Universiteit Utrecht

Double Bachelor's
Mathematics and Physics

From Electrodynamics to Grand Unified Theories:
A Mathematical Analysis of Gauge Theories

Bachelor Thesis

Jesse Straat

$$\mathcal{L} = -\frac{1}{2}\sqrt{\frac{G\hbar^3}{c}}\,\langle F_{\mu\nu}, F^{\mu\nu}\rangle_g + \hbar c\,\bar{\psi}\,\not{D}^{\omega}\psi - m_f c^2\,\bar{\psi}\,\psi$$

Supervisors:

Prof. Dr Thomas Grimm
Institute for Theoretical Physics

Prof. Dr Marius Crainic
Mathematical Institute

Arno Hoefnagels, MSc
Institute for Theoretical Physics

June 15, 2022

Abstract

Gauge theories are mathematical structures that form an integral part of modern physics, particularly through the standard model. It is therefore profitable for the modern physicist to gain a deeper knowledge of the mathematics that lie at the foundation of gauge theories. In this thesis, we develop the general notion of a gauge theory using principal bundles and connection forms thereon. We furthermore define fermions using spinor bundles that are twisted in order to interact with the gauge potential. We develop the Yang-Mills-Dirac Lagrangian, which is usable for all gauge theories with a compact structure group. Finally, we discuss further developments of gauge theories, with a focus on grand unified theories such as the Georgi-Glashow model and $SO(10)$, and their flaws and virtues.

The image on the title page depicts the Yang-Mills-Dirac Lagrangian, which is the final major result we provide in this thesis.

Contents

1 Introduction

In 1865, James Maxwell introduced the world to his new unified theory of electromagnetism [1]. In addition to being the first relativistic theory, it was the first physical theory that had a gauge symmetry, which meant that one could freely change the value of the potentials in electromagnetism in a certain way, being a gauge transformation

$$\varphi \to \varphi - \frac{\partial \Lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\Lambda \tag{1.1}$$

— with φ the electric potential, $\mathbf{A}$ the magnetic vector potential and Λ any real function — without influencing the physics of the system. With the development of quantum mechanics, and specifically quantum electrodynamics by Dirac, the electromagnetic potentials received a new role, causing a phase difference in wave functions. In QED, we identify the change in gauge with some phase factors $e^{i\frac{q}{\hbar}\Lambda}$ (with q the charge of the particle in question), which are elements of $U(1)$, which is the gauge group of electromagnetism.

In 1954, physicists Yang and Mills generalised gauge theory to non-abelian gauge groups, attempting to describe the strong interaction, in what is now know as Yang-Mills theory [2]. Originally, the new theory did not find much support, since it required massless gauge bosons, but when the notion of symmetry breaking was introduced, the theory regained popularity as it now provided an accurate model of the fundamental interactions, with the standard model as a result.

Meanwhile, major strides were made in the field of differential geometry by mathematicians such as Charles Ehresmann and Shiing-Shen Chern, which caused the development of fibre bundles. Following Dirac's research into QED, and specifically the magnetic monopole, physicists realised that the use of bundles was the correct way of describing gauge theories.

Modern physics is built upon gauge theories, and developments in physical fields are often mathematical in nature. Therefore, familiarizing oneself with gauge theory and its mathematical foundations is a very useful exercise. It should provide us with a mathematically justified theory, which can even make it possible to look beyond current physics and generalise our theories to, for example, a grand unified theory. But what are these mathematical foundations? In this thesis, we attempt to provide the reader with an answer to this question by developing the basic notions of gauge theories, with a focus on the underlying mathematical theory.

In chapter 2 of this thesis, we work through the mathematical process of developing tensor calculus on arbitrary manifolds, which is an important mathematical tool used in physical field theories. It will be necessary to be familiar with abstract algebra at a basic undergraduate level.

In chapter 3, we develop the notion of fibre bundles, which, as teased in the introductory paragraph, will form the mathematical outline that defines gauge theory. Basic knowledge of differential manifolds is presumed.

Then, using a physical dictionary as a segue, we apply our knowledge of bundles and move on to formally developing gauge theory in chapter 4 by returning to the base space.

To introduce fermions to our gauge theory, we develop spinors in chapter 5. To this end, we explore Clifford algebras and the spin group, and construct a vector bundle to represent the fermions. Finally, this vector bundle is “twisted” in order to account for interactions of fermions with the gauge field.

In order to derive some mechanics for our gauge theory, we introduce a Lagrangian in chapter 6 called the Yang-Mills-Dirac Lagrangian, which is developed as a generalisation of the Lagrangians found in QED.

Finally, in chapter 7, we shortly introduce the concept of a grand unified theory, to provide an outlook on potential future developments of gauge theory.

After a short chapter summarising our conclusions, we work out some long proofs in the appendix. These are omitted from the thesis to ensure a smooth reading experience.

The thesis has been split into two parts. Part I covers mostly mathematical theory. It is very important for the development of gauge theory, but can also be used outside of the context of this thesis as an introduction to the concepts. Part II, on the other hand, concerns theory mostly used in physical contexts.

1.1 Mathematical preliminaries

For this thesis, it is very important to have a basic knowledge of abstract algebra (such as in the book by Dummit & Foote [3]) and differential manifolds (such as in Kobayashi [4]). Specific algebraic topics used in this thesis include (but are not limited to!) group theory, (matrix) algebras, Lie algebras, short exact sequences, bilinear forms and quotients.

1.2 Notation and conventions

In mathematics, we tend to work with commutative diagrams, in order to explain how several functions work together in a rather intuitive way. These diagrams consist of sets (the vertices) and mappings between the sets (the edges), such that any composition of mappings (or “path” across the edges) will give you the same result. Below, two examples are provided: the first being trivial, and the second such that $g \circ f = \psi \circ \varphi$.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi"] & B \arrow[d, "g"] \\
C \arrow[r, "\psi"] & D
\end{tikzcd}
$$

Whenever, in the context of tensors, you see a variable with a superscript, such as x^j, the superscript denotes a (contravariant) index, not a power. If it is not clear from the context whether a superscript denotes a power or an index, we mention so explicitly.
In chapter 2.1, we often abbreviate summations over indices by ommiting all indices we are summing over. For example, for an n-dimensional manifold,

$$\sum K^{i_1 i_2 \dots i_n} e_{i_1 i_2 \dots i_n} = \sum_{i_1, i_2, \dots, i_n = 1}^{n} K^{i_1 i_2 \dots i_n} e_{i_1 i_2 \dots i_n}. \tag{1.2}$$

You can immediately see this convention will save a lot of space. We dub this the pseudo-Einsteinian summation convention, which is to be replaced by the Einstein Summation Convention in section 2.1.2, in which we also omit the summation symbol $\sum$.

1.2.1 A summary of mathematical notation

$X := Y$	Some object X is defined to be equal to Y.
$X \cong Y$	Some algebraic object X is isomorphic to Y.
$A \subseteq B$	A is a subset of B (this is equivalent to $B \supseteq A$).
$A \subset B$	A is a proper subset of B, so $A \subseteq B$ and $A \neq B$ (this is equivalent to $B \supset A$).
$A \hookrightarrow B$	For $A \subseteq B$, the inclusion map, which is the function $x \mapsto x$.
$\mathbb{N}$	The set of natural numbers, excluding zero: $\mathbb{N} = \{1, 2, 3, \dots\}$.
$\mathbb{N}_0$	The set of natural numbers, including zero: $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.
$\mathbb{Z}$	The set of integers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{H}$	The set of quaternions.
$\mathbb{K}$	Either the set of real numbers, or complex numbers, when it doesn't matter which field you choose.
A^*	The conjugate transpose of some matrix A.
I_n	The $n \times n$ identity matrix.
$\varepsilon_{i_1 \cdots i_n}$	The Levi-Civita symbol.
$f \circ g$	The composition of the functions f and g, i.e., $f \circ g(x) = f(g(x))$.
V^*	The dual space of some vector space V.
$V \oplus W$	The direct sum of vector spaces V and W.
$V \otimes W$	The tensor product of vector spaces V and W.
T_pM	The tangent space of the manifold M at the point p.
$\mathfrak{X}(M)$	The space of vector fields on the manifold M.
f^*, f_*	The pull-back and push-forward of some function f, respectively.
$\Omega^k(M)$	The space of k-forms on the manifold M.
$\mathbb{T}^m_n(V)$	The space of (m, n)-tensors on some vector space V.
$\Gamma(E)$	The set of (local) sections over some total space E.
$\star$	The Hodge star operation.
$F(E)$	The frame bundle of some vector bundle with total space E.
Δ_n	The spinor space related to a n-dimensional base space.

1.3 Acknowledgements

I would like to express special thanks to Prof. Dr Thomas Grimm, Prof. Dr Marius Crainic and Arno Hoefnagels for advising me on the shape of the thesis and providing me with a deeper understanding of certain advanced topics.
Secondly, I thank the members of the String Theory group at Utrecht University for suggesting relevant books and allowing me to join their weekly journal clubs.
Finally, I would like to extend my gratitude to Cisca Kalmijn for giving feedback on, and for providing company during the writing of my thesis. Without you, the process would have been be much lonelier.

Part I

Mathematical foundations of gauge theories

2 Tensors

Importantly, we need some mathematical tool to describe "fields" over manifolds. The objects that fit this need are tensor fields, used thoroughly by physicists in field theories.
We firstly define what we mean algebraically by a tensor, using the aptly-named tensor products. Subsequently, we construct use the algebraic definition to define tensor fields on manifolds, and finally look at the metric tensor, which is a concrete example of a tensor field that will prove to be very important in geometry.

2.1 Tensor algebra

In physics, tensors are often defined as physical quantities that transform in a certain way when you rotate your basis vectors. However, using our mathematical understanding of algebra and manifolds, we can define what a tensor is in a more rigorous manner.

2.1.1 Introduction to tensors

We first introduce the notion of tensors over any (by assumption, finite-dimensional) vector space using the tools from algebra.

Definition 2.1. *Let V be any real vector space, and $m, n \in \mathbb{N}_0$. We define the* ***contravariant tensors of rank m*** *over V to be the set*

$$\mathbb{T}^m(V) := \bigotimes_{i=1}^{m} V. \tag{2.1}$$

The ***covariant tensors of rank n*** *are the set*

$$\mathbb{T}_n(V) := \bigotimes_{i=1}^{n} V^*. \tag{2.2}$$

Recall, furthermore, the empty product

$$\mathbb{T}^0(V) = \mathbb{T}_0(V) = \mathbb{R}. \tag{2.3}$$

Together, they combine to form the ***tensors of type (m, n)***

$$\mathbb{T}^m_n(V) := \mathbb{T}^m(V) \otimes \mathbb{T}_n(V). \tag{2.4}$$

A ***tensor*** *of type (m, n) (or (m, n)-tensor for short) is an element of $\mathbb{T}^m_n(V)$.*

If we have some basis e_i for a V, and an induced dual basis e^j for V^*, the basis of $\mathbb{T}^m_n(V)$ is then given by $\{e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e^{j_1} \otimes \cdots \otimes e^{j_n}\}$, and any (m, n)-tensor K can then be written[1]

$$K = \sum K^{i_1 \dots i_m}_{j_1 \dots j_n}\ e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e^{j_1} \otimes \cdots \otimes e^{j_n}. \tag{2.5}$$

[1]This notation is very tiresome, and we actually get rid of the basis vectors in section 2.1.2, which will save us a lot of unnecessary work.

We follow the convention that superscripted indices of some tensor component correspond to an element of the basis, while subscripted indices correspond to an element of the dual basis. Moreover, here, we used the pseudo-Einsteinian summation convention, as introduced in chapter 1. This choice will prove to be essential when we introduce Ricci calculus.
Recall, in particular, that, if we define $L(X, Y)$ as the set of linear mappings $X \to Y$,

$$\mathbb{T}^m_n(V) \cong L(\mathbb{T}_n(V)^*, \mathbb{T}^m(V)) = L(\mathbb{T}^n(V)^{**}, \mathbb{T}^m(V)) = L(\mathbb{T}^n(V), \mathbb{T}^m(V)),$$

the last step being true because $\mathbb{T}^n(V)$ is finite-dimensional. As an intuitive interpretation of tensors, one may therefore think of them as linear mappings between contravariant tensor spaces.
We thus find that $(1, 1)$-tensors correspond to linear mappings $V \to V$, which we usually represent using square matrices. Indeed, given a basis $\{e_1, \ldots, e_n\}$ of V, one can uniquely represent any $(1, 1)$-tensor as an $n \times n$ square matrix, and any $n \times n$ square matrix induces a $(1, 1)$-tensor. where the matrix representations in different bases are related through the regular basis transformation matrices. For example, in $\mathbb{R}^2$, with natural basis $\{e_1, e_2\}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\ e_1 \otimes e^1 + b\ e_1 \otimes e^2 + c\ e_2 \otimes e^1 + d\ e_2 \otimes e^2. \tag{2.6}$$

Someone familiar with quantum mechanics might relate this notation to bra-ket notation, such that $e_i = |i\rangle$ and $e^i = \langle i|$. This relation is indeed true, as kets (resp. bras) are nothing more than the vectors (resp. dual vectors) of the state they represent.

Definition 2.2. *We give the following four types of tensors their own names:*
A $(0, 0)$-tensor is called a ***scalar****.*
A $(1, 0)$-tensor is called a ***vector****.*
A $(0, 1)$-tensor is called a ***covector*** *or* ***dual vector****.*
A $(1, 1)$-tensor is called a ***linear transformation****.*

We can also define some mapping $\Psi(a, b) : \mathbb{T}^m_n(V) \times \mathbb{T}^p_q(V) \to \mathbb{T}^{m+p-1}_{n+q-1}(V)$ for $m, q \in \mathbb{N}$ and $n, p \in \mathbb{N}_0$ by applying the a-th vector component of the first tensor to the b-th covector component of the second tensor. Explicitly,

$$\begin{aligned}
&\left(\sum A^{i_1 \ldots i_m}_{j_1 \ldots j_n}\ e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e^{j_1} \otimes \cdots \otimes e^{j_n}, \sum B^{k_1 \ldots k_p}_{l_1 \ldots l_q}\ e_{k_1} \otimes \cdots \otimes e_{k_p} \otimes e^{l_1} \otimes \cdots \otimes e^{l_q}\right) \\
&\mapsto \sum A^{i_1 \ldots i_{a-1} r i_{a+1} \ldots i_m}_{j_1 \ldots j_n} B^{k_1 \ldots k_p}_{l_1 \ldots l_{b-1} r l_{b+1} \ldots l_q}\ e_{i_1} \otimes \cdots \otimes e_{i_{a-1}} \otimes e_{i_{a+1}} \otimes \cdots \otimes e_{i_m} \otimes e_{k_1} \otimes \cdots \otimes e_{k_p} \\
&\qquad \otimes e^{j_1} \otimes \cdots \otimes e^{j_n} \otimes e^{l_1} \otimes \cdots \otimes e^{l_{b-1}} \otimes e^{l_{b+1}} \otimes \cdots \otimes e^{l_q}.
\end{aligned} \tag{2.7}$$

We can also define a similar mapping $\Psi'(a, b, \alpha, \beta) :\ \mathbb{T}^m_n(V) \times \mathbb{T}^p_q(V) \to \mathbb{T}^{m+p-2}_{n+q-2}(V)$ where we also apply the α-th covector component of the first tensor to the β-th vector component of the second tensor.
This mapping also induces a function $\mathbb{T}^m_n(V) \to \mathbb{T}^{m-1}_{n-1}(V)$, with

$$K \mapsto \Psi'(a, 1, \alpha, 1)\left(K, \sum e_i \otimes e^i\right). \tag{2.8}$$

We call this mapping the **contraction**. These three mappings together allow us to apply any number of vector and covector components of some two matrices to one another.
As mentioned before, writing the basis vectors explicitly can become very tiresome, as you can see in the explicit definition of $\Psi(a,b)$. Later, we will be able to neglect these, but for now, we're stuck with using the basis vectors.
Note that if we have some $(1,1)$-tensors, represented as matrices A, B, then their matrix product AB is given by $(AB)^i_j = \sum A^i_k B^k_j$. Consequently, if B is the inverse of A, we get $\sum A^i_k B^k_j = \delta^i_j$.
Let us now look at how a tensor transforms under a change of basis.

Lemma 2.3. *If our original basis of V is $\{e_i\}$, and our new basis is given by $\overline{e}_k = \sum A^i_k e_i$, we can write A^i_k as some matrix A (not a tensor!), with inverse B (basis transformations must be invertible). Then the components $K^{i_1\cdots i_m}_{j_1\cdots j_n}$ of any tensor $K \in \mathbb{T}^m_n(V)$ transform like*

$$\overline{K}^{k_1\ldots k_m}_{l_1\ldots l_n} = \sum B^{k_1}_{i_1}\cdots B^{k_m}_{i_m} A^{j_1}_{l_1}\cdots A^{j_n}_{l_n} K^{i_1\cdots i_m}_{j_1\cdots j_n}. \tag{2.9}$$

Proof. First, we find what $\overline{e}^l$ is, in terms of e^j. We know that the dual basis is chosen uniquely such that $\overline{e}^l\overline{e}_k = \delta^l_k$. Our guess is that $\overline{e}^l = \sum B^l_j e^j$, which we now check using this uniqueness property.

$$\begin{aligned}\overline{e}^l\overline{e}_k &= \sum B^l_j e^j A^i_k e_i\\ &= \sum B^l_j A^i_k \delta^j_i\\ &= \sum B^l_i A^i_k = \delta^l_k.\end{aligned}$$

Here we use that the components of the identity matrix are δ^l_k. We may conclude that, indeed, $\overline{e}^l = \sum B^l_j e^j$.
The proof is given simply by writing out the tensor in both bases:

$$\begin{aligned}&\sum \overline{K}^{k_1\ldots k_m}_{l_1\ldots l_n}\ \overline{e}_{k_1}\otimes\cdots\otimes\overline{e}_{k_m}\otimes\overline{e}^{l_1}\otimes\cdots\otimes\overline{e}^{l_n}\\ =&\sum B^{k_1}_{i_1}\cdots B^{k_m}_{i_m}A^{j_1}_{l_1}\cdots A^{j_n}_{l_n}K^{i_1\cdots i_m}_{j_1\cdots j_n}\ \overline{e}_{k_1}\otimes\cdots\otimes\overline{e}_{k_m}\otimes\overline{e}^{l_1}\otimes\cdots\otimes\overline{e}^{l_n}\\ =&\sum B^{k_1}_{i_1}\cdots B^{k_m}_{i_m}A^{j_1}_{l_1}\cdots A^{j_n}_{l_n}K^{i_1\cdots i_m}_{j_1\cdots j_n}A^{a_1}_{k_1}\cdots A^{a_m}_{k_m}B^{l_1}_{b_1}\cdots B^{l_n}_{b_n}\ e_{a_1}\otimes\cdots\otimes e_{a_m}\otimes e^{b_1}\otimes\cdots\otimes e^{b_n}\\ =&\sum K^{i_1\cdots i_m}_{j_1\cdots j_n}\delta^{a_1}_{i_1}\cdots\delta^{a_m}_{i_m}\delta^{j_1}_{b_1}\cdots\delta^{j_n}_{b_n}\ e_{a_1}\otimes\cdots\otimes e_{a_m}\otimes e^{b_1}\otimes\cdots\otimes e^{b_n}\\ =&\sum K^{i_1\cdots i_m}_{j_1\cdots j_n}\ e_{i_1}\otimes\cdots\otimes e_{i_m}\otimes e^{j_1}\otimes\cdots\otimes e^{j_n} = K.\end{aligned}$$

Therefore, the components do indeed transform in this way. □

Note that indices corresponding to vectors make the components of the tensor transform as the inverse of the transformation of the basis vectors, while the indices corresponding to covectors make them transform as the transformation of the basis vectors itself. This invites us to call indices relating to vectors **contravariant**, while we call indices relating to covectors **covariant**. This is also the exact reason why we separated the indices in our notation, since we can now see immediately whether some index makes the components transform covariantly or contravariantly.

2.1.2 Ricci calculus

As you may have seen, keeping track of unit (co)vectors is a very tedious task. Our claim is now that any meaningful equation between tensors can be described by only the components of the tensor, and is therefore independent on the choice of basis. Here, by a "meaningful equation", we mean equations where we only apply the previously defined Ψ, Ψ', take the (tensor) product of two tensors, or sum up two tensors of the same type. This claim would invite us to omit the basis vectors from any tensor equation, which would save us a lot of time. For example, equation (2.7) could be written like

$$\left(A^{i_1\cdots i_m}_{j_1\cdots j_n}, B^{k_1\cdots k_p}_{l_1\cdots l_q}\right) \mapsto \sum A^{i_1\cdots i_{a-1} r i_{a+1}\cdots i_m}_{j_1\cdots j_n} B^{k_1\cdots k_p}_{l_1\cdots l_{b-1} r l_{b+1}\cdots l_q}. \tag{2.10}$$

First, we must prove that it is valid to omit basis (co)vectors.

Theorem 2.4. *The definition of Ψ and Ψ' are independent of choice of basis. Furthermore, if S, T are both tensors of rank (p, q), and we define the tensor L, in some basis $\{e_1, \ldots, e_n\}$, such that*

$$L^{i_1\cdots i_p}_{j_1\cdots j_q} = S^{i_1\cdots i_p}_{j_1\cdots j_q} + T^{i_1\cdots i_p}_{j_1\cdots j_q}, \tag{2.11}$$

the transformed components of the tensors for a new basis $\{\overline{e}_1, \ldots, \overline{e}_n\}$ must satisfy an equivalent equation

$$\overline{L}^{i_1\cdots i_p}_{j_1\cdots j_q} = \overline{S}^{i_1\cdots i_p}_{j_1\cdots j_q} + \overline{T}^{i_1\cdots i_p}_{j_1,\ldots,j_q}. \tag{2.12}$$

Moreover, if A, B are tensors of rank (a_1, a_2) and (b_1, b_2), respectively, and we define the tensor C of rank $(a_1 + b_1, a_2 + b_2)$, in some basis $\{e_1, \ldots, e_n\}$, such that

$$C^{i_1\cdots i_{a_1+b_1}}_{j_1\cdots j_{a_1+b_2}} = A^{i_1\cdots i_{a_1}}_{j_1\cdots j_{a_2}} B^{i_{a_1+1}\cdots i_{a_1+b_1}}_{j_{a_2+1}\cdots j_{a_2+b_2}}, \tag{2.13}$$

the transformed components to $\{\overline{e}_1, \ldots, \overline{e}_n\}$ must satisfy

$$\overline{C}^{i_1\cdots i_{a_1+b_1}}_{j_1\cdots j_{a_1+b_2}} = \overline{A}^{i_1\cdots i_{a_1}}_{j_1\cdots j_{a_2}} \overline{B}^{i_{a_1+1}\cdots i_{a_1+b_1}}_{j_{a_2+1}\cdots j_{a_2+b_2}}. \tag{2.14}$$

Proof. The proof is rather easy to prove by writing out, and we only need the transformation rules introduced in lemma 2.3. □

This tells us that, as long as we limit ourselves to these operations, it is valid to omit the basis (co)vectors from our tensors. When we make use of this fact, know that "behind the scenes", we do choose a basis, otherwise the indices would not make any sense, but it does not matter what that basis is *exactly*, since any equations we get are equivalent in any basis we could choose.
In the rest of this section, we define the notation of tensors known as Ricci calculus, which makes use of the fact that we do not need to mention basis vectors.

Indices. Any tensor K is represented by its components, denoted by the tensor followed by some (lower-case) indices. As before, covariant indices are in subscript, while covariant indices are in superscript.[2] There is one big change, however, as indices have a certain defined order that does not necessarily separate covariant from contravariant indices. This is necessary for when we start "raising" and "lowering" indices on a pseudo-Riemannian manifold. For example, one could have the tensor $K^{ij}{}_{k}{}^{l}{}_{mn}$.
An exception to the rule is the Kronecker delta, which, as a tensor, corresponds to the identity mapping, and whose components don't change upon raising or lowering: $\delta^i{}_j = \delta_j{}^i = \delta^i_j = \delta^{ij} = \delta_{ij}$ (these equalities are component-wise — the entire tensors cannot be equal since they are of a different type).

Einstein summation convention. For some tensors A^i and B_i, it would not make any sense to view the element A^iB_i for some i, since this is not an operation that is "compatible" with leaving out the basis (co)vectors. Therefore, we define the Einstein summation convention where if a single index i appears as both a covariant and a contravariant index, a summation over i is implied. For example, $A^iB_i := \sum A^iB_i$.
Also note that the Einstein summation convention allows us to shorten contractions, such as $K^i{}_i$.
The Einstein summation convention will replace the previously defined pseudo-Einsteinian summation convention. The only difference is that we no longer include the summation $\sum$.

Multi-index shortening. When we have a tensor that consists of a high amount of indices, we can shorten a sequence of indices by a capital letter. For example, we have $K^I{}_J{}^K := K^{i_1\cdots i_n}{}_{j_1\cdots j_m}{}^{k_1\ldots k_l}$. This combines with the summation convention such that $A^IB_I := A^{i_1\ldots i_n}B_{i_1\ldots i_n}$.

Ricci calculus will serve to make tensor notation much easier, in a mathematically accurate manner. Usually, physics students get Ricci calculus "for free" in their first course concerning tensors, but we went through the relevant mathematics to prove that it makes sense.
From these operations, we can also create new operations that are compatible with Ricci calculus. We cover two of these, relating to the symmetricity of the tensor.

Definition 2.5. *For some tensor, we can* ***symmetrise*** *a collection of either covariant or contravariant (but not a mixture) indices* $(i_1, \ldots, i_n)$ *by summing all* $n!$ *permutations of indices, and then dividing the result by* $n!$*. The collection of indices to be symmetrised over is enclosed within braces* $(\cdot)$*. For example,*

$$T_{(i_1\ldots i_n)K}{}^L = \frac{1}{n!}\sum_{\sigma\in S_n} T_{i_{\sigma(1)}\cdots i_{\sigma(n)}K}{}^L. \tag{2.15}$$

[2] A useful mnemonic: when you need to decide whether an index is covariant or contravariant, you should look at the third letter of each word, so co**v**ariant or co**n**travariant. The letter "v" looks like an arrow pointing down, so co**v**ariant corresponds to a lower index, and the letter "n" looks like an arrow pointing up, so co**n**travariant corresponds to an upper index.

Here, S_n is the set of permutations of length n. Note that the tensor here is just an example, and the definition can be generalised to any collection of covariant or contravariant indices; they don't even have to be next to each other. We seperate a set of indices from the collection that is to be symmetrised by writing them within $|\cdot|$, and indices of different variance than the collection are also excluded, for example,

$$T_{(i_1}{}^{l_1}{}_{i_2 \dots i_j |K| i_{j+1} \dots i_n)} = \frac{1}{n!} \sum_{\sigma \in S_n} T_{i_{\sigma(1)}}{}^{l_1}{}_{i_{\sigma(2)} \dots i_{\sigma(j)} K i_{\sigma(j+1)} i_{\sigma(j+1)} \dots i_{\sigma(n)}}. \tag{2.16}$$

*We say that some tensor is **symmetric** over the indices $(i_1, \dots, i_n)$ if it is equal to the symetrization over those indices. If the tensor is symmetric over all its indices, we say it is **symmetric**.*
*Similarly, we can **anti-symmetrise** a tensor over some indices. We do this in the exact same manner, but we multiply each term in the sum by the sign of the permutation, and the anti-symmetrisation is denoted by square brackets $[\cdot]$, for example,*

$$T_{[i_1 \dots i_n]K}{}^{L} = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) T_{i_{\sigma(1)} \dots i_{\sigma(n)} K}{}^{L}. \tag{2.17}$$

*We say that a tensor is **anti-symmetric** (or skew-symmetric) over indices $(i_1, \dots, i_n)$ if it is equal to the anti-symmetrisation over those indices, and if it is anti-symmetric over all indices, we say the tensor is **anti-symmetric** (or skew-symmetric).*

2.2 Tensor fields

Now, we combine tensor algebra with differential manifolds to achieve the notion of tensor fields, similarly to how one defines vector fields.

Definition 2.6. *Let M be some manifold. A **(smooth) tensor field** T of rank (m, n) on M of rank (m, n) is some function that assigns to every point $p \in M$ a tensor $T_p \in \mathbb{T}^m_n(T_pM)$, such that in some neighbourhood of p, the components of the tensor at each point, under any fixed local coordinate system, are smooth functions.*
The set of all tensor fields of rank (m, n) on M is denoted $\mathcal{T}^m_n(M)$, and the set of all tensor fields of any rank is $\mathcal{T}(M)$.
*We say some tensor field is **symmetric** or **anti-symmetric** if it is symmetric or anti-symmetric at every point, respectively.*

In particular, $\mathfrak{X}(M) = \mathcal{T}^1_0(M)$ and $\Omega(M) = \mathcal{T}^0_1(M)$. More generally, the set of k-forms is exactly the set of all anti-symmetric $(0, k)$-tensors. If we take some anti-symmetric tensor

$\omega_{\mu_1\ldots\mu_k}$, we know that

$$\begin{aligned}
\omega &= \omega_{\mu_1\ldots\mu_k}\ e^{\mu_1}\otimes\cdots\otimes e^{\mu_k}\\
&= \sum_{\mu_1<\cdots<\mu_k} \varepsilon_{i_1\ldots i_k}\omega_{\mu_1\ldots\mu_k}\ e^{\mu_{i_1}}\otimes\cdots\otimes e^{\mu_{i_k}}\\
&= \sum_{\mu_1<\cdots<\mu_k} \omega_{\mu_1\ldots\mu_k}\ e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}\\
&= \frac{1}{k!}\omega_{\mu_1\ldots\mu_k}\ e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}.
\end{aligned}$$

From now on, we write any k-form in such a way using the factorial, such that we can easily turn forms into tensors (and vice versa). Furthermore, the total derivative is given by

$$\begin{aligned}
\mathrm{d}\omega &= \frac{1}{k!}\mathrm{d}\omega_{\mu_1\ldots\mu_k}\wedge e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}\\
&= \frac{1}{k!}\frac{\partial\omega_{\mu_1\ldots\mu_k}}{\partial x^\nu}\ e^\nu\wedge e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}\\
&= \frac{1}{(k+1)!}\left((k+1)\frac{\partial\omega_{\mu_1\ldots\mu_k}}{\partial x^\nu}\right)e^\nu\wedge e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}.
\end{aligned}$$

We now define the operation ∂_ν such that

$$\partial_\nu\omega_{\mu_1\ldots\mu_k} := \frac{\partial\omega_{\mu_1\ldots\mu_k}}{\partial x^\nu}. \tag{2.18}$$

We only have one issue left: $(k+1)\partial_\nu\omega_{\mu_1\ldots\mu_k}$ is not an anti-symmetric tensor, so we cannot yet easily translate the exterior derivative of some form to a tensor. We will therefore work on rewriting the equation into some tensor.

$$\begin{aligned}
\frac{1}{k!}\frac{\partial\omega_{\mu_1\ldots\mu_k}}{\partial x^\nu}\ e^\nu\wedge e^{\mu_1}\wedge\cdots\wedge e^{\mu_k} &= \frac{1}{k!}\,\partial_{\mu_0}\omega_{\mu_1\ldots\mu_k}\ e^{\mu_0}\wedge\cdots\wedge e^{\mu_k}\\
&= \frac{1}{k!}\,\partial_{\mu_0}\omega_{\mu_1\ldots\mu_k}\varepsilon_{i_0\ldots i_k}\ e^{\mu_{i_0}}\otimes\cdots\otimes e^{\mu_{i_k}}\\
&= \frac{1}{k!}\,\partial_{\mu_{j_0}}\omega_{\mu_{j_1}\ldots\mu_{j_k}}\delta^{j_0 i_0}\cdots\delta^{j_k i_k}\varepsilon_{i_0\ldots i_k}\ e^{\mu_0}\otimes\cdots\otimes e^{\mu_k}\\
&= (k+1)\,\partial_{[\mu_0}\omega_{\mu_1\ldots\mu_k]}\ e^{\mu_0}\otimes\cdots\otimes e^{\mu_k},
\end{aligned}$$

and we find that

$$(\mathrm{d}\omega)_{\mu_0\ldots\mu_k} = (k+1)\,\partial_{[\mu_0}\omega_{\mu_1\ldots\mu_k]}. \tag{2.19}$$

2.2.1 Pseudo-Riemannian manifolds

During our introduction of Ricci calculus, we stated that it was very important to keep track of indices, even between indices with different variance, so $T^i_{\ j}$ and $T_j^{\ i}$ are not necessarily equal. However, in our current formulation, there is no reason to do this, since covariant and contravariant indices are strictly separated. However, anyone familiar with tensors will probably know that we usually allow indices to be raised or lowered. This validates keeping track of the order, since if we lower the i-index of the two tensors we defined earlier, we would get T_{ij} and T_{ji}, respectively, which are clearly not the same.

