rough interval coefficient.

The year wise implementations of TP and STP with different variation are recorded at a glance in the follows Table 1.

Table 1. Year wise implementations of TP and STP

References'	Different kind of TP	Item	Fixed charge	Budget constraint	Different kind of environment
Hitchcock et al. [3] in 1941	2-dimensional	one	×	–	crisp
Schell [4] in 1955	3-dimensional	one	×	–	crisp
Haley [11] in 1962	3-dimensional	one	×	–	crisp
Hirsch and Dantzig [7] in 1968	2-dimensional	one	✓	×	crisp
Tao et al. [12] in 2012	3-dimensional	one	×	×	rough
Ojha et al. [10] in 2013	2-dimensional	more than one	×	✓	fuzzy
Yang et al. [13] in 2015	2-dimensional	one	✓	×	fuzzy
Giri et al. [9] in 2015	3-dimensional	more than one	✓	×	fuzzy
Kocken et al. [8] in 2016	3-dimensional	one	×	×	fuzzy
Das et al. [5] in 2017	3-dimensional	one	×	×	rough
Present Paper	4-dimensional	more than one	✓	✓	rough

Thus, the main finding of this work is as follows:

- Some researchers may consider 4D-TPs and 4D-MITPs. More over, 4D-MBITP with rough parameters is also an updated contribution.
- The items are complementary and substitutability in nature, that is the demand is negative on account of the effect of substitutability and is positive when the items are complementary types.
- The earlier researchers give attention to minimization of aggregate transportation expenditure and very few have realized the important of consideration of total profit instead to total cost.

2 Preliminary

In this section some definitions and properties on Rough Intervals (RI) are given. For more details see Hamzehee et al. [1]. An RI can be treated as a qualitative amount from vague concept specified on a variable $xin\mathbf{R}$ [14], which is abstracted in the following definition.

2.1 Rough Intervals and Its Algebra

The algebraic operations applied on Rough Intervals (RIs) are taken from Moore's Interval Arithmetic [13]. Rebolledo [12] discussed the rough interval arithmetic very significantly.

Let Q= $([\underline{a}_l, \underline{a}_u], [\overline{a}_l, \overline{a}_u])$ and T= $([\underline{b}_l, \underline{b}_u], [\overline{b}_l, \overline{b}_u])$ represents the rough intervals. Then the different operation on these are:

Addition: $Q + T = ([\underline{a}_l + \underline{b}_l, \underline{a}_u + \underline{b}_u], [\overline{a}_l + \overline{b}_l, \overline{a}_u + \overline{b}_u])$
Subtraction: $Q - T = ([\underline{a}_l - \underline{b}_u, \underline{a}_u - \underline{b}_l], [\overline{a}_l - \overline{b}_u, \overline{a}_u - \overline{b}_l])$
Negation: $-Q = ([-\underline{a}_u,\ \underline{a}_l],\ [-\underline{a}_u,\ \underline{a}_l])$.

2.2 Expected Value of a Rough Interval (Shu & Edmund [15]):

Let an event X be uttered by $\{x \mid \psi(x) \in T\}$ whereas ψ is a mathematical function in universe U in $\mathbb{R}$, $T \subseteq \mathbb{R}$ and X is estimated by $(\underline{X}, \overline{X})$ consorting to the similar relation in $\mathbb{R}$. The corresponding lower expected measure of X is given by

$$\underline{E}[\psi] = \int_0^{\infty} \underline{Appro}\{\psi \geq y\}dy - \int_{-\infty}^{0} \underline{Appro}\{\psi \leq y\}dy \ \ where \ \ \underline{Appro}(X) = \frac{|X \cap \underline{X}|}{\underline{X}}$$

and the corresponding upper expected measure of X is given by

$$\overline{E}[\psi] = \int_0^{\infty} \overline{Appro}\{\psi \geq y\}dy - \int_{-\infty}^{0} \overline{Appro}\{\psi \leq y\}dy \ \ where \ \ \overline{Appro}(X) = \frac{|X|}{\overline{X}}$$

And the corresponding expected value of X is given by

$$E[X] = \int_0^{\infty} Appro\{\psi \geq y\}dy - \int_{-\infty}^{0} Appro\{\psi \leq y\}dy$$

Proposition (Shu & Edmund [15]):
For any predetermined parameter η, choosed by the decision maker's (DM's) preference
$E(X) = \eta \underline{E}(X) + (1-\eta)\overline{E}(X)$

2.3 General Linear Programming Problem in Rough Interval Environment

A Linear Programming Problem (LPP) to determine the decision vector $x = (x_1, x_2, ..., x_n)^t$ in which the coefficients of the objective function and the constraints are rough intervals.

$$\left.\begin{array}{l} \text{Maximize } Z = \sum_{j=1}^{N}([\underline{c}_{jl}, \underline{c}_{ju}], [\overline{c}_{jl}, \overline{c}_{ju}])x_j \\ \quad \text{s.t } \sum_{j=1}^{N}([\underline{a}_{ijl}, \underline{a}_{iju}], [\overline{a}_{ijl}, \overline{a}_{iju}])x_j \leq ([\underline{b}_{il}, \underline{b}_{iu}], [\overline{b}_{il}, \overline{b}_{iu}]) \quad \forall \quad i = 1, 2, ...m, \\ \quad \sum_{j=1}^{N}([\underline{d}_{kjl}, \underline{d}_{kju}], [\overline{d}_{kjl}, \overline{d}_{kju}])x_j \geq ([\underline{e}_{kl}, \underline{e}_{ku}], [\overline{e}_{kl}, \overline{e}_{ku}]) \quad \forall \quad k = 1, 2, ...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \quad j = 1, 2, ...n \end{array}\right\} \tag{1}$$

Where $([\underline{c}_{jl}, \underline{c}_{ju}], [\overline{c}_{jl}, \overline{c}_{ju}])$, $([\underline{a}_{ijl}, \underline{a}_{iju}], [\overline{a}_{ijl}, \overline{a}_{iju}])$, $([\underline{b}_{il}, \underline{b}_{iu}], [\overline{b}_{il}, \overline{b}_{iu}])$, $([\underline{d}_{kjl}, \underline{d}_{kju}], [\overline{d}_{kjl}, \overline{d}_{kju}])$ and $([\underline{e}_{kl}, \underline{e}_{ku}], [\overline{e}_{kl}, \overline{e}_{ku}])$ $(i = 1, 2, ...m;\ j = 1, 2, ...n;\ k = 1, 2, ...p;)$ are rough intervals. Now according to Hamzehee et al. [1], here we state two theorems which help us to find the rough optimal range of the problem (1). Let us consider two LPIC problems which are as follows.

LPIC-1:

$$\left.\begin{array}{l} \text{Maximize } Z = \sum_{j=1}^{N}[\underline{c}_{jl}, \underline{c}_{ju}]x_j \\ \quad \text{s.t } \sum_{j=1}^{N}([\underline{a}_{ijl}, \underline{a}_{iju}])x_j \leq ([\underline{b}_{il}, \underline{b}_{iu}]) \quad \forall \quad i = 1, 2, ...m, \\ \quad \sum_{j=1}^{N}([\underline{d}_{kjl}, \underline{d}_{kju}])x_j \geq ([\underline{e}_{kl}, \underline{e}_{ku}]) \quad \forall \quad k = 1, 2, ...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \quad j = 1, 2, ...n \end{array}\right\} \tag{2}$$

LPIC-2:

$$\left.\begin{array}{l} \text{Maximize } Z = \sum_{j=1}^{N}([\overline{c}_{jl}, \overline{c}_{ju}])x_j \\ \quad \text{s.t } \sum_{j=1}^{N}([\overline{a}_{ijl}, \overline{a}_{iju}])x_j \leq ([\overline{b}_{il}, \overline{b}_{iu}]) \quad \forall \quad i = 1, 2, ...m, \\ \quad \sum_{j=1}^{N}([\overline{d}_{kjl}, \overline{d}_{kju}])x_j \geq ([\overline{e}_{kl}, \overline{e}_{ku}]) \quad \forall \quad k = 1, 2, ...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \quad j = 1, 2, ...n \end{array}\right\} \tag{3}$$

The best optimal solution of the problem (1) according to Hamzehee et al. [1] and Chinneck and Ramadan [2] is i.e. LPIC-1 is found by solving:
LP1:

$$\left.\begin{array}{l} \text{Maximize } Z=\sum_{j=1}^{N} \underline{c}_{ju} x_j \\ \quad \text{s.t } \sum_{j=1}^{N} \underline{a}_{ijl} x_j \leq \underline{b}_{iu} \quad \forall \;\; i=1,2,...m, \\ \quad \sum_{j=1}^{N} \underline{d}_{kju} x_j \geq \underline{e}_{kl}, \quad \forall \;\; k=1,2,...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \;\; j=1,2,...n \end{array}\right\} \tag{4}$$

The best optimal solution of the problem (1) i.e. LPIC-1 is found by solving:
LP2:

$$\left.\begin{array}{l} \text{Maximize } Z=\sum_{j=1}^{N} \underline{c}_{jl} x_j \\ \quad \text{s.t } \sum_{j=1}^{N} \underline{a}_{iju}) x_j \leq \underline{b}_{il} \quad \forall \;\; i=1,2,...m, \\ \quad \sum_{j=1}^{N} \underline{d}_{kjl} x_j \geq \underline{e}_{ku}) \quad \forall \;\; k=1,2,...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \;\; j=1,2,...n \end{array}\right\} \tag{5}$$

The best optimal solution of the problem (1) i.e. LPIC-2 is found by solving:
LP3:

$$\left.\begin{array}{l} \text{Maximize } Z=\sum_{j=1}^{N} \overline{c}_{ju} x_j \\ \quad \text{s.t } \sum_{j=1}^{N} \overline{a}_{ijl} x_j \leq \overline{b}_{iu} \quad \forall \;\; i=1,2,...m, \\ \quad \sum_{j=1}^{N} \overline{d}_{kju} x_j \geq \overline{e}_{kl} \quad \forall \;\; k=1,2,...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \;\; j=1,2,...n \end{array}\right\} \tag{6}$$

The best optimal solution of the problem (1) i.e. LPIC-2 is found by solving:
LP4:

$$\left.\begin{array}{l} \text{Maximize } Z=\sum_{j=1}^{N} \overline{c}_{jl} x_j \\ \quad \text{s.t } \sum_{j=1}^{N} \overline{a}_{iju}) x_j \leq \overline{b}_{il} \quad \forall \;\; i=1,2,...m, \\ \quad \sum_{j=1}^{N} \overline{d}_{kjl} x_j \geq \overline{e}_{ku} \quad \forall \;\; k=1,2,...p, \\ \quad \text{and } x_j \geq 0 \quad \forall \;\; j=1,2,...n \end{array}\right\} \tag{7}$$

Type and algorithm of solutions:
There are three possible types of solutions of the problem LPIC-1 and LPIC-2 which are listed below.

- If LP-1 and LP-2 (LP-3 and LP-4) have optimal solutions then the problem LPIC-1 (LPIC-2) has a finite bounded surely optimal (possibly optimal)

range. If the maximizing value of LP-1 and LP-2 (LP-3 and LP-4) are respectively $\underline{z}_l, \underline{z}_u (\overline{z}_l, \overline{z}_u)$ then the surely optimal range (possibly optimal range) of LPIC-1 (LPIC-2) is $[\underline{z}_l, \underline{z}_u]([\overline{z}_l, \overline{z}_u])$
- If LP-2 (LP-4) is unbounded then LPIC-1 (LPIC-2) is unbounded.
- If LP-1 (LP-3) is infeasible then LPIC-1 (LPIC-2) is infeasible.

The solution procedure i.e., optimization algorithm of the model can be depicted as (Fig. 1):

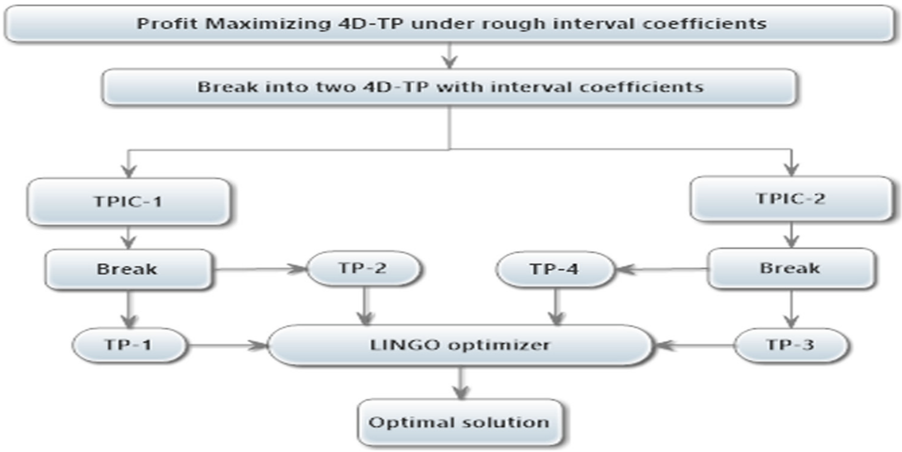

Fig. 1. Optimization algorithm flowchart

3 Model Description and Formulation

Notations
For r-th item,

- A_i^r: Quantity of homogeneous merchandise available at $i-th$ source.
- D_{0j}^r: Market demand at $j-th$ goal.
- D_j^r: Actual demand at $j-th$ goal.
- e_{kp}: Quantity of the merchandise which can be carried by $k-th$ conveyance along $p-th$ route.
- C_{ijpk}^r: Per unit transportation price from $i-th$ origin to $j-th$ goal by $k-th$ vehicle via $p-th$ route.
- S_j^r: Selling expenses at the $j-th$ destination.
- P_i^r: Purchasing price at the $i-th$ origin.
- x_{ijpk}^r: The transported quantity from $i-th$ source/origin to $j-th$ goal/destination by $k-th$ vehicle along $p-th$ route.
- f_{ijpk}: fixed transportation cost for shipping units from $i-th$ source/origin to $j-th$ goal/destination by $k-th$ vehicles along $p-th$ route.

- λ^r_{ijpk}: Rate of breakability per unit distance from $i-th$ source to $j-th$ goal via $p-th$ route and $k-th$ conveyance.
- $Budj$: Total budget at the $j-th$ goal point.
- ds_{ijp}: Distance from $i-th$ origin to $j-th$ goal along $p-th$ route.
- R: Total number of items.
- M: Total number of origin/sources.
- N: Total number of goal/destinations.
- L: Total number of route/paths.
- K: Total number of vehicles/conveyance.
- α: Power of the route length, related with the frangibility.
- β_j, θ^1_j, θ^{11}_j and τ: Price sensitivity of products.

