上周作业解答

Exercise 1: I choose a number uniformly at random from the range [1, 1000000].Using the inclusion-exclusion principle, determine the probability that the number chosen is divisible by one or more of 4, 6, and 9.

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Answer:

在区间$[1,1000000]$内，令

A=｛能被**4**整除的数｝，**B**=｛能被**6**整除的数｝，**C**=｛能被**9**整除的数｝，

A⋂**B**=｛能被**12**整除的数｝，**B**⋂**C**=｛能被**18**整除的数｝，**A**⋂**C**=｛能被**36**整除的数｝，

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依据容斥定理，有

$$\begin{aligned} P(A\cup B\cup C) &= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)\\ &\quad +P(A\cap B\cap C)\\ &= \frac{250000+166666+111111-83333-55555-27777+27777}{1000000}\\ &= \frac{388889}{1000000}. \end{aligned}$$

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Exercise 2: I have a fair coin and a two-headed coin. I choose one of the two coins randomly with equal probability and flip it. Given that the flip was heads, what is the probability that I flipped the two-headed coin?

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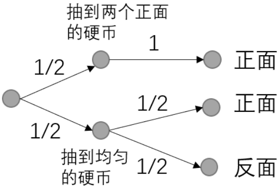

可以画出以上概率树，令

A_1={抽到两个正面的硬币}，A_2={抽到均匀的硬币}，**B**={投出正面}，有

$$P(A_1|B)=\frac{P(A_1B)}{P(B)}=\frac{P(A_1B)}{P(A_1)P(B)+P(A_2)P(B)}=\frac{\frac{1}{2}}{\frac{1}{2}\times 1+\frac{1}{2}\times\frac{1}{2}}=\frac{2}{3}.$$

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Exercise 3: The following problem is known as the Monty Hall problem, after the host of the game show "Let's Make a Deal ". There are three curtains. Behind one curtain is a new car and behind the other two are goats. The game is played as follows. The contestant chooses the curtain that she thinks the car is behind. Monty then opens one of the other curtains to show a goat. (Monty may have more than one goat to choose from; in this case, assume he chooses which goat to show uniformly at random.) The contestant can then stay with the curtain she originally chose or switch to the other unopened curtain. After that, the location of the car is revealed, and the contestant wins the car or the remaining goat. Should the contestant switch curtains or not, or does it make no difference?

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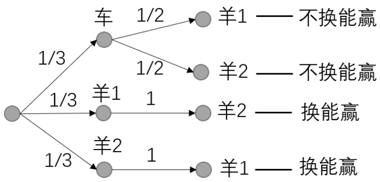
Answer:



可以画出以上概率树。第一步，选手从背后分别有车、羊1、羊2的3幅窗帘中随机选择1个，3类选择的概率相等（1/3）。第二步，当选手选择的是背后是车的窗帘时，主持人可以从背后是羊1、羊2的2幅窗帘中随机选择1个打开，2类选择的概率相等（1/2），这时选手不换选择能赢得车；当选手选择的是背后是羊1的窗帘时，主持人只能打开背后是羊2的窗帘，概率为1，这时选手更换选择能赢得车；当选手选择的是背后是羊2 的窗帘时，主持人只能打开背后是羊1 的窗帘，概率为1，这时选手更换选择能赢得车。所以，我们有

$P(\text{换能赢}) = 2 \times \frac{1}{3} \times 1 = \frac{2}{3}.$

$P(\text{不换能赢}) = \frac{1}{3} \times \frac{1}{2} \times 2 = \frac{1}{3}.$

换能赢得车的概率比不换的概率大，所以，选择换。

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Exercise 4: Suppose that we roll ten standard six-sided dice. What is the probability that their sum will be divisible by 6, assuming that the rolls are independent? (Hint: Use the principle of deferred decisions, and consider the situation after rolling all but one of the dice.)

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Exercise 4: Suppose that we roll ten standard six-sided dice. What is the probability that their sum will be divisible by 6, assuming that the rolls are independent? (Hint: Use the principle of deferred decisions, and consider the situation after rolling all but one of the dice.)

Answer:

令掷每枚骰子得到的点数为$A_i \in \{1,2,3,4,5,6\}, i=1,2,\cdots 10$，掷10枚骰子得到的点数和为$\sum_{i=1}^{10} A_i \in \{10,11,\cdots 60\}$。由于10枚骰子的投掷是相互独立的，假设现已掷骰子9枚，则可以分为6种情况：9枚骰子点数和除6余$0,1,2,\cdots 5$，用B_i表示，B_0表示9枚骰子点数和整除6，B_1表示9枚骰子点数和除6余1，如此类推。则投掷第10枚骰子，其点数与前9枚骰子的点数和被6整除的概率为

$P_X \triangleq P(\{\sum_{i=1}^{10} A_i \text{ 被6整除}\}) = P(B_0)P(A_{10}=6)+P(B_1)P(A_{10}=5)+P(B_2)P(A_{10}=4)+P(B_3)P(A_{10}=3)+P(B_4)P(A_{10}=2)+P(B_5)P(A_{10}=1)$

因为骰子是标准的，所以有

$P(A_i=1)=P(A_i=2)=P(A_i=3)=P(A_i=4)=P(A_i=5)=P(A_i=6)=1/6.$

所以$P_X=\frac{1}{6}\times(P(B_0)+P(B_1)+P(B_2)+P(B_3)+P(B_4)+P(B_5))=\frac{1}{6}\times 1=\frac{1}{6}$.