Definition 2.7. *Let M be some manifold. A* ***metric tensor*** *is some symmetric tensor field g_{ij} of rank $(0,2)$ that is non-degenerate, i.e., if $U^i \in T_p(M)$ for some $p \in M$, and ${g_p}_{ij} U^i V^j = 0$ for all $V^j \in T_p(M)$, then $U^i = 0$.*
A manifold equipped with a metric tensor is called a ***pseudo-Riemannian manifold****.*
On a pseudo-Riemannian manifold, we define lowering of indices such that

$$V_i := g_{ij} V^j, \tag{2.20a}$$

and, similarly, raising of indices,

$$V^j := g^{ij} V_j, \tag{2.20b}$$

where g^{ij} is the unique tensor field that at any point is the inverse of g_{ij}, or $g^{ij} g_{jk} = \delta^i_k$.
This definition is generalised to raise or lower an index of any tensor.

Now, we make use of the notation introduced to write

$$g_{ij} U^i V^j = U^i V_i, \tag{2.21}$$

and we see that $U_i = 0$ if and only if $U^i = 0$ (which actually proves bijectivity of the metric tensor as a map $T_pM \to T_p^*M$), due to the condition that the metric tensor must be non-degenerate.

Theorem 2.8 (Sylvester's law of inertia)**.** *Let g be some non-degenerate symmetric tensor in $\mathbb{T}^0_2(V)$, with V some n-dimensional vector space over the field $\mathbb{R}$. Then there is a unique $p \in \mathbb{N}_0$ such that for some basis, $g_{ij} = 0$ for $i \neq j$, $g_{ii} = -1$ for $1 \leq i \leq p$, and $g_{ii} = +1$ for $p < i \leq n$. [5, pp. 360–361]*

This theorem was paraphrased from [5] in a way to better fit this context. For those interested, the proof is also given there.
The process of finding these g_{ii} is actually equivalent to diagonalising the matrix $(g_{ij})_{i,j}$, and therefore these g_{ii} can be thought of as the eigenvalues of the matrix.
Due to smoothness of the metric tensor, this number p is unique on the entirety of any (connected) pseudo-Riemannian manifold.

Moreover, if we define the determinant $g = \det\big((g_{ij})_{1\leq i,j\leq n}\big)$, it must have constant sign in any basis at any point, since, in the basis we defined in the theorem, the determinant must be ± 1. In any other base with transformation matrix A, the metric transforms like ${g'}_{ij} = {A_i}^k {A_j}^l g_{kl}$, and therefore $g' = (\det A)^2 g$, and the sign doesn't change. Moreover, due to continuity, and the fact that g cannot be equal to 0 tells us that the sign also cannot change under translations over the manifold. In particular, $\operatorname{sign}(g) = (-1)^p$.

Definition 2.9. *Let M be some n-dimensionl (connected) pseudo-Riemannian manifold with metric tensor g_{ij}. We define the metric's* ***signature*** *to be $(n-p, p)$, with p defined as in theorem 2.8. In physical contexts, we typically refer to $n-p$ as the* ***space-like dimension****, and to p as the* ***time-like dimension****.*
If g_{ij} has signature $(n, 0)$, we refer to it as a ***Riemannian metric****, and M is a* ***Riemannian manifold****. If $M = \mathbb{R}^n$, we say that M is a* ***Euclidean space****.*

Furthermore, if g_{ij} has signature $(n-1,1)$, we say it is a ***Lorentzian metric****, and M is then a* ***Lorentzian manifold****. If $M=\mathbb{R}^n$, we say it is a* ***Minkowski space****.*

Conventionally, on Riemannian manifolds, we use indices from the Latin alphabet, so T_{ij}, while on Lorentzian manifold, we use indices from the Greek alphabet, so $T_{\mu\nu}$.
These different spaces also more-or-less separate our fields of study. Classical mechanics usually takes place on 3-dimensional Euclidean space, with time acting more like a parameter, special relativity takes place on 4-dimensional Minkowski space, and general relativity concerns itself with 4-dimensional Lorentzian manifolds.
The reason why relativistic theories are Lorentzian is because we want the speed of light in a vacuum c to be constant, so $ct=\|\mathbf{x}\|$, if the light particle is at the origin for $t=0$. Reordering and using $\mathbf{x}=(x^1,x^2,x^3)$ gives us $-c^2t^2+(x^1)^2+(x^2)^2+(x^3)^2=0$, with $(x^i)^2$ being the square of x^i. Therefore, this invites us to define $x^0:=ct$, and with $g_{\mu\nu}=\operatorname{diag}(-1,1,1,1)$ the metric, we get $x_\mu x^\mu=0$.
As for the scope of this thesis, we are primarily interested in physics on the spacetime manifold, which is Lorentzian. Therefore, from this point on, we generally use Greek letters for our indices, rather than Latin letters.
The metric tensor now induces a bilinear mapping $\langle\cdot,\cdot\rangle:\ \Omega(M)\times\Omega(M)\to\mathcal{C}^\infty(M)$

$$\langle e^\mu, e^\nu\rangle = g^{\mu\nu}, \tag{2.22}$$

equivalently,

$$\langle\omega,\eta\rangle=\omega_\mu\eta_\nu\langle e^\mu,e^\nu\rangle=\omega_\mu\eta^\mu. \tag{2.23}$$

Definition 2.10. *The* ***bilinear form on k-forms*** *is defined as the map*

$$\langle\cdot,\cdot\rangle:\ \Omega^k(M)\times\Omega^k(M)\to\mathcal{C}^\infty(M), \tag{2.24}$$

where

$$\langle e^{\mu_1}\otimes\cdots\otimes e^{\mu_k}, e^{\nu_1}\otimes\cdots\otimes e^{\nu_k}\rangle=\frac{1}{k!}g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}, \tag{2.25}$$

and therefore,

$$\begin{aligned}\langle e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}, e^{\nu_1}\wedge\cdots\wedge e^{\nu_k}\rangle &= \langle\varepsilon_{i_1\dots i_k}e^{\mu_{i_1}}\otimes\cdots\otimes e^{\mu_{i_k}},\varepsilon_{j_1\dots j_k}e^{\nu_{j_1}}\otimes\cdots\otimes e^{\nu_{j_k}}\rangle\\ &=\frac{1}{k!}\varepsilon_{i_1\dots i_k}\varepsilon_{j_1\dots j_k}g^{\mu_{i_1}\nu_{j_1}}\dots g^{\mu_{i_k}\nu_{j_k}}\\ &=\varepsilon_{j_1\dots j_k}g^{\mu_1\nu_{j_1}}\dots g^{\mu_k\nu_{j_k}}\\ &=\det\big((g^{\mu_i\nu_j})_{1\le i,j\le k}\big).\end{aligned} \tag{2.26}$$

2.2.2 The volume element and Hodge $\star$ operator

When working on a pseudo-Riemannian orientable manifold, we can define a canonical volume element of the manifold, using the fact that the metric tensor is nowhere-vanishing.

Lemma 2.11. *Let M be some pseudo-Riemannian orientable n-dimensional manifold with metric tensor $g_{\mu\nu}$. Define*

$$\mathrm{d}V = \sqrt{|g|}\, e^1 \wedge \cdots \wedge e^n, \tag{2.27}$$

with $e_1, \dots, e_n$ local basis vectors induced from an oriented local coordinate system, and

$$g := \det\big((g_{\mu\nu})_{1\leq\mu,\nu\leq m}\big) = \varepsilon_{\nu_1\nu_2\cdots\nu_n}\delta^{\nu_1\mu_1}\cdots\delta^{\nu_n\mu_n}g_{1\mu_1}g_{2\mu_2}\dots g_{n\mu_n}. \tag{2.28}$$

Then $\mathrm{d}V$ uniquely defines a volume form on M.

Proof. We first will have to prove that $\mathrm{d}V$ is indeed an m-form. At each point in M, it is clear that $\mathrm{d}V$ is an m-covector, i.e., an element of $\Lambda^m T_p^*M$, since it is just some scalar times the m-covector $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$. Moreover, $g_{\mu\nu}$ is a smooth expression, and so is taking the determinant. As discussed before, we know that $g \neq 0$ at all points, and therefore, $|g|$ is smooth, and not equal to zero. It is then easy to find that $\sqrt{|g|}$ is a smooth function, and therefore, $\mathrm{d}V$ is an m-form on M.
For $\mathrm{d}V$ to be a volume form, we only need to show that it is nowhere zero. This is trivial, since we have already shown that $\sqrt{|g|} \neq 0$ everywhere, and thus $\mathrm{d}V \neq 0$.
As for unicity of the volume form, consider any other local basis $f_1, \dots, f_n$ that is induced from a positively oriented coordinate system. Then there must be some matrix $A_\mu{}^\nu$ such that $f_\mu = A_\mu{}^\nu e_\nu$. In this basis, we know that g becomes

$$g'_{\mu\nu} = A_\mu{}^\rho A_\nu{}^\sigma g_{\rho\sigma},$$

and so the related matrix is

$$(g'_{\mu\nu})_{\mu,\nu} = A(g_{\mu\nu})_{\mu,\nu}A^T,$$

and thus

$$g' = g(\det A)(\det A^T).$$

The induced volume form then is

$$\mathrm{d}V' = \det A\sqrt{|g|}\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^n,$$

where we used that $\det A > 0$, since the two bases share orientation. Now, using the Jacobian,

$$\mathrm{d}V' = \det A\sqrt{|g|}\,\det A^{-1}\,\mathrm{d}e^1 \wedge \cdots \wedge \mathrm{d}e^n = \mathrm{d}V.$$

And thus, this volume form is the same for any basis, proving uniqueness. □

Definition 2.12. *Let M be some pseudo-Riemannian orientable n-dimensional manifold with canonical volume form $\mathrm{d}V$ and the induced bilinear form on differential forms as above. Then the* ***Hodge star operator*** *is the unique linear mapping $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ such that, for any $\omega, \eta \in \Omega^k(M)$,*

$$\omega \wedge \star\eta = \langle\omega, \eta\rangle\mathrm{d}V. \tag{2.29}$$

We say that $\star\eta$ is the ***Hodge dual*** *to η.*

Lemma 2.13. *The Hodge star operator is the function induced by*

$$\star(e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}) = \frac{\sqrt{|g|}}{(n-k)!} g^{\mu_1\nu_1} \cdots g^{\mu_k\nu_k} \varepsilon_{\nu_1\dots\nu_k\nu_{k+1}\dots\nu_n}\, e^{\nu_{k+1}} \wedge \cdots \wedge e^{\nu_n} \tag{2.30}$$

Proof. Let us first assume that for any k,

$$e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \wedge \star(e^{\nu_1} \wedge \cdots \wedge e^{\nu_k}) = \langle e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}, e^{\nu_1} \wedge \cdots \wedge e^{\nu_k} \rangle \mathrm{d}V, \tag{2.31}$$

with $\mu_i < \mu_{i+1}$ and $\nu_i < \nu_{i+1}$, then, for any k-forms ω, η, we have

$$\omega \wedge \star\eta = \frac{1}{k!}\frac{1}{k!} \omega_{\mu_1\dots\mu_k} \eta_{\nu_1\dots\nu_k} e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \wedge \star(e^{\nu_1} \wedge \cdots \wedge e^{\nu_k}) = \langle \omega, \eta \rangle\ \mathrm{d}V.$$

This tells us that we only have to show that equation (2.31) is true to prove the lemma. This is a rather straightforward process, since

$$\begin{aligned}\langle e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}, e^{\nu_1} \wedge \cdots \wedge e^{\nu_k} \rangle\ \mathrm{d}V &= \det\big((g^{\mu_i\nu_j})_{1\le i,j\le k}\big) \sqrt{|g|}\, e^1 \wedge \cdots \wedge e^n \\ &= \varepsilon_{i_1\dots i_k} g^{\nu_1\mu_{i_1}} \cdots g^{\nu_k\mu_{i_k}} \sqrt{|g|}\, e^1 \wedge \cdots \wedge e^n,\end{aligned}$$

now define μ_i for $i \ge k+1$ such that $\mu_i < \mu_{i+1}$, and $\mu_i \ne \mu_j$ for $i \ne j$ (including $i, j < k+1$), such that (no summation over μ_i is implied in the next equation)

$$= \varepsilon_{i_1\dots i_k} g^{\nu_1\mu_{i_1}} \cdots g^{\nu_k\mu_{i_k}} \sqrt{|g|} \varepsilon_{\mu_1\dots\mu_n}\, e^{\mu_1} \wedge \cdots \wedge e^{\mu_n},$$

if we now let $k+1 \le i_p \le n$ for $p > k$, and define $\varepsilon_{i_{k+1}\dots i_n} := \varepsilon_{(i_{k+1}-k)\dots(i_n-k)}$,

$$= e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \wedge \left(\frac{1}{(n-k)!} \varepsilon_{i_1\dots i_k} \varepsilon_{i_{k+1}\dots i_n} g^{\nu_1\mu_{i_1}} \cdots g^{\nu_k\mu_{i_k}} \sqrt{|g|} \varepsilon_{\mu_1\dots\mu_n}\, e^{\mu_{i_{k+1}}} \wedge \cdots \wedge e^{\mu_{i_n}} \right),$$

now, we furthermore replace i_p by $1 \le j_p \le n$, and by using that $e^j \wedge e^j = 0$ for any index j,

$$\begin{aligned}&= e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \wedge \left(\frac{\sqrt{|g|}}{(n-k)!} \varepsilon_{j_1\dots j_n} g^{\nu_1\mu_{j_1}} \cdots g^{\nu_k\mu_{j_k}} \varepsilon_{\mu_1\dots\mu_n}\, e^{\mu_{j_{k+1}}} \wedge \cdots \wedge e^{\mu_{j_n}} \right) \\ &= e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \wedge \left(\frac{\sqrt{|g|}}{(n-k)!} g^{\nu_1\rho_1} \cdots g^{\nu_k\rho_k} \varepsilon_{\rho_1\dots\rho_n}\, e^{\rho_{k+1}} \wedge \cdots \wedge e^{\rho_n} \right),\end{aligned}$$

hence proving equation (2.31), and thus the lemma. □

We provide some examples of Hodge duals,

$$\star 1 = \frac{\sqrt{|g|}}{n!} \varepsilon_{\nu_1\dots\nu_n}\, e^{\nu_1} \wedge \cdots \wedge e^{\nu_n} = \sqrt{|g|} e^1 \wedge \cdots \wedge e^n = \mathrm{d}V, \tag{2.32a}$$

$$\star \mathrm{d}V = |g| g^{1\nu_1} \cdots g^{m\nu_m} \varepsilon_{\nu_1\dots\nu_m} = \frac{|g|}{g} = \mathrm{sign}(g), \tag{2.32b}$$

$$\star\omega = \frac{1}{(n-k)!} \frac{\sqrt{|g|}}{k!} \omega^{\mu_1\dots\mu_k} \varepsilon_{\mu_1\dots\mu_n}\, e^{\mu_{k+1}} \wedge \cdots \wedge e^{\mu_n}, \tag{2.32c}$$

equivalently,

$$(\star\omega)_{\mu_1\dots\mu_{n-k}} = \frac{\sqrt{|g|}}{k!}\varepsilon_{\nu_1\dots\nu_k\mu_1\dots\mu_{n-k}}\omega^{\nu_1\dots\nu_k}. \tag{2.32d}$$

Lemma 2.14. *Let $g_{\mu\nu}$ be a metric tensor of signature $(n-p,p)$ on some oriented pseudo-Riemannian manifold. Then, for any k-form ω,*

$$\star\star\,\omega = (-1)^{k(n-k)+p}\omega. \tag{2.33}$$

Proof. We prove the lemma by showing first that

$$\star\star\,(e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}) = (-1)^{k(n-k)+p}e^{\mu_1}\wedge\cdots\wedge e^{\mu_k},$$

from which the general statement automatically follows. The result will be shown by writing it out:

$$\begin{aligned}
&\star\star(e^{\mu_1}\wedge\cdots\wedge e^{\mu_k})\\
&= \frac{|g|}{k!(n-k)!}g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}\varepsilon_{\nu_1\dots\nu_n}g^{\rho_{k+1}\nu_{k+1}}\cdots g^{\rho_n\nu_n}\varepsilon_{\rho_{k+1}\dots\rho_n\rho_1\dots\rho_k}e^{\rho_1}\wedge\cdots\wedge e^{\rho_k}\\
&= |g|g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}\varepsilon_{\nu_1\dots\nu_n}g^{\mu_{k+1}\nu_{k+1}}\cdots g^{\mu_n\nu_n}\varepsilon_{\mu_{k+1}\dots\mu_n\mu_1\dots\mu_k}e^{\mu_1}\wedge\cdots\wedge e^{\mu_k},
\end{aligned}$$

now we contract the two Levi-Civita symbols,

$$\begin{aligned}
&= |g|g^{1\nu_1}\cdots g^{k\nu_k}\varepsilon_{\nu_1\dots\nu_n}g^{k+1\nu_{k+1}}\cdots g^{n\nu_n}\varepsilon_{k+1\dots n\,1\dots k}e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}\\
&= \frac{|g|}{g}(-1)^{k(n-k)}e^{\mu_1}\wedge\cdots\wedge e^{\mu_k} = (-1)^{k(n-k)+p}e^{\mu_1}\wedge\cdots\wedge e^{\mu_k}.
\end{aligned}$$

□

Definition 2.15. *Let M be some oriented pseudo-Riemannian n-dimensional manifold with a metric tensor of signature $(n-p,p)$. The map*

$$\delta:\ \Omega^k(M)\to\Omega^{k-1}(M),\ \ \delta = (-1)^{n(k-1)+1+p}\star\mathrm{d}\star \tag{2.34}$$

is called the ***codifferential****.*

Note that now, in particular,

$$\star\delta = (-1)^{-k^2}\mathrm{d}\star = (-1)^k\mathrm{d}\star, \tag{2.35}$$

and, similarly.

$$\delta\star = (-1)^{n^2-n+1-k^2}\star\mathrm{d} = (-1)^{k+1}\star\mathrm{d}. \tag{2.36}$$

Lemma 2.16. *For any $k \in \mathbb{N}_0$, $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k+1}(M)$,*

$$\int_M \langle \mathrm{d}\omega, \eta \rangle \mathrm{d}V = \int_M \langle \omega, \delta\eta \rangle \mathrm{d}V. \tag{2.37}$$

Proof. Note that

$$\begin{aligned}(\langle \mathrm{d}\omega, \eta\rangle - \langle \omega, \delta\eta\rangle)\mathrm{d}V &= \mathrm{d}\omega \wedge \star\eta - \omega \wedge \star\delta\eta \\ &= \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\star\eta \\ &= \mathrm{d}(\omega \wedge \star\eta)\,.\end{aligned}$$

Therefore, through Stokes's theorem,

$$\int_M (\langle \mathrm{d}\omega, \eta\rangle - \langle \omega, \delta\eta\rangle)\ \mathrm{d}V = \int_M \mathrm{d}(\omega \wedge \star\eta) = 0. \tag{2.38}$$

$\square$

3 Fibre bundles

In this chapter, we discuss the mathematical objects that will later allow us to define gauge theories: fibre bundles. To this end, we first define the notion of Lie groups, which are very important geometric objects in modern mathematics. We then define fibre bundles, and in particular principle bundles, and look at some properties. We end the chapter by covering vector bundles, as well. For further information on bundles, in a context that also relates to gauge theory, one can refer to [6]. A more mathematical approach can be found in [7].

3.1 Lie groups

In this paragraph, we provide the basic notions on so-called Lie groups. A lot of theory on Lie groups was developed by Norwegian mathematician Sophus Lie in the nineteenth century, which has led to him being the namesake of many notions related to Lie groups.

3.1.1 Basic definitions

We start with defining the notion of Lie groups.

Definition 3.1. *Some set G is called an n-dimensional* ***Lie group*** *if it is a group, an n-dimensional smooth manifold, and the mapping $G \times G \to G,\ (g,h) \mapsto gh^{-1}$ is smooth. Some subset $H \subseteq G$ is an m-dimensional* ***Lie subgroup*** *of G if it is a subgroup and m-dimensional submanifold of G. Note that then $(g,h) \mapsto gh^{-1}$ is still smooth on H, and it maps from $H \times H$ to H through the subgroup criterion.*

Whenever we are working on a Lie group, we conventionally denote its identity by e.
Note that our definition ensures that the mappings $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth, through a similar process as the proof to the subgroup criterion.
Now it is known that the (special) unitary group is both a group and a smooth manifold. Using techniques from analysis, including a power series, one can show that the map $(A,B) \mapsto AB^{-1}$ is indeed smooth, and therefore, the (special) unitary groups are Lie groups. Moreover, the general linear group is also a Lie group through the same logic.

Theorem 3.2 (Cartan's theorem)**.** *Every closed subgroup of a Lie group is a Lie subgroup.*

The proof to Cartan's theorem is rather complicated, and will therefore be omitted from this thesis.
Now, recall that $U(n)$ and $SU(n)$ have very special tangent spaces, $T_AU(n) = AT_{I_n}U(n)$ and $T_ASU(n) = AT_{I_n}SU(n)$. This is actually a property that we can generalise to any Lie group.

Definition 3.3. *Let G be some Lie group. For some $g \in G$, we define the* ***left-translation*** *as $L_g:\ G \to G,\ h \mapsto gh$, which is automatically a diffeomorphism. We say that some vector field $X \in \mathfrak{X}(G)$ is* ***left-invariant*** *if $(\mathrm{d}L_g)_h X_h = X_{gh}$ for all $g,h \in G$, or, equivalently, $L_{g*}X = X \circ L_g$. The vector space of all left-invariant vector fields on G is denoted $\mathfrak{g}$ (the corresponding lowercase Fraktur).*

Note that since L_g is a diffeomorphism, $(\mathrm{d}L_g)_e$ is an isomorphism between T_eG and T_gG. Therefore, we can define an isomorphism between T_eG and $\mathfrak{g}$ through $X_e \mapsto (g \mapsto (\mathrm{d}L_g)_e X_e)$, and we find that the set of left-invariant vector fields is isomorphic to the tangent space at the identity. We will often identify the two.
Similarly to the special tangent spaces on $U(n)$ and $SU(n)$, we find for any Lie group G, by identifying $\mathfrak{g}$ with the tangent space at e,

$$T_gG = (\mathrm{d}L_g)_e\mathfrak{g}. \tag{3.1}$$

In particular, if G is the (special) unitary group, $(\mathrm{d}L_A)_{I_n} : B \mapsto AB$ for any $A \in G$, and we regain the original tangent spaces

$$T_AU(n) = A\mathfrak{u}(n), \qquad T_ASU(n) = A\mathfrak{su}(n). \tag{3.2}$$

Theorem 3.4. *For G some Lie group, the set of left-invariant vector fields on G, being $\mathfrak{g}$, together with the Lie bracket is a Lie algebra.*
*We say that $\mathfrak{g}$ is the **Lie algebra** of G.*

Proof. Recall that $\mathfrak{X}(M)$ is a Lie algebra, and since $\mathfrak{g}$ is a subspace of $\mathfrak{X}(M)$, the properties of the bracket are already satisfied. We only have to prove that the Lie bracket maps $\mathfrak{g}$ onto itself.
Take any $X, Y \in \mathfrak{g}$, and some $g \in G$. Then, recall that

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X \circ L_g, Y \circ L_g] = [X,Y] \circ L_g, \tag{3.3}$$

and therefore $\mathfrak{g}$ is indeed a Lie algebra under the Lie bracket. □

Finally, we define the adjoint representation.

Definition 3.5. *Let G be some Lie group with Lie algebra $\mathfrak{g}$. For any $g \in G$, we define the isomorphism*

$$\Psi_g : G \to G, \; h \mapsto ghg^{-1}. \tag{3.4}$$

*This Lie group isomorphism then induces an isomorphism $(\mathrm{d}\Psi_g)_h : T_hG \to T_{\Psi_g(h)}M$, which defines the **adjoint representation** of g*

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \; X \mapsto (\mathrm{d}\Psi_g)_e X, \tag{3.5}$$

where we identify $\mathfrak{g} = T_eG$.

3.1.2 Lie groups and manifolds

We can let some Lie group act on some set, particularly a manifold.

Definition 3.6. *For G some Lie group, and M some manifold, an **action** of G on M is some differentiable map $G \times M \to M$, $(g,x) \mapsto gx$ such that*

(i) $ex = x$ for all $x \in M$;

(ii) $g(hx) = (gh)x =: ghx$ for all $g, h \in G$ and $x \in M$.

*The action is **transitive** if for all $x, y \in M$, there is some $g \in G$ such that $y = gx$.*
*The action is **effective** if, for any $g \in G$ with $gx = x$ for all $x \in M$, we have $g = e$, i.e., the trivial action is unique.*
*The action is **free** if, for any $g \in G$ with $gx = x$ for any one $x \in M$, we have $g = e$, i.e., e is the only element of G without fixed points. Notice that if an action is free, then it must also be effective.*
*The actions is **regular** if it is both transitive and free.*

An example of an action is the left translation on G, as we defined before (here G is both the Lie group and the manifold). The left translation is, in fact, a transitive and free, and thus regular, action.
To end this section, we introduce the infinitesimal action.

Definition 3.7. *Let G be some Lie group that acts on a manifold M on the right. For any $x \in M$, we define the **orbit***

$$\sigma_x : \ G \to M, \ g \mapsto xg. \tag{3.6}$$

*For $\mathfrak{g}$ the Lie algebra of G, one defines the **infinitesimal action** $\# : \ \mathfrak{g} \to \mathfrak{X}(M)$ by*

$$\#(X) = X^{\#}, \ (X^{\#})_x = (\mathrm{d}\sigma_x)_e X \tag{3.7}$$

for any $x \in M$ and $\alpha \in \mathfrak{g}$.
We denote the function $\#_x$ being the mapping $X \mapsto (X^{\#})_x$, i.e., $\#_x = (\mathrm{d}\sigma_x)_e$.

The infinitesimal action is well-defined because the orbit is a smooth function.
Furthermore, notice that if $\#_x X = 0$, then there must be some curve γ over G with $X = \dot\gamma(0)$ and $\gamma(0) = e$, such that $\sigma_x \circ \gamma(t) = x\gamma(t)$ is (locally) constant. Therefore, $x\gamma(t) = x$, and if the action is free, we have $\gamma(t) = e$ around $t = 0$. We now find that $X = \dot\gamma(0) = 0$. In conclusion, if the action is free, we have $\ker \#_x = \{0\}$, and thus the infinitesimal action at any single point is injective.

Lemma 3.8. *The infinitesimal action as defined above is a Lie algebra homomorphism.*

Proof. Recall that $\mathfrak{X}(M)$ is a Lie algebra. To show that $\#$ is a homomorphism, we must show that it is a vector space homomorphism that conserves the bracket functions defined on the Lie algebras:

$$[X, Y]^{\#} = [X^{\#}, Y^{\#}].$$

We start by proving this is a vector space homomorphism. Take any $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
{(X+Y)^{\#}}_x &= (\mathrm{d}\sigma_x)_e(X+Y) \\
&= (\mathrm{d}\sigma_x)_e X + (\mathrm{d}\sigma_x)_e Y = (X^{\#} + Y^{\#})_x, \\
{(\lambda X)^{\#}}_x &= (\mathrm{d}\sigma_x)_e(\lambda X) \\
&= \lambda(\mathrm{d}\sigma_x)_e(X) = \lambda {X^{\#}}_x.
\end{aligned}$$

And thus, it is indeed a vector space homomorphism.
Now, let us observe $[X,Y]^\#$:

$$\begin{aligned}[X,Y]^\#{}_x &= (\mathrm{d}\sigma_x)_e[X,Y] \\ &= [(\mathrm{d}\sigma_x)_e X, (\mathrm{d}\sigma_x)_e Y] = [X^\#{}_x, Y^\#{}_x].\end{aligned}$$

Therefore, this is indeed a Lie algebra homomorphism. □

We end this section by describing a relationship between the infinitesimal action and the adjoint action.

Lemma 3.9. *Let G be some Lie group acting on some manifold M with corresponding Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$ and any $g \in G$,*

$$R_{g_*}X^\# = (\mathrm{Ad}_{g^{-1}}X)^\#. \tag{3.8}$$

Proof. We prove this by evaluating both sides of the equation at some point $x \in M$.

$$\begin{aligned}((R_{g_*})X^\#)_x &= (\mathrm{d}R_g)_{R_g{}^{-1}(x)} X^\#{}_{R_g{}^{-1}(x)} \\ &= (\mathrm{d}R_g)_{xg^{-1}}(\mathrm{d}\sigma_{xg^{-1}})_e X \\ &= \Big(\mathrm{d}\Big(R_g \circ \sigma_{xg^{-1}}\Big)\Big)_e X.\end{aligned}$$

Note now that

$$R_g \circ \sigma_{xg^{-1}} : \; h \mapsto xg^{-1}hg,$$

and, actually,

$$\sigma_x \circ \Psi_{g^{-1}} : \; h \mapsto xg^{-1}hg.$$

Therefore, these two functions are completely equivalent,

$$\begin{aligned}((\mathrm{d}R_g)X^\#)_x &= \Big(\mathrm{d}\Big(\sigma_x \circ \Psi_{g^{-1}}\Big)\Big)_e X \\ &= (\mathrm{d}\sigma_x)_e(\mathrm{d}\Psi_{g^{-1}})_e X \\ &= ((\mathrm{d}\Psi_{g^{-1}})_e X)^\#{}_x \\ &= (\mathrm{Ad}_{g^{-1}}X)^\#{}_x.\end{aligned}$$

□

3.2 Introduction to fibre bundles

In this section, we consider the basic theory concerning fibre bundles, and introduce a basic example to get used to the theory.

3.2.1 Definitions

Recall that one defines tangent spaces on manifolds by assigning a vector space to each point in the manifold. We generalise this notion to the following.

Definition 3.10. *Let E, M be any two sets, with some map $\pi : E \to M$. Then the triple (E, M, π) is called a* ***bundle****. We call E the* ***total space****, M the* ***base space*** *and π the* ***projection****. For any $x \in M$, we say that $\pi^{-1}(x)$ is the* ***fibre*** *at x.*

Assume now that E, M are some smooth manifolds, and π is a smooth surjection. If there is some smooth manifold F such that $\pi^{-1}(x)$ is diffeomorphic to F for all $x \in M$, we say that the 4-tuple (E, M, π, F) is a ***(smooth) fibre bundle****, and call F the* ***fibre****. We often represent fibre bundles as sequences of homeomorphisms*

$$F \longrightarrow E \xrightarrow{\pi} M. \tag{3.9}$$

If we have some Lie group G that acts on F (on the left), we call G the ***structure group****.*

If a fibre bundle has a structure group G, a ***G-atlas*** *is a collection $\{(U_i, \varphi_i)\}$ such that the U_i are open sets that cover M, and $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ are diffeomorphisms with $\varphi_i(p) = (\pi(p), f)$ for any $f \in F$. We furthermore require that the functions*

$$\varphi_{i,x} : \pi^{-1}(x) \to F, \; \varphi_i|_{\pi^{-1}(x)}(p) = (x, \varphi_{i,x}(p)) \tag{3.10}$$

are diffeomorphisms. We have the following commutative diagrams:

$$\begin{tikzcd} U_i & \pi^{-1}(U_i) \arrow[l, "\pi"'] \arrow[d, "\varphi_i"] \\ & U_i \times F \arrow[ul, "(x,f)\mapsto x"] \end{tikzcd} \qquad \begin{tikzcd} F & \pi^{-1}(x) \arrow[l, "\varphi_{i,x}"'] \arrow[d, "\varphi_i|_{\pi^{-1}(x)}"] \\ & \{x\} \times F \arrow[ul, "(x,f)\mapsto f"] \end{tikzcd} \tag{3.11}$$

Lastly, for $x \in U_i \cap U_j$, the ***transition functions***

$$t_{ij}(x) := \varphi_{i,x} \circ \varphi_{j,x}{}^{-1} : F \to F \tag{3.12}$$

must act like elements of G on F. This induces the function $t_{ij} : U_i \cap U_j \to G$. The tuple (U_i, φ_i) is called a ***local trivialization****.*

Intuitively, for a fibre bundle $F \longrightarrow E \xrightarrow{\pi} M$, the total space E looks *locally* like $M \times F$, though not necessarily globally. If $E = M \times F$, we also call the fibre bundle a **trivial bundle**. This is actually where the name of the local trivializations come from, since they make the fibre bundle look locally trivial.

The definition can be rather tough, so we will consider some examples and a metaphor. If you start with a base space and a fibre, the total space is practically a copy of the fibre at each point of the manifold.

Imagine a hairbrush; at each point (by approximation) of the "bald" brush, there is a bristle that is diffeomorphic to all others, though they may differ in length and thickness. The bald brush is the base space, and the bristles are the fibres. The complete hairbrush is then the total space.