Assumptions

(i) Particulars are breakable and carried from sources to goals using a vehicle through a path. Broken/ damaged amounts depend on conveyance and path.
(ii) Particulars are substitutable and complementary to each other. In case of substitute item, the demand is negative and is positive when the items are complementary nature.

$$\left.\begin{aligned} D^1_j &= D^1_{0j} - \beta_j S^1_j + \tau\beta_j S^2_j - \theta^1_j \beta_j S^3_j \\ D^2_j &= D^2_{0j} + \tau\beta_j S^1_j - \beta_j S^2_j - \theta^{11}_j \beta_j S^3_j \\ D^3_j &= D^3_{0j} - \theta^1_j \beta_j S^1_j - \theta^{11}_j \beta_j S^2_j - \beta_j S^3_j \end{aligned}\right\}$$

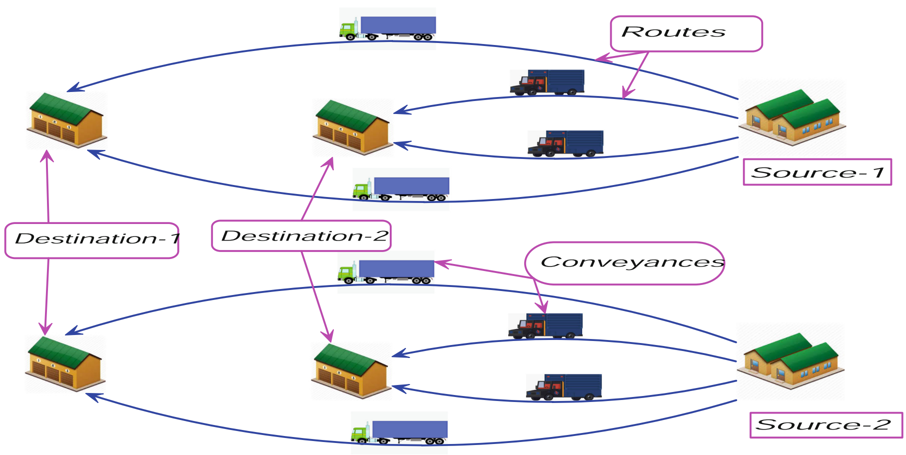

Fig. 2. Pictorial representation of 4-Dimensional Transportation Model

4 Proposed Model Formulation

The proposed model formulate for R substitutable particulars from M sources, transports those particulars through K vehicles along L routes to N goal point for sale. The schematic diagram of such transportation problem is depicted as Figure-2.

4.1 Model-I: Crisp Model

$$\left.\begin{array}{l}
\text{Maximize } Z = \sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\{S_j^r(1-\lambda_{ijpk}^r ds_{ijp}^{\alpha}) - P_i^r - C_{ijpk}^r ds_{ijp}\}x_{ijpk}^r \\
\qquad - \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K} w(x_{ijpk}^r) f_{ijpk}^r \\
\text{subject to } \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K} x_{ijpk}^r \le A_i^r, \quad \forall\ i=1,2,...,M, r=1,2,...,R \\
\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^1 ds_{ijp}^{\alpha})x_{ijpk}^1 \ge D_{0j}^1 - \beta_j s_j^1 + \tau\beta_j s_j^2 - \theta_j^1\beta_j s_j^3 \\
\qquad \forall\ j=1,2,...,N, r=1,2,...,R \\
\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^2 ds_{ijp}^{\alpha})x_{ijpk}^2 \ge D_{0j}^2 + \tau\beta_j S_j^1 - \beta_j S_j^2 - \theta_j^{11}\beta_j S_j^3 \\
\qquad \forall\ j=1,2,...,N, r=1,2,...,R \\
\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^3 ds_{ijp}^{\alpha})x_{ijpk}^3 \ge D_{0j}^3 - \theta_j^1\beta_j S_j^1 - \theta_j^{11}\beta_j S_j^2 - \beta_j S_j^3 \\
\qquad \forall\ j=1,2,...,N, r=1,2,...,R \\
\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N} x_{ijpk}^r \le e_{kp} \qquad \forall\ k=1,2,...,K, p=1,2,...,L \\
\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\left[P_i^r + C_{ijpk}^r ds_{ijp}\right]x_{ijpk}^r + \sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K} w(x_{ijpk}^r) f_{ijpk} \le Bud_j \\
\qquad \forall\ j=1,2,...N, \\
\text{and } x_{ijpk}^r \ge 0 \ \ \forall\ \ i=1,2,...,M, j=1,2,...,N, p=1,2,...,L, k=1,2,...,K, \\
\qquad r=1,2,...,R
\end{array}\right\} \tag{8}$$

4.2 Model-II: Rough Model

If the, selling expenses, cost of transportation, unit purchasing prices, fixed charges, availability, demand, capacities are taken as rough variable.

Let

$$\left.\begin{array}{l}
\check{S}_j^1 = ([\underline{S}_{jl}^1, \underline{S}_{ju}^1], [\overline{S}_{jl}^1, \overline{S}_{ju}^1]), \check{S}_j^2 = ([\underline{S}_{jl}^2, \underline{S}_{ju}^2], [\overline{S}_{jl}^2, \overline{S}_{ju}^2]), \check{S}_j^3 = ([\underline{S}_{jl}^3, \underline{S}_{ju}^3], [\overline{S}_{jl}^3, \overline{S}_{ju}^3]), \\
\check{P}_i^r = ([\underline{P}_{il}^r, \underline{P}_{iu}^r], [\overline{P}_{il}^r, \overline{P}_{iu}^r])\check{C}_{ijpk}^r = ([\underline{C}_{ijpkl}, \underline{C}_{ijpku}], [\overline{C}_{ijpkl}, \overline{C}_{ijpku}]), \\
\qquad \check{f}_{ijpk} = ([\underline{f}_{ijpkl}, \underline{f}_{ijpku}], [\overline{f}_{ijpkl}, \overline{f}_{ijpku}]), \\
\check{A}_i^r = ([\underline{A}_{il}^r, \underline{A}_{iu}^r], [\overline{A}_{il}^r, \overline{A}_{iu}^r])\check{D}_j^1 = ([\underline{D}_{0jl}^1, \underline{D}_{0ju}^1], [\overline{D}_{0jl}^1, \overline{D}_{0ju}^1]) \\
\qquad \check{D}_j^2 = ([\underline{D}_{0jl}^2, \underline{D}_{0ju}^2], [\overline{D}_{0jl}^2, \overline{D}_{0ju}^2]), \\
\check{D}_j^3 = ([\underline{D}_{0jl}^3, \underline{D}_{0ju}^3], [\overline{D}_{0jl}^3, \overline{D}_{0ju}^3])\check{e}_{kp} = ([\underline{e}_{kpl}, \underline{e}_{kpu}], [\overline{e}_{kpl}, \overline{e}_{kpu}]), \\
\qquad B\check{u}d_j = ([\underline{Bud}_{jl}, \underline{Bud}_{ju}], [\overline{Bud}_{jl}, \overline{Bud}_{ju}]) \\
\forall\ i=1,2,...,M,\ j=1,2,...,N,\ k=1,2,...,K,\ r=1,2,...,L.
\end{array}\right\} \tag{9}$$

So, after introducing of these rough parameters, the above crisp model (10) is translated to the following rough model.

$$\left.\begin{array}{l}
\text{Maximize } Z=\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\Bigg\{\Bigg(\big[\underline{S}^r_{jl},\underline{S}^r_{ju}\big],\big[\overline{S}^r_{jl},\overline{S}^r_{ju}\big]\Bigg)(1-\lambda^r_{ijpk}ds^{\alpha}_{ijp})\\
\quad-\Bigg(\big[\underline{P}^r_{il},\underline{P}^r_{iu}\big],\big[\overline{P}^r_{il},\overline{P}^r_{iu}\big]\Bigg)-\Bigg(\big[\underline{C}^r_{ijpkl},\underline{C}^r_{ijpku}\big],\big[\overline{C}^r_{ijpkl},\overline{C}^r_{ijpku}\big]\Bigg)ds_{ijp}\Bigg\}x^r_{ijpk}\\
\quad-\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x^r_{ijpk})\Bigg(\big[\underline{f}_{ijpkl},\underline{f}_{ijpku}\big],\big[\overline{f}_{ijpkl},\overline{f}_{ijpku}\big]\Bigg)\\
\text{subject to } \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}x^r_{ijpk}\le\Bigg(\big[\underline{A}^r_{il},\underline{A}^r_{iu}\big],\big[\overline{A}^r_{il},\overline{A}^r_{iu}\big]\Bigg)\\
\qquad\qquad\qquad\qquad\forall\ \ i=1,2,...,M,r=1,2,...,R\\
\quad\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^1_{ijpk}ds^{\alpha}_{ijp})x^1_{ijpk}\ge\Bigg\{\Bigg(\big[\underline{D}^1_{0jl},\underline{D}^1_{0ju}\big],\big[\overline{D}^1_{0jl},\overline{D}^1_{0ju}\big]\Bigg)\\
\quad-\beta_j\Bigg(\big[\underline{S}^1_{jl},\underline{S}^1_{ju}\big],\big[\overline{S}^1_{jl},\overline{S}^1_{ju}\big]\Bigg)+\tau\beta_j\Bigg(\big[\underline{S}^2_{jl},\underline{S}^2_{ju}\big],\big[\overline{S}^2_{jl},\overline{S}^2_{ju}\big]\Bigg)\\
\quad-\theta^1_j\beta_j\Bigg(\big[\underline{S}^3_{jl},\underline{S}^3_{ju}\big],\big[\overline{S}^3_{jl},\overline{S}^3_{ju}\big]\Bigg)\Bigg\},\quad\forall\ \ j=1,2,...,N,r=1,2,...,R\\
\quad\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^2_{ijpk}ds^{\alpha}_{ijp})x^2_{ijpk}\ge\Bigg\{\Bigg(\big[\underline{D}^2_{0jl},\underline{D}^2_{0ju}\big],\big[\overline{D}^2_{0jl},\overline{D}^2_{0ju}\big]\Bigg)\\
\quad+\tau\beta_j\Bigg(\big[\underline{S}^1_{jl},\underline{S}^1_{ju}\big],\big[\overline{S}^1_{jl},\overline{S}^1_{ju}\big]\Bigg)-\beta_j\Bigg(\big[\underline{S}^2_{jl},\underline{S}^2_{ju}\big],\big[\overline{S}^2_{jl},\overline{S}^2_{ju}\big]\Bigg)\\
\quad-\theta^{11}_j\beta_j\Bigg(\big[\underline{S}^3_{jl},\underline{S}^3_{ju}\big],\big[\overline{S}^3_{jl},\overline{S}^3_{ju}\big]\Bigg)\Bigg\},\quad\forall\ \ j=1,2,...,N,r=1,2,...,R\\
\quad\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^3_{ijpk}ds^{\alpha}_{ijp})x^3_{ijpk}\ge\Bigg\{\Bigg(\big[\underline{D}^3_{0jl},\underline{D}^3_{0ju}\big],\big[\overline{D}^3_{0jl},\overline{D}^3_{0ju}\big]\Bigg)\\
\quad-\beta_j\theta^1_j\Bigg(\big[\underline{S}^1_{jl},\underline{S}^1_{ju}\big],\big[\overline{S}^1_{jl},\overline{S}^1_{ju}\big]\Bigg)-\beta_j\theta^{11}_j\Bigg(\big[\underline{S}^2_{jl},\underline{S}^2_{ju}\big],\big[\overline{S}^2_{jl},\overline{S}^2_{ju}\big]\Bigg)\\
\quad-\beta_j\Bigg(\big[\underline{S}^3_{jl},\underline{S}^3_{ju}\big],\big[\overline{S}^3_{jl},\overline{S}^3_{ju}\big]\Bigg)\Bigg\},\quad\forall\ \ j=1,2,...,N,r=1,2,...,R\\
\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}x^r_{ijpk}\le\Bigg(\big[\underline{e}_{kpl},\underline{e}_{kpu}\big],\big[\overline{e}_{kpl},\overline{e}_{kpu}\big]\Bigg)\ \ \forall\ \ k=1,2,...,K,p=1,2,...,L\\
\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\Bigg\{\Bigg(\big[\underline{P}^r_{il},\underline{P}^r_{iu}\big],\big[\overline{P}^r_{il},\overline{P}^r_{iu}\big]\Bigg)\\
+\Bigg(\big[\underline{C}^r_{ijpkl},\underline{C}^r_{ijpku}\big],\big[\overline{C}^r_{ijpkl},\overline{C}^r_{ijpku}\big]\Bigg)ds_{ijp}\Bigg\}x^r_{ijpk}+\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x^r_{ijpk})\\
\Bigg(\big[\underline{f}_{ijpkl},\underline{f}_{ijpku}\big],\big[\overline{f}_{ijpkl},\overline{f}_{ijpku}\big]\Bigg)\le\Bigg(\big[\underline{Bud}_{jl},\underline{Bud}_{ju}\big],\big[\overline{Bud}_{jl},\overline{Bud}_{ju}\big]\Bigg)\\
\quad\forall\ \ j=1,2,...N\\
\text{and } x^r_{ijpk}\ge 0\ \ \forall\ \ i=1,2,...,M,j=1,2,...,N,\\
p=1,2,...,L,k=1,2,...,K,r=1,2,...,R
\end{array}\right\}\qquad(10)$$

According to the discussion of general LPP with rough interval coefficient (LPRIC) in the preliminary section, at first we get two different transportation problems (TPIC-1, TPIC-2) with interval coefficients. Again as discussed TPIC-1 and TPIC-2 can be braked as TP-1, TP-2 and TP-3, TP-4.

TP-1:

$$
\left.\begin{aligned}
&\text{Maximize } \underline{Z}^l = \sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\underline{S}_{jl}^{r}(1-\lambda_{ijpk}^{r}ds_{ijp}^{\alpha})-\underline{P}_{iu}^{r}-\underline{C}_{ijpku}^{r}ds_{ijp}\right\}x_{ijpk}^{r}\\
&\qquad -\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x_{ijpk}^{r})\underline{f}_{ijpku}\\
&\text{s.t} \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}x_{ijpk}^{r}\le \underline{A}_{il}^{r} \quad \forall\ \ i=1,2,...,M, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^{1}ds_{ijp}^{\alpha})x_{ijpk}^{1}\ge\left\{\underline{D}_{0ju}^{1}-\beta_j\underline{S}_{jl}^{1}+\tau\beta_j\underline{S}_{ju}^{2}-\theta_j^{1}\beta_j\underline{S}_{jl}^{3}\right\},\\
&\qquad\qquad \forall\ \ j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^{2}ds_{ijp}^{\alpha})x_{ijpk}^{2}\ge\left\{\underline{D}_{0ju}^{2}+\tau\beta_j\underline{S}_{ju}^{1}-\beta_j\underline{S}_{jl}^{2}-\theta_j^{11}\beta_j\underline{S}_{jl}^{3}\right\},\\
&\qquad\qquad \forall\ \ j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^{3}ds_{ijp}^{\alpha})x_{ijpk}^{3}\ge\left\{\underline{D}_{0ju}^{3}-\beta_j\theta_j^{1}\underline{S}_{jl}^{1}-\beta_j\theta_j^{11}\underline{S}_{jl}^{2}-\beta_j\underline{S}_{jl}^{3}\right\},\\
&\qquad\qquad \forall\ \ j=1,2,...,N, r=1,2,...,R\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}x_{ijpk}^{r}\le \underline{e}_{kpl} \quad k=1,2,...,K, p=1,2,...,L\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\underline{P}_{iu}^{r}+\underline{C}_{ijpku}^{r}ds_{ijp}\right\}x_{ijpk}^{r}+\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x_{ijpk}^{r})\underline{f}_{ijpku}\Big)\le \underline{Bud}_{jl},\\
&\qquad\qquad \forall\ \ j=1,2,...N\\
&\text{and } x_{ijpk}^{r}\ge 0 \ \ \forall\ \ i=1,2,...,M, j=1,2,...,N, p=1,2,...,L,\\
&\qquad\qquad k=1,2,...,K, r=1,2,...,R
\end{aligned}\right\}\quad(11)
$$

TP-2:

$$
\left.\begin{aligned}
&\text{Maximize } \underline{Z}^u = \sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\underline{S}_{ju}^{r}(1-\lambda_{ijpk}^{r}ds_{ijp}^{\alpha})-\underline{P}_{il}^{r}-\underline{C}_{ijpkl}^{r}ds_{ijp}\right\}x_{ijpk}^{r}\\
&\qquad -\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x_{ijpk}^{r})\underline{f}_{ijpkl}\\
&\text{s.t} \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}x_{ijpk}^{r}\le \underline{A}_{iu}^{r} \quad \forall\ \ i=1,2,...,M, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^{1}ds_{ijp}^{\alpha})x_{ijpk}^{1}\ge\left\{\underline{D}_{0jl}^{1}-\beta_j\underline{S}_{ju}^{1}+\tau\beta_j\underline{S}_{jl}^{2}-\theta_j^{1}\beta_j\underline{S}_{ju}^{3}\right\},\\
&\qquad\qquad \forall\ \ j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^{2}ds_{ijp}^{\alpha})x_{ijpk}^{2}\ge\left\{\underline{D}_{0jl}^{2}+\tau\beta_j\underline{S}_{jl}^{1}-\beta_j\underline{S}_{ju}^{2}-\theta_j^{11}\beta_j\underline{S}_{ju}^{3}\right\},\\
&\qquad\qquad \forall\ \ j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda_{ijpk}^{3}ds_{ijp}^{\alpha})x_{ijpk}^{3}\ge\left\{\underline{D}_{0jl}^{3}-\beta_j\theta_j^{1}\underline{S}_{ju}^{1}-\beta_j\theta_j^{11}\underline{S}_{ju}^{2}-\beta_j\underline{S}_{ju}^{3}\right\},\\
&\qquad\qquad \forall\ \ j=1,2,...,N, r=1,2,...,R\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}x_{ijpk}^{r}\le \underline{e}_{kpu} \quad \forall\ \ k=1,2,...,K, p=1,2,...,L\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\underline{P}_{il}^{r}+\underline{C}_{ijpkl}^{r}ds_{ijp}\right\}x_{ijpk}^{r}+\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x_{ijpk}^{r})\underline{f}_{ijpkl}\le \underline{Bud}_{ju}\\
&\quad \forall\ \ j=1,2,...N\\
&\text{and } x_{ijpk}^{r}\ge 0 \ \ \forall\ \ i=1,2,...,M, j=1,2,...,N, p=1,2,...,L,\\
&\qquad\qquad k=1,2,...,K, r=1,2,...,R
\end{aligned}\right\}\quad(12)
$$

TP-3:

$$\left.\begin{aligned}
&\text{Maximize } \overline{Z}^l = \sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\overline{S}^r_{jl}(1-\lambda^r_{ijpk}ds^{\alpha}_{ijp}) - \overline{P}^r_{iu} - \overline{C}^r_{ijpku}ds_{ijp}\right\}x^r_{ijpk}\\
&\qquad\qquad - \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K} w(x^r_{ijpk})\overline{f}_{ijpku}\\
&\text{subject to } \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K} x^r_{ijpk} \le \overline{A}^r_{il} \quad \forall \;\; i=1,2,...,M, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^1_{ijpk}ds^{\alpha}_{ijp})x^1_{ijpk} \ge \left\{\overline{D}^1_{0ju} - \beta_j\overline{S}^1_{jl} + \tau\beta_j\overline{S}^2_{ju} - \theta^1_j\beta_j\overline{S}^3_{jl}\right\},\\
&\qquad\qquad\qquad j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^2_{ijpk}ds^{\alpha}_{ijp})x^2_{ijpk} \ge \left\{\overline{D}^2_{0ju} + \tau\beta_j\overline{S}^1_{ju} - \beta_j\overline{S}^2_{jl} - \theta^{11}_j\beta_j\overline{S}^3_{jl}\right\},\\
&\qquad\qquad\qquad \forall \;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^3_{ijpk}ds^{\alpha}_{ijp})x^3_{ijpk} \ge \left\{\overline{D}^3_{0ju} - \beta_j\theta^1_j\overline{S}^1_{jl} - \beta_j\theta^{11}_j\overline{S}^2_{jl} - \beta_j\overline{S}^3_{jl}\right\},\\
&\qquad\qquad\qquad \forall \;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N} x^r_{ijpk} \le \overline{e}_{kpl} \quad k=1,2,...,K, p=1,2,...,L\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\overline{P}^r_{iu} + \overline{C}^r_{ijpku}ds_{ijp}\right\}x^r_{ijpk} + \sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K} w(x^r_{ijpk})\overline{f}_{ijpku} \le \overline{Bud}_{jl}\\
&\qquad\qquad\qquad \forall \quad j=1,2,...N\\
&\text{and } x^r_{ijpk} \ge 0 \;\; \forall \;\; i=1,2,...,M, j=1,2,...,N, p=1,2,...,L,\\
&\qquad\qquad\qquad k=1,2,...,K, r=1,2,...,R
\end{aligned}\right\} \quad (13)$$

TP-4:

$$\left.\begin{aligned}
&\text{Maximize } \overline{Z}^u = \sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\overline{S}^r_{ju}(1-\lambda^r_{ijpk}ds^{\alpha}_{ijp}) - \overline{P}^r_{il} - \overline{C}^r_{ijpkl}ds_{ijp}\right\}x^r_{ijpk}\\
&\qquad\qquad - \sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K} w(x^r_{ijpk})\overline{f}_{ijpkl}\\
&\text{subject to } \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K} x^r_{ijpk} \le \overline{A}^r_{iu} \quad \forall \;\; i=1,2,...,M, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^1_{ijpk}ds^{\alpha}_{ijp})x^1_{ijpk} \ge \left\{\overline{D}^1_{0jl} - \beta_j\overline{S}^1_{ju} + \tau\beta_j\overline{S}^2_{jl} - \theta^1_j\beta_j\overline{S}^3_{ju}\right\},\\
&\qquad\qquad\qquad \forall \;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^2_{ijpk}ds^{\alpha}_{ijp})x^2_{ijpk} \ge \left\{\overline{D}^2_{0jl} + \tau\beta_j\overline{S}^1_{jl} - \beta_j\overline{S}^2_{ju} - \theta^{11}_j\beta_j\overline{S}^3_{ju}\right\},\\
&\qquad\qquad\qquad \forall \;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^3_{ijpk}ds^{\alpha}_{ijp})x^3_{ijpk} \ge \left\{\overline{D}^3_{0jl} - \beta_j\theta^1_j\overline{S}^1_{ju} - \beta_j\theta^{11}_j\overline{S}^2_{ju} - \beta_j\overline{S}^3_{ju}\right\},\\
&\qquad\qquad\qquad \forall \;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N} x^r_{ijpk} \le \overline{e}_{kpu} \quad k=1,2,...,K, p=1,2,...,L\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\left\{\overline{P}^r_{il} + \overline{C}^r_{ijpkl}ds_{ijp}\right\}x^r_{ijpk} + \sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K} w(x^r_{ijpk})\overline{f}_{ijpkl} \le \overline{Bud}_{ju}\\
&\qquad\qquad\qquad \forall \;\; j=1,2,...N\\
&\text{and } x^r_{ijpk} \ge 0 \;\; \forall \;\; i=1,2,...,M, j=1,2,...,N, p=1,2,...,L,\\
&\qquad\qquad\qquad k=1,2,...,K, r=1,2,...,R
\end{aligned}\right\} \quad (14)$$

Approach-2:

After use of Expected Value Method the Model-II (Rough Model) mathematically reduces to:

$$
\left.
\begin{aligned}
&\text{Maximize } E\Big[Z\Big] = E\Bigg[\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\Bigg\{\Big([\underline{S}^r_{jl},\underline{S}^r_{ju}],[\overline{S}^r_{jl},\overline{S}^r_{ju}]\Big)(1-\lambda^r_{ijpk}ds^{\alpha}_{ijp})\\
&\quad -\Big([\underline{P}^r_{il},\underline{P}^r_{iu}],[\overline{P}^r_{il},\overline{P}^r_{iu}]\Big)-\Big([\underline{C}^r_{ijpkl},\underline{C}^r_{ijpku}],[\overline{C}^r_{ijpkl},\overline{C}^r_{ijpku}]\Big)ds_{ijp}\Bigg\}x^r_{ijpk}\\
&\quad -\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x^r_{ijpk})\Big([\underline{f}_{ijpkl},\underline{f}_{ijpku}],[\overline{f}_{ijpkl},\overline{f}_{ijpku}]\Big)\Bigg]\\
&\text{subject to } \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}x^r_{ijpk}\le E\Big[\Big([\underline{A}^r_{il},\underline{A}^r_{iu}],[\overline{A}^r_{il},\overline{A}^r_{iu}]\Big)\Big]\\
&\qquad\qquad\qquad\qquad \forall\;\; i=1,2,...,M, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^1_{ijpk}ds^{\alpha}_{ijp})x^1_{ijpk}\ge E\Bigg[\Bigg\{\Big([\underline{D}^1_{0jl},\underline{D}^1_{0ju}],[\overline{D}^1_{0jl},\overline{D}^1_{0ju}]\Big)\\
&\quad -\beta_j\Big([\underline{S}^1_{jl},\underline{S}^1_{ju}],[\overline{S}^1_{jl},\overline{S}^1_{ju}]\Big)+\tau\beta_j\Big([\underline{S}^2_{jl},\underline{S}^2_{ju}],[\overline{S}^2_{jl},\overline{S}^2_{ju}]\Big)\\
&\quad -\theta^1_j\beta_j\Big([\underline{S}^3_{jl},\underline{S}^3_{ju}],[\overline{S}^3_{jl},\overline{S}^3_{ju}]\Big)\Bigg\}\Bigg],\quad \forall\;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^2_{ijpk}ds^{\alpha}_{ijp})x^2_{ijpk}\ge E\Bigg[\Bigg\{\Big([\underline{D}^2_{0jl},\underline{D}^2_{0ju}],[\overline{D}^2_{0jl},\overline{D}^2_{0ju}]\Big)\\
&\quad +\tau\beta_j\Big([\underline{S}^1_{jl},\underline{S}^1_{ju}],[\overline{S}^1_{jl},\overline{S}^1_{ju}]\Big)-\beta_j\Big([\underline{S}^2_{jl},\underline{S}^2_{ju}],[\overline{S}^2_{jl},\overline{S}^2_{ju}]\Big)\\
&\quad -\theta^{11}_j\beta_j\Big([\underline{S}^3_{jl},\underline{S}^3_{ju}],[\overline{S}^3_{jl},\overline{S}^3_{ju}]\Big)\Bigg\}\Bigg],\quad \forall\;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^3_{ijpk}ds^{\alpha}_{ijp})x^3_{ijpk}\ge E\Bigg[\Bigg\{\Big([\underline{D}^3_{0jl},\underline{D}^3_{0ju}],[\overline{D}^3_{0jl},\overline{D}^3_{0ju}]\Big)\\
&\quad -\beta_j\theta^1_j\Big([\underline{S}^1_{jl},\underline{S}^1_{ju}],[\overline{S}^1_{jl},\overline{S}^1_{ju}]\Big)-\beta_j\theta^{11}_j\Big([\underline{S}^2_{jl},\underline{S}^2_{ju}],[\overline{S}^2_{jl},\overline{S}^2_{ju}]\Big)\\
&\quad -\beta_j\Big([\underline{S}^3_{jl},\underline{S}^3_{ju}],[\overline{S}^3_{jl},\overline{S}^3_{ju}]\Big)\Bigg\}\Bigg],\quad \forall\;\; j=1,2,...,N, r=1,2,...,R\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}x^r_{ijpk}\le E\Big[\Big([\underline{e}_{kpl},\underline{e}_{kpu}],[\overline{e}_{kpl},\overline{e}_{kpu}]\Big)\Big]\\
&\qquad\qquad\qquad \forall\;\; k=1,2,...,K, p=1,2,...,L\\
&\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}E\Bigg[\Bigg\{\Big([\underline{P}^r_{il},\underline{P}^r_{iu}],[\overline{P}^r_{il},\overline{P}^r_{iu}]\Big)\\
&\quad +\Big([\underline{C}^r_{ijpkl},\underline{C}^r_{ijpku}],[\overline{C}^r_{ijpkl},\overline{C}^r_{ijpku}]\Big)ds_{ijp}\Bigg\}x^r_{ijpk}\\
&\quad +\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x^r_{ijpk})\Big([\underline{f}_{ijpkl},\underline{f}_{ijpku}],[\overline{f}_{ijpkl},\overline{f}_{ijpku}]\Big)\Bigg]\\
&\quad \le E\Big[\Big([\underline{Bud}_{jl},\underline{Bud}_{ju}],[\overline{Bud}_{jl},\overline{Bud}_{ju}]\Big)\Big]\\
&\qquad\qquad\qquad \forall\;\; j=1,2,...N\\
&\text{and } x^r_{ijpk}\ge 0\;\;\forall\;\; i=1,2,...,M, j=1,2,...,N, p=1,2,...,L,\\
&\qquad\qquad\qquad k=1,2,...,K, r=1,2,...,R
\end{aligned}
\right\}\qquad (15)
$$