We now return to the tangent space on some manifold M. We define the fibre $\mathbb{R}^n$, where n is the dimension of M. For any x, we can find some diffeomorphism between a neighbourhood in $\mathbb{R}^n$ and a neighbourhood of x, whose total differential is a diffeomorphism between $\mathbb{R}^n$ and T_xM. Now define the **tangent bundle**

$$TM := \cup_{x \in M} T_xM, \tag{3.13}$$

and a projection map $\pi : TM \to M$, where $\pi(X_x) = x$ if $X_x \in T_xM$. Clearly, $\pi^{-1}(x) = T_xM$. This defines a fibre bundle

$$\mathbb{R}^n \longrightarrow TM \xrightarrow{\pi} M. \tag{3.14}$$

Now, the Lie group that acts on $\mathbb{R}^n$ is the general linear group $GL(n)$. Furthermore, we can define an open covering $\{U_i\}$ with some local coordinate systems. Through the local coordinate system, we can define $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ in the following manner: let us say we want to find $\varphi_i(X_x)$ for $X_x \in T_xM$, then, using the local coordinate system x^j, we first want to decompose

$$X_x = \sum_{j=1}^{n} X^j \left(\frac{\partial}{\partial x^j}\right)_x,$$

where $X^j = (\mathrm{d}x^j)_x X_x$. Then, we can define

$$\varphi_i(X_x) = \left(\pi(X_x), \sum_{j=1}^{n} X^j \hat{e}_j\right) = \left(\pi(X_x), \sum_{j=1}^{n} (\mathrm{d}x^j)_x X_x \hat{e}_j\right) \tag{3.15}$$

with $\{\hat{e}_j\}$ the natural basis of $\mathbb{R}^n$. The transition functions are then equivalent to the matrices that transform vectors between two bases $\left\{\left(\frac{\partial}{\partial x^j}\right)_x\right\}$ and $\left\{\left(\frac{\partial}{\partial x'^j}\right)_x\right\}$ for the two coordinate systems of the two open sets, which are indeed always in $GL(n)$.
Similarly, we can define the dual tangent bundle T^*M.
Fibre bundles give us the notion of a physical field, which assigns some value to every point in space. Of course, vector fields would be represented by the tangent bundle, scalar fields would have a fibre equal to $\mathbb{R}$, etcetera.

Definition 3.11. *Let $F \longrightarrow E \xrightarrow{\pi} M$ be some fibre bundle. A **section** is some smooth map $s : M \to E$ such that $\pi \circ s(x) = x$ for all $x \in M$, and thus, clearly, $s(x) \in \pi^{-1}(x)$.*
The set of all sections is denoted $\Gamma(M, E)$.
*We can also generalise the notion of sections to **local sections**, which are sections that are only defined on some neighbourhood in M. The local sections defined on some neighbourhood U are denoted $\Gamma(U, E)$, and the set of all local sections is $\Gamma(E)$.*

A section is now exactly our physical understanding of a field. For example, $\Gamma(M, TM)$ is exactly $\mathfrak{X}(M)$, the space of vector fields.
As we have done many times before, we develop a method of comparing fibre bundles by defining a certain kind of mappings. However, since fibre bundles are not generally considered algebraic objects, we do not call the mappings "homomorphisms".

Definition 3.12. *Let* $F \longrightarrow E \xrightarrow{\pi} M$ *and* $F' \longrightarrow E' \xrightarrow{\pi'} M'$ *be some bundles. A smooth mapping* $\tilde{f}: E \to E'$ *is called a* ***bundle map*** *if* $\tilde{f}(\pi^{-1}(x)) \subseteq \pi'^{-1}(q_x)$ *for all* $x \in M$ *and some* $q_x \in M'$.
$\tilde{f}$ *then induces a smooth map* $f: M \to M'$ *through* $x \mapsto q_x$. *In other words, the following diagram commutes.*

$$\begin{tikzcd} F \arrow[r] & E \arrow[r, "\pi"] \arrow[d, "\tilde{f}"] & M \arrow[d, "f"] \\ F' \arrow[r] & E' \arrow[r, "\pi'"] & M' \end{tikzcd} \tag{3.16}$$

Moreover, if $M = M'$ *and* $f(x) = x$ *for all* $x \in M$, *we say that the fibre bundles are* ***equivalent***.

Now, we define a special kind of fibre bundle.

Definition 3.13. *Let* $F \longrightarrow P \xrightarrow{\pi} M$ *be some fibre bundle with structure group* G. *If the fibre is the structure group,* $F = G$, *and there is a smooth right action of* G *on* P *that is regular on* $\pi^{-1}(x)$ *for any* $x \in M$, *we say that the fibre bundle is a* ***principal bundle***.

Now, for any principal bundle $G \longrightarrow P \xrightarrow{\pi} M$, we define a canonical G-atlas. For any point $x \in M$, there must be some open neighbourhood U that admits a local section $s \in \Gamma(U, P)$ (existence is implied by the diffeomorphism with $\mathbb{R}^n$). Then define the map $\varphi: \pi^{-1}(U) \to U \times G$ by

$$\varphi(s(x)g) := (x, g), \tag{3.17}$$

which is defined on the entire domain, since the action is transitive on each $\pi^{-1}(x)$, and it is well-defined, since the action is free. Then, for any other local section $s' \in \Gamma(U', P)$ with corresponding map φ', the transition map between the sections is

$$t(x): G \to G, \; t(x)g = s(x)^{-1}(s'(x)g) = s(x)^{-1}s'(x)g.$$

$s(x), s'(x) \in \pi^{-1}(x) \cong G$ can clearly be seen as elements in the Lie group G. Therefore, $t(x) = s(x)^{-1}s'(x)$ can also be seen as an element in G. We can thus conclude that this method induces a G-atlas over the principal bundle.
This also implies that for any local section $s \in \Gamma(U, P)$, there is some local trivialization chart (U, φ) such that

$$s(x) = \varphi^{-1}(x, e). \tag{3.18}$$

We also see that, if $p = s(x)$ for some $x \in M$ and $p \in P$, we can conclude that $ph = \varphi^{-1}(x, h)$ for any $h \in G$. We generalise this to

$$ph := {\varphi_i}^{-1}(x, g_i h), \quad \text{where } (x, g_i) = {\varphi_i}^{-1}(p) \text{ for } p \in \pi^{-1}(U_i), h \in G, \tag{3.19}$$

for (U_i, φ_i) any local trivialization chart, which allows us to canonically define the right-action given some G-atlas. Defining a G-atlas and defining the right action is therefore completely equivalent. If you have a G-atlas on some fibre bundle with structure group equal to the fibre, it generates a right action, and if you have a principle bundle, the right action generates a G-atlas; the only difference being that the G-atlas generated by some right action is the most general extension to any G-atlas that would generate the right action, so they are not necessarily equal.

From this point on, we may assume that any principal bundle has the G-atlas as defined above.

Definition 3.14. *Take principal bundles* $G \longrightarrow P \xrightarrow{\pi} M$ *and* $G' \longrightarrow P' \xrightarrow{\pi} M$ *, where* G *is a subgroup of* G'*. We say that a bundle map* $f: P \to P'$ *is a* ***G-reduction*** *of* P' *if* $f(pg) = f(p)g$ *and the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r] & P \arrow[d, "f"] \arrow[dr, "\pi"] & \\ & & M \\ G' \arrow[r] & P' \arrow[ur, "\pi'"] & \end{tikzcd} \tag{3.20}$$

3.2.2 Example: the Hopf fibration

Possibly the most famous example of a fibre bundle is the Hopf fibration, which relates the 3-sphere to the 2-sphere and the unit circle. We will first define some of these notions. We have, respectively, the 3-sphere and 2-sphere,

$$S^3 := \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}, \tag{3.21}$$

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}, \tag{3.22}$$

which are 3-dimensional and 2-dimensional smooth manifolds, respectively. Furthermore, since $U(1)$ is identifiable with the unit circle, we use it as such.

We now want to find a projection π such that $U(1) \longrightarrow S^3 \xrightarrow{\pi} S^2$ is a fibre bundle, or, perhaps, even a principle bundle.

Proof. We start by identifying $\mathbb{R}^4 = \mathbb{C}^2$ and $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$, such that

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$S^2 = \{(x, z) \in \mathbb{R} \times \mathbb{C} \mid x^2 + |z|^2 = 1\}.$$

We now define the Hopf fibration as the map

$$\pi : S^3 \to S^2, \ (z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1{z_2}^*). \tag{3.23}$$

Equivalently, if we had stayed in real space, the Hopf fibration would read

$$\pi : S^3 \to S^2, \ (x, y, z, t) \mapsto (x^2 + y^2 - z^2 - t^2, 2(xz + yt), 2(yz - xt)).$$

Now, to prove that the map is well-defined, we already knows it maps to $\mathbb{R} \times \mathbb{C}$, and only need to prove the condition that defines S^2:

$$\begin{aligned}\pi_1(z_1, z_2)^2 + |\pi_2(z_1, z_2)|^2 &= \left(|z_1|^2 - |z_2|^2\right)^2 + 4|z_1 {z_2}^*|^2 \\ &= |z_1|^4 - 2|z_1|^2|z_2|^2 + |z_2|^4 + 4|z_1|^2|z_2|^2 \\ &= (|z_1|^2 + |z_2|^2)^2 = 1.\end{aligned}$$

Therefore, the Hopf fibration indeed maps from S^3 to S^2. We now need to prove that π is smooth and surjective. Smoothness is rather trivial, since each component of the function is smooth. Now take any point $(x, z) \in S^2$. For surjectivity, note that

$$\pi(z_1, z_2) = (2|z_1|^2 - 1, 2z_1 {z_2}^*), \tag{3.24a}$$

$$\pi(z_1, z_2) = (1 - 2|z_2|^2, 2z_1 {z_2}^*), \tag{3.24b}$$

and therefore, if we want to find some (z_1, z_2) such that $\pi(z_1, z_2) = (x, z)$, we are free to choose z_2 to be any number such that $|z_2|^2 = \frac{1}{2}(1-x)$. Assume $x = 1$, then a trivial solution is $(1, 0) \in S^3$. In other cases, we choose the simplest case in which the identity is true: $z_2 = \sqrt{\frac{1}{2}(1-x)}$, and we define $z_1 = \frac{z}{2z_2}$, in which case, obviously, $\pi(z_1, z_2) = (x, z)$. We only need to check now that $(z_1, z_2) \in S^3$:

$$\begin{aligned}|z_1|^2 + |z_2|^2 &= \frac{1}{2}(1-x) + \frac{|z|^2}{2(1-x)} \\ &= \frac{1}{2}(1-x) + \frac{1-x^2}{2(1-x)} \\ &= \frac{1}{2}(1-x) + \frac{1}{2}(1+x) = 1.\end{aligned}$$

Thus, π indeed defines a projection from S^3 to S^2.
We now characterise $\pi^{-1}(x, z)$ for some $(x, z) \in S^2$. Note that if $(z_1, z_2), (z_1', z_2') \in \pi^{-1}(x, z)$, then, through equations (3.24), $|z_1| = |z_1'|$ and $|z_2| = |z_2'|$, so then there exist some $\lambda, \mu \in U(1)$ such that $z_1' = \lambda z_1$ and $z_2' = \mu z_2$. We see that[3]

$$2\lambda\mu^* z_1 {z_2}^* = 2z_1' {z_2'}^* = z = 2z_1 {z_2}^*,$$

and thus $\lambda\mu^* = 1 = \lambda\lambda^*$, and we can conclude that $\lambda = \mu$. Therefore, $\pi^{-1}(x, z) \subseteq U(1)(z_1, z_2)$, with $U(1)(z_1, z_2)$ defined through scalar multiplication of (z_1, z_2).
Furthermore, if we define $(z_1'', z_2'') = \lambda(z_1, z_2)$ for some $\lambda \in U(1)$, then

$$\pi(z_1'', z_2'') = \left(|z_1''|^2 - |z_2''|^2, 2z_1'' {z_2''}^*\right) = (|z_1|^2 - |z_2|^2, 2z_1 {z_2}^*) = \pi(z_1, z_2),$$

and we may conclude that $\pi^{-1}(x, z) = U(1)(z_1, z_2)$. Therefore, there exists a clear diffeomorphism between $\pi^{-1}(x, z)$ and $U(1)$ for any $(x, z) \in S^2$.

We conclude that $U(1) \longrightarrow S^3 \xrightarrow{\pi} S^2$ is a fibre bundle. Furthermore, since $U(1)$ is a

[3] We assume that $z \neq 0$, but if $z = 0$, the result is still true, since either of the z_i should be zero.

Lie group, it is automatically also the structure group of the fibre bundle.

We now work towards proving that this fibre bundle is actually a principal bundle. We start by defining the obvious right action

$$S^3 \times U(1) \to S^3,\ ((z_1, z_2), \lambda) \mapsto (z_1, z_2)\lambda := \lambda(z_1, z_2) = (\lambda z_1, \lambda z_2), \tag{3.25}$$

which is trivially smooth. Furthermore, if there is some $(z_1, z_2) \in S^3$ and a $\lambda \in U(1)$ such that $(z_1, z_2) = (z_1, z_2)\lambda$, then $(\lambda - 1)z_1 = (\lambda - 1)z_2 = 0$, which implies that either $z_1 = z_2 = 0$ or $\lambda = 1$. The first option is impossible, and it is thus implied that $\lambda = 1$, and therefore, this action is free.
Furthermore, since $\pi^{-1}(x, z) = U(1)(z_1, z_2)$ for any single $(z_1, z_2) \in \pi^{-1}(x, z)$, it follows immediately that this action is also transitive on $\pi^{-1}(x, z)$. Therefore, it is regular, and we may conclude that the fibre bundle is a principal bundle.

Now, we define a local section on this principal bundle,

$$s:\ S^2 \setminus \{(1, 0)\} \to S^3,\ (x, z) \mapsto \left(\sqrt{\frac{1}{2}(1 - x)}, \frac{z}{\sqrt{2(1 - x)}}\right),$$

for which we have already proven that $\pi \circ s(x, z) = (x, z)$, and it is clearly smooth for $x \neq 1$. Therefore, this is indeed a local section. You could actually see the point $(1, 0)$ as the "North pole", which gets left out of this section. We can now define a canonical G-chart

$$\varphi^{-1}:\ S^2 \setminus \{(1, 0)\} \times U(1) \to \pi^{-1}(S^2 \setminus \{(1, 0)\}),\ ((x, z), \lambda) \mapsto \left(\lambda\sqrt{\frac{1}{2}(1 - x)}, \frac{\lambda z}{\sqrt{2(1 - x)}}\right).$$

□

In gauge theory, the Hopf fibration is an especially important principal bundle, since it corresponds to the so-called Dirac magnetic monopole. However, the theory necessary for the Dirac monopole is outside the scope of this thesis, and will therefore not be covered. Nonetheless, we can conclude that the Hopf fibration is very much physically relevant.

3.2.3 Connections on principal bundles

In order to make the step from principal bundles to gauge theories, we need the concept of connections. The purpose of connections is to split the tangent space T_pP of the total space at some point p into a direct sum of two certain subspaces.

Definition 3.15. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle. A* ***connection*** *on* P *is some function* HP *that assigns to each point* $p \in P$ *a subspace* H_pP *of* T_pP *that is a horizontal subspace, i.e.,*

$$(\mathrm{d}\pi)_p|_{H_pP}:\ H_pP \to T_{\pi(p)}M \quad \text{is an isomorphism.} \tag{3.26}$$

Furthermore, we require for any $g \in G$ that

$$H_{pg}P = (\mathrm{d}R_g)_p(H_pP), \tag{3.27}$$

*where $R_g:\ P \to P,\ p \mapsto pg$ is the **right-translation**.*

Now, given some connection HP, for each horizontal subspace H_pP, there must also be some vertical subspace V_pP such that $T_pP = H_pP \oplus V_pP$. In fact, from the definition, we may specifically define $V_pP = \ker(\mathrm{d}\pi)_p$.
A more natural, or useful, way of defining a connection is through the connection one-form. To introduce it, first observe that we have the following short sequence of vector spaces:

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\#_p} T_pP \xrightarrow{(\mathrm{d}\pi)_p} T_{\pi(p)}M \longrightarrow 0. \tag{3.28}$$

Lemma 3.16. *The above sequence is exact.*

Proof. First note that $\#_p$ is injective, since our action is free. As mentioned before, the right action induces a G-atlas, so let (U, φ) be some local trivialization around p, then $\pi|_U = \pi_1 \circ \varphi^{-1}$, where $\pi_1:\ U \times G \to U,\ (x, g) \mapsto x$. We now find $(\mathrm{d}\pi)_p = (\mathrm{d}\pi_1)_{\varphi^{-1}(p)} \circ (\mathrm{d}\varphi^{-1})_p$, where both parts of the composition are surjective, and therefore $(\mathrm{d}\pi)_p$ is surjective.
It remains to be shown that $\ker(\mathrm{d}\pi)_p = \#_p(\mathfrak{g})$. Note that $\pi(\sigma_p(g)) = \pi(pg) = \pi(p)$ is constant, and thus $(\mathrm{d}\pi)_p \circ \#_p = \mathrm{d}\big(\pi \circ \sigma_p\big)_e = 0$. Therefore, $\ker(\mathrm{d}\pi)_p \supseteq \#_p(\mathfrak{g})$.
Linear algebra tells us that

$$\dim(\ker(\mathrm{d}\pi)_p) = \dim(T_pP) - \dim(\mathrm{im}(\mathrm{d}\pi)_p),$$

where $\dim(T_pP) = \dim(P)$ and $\dim(\mathrm{im}(\mathrm{d}\pi)_p) = \dim(M)$ because of surjectivity. The local trivializations now tells us that $\ker(\mathrm{d}\pi)_p \cong G$, and thus $\dim(\ker(\mathrm{d}\pi)_p) = \dim(G) = \dim(\mathfrak{g})$, and because of injectivity, we conclude that $\dim(\ker(\mathrm{d}\pi)_p) = \dim(\#_p(\mathfrak{g}))$. Now, since $\#_p(\mathfrak{g})$ is a linear subspace of $\ker(\mathrm{d}\pi)_p$ with equal dimension, we may conclude that the two are equal. Thus, the sequence is indeed exact. □

Because this is a short exact sequence, we may find that $V_pP = \ker(\mathrm{d}\pi)_p = \#_p(\mathfrak{g})$. Do keep in mind that the fact that V_pP is now uniquely defined does not necessarily imply that H_pP is uniquely defined.
Now, the splitting lemma ([3, pp. 384–385], propositions 25 and 26) tells us that there are some unique homomorphisms $h_p:\ T_{\pi(p)}M \to T_pP$ and $\omega_p:\ T_pP \to \mathfrak{g}$ such that $\omega_p \circ \#_p = (\mathrm{d}\pi)_p \circ h_p = \mathrm{id}$. For π_V, π_H the projections of T_pP onto the vertical subspace and horizontal subspace, respectively, the following diagram commutes:

$$\begin{tikzcd}
 & & V_pP & & \\
0 \arrow[r] & \mathfrak{g} \arrow[ur, "\#_p"] & T_pP \arrow[l, "\omega_p"] \arrow[u, "\pi_V"'] \arrow[r, "(\mathrm{d}\pi)_p"] \arrow[d, "\pi_H"'] & T_{\pi(p)}M \arrow[r] \arrow[dl, "h_p"] & 0. \\
 & & H_pP & &
\end{tikzcd} \tag{3.29}$$

In fact, if one has either of these homomorphisms, we equivalently find a corresponding horizontal subspace $H_p = h_p(T_{\pi(p)}M) = \ker \omega_p$. We are therefore free to choose any of the three objects, giving us equivalent results.
Usually, it is most useful to consider the map ω_p. In fact, using any connection, we can define some (smooth, since the right action is transitive) map ω, which assigns to each point p a linear map $\omega_p : T_pP \to \mathfrak{g}$. We say that ω is a $\mathfrak{g}$-valued 1-form over P, the set of which we denote by $\Omega^1(P, \mathfrak{g})$. In general, a $\mathfrak{g}$-valued k-form is nothing more than what its name implies: a k-form that maps to $\mathfrak{g}$ instead of $\mathbb{R}$, explicitly, $\Omega^k(P, \mathfrak{g}) = \Omega^k(P) \otimes \mathfrak{g}$.
For A_ς a basis of $\mathfrak{g}$, any $\mathfrak{g}$-valued form can be written like $\omega = \omega^\varsigma A_\varsigma$, with ω^ς a regular form. From this notation, we can generalise many operations that we previously only defined on k-forms, like the Hodge star operation, such that $\star\omega = \star\omega^\varsigma A_\varsigma$.
Moreover, for $\mathfrak{g}$-valued 1-forms ω, η, we may generalise the Lie bracket to produce a $\mathfrak{g}$-valued 2-form:

$$[\omega, \eta](X, Y) = [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)]. \tag{3.30}$$

This Lie bracket actually behaves similarly to the wedge product between two differential forms. In fact, the exterior derivative of the Lie bracket of $\mathfrak{g}$-valued forms works equivalently to the exterior derivative of the wedge product.
Returning to our 1-form ω, notice that, for any $X \in \mathfrak{X}(P)$, which can be split into $X = X^H + X^V$ with ${X^H}_p \in H_pP$ and ${X^V}_p \in V_pP$ for all u,

$$\begin{aligned}
({R_g}^*\omega)_p(X_p) &= ({R_g}^*\omega)_p({X^H}_p) + ({R_g}^*\omega)_p({X^V}_p) \\
&= \omega_{pg}((\mathrm{d}R_g)_p {X^H}_p) + \omega_{pg}((\mathrm{d}R_g)_p {X^V}_p),
\end{aligned}$$

per definition, $(\mathrm{d}R_g)_p {X^H}_p \in H_{pg}P = \ker \omega_{pg}$, and therefore

$$\begin{aligned}
&= 0 + \omega_{pg}((\mathrm{d}R_g)_p {X^V}_p) \\
&= \mathrm{Ad}_{g^{-1}}\omega_p({X^H}_p) + \omega_{pg}((\mathrm{d}R_g)_p {X^V}_p),
\end{aligned}$$

writing ${X^V}_p = {A^\#}_p$ for $A = \omega_p({X^V}_p) \in \mathfrak{g}$, and using lemma 3.9,

$$\begin{aligned}
&= \mathrm{Ad}_{g^{-1}}\omega_p({X^H}_p) + \omega_{pg}({(\mathrm{Ad}_{g^{-1}}A)^\#}_{pg}) \\
&= \mathrm{Ad}_{g^{-1}}\omega_p({X^H}_p) + \mathrm{Ad}_{g^{-1}}A \\
&= \mathrm{Ad}_{g^{-1}}\omega_p({X^H}_p) + \mathrm{Ad}_{g^{-1}}\omega_p({X^V}_p) \\
&= \mathrm{Ad}_{g^{-1}}\omega_p(X_p),
\end{aligned}$$

and we can see that ${R_g}^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ if the H_pP define a connection.

Definition 3.17. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle. Then some* $\mathfrak{g}$*-valued* 1*-form* ω *over* P *is a* ***connection 1-form*** *if it satisfies the following requirements:*

(i) It projects onto $V_pP \cong \mathfrak{g}$*:* $\omega \circ \# = \mathrm{id}$*;*

(ii) It is G*-invariant:* ${R_g}^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$.

*Then the **horizontal subspace** at $p \in P$ is*

$$H_p P := \ker \omega_p. \tag{3.31}$$

Clearly, for any connection, the associated 1-form ω as defined before satisfies these conditions. However, we want to show that the two are actually equivalent, so we want to define a connection from the connection 1-form.

Lemma 3.18. *The horizontal subspaces defined by a connection 1-form induce a connection on P.*

Proof. It is already clear from the splitting lemma that the induced HP assigns a horizontal subspace to each p. It remains to be shown that

$$H_{pg} P = (\mathrm{d}R_g)_p (H_p P).$$

We have seen that, for $X_p \in H_p P$,

$$\omega_{pg}((\mathrm{d}R_g)_p X_p) = \mathrm{Ad}_{g^{-1}} \omega_p(X_p).$$

Therefore, $\omega_p(X_p) = 0$ if and only if $\omega_{pg}((\mathrm{d}R_g)_p X_p) = 0$. Thus, $X_p \in H_p P = \ker \omega_p$ if and only if $(\mathrm{d}R_g)_p X_p \in H_{pg} P = \ker \omega_{pg}$, and therefore

$$H_{pg} P = (\mathrm{d}R_g)_p (H_p P),$$

which finally proves that HP is a connection. □

We may now talk about connection 1-forms instead of connections, which will prove incredibly useful to us in defining gauge theories in chapter 4.
Before moving on from horizontal spaces, we want to define the notion of horizontal lift of curves.

Definition 3.19. *Observe some principal bundle $G \longrightarrow P \xrightarrow{\pi} M$ with some connection HP, and let $\gamma : \mathbb{R} \supseteq I \to M$ be a smooth curve over M. A **horizontal lift** of γ to P is some curve $\tilde{\gamma} : I \to P$ such that*

1. $\pi \circ \tilde{\gamma} = \gamma$;
2. $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)} P$.

Lemma 3.20. *Let $\gamma :]-\varepsilon, \varepsilon[\to M$ be any curve over M. Then, for every $p \in \pi^{-1}(\gamma(0))$, there exists a unique horizontal lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) = p$.*

The proof is found in [7], lemma 2.55.

Corollary 3.21. *If $\tilde{\gamma}$ is one horizontal lift of some curve γ, and $\tilde{\gamma}'$ is another, there exists some $g \in G$ such that $\tilde{\gamma}'(t) = \tilde{\gamma}(t) g$.*

The horizontal lift will prove useful later on, but for now, we can move on from it.

Definition 3.22. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle with some connection 1-form* ω*. The* ***curvature*** *of the connection* ω *is now defined to be the* $\mathfrak{g}$*-valued 2-form*

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]. \tag{3.32}$$

On an Abelian gauge group, $[\omega, \omega] = 0$, and then

$$\Omega = \mathrm{d}\omega. \tag{3.33}$$

Lemma 3.23. *Let* Ω *be any curvature of some connection* ω *on a principal bundle* $G \longrightarrow P \xrightarrow{\pi} M$ *. Then for any* $g \in G$*, we have*

$$R_g{}^*\Omega = \mathrm{Ad}_{g^{-1}} \circ \Omega. \tag{3.34}$$

Proof. This proof is relatively simple, and only requires some writing out.

$$\begin{aligned} R_g{}^*\Omega &= R_g{}^*\mathrm{d}\omega + \frac{1}{2}R_g{}^*[\omega, \omega] \\ &= \mathrm{d}\big(R_g{}^*\omega\big) + \frac{1}{2}[R_g{}^*\omega, R_g{}^*\omega] \\ &= \mathrm{d}\Big(\mathrm{Ad}_{g^{-1}} \circ \omega\Big) + \frac{1}{2}[\mathrm{Ad}_{g^{-1}} \circ \omega, \mathrm{Ad}_{g^{-1}} \circ \omega] \\ &= \mathrm{Ad}_{g^{-1}} \circ \Omega. \end{aligned}$$

We used here that $f^*[\omega, \omega] = [f^*\omega, f^*\omega]$ for any function f that maps into P, which is a rather simple result to check. □

Now, we formulate a very special equation for the connection.

Definition 3.24. *Let* $\eta \in \Omega^k(P) \otimes V$ *be some* V*-valued* k*-form on* P*. We then define the* ***covariant derivative*** *as the map* $D : \Omega^k(P) \otimes V \to \Omega^{k+1}(P) \otimes V$ *such that*

$$D\eta(X_1, \ldots, X_{k+1}) := \mathrm{d}\eta(X_1{}^H, \ldots, X_{k+1}{}^H), \tag{3.35}$$

where $X^H = \pi_H(X) = X - \omega(X)^\#$ *is the horizontal component of* X*.*

Theorem 3.25 (The Bianchi Identity)**.** *For* Ω *a curvature form,*

$$D\Omega = 0. \tag{3.36}$$

Proof. Firstly, we write out the differential of Ω

$$\mathrm{d}\Omega = \frac{1}{2}\mathrm{d}[\omega, \omega].$$

From differential geometry, we know furthermore that

$$\begin{aligned}2\mathrm{d}\Omega(X,Y,Z) = L_X([\omega,\omega](Y,Z)) + L_Y([\omega,\omega](Z,X)) + L_Z([\omega,\omega](X,Y)) \\ - [\omega,\omega]([X,Y],Z) - [\omega,\omega]([Y,Z],X) - [\omega,\omega]([Z,X],Y).\end{aligned}$$

We already know that $\omega = 0$ on HP. Therefore, $[\omega,\omega] = 0$ on $TP \times HP$, and we conclude that, indeed,

$$\mathrm{d}\Omega(X^H, Y^H, Z^H) = 0,$$

and therefore,

$$D\Omega = 0.$$

□

3.3 Vector bundles

Another special set of fibre bundles are the vector bundles. These will return later in chapter 5, and are also very important in general relativity.

Definition 3.26. *A* ***vector bundle*** *is a fibre bundle* $F \longrightarrow E \xrightarrow{\pi} M$ *such that the fibre* F *is a vector space, and the fibres* $\pi^{-1}(x)$ *are linearly isomorphic to* F *for all* $x \in M$*. We say that it is a real (or complex) vector bundle of rank* k *if* F *is a real (or complex) vector space of dimension* k*.*

Of course, examples of vector bundles include the tangent bundle TM and the "scalar bundle", which is also a principal bundle, $\mathbb{R} \longrightarrow E \xrightarrow{\pi} M$.
As to not differentiate between the real and complex space, we define the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. When it does not matter whether we choose the real or complex numbers, we use $\mathbb{K}$, instead. Note that we are free to identify $F = \mathbb{K}^k$, and that we can always find the structure group $G = \mathrm{Aut}(\mathbb{K}^k)$, being the group of automorphisms over $\mathbb{K}^k$. Now, we explicitly find a way to relate principal bundles and vector bundles. We first observe a way of generating a principal bundle from any vector bundle (the frame bundle), then show a way of doing the opposite (the associated bundle).

Definition 3.27. *Let* $\mathbb{K}^k \longrightarrow E \xrightarrow{\pi} M$ *be some vector bundle. A* ***frame*** *at* $x \in M$ *is some linear isomorphism* $\varphi : \mathbb{K}^k \to \pi^{-1}(x)$ *(equivalently, it is some choice of basis). The set of frames at* x *is denoted* $F_x(E)$*. The* ***frame bundle*** *is then* $F(E) = \cup_{x\in M} F_x(E)$*, together with the obvious projection* $\pi' : F(E) \to M$*.*
By identifying $\pi^{-1}(x)$ *with* $\mathbb{K}^k$*, we note that* $F_x(E)$ *is exactly the set of invertible matrices* $GL(\mathbb{K}^k)$*, which shows that the frame bundle defines the principal bundle*

$$GL(\mathbb{K}^k) \longrightarrow F(E) \xrightarrow{\pi'} M. \tag{3.37}$$

Definition 3.28. *Let G be some group. A* ***representation*** *of G on some vector space V is a group homomorphism*

$$\rho: \; G \to GL(V). \tag{3.38}$$

We say that the representation is ***irreducible*** *if the only linear subspaces $W \subseteq V$ such that $\rho(g)W \subseteq W$ for all $g \in G$ are $W = \{0\}$ and $W = V$.*
In the case that G is an algebra, a representation is instead some algebra homomorphism

$$\rho: \; G \to \mathrm{End}(V). \tag{3.39}$$

Definition 3.29. *Assume we have some principal bundle $G \longrightarrow P \xrightarrow{\pi} M$ and some representation $G \to GL(\mathbb{K}^k)$. There is then some induced action of G on the manifold $P \times \mathbb{K}^k$, through*

$$(p, v)g := (pg, g^{-1}v). \tag{3.40}$$

We now define the ***associated vector bundle*** *as the set $(P \times \mathbb{K}^k)/G$, with induced projection $\pi' : \; (P \times \mathbb{K}^k)/G \to M$, with $\pi'(p, v) = \pi(p)$.*
Notice now that, for $x \in M$ and any one e_x such that $\pi(e_x) = x$,

$$\pi'^{-1}(x) = \{(e_x g, v) \mid g \in G, \, v \in \mathbb{K}^k\} = \{(e_x, g^{-1}v) \mid g \in G, \, v \in \mathbb{K}^k\} \cong \mathbb{K}^k. \tag{3.41}$$

This gives us the vector bundle

$$\mathbb{K}^k \longrightarrow (P \times \mathbb{K}^k)/G \xrightarrow{\pi'} M. \tag{3.42}$$

As you can see, there is now a one-to-one correspondence between vector bundles of rank k and principal bundles over the structure group $GL(\mathbb{K}^k)$.