The problem (15) can be written as:

$$\left.\begin{array}{l}
\text{Maximize } Z = \sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}\Big[\{\eta(\underline{S}^{r}_{jl}+\underline{S}^{r}_{ju})+(1-\eta)(\overline{S}^{r}_{jl}+\overline{S}^{r}_{ju})\}/2*(1-\lambda^{r}_{ijpk}ds^{\alpha}_{ijp})\\
\quad -\{\eta(\underline{P}^{r}_{il}+\underline{P}^{r}_{iu})+(1-\eta)(\overline{P}^{r}_{il}+\overline{P}^{r}_{iu})\}/2-\{\eta(\underline{C}^{r}_{ijpkl}+\underline{C}^{r}_{ijpku})\\
\quad +(1-\eta)(\overline{C}^{r}_{ijpkl}+\overline{C}^{r}_{ijpku})\}/2*ds_{ijp}\Big]x^{r}_{ijpk}\\
\quad -\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x^{r}_{ijpk})\Big[\{\eta(\underline{f}_{ijpkl}+\underline{f}_{ijpku})+(1-\eta)(\overline{f}_{ijpkl}+\overline{f}_{ijpku})\}/2\Big]\\
\text{subject to } \sum_{j=1}^{N}\sum_{p=1}^{L}\sum_{k=1}^{K}x^{r}_{ijpk}\le\Big[\{\eta(\underline{A}^{r}_{il}+\underline{A}^{r}_{iu})+(1-\eta)(\overline{A}^{r}_{il}+\overline{A}^{r}_{iu})\}/2\Big]\\
\qquad\qquad\qquad\qquad \forall\ \ i=1,2,...,M, r=1,2,...,R\\
\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^{1}_{ijpk}ds^{\alpha}_{ijp})x^{1}_{ijpk}\ge\Big[\{\eta(\underline{D}^{1}_{0jl}+\underline{D}^{1}_{0ju})+(1-\eta)(\overline{D}^{1}_{0jl}+\overline{D}^{1}_{0ju})\}/2\\
\quad -\beta_j\{\eta(\underline{S}^{1}_{jl}+\underline{S}^{1}_{ju})+(1-\eta)(\overline{S}^{1}_{jl}+\overline{S}^{1}_{ju})\}/2+\tau\beta_j\{\eta(\underline{S}^{2}_{jl}+\underline{S}^{2}_{ju})+(1-\eta)(\overline{S}^{2}_{jl}+\overline{S}^{2}_{ju})\}/2\\
\quad -\theta^{1}_{j}\beta_j\{(\underline{S}^{3}_{jl}+\underline{S}^{3}_{ju})+(1-\eta)(\overline{S}^{3}_{jl}+\overline{S}^{3}_{ju})\}/2\Big],\quad\forall\ \ j=1,2,...,N, r=1,2,...,R\\
\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^{2}_{ijpk}ds^{\alpha}_{ijp})x^{2}_{ijpk}\ge\Big[\{\eta(\underline{D}^{2}_{0jl}+\underline{D}^{2}_{0ju})+(1-\eta)(\overline{D}^{2}_{0jl}+\overline{D}^{2}_{0ju})\}/2\\
\quad +\tau\beta_j\{\eta(\underline{S}^{1}_{jl}+\underline{S}^{1}_{ju})+(1-\eta)(\overline{S}^{1}_{jl}+\overline{S}^{1}_{ju})\}/2-\beta_j\{\eta(\underline{S}^{2}_{jl}+\underline{S}^{2}_{ju})+(1-\eta)(\overline{S}^{2}_{jl}+\overline{S}^{2}_{ju})/2\\
\quad -\theta^{11}_{j}\beta_j\{\eta(\underline{S}^{3}_{jl}+\underline{S}^{3}_{ju})+(1-\eta)(\overline{S}^{3}_{jl}+\overline{S}^{3}_{ju})\}/2\Big],\quad\forall\ \ j=1,2,...,N, r=1,2,...,R\\
\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}(1-\lambda^{3}_{ijpk}ds^{\alpha}_{ijp})x^{3}_{ijpk}\ge\Big[\{\eta(\underline{D}^{3}_{0jl}+\underline{D}^{3}_{0ju})+(1-\eta)(\overline{D}^{3}_{0jl}+\overline{D}^{3}_{0ju})\}/2\\
\quad -\beta_j\theta^{1}_{j}\{\eta(\underline{S}^{1}_{jl}+\underline{S}^{1}_{ju})+(1-\eta)(\overline{S}^{1}_{jl}+\overline{S}^{1}_{ju})\}/2-\beta_j\theta^{11}_{j}\{\eta(\underline{S}^{2}_{jl}+\underline{S}^{2}_{ju})+(1-\eta)(\overline{S}^{2}_{jl}+\overline{S}^{2}_{ju})\}/2\\
\quad -\beta_j\{\eta(\underline{S}^{3}_{jl}+\underline{S}^{3}_{ju})+(1-\eta)(\overline{S}^{3}_{jl}+\overline{S}^{3}_{ju})\}/2\Big],\ \ j=1,2,...,N, r=1,2,...,R\\
\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{j=1}^{N}x^{r}_{ijpk}\le\Big[\eta(\underline{e}_{kpl}+\underline{e}_{kpu})+(1-\eta)(\overline{e}_{kpl}+\overline{e}_{kpu})]/2)\Big]\\
\qquad\qquad\qquad\qquad\qquad\qquad \forall\ \ k=1,2,...,K, p=1,2,...,L\\
\sum_{r=1}^{R}\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}\Big[\{\eta(\underline{P}^{r}_{il}+(1-\eta)\underline{P}^{r}_{iu})+(\overline{P}^{r}_{il}+\overline{P}^{r}_{iu})\}/2+\{\eta(\underline{C}^{r}_{ijpkl}+\underline{C}^{r}_{ijpku})\\
\quad +(1-\eta)(\overline{C}^{r}_{ijpkl}+\overline{C}^{r}_{ijpku})\}/2*ds_{ijp}\Big]x^{r}_{ijpk}+\sum_{i=1}^{M}\sum_{p=1}^{L}\sum_{k=1}^{K}w(x^{r}_{ijpk})\Big[\{\eta(\underline{f}_{ijpkl}+\underline{f}_{ijpku})\\
\quad +(1-\eta)(\overline{f}_{ijpkl}+\overline{f}_{ijpku})\}/2\Big]\le\Big[\{\eta(\underline{Bud}_{jl}+\underline{Bud}_{ju})+(1-\eta)(\overline{Bud}_{jl}+\overline{Bud}_{ju})\}/2\Big]\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ \ j=1,2,...N\\
\text{and } x^{r}_{ijpk}\ge 0\ \ \forall\ \ i=1,2,...,M, j=1,2,...,N, p=1,2,...,L, k=1,2,...,K, r=1,2,...,R
\end{array}\right\}\tag{16}$$

5 Numerical Experiments

Let us consider a real life transportation system where, number of sources = 2 (i.e. M = 2), number of goal = 2 (i.e. N = 2), number of vehicles = 2 (i.e. K = 2) and number of routes = 2 (L = 2), number of items = 3 (i.e. R = 3) are considered. The input values crisp and rough interval form are presented in Tables 2, 3 and 4. The amount of breakability is crisp environment is also presented in Table 2.
The other parametric values are given in Tables 4 and 5.

Table 2. Transportation costs (C^r_{ijpk}), fixed charges (f_{ijpk}), breakability (λ^r_{ijpk})

	p	1				2			
	k	1		2		1		2	
	i/j	1	2	1	2	1	2	1	2
Model-I									
(Crisp Model)									
C^1_{ijkp}	1	0.5	2.1	1.7	1.12	0.85	0.5	1.38	1.28
	2	1.21	1.28	2.05	0.6	1.65	2.72	1.66	1.78
C^2_{ijkp}	1	1.5	2.25	2.0	1.3	0.08	0.7	1.05	1.25
	2	1.05	2.15	2.2	0.9	1.35	2.25	1.85	0.98
C^3_{ijkp}	1	1.3	1.2	1.91	1.2	0.9	0.94	0.73	0.93
	2	1.22	1.3	2	0.7	1.8	3	1.8	1.8
λ^1_{ijkp}	1	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
	2	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
λ^2_{ijkp}	1	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
	2	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
λ^3_{ijkp}	1	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
	2	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02
f_{ijkp}	1	1.9	1.7	1.1	1.3	1.2	1.4	0.3	1.8
	2	1.2	1.85	0.9	1.5	1.5	2.1	2.1	1.6

Table 3. Transportation costs (C^r_{ijpk}), fixed charges (f_{ijpk}), breakability (λ^r_{ijpk})

Model-II									
(Rough Model)									
$\check{C}^1_{ijkp}$	1	([1,2], [0.7,2.3])	([0.5,1.5], [0.5,2])	([1.1,2.3], [1,2.5])	([0.8,1.3], [0.5,2])	([1.4,2.3], [1.2,2.8])	([0.7,1], [0.5,1.5])	([1.2,2], [1.2,2.3])	([1.2,2.4], [1,3])
	2	([1.2,2.5], [1,3])	([0.9,2], [0.5,3.2])	([1.1,2], [0.7,2.5])	([1,2], [0.9,2.6])	([1.1,2.2], [1,2.7])	([1,2.5], [0.3,3])	([0.9,2.1], [0.7,2.8])	([1.3,2.5], [1.1,3.5])
$\check{C}^2_{ijkp}$	1	([1,2], [0.7,2.3])	([0.5,1.5], [0.5,2])	([1.1,2.3], [1,2.5])	([0.8,1.3], [0.5,2])	([1.4,2.3], [1.2,2.8])	([0.7,1], [0.5,1.5])	([1.2,2], [1.2,2.3])	([1.2,2.4], [1,3])
	2	([1.2,2.5], [1,3])	([0.9,2], [0.5,3.2])	([1.1,2], [0.7,2.5])	([1,2], [0.9,2.6])	([1.1,2.2], [1,2.7])	([1,2.5], [0.3,3])	([0.9,2.1], [0.7,2.8])	([1.3,2.5], [1.1,3.5])
$\check{C}^3_{ijkp}$	1	([1.3,2.7], [1,3])	([1.7,2.8], [1.2,3.1])	([1.5,2.5], [1.4,3.1])	([1.6,2.4], [1.5,3.5])	([1.3,3], [1.2,3.5])	([1.4,2], [1.1,2.5])	([1.5,2.6], [1.3,3])	([1.8,2.7], [1.6,3.4]
	2	([1.5,2.5], [1,3.5])	([1.3,3], [1.1,3.7])	([1.7,2.7], [1.5,3])	([1.4,3], [1.2,3.2])	([2,2.7], [1.8,3])	([1.3,2.2], [1,2.5])	([1.6,2.1], [1.5,2.8])	([1.6,2.5], [1.4,2.8])
$\check{f}_{ijkp}$	1	([0.5,1.5], [0.5,2])	([0.7,1], [0.6,1.5])	([0.7,1.3], [0.4,2])	([0.8,1.6], [0.3,1.7])	([0.7,1.4], [0.6,2])	([0.8,1.8], [0.3,2.3])	([1,1.5], [0.8,1.8])	([0.9,1.8], [0.7,2.1])
	2	([0.2,0.5], [0.1,1])	([0.3,0.6], [0.2,1])	([0.9,1.7], [0.6,1.8])	([1,1.5], [0.5,1.6])	([0.9,1.5], [0.4,1.9])	([0.5,1.6], [0.2,2])	([0.4,0.9], [0.2,1])	([1,1.7], [0.5,2])

Table 4. Parametric values for the Models.

Models	Source	Demand	Capacities of conveyance
	$(A_1^1, A_1^2, A_1^3$	$(D_{01}^1, D_{01}^2, D_{01}^3$	(e_1^1, e_1^2, e_1^3)
	$A_2^1, A_2^2, A_2^3)$	$D_{02}^1, D_{02}^2, D_{02}^2)$	(e_2^1, e_2^2, e_2^3)
-I	(90, 80, 85	(75, 73, 70,	(90, 85, 80
	85, 75, 70)	65, 63, 60)	85, 70, 75)
-II	{([89.5, 90],[88.6, 91]),	{([74, 75],[73.6, 76]),	{([89, 90],[88, 91]),
	([79.6, 80],[78.5, 81.3]),	([72.6, 73],[72, 74]),	([84, 85],[83, 86.3]),
	([84.7, 85],[83.5, 86]),	([69.4, 70],[68.7, 71]),	([79, 80],[78, 81.3]),
	([84.4, 85],[83.6, 86]),	([64.5,65],[64, 66]),	([84, 85],[83, 86.3]),
	([74, 75],[73.6, 76]),	([62.4, 63],[62, 64]),	([69, 70],[68, 71]),
	([69.4, 70],[68.7, 71])}	([59.4, 60],[59, 61])}	([74, 75],[73, 76])}

Table 5. Parametric values for the Models.

Models	Purchasing costs	Unit selling price	Budget
	$(P_1^1, P_1^2, P_1^3$	$(S_1^1, S_1^2, S_1^3$	(Bud_1, Bud_2, Bud_3)
	$P_2^1, P_2^2, P_2^3)$	$S_2^1, S_2^2, S_2^3)$	
-I	(9, 8, 10	(52, 30, 24,	(3000, 2500, 2700)
	6, 7, 8.5)	46, 28, 20)	
-III	{([8.3, 9],[8, 10]),	{([51.5, 52],[51, 53]),	{([8250,5000],
	([7.5, 8],[7, 9]),	([29, 30],[28, 31]),	[7500,8600]),
	([9, 10],[8, 10.5]),	([23.5, 24],[23, 25]),	[7500,8600]),
	([5.5, 6],[5, 7]),	([45.5,45],[45.3, 46]),	([4500,4600],
	([6.5, 7],[6, 7.8]),	([27.4, 28],[27, 29]),	[4400,4700]),
	([8, 8.5],[7.5, 9])}	([19.5, 20],[19.1, 21])}	[4400,4700])}

The following Table 6, represented the distances from different origins to destinations by route-1 and route-2.

Then the problem is to determined the optimal policy of transportation to maximize the profit. Under the objective maximization profit subject to the traditional constrain and budget constrain for the given input data are

Table 6. Distances (ds_{ijp}) of the routes for the Models.

Route	Destination	Origin-1	Origin-2
1	1	35	30
	2	40	25
2	1	30	35
	2	25	30

5.1 2-Dimensional TP Model

If we take L = 1 and K = 1 in Rough Model, then we get the 2D-TP Model. In our example, it is assumed that there are three types of products. The 1st and 2nd ones are the substitute, the 1st and 3rd are complementary, the 2nd and 3rd are also complimentary.