3.3.1 The frame bundle of the tangent bundle and its reductions

If we consider some n-dimensional manifold M, the tangent bundle of course induces a n-dimensional frame bundle

$$GL(\mathbb{R}^n) \longrightarrow F(TM) \xrightarrow{\pi} M. \tag{3.43}$$

If no vector bundle is specified, the frame bundle of some manifold is the frame bundle of its tangent bundle.
Assume now that our manifold is a pseudo-Riemannian manifold with metric g of signature $(n-p, p)$. A frame $\varphi : \; \mathbb{R}^n \to T_xM$ is called orthonormal if

$$g_{x\mu\nu}\varphi(e_i)^\mu \varphi(e_j)^\nu = \begin{cases} -1 & \text{if } i = j \leq p \\ 1 & \text{if } i = j > p, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.44}$$

Similarly to before, we denote the set of orthonormal frames at x by $O_x(TM)$, and $O(TM) = \cup_{x \in M} O_x(TM)$ is the orthonormal frame bundle with the obvious projection $\pi : \; O(TM) \to$

M. Clearly, there is now an inclusion $\iota: \ O(TM) \to F(TM)$.
Now, if we once again identify T_xM with $\mathbb{R}^n$, we discover that these orthogonal frames precisely coincide with matrices $A \in GL(\mathbb{R}^n)$ such that

$$AGA^t = G, \tag{3.45}$$

with G the diagonal matrix that represents g under the basis as in Sylvester's law of inertia, i.e.,

$$G = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{3.46}$$

Note that this is a group, which we call the orthogonal group, and denote by $O(n-p,p)$. This induces the principal bundle

$$O(n-p,p) \longrightarrow O(TM) \xrightarrow{\ \pi\ } M. \tag{3.47}$$

Now, the inclusion ι is, trivially, a $O(n-p,p)$-reduction, and shows that the frame bundle of any pseudo-Riemannian manifold can be reduced to the orthogonal frame bundle.
Actually, in some cases, the manifold satisfies certain properties that allow the frame bundle to be reduced to other principal bundles. Some more examples of reductions can be found in chapter 5.

Part II

Gauge theory and its physical applications

A note on physical nomenclature

In physical contexts, one may often find different names for some mathematical notions that we have defined up to this point. In this short chapter, we will provide a "dictionary", as it were, to explain some physical terms used in this thesis in a mathematical sense.

Fermion - The particle associated to a spinor.

Gauge - A section. "Fixing a gauge": choosing a section.

Gauge boson - The particle associated to a gauge field.

Gauge field - See "Gauge potential".

Gauge potential - Some connection form ω pulled back by a section s: $s^*\omega$.

Gauge field strength - Some curvature form Ω pulled back by a section s: $s^*\Omega$.

Gauge group - The structure group of a principal bundle.

Gauge invariant - Invariant under change of section.

Gauge theory - The study of gauge fields and gauge field strengths on a certain principal bundle.

Gauge transformation - Choosing a different section.

Spinor - A section of the spinor bundle.

Structure group - See "Gauge group".

Vielbein - A local choice of orthonormal basis of the tangent space of some manifold.

4 Gauge theories

In the previous chapter, we looked at principal bundles, and defined the notions of connections and curvatures on the total space. In this chapter, we apply these notions and define what we mean by a gauge theory, such that we can later apply it to represent some physical theories. To this end, we first use the pull-back to return the notions from the previous chapter to the base space of some principle bundle. As an example, we look at the principle bundle with structure group $U(1)$, from which we can derive some of Maxwell's equations. We furthermore define connections on vector bundles, and the most important example of one: the Levi-Civita connection.

4.1 Return to the base space

There is one large issue with the notions up to now: they are defined on the total space. However, in "real life", which is generally what physics focuses on, we live in the base space, not the total space. Therefore, we try to pull these notions back onto the base space, which will make more sense to talk about.

Definition 4.1. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle with a connection* 1*-form* ω*, and let* U *be some open subset of* M*. Given a local section* $s \in \Gamma(U, P)$*, the* ***(local) gauge potential*** *or* ***(local) gauge field*** *on* U *is defined*

$$\mathcal{A} := s^*\omega, \tag{4.1}$$

which is a $\mathfrak{g}$*-valued* 1*-form on* U*.*
For Ω *the curvature of* ω*, the* ***(local) gauge field strength*** *is the* $\mathfrak{g}$*-valued* 2*-form*

$$\mathcal{F} := s^*\Omega = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]. \tag{4.2}$$

Clearly, the explicit definition of the gauge potential is dependent on your choice of section, which is what physicists call "fixing a gauge". A **gauge theory** is the study of these two forms on some principal bundle.

4.1.1 The compatibility condition

In general, you cannot define a gauge potential on the entire base space, since non-trivial bundles do not necessarily admit global sections. Therefore, one will have several local gauge potentials on the manifold, that should be induced from the same connection 1-form ω. We would now like to find a way of comparing some collection of given gauge potentials in order to judge whether they are "compatible", i.e., follow from the same connection form.

Theorem 4.2 (The compatibility condition)**.** *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle, and* $\{(U_i, \varphi_i)\}$ *some* G*-atlas of* M *for* $i \in I$*.*
Define for each $i \in I$ *the* ***canonical section*** s_i *such that*

$$s_i(x) = {\varphi_i}^{-1}(x, e) \qquad \text{for all } x \in U_i. \tag{4.3}$$

Now, let $\mathcal{A}_i$ be a $\mathfrak{g}$-valued 1-form on U_i for all i. Assume that $U_i \cap U_j \neq \emptyset$ for some i, j. If, for all such i, j, we have that

$$\mathcal{A}_j = \mathrm{Ad}_{t_{ji}} \circ \mathcal{A}_i + {t_{ij}}^* \Theta, \tag{4.4}$$

where Θ is the ***Cartan 1-form*** *on G, defined by*

$$\Theta_g : \; T_g G \to T_e G = \mathfrak{g}, \;\; X_g \mapsto (\mathrm{d}L_{g^{-1}})_g X_g, \tag{4.5}$$

then there is a unique connection 1-form ω on P such that

$$\mathcal{A}_i = {s_i}^* \omega. \tag{4.6}$$

In particular, these gauge potentials are then compatible with one another.

The proof of this theorem can be found in section A.1.

Corollary 4.3. *If G is a matrix group, the compatibility condition reduces to*

$$\mathcal{A}_j = t_{ji} \mathcal{A}_i t_{ij} + t_{ji} \mathrm{d} t_{ij}. \tag{4.7}$$

This compatibility condition now gives us a way of checking whether two gauge potentials agree. Alternatively, it allows us to perform **gauge transformations**: creating a new gauge potential from an old one by switching gauge (or section). You do not even need to explicitly choose two sections, since the compatibility condition allows us to transform the gauge potential using only the transition function, which can be any function $t : \; M \to G$.

4.1.2 The field strength

We now return to the gauge field strength. We first observe the effect of the compatibility condition on the field strength. Assume that we have two compatible local gauge potentials $\mathcal{A}_i$ and $\mathcal{A}_j$ with corresponding sections $s_j = s_i t_{ij}$, i.e., there exists some connection 1-form ω such that

$$\mathcal{A}_i = {s_i}^* \omega, \qquad \mathcal{A}_j = {s_j}^* \omega.$$

Then, if Ω is the curvature of ω,

$$\mathcal{F}_i = {s_i}^* \Omega, \qquad \mathcal{F}_j = {s_j}^* \Omega,$$

and we find for $X_x, Y_x \in T_x M$, with any $x \in M$,

$$\begin{aligned}
(\mathcal{F}_j)_x (X_x, Y_x) &= \Omega_{s_j(x)} ((\mathrm{d}s_j)_x X_x, (\mathrm{d}s_j)_x Y_x) \\
&= \Omega_{R_{t_{ij}}(s_i(x))} ((\mathrm{d}R_{t_{ij}})_{s_i(x)} (\mathrm{d}s_i)_x X_x, (\mathrm{d}R_{t_{ij}})_{s_i(x)} (\mathrm{d}s_i)_x Y_x) \\
&= {R_{t_{ij}}}^* \Omega_{s_i(x)} ((\mathrm{d}s_i)_x X_x, (\mathrm{d}s_i)_x Y_x) \\
&= \mathrm{Ad}_{t_{ji}} \circ \Omega_{s_i(x)} ((\mathrm{d}s_i)_x X_x, (\mathrm{d}s_i)_x Y_x) \\
&= \mathrm{Ad}_{t_{ji}} \circ (\mathcal{F}_i)_x (X_x, Y_x).
\end{aligned}$$

Corollary 4.4 (Gauge transformation of the field strength)**.** *Take sections $s_j = s_i t_{ij}$, and corresponding field strengths $\mathcal{F}_j$ and $\mathcal{F}_i$. If the gauge fields that induce the field strengths are compatible, we have*

$$\mathcal{F}_j = \mathrm{Ad}_{t_{ji}} \circ \mathcal{F}_i. \tag{4.8}$$

If G is a matrix group,

$$\mathcal{F}_j = t_{ji}\mathcal{F}_i t_{ij}. \tag{4.9}$$

In a gauge theory, we could only measure objects that are invariant under gauge transformations (gauge invariant). After all, the choice of section is arbitrary, and shouldn't influence the physics. Mathematically, however, we would be studying consequences of the connection form, which is also gauge invariant, but it does not "live" on the base space, so should not be measurable.
Within gauge theories, the group of gauge transformations exactly corresponds to the structure group, which is also why it is called the gauge group, or symmetry group.
We now return to theorem 3.25, the Bianchi identity, and use it to make a similar identity for $\mathcal{F}$.

Lemma 4.5 (The Bianchi identity on the base space)**.** *Let ω be any connection form of P, with corresponding gauge field $\mathcal{A}$ and gauge field strength $\mathcal{F}$. Then we find*

$$\mathrm{d}_{\mathcal{A}}\mathcal{F} := \mathrm{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0. \tag{4.10}$$

Proof. We start with using that, for Ω the curvature form,

$$\mathrm{d}\Omega = \frac{1}{2}\mathrm{d}[\omega, \omega],$$

and therefore

$$\begin{aligned}\mathrm{d}\mathcal{F} &= \mathrm{d} \circ s^*\Omega \\ &= \frac{1}{2}\mathrm{d} \circ s^*[\omega, \omega] \\ &= \frac{1}{2}\mathrm{d}[\mathcal{A}, \mathcal{A}],\end{aligned}$$

since $\mathcal{A}$ is a 1-form,

$$\begin{aligned}&= \frac{1}{2}[\mathrm{d}\mathcal{A}, \mathcal{A}] - \frac{1}{2}[\mathcal{A}, \mathrm{d}\mathcal{A}] \\ &= [\mathcal{F}, \mathcal{A}] = -[\mathcal{A}, \mathcal{F}].\end{aligned}$$

□

On an Abelian structure group, we would find $[\mathcal{F}, \mathcal{A}] = 0$, and thus the identity would become $\mathrm{d}\mathcal{F} = 0$.

4.2 Example: $U(1)$ gauge theory

The earliest and most famous example of a gauge theory is electromagnetism. In electromagnetism, there is a freedom in choosing the potentials of the electric and magnetic field, respectively, which we will later find to be a gauge transformation.
The gauge group that describes electromagnetism is $U(1)$, which we show by working through the results of using $U(1)$ as a gauge group.

4.2.1 $U(1)$ gauge theory

We first construct a principle bundle. Of course, we have to choose $U(1)$ as the gauge group, and we choose M to be some 4-dimensional Lorentzian manifold, representing spacetime. Following convention, Greek indices represent any index in $\{0,1,2,3\}$ and Latin indices represent any index in $\{1,2,3\}$. We find now some principal bundle

$$U(1) \longrightarrow P \xrightarrow{\pi} M, \tag{4.11}$$

and choosing any section, or gauge, $s: M \to P$, we find the gauge potential $\mathcal{A}$ and field strength $\mathcal{F}$. The Lie algebra here is given by the imaginary numbers $\mathfrak{u}(1) = i\mathbb{R}$, and therefore, we may define some (real) forms A and F such that $\mathcal{A} = iA$ and $\mathcal{F} = iF$. In the notation introduced in section 2.1.2, these forms are A_μ and $F_{\mu\nu}$.
Now, $U(1)$ is actually Abelian, so we find that

$$\mathcal{F} = \mathrm{d}\mathcal{A} \;\Leftrightarrow\; F = \mathrm{d}A \tag{4.12a}$$

therefore,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.12b}$$

and, furthermore,

$$\mathrm{d}\mathcal{F} = 0 \;\Leftrightarrow\; \mathrm{d}F = 0, \tag{4.13a}$$

which, we find, implies

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0. \tag{4.13b}$$

Equivalently, we could have written $\mathrm{d}F = 0$ like

$$\delta(\star F) = -\star(\mathrm{d}F) = 0. \tag{4.13c}$$

Now, $F_{\mu\nu}$ is a 2-form, and therefore antisymmetric. We may define its components

$$F_{i0} := E_i/c, \qquad F_{ij} := \varepsilon_{ijk'}\delta^{k'k}B_k, \tag{4.14}$$

for any real E_i, B_i and c some non-zero real constant, and the rest of $F_{\mu\nu}$ is induced uniquely from these expressions. This $F_{\mu\nu}$ is called the **Faraday tensor**, and is sometimes written

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}. \tag{4.15}$$

Let us put this definition into equation (4.13b), as this Bianchi identity seems to imply some "law" on the Faraday tensor. The non-trivial (i.e., $\mu \neq \nu \neq \sigma$) equations are

$$\partial_0 B_3 + \frac{1}{c}\partial_1 E_2 - \frac{1}{c}\partial_2 E_1 = 0, \qquad -\partial_0 B_2 + \frac{1}{c}\partial_1 E_3 - \frac{1}{c}\partial_3 E_1 = 0,$$
$$\partial_0 B_1 + \frac{1}{c}\partial_2 E_3 - \frac{1}{c}\partial_3 E_2 = 0, \qquad \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0,$$

or, equivalently, if we identify $x_0 = ct$ and write E_i, B_i as vectors $\mathbf{E}, \mathbf{B}$,

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{4.16}$$

Anyone familiar with electromagnetism should immediately recognise these two equations as the Maxwell-Faraday law and Gauss's law for a magnetic field. This strengthens our argument that electromagnetism is indeed a $U(1)$ gauge theory.
Now, returning to equation (4.12b), we see that $E_i/c = \partial_i A_0 - \partial_0 A_i$. If we now guess that $A_0 = -\frac{1}{c}\phi$, with ϕ the electric scalar potential, and A_i is the magnetic vector potential, we find the equation

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.17}$$

as one would expect from electromagnetism. We test this guess by also formulating an expression for $\mathbf{B}$ from A_μ,

$$B_i = \frac{1}{2}\varepsilon_i^{\ jk} F_{jk} = \frac{1}{2}\varepsilon_{ij'k'}\delta^{j'j}\delta^{k'k}(\partial_j A_k - \partial_k A_j) = \varepsilon_{ij'k'}\delta^{j'j}\delta^{k'k}\partial_j A_k = (\nabla \times \mathbf{A})_i, \tag{4.18}$$

which confirms that this choice of A_μ does indeed generate the Faraday tensor. One sometimes writes

$$A_\mu = \left(-\frac{1}{c}\phi, \mathbf{A}\right). \tag{4.19}$$

The choice of $U(1)$ as the gauge group therefore was a very good choice, and did produce electromagnetism as a physical theory.
Note that through the necessity of choosing a basis implies that the definition of the Faraday tensor is a local phenomenon, and therefore, so is the "separation" of the electric and magnetic field. Only when the tangent bundle of the manifold is trivial, one can choose a global basis, and therefore separate the two fields on the whole manifold.

4.2.2 $U(1)$ versus $\mathbb{R}$

Note that in the derivation of the equations in the last section, we never used the exact gauge group, only the related Lie algebra. In truth, however, we would have gotten the same results for any other Lie algebra that is isomorphic to $\mathbb{R}$. In other words, we could have chose any 1-dimensional Lie group. Thankfully, there are only two 1-dimensional connected Lie groups, being $U(1)$, which was the Lie group we chose, and $\mathbb{R}$ itself.
The important distinction between $U(1)$ and $\mathbb{R}$ happens to be that $U(1)$ is compact, being necessary for the Yang-Mills Lagrangian to work, which we introduce in chapter 6.
Finally, in quantum electrodynamics, the gauge field gets a second role: that of a phase

difference. Here, gauge transformations induced by some function Λ correspond to a phase difference of $\exp\left(i\frac{q}{\hbar c}\Lambda\right)$, where q is the charge, which exactly corresponds to some element of $U(1)$. This suggests that the gauge transformations correspond to a structure group $U(1)$.

4.2.3 Gauge transformations

The rest of this section will be dedicated to explicitly providing an expression for the gauge transformation in a $U(1)$ gauge theory.
Take any open subset V of M such that $\pi^{-1}(V) \cong V \times U(1)$, and assume we have some section s defined on V. We can now write this section as some smooth map

$$s: \ V \to V \times U(1), \ x \mapsto (x, S(x)), \tag{4.20}$$

with $S: \ V \to U(1)$. If we define any other section s' on V with corresponding $S': \ V \to U(1)$, the transition function is then given by

$$t: \ V \to U(1), \ x \mapsto S(x)^{-1}S'(x), \tag{4.21}$$

although the precise definition won't matter to us a lot — we just need the sets it maps between. In fact, we can rewrite it for some $\Lambda: \ V \to \mathbb{R}$

$$t(x) = e^{i\Lambda(x)}. \tag{4.22}$$

We now attempt to find $\mathrm{Ad}_{t^{-1}(x)}$. Trivially, we have that

$$\Psi_{t^{-1}}: \ U(1) \to U(1), \ g \mapsto t^{-1}gt = g, \tag{4.23}$$

since $U(1)$ is Abelian. Therefore, we immediately find that $\mathrm{Ad}_{t^{-1}}$ is the identity mapping.
We already know that $T_gU(1) = g\mathfrak{u}(1) = gi\mathbb{R}$, and therefore, the Cartan one-form is in this case defined like

$$\Theta_g: \ gi\mathbb{R} \to i\mathbb{R}, \ z \mapsto g^{-1}z, \tag{4.24}$$

and therefore

$$(t^*\Theta)_x X_x = t(x)^{-1}(\mathrm{d}t)_x X_x = i(\mathrm{d}\Lambda)_x X_x. \tag{4.25}$$

We thus find, through theorem 4.2, that

$$\mathcal{A}' = \mathcal{A} + i\mathrm{d}\Lambda, \tag{4.26}$$

or,

$$A' = A + \mathrm{d}\Lambda, \tag{4.27}$$

in components:

$$A'_\mu = A_\mu + \partial_\mu\Lambda. \tag{4.28}$$

Furthermore, we trivially find that

$$\mathcal{F}' = \mathcal{F}, \tag{4.29}$$

and thus

$$F' = F. \tag{4.30}$$

These are precisely the gauge transformations we normally find in electromagnetism:

$$\frac{1}{c}\phi \to \frac{1}{c}\phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\Lambda, \tag{4.31a}$$

$$\mathbf{E} \to \mathbf{E}, \qquad \mathbf{B} \to \mathbf{B}. \tag{4.31b}$$

4.3 Digression: Connections on vector bundles

Similarly to connections on principal bundles, it is also possible to define connections on vector bundles. In fact, there exists a way of turning these connections into one another.

Definition 4.6. *Let* $V \longrightarrow E \xrightarrow{\pi} M$ *be any vector bundle. A* ***connection*** *is the some bilinear map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \tag{4.32}$$

where we denote

$$\nabla_X s := \nabla(X, s), \tag{4.33}$$

that satisfies

$$\nabla_{fX} s = f \nabla_X s, \qquad \nabla_X(fs) = f\nabla_X s + L_X(f)s \tag{4.34}$$

for all smooth functions $f : M \to \mathbb{R}$*, and where* L *denotes the Lie derivative.*

With this connection, we can find its action on a section along some path $\gamma : \mathbb{R} \supseteq I \to M$ as inspired by [7],

$$\frac{\nabla(s \circ \gamma)}{\mathrm{d}t}(t) := \nabla_{\dot\gamma} s(\gamma(t)). \tag{4.35}$$

Actually, this definition naturally expands to some operator over any path $u : I \to E$. Locally, we know we can write u like

$$u(t) = (\gamma(t), v(t)),$$

where $\gamma = \pi \circ u$ and $v : I \to V$. We then define

$$\frac{\nabla u}{\mathrm{d}t}(t) = \left(\gamma(t), \frac{\mathrm{d}v}{\mathrm{d}t}(t)\right) + v^i(t) \nabla_{\dot\gamma(t)} e_i(\gamma(t)), \tag{4.36}$$

where e_i is a local basis for the vector bundle and the linear operations (additions, scalar multiplication) are defined as on the vector component of the paths.

4.3.1 The covariant derivative of associated bundles

Assume we have some principal bundle $G \longrightarrow P \xrightarrow{\pi} M$ with some connection and a representation $\rho : G \to GL(V)$. Consider the associated bundle $V \longrightarrow (P \times V)/G \xrightarrow{\pi'} M$. Note that any local section $s \in \Gamma((P \times V)/G)$ can be written like

$$s(x) = (s'(x), \varphi(x)), \tag{4.37}$$

where $s' \in \Gamma(P)$ and $\varphi : D(s) \to V$ is smooth ($D(s)$ denotes the domain of s). Now, for any smooth curve $\gamma : I \to M$ with horizontal lift $\tilde\gamma : I \to P$, we may assume, without loss of generality, that

$$s(\gamma(t)) = (\tilde\gamma(t), \varphi(\gamma(t))), \tag{4.38}$$

since we recall that $(pg, g^{-1}v) = (p, v)$. Now, we choose some $X \in \mathfrak{X}(M)$. We furthermore set a $x \in M$, and define some curve $\gamma :]-\varepsilon, \varepsilon[$, for $\varepsilon > 0$, such that $\gamma(0) = x$ and $\dot\gamma(0) = X_x$.

The **covariant derivative** of s along X with respect to the curve γ and horizontal lift $\tilde{\gamma}$ is given by

$$(\nabla_X)^{\tilde{\gamma}}_{\gamma} s(x) = \left(\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\gamma(t))\bigg|_{t=0}\right). \tag{4.39}$$

We would now like to show that the choice of γ and $\tilde{\gamma}$ does not matter. The first is rather easy to see, since the covariant derivative only depends on $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$. Furthermore, if we choose some other horizontal lift $\tilde{\gamma}'(t)$, there must exists some $g \in G$ such that $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$, and then $s(\gamma(t)) = (\tilde{\gamma}'(t), g^{-1}\varphi(\gamma(t)))$, such that

$$(\nabla_X)^{\tilde{\gamma}'}_{\gamma} s(x) = \left(\tilde{\gamma}(0)g, \frac{\mathrm{d}}{\mathrm{d}t} g^{-1}\varphi(\gamma(t))\bigg|_{t=0}\right) = (\nabla_X)^{\tilde{\gamma}}_{\gamma} s(x). \tag{4.40}$$

Therefore, we no longer require the curve or its horizontal lift, and from now on will denote the covariant derivative of s along X at the point x

$$\nabla_X s(x) := (\nabla_X)^{\tilde{\gamma}}_{\gamma} s(x). \tag{4.41}$$

Finally, note that $\pi \circ \nabla_X s(x) = x$, and therefore, $\nabla_X s : U \to (P \times V)/G$ defines a section. Thus, the covariant derivative is a function $\nabla_X : \Gamma((P \times V)/G)) \to \Gamma((P \times V)/G))$. Alternatively, it is a function $\nabla : \Gamma((P \times V)/G)) \to T^*M \otimes \Gamma((P \times V)/G))$

Lemma 4.7. *Assume we have some connection form ω on the principal bundle. Then, for any $s \in \Gamma((P \times V)/G)$, written as in equation (4.37), the covariant derivative is given by*

$$\nabla_X s(x) = (s'(x), (\mathrm{d}\varphi)_x X_x + (s'^*\omega)_x X_x \varphi(x)). \tag{4.42}$$

Proof. Choose some curve $\gamma :]-\varepsilon, \varepsilon[\to M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_x$ that does not intersect itself, and some horizontal lift $\tilde{\gamma}$. Define furthermore $g : I \to G$ such that $\tilde{\gamma}(t) = s'(\gamma(t))g(t)$. Then,

$$s(\gamma(t)) = (\tilde{\gamma}(t), g^{-1}(t)\varphi(\gamma(t))),$$

and, therefore,

$$\begin{aligned}
\nabla_X s(x) &= \left(\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t} g^{-1}(t)\varphi(\gamma(t))\bigg|_{t=0}\right) \\
&= \left(\tilde{\gamma}(0), g^{-1}(0)(\mathrm{d}\varphi)_x X_x + \dot{g^{-1}}(0)\varphi(x)\right),
\end{aligned}$$

and

$$\begin{aligned}
\dot{g^{-1}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t'}\left(g^{-1}(t')g(t')g^{-1}(t')\right)\bigg|_{t'=t} \\
&= 2\dot{g^{-1}}(t) + g^{-1}(t)\dot{g}(t)g^{-1}(t), \\
\dot{g^{-1}}(t) &= -g^{-1}(t)\dot{g}(t)g^{-1}(t),
\end{aligned}$$

and hence,

$$\begin{aligned}
\nabla_X s(x) &= \left(\tilde{\gamma}(0), g^{-1}(0)(\mathrm{d}\varphi)_x X_x - g^{-1}(0)\dot{g}(0)g^{-1}(0)\varphi(x)\right) \\
&= \left(s'(x), (\mathrm{d}\varphi)_x X_x - \dot{g}(0)g^{-1}(0)\varphi(x)\right).
\end{aligned}$$

All we have to do now, is find what exactly $\dot{g}(0)$ is. First, we generate a new principal bundle $G \longrightarrow \pi^{-1}(\gamma(I)) \xrightarrow{\gamma^{-1}\circ\pi} I$ with sections $\tilde{\gamma}$ and $s' \circ \gamma$. Then lemma A.5 tells us that, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
(\mathrm{d}\tilde{\gamma})_0\alpha &= (\mathrm{d}R_{g(0)})_{s'(x)}(\mathrm{d}s')_x(\mathrm{d}\gamma)_0\alpha + ((g(0)^*\Theta)_0\alpha)^{\#}_{\tilde{\gamma}(0)} \\
&= (\mathrm{d}R_{g(0)})_{s'(x)}(\mathrm{d}s')_x(\mathrm{d}\gamma)_0\alpha + (g^{-1}(0)(\mathrm{d}g)_0\alpha)^{\#}_{\tilde{\gamma}(0)} \\
\dot{\tilde{\gamma}}(0) &= (\mathrm{d}R_{g(0)})_{s'(x)}(\mathrm{d}s')_x X_x + (g^{-1}(0)\dot{g}(0))^{\#}_{\tilde{\gamma}(0)}.
\end{aligned}$$

Now, $\dot{\tilde{\gamma}}(0)$ is horizontal, and therefore,

$$\begin{aligned}
0 &= \omega((\mathrm{d}R_{g(0)})_{s'(x)}(\mathrm{d}s')_x X_x) + g^{-1}(0)\dot{g}(0) \\
&= \mathrm{Ad}_{g^{-1}(0)}(s'^*\omega)_x X_x + g^{-1}(0)\dot{g}(0) \\
&= g^{-1}(0)(s'^*\omega)_x X_x g(0) + g^{-1}(0)\dot{g}(0).
\end{aligned}$$

We conclude that

$$\dot{g}(0) = -(s'^*\omega)_x X_x g(0).$$

Putting this into our covariant derivative lets us conclude that

$$\nabla_X s(x) = \big(s'(x), (\mathrm{d}\varphi)_x X_x + (s'^*\omega)_x X_x \varphi(x)\big).$$

□

Lemma 4.8. *The covariant derivative is a connection on the associated bundle.*

Proof. Looking at the previous lemma, it is rather obvious that ∇ is bilinear, where summation of two sections is realised through summation of the V-components under the same section $s' \in \Gamma(P)$. As for the other properties,

$$\begin{aligned}
\nabla_{fX} s(x) &= (s'(x), (\mathrm{d}\varphi)_x f(x) X_x + (s'^*\omega)_x f(x) X_x \varphi(x)) = f(x)\nabla_X s(x), \\
\nabla_X (fs)(x) &= (s'(x), (\mathrm{d}f\varphi)_x X_x + (s'^*\omega)_x X_x f(x)\varphi(x)) \\
&= (s'(x), (\mathrm{d}f)_x X_x \varphi(x) + (\mathrm{d}\varphi)_x X_x f(x) + (s'^*\omega)_x X_x f(x)\varphi(x)) \\
&= (s'(x), (\mathrm{d}f)_x X_x \varphi(x)) + f(x)\nabla_X s(x) \\
&= (\mathrm{d}f)_x X_x s(x) + f(x)\nabla_X s(x) \\
&= L_X f(x) s(x) + f(x)\nabla_X s(x).
\end{aligned}$$

□

We therefore see that, starting with some connection form on a principal bundle, we can define a related connection on the vector bundle.

4.3.2 Inducing a connection form on the frame bundle

Now, we do the opposite of what we did previously. In this section, we show a way of developing a connection form on the frame bundle starting from a connection on some vector bundle.

Let us say we have some vector bundle $\mathbb{R}^k \longrightarrow E \xrightarrow{\pi} M$ with some connection ∇ and the frame bundle $GL(\mathbb{R}^K) \longrightarrow F(E) \xrightarrow{\pi'} M$. From here, take any path $e: \mathbb{R} \to F(E)$, which we can separate into some collection of linearly independent vectors $e(t) = (e_1(t), \dots, e_k(t))$ with $e_i: \mathbb{R} \to E$. Then,

$$\frac{\nabla e_i}{\mathrm{d}t}(t) = \omega_i^j(t) e_j(t) \tag{4.43}$$

for some $\omega_i^j(t) \in \mathbb{R}$. Note now that we can form a matrix

$$\omega(t) = (\omega_i^j(t))_{1 \leq i,j \leq k} \in \mathfrak{gl}(\mathbb{R}^k), \tag{4.44}$$

such that

$$\frac{\nabla e_i}{\mathrm{d}t}(t) = e_i(t)\omega(t). \tag{4.45}$$

Therefore, for any path e, we have found a representation of the connection using a matrix, which is in the Lie algebra of the frame bundle. We therefore define the 1-form $\omega_\nabla \in \Omega^1(F(E), \mathfrak{gl}(\mathbb{R}^k))$

$$\omega_\nabla(\dot{e}(t)) := \omega(t). \tag{4.46}$$

Lemma 4.9. *The* 1*-form* ω_∇ *induced by the connection* ∇ *is a connection* 1*-form on the frame bundle. We call it the* ***frame connection form*** *induced by* ∇*.*

Proof. To prove that this function is a connection 1-form, there are of course two properties to be proven:

(i) $\omega_\nabla \circ \# = \mathrm{id}$;

(ii) $R_g{}^*\omega_\nabla = \mathrm{Ad}_{g^{-1}} \circ \omega_\nabla$.

To prove the first property, we first want an explicit expression for $\#$. Recall that, for $A \in \mathfrak{gl}(\mathbb{R}^k)$ and $p \in F(E)$,

$$(A^\#)_p = (\mathrm{d}\sigma_p)_e A,$$

and since we have the path e^{tA} over $\mathfrak{gl}(\mathbb{R}^k)$ that A is tangent to, if we have any smooth f around p,

$$\begin{aligned}(A^\#)_p(f) &= \left.\frac{\mathrm{d}f(\sigma_p(e^{tA}))}{\mathrm{d}t}\right|_{t=0} \\ &= \left.\frac{\mathrm{d}f(pe^{tA})}{\mathrm{d}t}\right|_{t=0} \\ &= \left.\frac{\mathrm{d}(pe^{tA})}{\mathrm{d}t}\right|_{t=0}(f).\end{aligned}$$

Thus, we find that

$$(A^{\#})_p = \left.\frac{\mathrm{d}(pe^{tA})}{\mathrm{d}t}\right|_{t=0}.$$

We can therefrom derive that

$$[\omega_{\nabla}((A^{\#})_p)]_i^j p_j = \frac{\nabla p_i e^{tA}}{\mathrm{d}t}(0) = p_i A,$$

and hence,

$$\omega_{\nabla}((A^{\#})_p) = A.$$

As for the second property, notice that, for some path $p: I \to F(E)$, we have that $(\mathrm{d}R_g)_{p(0)}\frac{\mathrm{d}p}{\mathrm{d}t}(0) = \frac{\mathrm{d}pg}{\mathrm{d}t}(0)$, and thus

$$\begin{aligned}
p_i(0) g {R_g}^* \omega_{\nabla}(\dot{p}(0)) &= \frac{\nabla(p_i g)}{\mathrm{d}t}(0) \\
&= \frac{\nabla p_i}{\mathrm{d}t}(0) g \\
&= p_i(0)\omega_{\nabla}(\dot{p}(0)) g.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g {R_g}^* \omega_{\nabla}(\dot{p}(0)) &= \omega_{\nabla}(\dot{p}(0)) g, \\
{R_g}^* \omega_{\nabla}(\dot{p}(0)) &= \mathrm{Ad}_{g^{-1}} \omega_{\nabla}(\dot{p}(0)).
\end{aligned}$$

We may conclude that ω_{∇} is indeed a connection form. □

4.3.3 The Levi-Civita connection

If the vector bundle we consider is a tangent bundle over some manifold M, we get a special case for the connection ∇, being a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M). \tag{4.47}$$

For pseudo-Riemannian manifolds, we want to define some "canonical" connection over its tangent bundle, being one which acts "nicely" with respect to the metric tensor.

Definition 4.10. *Assume we have some pseudo-Riemannian manifold M with metric g and some connection ∇ over its tangent bundle TM. We say the connection is **compatible with the metric g** if*

$$L_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{4.48}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Another nice property we could ask of a connection would be that of being torsion-less.

Definition 4.11. *Let ∇ be some connection over the tangent bundle of some manifold M. We say it is **torsion-free** if, for all $X, Y \in \mathfrak{X}(M)$,*

$$\nabla_X Y - \nabla_Y X = [X, Y]. \tag{4.49}$$

Theorem 4.12 (Fundamental theorem of Riemannian geometry)**.** *On the tangent bundle, there exists a unique connection ∇ that is both compatible with the metric and torsion-free. We call this the **Levi-Civita connection**.*

Proof. Let us first observe what happens if we assume ∇ to be such a connection. Take any three vector fields X, Y, Z over M. Then the compatibility with the metric gives us that

$$\begin{aligned} L_X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\ L_Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \\ L_Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \end{aligned}$$

Therefore, in particular, using that ∇ must be torsion-free,

$$\begin{aligned} L_X(g(Y,Z)) - L_Z(g(X,Y)) &= g(Y, \nabla_X Z - \nabla_Z X) + g(Z, \nabla_X Y) - g(X, \nabla_Z Y) \\ &= g(Y, [X,Z]) + g(Z, \nabla_X Y) - g(X, \nabla_Z Y), \end{aligned}$$

and furthermore,

$$\begin{aligned} &L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y)) \\ &\qquad = g(Y,[X,Z]) + g(Z, \nabla_X Y + \nabla_Y X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &\qquad = g(Y,[X,Z]) + g(X,[Y,Z]) + g(Z,[Y,X]) + 2g(Z, \nabla_X Y). \end{aligned}$$

This equation implies the so-called **Koszul formula**

$$\begin{aligned} 2g(Z, \nabla_X Y) = g(Y,[Z,X]) &+ g(X,[Z,Y]) + g(Z,[X,Y]) \\ &+ L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y)). \end{aligned} \tag{4.50}$$

Now, back to our proof, this Koszul formula will generate our connection. Note that, since the equation is true for all Z, we can uniquely define our $\nabla_X Y$ such that the Koszul formula holds, which induces our candidate for the Levi-Civita connection ∇.
The condition under which ∇ is now defined is weaker than the separate conditions of compatibility and being torsion-free. Therefore, we need to check the conditions, explicitly.
Firstly, through simple writing out, we see that for all $X, Y, Z \in \mathfrak{X}(M)$,

$$2g(\nabla_X Y, Z) + 2g(Y, \nabla_X Z) = 2L_X(g(Y,Z)).$$

Furthermore, note that for all $X, Y, Z \in \mathfrak{X}(M)$,

$$2g(Z, \nabla_X Y) - 2g(Z, \nabla_Y X) = 2g(Z, [X,Y]),$$

and thus,

$$g(Z, \nabla_X Y - \nabla_Y X) = g(Z, [X,Y]).$$

Since this is true for all Z, we find that $\nabla_X Y - \nabla_Y X = [X, Y]$, and therefore, ∇ is torsion-free. Therefore, this connection indeed satisfies the properties we were looking for. □

The uniqueness and existence of this Levi-Civita connection on any pseudo-Riemannian manifold gives us a "natural" choice of connection whenever we are considering such manifolds. Now, if we have the frame bundle $GL(\mathbb{R}^k) \longrightarrow F(TM) \xrightarrow{\pi'} M$, we can induce, as introduced before, the frame connection form ω_∇ from the Levi-Civita connection. This specific connection form will return later.

5 Spinors

The theory as described in the previous chapter, on gauge fields, describes so-called "gauge bosons", particles which mediate the interaction related to the gauge theory. For example, photons are the gauge boson of the electromagnetism. However, anyone familiar with particle physics will know that there also exists another type of particle: the "fermions". To describe these fermion fields, we require the so-called spin group, which is a group constructed from a Clifford algebra. For a complete analysis of all these groups, one may refer to [8] (do note that they define the Clifford algebra slightly differently), or [9].
In this chapter, we start by looking at Clifford algebras and the spin group. Following the algebraic introduction, we look at spinor space by choosing a representation for the spin group. By constructing vector bundles out of these spinor spaces, the spinor bundles, we can define fermion fields on any manifold.

5.1 The Clifford algebra and spin

To start off this chapter, we first sketch the setting. We have some real n-dimensional vector space V with a quadratic form, i.e., a non-degenerate symmetric $(0, 2)$-tensor, g. In general, we may identify V with $\mathbb{R}^n$, and, using theorem 2.8, g must have some signature $(n-p, p)$. We may then define some orthonormal basis, i.e., a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that

$$g(e_i, e_j) = \begin{cases} -\delta_{ij} & \text{for } i \leq p, \\ \delta_{ij} & \text{for } i > p. \end{cases} \tag{5.1}$$

5.1.1 Clifford algebras

Definition 5.1. *The* ***tensor algebra*** $\mathfrak{T}(V)$ *over* V *is given by*

$$\mathfrak{T}(V) = \bigoplus_{k=0}^{\infty} \bigotimes_{l=1}^{k} V, \tag{5.2}$$

with multiplication operator $\otimes$.
Define now the ideal

$$\mathfrak{I}(V, g) := \langle v \otimes v - g(v, v) \mid v \in V \rangle. \tag{5.3}$$

We define the ***Clifford algebra on V with respect to g*** *to be the quotient*

$$Cl(V, g) := \mathfrak{T}(V)/\mathfrak{I}(V, g), \tag{5.4}$$

which is the (associative) algebra that consists of elements in $\mathfrak{T}(V)$*, under the equality*

$$v \otimes v = g(v, v) \tag{5.5}$$

for all $v \in V$*.*

Following convention, we actually define $vw := v \otimes w$ for $v, w \in Cl(V, q)$. Moreover, by identifying V with $\mathbb{R}^n$, and using that the signature of g is $(n-p, p)$, we define

$$Cl_{n-p,p} := Cl(V, g), \tag{5.6}$$

which gives an equivalency for all Clifford algebras over some n-dimensional real vector space with respect to a quadratic form of signature $(n-p, p)$, since a change in quadratic form is just a change in orthonormal basis induced by the form.

Lemma 5.2. *Consider some Clifford algebra $Cl_{n-p,p}$. The identity in equation (5.5) is equivalent to the equality*

$$\{v, w\} = 2g(v, w) \tag{5.7}$$

for any $v, w \in V$ and $\{\cdot, \cdot\}$ denotes the anti-commutator

$$\{v, w\} := vw + wv. \tag{5.8}$$

Proof. It is clear that this equation implies equation (5.5), by setting $w = v$.
We note that

$$g(v+w, v+w) = g(v, v) + g(v, w) + g(w, v) + g(w, w) = g(v, v) + 2g(v, w) + g(w, w).$$

Therefore,

$$\begin{aligned} 2g(v, w) &= g(v+w, v+w) - g(v, v) - g(w, w) \\ &= (v+w)(v+w) - vv - ww \\ &= wv + vw = \{v, w\}. \end{aligned}$$

Therefore, these equations are equivalent. □

Corollary 5.3. *From the past lemma, the basis of $Cl_{n-p,p}$ is given by*

$$e_{i_1} \otimes \cdots \otimes e_{i_k}, \tag{5.9}$$

where $1 \leq i_1 < \cdots < i_k \leq n$, and the element 1. Therefore,

$$\dim Cl_{n-p,p} = \sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{5.10}$$

We now work towards finding some general classification of any Clifford algebra, by working out some fundamental examples.

Lemma 5.4. *The Clifford algebras $Cl_{n-p,p}$ for $n \leq 2$ are given by*

$$\begin{aligned} &Cl_{0,0} = \mathbb{R}, \quad && Cl_{0,1} = \mathbb{C}, \quad && Cl_{1,0} = \mathbb{R} \oplus \mathbb{R}, \\ &Cl_{0,2} = \mathbb{H}, \quad && Cl_{1,1} = M_2(\mathbb{R}), \quad && Cl_{2,0} = M_2(\mathbb{R}). \end{aligned} \tag{5.11}$$

Proof. The first Clifford algebra is rather trivial, since $\mathbb{R}^0 = \{0\}$, and therefore, $\mathfrak{T}(\mathbb{R}^0) = \mathbb{R}$, and we can easily see that $Cl_{0,0} = \mathbb{R}$.
Now, if we denote $\mathbf{i}$ to be the unit vector in $\mathbb{R}$, the tensor algebra is given by elements of the form

$$\sum_{n=0}^{\infty} x_n \mathbf{i}^n,$$

where $\mathbf{i}^n := \bigotimes_{m=1}^{n} \mathbf{i}$. We then immediately find the Clifford algebras

$$Cl_{1,0} = Cl_{0,1} = \{a + b\mathbf{i} \mid a, b \in \mathbb{R}\}.$$

The difference between the two algebras being that in $Cl_{1,0}$, we have that $\mathbf{i}^2 = 1$, which therefore behaves like 1 in $\mathbb{R}$, while in $Cl_{0,1}$, we have that $\mathbf{i}^2 = -1$, which therefore behaves like the imaginary unit i. We conclude that

$$Cl_{1,0} = \mathbb{R} \oplus \mathbb{R}, \qquad Cl_{0,1} = \mathbb{C}.$$

Now, if we go to $\mathbb{R}^2$, we denote the two unit vectors by $\mathbf{i}$ and $\mathbf{j}$. The Clifford algebras then all consist of elements

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $\mathbf{k} := \mathbf{ij}$. In all cases, we find

$$\mathbf{k}^2 = \mathbf{ijij} = \mathbf{ij}(\{\mathbf{i}, \mathbf{j}\} - \mathbf{ji}) = -\mathbf{ijji}.$$

For $Cl_{0,2}$, we have that $\mathbf{i}^2 = \mathbf{j}^2 = -1$, and therefore $\mathbf{k}^2 = -1$, as well. We furthermore derive the following equations

$$\mathbf{ji} = -\mathbf{k}, \qquad \mathbf{ik} = -\mathbf{j}, \qquad \mathbf{ki} = \mathbf{j}, \qquad \mathbf{kj} = -\mathbf{i}, \qquad \mathbf{jk} = \mathbf{i}.$$

Thus, indeed, $Cl_{0,2} = \mathbb{H}$.
Now, in the case $Cl_{1,1}$, we find that $\mathbf{i}^2 = \mathbf{k}^2 = 1$ and $\mathbf{j}^2 = -1$, with the equations

$$\mathbf{ji} = -\mathbf{k}, \qquad \mathbf{ik} = \mathbf{j}, \qquad \mathbf{ki} = -\mathbf{j}, \qquad \mathbf{kj} = -\mathbf{i}, \qquad \mathbf{jk} = \mathbf{i}.$$

Actually, for $Cl_{2,0}$, we find $\mathbf{i}^2 = \mathbf{j}^2 = 1$ and $\mathbf{k}^2 = -1$, with

$$\mathbf{ji} = -\mathbf{k}, \qquad \mathbf{ik} = \mathbf{j}, \qquad \mathbf{ki} = -\mathbf{j}, \qquad \mathbf{kj} = \mathbf{i}, \qquad \mathbf{jk} = -\mathbf{i},$$

which is equivalent to the relations for $Cl_{1,1}$ if we let $\mathbf{j} \leftrightarrow \mathbf{k}$. We thus find that $Cl_{2,0} = Cl_{1,1}$.
Acting on our hunch that these Clifford algebras are the matrix algebra $M_2(\mathbb{R})$, we define for $Cl_{2,0}$

$$1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{i} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{j} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{k} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which indeed forms a basis for $M_2(\mathbb{R})$ and satisfies the equations for $Cl_{2,0}$. Therefore, we conclude that

$$Cl_{1,1} = Cl_{2,0} = M_2(\mathbb{R}).$$

□

It will later turn out that these six Clifford algebras are the only ones we need to derive explicitly, and all others can be derived from these ones.

Theorem 5.5 (Universality property)**.** *Define A to be some real algebra with unit 1, and let the function*

$$f:\ V \to A \tag{5.12}$$

be some linear map such that

$$f(v)f(v) = g(v,v)1 \tag{5.13}$$

for all $v \in V$. Then there is a unique extension of f to an algebra homomorphism

$$\tilde{f}:\ Cl_{n-p,p} \to A. \tag{5.14}$$

$Cl_{n-p,p}$ is the only associative real algebra that has this property.

Proof. We first extend f to a homomorphism $\tilde{f}':\ \mathfrak{T}(V) \to A$, which is uniquely defined through

$$\tilde{f}'(v_1 \cdots v_k) = f(v_1) \cdots f(v_k),$$

and

$$\tilde{f}'(x) = x\ 1$$

for $x \in \mathbb{R}$. We see that for any $v \in V$,

$$\tilde{f}'(vv - g(v,v)) = f(v)f(v) - g(v,v)1 = 0,$$

and thus $\tilde{f}'|_{\mathfrak{I}(V,g)} = 0$. We therefore find that $\tilde{f}'$ uniquely reduces to a function $\tilde{f}$ on the quotient $Cl_{n-p,p}$, which is the unique extension we were looking for.
Take now any real associative algebra $\mathcal{A}$, and assume f extends to a unique $\tilde{g}:\ \mathcal{A} \to A$. Then there must be some linear inclusion map $\iota:\ V \to \mathcal{A}$ (otherwise, extending would not even make sense), with the property $\iota(v)\iota(v) = vv = g(v,v)$ for all $v \in V$.
The uniqueness of $\tilde{g}$ tells us that any element of $\mathcal{A}$ must be written as a unique linear combination of products of $\iota(V) \cong V$ under the equality $vv = g(v,v)$, which is exactly $Cl_{n-p,p}$, and therefore $\mathcal{A} \cong Cl_{n-p,p}$. □

Therefore, to show that some associative algebra is the Clifford algebra, we only need to show it satisfies the above condition.

Lemma 5.6. *We have the following isomorphisms for the Clifford algebra:*

$$Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_{2,0}, \tag{5.15}$$
$$Cl_{0,n+2} \cong Cl_{n,0} \otimes Cl_{0,2}, \tag{5.16}$$
$$Cl_{n-p+1,p+1} \cong Cl_{n-p,p} \otimes Cl_{1,1}. \tag{5.17}$$

Proof. We start with proving the first isomorphism. Define the linear function

$$f:\ \mathbb{R}^{n+2} \to Cl_{0,n} \otimes Cl_{2,0},\ x + y \mapsto x \otimes \mathbf{k} + 1 \otimes y,$$

where we use that $\mathbb{R}^{n+2} = \mathbb{R}^n \oplus \mathbb{R}^2$. Then

$$\begin{aligned} f(x+y)f(x+y) &= x^2 \otimes \mathbf{k}^2 + x \otimes \{\mathbf{k}, y\} + 1 \otimes y^2 \\ &= (-g_{0,n}(x,x) + g_{2,0}(y,y))1 \otimes 1 = g_{n+2,0}(x+y, x+y)1 \otimes 1. \end{aligned}$$

Therefore, theorem 5.5 tells us there is some unique extension of f to an algebra homomorphism

$$\tilde{f}: \ Cl_{n+2,0} \to Cl_{0,n} \otimes Cl_{2,0}.$$

Notice now that

$$\begin{aligned} \tilde{f}((e_i + 0)(0 + \mathbf{j})) &= (e_i \otimes \mathbf{k})(1 \otimes \mathbf{j}) = e_i \otimes \mathbf{i}, \\ \tilde{f}((e_i + 0)(0 - \mathbf{i})) &= e_i \otimes \mathbf{j}, \end{aligned}$$

and we see that all generators of $Cl_{0,n} \otimes Cl_{2,0}$ are reached by $\tilde{f}$, which is therefore surjective. Moreover, since the dimension of both algebras is 2^n, we conclude that it must actually be bijective. Therefore, $\tilde{f}$ is an isomorphism.
The proofs for the second and third isomorphism are analogous to the proof of the first one. □

Corollary 5.7. *The following isomorphisms automatically follow:*

$$Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0}, \tag{5.18}$$

$$Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8}, \tag{5.19}$$

and

$$Cl_{8,0} \cong Cl_{0,8} \cong M_{16}(\mathbb{R}) \tag{5.20}$$

Corollary 5.8 (Characterisation of the Clifford algebra)**.** *One can derive from the previous corollary and lemma that all Clifford algebras are as given in table 5.1.*

Table 5.1: The characterisation of all Clifford algebras.

$(n-p)-p \bmod 8$	$Cl_{n-p,p}$	Representation space
0	$M_{2^{n/2}}(\mathbb{R})$	$\mathbb{R}^{2^{n/2}}$
1	$M_{2^{(n-1)/2}}(\mathbb{R}) \oplus M_{2^{(n-1)/2}}(\mathbb{R})$	$\mathbb{R}^{2^{(n+1)/2}}$
2	$M_{2^{n/2}}(\mathbb{R})$	$\mathbb{R}^{2^{n/2}}$
3	$M_{2^{(n-1)/2}}(\mathbb{C})$	$\mathbb{C}^{2^{(n-1)/2}}$
4	$M_{2^{(n-2)/2}}(\mathbb{H})$	$\mathbb{H}^{2^{(n-2)/2}}$
5	$M_{2^{(n-3)/2}}(\mathbb{H}) \oplus M_{2^{(n-3)/2}}(\mathbb{H})$	$\mathbb{H}^{2^{(n-1)/2}}$
6	$M_{2^{(n-2)/2}}(\mathbb{H})$	$\mathbb{H}^{2^{(n-2)/2}}$
7	$M_{2^{(n-1)/2}}(\mathbb{C})$	$\mathbb{C}^{2^{(n-1)/2}}$

The isomorphisms in this corollary clearly define irreducible representations for all Clifford algebras, as can be found in the third column of table 5.1.

5.1.2 The group of units of the Clifford algebra

We now want to turn the Clifford algebra into a multiplicative group. Since it contains the number 0, it cannot possibly be a multiplicative group itself, so we need to define a new set.

Definition 5.9. *The **group of units of the Clifford algebra** over V with respect to some quadratic form with signature $(n-p,p)$ is the group*

$$Cl^{\times}_{n-p,p} := \{x \in Cl(n-p,p) \mid \text{there is an } x^{-1} \in Cl(n-p,p) \text{ with } xx^{-1} = x^{-1}x = 1\} \tag{5.21}$$

with respect to the multiplication operator $\otimes$.

Note that, since $vv = g(v,v)$, we must have that $v^{-1} = \frac{1}{g(v,v)}v$ for all $v \in V \cap Cl^{\times}_{n-p,p}$ (note that if $g(v,v) = 0$, v is not invertible).
It is actually very easy to construct these groups by observing the matrix algebras in table 5.1, and replace them by their respective general linear group. We furthermore see that the multiplicative Clifford algebra is an open subset of the Clifford algebra, and therefore a 2^n-dimensional Lie group, with Lie algebra

$$\mathfrak{cl}^{\times}_{n-p,p} = Cl_{n-p,p}. \tag{5.22}$$

This Lie algebra induces the related adjoint representation map

$$Ad : \; Cl^{\times}_{n-p,p} \to \mathrm{Aut}(Cl_{n-p,p}), \tag{5.23}$$

where $\mathrm{Aut}(Cl_{n-p,p})$ denotes the set of bijective linear mappings from $Cl_{n-p,p}$ to itself. Since we are considering matrices, the adjoint representation is given explicitly by

$$Ad_x y = xyx^{-1}. \tag{5.24}$$

Note, in particular, that,

$$Ad_v w = -w + 2\frac{g(v,w)}{g(v,v)}v \tag{5.25}$$

for any $v, w \in V$. We see that $-Ad_v$ is precisely the reflection map about the hyperplane that is orthogonal to v. Later on, we will need some homomorphism $\widetilde{Ad} : \; Cl^{\times}_{n-p,p} \to \mathrm{Aut}(Cl_{n-p,p})$ such that $\widetilde{Ad}_v = -Ad_v$ is the reflection map. This induces the entire function

$$\widetilde{Ad}_x y = \alpha(x)yx^{-1}. \tag{5.26}$$

We call $\widetilde{Ad}$ the **twisted adjoint representation** of the Clifford algebra.
On a Lorentzian 4-dimensional manifold, the Clifford algebra and its multiplicative group of units on the tangent space at any point with respect to the metric tensor is

$$Cl_{3,1} = M_4(\mathbb{R}), \qquad Cl^{\times}_{3,1} = GL_4(\mathbb{R}). \tag{5.27}$$

5.1.3 Pin and spin groups

From the Clifford algebra, we can define two new groups: the pin group and the spin group. Their names reflects their physical relevance in particle physics, which is exactly why we will be covering them here.

Definition 5.10. *The **pin group** over V is given by*

$$\mathrm{Pin}(n-p,p) := \{v_1 \cdots v_k \mid k \geq 0,\, v_i \in V,\, g(v_i,v_i) = \pm 1 \text{ for all } i\}. \tag{5.28}$$

*The **spin group** over V is then*

$$\mathrm{Spin}(n-p,p) := \{v_1 \cdots v_k \mid k \text{ is even},\, v_i \in V,\, g(v_i,v_i) = \pm 1 \text{ for all } i,\, v_1 \cdots v_k v_k \cdots v_1 = 1\}. \tag{5.29}$$

Moreover, if g is either positive definite or negative definite,

$$\mathrm{Pin}(n) = \mathrm{Pin}(n,0) = \mathrm{Pin}(0,n), \qquad \mathrm{Spin}(n) = \mathrm{Spin}(n,0) = \mathrm{Spin}(0,n). \tag{5.30}$$

Note that the spin group consists exactly of elements in the pin group such that there is exactly an even amount of i such that $g(v_i,v_i) = +1$ and an even amount of j such that $g(v_j,v_j) = -1$. This "symmetry" between positive and negative signs also implies directly that

$$\mathrm{Spin}(n-p,p) = \mathrm{Spin}(p,n-p). \tag{5.31}$$

Both of these sets are indeed groups under multiplication, since they are clearly closed under multiplication, contain 1, and they are closed under inversion, as

$$g({v_i}^{-1},{v_i}^{-1}) = \frac{1}{g(v_i,v_i)} = g(v_i,v_i).$$

Actually, both these groups are closed subgroups of $Cl^{\times}_{n-p,p}$, and are therefore Lie groups.

Definition 5.11. *We define the **orthogonal group** over V with respect to g as*

$$O(n-p,p) := \{A \in GL(V) \mid g(Av,Av) = g(v,v) \text{ for all } v \in V\}, \tag{5.32}$$

*and the **special orthogonal group** as*

$$SO(n-p,p) := \{A \in O(n-p,p) \mid \det A = 1\}. \tag{5.33}$$

Moreover, we define $SO^0(n-p,p)$ to be the largest connected subset of $SO(n-p,p)$ that contains the identity.

Keep in mind that we identified V with $\mathbb{R}^n$, so taking the determinant is defined. The (special) orthogonal groups are now clearly closed subgroups of $GL(V)$, which is a Lie group, and therefore, both of these groups are actually Lie groups.

Later, we will want to find some double covering of the (s)pin group on the (special) orthogonal matrices.
Similarly to the usual (special) orthogonal groups, we have

$$\dim(O(n-p,p)) = \dim(SO(n-p,p)) = n(n-1)/2. \tag{5.34}$$

Lemma 5.12. *For any $A \in O(n-p,p)$, we have that*

$$\det A = \pm 1. \tag{5.35}$$

Furthermore,

$$O^0(n-p,p) = SO^0(n-p,p). \tag{5.36}$$

Proof. First, through writing out $g(Av+Aw, Av+Aw)$, we note that $g(Av,Aw) = g(v,w)$ for all $v,w \in V$ and $A \in O(n-p,p)$. Moreover, if we write g as the matrix $G := (g_{ij})_{1\leq i,j\leq n}$, with $g_{ij} := g(e_i, e_j)$, notice that

$$g(Av,Aw) = (Av)^T GAw = v^T(A^T GA)w,$$

from which we conclude that

$$A^T GA = G.$$

Now, when we take the determinant on both sides, we find that

$$(\det A)^2 \det G = \det G \neq 0,$$

and thus, indeed,

$$\det A = \pm 1.$$

Now, the determinant is a continuous function. Therefore, it separates the orthogonal group in two disconnected parts: one where the determinant is $+1$ and one where the determinant is -1. Since, trivially, $\det I_n = 1$, we find that $O^0(n-p,p)$ must be contained in the part where the determinant is $+1$. Note that this is exactly the special orthogonal group:

$$O^0(n-p,p) \subseteq SO(n-p,p).$$

This lets us conclude that

$$O^0(n-p,p) = SO^0(n-p,p).$$

□

Theorem 5.13. *We have the following two short exact sequences of Lie groups:*

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Pin}(n-p,p) \xrightarrow{\widetilde{Ad}} O(n-p,p) \longrightarrow 1, \tag{5.37}$$

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(n-p,p) \xrightarrow{\widetilde{Ad}} SO^0(n-p,p) \longrightarrow 1, \tag{5.38}$$

where $\mathbb{Z}_2 := \{-1,1\}$ is a zero-dimensional Lie group under multiplication.
In particular, $\mathrm{Pin}(n-p,p)$ *and* $\mathrm{Spin}(n-p,p)$ *form a double cover of* $O(n-p,p)$ *and* $SO^0(n-p,p)$*, respectively, and*

$$\dim(\mathrm{Pin}(n-p,p)) = \dim(\mathrm{Spin}(n-p,p)) = n(n-1)/2. \tag{5.39}$$

The proof can be found in section A.2.
This double covering is exactly the reason why we are so interested in the spin group. Without going into too much detail, in quantum mechanics, we generally consider so-called projective unitary representations of symmetry groups. In the case of spin, the symmetry group is the group of continuous rotations, which exactly corresponds to $SO^0(n-p,p)$, a subgroup of the so-called Poincaré group. Bargmann's theorem now implies that projective unitary representations of the Poincaré group exactly correspond to regular unitary representations of the double cover. This warrants an interest in the double cover of $SO^0(n-p,p)$, being $\mathrm{Spin}(n-p,p)$, and its unitary representations.

5.1.4 Explicit computation of the spin group in 4-dimensional Lorentzian spacetime

Now, we explicitly evaluate the groups introduced in this section for signature $(3,1)$, i.e., a 4-dimensional Lorentzian manifold, such as our spacetime. Referring to table 5.1, we know that

$$Cl_{3,1} \cong M_4(\mathbb{R}), \qquad Cl_{3,1}^{\times} \cong GL(\mathbb{R}^4). \tag{5.40}$$

We know that the unit vectors behave like

$$e_0 e_0 = -1, \qquad e_i e_i = 1, \qquad \{e_\mu,\ e_\nu\} = 0 \quad \text{for } \mu \neq \nu. \tag{5.41}$$

Our goal is now to find some representation $\rho :\ Cl_{3,1} \to M_4(\mathbb{R})$ that satisfies these identities. Trivially, we choose $\rho(1) = I_4$. If we are now able to define $\gamma_\mu := \rho(e_\mu)$ for all μ, the rest of the representation follows automatically. An educated guess gives us that

$$\begin{aligned}
\gamma_0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\
\gamma_2 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{5.42}$$

The induced basis is then given by I_4, γ_μ, and

$$\begin{aligned}
\gamma_0\gamma_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_0\gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_0\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\gamma_1\gamma_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma_1\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2\gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\gamma_0\gamma_1\gamma_2 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_0\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\gamma_0\gamma_1\gamma_3 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{5.43}$$

It is easy to check that these 16 matrices do indeed form a basis of $M_4(\mathbb{R})$.
As we now move towards the spin group, we only be observing the even Clifford algebra $Cl^0_{n-p,p}$, which is generated by the basis $I_4, \gamma_\mu\gamma_\nu, \gamma_0\gamma_1\gamma_2\gamma_3$. Through these, we can make a representation for $Cl^0_{n-p,p}$, through

$$\rho:\ Cl^0_{n-p,p} \to M_2(\mathbb{C})$$
$$\rho(I_4) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \rho(\gamma_0\gamma_1) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \rho(\gamma_0\gamma_2) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \rho(\gamma_0\gamma_3) = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$
$$\rho(\gamma_1\gamma_2) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \rho(\gamma_1\gamma_3) = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix},\quad \rho(\gamma_2\gamma_3) = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix},\quad \rho(\gamma_0\gamma_1\gamma_2\gamma_3) = \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}. \tag{5.44}$$

This is actually the reverse of the representation of $M_1(\mathbb{C})$ into $M_2(\mathbb{R})$, identifying

$$1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad i = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{5.45}$$

If we take now any two $v, w \in V$, expanded like $v = v^m u\gamma_\mu$ and $w = w^\mu\gamma_\mu$, it is clear that $vw \in Cl^0_{3,1}$. Taking the determinant of its representation gives us

$$\det\rho(vw) = g(v,v)g(w,w). \tag{5.46}$$

Therefore, for any $v_1 \cdots v_k \in \mathrm{Spin}(3,1)$, we find

$$\det\rho(v_1\cdots v_k) = g(v_1,v_1)\cdots g(v_k,v_k) = 1. \tag{5.47}$$

We find that — under the representation ρ, which we omit from this point on — $\mathrm{Spin}(3,1)$ is a Lie subgroup of $SL(\mathbb{C}^2)$. Note, moreover, that both of these Lie groups have a dimension of 6. Through the inverse function theorem, this implies that $\mathrm{Spin}(3,1)$ is open in $SL(\mathbb{C}^2)$.
Now, taking any $A \notin \mathrm{Spin}(3,1)$, we find that the set $A\,\mathrm{Spin}(3,1)$ is diffeomorphic to $\mathrm{Spin}(3,1)$, and therefore also open. Moreover, $A\,\mathrm{Spin}(3,1) \cap \mathrm{Spin}(3,1)$ must be the empty set, since otherwise, if $AG \in \mathrm{Spin}(3,1)$ for some $G \in \mathrm{Spin}(3,1)$, then $A = (AG)G^{-1}$ must also be in $\mathrm{Spin}(3,1)$, which gives a contradiction.
We now take the union over all such A: $U = \cup_{A\notin\mathrm{Spin}(3,1)} A\,\mathrm{Spin}(3,1)$, we note that U must be open, $U \cap \mathrm{Spin}(3,1) = \emptyset$ and $U \cup \mathrm{Spin}(3,1) = SL(\mathbb{C}^2)$. Since $\mathrm{Spin}(3,1)$ is non-empty, and $SL(\mathbb{C}^2)$ is connected, this implies directly that $U = \emptyset$, and therefore,

$$\mathrm{Spin}(3,1) = SL(\mathbb{C}^2). \tag{5.48}$$

5.1.5 Spin structures

Now that we have developed the required algebraic outline for the spin group, we return to geometry by introducing principal bundles called spin structures, similarly to the previous chapters.
In addition to the notion of orientability of manifolds, which ensures two observers on the manifold will always agree on the orientation of some basis vectors (i.e., one observer is not in the "mirror image" of the universe that the other observer is in), we can define a notion of spacetime-orientability, which ensures that the two observers will always agree on the "direction" of time.

Definition 5.14. *Let M be some pseudo-Riemannian manifold with signature $n-p, p$, and observe the reduction of the frame bundle* $O(n-p,p) \longrightarrow P \xrightarrow{\pi} M$.
*We say that M is **orientable** if this principal bundle can be further reduced to some principal bundle with fibre $SO(n-p,p)$.*
*We say that M is **spacetime-orientable** if the principal bundle can be reduced to a principal bundle with fibre $SO^0(n-p,p)$.*

What these two notions mean, is that when an observer *chooses* some basis vectors of the tangent space at any point of the manifold, then follows a path through the manifold that returns to the same point, the observer's "transformed" basis must agree with the original basis to some extent.
If some manifold is orientable, the bases are transformed into one another by a matrix in $SO(n-p,p)$, which is exactly the group of orientation-preserving transformations. Therefore, the two bases would agree on orientation, which also shows that this definition is equivalent to the "classical" definition of orientability.
If a manifold is spacetime-orientable, the new basis agrees with the original one in such a way that the transformation matrix can be found in the identity connected component. In Lorentzian manifolds, one can see this as the direction of time not being "reversed".

Definition 5.15. *Let M be some spacetime-orientable manifold with signature $(n-p,p)$ and principal bundle* $SO^0(n-p,p) \longrightarrow SO^0(M) \xrightarrow{\pi_{SO^0}} M$. *A **spin structure** on M is some pair $(\mathrm{Spin}(M), f)$ such that*

1. *$f: \mathrm{Spin}(M) \to SO^0(M)$ is a double covering bundle map;*
2. *We have a principal bundle* $\mathrm{Spin}(n-p,p) \longrightarrow \mathrm{Spin}(M) \xrightarrow{\pi} M$ *for $\pi = \pi_{SO^0} \circ f$;*
3. *The following diagram commutes:*

$$\begin{tikzcd}
SO^0(M) \times SO^0(n-p,p) \arrow[r, "\Psi_{SO^0}"] & SO^0(M) \arrow[dr, "\pi_{SO^0}"] & \\
 & & M \\
\mathrm{Spin}(M) \times \mathrm{Spin}(n-p,p) \arrow[uu, "f\times\widetilde{Ad}"] \arrow[r, "\Psi"] & \mathrm{Spin}(M) \arrow[uu, "f"] \arrow[ur, "\pi"] &
\end{tikzcd} \tag{5.49}$$

Here, Ψ and Ψ_{SO^0} are the right actions of the structure groups on the total spaces.