6 Discussion

Tables 7, 8 and 9, represents the optimal solutions for given assumed values, which are self explained. From the above observation of TP-1 and TP-2, we can conclude that it yields satisfactory results and that of TP-3 and TP-4 yield almost satisfactory results.

The profit is varied for the substitute and complementary nature of the items observed from Table 9. When items are independent then profit is maximum that is $ 7388.841. When item-1 and item-2 are substitute in nature with each other and item-3 is complementary nature then profit is $ 6966.775. But when the item-1 and item-2 are the substitute nature to each other then profit is minimum that is $5499.527.

Table 9 shown the optimal profit of model-1 for the nature of the items. This table helps manager to make the decision of the type of items he/she should use for his/her business policy. For the current consideration, it is observed that the existence of the complementary item is more profitable than the existence of the substitute item.

Table 7. Optimum results for Models

Models	Crisp model	4D Rough model			
		TP-1	TP-2	TP-3	TP-4
Optimal profit	6966.775	5293.95	8058.61	4109.11	9130.81
		Set-1	Set-2	Set-3	Set-4
	x11111 = 67.07	x11221 = 12.04	x11122 = 58.7	x11111 = 64.4	x11111 = 62.44
	x11122 = 51.75	x12112 = 48	x11221 = 65.3	x11221 = 32.2	x11112 = 15.41
	x11222 = 8.75	x12211 = 27.96	x11222 = 39.6	x11222 = 7.9	x11222 = 29.9
	x21121 = 66.15	x21121 = 52.46	x12211 = 12.1	x12211 = 0.2	x12112 = 9.09
	x21122 = 57.1	x21122 = 11.84	x12221 = 42.2	x21121 = 30.3	x12211 = 52.24
	x22211 = 17.74	x21211 = 43.04	x21111 = 54	x21122 = 9.9	x21121 = 64.71
	x22212 = 49.9	x21222 = 35.16	x21121 = 9.80	x22211 = 41.31	x21122 = 45.91
	x22213 = 16.9	x21223 = 16.07	x21123 = 14.80	x22213 = 1.23	x21123 = 27.09
	x11213 = 7.9	x21123 = 27.16	x22123 = 23.80	x22113 = 52.04	x22123 = 15.09
	x11123 = 15.91	x11123 = 15.91	x11123 = 15.91	x22212 = 28.65	x21113 = 18.51
	others are zero	others are zero	x21122 = 33.4	other are zero	x21211 = 18.29
			x22212 = 14.9		x22212 = 31.21
			other are zero		other are zero

Table 8. Optimum results for 2D-TP Model

4113.06	8093.95	453.61	8243.93
x111 = 61	x111 = 65.82	x111 = 65.52	x111 = 61
x112 = 54	x112 = 17.26	x112 = 2.62	x112 = 17.2
x121 = 52.0	x121 = 15.36	x121 = 0.67	x211 = 8
x122 = 3.0	x122 = 0.70	x122 = 0.32	x223 = 21
x211 = 7	x211 = 24.8	x211 = 2.18	x222 = 26.21
x213 = 5.5	x213 = 5.0	x213 = 2.23	other zero
x221 = 16.5	x221 = 8.0	x221 = 2.15	
x223 = 21.5	x223 = 29.4	x223 = 3.32	
other zero	other zero	other zero	

Table 9. Comparison of substitute and complementary items

Problem	Independent	Substitute	Complementary	Profit for model-1
1	Item-1, Item-2, Item-3			7388.841
1		Item-1, Item-2	Item-3	6966.775
2		Item-1, Item-2		5497.527
3			Item-1, Item-3	6021.859
4			Item-2, Item-3	5952.067

7 Conclusion

Renowned scientific discoveries and Research always had a practical application in the real world. Standing in the 21st century where machines, technology and industries govern the civilization, the need for sophisticated and easy techniques in the industry for optimizing products and profit is of immense importance. To further proceed and bring new ideas we investigate a four-dimensional multifarious item transportation issues with rough interval coefficient, which maximizes the profit considering substitute and complementary products. The concept of rough interval and its properties are discussed briefly. The problem is converted in two different transportation problem having interval coefficient. These two transportation problem is disintegrated into four classical transportation cases. Few examples are illustrated using LINGO 14.0 software showcasing the smooth working of the technique.

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Job Scheduling in Computational Grid Using a Hybrid Algorithm Based on Genetic Algorithm and Particle Swarm Optimization

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Abstract. Grid computing has been treated as a new paradigm for solving large and complex scientific problems using resource sharing technique through many distributed administrative domains. The dynamic nature of Grid resources and the demands of users create challenge in the Grid scheduling problem that cannot be addressed by deterministic algorithms with polynomial time complexity. Thus, the use of meta-heuristic is more appropriate option in obtaining optimal results. The Genetic Algorithm (GA) has been proven as one of the best methods for Grid scheduling. The GA explores the problem space globally, but is sometimes unable to search locally. Thus, a hybrid algorithm is proposed which combines intelligently the GA with Particle Swarm Optimization (PSO) for the Grid job scheduling. The hybrid GA-PSO aims to reduce the schedule makespan and flowtime. The proposed hybrid algorithm is compared with the standard GA and PSO on both parameters. The comparison results exhibit that the proposed algorithm outperforms other two algorithms.

Keywords: Computational Grid · Job scheduling · Makespan · Flowtime · GA and PSO

1 Introduction

Huge improvements in performance of wide-area network and fast yet low-cost computers drive the Grid computing to appear as a potential attractive computing platform. One type of Grid computing known as computational Grid aims to cumulate the power of heterogeneous, geographically distributed, multiple-domain computational resources to present high performance or high-throughput computing. To realize the attractive potentials of computational Grids, an effective and efficient job scheduling system is essentially fundamental. Job scheduling in computational Grid, which means the allocation of user submitted jobs to appropriate distributed computational resources, is one of the most challenging and complex tasks (Nabrzyski 2004).

The first phase of job scheduling in computational Grid is resource discovery. A list of all available resources is prepared. The second phase collects information about these

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resources and selects the most suitable resource for each job. In the third phase, the job is executed, which requires file staging and cleanup (Yan-ping 2008).

A Grid system consists of many heterogeneous resources that are geographically dispersed under different ownerships each having its own access policy, cost and various constraints as well as many user-applications with varied requirements. Since these resources are heterogeneous and geographically scattered and are used for common purposes, efficiency of a Grid system is highly dependent on an effective and efficient design of its scheduler. The job scheduling in computational Grid is an NP-hard problem which cannot be solved by deterministic algorithms (Ma 2011). Therefore, the use of a heuristic/meta-heuristic method is more fitting option in obtaining optimal results. The job scheduling problem is multi-objective in its general formulation. The two most significant objectives are minimization of makespan and flowtime of the system. Makespan is the time when the Grid completes the latest job and flowtime is the sum of finalization times of all the jobs.

In this paper, a hybrid algorithm based on the Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) is proposed. This hybrid algorithm cleverly utilizes the advantages of GA and PSO to concurrently minimize two key performance issues, viz. makespan and flowtime, of a computational Grid system. The GA has been one of the widely used evolutionary heuristic algorithms for constrained optimization problems, but the weakness of the algorithm is that it can easily be trapped in local minima. To evade such local minima problem, the PSO can be used to execute the local search more efficiently. The hybrid algorithm uses the mutation, crossover operators of the GA and PSO formula. A fuzzy probability is used to select the GA and PSO operators at each iteration for each particle or chromosome.

This paper is organized as follows. Section 2 briefly outlines the relevant past works done on job scheduling in computational Grid environment. In Sect. 3, the framework of Grid job scheduling problem has been defined. Section 4 presents the GA. Section 5 provides the PSO. Section 6 states mutation and crossover probabilities used at each iteration. Section 7 describes the proposed hybrid technique for scheduling jobs in computational Grid systems. Section 8 exhibits the results obtained in this study. Finally, Sect. 9 concludes the paper.

2 Related Works

Due to heterogeneous and complex characteristics of resources, the job scheduling in Grid is treated as an NP-hard problem. Much research has been channelized towards heuristic/meta-heuristic methods to solve the Grid scheduling problem. Meta-heuristic methods formulate realistic assumptions based on a priori knowledge of the related environment and of the system load characteristics. The most commonly used meta-heuristic algorithms are Genetic Algorithm (GA), Particle Swarm Optimization (PSO), Simulated Annealing (SA), Ant Colony Optimization (ACO) and Cuckoo Search (CS).

The GA is an adaptive technique that can be used to solve combinatorial optimization problems, based on the genetic process of biological organisms. Job scheduling in computational Grid using GA has been addressed by Aggarwal and Kent (2005), Buyya et al. (2000), Braun et al. (2001), Gao et al. (2005), Martino and Mililotti (2004), Page and

Naughton (2005), Xhafa et al. (2008) and Zomaya and Teh (2001). A new method based on GA has been proposed by Prakash and Vidyarthi (2015) to maximize the availability of resources for job scheduling in computational Grid. An improved genetic-based scheduling for Grid computing has been proposed Kolodziej and Xhafa (2011). The PSO has been used for the job scheduling in computational Grid by Izakian et al. (2009), Zhang et al. (2008) and Salman et al. (2002). A new algorithm based on fuzzy PSO has been proposed by Abraham et al. (2010) for scheduling jobs on computational Grids. Job scheduling in computational Grid using Simulated Annealing (SA) has been studied by Buyya et al. (2000). A local search based approach using SA has been proposed by Goswami et at. (2011). A hybrid algorithm which combines GA with SA has been proposed by Wang et al. (2010) for Grid job scheduling. An Ant Colony Optimization (ACO) based method for the problem has been studied by Ritchie (2003). Lorpunmanee et al. (2007) have also implemented an ACO algorithm for dynamic job scheduling in Grid environment. The effect of inter-process communication in auto-controlled ACO based scheduling on computational Grid has been investigated by Tiwari and Vidyarthi (2014). A recently developed meta-heuristic technique called Cuckoo Search (CS) has been studied by Prakash et al. (2012), Rabiee and Sajedi (2013), and Ghosh et al. (2017) for the job scheduling in computational Grids. A recent trend is to propose some hybrid techniques which combine two or more meta-heuristic algorithms in order to achieve better results compared to a single algorithm (Xhafa 2009; Ghosh 2016).

3 Problem Definition

A computational Grid system is represented by a pool of decentralized heterogeneous resources and a large number of jobs. Let $J = \{J_1, J_2, \ldots.., J_n\}$ denote the set of n jobs that are independent of each other to be scheduled on m resources $R = \{R_1, R_2, \ldots, R_m\}$ within the Grid system. In this paper, the problem of scheduling n jobs on m computing resources with objectives of minimizing the makespan and flowtime has been considered. If the number of jobs is less than the number of resources in the Grid, then it is easy to assign the jobs to the appropriate resources. If the number of jobs is more than the number of resources, then an efficient scheduling algorithm is required for allocation of jobs. So, it is considered that the number of jobs is more than the number of computing resources in this work. Since one job cannot be assigned to different resources, no job migration is allowed.

Since the Grid scheduling is performed statically, an estimation of the computational load of each job and the computing capacity of each resource in the system is assumed to be available a priori. This formulation is practically applicable as the computing capacity and computational needs can be known from resource specifications, job characteristics, historic data or prediction. This information can be represented in an Expected Time to Compute (ETC) matrix (Braun 2001) where the expected time to compute job J_i on resource R_j is specified by each entry $ETC(J_i, R_j)$. The size of the ETC matrix is $n \times m$. For the simulation studies, characteristics of the ETC matrices are varied in an attempt to represent a series of possible heterogeneous environments.

To measure the quality of solutions for the problem of scheduling jobs in computational Grids, many objective parameters are available. Thus, this problem is a multi-objective in its general formulation. In this study, the simultaneous minimization of makespan and flowtime are considered. The most popular and extensively studied optimization criterion is the minimization of the makespan. The total application execution time is known as makespan. The total application execution time is measured from the time the first job is sent to the Grid system until the last job comes out of the Grid. The makespan is given by

$$Makespan = max\,\{RT\big(R_j\big)\}, \qquad \forall j \in m \tag{1}$$

where, $RT\ (R_j)$ is known as the ready time of resource R_j and is calculated as

$$RT(R_j) = \sum_{i=1}^{n} ETC(J_i, R_j) \tag{2}$$

Another important optimization criterion is the minimization of the flowtime which is the sum of finalization times of all the jobs. Therefore, minimizing the value of flowtime means reducing the average response time of the Grid system. Formally it can be defined as:

$$Flowtime = \sum_{j=1}^{m} RT(R_j) \tag{3}$$

It is worth mentioning that in reality, makespan and flowtime are contradictory criteria. The flowtime has a higher magnitude order compared to the makespan and their difference increases as more jobs and resources are used. For this reason, the value of mean flowtime is used to evaluate flowtime. Besides, both values are weighted in order to balance their importance. Fitness value of a given solution S is thus calculated as:

$$fitness(S) = \lambda \times makespan(S) + (1 - \lambda) \times mean\ flowtime(S) \tag{4}$$

where, $mean\ flowtime(S) = \frac{flowtime(S)}{no.\ of\ resources}$ and λ is used to control the efficacy of parameters used in this equation. In this study, λ is taken as 0.7 because the makespan is considered as the primary objective.

4 Genetic Algorithm (GA)

Developed in 1975 by John Holland, Genetic Algorithm (GA) is a heuristic search tool designed to mimic the natural process of evolution (Holland 1975). The GA is generally used to produce useful solutions for optimization and search problems employing the natural processes of evolution such as selection, inheritance, mutation and crossover. The GA starts with a set of solutions represented by a group of chromosomes called the population. The size of population (n) is the number of chromosomes in a population. In the context of job scheduling in Grids, a solution is a mapping sequence between jobs and resources. The GA can perform with coded variables. Usually, the binary coding is the widely accepted technique of encoding the algorithm. The GA has to encode the total parameters as binary numbers when the initial population is created. Hence, while working over a set of binary solutions, the GA must decode all the solutions to report the optimal solutions. A fitness value is assigned to each solution (chromosome) representing the abilities of an individual (chromosome) to compete. The fitness value indicates degree of goodness of individual chromosome compared to others in the population. The individual with the optimal (or near optimal) fitness value is aimed. The GA employs selective breeding of the chromosomes to produce offspring having better fitness value than the parents by mixing information from the chromosomes. A new population is generated using the genetic operators, namely selection, crossover and mutation (Haupt 2004). Then, the chromosomes of this new population are evaluated again. This completes one iteration of the GA. The GA stops when a predefined number of iterations are reached or all chromosomes converge to the same mapping. Finally, the chromosome with the best fitness value in the last iteration is treated as the optimal solution. In the present study, crossover and mutation are hybridized with the Particle Swarm Optimization (PSO). The two genetic operators are described in the following sections.

4.1 Crossover Operator

The crossover operator is the most important component of the GA. The function of crossover operator is to produce new individuals (i.e. offspring) by selecting individuals from the parental generation and interchanging their *genes*. The plan is to obtain two offspring of better quality that will be put in the place of the parents and enable the search to explore new regions of solution space not explored yet. Furthermore, a number of pair of parents can produce offspring using this operator (Chang 2007). This number is calculated as $\frac{P_c \times n}{2}$, where P_c and n stand for the crossover probability and population size respectively. Assume that $x_i\ (t)$ and $x_j\ (t)$ are two randomly selected chromosomes, where $x_i\ (t)$ has a smaller fitness value than $x_j\ (t)$. Then the crossover formula can be written as

$$\begin{aligned} x_i(t+1) &= x_i(t) + \lambda_1[x_i(t) - x_j(t)] \\ x_j(t+1) &= x_j(t) + \lambda_2[x_i(t) - x_j(t)] \end{aligned} \tag{5}$$

where, λ_1 and λ_2 are random numbers in the range [0, 1].
Between $x(t)$ and $x(t+1)$, whichever has the better fitness value should be chosen, when Eq. (5) is computed.

4.2 Mutation Operator

According to the searching policy of the GA, it may easily fall into the local or suboptimal solutions when the chromosomes try to find the global optimum solution. In order to avoid such premature convergence, the mutation operator is used. It lets the chromosomes explore new regions of the parameter space to produce more potential solutions. The function of this operator is to modify the value of the number of chromosomes in the population. This number is calculated as $P_m \times n$, where P_m is the probability of mutation and n is the size of population. This operator ensures maintenance of diversity in evolving populations of GAs. If chromosomes are chosen randomly, the mutation formula is calculated as (Mahmoodabadi 2013):

$$x_i(t + 1) = x_{\min}(t) + \alpha[x_{\max}(t) - x_{\min}(t)] \tag{6}$$

where, x_i (t) is a randomly chosen chromosome, and x_{max} (t) and x_{min} (t) are the upper bound and lower bound with respect to the search domain respectively. Also, α is a random number in the range [0, 1]. When Eq. (6) is computed, between $x(t)$ and $x(t+1)$, whichever has the better fitness value should be chosen.

5 Particle Swarm Optimization (PSO)

The PSO, introduced by Kennedy and Eberhart (1995), has been inspired by the behavior of social species such as fish schooling, birds flocking and insects swarming. It employs a number of particles which constitute a swarm. Each particle (individual) of the swarm corresponds to a potential solution of the optimization problem. In PSO, the solution space of the problem is formulated as a search space. A swarm is initialized randomly in the search space of an objective function. Particles are flying through hyper-dimensional search space and their positions are updated based upon the social-psychological propensity of individuals to mimic the success of other individuals. Therefore, the positions of particles are updated based upon their own and neighbors' experience. The position of a particle which is denoted by $x_i(t)$ is updated using

$$x_i(t+1) = x_i(t) + v_i(t+1) \tag{7}$$

Each particle moves according to its velocity, which is denoted by $v_i(t)$, defined as follows

$$v_i(t+1) = \omega v_i(t) + c_1 r_1[x_{pbest_i} - x_i(t)] + c_2 r_2[x_{gbest} - x_i(t)] \tag{8}$$

where, c_1 is the cognitive learning coefficient and represents the attraction that a particle has toward its own success; whereas c_2 is the social learning coefficient and represents the attraction that a particle has toward the success of the entire swarm. x_{pbesti} specifies the personal best location in the search space ever visited by particle i and x_{gbest} denotes the best location discovered so far in the entire swarm. ω denotes the inertia weight that controls the impact of previous velocities of a particle on its current velocity. r_1 and r_2 are independently uniformly distributed random values in the range [0, 1].

6 Mutation Probability and Crossover Probability

The mutation probability maximizes the probability that the algorithm finds the optimum value of the objective function under simple assumptions. The mutation probability (P_m) at each iteration is obtained as

$$P_m = \mu_m \times Max\_It \tag{9}$$

where, μ_m represents a positive number and *Max_It* is the maximum number of iteration limiting the changes in positions of the chromosomes or particles of the entire population or swarm.

The crossover probability specifies how often the crossover will be performed. If there is no crossover, offspring is exact the same copy of parents. The crossover probability (P_c) at each iteration is calculated as

$$P_c = \mu_c \times P_f \tag{10}$$

where, μ_c is a positive constant. P_f represents a fuzzy variable and its membership functions and fuzzy rules are given in Fig. 1 and Table 1. The min-max-gravity method, or the simplified inference method, or the product-sum-gravity method can be used to determine the inference result P_f of the consequence variable (Mizumoto 1996).

7 Hybrid Algorithm

A hybrid algorithm based on the GA and PSO is proposed to solve the job scheduling problem in computational Grid. This hybrid algorithm is based on a hybrid of genetic operators and PSO formula to renew the chromosomes and particle positions. Here, each chromosome is a particle and the collection of chromosomes is considered as a swarm. At the start of the algorithm, a population is randomly chosen. At each iteration, the crossover and mutation probabilities are calculated. Then x_{pbesti} and x_{gbest} are represented when the fitness values of all particles are determined. The GA operators, viz. crossover and mutation, are involved to regulate the chromosomes. It is worth to mention that the chromosomes which are not selected for genetic operations will be assigned to particles and enhanced by the use of the PSO. The whole process is repeated until the terminating criteria are fulfilled. The pseudo-code of the hybrid GA-PSO algorithm is given in Fig. 2.

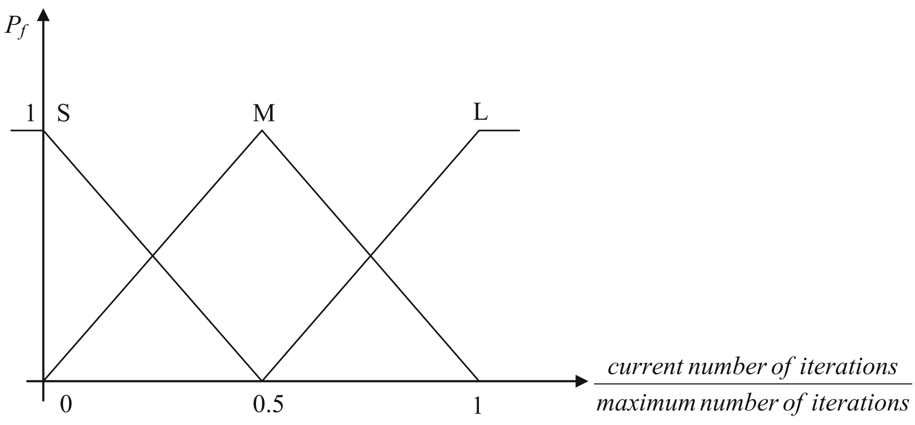

Fig. 1. Membership functions of fuzzy variable P_f

Table 1. Fuzzy rules of fuzzy variable P_f

Antecedent variable	Consequence variable
S	0
M	0.5
L	1

```
Initialize population of n chromosomes.
While (terminating conditions are not met) do
Begin
        Calculate P_m and P_c using Eqs. (9) and (10) respectively.
        Evaluate the objective function value for each member, and update pbest_id (i = 1, 2, ….., n)
        and gbest_d.
        If rand < P_c then
        Randomly choose two chromosomes from population and update them using
        crossover operator;
        Else if rand < P_m then
        Randomly choose a chromosome from population and update it using mutation
        operator;
        Else
        Randomly choose a particle from swarm and update its position using Eq. (7);
End
Output the optimal solution and the optimal objective function value.
```

Fig. 2. Pseudo-code of the hybrid GA-PSO algorithm

8 Experiments & Results

In order to assess the performance of hybrid GA-PSO algorithm, it is compared with standard GA and PSO methods for the job scheduling problem in computational Grids. All algorithms have common set of objectives, viz. minimization of both makespan and flowtime. The tests are performed on a Pentium IV 2.6 GHz, 4 GB RAM desktop, and the algorithms are realized using MATLAB Release 2013A. To enhance the performances of the proposed algorithm as well as standard GA and PSO, fine alteration is made and the best values for their parameters are picked up which are given in Table 2. Also the benchmark by Ali et al. (2000) is considered for the realistic simulations in this work.

Like the simulation model proposed by Ali et al., expected time to compute (ETC) matrix is used for 512 jobs and 16 resources. To assess the algorithms for diverse mapping situations, the characteristics of the ETC matrix were varied based on three metrics: job heterogeneity, resource heterogeneity and consistency. Job heterogeneity refers to the amount of difference among the execution times of jobs for a particular resource/machine. Resource heterogeneity defines the difference that is possible among the execution times for a given job across all the resources. An ETC matrix is said to be consistent if whenever a resource R_j executes any job J_i faster than resource R_k, then resource R_j executes all jobs faster than resource R_k. Consistent matrices are created by sorting each row of the ETC matrix independently, with resource R_0 always being the fastest. On the other hand, inconsistent matrices describe the situation where resource R_j may be faster than resource R_k for some jobs and slower for others. These matrices are kept in the unordered, random condition in which they are created. Partially consistent matrices are inconsistent matrices that contain a consistent sub-matrix of a predefined size. Following four cases are considered in this simulation for the consistent matrices.

Case-1: Low job heterogeneity and low resource heterogeneity
Case-2: Low job heterogeneity and high resource heterogeneity
Case-3: High job heterogeneity and low resource heterogeneity
Case-4: High job heterogeneity and high resource heterogeneity.

Table 2. Parameter settings for the algorithms

Algorithm	Parameter name	Parameter value
GA	Population size	25
	Maximum number of iterations	500
	Crossover probability (P_c)	0.8
	Mutation probability (P_m)	0.07
	Scale for mutations	0.1
PSO	Population size	25
	Maximum number of iterations	500
	$c_1 = c_2$	1.49
	$r_1 = r_2$	0.8
	Inertia weight (ω)	$0.9 \rightarrow 0.4$
Hybrid GA-PSO	Population size	25
	Maximum number of iterations	500
	μ_m	0.002
	μ_c	0.2
	Inertia weight (ω)	$0.9 \rightarrow 0.4$

The makespan and flowtime obtained using hybrid GA-PSO, standard GA and PSO are compared in Tables 3 and 4 respectively. The results are averaged over 10 independent runs. In Table 3, the first column indicates the case number, the second, third and fourth columns specify the mean makespan (in second) obtained by GA, PSO and hybrid GA-PSO algorithms respectively. The results show that the hybrid GA-PSO has minimized makespan than other algorithms. The performances based on average flowtime (in second) of all the algorithms are tabulated in Table 4. Table 4 depicts the minimum average flowtime for the proposed hybrid GA-PSO algorithm in comparison with the classical GA and PSO algorithms for all four cases. The statistical results of all algorithms in terms of the mean makespan and flowtime for the four cases are summarized in Figs. 3 and 4 respectively. From the figures, it can be seen that the proposed hybrid GA-PSO algorithm produces reduced mean makespan and flowtime values. Therefore, the proposed hybrid GA-PSO algorithm outperforms the standard GA and PSO algorithms for the problem.

Table 3. Comparison of statistical results for average makespan (in second) obtained by GA, PSO and hybrid GA-PSO algorithms

Case No.	GA	PSO	Hybrid GA-PSO
1	6102	7567	**5879**
2	426122	464149	**393383**
3	215654	225412	**205121**
4	12507697	14660686	**11744238**

Table 4. Comparison of statistical results for average flowtime (in second) obtained by GA, PSO and hybrid GA-PSO algorithms

Case No.	GA	PSO	Hybrid GA-PSO
1	127528	135698	**120136**
2	8302547	8245587	**7942282**
3	4635376	3635254	**3514250**
4	238630142	205944724	**197566263**

9 Conclusions

This paper investigates the job scheduling algorithms in Grid environments as optimization problems. For scheduling the jobs in computational Grids, a hybrid scheduling approach using Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) is proposed. This hybrid algorithm cleverly combines the advantages of GA and PSO. The objectives of the scheduler in this study are to minimize the makespan and flowtime concurrently. The performance of the hybrid GA-PSO method has been compared with standard GA and PSO approaches through carrying out exhaustive simulation tests on diverse settings. Experimental results show that the hybrid GA-PSO algorithm outperforms other popular heuristic techniques, viz. GA and PSO in different circumstances. The future studies will be directed towards other performance parameters such as CPU utilization, processing cost and fault rate.

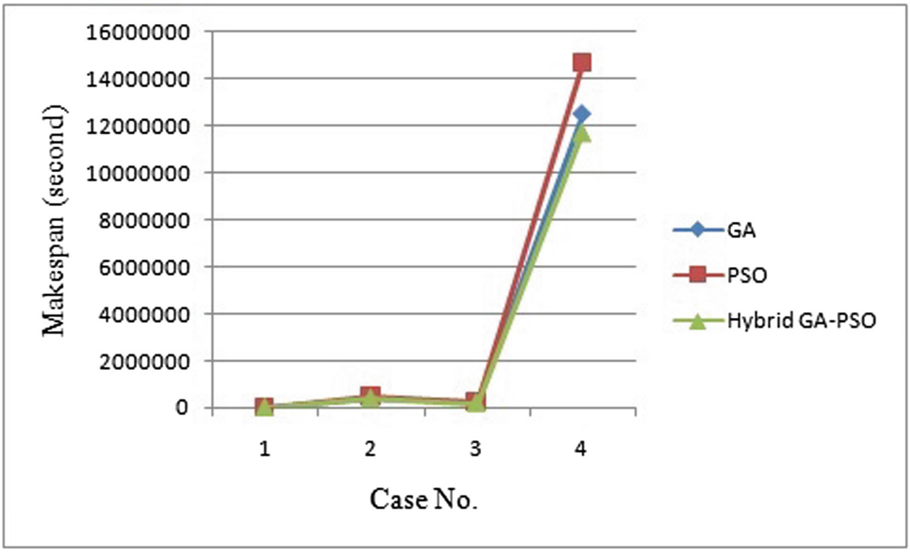

Fig. 3. Mean makespan (in second) comparison among GA, PSO and hybrid GA-PSO algorithms

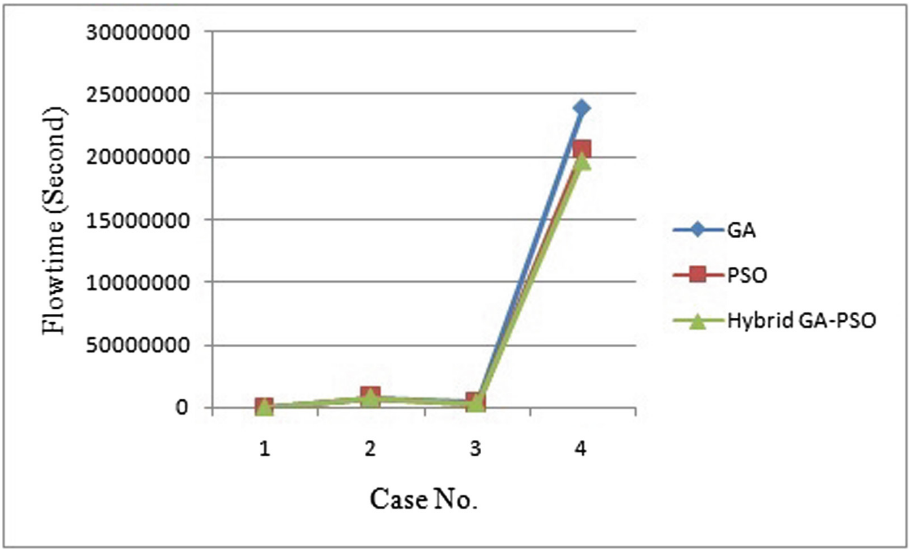

Fig. 4. Average flowtime (in second) comparison among GA, PSO and hybrid GA-PSO algorithms

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Measuring the Effect of Yoga-Lifestyle on the Employees of Higher Education Institutions of West Bengal Through Structure Equation Modelling (SEM): A New Approach Towards Human Resource Management

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Abstract. In this world of cut throat competition, individuals compete to emerge at the top as being a professionally successful individual. In doing so, the individuals' neglect their own health as they do not get time to hone and nourish their health, be it mental or physical. After a stipulated period of time, this backfires on them and they face a multitude of physical health as well as mental health problems. The pace of an industry is an extrinsic factor and it is not under the control of the individual, however, the particular individual can take care of his health as much as possible to prevent a fall out later onwards. Yoga intervention can be a crucial remedial factor that aids the individual to become physically as well as mentally capable of functioning in an organisation (Maharana et al. 2014). Practicing yoga equips the persons with confidence and knowledge, ultimately, working wonders at the workplace. It might be said that the organisational effectiveness of the human resources increases exponentially if the individual adapts to the environment very well and gel with the other employees in order to produce the best results. The employee productivity of an organisation will also increase and errors would be minimized.