In other words, a spin structure is some canonical "elevation" of the reduced frame bundle to a principal bundle over the Spin group.
Not all manifolds admit a spin structure. In fact, a manifold admits a spin structure if and only if the so-called second Stiefel-Whitney class of TM, denoted by $w_2(TM)$, is equal to

zero. We will not elaborate on the second Stiefel-Whitney class in this thesis, but one may refer to chapter II of [8] for more information.

5.2 Spinors

Now, we make use of the third column in table 5.1 to construct an associated vector bundle from the spin structure. To this end, we want to find some representation of the spin group. Note that, through the representations

$$\mathbb{R} \to \mathbb{C}, \qquad \mathbb{H} \to M_2(\mathbb{C}), \tag{5.50}$$

we can make a representation for the Clifford algebra for the case that $n - 2p \bmod 4 \neq 1$:

$$\rho:\ Cl_{n-p,p} \to \mathrm{End}(\Delta_n), \tag{5.51}$$

where Δ_n is the **spinor space**, defined by

$$\Delta_n := \begin{cases} \mathbb{C}^{2^{n/2}} & \text{if } n \text{ is even,} \\ \mathbb{C}^{2^{(n-1)/2}} & \text{if } n \text{ is odd.} \end{cases} \tag{5.52}$$

Now, since we are mainly interested in representing the spin group in such a way, we define the representation

$$\rho:\ \mathrm{Spin}(n-p,p) \to \mathrm{End}(\Delta_n) \tag{5.53}$$

for all n, p, dealing with the previously excluded cases by using that $\mathrm{Spin}(n-p,p) = \mathrm{Spin}(p,n-p)$. This representation is called the **spinor representation**.

Definition 5.16. *Let M be some spacetime-orientable manifold with signature $(n-p,p)$ and some spin structure $(\mathrm{Spin}(M), f)$. A **spinor bundle** is then the vector bundle associated to the spin structure, i.e.,*

$$\Delta_n \longrightarrow S(M) \xrightarrow{\ \pi\ } M, \tag{5.54}$$

where $S(M) := (\mathrm{Spin}(M) \times \Delta_n)/\mathrm{Spin}(n-p,p)$.
*(Local) sections of the spinor bundle are called **spinors**.*

These spinors are now the mathematical objects that represent fermions in gauge theory, and conventionally, we denote spinors by the Greek letter ψ, in reference to the quantum mechanical wave function. The correspondence to fermions can be easily seen by working out the case where the metric has signature $(3,0)$, i.e., the non-relativistic case, such that $\Delta_3 = \mathbb{C}^2$, which indeed corresponds to the two-dimensional wave function of fermions, and $\mathrm{Spin}(3,0)$ consists exactly of elements $aI_2 + bi\sigma_x + ci\sigma_y + di\sigma_z$, where $a^2 + b^2 + c^2 + d^2 = 1$ and σ_i are the Pauli matrices.

5.2.1 Sections and the covariant derivative

We now discuss a way of generating a section of the spinor bundle. Remember that the spinor bundle is generated from a spin structure, which, in turn, stems from the frame bundle. Therefore, we start with a section on the frame bundle, from which we work to the spinor bundle.

Definition 5.17. *A section $e = (e_1, \ldots, e_n)$, being a choice of orthonormal basis, on the reduced frame bundle* $SO^0(n-p,p) \longrightarrow P \xrightarrow{\pi} M$ *is called a* ***vielbein****.*

Note that vielbeins diagonalise the metric tensor to only contain elements equal to ± 1 on its diagonal.
Now, if we take a specific local section $s \in \Gamma(U, \mathrm{Spin}(M))$, any local section $\psi \in \Gamma(U, S(M))$ can be written like

$$\psi = (s, \varphi), \tag{5.55}$$

where $\varphi : \ U \to \Delta_n$.
Now, we call the covariant derivative of the spinor bundle the **spin covariant derivative**

$$\nabla_X(s,\varphi)(x) = (s, (\mathrm{d}\varphi)_x X_x + (s^* \omega_{\mathrm{Spin}})_x X_x \varphi(x)). \tag{5.56}$$

Here, ω_{Spin} is the canonical connection form on the $\mathrm{Spin}(n-p,p)$-bundle induced by the Levi-Civita connection on the tangent bundle. As we discussed before, the Levi-Civita connection induces a connection form ω_{SO} on the frame bundle, which allows us to canonically define $\omega_{\mathrm{Spin}} = (\widetilde{\mathrm{Ad}}_*)^{-1} \circ (f^* \omega_{\mathrm{SO}})$.

Definition 5.18. *For $(n-p)-p \bmod 4 \neq 1$, there is a clear mapping $\Phi : \ \mathbb{R}^n \to \mathrm{End}(\Delta_n)$, through the complex representation of the Clifford algebra. If we, moreover, compose the representation with the projection map $M \oplus M \to M$, $a + b \mapsto a$, we can extend the mapping to $\Phi : \ \mathbb{R}^n \to \mathrm{End}(\Delta_n)$ for all p.*
We now define ***Clifford multiplication*** *through the map*

$$\mu : \ \mathbb{R}^n \times \Delta_n \to \Delta_n, \ (v, \varphi) \mapsto \Phi(v)\varphi =: v \cdot \varphi. \tag{5.57}$$

This can be expanded to a map

$$\mu : \ TM \otimes S(M) \to S(M), \tag{5.58}$$

where we denote

$$v \cdot p := \mu(v, p). \tag{5.59}$$

Now, we have the sequence

$$\Gamma(U, S(M)) \xrightarrow{\nabla} \Gamma(U, T^*M \otimes S(M)) \xrightarrow{g} \Gamma(U, TM \otimes S(M)) \xrightarrow{\mu} \Gamma(U, S(M)). \tag{5.60}$$

Definition 5.19. *The* ***Dirac operator***[10] *is the mapping*

$$\not{D}: \ \Gamma(U, S(M)) \to \Gamma(U, S(M)), \ \not{D} = \mu \circ g \circ \nabla. \tag{5.61}$$

In tensor notation, if we take some spinor ψ, the Dirac operator acts on it such that

$$\not{D}\psi = g^{\mu\nu} e_\mu \cdot \nabla_{e_\nu} \psi, \tag{5.62}$$

where e_μ is any vielbein.
In addition to the Dirac operator, we want to define some bilinear form $\langle \cdot, \cdot \rangle : \ \Delta_n \times \Delta_n \to \mathbb{C}$, which we will use when we develop a Lagrangian in chapter 6. The following definition is from [11].

Definition 5.20. *Fix* $\varepsilon = \pm 1$. *A* ***Dirac form*** *is some non-degenerate* $\mathbb{R}$*-bilinear form*

$$\langle \cdot, \cdot \rangle : \ \Delta_n \times \Delta_n \to \mathbb{C} \tag{5.63}$$

such that for $\varphi, \varphi' \in \Delta_n$,

(i) $\langle \varphi', v \cdot \varphi \rangle = \varepsilon \langle v \cdot \varphi', \varphi \rangle$ *for all* $v \in \mathbb{R}^n$;

(ii) $\langle \varphi, \varphi' \rangle = \langle \varphi', \varphi \rangle^*$;

(iii) $\langle \varphi', z\varphi \rangle = z \langle \varphi', \varphi \rangle = \langle z^* \varphi', \varphi \rangle$ *for all* $z \in \mathbb{C}$.

We can furthermore define for any $\varphi \in \Delta_n$ *the dual vector* $\overline{\varphi} = \langle \varphi, \cdot \rangle \in \Delta_n{}^*$, *called the* ***Dirac conjugate***.

In the case that the Dirac form is invariant under $\mathrm{Spin}(n-p, p)$, we can generalise the Dirac form to some bilinear function $\langle \cdot, \cdot \rangle_{S(M)}$ mapping sections over the spinor bundle $\Delta_n \longrightarrow S(M) \xrightarrow{\pi} M$ to scalar functions over the base space, since then fiberwisely applying the Dirac form is well-defined.

Lemma 5.21. *The Dirac form is invariant under* $\mathrm{Spin}(n-p, p)$, *i.e.,*

$$\langle xv, xw \rangle = \langle v, w \rangle \tag{5.64}$$

for all $v, w \in \Delta_n$ *and* $x \in \mathrm{Spin}(n-p, p)$

Proof. For this lemma, we only need the first property of the Dirac form. Note that $x = v_1 \cdots v_k$ for some even k and $v_i \in \mathbb{R}^n$, and therefore,

$$\begin{aligned} \langle xv, xw \rangle &= \langle v_1 \cdots v_k v, v_1 \cdots v_k w \rangle \\ &= \varepsilon \langle v_2 \cdots v_k v, v_1 v_1 \cdots v_k w \rangle \\ &= \langle v, v_k \cdots v_1 v_1 \cdots v_k w \rangle \\ &= \langle v, w \rangle. \end{aligned}$$

This proves the lemma. □

5.2.2 Interaction of the spinor with a gauge field

Now, we have developed both gauge fields, describing gauge bosons and the fundamental forces, and spinors, describing fermions. However, we know from experiment that fermions and gauge fields interact with one another. For example, electrons are electrically charged, and therefore interact with the electromagnetic field. The objects that combine the two fields are called “twisted spinor bundles”. This section is based largely on Hamilton's book[11], since it offers a very clear mathematical formulation of the twisted bundles.
Before we define the twisted spinor bundles, however, we want to find a general way of “mixing” vector bundles such that they can interact with one another. In quantum mechanics, interacting particles are realized by taking the tensor product of the Hilbert spaces, which gives us a hunch that we should do something similar to the vector bundles.

Lemma 5.22. *Let* $V \longrightarrow E \xrightarrow{\pi_V} M$ *and* $W \longrightarrow F \xrightarrow{\pi_W} M$ *be some vector bundles. There is a well-defined operation* $\cdot \otimes \cdot$ *such that* $E \otimes F$ *is a vector bundle with fibre* $V \otimes W$*.* $E \otimes F$ *is called the* ***tensor product*** *of* E *and* F*.*

Proof. In this proof, we describe a way of constructing the tensor product such that it is indeed a vector bundle with fibre $V \otimes W$.
Start with some $GL(V)$-atlas $\{(A_i, \varphi_i)\}$ and $GL(W)$-atlas $\{(B_j, \psi_j)\}$ of the V-bundle and W-bundle, respectively. By intersecting the domains, we get new atlases $\{(U_i, \varphi_i)\}$ and $\{(U_i, \psi_i)\}$ (explicitly, they are $\{(A_i \cap B_j, \varphi_i)\}$ and $\{(A_i \cap B_j, \psi_j)\}$), such that they agree on their domains.
These atlases induce the function

$$\Phi_i := \varphi_i \otimes \psi_i : \ {\pi_V}^{-1}(U_i) \otimes {\pi_W}^{-1}(U_i) := \cup_{x \in U_i}({\pi_V}^{-1}(x) \otimes {\pi_W}^{-1}(x)) \to U_i \times (V \otimes W).$$

Then the tensor product is

$$E \otimes F = \cup_{x \in M}({\pi_V}^{-1}(x) \otimes {\pi_W}^{-1}(x)),$$

together with the topology induced by the local trivializations $\{(U_i, \Phi_i)\}$. This tensor product now clearly induces a vector bundle over the fibre $V \otimes W$. ☐

An example of a tensor product of vector bundles is any tensor bundle $T^a_b M = \bigotimes_{i=1}^a TM \otimes \bigotimes_{j=1}^b T^*M$, which is the vector bundle corresponding to (a, b)-tensor fields.

Definition 5.23. *Let* M *be some spacetime-orientable manifold and* $S(M)$ *be a related spinor bundle. Furthermore, let* $G \longrightarrow P \xrightarrow{\pi'} M$ *be some principal bundle (the gauge bundle) with some representation* $\rho : \ G \to \mathrm{End}(\mathbb{C}^k)$*, for some* k*. The* ***twisted spinor bundle*** *is then the associated vector bundle*

$$\Delta_n \otimes \mathbb{C}^k \longrightarrow S(M) \otimes (P \times \mathbb{C}^k)/G \xrightarrow{\pi} M \ . \tag{5.65}$$

Given local sections s on P and ε on $\mathrm{Spin}(M)$ over U, any section on the twister spinor bundle, dubbed a **twisted spinor**, can be written

$$\psi := \sum_{i=1}^{k} (\varepsilon, v_i) \otimes (s, e_i), \tag{5.66}$$

with $\{e_i\}$ the Euclidean basis of $\mathbb{C}^k$ and $v_i : U \to \Delta_n$. Alternatively, we introduce the notation

$$\psi = \left(\varepsilon \times s, \sum_{i=1}^{k} v_i \otimes e_i\right) = (\varepsilon \times s, \varphi), \tag{5.67}$$

where $\varphi : U \to \Delta_n \otimes \mathbb{C}^k$. Note now that, for $g \in \mathrm{Spin}(n-p,p)$ and $h \in G$,

$$((\varepsilon g) \times (sh), \varphi) = (\varepsilon \times s, (g \otimes h)\varphi). \tag{5.68}$$

This twisted spinor bundle expands the spinors to not only separate into spin up and spin down, but also introduces a "charge" associated to the principal bundle. For example, considering the $SU(3)$ gauge theory, corresponding to the strong interaction, together with spinors in three-dimensional Riemannian space, we note that $SU(3)$ can clearly be represented to some matrix group over $\mathbb{C}^3$, and thus we can (locally) write the φ-component of such a twisted spinor like

$$\varphi = \begin{pmatrix} \psi_{u,r} \\ \psi_{u,g} \\ \psi_{u,b} \\ \psi_{d,r} \\ \psi_{d,g} \\ \psi_{d,b} \end{pmatrix}, \tag{5.69}$$

where $\psi_{i,j} : U \to \mathbb{C}$, and r, g, b correspond to the red, green and blue colour charge, respectively. In the full 4-dimensional standard model (with gauge group $G = U(1) \times SU(2) \times SU(3)$), we get a spinor with no less than $4 \times 6 = 24$ components following the same method. Now, the gauge field "relates" the components of these vectors, and "transforms" them into one another. For example, once again in the gauge theory of the 8-dimensional $SU(3)$ group, there is a basis of its Lie algebra $\mathfrak{su}(3)$, known as the Gell-Mann matrices, which are written in terms of the eight colours of the gluon, the gauge boson of this gauge theory. To give an example, one of the generators is given by

$$(r\bar{b} + b\bar{r})/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{5.70}$$

where $r\bar{b}$ rotates a "blue" state into a "red" state, and vice versa for $b\bar{r}$. For spinors in three dimensions, this generator then becomes

$$\frac{1}{\sqrt{2}} I_2 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.71}$$

Note that in our definition, we put no restriction on ρ, and it can therefore be any representation. One should, however, always specify what representation is used in the definition of some fermion, as different representations corresponds to different fermion species. For example, within the $SU(3)$ gauge theory, electrons are colourless and correspond to the trivial representation $\rho : SU(3) \to \{I_1\}$. Quarks, on the other hand, can have any of three colours

and correspond to the adjoint representation Ad : $SU(3) \to \mathfrak{su}(3)$. We will not go into further depth regarding the specific representations corresponding to the different species of fermions in this thesis.
As we did in the previous section, we define a covariant derivative and, subsequently, a Dirac operator for the twisted spinor bundle.

Definition 5.24. *Let* $G \longrightarrow P \xrightarrow{\pi'} M$ *be the gauge bundle with some connection form* ω *and induced covariant derivative* ∇' *of the associated vector bundle, and* $S(M)$ *a spinor bundle over some* n*-dimensional manifold* M *with spin covariant derivative* ∇*. The* ***twisted covariant derivative*** *over the twisted spinor bundle* $S(M) \otimes (P \times \mathbb{C}^k)/G$ *is then*

$$\nabla^\omega = \nabla \otimes 1 + 1 \otimes \nabla'. \tag{5.72}$$

Explicitly,

$$\begin{aligned}\nabla^\omega_X \left(\varepsilon \times s, \sum_{i=1}^k v_i \otimes e_i\right)(x) = \Bigg(\varepsilon \times s, \sum_{i=1}^k &[(\mathrm{d}v_i)_x X_x + (\varepsilon^* \omega_{\mathrm{Spin}})_x X_x v_i(x)] \otimes e_i \\ &+ v_i \otimes [(s^*\omega)_x X_x e_i(x)]\Bigg).\end{aligned} \tag{5.73}$$

Now, we redefine the Dirac operator for the twisted spinor bundle.

Definition 5.25. *The* ***twisted Dirac operator*** *is the mapping*

$$\not{D}^\omega : \Gamma(U, S(M) \otimes (P \times \mathbb{C}^k)/G) \to \Gamma(U, S(M) \otimes (P \times \mathbb{C}^k)/G), \; \not{D}^\omega = \mu \circ g \circ \nabla^\omega. \tag{5.74}$$

Again, in tensor notation,

$$\not{D}^\omega \psi = g^{\mu\nu} e_\mu \cdot \nabla^\omega_{e_\nu} \psi, \tag{5.75}$$

with e_μ some vielbein.
Finally, we also generalise the Dirac form to the twisted spinor bundle. If we have some Dirac form $\langle \cdot, \cdot \rangle$ on Δ_n and any positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle'$ on $\mathbb{C}^k$ (i.e., an inner product), we induce the **twisted Dirac form** $\langle \cdot, \cdot \rangle^t$ on $\Delta_n \otimes \mathbb{C}^k$ through

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle^t = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle'. \tag{5.76}$$

If we apply the twisted Dirac form fiberwisely, we once again get a bilinear function $\langle \cdot, \cdot \rangle^t_{S(M)}$ over sections on the twisted spinor bundle that maps to scalar functions over the base space. Furthermore, we once again introduce the **Dirac conjugate** as $\overline{\varphi} = \langle \varphi, \cdot \rangle^t_{S(M)} \in (\Delta_n \otimes \mathbb{C}^k)^*$.

5.3 Example: Explicit calculations in 4-dimensional Lorentz spacetime

As we concluded in section 5.1.4, we know that the spin group in 4-dimensional Lorentzian spacetime is given by

$$\mathrm{Spin}(n-p, p) = SL(\mathbb{C}^2). \tag{5.77}$$

Now, we would like to define a Dirac form for this case. To do so, we prove the following claim.

Claim 5.26. *The function* $\langle \cdot, \cdot \rangle : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$, *defined by*

$$\langle \varphi', \varphi \rangle = i\varphi'^* \gamma_0 \varphi, \tag{5.78}$$

is a Dirac form. Here, γ_0 *is as in section 5.1.4.*

Proof. Firstly, we have to show the function $\langle \cdot, \cdot \rangle$ is a non-degenerate $\mathbb{R}$-bilinear form. Bilinearity is trivial, so we only need to check that the function is non-degenerate. Note, for any $\varphi \in \mathbb{C}^4$, that

$$\langle \gamma_0 \varphi, \varphi \rangle = i\varphi^* \varphi = i\|\varphi\|^2,$$

from which it directly follows that this bilinear form is indeed non-degenerate.
Now, we continue on to the proof, proving each property of the Dirac form.

(i) Take any $v \in \mathbb{R}^4$. Then, for $\varphi, \varphi' \in \mathbb{C}^4$, we have that

$$\begin{aligned}\langle \varphi', v \cdot \varphi \rangle &= i\varphi'^* \gamma_0 v^\mu \gamma_\mu \varphi \\ &= i\varphi'^* \gamma_0 v^0 \gamma_0 \varphi - i\varphi'^* \gamma_i v^i \gamma_0 \varphi \\ &= -i(v^0 \gamma_0 \varphi')^* \gamma_0 \varphi - i(v^i \gamma_i \varphi')^* \gamma_0 \varphi \\ &= -\langle v \cdot \varphi', \varphi \rangle.\end{aligned}$$

This proves the first property.

(ii) Take once more any $\varphi, \varphi' \in \mathbb{C}^4$, then

$$\begin{aligned}\langle \varphi, \varphi' \rangle &= i\varphi^* \gamma_0 \varphi' \\ &= (-i\varphi'^* (-\gamma_0) \varphi)^* \\ &= \langle \varphi', \varphi \rangle^*,\end{aligned}$$

and we find the second property.

(iii) Take $\varphi, \varphi' \in \mathbb{C}^4$ and $z \in \mathbb{C}$, and notice

$$\begin{aligned}\langle \varphi', z\varphi \rangle &= i\varphi'^* \gamma_0 z \varphi \\ &= z i \varphi'^* \gamma_0 \varphi = z\langle \varphi', \varphi \rangle \\ &= i(z^* \varphi')^* \gamma_0 \varphi = \langle z^* \varphi', \varphi \rangle.\end{aligned}$$

We conclude that this function is indeed a valid Dirac form. □

From our Dirac form, we also get the Dirac conjugate

$$\overline{\varphi} = i\varphi^* \gamma_0. \tag{5.79}$$

In physical literature, where instead of a $(3, 1)$-signature, a $(1, 3)$-signature is used instead, you may see an altered convention where $\overline{\varphi} = \varphi^* \gamma_0$.

6 Lagrangian mechanics

Up to this point, we have considered gauge theories as a purely mathematical theory — a mathematical framework for physical theories. Despite introducing some laws, in particular, the Bianchi identity, there is no physics explicitly present in our theory. In order to "inject" some physics, we might try to define some Lagrangian, and use the it to derive more physical laws.

In this chapter, we first introduce Lagrangian mechanics to readers unfamiliar with it. Afterwards, we try to construct a Lagrangian to be used in gauge theories, starting with the Yang-Mills Lagrangian for the gauge field, and then introducing the Dirac Lagrangian for spinor fields. We finally look at the explicit example for the standard model, where the structure group is $U(1) \times SU(2) \times SU(3)$.

6.1 An introduction to Lagrangian mechanics

In classical mechanics, Lagrangian mechanics is a reformulation of Newtonian mechanics. If, at some position $x = (x_1, x_2, x_3)$, some particle has kinetic energy $T(\dot{x})$ and potential energy $V(x)$, the **Lagrangian** is given by

$$L(x(t), \dot{x}(t), t) = T(\dot{x}(t)) - V(x(t)). \tag{6.1}$$

We then define some **action**

$$S[x] = \int L(x(t))\, \mathrm{d}t\,. \tag{6.2}$$

In a classical mechanics course, you will learn that Newton's laws of motion are actually equivalent to S being (locally) stationary for some path x (in practice, it is usually minimised or maximised), which is known as the **stationary-action principle**. Using functional derivatives:

$$\frac{\delta S}{\delta x} = 0 \tag{6.3}$$

at the "correct" path x. This is, through integration by parts, equivalent to the Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}. \tag{6.4}$$

We can generalise this notion to a field theory over any pseudo-Riemannian orientable manifold. Let M be any such manifold with induced volume form $\mathrm{d}V$, and E some vector space, and define some function $\phi : M \to E$, which we call the "field". Using the basis of E, we can actually write the field as $\phi^i : M \to \mathbb{R}$ for some indices i. Then, we can define a Lagrangian functional

$$\mathcal{L}[\phi] : M \to \mathbb{C}, \tag{6.5}$$

and a fitting action (recall the volume element $\mathrm{d}V$)

$$S[\phi] = \int \mathcal{L}[\phi] \mathrm{d}V. \tag{6.6}$$

To then find a restriction to the field, we require, for all i, that

$$\frac{\delta S}{\delta \phi^i} = 0. \tag{6.7}$$

Sadly, though, we cannot define a Lagrange equation as easily as we did on $\mathbb{R}^n$, since we can't use integration by parts as easily on a general manifold as we could on a flat spacetime.

6.2 Lagrangian mechanics in gauge theories: the Yang-Mills Lagrangian

In gauge theories, the "fields" in consideration are — perhaps obviously — the gauge potential $\mathcal{A}$ and the twisted spinor ψ. The total Lagrangian should therefore be some combination

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_S, \tag{6.8}$$

where $\mathcal{L}_G$ is the Lagrangian for the gauge potential and $\mathcal{L}_S$ is the Lagrangian for the spinor, including interactions of the spinor with the gauge potential. In this first section, we study $\mathcal{L}_G$, which we will later choose to be the so-called Yang-Mills Lagrangian.

First off, we once again consider electromagnetism: the $U(1)$ gauge theory. In electromagnetism, the Lagrangian density of the field (which is all kinetic!) is given by

$$\mathcal{L} = \frac{1}{2\mu_0}\left(\frac{1}{c^2}\mathbf{E}^2 - \mathbf{B}^2\right) = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4\mu_0}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4\mu_0}\mathrm{Tr}({\mathcal{F}_{\mu\nu}}^{*}\mathcal{F}^{\mu\nu}). \tag{6.9}$$

Note that the map $(A, B) \mapsto \frac{1}{2}\mathrm{Tr}(A^*B)$ behaves like a positive-definite symmetric bilinear form. This gives us the hunch that the kinetic part of the gauge Lagrangian is

$$\mathcal{L}_{G_K} = -\frac{1}{2}\langle \mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}\rangle_{\mathfrak{g}}, \tag{6.10}$$

with $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ some positive-definite symmetric bilinear form on the Lie algebra $\mathfrak{g}$. The requirement of positive-definiteness is kept to ensure that the gauge field has a positive kinetic energy. We furthermore want the Lagrangian to be gauge invariant, i.e.,

$$\langle \mathrm{Ad}_t \circ \mathcal{F}_{\mu\nu}, \mathrm{Ad}_t \circ \mathcal{F}^{\mu\nu}\rangle_{\mathfrak{g}} = \langle \mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}\rangle_{\mathfrak{g}} \tag{6.11}$$

for all transition functions t. It must be gauge invariant because our arbitrary choice in section, which is a non-physical mathematical object, should not influence our physical system, which is described by the Lagrangian. It is a sufficient condition to require

$$\langle \mathrm{Ad}_g A, \mathrm{Ad}_g B\rangle_{\mathfrak{g}} = \langle A, B\rangle_{\mathfrak{g}} \tag{6.12}$$

for all $g \in G$ and $A, B \in \mathfrak{g}$. We are therefore looking for a so-called Ad-**invariant** positive-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$.

Lemma 6.1. *Let G be some Lie group with Lie algebra $\mathfrak{g}$. If G is compact, then there exists an Ad-invariant positive-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$.*

Proof. Note first that $\mathfrak{g}$ is a real vector space, and therefore it can be identified with $\mathbb{R}^d$ for some d. Hence, there must exist some positive-definite symmetric bilinear form $\langle\cdot,\cdot\rangle'$. Define now $\langle\cdot,\cdot\rangle_{\mathfrak{g}}:\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ through

$$\langle A, B\rangle_{\mathfrak{g}} = \int_G \langle \mathrm{Ad}_g\, A, \mathrm{Ad}_g\, B\rangle' \mathrm{d}g.$$

Here, we use that we can integrate over any compact Lie group using the (right) Haar measure (see further: [12, p. 182] or [13, pp. 239–240]). This function is still a positive-definite symmetric bilinear form, as inherited from the original bilinear form. Moreover, for any $h\in G$, we use the properties of the Haar measure and find that

$$\langle \mathrm{Ad}_h A,\ \mathrm{Ad}_h\, B\rangle_{\mathfrak{g}} = \int_G \langle \mathrm{Ad}_{gh}\, A, \mathrm{Ad}_{gh}\, B\rangle'\mathrm{d}g = \int_G \langle \mathrm{Ad}_g\, A, \mathrm{Ad}_g\, B\rangle'\mathrm{d}g = \langle A, B\rangle_{\mathfrak{g}}.$$

This proves the lemma. □

Definition 6.2. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle with compact structure group. Locally, the* ***Yang-Mills Lagrangian*** *is defined to be*

$$\mathcal{L}_{YM}[\mathcal{A}] = -\frac{1}{2}\sqrt{\frac{G\hbar^3}{c}}\langle \mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}\rangle_{\mathfrak{g}}. \tag{6.13}$$

The factor $\sqrt{\frac{G\hbar^3}{c}}$ here is added such that the Lagrangian has the dimension of energy. In most physical contexts, the use of natural units eliminates this prefactor.
Note that, since the Yang-Mills Lagrangian is actually gauge invariant, the local definition extends naturally to a global definition $\mathcal{L}_{YM}[\omega]:\ M\to\mathbb{R}$, where ω is a connection form.
Now, recall the discussion in section 4.2.2 — our condition for the structure group to be compact is exactly why we prefer to use $U(1)$ over $\mathbb{R}$ in electromagnetism.

6.2.1 Adding a mass term

In addition to the Yang-Mills Lagrangian, the gauge Lagrangian gains a mass term if the field is massive, i.e., the associated boson is massive,

$$\mathcal{L}_G = \mathcal{L}_{YM} + \frac{1}{2}{m_G}^2\langle \mathcal{A}_\mu, \mathcal{A}^\mu\rangle_{\mathfrak{g}}. \tag{6.14}$$

If we now apply a gauge transformation

$$\mathcal{A}\to \mathrm{Ad}_{g^{-1}}\circ\mathcal{A} + g^*\Theta, \tag{6.15}$$

with $g:\ M\to G$ a transition function, the Lagrangian transforms like

$$\begin{aligned}\mathcal{L}_G &\to \mathcal{L}_{YM} + \frac{1}{2}{m_G}^2\langle \mathcal{A}_\mu, \mathcal{A}^\mu\rangle_{\mathfrak{g}} + {m_G}^2\langle (g^*\Theta)_\mu, \mathcal{A}^\mu\rangle_{\mathfrak{g}} + \frac{1}{2}{m_G}^2\langle (g^*\Theta)_\mu, (g^*\Theta)^\mu\rangle_{\mathfrak{g}} && (6.16)\\ &= \mathcal{L}_G + {m_G}^2\langle (g^*\Theta)_\mu, \mathcal{A}^\mu\rangle_{\mathfrak{g}} + \frac{1}{2}{m_G}^2\langle (g^*\Theta)_\mu, (g^*\Theta)^\mu\rangle_{\mathfrak{g}}. && (6.17)\end{aligned}$$

The action transforms like

$$S_G[\mathcal{A}] \to S_G[\mathcal{A}] + {m_G}^2 \int_M \langle (g^*\Theta)_\mu, \mathcal{A}^\mu \rangle_{\mathfrak{g}} + \frac{1}{2} \langle (g^*\Theta)_m u, (g^*\Theta)_\mu \rangle_{\mathfrak{g}} \mathrm{d}V, \tag{6.18}$$

which we vary

$$\frac{\mathrm{d}}{\mathrm{d}t} S_G[\mathcal{A} + t\delta\mathcal{A}]\bigg|_{t=0} \to \frac{\mathrm{d}}{\mathrm{d}t} S_G[\mathcal{A} + t\delta\mathcal{A}]\bigg|_{t=0} + {m_G}^2 \int_M \langle (g^*\Theta)_\mu, \delta\mathcal{A}^\mu \rangle_{\mathfrak{g}} \mathrm{d}V. \tag{6.19}$$

Due to gauge invariance, the added term in the variation should not influence the physics, and therefore must be zero, implying that either $m_G = 0$ or $g^*\Theta = 0$. The last equality would imply that $\Theta = 0$, which is impossible for higher-than-zero-dimensional Lie groups. Therefore, we find that $m_G = 0$: the gauge field has no mass. We conclude that the gauge Lagrangian is given by

$$\mathcal{L}_G = \mathcal{L}_{YM}. \tag{6.20}$$

In the early years of particle physics, this result was actually extremely problematic. Some gauge bosons, such as the photon, are indeed massless. However, other gauge bosons, specifically the W and Z particles, are actually massive. In fact, they are some of the heaviest particles in the standard model! Eventually, the issue was fixed through the introduction of symmetry breaking and the Higgs field, which is a scalar field. The particle associated to the field, the Higgs boson, was finally discovered in 2012 at the Large Hadron Collider at CERN[14], "solving" the problem.
The Higgs field falls outside the scope of this thesis, and will therefore not be covered. However, the interested reader may refer to [11] for more information.