Keywords: Human resource management · Yoga intervention · Organisational effectiveness · Employee productivity

1 Introduction

Human resource is an often neglected topic across various organisations and many consider it as obsolete. However, without the presence of skilled workforce, it is impossible

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for an organisation to operate (Vermeeren et al. 2014). Thus, it is very much imperative to design an environment within the organisation that motivates the employees intrinsically as well as extrinsically to work to the best of their ability and contribute positively towards the organisation (Mahadevan and Mohamed 2014). The competency of this strategy would more or less determine the output of the organisation in terms of its value. When talking about the external environment in which an organisation functions, it is of primary importance to mention the volatility of the market. Each and every moment within the market spurs a new change in the market for better or for the worse. This makes it extremely strenuous for the organisation to forecast the future situation of the firm. Though contingency measures are accounted for, it is not very helpful. These situations could be managed with the superior skills of the employees who are working for the organisation at the precise time in which the crisis hits the organisation. Due to this, the employees are exposed to varying degrees of stress and anxiety (Asthana and Asthana 2012). This holds especially true for the education industry as the mode of education is constantly changing with the changing times. Over a decade ago, the sole means of teaching would be in a classroom setting. However, now, the concept of smart classrooms is evolving very expeditiously, especially in the higher educational institutions. A teacher sitting in Singapore could take a class in India through the utilisation of Smart Classrooms. This puts immense pressure on the teachers to learn the needed innovations in time so that they do not fall behind in learning the unfolding technology. This can put a toll on them, further deteriorating their mental and physical health. Modern research indicates that the employees need to be physically fit as well as mentally alert if they wish to succeed (Maddux et al. 2017). This could be done with the help of Yoga (Shivajirao 2017) which is very cost friendly as well as time friendly. Yoga can be basically defined as a set of physical, mental and spiritual practices or disciplines. It has its roots in India. It is one of the most orthodox methods which urge an individual to look within themselves and transform themselves into a version of themselves who are optimally functional in all segments of life. Yoga has already been implemented within the school course curriculum across India as it has been seen that the children can benefit a lot from practicing Yoga from a very early age (Butzer et al. 2016). However, the teachers of the institutions are not trained in this matter. The aim of this paper is to point out the important factors that can be obtained through the help of Yoga and which can be advantageous to the employees of the higher educational institutes. This would help in creating a happy employee who would be responsible for educating the youth of India who will drive the future growth of the country as a whole (Betchoo 2013).

2 Literature Review

Yoga is a system of mental as well as the physical discipline and the spectrum of activity of yoga consists of practitioners from charlatans to practiced yogis, even so, the research on this topic takes some time as it is a prolonged process involving lots of field work (Anand et al. 1991). The overall themes that are very relevant in describing yoga are given as follows: the value of yoga, the body is just a vehicle to the self, relationships, the individual self as compared to the true self and finally transformation which is the most important. It has been defined through the past decades that yoga is a means for mindfulness and it automatically activates the body's natural mechanisms to manage stress. Yoga is considered to be one of the best practices to balance the body and the mind proportionately as it is helpful in boosting the fitness, motor skills and health of the individual by equipping them with appropriate knowledge, skills and capacity consecutively (Galan et al. 2017). Yoga is a booming topic nowadays as a lot of research is being carried out for treating various health conditions with the use of yoga as well as treating various medical conditions with the help of yoga (Groessl et al. 2015). The present system of education has drastically failed in achieving the integrated objective of developing the mind, body and soul of the learners, however, a majority of the people are considering introducing Yoga education in the curriculum to a certain extent in order to find a remedy to this particular problem as it has been observed that Yoga is helpful in the all round development of the learner in terms of various dimensions like individual, social, cognitive, emotional, psychomotor, behavioural, moral as well as spiritual (Bera 2017). Significant improvement was observed in the mental ability and memory of the students who practice yoga regularly as compared to those who do not (Verma et al. 2014). The learners who practiced yoga earned higher grades as compared to learners who did not practice yoga as the stress level of the learners who practiced yoga were very low which improved their competitive performance drastically (Jeba 2018). Yoga inculcates physical as well as mental discipline in the students and it plays a vital role in increasing the concentration and the stamina of the students. The present generation is not much involved in physical activities in this digital age which showcases booming rise of chronic diseases such as diabetes and obesity and one of the most fastest ways of shutting down this problem is to inculcate yoga which will go a long way in developing the mental well being, physical well being, social well being and daily behaviours of the child (Chen and Pauwels 2014). Yoga helps the school students in coping with the academic stress as well as mental health problems which are a resultant of the increasingly competitive nature of the surrounding environment right from a tender age (Sharma et al. 2010).

It is seen that children use the tools that they are taught in order to cope with anxiety and stress and it enables them to listen to their own bodies which enable right diagnosis at the right time in times of need and helps them to adapt to varying situations within a stipulated period of time (Geldert 2017). A study on yoga reveals that yoga practice has a most direct impact on the attention of the individual (Carlin et al. 2009). Researchers can attest to the fact that meditation enhances performance of any individual drastically by boosting their physical as well as psychological self (Kaur et al. 2016). Yoga has been seen to incorporate self-compassion amongst adolescents in this era of cut throat competition where a lot is expected out of the adolescents. Schools and colleges can encourage the students to build a higher emotional resilience by providing a more holistic mind and body approach with the adoption of yoga practices (Khalsa 2015). Research has been done about the effect of yoga for the students and it is very evident that yoga does play a positive role, but, no significant research has been done for the employees working in the higher educational institutions, a fact which must be remedied as soon as possible which triggered the idea of writing this paper with this same objective. The Indian corporate sector is a battlefield and the only way of surviving it by balancing the work life and personal life by being hardworking, disciplined, punctual and mentally alert as they are exposed to prolong periods of stress and tension which manifests themselves in the form of hypertension, insomnia, high or low blood pressure, backaches, depression, spondylitis, migraines, etc.; the remedy of this to a certain extent is yoga as it acts as a reviver of mind, body and soul by enhancing the energy of the persons and simultaneously developing a positive attitude. The research demonstrated that yoga classes can be fruitful in providing short term mental health benefits in chronically stressed women, however, the benefits are maintained only within the engagement with the yoga classes. Yoga can be treated as a treatment mechanism for people suffering from depression and anxiety as preliminary evidence suggest that yoga is efficacious in reducing the symptoms for unipolar depression as yoga has an plausible effect on the cognitive as well as biological mechanisms of the individual (Uebelacker and Broughton 2016). Practicing yoga can work miracles in reducing the stress and anxiety and its effect is immediate and more pronounced than any other form of physical training. If a comparative study is done between yoga and other physical exercise, yoga will emerge to be as effective as or perhaps, more effective than physical exercise in both the healthy and diseased populations, at the same time, improving various outcomes related to the health (Ross and Thomas 2010). Yoga is an embodiment of a mixture of physical postures and movement, breath exercises and mindfulness and it is very useful in the treatment of trauma amongst the adults who have been through a traumatic accident (Rhodes 2014). Anxiety and stress have become the pall bearers of the human population due to chronic diseases and yoga can be used as an non-pharmacological therapy to aid in the suppression of this ailment if not completely cure it (Li and Goldsmith 2012). Stress is a complex response that affects an individual's health, work performance, family life and social life as well as certain physiological triggers that prepares the body for the fight or flight response and if it is not controlled through yoga or certain other means, it would have alarming repercussions on the individual as well as the organisational level which will lead to decrease productivity (Deshpande 2012). Yoga is seen to be having a positive effect not only in curing an individual's depression but also it has a positive effect on the functioning of a person. Teachers having knowledge of yoga can assist themselves as well as their students in dealing with everyday stress and anxiety and living a better life altogether. The extrinsic factors which cause distress amongst the academic staff members are the financial constraints, workload pressure, job insecurity, management styles and policies, poor workplace relationships, etc. The intrinsic factors responsible for the same are reward and recognition issues, lack of individual control, poor promotional prospects, etc. (Kelley 2017). Yoga is seen to have a direct relevance to job burnout which is a major hazard of the 21st century which affects the majority of the population by conjuring psychological problems in the minds of the employees which has serious repercussions on the personal health of the employee and also on the organisational effectiveness of the individual (Vinod and Sudhakar 2011). Job burnout may be determined by three dimensional parameters of exhaustion, cynicism and inefficacy and it can be controlled to a certain extent by adopting the yoga way of living. A research carried out in the Birbhum district of West Bengal indicates yoga education is very helpful to the trainee teachers as they are the future teachers of our nation as yoga helps them to cater to their problems of irritation, stress, anxiety, loneliness, depression, egoistic nature, etc. which further helps them in improving their productivity (Pal 2018). The top most countries investing in yoga research is USA, followed by India and UK as studies are showing irrefutable facts that yoga is a good method of providing therapy to the healthy as well as the diseased population (Gupta et al. 2018). Yoga can be treated as a supportive or adjunct therapy for certain pain associated disability and mental health as it is highly cost effective and it can be used to develop self care behaviour treatment, self skill as well as self confidence (Bussing et al. 2012).

3 Research Gap

From the literature review, it is clearly understood that insufficient studies were executed in India as well as West Bengal, relating to the effect of Yoga-Lifestyle on the employees of Higher Education Institutions of West Bengal through empirical analysis. Previous studies were conducted theoretically and empirical analysis was absent in those cases. In previous studies, all required and relevant factors related to the present study were not emphasized. Therefore, it is needed to examine different factors and variables/items related to the present study and their level of influences through a structural model in the scenario of West Bengal.

4 Hypotheses of the Study

$\mathbf{H_1}$: 'Relieving Stress & Anxiety' positively influences the 'Organizational Effectiveness'.
$\mathbf{H_2}$: 'Emotional Resilience' positively influences the 'Organizational Effectiveness'.
$\mathbf{H_3}$: 'Concentration' positively influences the 'Organizational Effectiveness'.
$\mathbf{H_4}$: 'Physical Health' positively influences the 'Organizational Effectiveness'.
$\mathbf{H_5}$: 'Adaptability' positively influences the 'Organizational Effectiveness'.
$\mathbf{H_6}$: 'Organizational Effectiveness' positively influences the 'Employee Productivity'.

5 Research Methodology

Here, our research model has been established with the help of both primary and secondary data. The 'Hypothesized Research Model' (Fig. 1) was developed by collected factors from previous literature. A structure questionnaire was framed with related variables for survey. Five point Likert scale has been used for measuring the responses of employees of higher education institutions of West Bengal related with the effect of Yoga lifestyle on employee productivity. 292 responses were finally collected for this study. The survey period was from 20th April, 2018 to 25th June, 2018. As sampling technique, convenience sampling has been used for selecting colleges/universities at the first stage. Then simple random sampling has been used to select the respondents as employees from each college/university for this study.

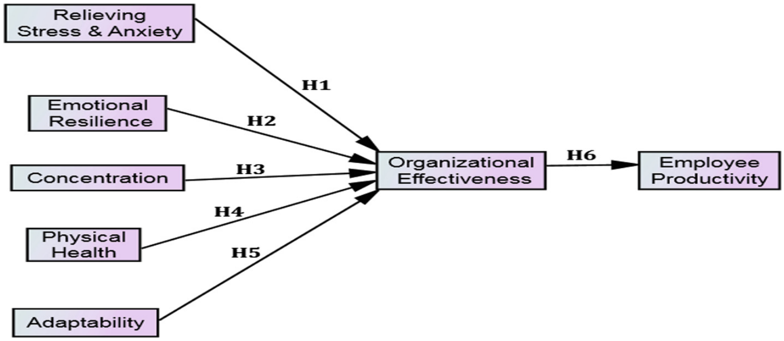

Fig. 1. Hypothesized research model

6 Analysis and Results

Structure Equation Modeling (SEM) has been used for developing the model and establishing the hypotheses by the help of AMOS 21.0 software. Validity and model fitness have been judged through measurement and structural model. Exploratory Factor Analysis (EFA) by the help of SPSS-21 describes the validation of questionnaire through data reduction method.

In this study, KMO and Bartlett's Test (Table 1) shows the appropriateness of Exploratory Factor Analysis (EFA).

Cronbach's alpha for all items (Table 2) which is greater than 0.70, shows the satisfactory range of reliability. Variables with factor loading of above 0.5 have created 7 different factors which are extracted from Rotated Component Matrix. These factors explain total 82.861% of the variations (Table 3).

Then, the fitness indexes have been verified as follows and hypotheses have been tested. Confirmatory Factor Analysis (CFA) has been executed for giving attention on testing how well defined variables represent different factors.

Table 1. KMO and Bartlett's Test

Kaiser-Meyer-Olkin measure of sampling adequacy		.713
Bartlett's test of sphericity	Approx. Chi-square	5056.022
	Df	190
	Sig.	**<0.001**

Table 2. Overall reliability statistics

Cronbach's alpha	No of items
0.871	**20**

Table 3. Result of factor analysis - rotated component matrix (a)

Rotated component matrix[a]							
	Component						
	Relieving stress & anxiety	Adaptability	Organizational effectiveness	Employee productivity	Emotional resilience	Physical health	Concentration
SA1	**.869**	.054	.210	.118	.076	.061	.051
SA4	**.854**	.026	.160	.108	.050	−.054	.103
SA2	**.845**	.091	.231	.051	.073	.109	.071
SA3	**.718**	.013	.167	.004	.104	.107	.229
A2	.011	**.966**	.090	.107	.084	.003	.016
A1	.095	**.910**	.039	.058	.007	−.006	−.011
A3	.052	**.889**	.040	.177	.074	.049	.121
OE2	.275	.086	**.887**	.176	.222	.092	.122
OE3	.282	.092	**.846**	.156	.232	.137	.155
OE1	.371	.041	**.828**	.116	.176	.083	.066
EP3	.084	.131	.091	**.919**	.062	.097	.048
EP1	.257	.077	.064	**.859**	.012	.144	.050
EP2	−.052	.135	.183	**.805**	−.055	−.088	−.055
ER2	.045	−.017	.094	−.006	**.872**	−.056	−.076
ER3	.086	.066	.225	.000	**.868**	.021	.144
ER1	.140	.124	.155	.021	**.798**	.164	.089
PH1	.064	−.019	.005	.033	.063	**.909**	.016
PH2	.091	.058	.215	.085	.027	**.871**	.057
C2	.116	.026	.121	−.068	.011	.015	**.918**
C1	.409	.121	.149	.159	.160	.086	**.719**
% of variance explained	**16.584**	**13.306**	**12.808**	**12.062**	**11.792**	**8.637**	**7.672**
Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalization							
(a) Rotation converged in 6 iterations							

Here, the fit indices (Table 4) of structural model (Fig. 2) indicate the acceptable range and prove a good model fit.

Table 4. Fit indices of confirmatory factor analysis for structural model

Fit index	Acceptable threshold levels	Structural model values
$\chi^{2/\mathrm{df}}$ (Chi-square/degree of freedom)	Values less than 3	**1.481**
RMSEA (Root mean-square error of approximation)	Values less than 0.06	**0.041**
GFI (Goodness of fit index)	Values greater than 0.90	**0.994**
AGFI (Adjusted goodness of fit index)	Values greater than 0.90	**0.960**
NFI (Normed fit index)	Values greater than 0.90	**0.982**
CFI (Comparative fit index)	Values greater than 0.90	**0.994**

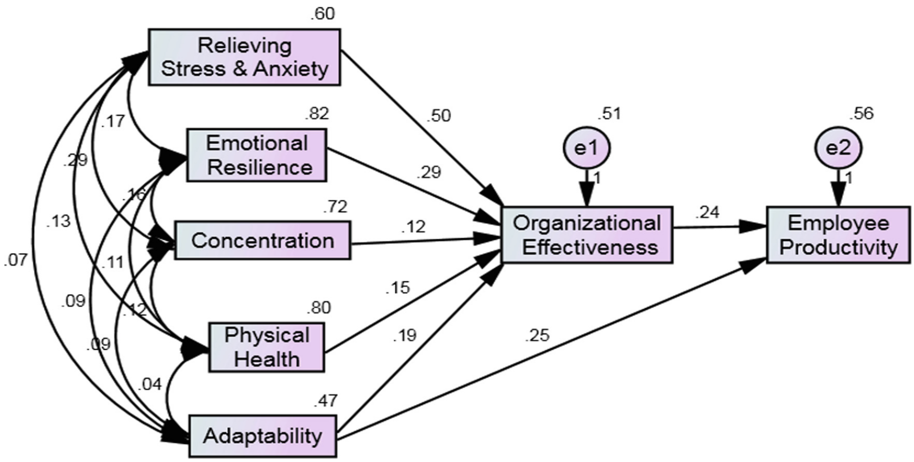

Fig. 2. Path diagram of hypothesized structural model

Higher-Standardized-Regression-Estimates (Greater than 0.7) show the higher reliability of variables. Construct-Reliabilities (Greater than 0.7) show the internal consistency among the variables. Here, AVE values are also higher than corresponding squared inter-construct correlation (SIC), so it supports discriminant validity (Tables 5 and 6). As per Hair et al. (2010) and Field (2009), the following conditions prove the convergent and discriminant validity in the measurement model (Table 7).

1. AVE $>$ 0.5; **2.** CR $>$ AVE; **3.** MSV $<$ AVE; **4.** ASV $<$ AVE

7 Path Analysis for Hypotheses Testing and Research Findings

H_1: 'Relieving Stress & Anxiety' positively influences the 'Organizational Effectiveness'

Structural model supports this hypothesis. The path coefficient is significant ($p < 0.001$) statistically and it has the expected positive sign (+0.498) which means 'Relieving Stress & Anxiety' positively influences the 'Organizational Effectiveness'. It is seen that Stress and anxiety is a source of concern for majority of the employees as they are forced to

Table 5. Squared correlations between factors in measurement model

Factors	PHY_HEALTH	ORG_EFF	STRESS_ANXIETY	EMP_PROD	EMO_RES	CONCEN	ADAPT
PHY_HEALTH	**0.787**						
ORG_EFF	0.219	**0.839**					
STRESS_ANXIETY	0.176	0.457	**0.750**				
EMP_PROD	0.080	0.208	0.172	**0.742**			
EMO_RES	0.061	0.431	0.219	0.016	**0.751**		
CONCEN	0.063	0.339	0.365	0.058	0.138	**0.747**	
ADAPT	-0.009	0.156	0.108	0.182	0.131	0.118	**0.734**

*Diagonal elements are **Average Variance Extracted (AVE).**

deal with unplanned situations again and again. Yoga helps significantly in working out the stress and frustration out of the body and increasing the mental as well as the physical health of the individual (Kapoor 2011).

H_2: 'Emotional Resilience' positively influences the 'Organizational Effectiveness'
The P-value for the path co-efficient from 'Emotional Resilience' to 'Organizational Effectiveness' is positive (+0.289) and significant ($p < 0.001$), indicating that 'Emotional Resilience' positively influences the 'Organizational Effectiveness'. Therefore hypothesis is supported. Emotional Quotient (EQ) has become as important for the employee as the Intelligence Quotient (IQ). IQ helps an employee working in a higher education institute to tap into the opportunities in order to succeed; however, EQ helps the employee to utilize this opportunity to the best of the employee's capability. The more emotionally stable the employee; it will have a strong positive impact on the organizational effectiveness of the employee (Shannon 2011).

H_3: 'Concentration' positively influences the 'Organizational Effectiveness'
Structural model supports this hypothesis. The path coefficient is ($p < 0.05$) statistically significant and it has the expected positive sign (+0.123) which means 'Concentration' positively influences the 'Organizational Effectiveness'. Time management is a very important factor for an academician as he or she has to do a multitude of activities. Time management can be greatly improved if effective concentration is given to a particular activity. This ultimately has a positive impact on the organizational effectiveness. Proper concentration is a key determinant which determines the level of organizational effectiveness of an individual (Oladimeji and Akingbade 2012).

H_4: 'Physical Health' positively influences the 'Organizational Effectiveness'
The P-value for the path co-efficient from 'Physical Health' to 'Organizational Effectiveness' is positive (+0.145) and significant ($p < 0.01$), indicating that 'Physical Health' positively influences the 'Organizational Effectiveness'. Therefore, hypothesis is supported. Physical health plays an effective role in organizational effectiveness. A diseased or a frail employee will always have some restraints which would not allow them to work freely in the organization, thus, decreasing the organizational effectiveness (Lin and Lin 2011).

Table 6. Measurement model results

Constructs/Factors	Variables	Standardized regression estimate	Construct reliability (CR)	Average variance extracted (AVE)	Maximum shared variance (MSV)	Average shared variance (ASV)
Organizational effectiveness (ORG_EFF)	OE1	0.81	**0.765**	**0.619**	**0.048**	**0.016**
	OE3	0.862				
	OE2	0.845				
Relieving stress & anxiety (STRESS_ANXIETY)	SA1	0.802	**0.877**	**0.704**	**0.209**	**0.104**
	SA4	0.751				
	SA2	0.762				
	SA3	0.681				
Employee productivity (EMP_PROD)	EP1	0.746	**0.837**	**0.563**	**0.209**	**0.077**
	EP3	0.792				
	EP2	0.683				
Emotional resilience (EMO_RES)	ER1	0.743	**0.785**	**0.550**	**0.043**	**0.019**
	ER3	0.788				
	ER2	0.719				
Concentration (CONCEN)	C1	0.734	**0.794**	**0.563**	**0.186**	**0.046**
	C2	0.759				
Physical health (Physical _Factor)	PH1	0.784	**0.716**	**0.557**	**0.133**	**0.048**
	PH2	0.79				
Adaptability (ADAPT)	A1	0.698	**0.778**	**0.539**	**0.033**	**0.017**
	A3	0.729				
	A2	0.774				

CR = (Σ Standardized loadings)^2/[(Σ Standardized loadings)^2 + Σ (measurement error for each indicator)]
AVE = Σ (Standardized loadings^2)/[Σ (Standardized loadings^2) + Σ (measurement error for each indicator)]
MSV = (Maximum value of correlation estimate between a particular construct and other related factors)^2
ASV = Average value of [(correlation estimate between a particular construct and other related factors)^2]

H_5: 'Adaptability' positively influences the 'Organizational Effectiveness'
Structural model supports this hypothesis. The path coefficient is significant ($p < 0.01$) statistically and it has the expected positive sign (+0.187) which means 'Adaptability' positively influences the 'Organizational Effectiveness'. Work is done in teams and hence, teamwork is very necessary. Adaptability is the trait by which an employee

Table 7. Path analysis of structural model

Measurement path			Hypothesis	Estimate	S.E.	C.R.	P	Assessment
Organizational effectiveness	←	Relieving stress & anxiety	**H1**	**0.498**	.062	8.025	**<0.001***	Supported
Organizational effectiveness	←	Emotional resilience	**H2**	**0.289**	.048	5.959	**<0.001***	Supported
Organizational effectiveness	<--	Concentration	**H3**	**0.123**	.056	2.184	**0.029****	Supported
Organizational effectiveness	←	Physical health	**H4**	**0.145**	.048	3.029	**0.002***	Supported
Organizational effectiveness	←	Adaptability	**H5**	**0.187**	.052	3.004	**0.005***	Supported
Employee productivity	←	Organizational effectiveness	**H6**	**0.237**	.047	5.051	**<0.001***	Supported
Employee productivity	←	Adaptability	**New**	**0.248**	.065	3.833	**<0.001***	Supported

*1% Level of Significance ($P < 0.01$) & **5% Level of Significance ($P < 0.05$)

would gel with the other team members and brainstorm about bright ideas that can lead to certain innovative changes. Yoga positively builds the sense of adaptability in the employees (Amah and Baridam 2012).

H_6: 'Organizational Effectiveness' positively influences the 'Employee Productivity' Structural model supports this hypothesis. The path coefficient is significant ($p < 0.01$) statistically and it has the expected positive sign (+0.237) which means 'Organizational Effectiveness' positively influences the 'Employee Productivity'. Organizational effectiveness leads to an increasing amount of output generated within the system within a stipulated period of time and quality work is encouraged and awarded. This will automatically lead to the employee productivity (Kataria et al. 2014).

8 Managerial Implications

Yoga is an ancient science which has a holistic approach on the body as well as the soul. The concept of Yoga is ancient spanning over a few centuries. It has many managerial implications on the human resources of an organisation which is perhaps could be considered as the most vital resource of any organisation. However, it becomes extremely difficult to manage the human resource because of their varying needs and wants, most important of which is their emotional aspect. Yoga helps the employees to build the inner strength of the employees by exploring their own selves by using out of the box mechanisms. It builds their physical as well as their mental form and hones their skills so that they may use them optimally in their workplaces (Kujur and Shah 2016). Various Yoga strategies can be used in a classroom setting to teach the students in such a manner that is optimally useful. From an employee point of view, many lessons could be learnt. Factors like relieving of stress and anxiety, emotional resilience, adaptability; concentration, etc. aid the employees to strengthen their intrinsic health. This way of thinking motivates the employees to bend their minds towards knowledge acquisition which urges the employees to delve deeper within the organisational innovation (Ozigbo 2012). This further moves on to positively influence the employee productivity. The Harvard Business School Study which was propagated over a span of 11 years, showed a marked relationship between the strength of the culture of the organisation with that of its profitability. Implementation of Yoga into an employee's life helps in the cultivation of an individual's competitive advantage in his or her relevant field which makes the person an invaluable asset in the company. However, in order to achieve the competitive advantage, one must not ignore the health. The findings of another study showcases that there is a most significant correlation between the factors of job satisfaction, job stress and health (Balaram et al. 2014). Hence, organisations must strive to motivate the employees and work towards building job satisfaction amongst them in order to get optimal amount of output from them without compromising on their quality of work. Thus, self awareness amongst the employees enshrined in ancient wisdom would help the employees to shine both on a personal and professional level. The Yoga way of life teaches an individual on the correct attitude of life by which the individual must abide in order to live a satisfying life. Only when an individual is at peace, he or she will be able to out-perform them at their workplaces working miracles. So, the organisations must help the individuals achieve this by supporting them in any manner possible within the scope of the organisation.

9 Conclusion

Yoga can be termed as a multi-dimensional approach. Also, there is a multi-faceted scope of Yoga in the twenty first century. Yoga has been successful in achieving international acclaim by providing relief to the individuals suffering from stress and anxiety due to their work pressure. 21st June is celebrated as the World Yoga Day to celebrate this innovative way of healing and inculcate this good habit of practicing yoga amongst the common mob. Thus, it must be made a part of the educational sector as well as that of the healthcare sector as these sectors involve a lot of stress for their employees. The study shows that stress and anxiety are very common within the workforce and they prove to be quite burdensome as well. Due to this, relieving stress and anxiety has a positive impact on the organisational effectiveness which furthermore has a positive impact on the employee productivity. On the basis of this study, a number of factors have been discovered which has a positive impact on the organisational effectiveness (Hussain and Ahmad 2012) and these factors can be build within an organisation through the help of Yoga. The leaders of various top notch companies are contemplating using Yoga to boost the employee's morale. The toughest challenge now is to find suitable trainers who would train the teachers in the higher educational institutions so that they might emerge as ideal employees of the Institute.

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