6.2.2 The Yang-Mills equation

We would now like to find a local equation for the gauge field that ensures that the action is stationary. To this end, we vary the gauge field. Take some $\mathfrak{g}$-valued 1-form ω and $t \in \mathbb{R}$, and notice that

$$\mathcal{F}[\mathcal{A} + t\omega] = \mathcal{F}[\mathcal{A}] + t(\mathrm{d}_{\mathcal{A}}\omega) + \frac{1}{2}t^2[\omega, \omega]. \tag{6.21}$$

Implementing this in the Yang-Mills action gives us

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} S_{YM}[\mathcal{A} + t\delta\mathcal{A}]\bigg|_{t=0} = -\int_M \langle (\mathrm{d}_{\mathcal{A}}\delta\mathcal{A} + t[\delta\mathcal{A}, \delta\mathcal{A}])_{\mu\nu}, \mathcal{F}^{\mu\nu}[\mathcal{A} + t\delta\mathcal{A}] \rangle_{\mathfrak{g}} \mathrm{d}V \bigg|_{t=0} \tag{6.22}$$

$$= -\int_M \langle (\mathrm{d}_{\mathcal{A}}\delta\mathcal{A})_{\mu\nu}, \mathcal{F}^{\mu\nu}[\mathcal{A}] \rangle_{\mathfrak{g}} \mathrm{d}V \tag{6.23}$$

$$= -\int_M \langle (\delta\mathcal{A})_{\mu\nu}, ({\mathrm{d}_{\mathcal{A}}}^*\mathcal{F})^{\mu\nu}[\mathcal{A}] \rangle_{\mathfrak{g}} \mathrm{d}V, \tag{6.24}$$

where ${\mathrm{d}_{\mathcal{A}}}^*$ is defined uniquely such that $\int_M \langle \mathrm{d}_{\mathcal{A}} A, B \rangle \mathrm{d}V = \int_M \langle A, {\mathrm{d}_{\mathcal{A}}}^* B \rangle \mathrm{d}V$. Since $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is positive-degenerate, this gives us that

$${\mathrm{d}_{\mathcal{A}}}^*\mathcal{F} = 0. \tag{6.25}$$

It actually turns out that $\mathrm{d}_{\mathcal{A}}{}^* = (-1)^{n(k-1)+1+p} \star \mathrm{d}_{\mathcal{A}} \star$[11], similarly to the codifferential, and so we get the **Yang-Mills equation**

$$\mathrm{d}_{\mathcal{A}} \star \mathcal{F} = 0. \tag{6.26}$$

In the case of the $U(1)$ gauge group, this equation exactly gives us Gauss's law of electric fields and Ampère's law in a vacuum, as

$$(\star F)_{\mu\nu} = \sqrt{|g|} \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & \frac{E_3}{c} & -\frac{E_2}{c} \\ -B_2 & -\frac{E_3}{c} & 0 & \frac{E_1}{c} \\ -B_3 & \frac{E_2}{c} & -\frac{E_1}{c} & 0 \end{pmatrix}, \tag{6.27}$$

and we find

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mathbf{0}, \qquad \nabla \cdot \mathbf{E} = 0. \tag{6.28}$$

6.3 Lagrangian mechanics in gauge theories: the Dirac Lagrangian

As for the Lagrangian for the spinor, we once again take inspiration from an existing Lagrangian. For fermions in four-dimensional Minkowski space, the Dirac Lagrangian is given by

$$\mathcal{L} = \hbar c \overline{\psi} \gamma^\mu \partial_\mu \psi - m_F c^2 \overline{\psi} \psi \tag{6.29}$$

for some fermion field ψ with mass m_F. In Minkowski space, have the trivial tangent bundle

$$\mathbb{R}^4 \longrightarrow \mathbb{R}^4 \times \mathbb{R}^4 \xrightarrow{\pi} \mathbb{R}^4. \tag{6.30}$$

On this, we define the function

$$\nabla : \mathfrak{X}(\mathbb{R}^4) \times \mathfrak{X}(\mathbb{R}^4) \to \mathfrak{X}(\mathbb{R}^4), \; \nabla_X Y(x) = \lim_{t \to 0} \frac{Y(x + tX(x)) - Y(x)}{t}. \tag{6.31}$$

Lemma 6.3. *The function ∇ as defined above is the Levi-Civita connection.*

Proof. Firstly, we have to show that ∇ is even a connection. From any course on analysis in several variables, we know that the function is bilinear. Moreover, for any $f : \mathbb{R}^4 \to \mathbb{R}$, it is trivial that

$$\nabla_X(fY) = f \nabla_X Y,$$

and, under the assumption that $f(x) \neq 0$ (otherwise, it would be trivial),

$$\begin{aligned} \nabla_{fX} Y(x) &= \lim_{t \to 0} \frac{Y(x + tf(x)X(x)) - Y(x)}{t} \\ &= f(x) \lim_{f(x)t \to 0} \frac{Y(x + tf(x)X(x)) - Y(x)}{f(x)t} \\ &= f(x) \nabla_X Y(x). \end{aligned}$$

Thus, ∇ is indeed a connection.
As for the rest of the properties of the Levi-Civita connection, we have

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

by definition, and thus, the connection is torsion-free. Finally,

$$\begin{aligned} L_X(g(Y,Z)) &= \lim_{t\to 0} \frac{g(Y,Z)(x+X(x)t) - g(Y,Z)(x)}{t} \\ &= \lim_{t\to 0} \left[\frac{g(Y(x+X(x)t), Z(x)) - g(Y(x), Z(x))}{t} \right. \\ &\qquad \left. + \frac{g(Y(x), Z(x+X(x)t)) - g(Y(x), Z(x))}{t} \right] \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \end{aligned}$$

We therefore find that ∇ is compatible with the metric, and hence, it is, indeed, the Levi-Civita connection. □

Now, the induced connection form is given by

$$\omega_\nabla X(x) = \frac{\nabla X(x + X_x t)}{\mathrm{d}t}(0) = \nabla_{X_x} X(x). \tag{6.32}$$

From there, we can find the induced covariant derivative on the spinor bundle

$$\nabla_{e_\mu}(s, \varphi)(x) = (s(x), \partial_\mu \varphi(x)). \tag{6.33}$$

Actually, in flat space, we can simply ignore the section s, as we may choose a global basis, and let s be the identity at every point. Thus, if we take some vector field $\psi \in \mathbb{C}^4$ over M,

$$\nabla_{e_\mu}\psi = \partial_\mu \psi. \tag{6.34}$$

Therefore, we can find the Dirac operator in this case to be

$$\not{D}\psi = g^{\mu\nu} e_\mu \cdot \partial_\nu \psi = \gamma^\mu \partial_\mu \psi. \tag{6.35}$$

The Dirac Lagrangian is therefore

$$\mathcal{L} = \hbar c \overline{\psi} \not{D} \psi - m_F c^2 \overline{\psi}\psi. \tag{6.36}$$

This is easily generalized to any (twisted) spinor bundle.

Definition 6.4. *Let* $\Delta_n \longrightarrow S(M) \xrightarrow{\pi} M$ *be some spinor bundle. Then the* ***Dirac Lagrangian*** *is given by*

$$\mathcal{L}_D[\psi] = \hbar c \overline{\psi} \not{D} \psi - m_F c^2 \overline{\psi}\psi. \tag{6.37}$$

If, instead, we are working with some twister spinor bundle

$$\Delta_n \otimes \mathbb{C}^k \longrightarrow S(M) \otimes (P \times \mathbb{C}^k)/G \xrightarrow{\pi} M$$

with some connection form ω on the principal bundle P, then the Dirac Lagrangian is

$$\mathcal{L}_D[\psi,\omega] = \hbar c\overline{\psi}\not{D}^\omega\psi - m_F c^2\overline{\psi}\psi. \tag{6.38}$$

Of course, we would like our Lagrangian to be gauge-invariant. This is trivially true, since the twisted Dirac operator does not depend on our choice of section.
Combining these two Lagrangians gives us our total Lagrangian.

Definition 6.5. *The* ***Yang-Mills-Dirac Lagrangian*** *is given by*

$$\mathcal{L}_{YMD}[\psi,\omega] = \mathcal{L}_{YM}[\omega] + \mathcal{L}_D[\psi,\omega] = -\frac{1}{2}\langle\mathcal{F}_{\mu\nu},\mathcal{F}^{\mu\nu}\rangle_{\mathfrak{g}} + \hbar c\overline{\psi}\not{D}^\omega\psi - m_F c^2\overline{\psi}\psi. \tag{6.39}$$

This Lagrangian is the same Lagrangian that is used in the standard model, save the missing Higgs terms.

6.4 The standard model in 4-dimensional Lorentzian spacetime

Now, finally, we can use all our knowledge to construct the (Higgs-less) standard model. Our first "assumption" is that the universe is a spacetime-orientable 4-dimensional Lorentzian manifold M, which is guaranteed by causality. Furthermore, we assume that the three fundamental forces (excluding gravity) are described by the gauge groups $U(1)$, $SU(2)$ and $SU(3)$ for the electromagnetic, weak and strong force, respectively. We'll combine these three gauge groups into a single Lie group $G := U(1)\times SU(2)\times SU(3)$. We then get two bundles: the "gauge bundle", which is a principal bundle

$$U(1)\times SU(2)\times SU(3) \longrightarrow P \xrightarrow{\pi_G} M \tag{6.40}$$

and the spinor bundle, which is the vector bundle

$$\mathbb{C}^{24} \longrightarrow S(M) \xrightarrow{\pi_S} M. \tag{6.41}$$

To now define the Lagrangian, we would like to find an Ad-invariant positive-definite symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$.

Lemma 6.6. *On G, a sufficient bilinear form would be*

$$\langle A,B\rangle_{\mathfrak{g}} := \mathrm{Tr}(A^*B). \tag{6.42}$$

Proof. Clearly, the function is bilinear, and it is symmetric since all elements of $\mathfrak{g}$ are anti-Hermitian, and therefore, $A^*B = AB^*$, and therefore $\mathrm{Tr}(A^*B) = \mathrm{Tr}(B^*A)$. To finish this proof, we only need to show that the function is positive-definite and Ad-invariant. Note that, for $A\in\mathfrak{g}$, it must have 6 imaginary eigenvalues λ_i, and therefore,

$$\langle A,A\rangle_{\mathfrak{g}} = \mathrm{Tr}(A^*A) = -\sum_{i=1}^{6}(\lambda_i)^2 \geq 0.$$

If now, $\langle A, A\rangle_{\mathfrak{g}} = 0$, we would have to require that all of the eigenvalues of A must be zero, and therefore, $A = 0$. We conclude that $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is indeed positive determinate.
For $X \in G$, notice that it is unitary, and therefore,

$$\begin{aligned} \langle \mathrm{Ad}_X A, \mathrm{Ad}_X B\rangle_{\mathfrak{g}} &= \mathrm{Tr}((XAX^{-1})^*(XBX^{-1})) \\ &= \mathrm{Tr}(XA^*X^{-1}XBX^{-1}) \\ &= \mathrm{Tr}(XA^*BX^{-1}) = \mathrm{Tr}(A^*B). \end{aligned}$$

This proves Ad-invariance, and therefore the lemma. □

The Yang-Mills-Dirac Lagrangian for the standard model is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}((\mathcal{F}_{\mu\nu})^*\mathcal{F}^{\mu\nu}) + \hbar c\overline{\psi}\not{D}^{\omega}\psi - m_F c^2\overline{\psi}\psi. \tag{6.43}$$

We will not go into much more detail of the standard model in this thesis, since it requires the Higgs mechanism and knowledge about quantum field theory. Nonetheless, I believe the Lagrangian is quite a satisfying result to end this chapter with.

7 Beyond the standard model: grand unified theory

Of course, physics does not end at the standard model. The standard model is not complete, since it cannot explain some important phenomena, such as gravity or dark matter. Modern theoretical physicists are attempting to improve upon the standard model, in order to provide a more complete physical theory. A theory that is complete, i.e., explains every phenomena that the standard model is missing, is called a **theory of everything**. The most famous (potential) example of such a theory is string theory.
Of course, going into string theory in the last chapter of this thesis is not a realistic process. Instead, we look at a small improvement upon the standard model: grand unified theories.

The three fundamental forces of nature that we can describe with the standard model are inherently "separated". They are all described by their own "coupling constant", which determines the strength of forces during an interaction. Ideally, however, we would be able to describe particle physics using only one unified force, with only one coupling constant such that, at high energies, the three forces become one. To achieve this, we can use a representation to "include" the standard model's structure group into a new Lie group that only allows one coupling constant.

Definition 7.1. *A* ***grand unified theory*** *(or GUT) is a gauge theory with a principal bundle* $G \longrightarrow P \xrightarrow{\pi} M$ *such that:*

1. *G is a simple Lie group (since this ensures there is only one coupling constant);*
2. *There exists a representation $\rho : U(1) \times SU(2) \times SU(3) \to G$ such that $\rho(U(1) \times SU(2) \times SU(3))$ is isomorphic to some finite quotient of $U(1) \times SU(2) \times SU(3)$.*

Since finite groups are always 0-dimensional Lie groups, finite quotients of $U(1) \times SU(2) \times SU(3)$ will produce the same Lie algebra as $U(1) \times SU(2) \times SU(3)$ itself. Since the Lie algebra is integral to gauge theory, this is a necessary condition to "conserve" the workings of the standard model in our representation.
Through the process of "symmetry breaking" (the details of which are outside the scope of this thesis), the grand unified theory groups can "break" into some smaller subgroup. In our case, this subgroup is usually $U(1) \times SU(2) \times SU(3)$, although it is allowed for a group to break into some other group that may subsequently break into the standard model group in a "cascade" of breaks.

Since a valid grand unified theory must be able to break into the standard model, it must have representations that agree with the representations of the standard model corresponding to the defined fermion species. However, as stated earlier, we will not go into detail regarding the representations of the standard model.

For the rest of this chapter, we consider some specific examples of grand unified theories. For a more general and detailed approach, I highly recommend chapter 3 of the paper by Langacker[15].

7.1 The Georgi–Glashow model

The Georgi–Glashow model is the simplest example of a grand unified theory. It is based on the structure group $SU(5)$, which is, in fact, compact, and so it allows us to define a Yang-Mills Lagrangian upon the bundle.
As the name implies, this grand unified theory was discovered by Howard Georgi and Sheldon Glashow in 1974[16].
Actually, all compact simple Lie groups can be classified in a certain amount of ways (see theorem 2.4.23 in [11]), and in this classification, $SU(5)$ is known as A_4.

Lemma 7.2. *There exists some representation*

$$\rho: \ U(1) \times SU(2) \times SU(3) \to GL(\mathbb{C}^5) \tag{7.1}$$

such that

$$\rho(U(1) \times SU(2) \times SU(3)) \subseteq SU(5) \tag{7.2}$$

and

$$\rho(U(1) \times SU(2) \times SU(3)) \cong U(1) \times SU(2) \times SU(3)/(\mathbb{Z}/6\mathbb{Z}). \tag{7.3}$$

Proof. Any element A of $U(1) \times SU(2) \times SU(3)$ can, clearly, be written as $A = (A_E, A_W, A_S)$, where we treat A_E as a complex number. We then define the mapping

$$\rho(A) = \begin{pmatrix} (A_E)^3 A_W & 0 \\ 0 & (A_E)^{-2} A_S \end{pmatrix},$$

which is, trivially, a group homomorphism, and therefore a representation. Notice that

$$\det \rho(A) = (A_E)^6 \det A_W (A_E)^{-6} \det A_S = 1,$$

which, indeed, lets us conclude that $\rho(U(1) \times SU(2) \times SU(3)) \subseteq SU(5)$. Finally, assume that $(A_E, A_W, A_S) \in \ker \rho$. Then we know that $A_W = (A_E)^{-3} I_2$ and $A_S = (A_E)^2 I_3$. Furthermore,

$$1 = (\det A_W)^{-1} = \det A_S = (A_E)^6,$$

and thus,

$$\ker \rho = \{(A_E, (A_E)^{-3} I_2, (A_E)^2 I_3) \mid (A_E)^6 = 1\} \cong \{\exp(k\pi/3) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}/6\mathbb{Z}.$$

Hence,

$$\rho(U(1) \times SU(2) \times SU(3)) \cong U(1) \times SU(2) \times SU(3)/(\mathbb{Z}/6\mathbb{Z}).$$

□

Note that $SU(5)$ is 24-dimensional, while the gauge group for the standard model was only 12-dimensional, and so the gauge field gains 12 extra dimensions. These new 12 generators of the gauge group relate aspects of fermions that were previously unrelated. For example, they turn quarks into leptons and relate quark flavour with colour.[15]
One of the aspects of the Georgi-Glashow model (and many other grand unified theories) is that it predicts the existence of an 't Hooft-Polyakov magnetic monopole with mass on the

order of TeV[17], which is much more massive than all other particles in the standard model. However, we have never been able to detect a magnetic monopole, let alone such a massive one.
Another aspect of Georgi-Glashow, and indeed, other grand unified theories, is that it predicts a decay process for protons

$$\mathrm{p} \to \mathrm{e}^+ + \pi^0 \tag{7.4}$$

with a lifetime of around 10^{30} years[15]. The standard model forbids this reaction, as it violates baryon number, and therefore, we should not be able to observe this reaction. However, experimentally, the lifetime of 10^{30} years is too short not to be observed, which rules out the Georgi-Glashow model.

7.2 $SO(10)$

Another often-used grand unified theory is $SO(10)$. Its name is quite misleading, however, since the actual gauge group that is considered is Spin(10), the double cover of $SO(10)$. However, it is more justified when you consider that the two groups share a Lie Algebra $\mathfrak{so}(10)$. Within the compact simple Lie group classification, as shortly discussed in the previous section, Spin(10) is classified as D_5.
Sadly, $SO(10)$ has no non-mathematical name, like the Georgi-Glashow model does. However, it was discovered by Fritzsch and Minkowski in 1975[18], and independently by Howard Georgi himself mere hours before discovering the Georgi-Glashow model.
To show that $SO(10)$ is indeed a grand unified theory, we use the fact that we already know some representation

$$U(1) \times SU(2) \times SU(3) \to SU(5). \tag{7.5}$$

Lemma 7.3. *There exists some representation*

$$\rho : \ SU(5) \to GL(\mathbb{R}^{10}) \tag{7.6}$$

such that

$$\rho(SU(5)) \subseteq \mathrm{Spin}(10) \tag{7.7}$$

and

$$\rho(SU(5)) \cong SU(5). \tag{7.8}$$

Proof. For any $A \in SU(5)$, there must exist some unique $A_R, A_I \in GL(\mathbb{R}^5)$ such that

$$A = A_R + iA_I.$$

We then naturally define the (homomorphic) embedding

$$\rho : \ SU(5) \hookrightarrow GL(\mathbb{R}^{10}), \ A \mapsto \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix}.$$

Being an embedding, this automatically implies that

$$\rho(SU(5)) \cong SU(5).$$

Furthermore,

$$\rho(A)^T\rho(A) = \begin{pmatrix} {A_R}^T & {A_I}^T \\ -{A_I}^T & {A_R}^T \end{pmatrix} \begin{pmatrix} A_R & -A_I \\ A_I & A_R \end{pmatrix} = \rho(A^*)\rho(A) = \rho(A^*A) = I_{10},$$

and we find similar results for $\rho(A)\rho(A)^T$. Hence, $\rho(SU(5)) \subseteq O(10)$. Finally, note that $SU(5)$ is a connected Lie group, and since ρ is (trivially) continuous, we should find, also, that $\rho(SU(5))$ is connected. As we have found in appendix A.2, the identity connected component of $O(10)$ is exactly $SO(10)$, and therefore, we may conclude that

$$\rho(SU(5)) \subseteq SO(10) = \mathrm{Spin}(10)/\mathbb{Z}_2 \subseteq \mathrm{Spin}(10).$$

This proves the lemma. □

Since not only the standard model is contained within $SO(10)$, but also the Georgi-Glashow model (and, in fact, another grand unified theory called the Pati-Salam model[19]), we can have a multi-staged "cascade" of symmetry breaks

$$SO(10) \to SU(5) \to U(1) \times SU(2) \times SU(3). \tag{7.9}$$

Within the classification of simple Lie groups, there actually is another group we can add to this cascade, being some Lie group that corresponds to E_6, which is a 156-dimensional Lie group.

Now, Spin(10) is 45-dimensional, which tells us that, in total, we have 33 extra generators in $SO(10)$ than we had in the standard model. Again, these represent interactions between previously unrelated fermion properties.

The proton decay, as mentioned in the previous section, has a lifetime on the order of 10^{33} years[20], which is a much more realistic time, and even experimentally viable. This therefore gives us a more viable grand unified theory than the Georgi-Glashow model.

Conclusion

In this thesis, we set out to develop a mathematical foundation for constructing gauge theories, in order to better understand the mathematics that underlies modern physics, and even look beyond the current constraints of physics by virtue of generalising the mathematical methods of current physics.

Firstly, we developed tensor fields in 2, which proved to be an important mathematical basis for performing various calculations on manifolds. In fact, we defined a very important tensor field, the metric tensor, which would later be prove to be essential, such as for integrating over the base space manifold.

In chapter 3, we introduced the reader to Lie group and Lie algebras. Furthermore, we defined fibre bundles, which would later be shown to form the foundation of gauge theories. We in particular looked at principal bundles and vector bundles, and even found ways of turning them into one another through frame bundles and associated bundles. Most importantly, though, we showed that a connection on a principal bundle can be equivalently defined through a so-called connection form.

Equipped with the connection form, we returned to the base space in chapter 4 by pulling the form back using any section. This gave us the so-called gauge potential, which proved fundamental to describing gauge theories. We also defined the field strength, and found that the Bianchi identity posed a restriction on the field strength. We furthermore applied our new theory to electromagnetism by choosing $U(1)$ as our structure group, identifying the gauge potential with the electromagnetic potentials, which gave us two out of four Maxwell equations.

After introducing the gauge potential, we moved on to describing fermions using spinors in chapter 5. In order to do this, we started by discussing Clifford algebras, and, specifically, the spin group. Using a representation, we managed to define the spinor bundle, sections on which were called spinors. We also defined Dirac forms and the Dirac operator, which would later be necessary for the Lagrangian. Moreover, through defining the tensor product of vector bundles, we were able to twist the spinor bundle, in order to have it interact with the gauge potential.

We introduced a Lagrangian for gauge theories in chapter 6, which allows us to derive more laws from our theory. For the gauge potential, we defined the Yang-Mills Lagrangian, and for spinors, we had the Dirac Lagrangian. We combined these two into a Yang-Mills-Dirac Lagrangian, which accounted for the gauge potential, the spinors, and the interactions therebetween. In particular, we derived the Yang-Mills equation for a spinor-less gauge theory, which, in the case of electromagnetism, gives us the last two Maxwell equations. The Yang-Mills-Dirac Lagrangian is valid for any gauge theory with a compact structure group.

Finally, we discussed a way of looking beyond the standard model in chapter 7. We looked at the general idea of grand unified theories, and specifically looked at the two most famous grand unified theories, being the Georgi-Glashow model and $SO(10)$. We also looked at some flaws in these grand unified theories, specifically a decay process of protons that should, experimentally, not be possible at all. This rules out the Georgi-Glashow model, since the proton lifetime is much too short.

There is much more to be discovered about gauge theories than what was covered in this thesis. For example, the standard model also requires the Higgs mechanism, which gives the weak force's bosons their mass and describes the "rules" of symmetry breaking as introduced in chapter 7. There are also many other ways of going beyond the standard model than just grand unified theories, such as theories of everything. I hope this thesis has granted the reader a strong mathematical foundation for studying modern physical theories using rigorous, mathematical methods.

A Some long proofs

A.1 Theorem 4.2

Theorem 4.2. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle, and* $\{(U_i, \varphi_i)\}$ *some* G*-atlas of* M *for* $i \in I$.
Define for each $i \in I$ *the* ***canonical section*** s_i *such that*

$$s_i(x) = {\varphi_i}^{-1}(x, e) \qquad \text{for all } x \in U_i. \tag{A.1}$$

Now, let $\mathcal{A}_i$ *be a* $\mathfrak{g}$*-valued* 1*-form on* U_i *for all* i. *Assume that* $U_i \cap U_j \neq \emptyset$ *for some* i, j. *If, for all such* i, j, *we have that*

$$\mathcal{A}_j = \mathrm{Ad}_{t_{ji}} \circ \mathcal{A}_i + {t_{ij}}^* \Theta, \tag{A.2}$$

where Θ *is the* ***Cartan 1-form*** *on* G, *defined by*

$$\Theta_g : \ T_g G \to T_e G = \mathfrak{g}, \ X_g \mapsto (\mathrm{d}L_{g^{-1}})_g X_g, \tag{A.3}$$

then there is a unique connection 1*-form* ω *on* P *such that*

$$\mathcal{A}_i = {s_i}^* \omega. \tag{A.4}$$

In particular, these gauge potentials are then compatible with one another.

For the proof of the theorem, we follow the structure suggested in [21, pp. 332–333], by proving some lemmas, then combining them to find the theorem as a result.

Lemma A.1. *Let* $G \longrightarrow P \xrightarrow{\pi} M$ *be some principal bundle, and let* (U, φ) *be any local trivialization, with canonical section* $s(x) = \varphi^{-1}(x, e)$. *Then, for any* $x \in U$ *and* $(X_x, A) \in T_x(U) \times \mathfrak{g}$, *we have*

$$(\mathrm{d}\varphi^{-1})_{(x,e)}(X_x, A) = (\mathrm{d}s)_x X_x + {A^{\#}}_{s(x)}. \tag{A.5}$$

Proof. The function $\varphi^{-1}(x', g)$ depends on two variables, so to calculate its total differential, we first split it into two functions, which are restricted to each variable.

$$\varphi^{-1}(x', e) = s(x'),$$

$$\varphi^{-1}(x, g) = s(x)g = \sigma_{s(x)} g.$$

Therefore,

$$(\mathrm{d}\varphi^{-1})_{(x,e)}(X_x, A) = (\mathrm{d}s)_x X_x + (\mathrm{d}\sigma_{s(x)})_e A = (\mathrm{d}s)_x X_x + {A^{\#}}_{s(x)}.$$

□

Corollary A.2. *In particular, every vector* $Y_{s(x)}$ *in* $T_{s(x)}\pi^{-1}(U)$ *can be (uniquely) written like*

$$(\mathrm{d}s)_x X_x + {A^{\#}}_{s(x)}$$

for $X_x = (\mathrm{d}\pi)_{s(x)} Y_{s(x)}$ *and some* $A \in \mathfrak{g}$.

Now, choose some G-atlas of M with corresponding canonical sections as in theorem 4.2.

Lemma A.3. *For any i, define ω_i on $\pi^{-1}(U_i)$ such that*

$$(\omega_i)_{s_i(x)}((\mathrm{d}s_i)_x X_x + A^{\#}{}_{s_i(x)}) = (\mathcal{A}_i)_x X_x + A, \tag{A.6a}$$

$$(\omega_i)_{s_i(x)g}(Y_{s_i(x)g}) = \mathrm{Ad}_{g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{g^{-1}})_{s_i(x)g} X_{s_i(x)g}). \tag{A.6b}$$

This defines a local connection 1-form on $\pi^{-1}(U_i)$.

Proof. We check that this ω_i is well-defined and defined over the entire domain. Firstly, corollary A.2 shows that the entire domain of $(\omega_i)_{s_i(x)}$ can be written (uniquely) as we did in equation (A.6a), so it covers its whole domain, and is well-defined. Moreover, due to the regularity of the action of G on P, we know that for any $p \in \pi^{-1}(U_i)$, there must be some unique $x \in U_i$ and $g \in G$ such that $p = s_i(x)g$, which shows that ω_i is defined and well-defined over the entire domain.
Now, we can trivially see that $(\omega_i)_p$ is linear and maps to $\mathfrak{g}$, so it must be a $\mathfrak{g}$-valued 1-form on $\pi^{-1}(U_i)$. We now refer to definition 3.17 to finalize the proof.

(i) Take any $s_i(x)g \in \pi^{-1}(U_i)$, and some $A \in \mathfrak{g}$. Then we have

$$\begin{aligned}
(\omega_i)_{s_i(x)g}(A^{\#}{}_{s_i(x)g}) &= \mathrm{Ad}_{g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{g^{-1}})_{s_i(x)g} A^{\#}{}_{s_i(x)g}) \\
&= \mathrm{Ad}_{g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{g^{-1}})_{s_i(x)g} (\mathrm{d}\sigma_{s_i(x)g})_e A) \\
&= \mathrm{Ad}_{g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{d}\sigma_{s_i(x)})_e (\mathrm{d}L_g)_{g^{-1}} (\mathrm{d}R_{g^{-1}})_e A) \\
&= \mathrm{Ad}_{g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{d}\sigma_{s_i(x)})_e \mathrm{Ad}_g A) \\
&= \mathrm{Ad}_{g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{Ad}_g A)^{\#}{}_{s_i(x)}) \\
&= \mathrm{Ad}_{g^{-1}} \circ \mathrm{Ad}_g A = A.
\end{aligned}$$

(ii) Take any $s_i(x)g \in \pi^{-1}(U_i)$, some $X_{s_i(x)g} \in T_{s_i(x)g}M$ and any $h \in G$. Then

$$\begin{aligned}
({R_h}^* \omega_i)_{s_i(x)g} X_{s_i(x)g} &= (\omega_i)_{s_i(x)gh}((\mathrm{d}R_h)_{s_i(x)g} X_{s_i(x)g}) \\
&= \mathrm{Ad}_{h^{-1}g^{-1}} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{h^{-1}g^{-1}})_{s_i(x)gh} (\mathrm{d}R_h)_{s_i(x)g} X_{s_i(x)g}) \\
&= \mathrm{Ad}_{h^{-1}} \circ \mathrm{Ad}_g^{-1} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{g^{-1}})_{s_i(x)g} X_{s_i(x)g}) \\
&= \mathrm{Ad}_{h^{-1}} \circ (\omega_i)_{s_i(x)g}(X_{s_i(x)g}).
\end{aligned}$$

And thus, ${R_h}^* \omega_i = \mathrm{Ad}_{h^{-1}} \circ \omega_i$.

Thus, this ω_i does indeed define a connection 1-form on $\pi^{-1}(U_i)$. □

Lemma A.4. *Let Θ be the Cartan 1-form as defined before on some manifold P with a Lie group G that acts upon it. Take any $p \in P$ and $g \in G$, then the following formula holds for $X_g \in T_g G$:*

$$(\mathrm{d}\sigma_p)_g X_g = (\Theta_g X_g)^{\#}{}_{pg}. \tag{A.7}$$

Proof.

$$\begin{aligned}(\mathrm{d}\sigma_p)_g X_g &= (\mathrm{d}\sigma_p)_g (\mathrm{d}L_g)_e (\mathrm{d}L_{g^{-1}})_g X_g \\ &= (\mathrm{d}\sigma_{pg})_e (\mathrm{d}L_{g^{-1}})_g X_g \\ &= (\mathrm{d}\sigma_{pg})_e \Theta_g X_g \\ &= (\Theta_g X_g)^{\#}{}_{pg}.\end{aligned}$$

□

Lemma A.5. *Let (U_i, φ_i) and (U_j, φ_j) be any two local trivializations of a principal bundle $G \longrightarrow P \xrightarrow{\pi} M$. Take any $x \in U_i \cap U_j$, and some $X_x \in T_x M$. Then we have*

$$(\mathrm{d}s_j)_x X_x = (\mathrm{d}R_{t_{ij}(x)})_{s_i(x)} (\mathrm{d}s_i)_x X_x + (({t_{ij}}^*\Theta)_x X_x)^{\#}{}_{s_j(x)}. \tag{A.8}$$

Proof. Firstly, we define the mapping

$$\Phi : \ P \times G \to P, \ (p, g) \mapsto pg,$$

and we see that

$$\begin{aligned}(\mathrm{d}\Phi)_{p,g}(X_p, Y_g) &= (\mathrm{d}\Phi)_{p,g}(X_p, 0) + (\mathrm{d}\Phi)_{p,g}(0, Y_g) \\ &= (\mathrm{d}R_g)_p X_p + (\mathrm{d}\sigma_p)_g Y_g \\ &= (\mathrm{d}R_g)_p X_p + (\Theta_g Y_g)^{\#}{}_{pg}.\end{aligned}$$

Then, using $s_j(x) = s_i(x) t_{ij}(x)$,

$$\begin{aligned}(\mathrm{d}s_j)_x X_x &= (\mathrm{d}s_i t_{ij})_x X_x \\ &= (\mathrm{d}\Phi(s_i, t_{ij}))_x X_x \\ &= (\mathrm{d}\Phi)_{s_i(x), t_{ij}(x)}((\mathrm{d}s_i)_x X_x, (\mathrm{d}t_{ij})_x X_x) \\ &= (\mathrm{d}R_{t_{ij}(x)})_{s_i(x)} (\mathrm{d}s_i)_x X_x + (\Theta_{t_{ij}(x)} (\mathrm{d}t_{ij})_x X_x)^{\#}{}_{s_i(x) t_{ij}(x)} \\ &= (\mathrm{d}R_{t_{ij}(x)})_{s_i(x)} (\mathrm{d}s_i)_x X_x + (({t_{ij}}^*\Theta)_x X_x)^{\#}{}_{s_j(x)}.\end{aligned}$$

□

Lemma A.6. *Define ω_i as in lemma A.3. Then for any i, j such that $U_i \cap U_j \neq \emptyset$, we have that $\omega_i|_{\pi^{-1}(U_i \cap U_j)} = \omega_j|_{\pi^{-1}(U_i \cap U_j)}$.*

Proof. Firstly, assume that $\omega_i|_{s_j(U_i \cap U_j)} = \omega_j|_{s_j(U_i \cap U_j)}$. Then, for any $p \in \pi^{-1}(U_i \cap U_j)$ and $X_p \in T_p P$, there must be some $x, g \in U_i \cap U_j \times G$ such that $p = s_j(x) g$, and then

$$\begin{aligned}(\omega_i)_p(X_p) &= (\omega_i)_{s_i(x) t_{ij}(x) g}(X_p) \\ &= \mathrm{Ad}_{g^{-1}} \circ \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{g^{-1} t_{ji}(x)})_p X_p) \\ &= \mathrm{Ad}_{g^{-1}} \circ \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_j)_{s_i(x)}((\mathrm{d}R_{g^{-1} t_{ji}(x)})_p X_p) \\ &= \mathrm{Ad}_{g^{-1}} \circ \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_j)_{s_j(x) t_{ji}(x)}((\mathrm{d}R_{g^{-1} t_{ji}(x)})_p X_p) \\ &= \mathrm{Ad}_{g^{-1}} \circ (\omega_j)_{s_j(x)}((\mathrm{d}R_{g^{-1}})_p X_p) \\ &= (\omega_j)_p(X_p).\end{aligned}$$

Therefore, we only need to prove the assumption.
Take now any $x \in U_i \cap U_j$ and $Y_{s_j(x)} \in T_{s_j(x)}P$. As we did before, we split it like

$$Y_{s_j(x)} = (\mathrm{d}s_j)_x X_x + {A^\#}_{s_j(x)},$$

with $X_x \in T_xM$ and $A \in \mathfrak{g}$. Then

$$(\omega_i)_{s_j(x)}(Y_{s_j(x)}) = (\omega_i)_{s_j(x)}((\mathrm{d}s_j)_x X_x) + (\omega_i)_{s_j(x)}({A^\#}_{s_j(x)}).$$

We write out both components

$$\begin{aligned}(\omega_i)_{s_j(x)}((\mathrm{d}s_j)_x X_x) &= (\omega_i)_{s_j(x)}((\mathrm{d}R_{t_{ij}(x)})_{s_i(x)}(\mathrm{d}s_i)_x X_x + ((t_{ij}{}^*\Theta)_x X_x)^\#{}_{s_j(x)}) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}s_i)_x X_x + (\mathrm{d}R_{t_{ji}(x)})_{s_j(x)}((t_{ij}{}^*\Theta)_x X_x)^\#{}_{s_j(x)}).\end{aligned}$$

For any $B \in \mathfrak{g}$,

$$\begin{aligned}(\mathrm{d}R_{t_{ji}(x)})_{s_j(x)} {B^\#}_{s_j(x)} &= (\mathrm{d}R_{t_{ji}(x)})_{s_j(x)}(\mathrm{d}\sigma_{s_j(x)})_e B \\ &= (\mathrm{d}\sigma_{s_j(x)})_{t_{ji}(x)}(\mathrm{d}R_{t_{ji}(x)})_e B \\ &= (\mathrm{d}\sigma_{s_i(x)})_e(\mathrm{d}L_{t_{ij}(x)})_{t_{ji}(x)}(\mathrm{d}R_{t_{ji}(x)})_e B \\ &= (\mathrm{d}\sigma_{s_i(x)})_e(\mathrm{d}\Psi_{t_{ij}(x)})B \\ &= (\mathrm{Ad}_{t_{ij}(x)} \circ B)^\#{}_{s_i(x)},\end{aligned}$$

so,

$$\begin{aligned}(\omega_i)_{s_j(x)}((\mathrm{d}s_j)_x X_x) &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}s_i)_x X_x + (\mathrm{Ad}_{t_{ij}(x)} \circ (t_{ij}{}^*\Theta)_x X_x)^\#{}_{s_i(x)}) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ ((\mathcal{A}_i)_x X_x + \mathrm{Ad}_{t_{ij}(x)} \circ (t_{ij}{}^*\Theta)_x X_x) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ (\mathcal{A}_i)_x X_x + (t_{ij}{}^*\Theta)_x X_x \\ &= (\mathcal{A}_j)_x X_x = (\omega_j)_{s_j(x)}((\mathrm{d}s_j)_x X_x).\end{aligned}$$

Furthermore,

$$\begin{aligned}(\omega_i)_{s_j(x)}({A^\#}_{s_j(x)}) &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{t_{ji}(x)})_{s_j(x)}{A^\#}_{s_j(x)}) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}R_{t_{ji}(x)})_{s_j(x)}(\mathrm{d}\sigma_{s_j(x)})_e A) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}\sigma_{s_j(x)})_{t_{ji}(x)}(\mathrm{d}R_{t_{ji}(x)})_e A) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{d}\sigma_{s_i(x)})_e(\mathrm{d}\Psi_{t_{ij}(x)})_e A) \\ &= \mathrm{Ad}_{t_{ji}(x)} \circ (\omega_i)_{s_i(x)}((\mathrm{Ad}_{t_{ij}(x)} \circ A)^\#{}_{s_i(x)}) \\ &= A = (\omega_j)_{s_j(x)}({A^\#}_{s_j(x)}).\end{aligned}$$

And thus we conclude

$$(\omega_i)_{s_j(x)}(Y_{s_j(x)}) = (\omega_j)_{s_j(x)}((\mathrm{d}s_j)_x X_x) + (\omega_j)_{s_j(x)}({A^\#}_{s_j(x)}) = (\omega_j)_{s_j(x)}(Y_{s_j(x)}).$$

And thus, indeed, $\omega_i|_{\pi^{-1}(U_i \cap U_j)} = \omega_j|_{\pi^{-1}(U_i \cap U_j)}$. $\square$

Now finally, we prove the theorem.

Proof of theorem 4.2. Define the connection 1-form ω on P such that $\omega|_{\pi^{-1}(U_i)} = \omega_i$. Through the last lemma, this is clearly well-defined. Furthermore,

$$({s_i}^*\omega)_x X_x = \omega_{s_i(x)}((\mathrm{d}s_i)_x X_x) = (\mathcal{A}_i)_x X_x,$$

and thus $(s_i)^*\omega = \mathcal{A}_i$. This leaves us to show that this connection 1-form is unique. Assume there exists another connection 1-form η such that ${s_i}^*\eta = \mathcal{A}_i$, and therefore

$$\omega_{s_i(x)}((\mathrm{d}s_i)_x X_x) = \eta_{s_i(x)}((\mathrm{d}s_i)_x X_x).$$

Since they are both connection 1-forms, we furthermore find that

$$\omega_{s_i(x)}({A^\#}_{s_i(x)}) = A = \eta_{s_i(x)}({A^\#}_{s_i(x)}),$$

and thus $\omega_{s_i(x)} = \eta_{s_i(x)}$. Now, for any $p \in P$, there exists some i and unique $x \in M$ and $g \in G$ such that $p = s_i(x)g$. Take some $Y_p \in T_pP$, and we find

$$\begin{aligned}
\omega_p(Y_p) &= \mathrm{Ad}_{g^{-1}} \circ \omega_{s_i(x)}((\mathrm{d}R_{g^{-1}})_p Y_p) \\
&= \mathrm{Ad}_{g^{-1}} \circ \eta_{s_i(x)}((\mathrm{d}R_{g^{-1}})_p Y_p) \\
&= ({R_g}^*\eta)_{s_i(x)}((\mathrm{d}R_{g^{-1}})_p Y_p) \\
&= \eta_{s_i(x)g}(Y_p).
\end{aligned}$$

And thus, $\omega = \eta$, and we conclude that this connection 1-form is uniquely defined. □

A.2 Theorem 5.13

Theorem 5.13. *We have the following two short exact sequences of Lie groups:*

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Pin}(n-p,p) \xrightarrow{\widetilde{Ad}} O(n-p,p) \longrightarrow 1, \tag{A.9}$$

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(n-p,p) \xrightarrow{\widetilde{Ad}} SO^0(n-p,p) \longrightarrow 1, \tag{A.10}$$

where $\mathbb{Z}_2 := \{-1, 1\}$ is a zero-dimensional Lie group under multiplication.
In particular, $\mathrm{Pin}(n-p,p)$ and $\mathrm{Spin}(n-p,p)$ form a double cover of $O(n-p,p)$ and $SO^0(n-p,p)$, respectively, and

$$\dim(\mathrm{Pin}(n-p,p)) = \dim(\mathrm{Spin}(n-p,p)) = n(n-1)/2. \tag{A.11}$$

We prove this theorem using a few lemmas. First, we shall show that the first short sequence is exact. Then, we find an explicit evaluation of $SO^0(n-p,p)$, which we use to prove the second short exact sequence. However, perhaps most importantly, we want to know that $SO^0(n-p,p)$ is a Lie group.

Lemma A.7. *Let G be any Lie group of dimension n, and denote by G^0 the largest connected subset of G that contains the identity element. Then G^0 is also a Lie group of dimension n.*

Proof. To prove this, we only need to show that G^0 is a submanifold of G, and that the subgroup criterion holds.
Any point in G^0 has some open neighbourhood in G that is diffeomorphic to $\mathbb{R}^n$. In particular, that open neighbourhood is connected, and therefore, G^0 is the union of all such neighbourhoods. This implies that G^0 is open in G, and therefore an n-dimensional submanifold.
Trivially, G^0 is non-empty, since it must contain the identity. This leaves us to prove that $xy^{-1} \in G^0$ for all $x, y \in G^0$. If we define $f:\ G^0 \times G^0 \to G$ by $f(x,y) = xy^{-1}$, we note immediately that it is a continuous function. Therefore, $f(G^0 \times G^0)$ must be connected, and since it, trivially, contains the identity element, we may conclude that $f(G^0 \times G^0) \subseteq G^0$, and thus, the subgroup criterion is satisfied. We have therefore proven that G^0 is an n-dimensional Lie subgroup of G. $\square$

Lemma A.8. *The short sequence*

$$1 \longrightarrow \mathbb{Z}_2 \hookrightarrow \mathrm{Pin}(n-p,p) \xrightarrow{\widetilde{Ad}} O(n-p,p) \longrightarrow 1, \tag{A.12}$$

is exact.

Proof. To prove any short exact sequence

$$1 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 1,$$

with ϕ, ψ homomorphisms, we only have to show that ϕ is injective, ψ is surjective and $\ker\psi = \phi(A)$.

We first show that $\widetilde{Ad}$ maps $\mathrm{Pin}(n-p,p)$ to $O(n-p,p)$. Note, through rearranging and using that $\{v,w\} = 2g(v,w)$, that $\widetilde{Ad}_{v_1 \cdots v_k}(V) \subseteq V$ for $v_1 \cdots v_k \in \mathrm{Pin}(n-p,p)$, and that therefore $\widetilde{Ad}_{v_1 \cdots v_k}$ is an endomorphism on V.
Take now any $x \in \mathrm{Pin}(n-p,p)$, and notice that $\widetilde{Ad}_{x^{-1}} = \widetilde{Ad}_x^{-1}$, and hence, $\widetilde{Ad}_x$ is invertible, and therefore contained within $GL(V)$. Now, for any $v \in V$,

$$g(\widetilde{Ad}_x v, \widetilde{Ad}_x v) = \widetilde{Ad}_x v \widetilde{Ad}_x v = \alpha(x) v x^{-1} \alpha(x) v x^{-1} = x v v x^{-1} = x g(v,v) x^{-1} = g(v,v),$$

and thus, indeed, $\widetilde{Ad}_x \in O(n-p,p)$.

Now, we require the Cartan-Dieudonné theorem (see, for example, [9, p. 77]), which states that any orthogonal matrix can be represented as a product of at most n reflections. Therefore, every $A \in O(n-p,p)$ can be written like

$$A = \widetilde{Ad}_{v_1} \cdots \widetilde{Ad}_{v_k},$$

where $v_i \in V$ and $k \leq n$. Note that we can assume that $g(v_i, v_i) = 1$ without loss of generality, since we can replace each v_i by $\frac{1}{\sqrt{g(v_i,v_i)}} v_i$, while the twisted adjoint remains invariant.
Now, we know that

$$A = \widetilde{Ad}_{v_1} \cdots \widetilde{Ad}_{v_k} = \widetilde{Ad}_{v_1 \cdots v_k},$$

and that $v_1 \cdots v_k \in \mathrm{Pin}(n-p,p)$. Therefore, $\widetilde{Ad}$ maps surjectively from $\mathrm{Pin}(n-p,p)$ onto $O(n-p,p)$.

To prove the short exact sequence, we only need to show that the map $\mathbb{Z}_2 \to \mathrm{Pin}(n-p,p)$ is injective — which is trivial, since it is the inclusion map — and that $\ker \widetilde{Ad} = \mathbb{Z}_2$. Take any $v_1 \cdots v_k \in \mathrm{Pin}(n-p,p)$, and assume that

$$\widetilde{Ad}_{v_1 \cdots v_k} = I_n.$$

Now, take some orthonormal basis $e_1, \dots, e_n$ of V, and note that $v_1 \cdots v_k$ is some polynomial

$$v_1 \cdots v_k = \sum_{\substack{i_1 < \cdots < i_m,\\ m \leq k}} a_{i_1 \dots i_m} e_{i_1} \cdots e_{i_m}.$$

For any j, we can now separate this sum into the following:

$$v_1 \cdots v_k = \sum_{\substack{i_1 < \cdots < i_m,\\ m \leq k-1\\ i_l \neq j \text{ for all } l}} (a_{i_1 \dots i_m} + b_{i_1 \dots i_m} e_j) e_{i_1} \cdots e_{i_m} = a + b e_j,$$

where

$$a = \sum_{\substack{i_1 < \cdots < i_m,\\ m \leq k-1\\ i_l \neq j \text{ for all } l}} a_{i_1 \dots i_m} e_{i_1} \cdots e_{i_m}, \qquad b = \sum_{\substack{i_1 < \cdots < i_m,\\ m \leq k-1\\ i_l \neq j \text{ for all } l}} b_{i_1 \dots i_m} e_{i_1} \cdots e_{i_m}.$$

Our assumption then gives that, for all $v \in V$,

$$(-1)^k (a + b e_j) v (a + b e_j)^{-1} = \alpha(a + b e_j) v (a + b e_j)^{-1} = v,$$

and thus,

$$(-1)^k (a + b e_j) v = v (a + b e_j).$$

If we now choose $v = e_j$, we find

$$(-1)^k a + (-1)^k b = e_j a + e_j b e_j = (-1)^k a e_j + (-1)^{k-1} b.$$

Therefore, we may conclude that $b = 0$, and thus $v_1 \cdots v_k$ has no polynomial terms that are multiples of e_j for all j, and hence, in particular, it must be some scalar in $\mathbb{R}$. Now, $\mathbb{R} \cap \mathrm{Pin}(n-p,p) = \pm 1 = \mathbb{Z}_2$, which proves the short exact sequence. □

Lemma A.9. *Assume $p(n-p) > 0$. Any matrix $A \in O(n-p,p)$ under the basis as in theorem 2.8 is decomposed like*

$$A = \begin{pmatrix} A^{--} & A^{-+} \\ A^{+-} & A^{++} \end{pmatrix}. \tag{A.13}$$

Now, the mapping

$$\xi : \; O(n-p,p) \to (\mathbb{Z}_2)^2, \; A \mapsto (\mathrm{sign}(\det A^{--}), \mathrm{sign}(\det A^{++})) \tag{A.14}$$

is well-defined and continuous.
In particular, $O(n-p,p)$ can be separated into the following four subspaces that are pairwise disconnected:

$$O(n-p,p)^{\pm_1\pm_2} = \{A \in O(n-p,p) \mid \xi(A) = (\pm_1 1, \pm_2 1)\}, \tag{A.15}$$

where $\pm_1, \pm_2$ are independently either $+$ or $-$.

Of course, for the case $p(n-p) = 0$, $O(n-p,p)$ can instead be separated into two subspaces $O(n-p,p)^{\pm} = \{A \in O(n-p,p) \mid \det A = \pm 1\}$.

Proof. To prove the lemma, we only need to show that ξ is well-defined and continuous, for which it is sufficient to check that there is no $A \in O(n-p,p)$ such that $\det A^{++} = 0$ or $\det A^{--} = 0$.
Of course, in the basis of theorem 2.8, we know that

$$A^T G A = G,$$

where

$$G = \begin{pmatrix} -I_p & 0 \\ 0 & I_{n-p} \end{pmatrix}.$$

Explicitly, then,

$$\begin{pmatrix} -(A^{--})^T A^{--} + (A^{+-})^T A^{+-} & -(A^{--})^T A^{-+} + (A^{+-})^T A^{++} \\ -(A^{-+})^T A^{--} + (A^{++})^T A^{+-} & -(A^{-+})^T A^{-+} + (A^{++})^T A^{++} \end{pmatrix} = \begin{pmatrix} -I_p & 0 \\ 0 & I_{n-p} \end{pmatrix}.$$

This gives us, in particular, two equations

$$(A^{--})^T A^{--} = I_p + (A^{+-})^T A^{+-}, \quad (A^{++})^T A^{++} = I_{n-p} + (A^{-+})^T A^{-+}.$$

Take now any vector $\mathbf{v} \in \mathbb{R}^p \setminus \{0\}$, and note that

$$\mathbf{v}^T (A^{--})^T A^{--} \mathbf{v} = \mathbf{v}^T \mathbf{v} + (A^{+-}\mathbf{v})^T (A^{+-}\mathbf{v}) \geq \|\mathbf{v}\|^2 > 0.$$

It follows that $(A^{--})^T A^{--}$ is positive definite, and therefore

$$(\det A^{--})^2 = \det\Big((A^{--})^T A^{--}\Big) > 0,$$

which implies that A^{--} always has non-zero determinant. The proof is the same for A^{++}, proving the lemma. $\square$

Lemma A.10. *For $p(n-p) > 0$, the space $O(n-p,p)^{++}$ is path-connected.*
For $p(n-p) = 0$, the space $O(n-p,p)^{+} = SO(n-p,p)$ is path-connected.

Proof. We separate this proof into two parts.

$\boldsymbol{p(n-p) = 0.}$ This case is trivial, since $SO(n,0) = SO(0,n) = SO(n)$, which we already know to be path-connected.

$p(n-p) > 0$. Define the orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ such that $g(\mathbf{e}_i, \mathbf{e}_j) = -\delta_{ij}$ for $i \leq p$ and $g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$, otherwise.
Assume $p > 1$. Take any $\mathbf{v} \in \mathbb{R}^n$ with $g(\mathbf{v}, \mathbf{v}) = -1$ and any $A \in O(n-p,p)^{++}$. Define some $\mathbf{w}$ such that $A\mathbf{v} \in \mathrm{Span}\{\mathbf{v}, \mathbf{w}\}$, $g(\mathbf{w}, \mathbf{w}) = -1$ and $g(\mathbf{v}, \mathbf{w}) = 0$, then complete the orthonormal basis arbitrarily. Now, there must be some $\varphi \in [0, 2\pi[$ such that

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \mathbf{v} = A\mathbf{v},$$

and define the continuous path

$$\mathscr{A}_{\mathbf{v}}^{A} : [0,1] \to O(n-p,p)^{++}, \ t \mapsto \begin{pmatrix} \cos\varphi t & \sin\varphi t & 0 \\ -\sin\varphi t & \cos\varphi t & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}.$$

If instead $p = 1$, the path is trivial, since then $A\mathbf{v} = \mathbf{v}$, and we choose $\mathscr{A}_{\mathbf{v}}^{A}(t) = I_n$.
Using the same process, we can find such paths for $g(\mathbf{v}, \mathbf{v}) = 1$, as well.
Notice that $\mathscr{A}_{\mathbf{v}}^{A}(t)\mathbf{a} = \mathbf{a}$ for $\mathbf{a} \perp \mathrm{Span}(\mathbf{v}, \mathbf{w})$.

Now, we move on to the actual proof. Take again any $A \in O(n-p,p)^{++}$ and define $\mathscr{A}_1 = \mathscr{A}_{\mathbf{e}_1}^{A}$. Then, for any $2 \leq k \leq n-1$, define

$$\mathscr{A}_k := \mathscr{A}_{\mathscr{A}_{k-1}(1)\circ\cdots\circ\mathscr{A}_1(1)\mathbf{e}_k}^{A\circ(\mathscr{A}_{k-1}(1)\circ\cdots\circ\mathscr{A}_1(1))^{-1}}.$$

Then $\mathscr{A}_k(1) \circ \cdots \circ \mathscr{A}_1(1)\mathbf{e}_i = A\mathbf{e}_i$ for $i \leq k-1$, since we may choose $A\mathbf{e}_i \perp \mathbf{w}$ (because $A\mathbf{e}_i \perp A\mathbf{e}_k$) and, furthermore, $A\mathbf{e}_i = \mathscr{A}_{k-1}(1) \circ \cdots \circ \mathscr{A}_1(1)\mathbf{e}_i \perp \mathscr{A}_{k-1}(1) \circ \cdots \circ \mathscr{A}_1(1)\mathbf{e}_k$. We now find the continuous path

$$\mathscr{A}(t) = \mathscr{A}_{n-1}(t) \circ \cdots \circ \mathscr{A}_1(t),$$

such that $\mathscr{A}(1)e_i = Ae_i$ for $i \leq n-1$. Now, $\mathscr{A}(1)e_n \perp \mathrm{Span}(\mathscr{A}(1)e_1, \ldots, \mathscr{A}(1)e_{n-1})$, and thus $\mathscr{A}(1)e_n = \pm Ae_n$. Since orientation is preserved, after all, $\det \mathscr{A}(1) = 1$, we find that the only solution is $\mathscr{A}(1)e_n = Ae_n$, and thus $\mathscr{A}(1) = A$. In conclusion, $O(n-p,p)^{++}$ is path-connected. □

Corollary A.11. *For $p(n-p) > 0$, we have $SO^0(n-p,p) = O(n-p,p)^{++}$.*
For $p(n-p) = 0$, we have $SO^0(n-p,p) = O(n-p,p)^{+} = SO(n-p,p)$.

Now, we can prove the theorem.

Proof of theorem 5.13. The first short exact sequence is already proven. As for the second short sequence, $\mathbb{Z}^2 \to \mathrm{Spin}(n-p,p)$ is still the inclusion and $\ker \widetilde{Ad} = \mathbb{Z}^2$. We only have to show that $\widetilde{Ad}(\mathrm{Spin}(n-p,p)) = SO^0(n-p,p)$. This proof will be separated into two cases, depending on the value of $p(n-p)$.

$p(n-p)=0$. It is more than sufficient to show that, for any $v_1\cdots v_k \in \mathrm{Pin}(n-p,p)$, we have $\widetilde{Ad}(v_1\cdots v_k)\in SO(n-p,p)$ if and only if $v_1\cdots v_k\in \mathrm{Spin}(n-p,p)$, since we already have the first short exact sequence. Indeed, using that reflections have determinant -1,

$$\det \widetilde{Ad}(v_1\cdots v_k) = \det \widetilde{Ad}(v_1)\cdots \det \widetilde{Ad}(v_k) = (-1)^k.$$

Now, the condition $v_1\cdots v_k v_k\cdots v_1 = 1$ is always met if k is even, and therefore, $\mathrm{Spin}(n-p,p)$ consists exactly of $v_1\ldots v_k$ with even k. Thus, it follows that $\det \widetilde{Ad}(v_1\cdots v_k)=1$ if and only if $v_1\cdots v_k\in \mathrm{Spin}(n-p,p)$, proving the statement.

$p(n-p)>0$. Refer to the proof of the previous lemma. There, we showed that any $A\in SO^0(n-p,p)$ can be decomposed as a series of rotations A_i over planes spanned by two vectors v_i, w_i with $g(v_i,v_i)=g(w_i,w_i)=\pm 1$. Now, from linear algebra, we know that there must be two vectors $u_1,u_2\in \mathrm{Span}(v_i,w_i)$ with $g(u_1,u_1)=g(u_2,u_2)=\pm1$ such that said rotation is equal to the composition of reflections across u_1 and u_2, or

$$A_i = \widetilde{\mathrm{Ad}}(u_1)\widetilde{\mathrm{Ad}}(u_2) = \widetilde{\mathrm{Ad}}(u_1u_2).$$

Through induction, it follows that there must be some element $u_1\cdots u_k\in\mathrm{Spin}(n-p,p)$ such that $\widetilde{\mathrm{Ad}}(u_1\cdots u_k)=A$. Therefore, $\widetilde{\mathrm{Ad}}(\mathrm{Spin}(n-p,p))\supseteq SO^0(n-p,p)$.
For the proof that $\widetilde{\mathrm{Ad}}(\mathrm{Spin}(n-p,p))\subseteq SO^0(n-p,p)$, we want to use that the previously defined function ξ is a homomorphism. For this, take any $A\in O(n-p,p)^{\pm_1\pm_2}$, and notice that

$$AI^{\pm_1\pm_2} := A\begin{pmatrix} \pm_1 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \pm_2 1\end{pmatrix} \in O(n-p,p)^{++}.$$

Thus, for every $A\in O(n-p,p)^{\pm_1\pm_2}$, there exists a unique $A'\in O(n-p,p)^{++}$ such that $A=A'I^{\pm_1\pm_2}$. Similarly, there exists a unique $A''\in O(n-p,p)^{++}$ such that $A=I^{\pm_1\pm_2}A''$.
Now, take some matrices $X\in O(n-p,p)^{\pm_1\pm_2}$ and $Y\in O(n-p,p)^{\pm_3\pm_4}$. Then

$$XY = I^{\pm_1\pm_2}X''Y'I^{\pm_3\pm_4}.$$

Since $O(n-p,p)^{++}$ is a group (we showed that earlier), $X''Y'\in O(n-p,p)^{++}$. Therefore, $Z:=I^{\pm_1\pm_2}X''Y'\in O(n-p,p)^{++}$ is in $O(n-p,p)^{\pm_1\pm_2}$, and there must exist some $Z'\in O(n-p,p)^{++}$ such that

$$XY = Z'I^{\pm_1\pm_2}I^{\pm_3\pm_4},$$

such that, clearly,

$$\xi(XY) = ((\pm_1 1)(\pm_3 1),(\pm_2 1)(\pm_4 1)) = \xi(X)\xi(Y).$$

Now, take any $v_1\cdots v_k\in\mathrm{Spin}(n-p,p)$. To prove the theorem, we have to show that $\xi(\widetilde{\mathrm{Ad}}(v_1\cdots v_k))=(1,1)$. Since k is even, we can separate these v_i into pairs, such that

$$\xi(\widetilde{\mathrm{Ad}}(v_1\cdots v_k)) = \prod_{i=1}^{k/2}\xi(\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})).$$

Assume now that $g(v_{2i-1}, v_{2i-1}) = g(v_{2i}, v_{2i})$ for some i, and assume, without loss of generality, that $g(v_{2i}, v_{2i}) = -1$. Then $\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})$ is some rotation over the plane spanned by v_{2i-1}, v_{2i}. Specifically, if we choose our orthonormal basis such that $e_1 = v_{2i}$, e_2 is orthonormal to e_1 such that $v_{2i-1} \in \mathrm{Span}(e_1, e_2)$ and $g(e_2, e_2) = -1$, and choose the other basis vectors arbitrarily, we find that there must be some $\varphi \in [0, 2\pi[$ such that

$$\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i}) = \begin{pmatrix} \cos\varphi & -\sin\varphi & & & \\ \sin\varphi & \cos\varphi & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

and, clearly, $\xi(\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})) = (1,1)$. The same is true for when $g(v_{2i}, v_{2i}) = 1$.
Now, we observe the case that $g(v_{2i-1}, v_{2i-1}) \neq g(v_{2i}, v_{2i})$, and assume without loss of generality that $g(v_{2i}, v_{2i}) = -1$. We define e_1 as before, and define, moreover, e_n orthogonal to e_1 such that $v_{2i-1} \in \mathrm{Span}(e_1, e_n)$ and $g(e_n, e_n) = 1$. Finish the orthonormal basis e_j arbitrarily. We then find that

$$\begin{aligned}
\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})e_j &= v_{2i-1}v_{2i}e_jv_{2i}v_{2i-1} \\
&= \begin{cases} -v_{2i-1}e_1v_{2i-1} & \text{if } j = 1, \\ v_{2i-1}(2g(v_{2i}, e_j) - e_jv_{2i})v_{2i}v_{2i-1} & \text{otherwise} \end{cases} \\
&= \begin{cases} -2g(v_{2i-1}, e_1)v_{2i-1} + e_1 & \text{if } j = 1, \\ v_{2i-1}e_jv_{2i-1} & \text{otherwise} \end{cases} \\
&= \begin{cases} -2g(v_{2i-1}, e_1)v_{2i-1} + e_1 & \text{if } j = 1, \\ 2g(v_{2i-1}, e_j)v_{2i-1} - e_j & \text{otherwise} \end{cases} \\
&= \begin{cases} -2g(v_{2i-1}, e_1)v_{2i-1} + e_1 & \text{if } j = 1, \\ 2g(v_{2i-1}, e_n)v_{2i-1} - e_n & \text{if } j = n, \\ -e_j & \text{otherwise.} \end{cases}
\end{aligned}$$

Now, there must be some $\alpha \in \mathbb{R}$ such that $v_{2i-1} = \alpha e_1 + \sqrt{1+\alpha^2}e_n$, and thus,

$$\begin{aligned}
&= \begin{cases} 2\alpha v_{2i-1} + e_1 & \text{if } j = 1, \\ 2\sqrt{1+\alpha^2}v_{2i-1} - e_n & \text{if } j = n, \\ -e_j & \text{otherwise} \end{cases} \\
&= \begin{cases} (1+2\alpha^2)e_1 + 2\alpha\sqrt{1+\alpha^2}e_n & \text{if } j = 1, \\ (1+2\alpha^2)e_n + 2\alpha\sqrt{1+\alpha^2}e_1 & \text{if } j = n, \\ -e_j & \text{otherwise.} \end{cases}
\end{aligned}$$

We therefore conclude that

$$\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i}) = \begin{pmatrix} 1+2\alpha^2 & & & & 2\alpha\sqrt{1+\alpha^2} \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ 2\alpha\sqrt{1+\alpha^2} & & & & 1+2\alpha^2 \end{pmatrix}.$$

Therefore,

$$\det\Big(\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})\Big)^{--} = (1+2\alpha^2)(-1)^{p-1},$$

and,

$$\det\Big(\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})\Big)^{++} = (1+2\alpha^2)(-1)^{n-p-1}.$$

This gives us that $\xi(\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})) = ((-1)^{p-1}, (-1)^{n-p-1})$.
Now, let us return to the equation

$$\xi(\widetilde{\mathrm{Ad}}(v_1 \cdots v_k)) = \prod_{i=1}^{k/2} \xi(\widetilde{\mathrm{Ad}}(v_{2i-1}v_{2i})).$$

Using the two cases we outlined above, and using that there must always be an even amount of duos $v_{2i-1}v_{2i}$ as in the second case, we find exactly that

$$\xi(\widetilde{\mathrm{Ad}}(v_1 \cdots v_k)) = (1,1),$$

and therefore,

$$\widetilde{\mathrm{Ad}}(\mathrm{Spin}(n-p,p)) \subseteq SO^0(n-p,p).$$

This proves the theorem. $\square$

References

[1] J. Maxwell and T. Torrance, *A Dynamical Theory of the Electromagnetic Field*, Torrance collection (Wipf and Stock, 1996).

[2] C. N. Yang and R. L. Mills, "Conservation of Isotopic Spin and Isotopic Gauge Invariance", Phys. Rev. **96**, 191–195 (1954).

[3] D. Dummit and R. Foote, *Abstract Algebra* (Wiley, 2003).

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Volume 1*, Wiley Classics Library (Wiley, 1996).

[5] C. Norman, *Undergraduate Algebra: A First Course*, Oxford science publications (Clarendon Press, 1986).

[6] M. Nakahara, *Geometry, Topology and Physics, Second Edition*, Graduate student series in physics (Taylor & Francis, 2003).

[7] M. Crainic, *Mastermath course Differential Geometry 2015/2016*, (2015) `https://webspace.science.uu.nl/~crain101/DG-2015/main10.pdf` (visited on 2022-06-01).

[8] H. Lawson and M. Michelsohn, *Spin Geometry (PMS-38), Volume 38*, Princeton Mathematical Series (Princeton University Press, 1989).

[9] D. Garling, *Clifford Algebras: An Introduction*, London Mathematical Society Student Texts (Cambridge University Press, 2011).

[10] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, Seminarbericht: Sektion Mathematik (Teubner, 1981).

[11] M. Hamilton, *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*, Universitext (Springer International Publishing, 2017).

[12] J. Duistermaat and J. Kolk, *Lie Groups*, Universitext (Springer Berlin Heidelberg, 1999).

[13] A. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics (Birkhäuser Boston, 2013).

[14] G. Aad et al., "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC", Physics Letters B **716**, 1–29 (2012).

[15] P. Langacker, "Grand unified theories and proton decay", Physics Reports **72**, 185–385 (1981).

[16] H. Georgi and S. L. Glashow, "Unity of All Elementary-Particle Forces", Phys. Rev. Lett. **32**, 438–441 (1974).

[17] G. 't Hooft, "Magnetic monopoles in unified theories", Nucl. Phys. B **79**, 276–284 (1974).

[18] H. Fritzsch and P. Minkowski, "Unified interactions of leptons and hadrons", Annals of Physics **93**, 193–266 (1975).

[19] J. C. Pati and A. Salam, "Lepton number as the fourth "color"", Phys. Rev. D **10**, 275–289 (1974).

[20] J. C. PATI, "PROBING GRAND UNIFICATION THROUGH NEUTRINO OSCILLATIONS, LEPTOGENESIS, AND PROTON DECAY", International Journal of Modern Physics A **18**, 4135–4155 (2003).

[21] G. Naber, *Topology, Geometry and Gauge fields: Foundations*, Texts in Applied Mathematics (Springer New York, 2010).

Index