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Exercise 5: Let X be a random variable with distribution function $F_X(x)$. Find the distribution function of the following random variables.
a) The maximum of n IID random variables with distribution function $F_X(x)$.
b) The minimum of n IID random variables with distribution $F_X(x)$.
c) The difference of the random variables defined in (a) and (b); assume X has a density $f_X(x)$.

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Answer:

a)令M_+为n个随机变量$X_1, X_2, \cdots X_n$中的最大值。请注意，对于任意实数x，$M_+ \leqslant x$当且仅当$X_j \leqslant x$ 对于所有$1 \leqslant j \leqslant n$ 成立。因此，我们有

$\mathrm{Pr}\{M_+ \leqslant x\} = \mathrm{Pr}\{X_1 \leqslant x, X_2 \leqslant x, \cdots, X_n \leqslant x\} = \prod_{j=1}^{n} \mathrm{Pr}\{X_j \leqslant x\}$，

以上第二步等式依据 X_j的独立性。因为$\mathrm{Pr}\{X_j \leqslant x\} = F_X(x)$对于所有$j$成立，所以，我们有$F_{M_+}(x) = \mathrm{Pr}\{M_+ \leqslant x\} = (F_X(x))^n$。

b)令M_-为n个随机变量$X_1, X_2, \cdots X_n$中的最小值。同样地，对于任意实数y，$M_- > y$当且仅当$X_j > y$ 对于所有$1 \leqslant j \leqslant n$ 成立。我们也可以用$\geqslant$代替严格的$>$，但因累积分布函数的严格不等式的需要，我们使用$>$。因此，我们有

$\mathrm{Pr}\{M_- > y\} = \mathrm{Pr}\{X_1 > y, X_2 > y, \cdots, X_n > y\} = \prod_{j=1}^{n} \mathrm{Pr}\{X_j > y\}$，

于是，我们有$F_{M_-}(y) = 1 - \mathrm{Pr}\{M_- > y\} = 1 - (1 - F_X(y))^n$。

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Answer:

c)有很多种办法可以解决这个问题，但其中一个比较简单的办法是基于第一个条件事件$X_1 \leqslant x$。然后$X_1 = M_+$当且仅当 $X_j \leqslant x$ 对于所有$2 \leqslant j \leqslant n$ 成立。同时，给定 $X_1 = M_+ = x$，我们有$R = M_+ - M_- \leqslant r$当且仅当 $X_j > x - r$ 对于所有$2 \leqslant j \leqslant n$ 成立。因此，我们有

$\mathrm{Pr}\{M_+ = x, R \leqslant r | X_1 = x\} = \prod_{j=2}^{n} \mathrm{Pr}\{x - r < X_j \leqslant x\} = [\mathrm{Pr}\{x - r < X \leqslant x\}]^{n-1} = [F_X(x) - F_X(x - r)]^{n-1}$。

现在，我们可以通过对事件$X_1 = x$取平均而把条件概率转化为全概率。假设X的概率密度函数是$f_X(x)$，我们有

$\mathrm{Pr}\{X_1 = M_+, R \leqslant r\} = \int_{-\infty}^{+\infty} f_X(x)[F_X(x) - F_X(x - r)]^{n-1} dx$。

最后，由于 X_j中任意两个随机变量相等的概率为**0**，所以$X_j = M_+$是不相交事件（除了一个概率为**0**的事件）。利用对称性，$X_j = x$与$X_1 = x$的论证情形相同，所以

$\mathrm{Pr}\{X_j = M_+, R \leqslant r\} = \mathrm{Pr}\{X_1 = M_+, R \geqslant r\}$。

于是我们有

$\mathrm{Pr}\{R \leqslant r\} = \int_{-\infty}^{+\infty} n f_X(x)[F_X(x) - F_X(x - r)]^{n-1} dx$。

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Exercise 6: Let $X_1, X_2, ..., X_n, ...$ be a sequence of independent identically distributed (IID) continuous random variables with the common probability density function $f_X(x)$; note that $P(X = \alpha) = 0$ for all α and that $P(X_1 = X_2) = 0$.
a) Find $P(X_1 \leq X_2)$ (give a numerical answer, not an expression; no computation is required and a one or two line explanation should be adequate).
b) Find $P(X_1 \leq X_2; X_1 \leq X_3)$ (in other words, find the probability that X_1 is the smallest of X_1, X_2, X_3; again, think–don't compute).

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b) Find $P(X_1 \leq X_2; X_1 \leq X_3)$ (in other words, find the probability that X_1 is the smallest of X_1, X_2, X_3; again, think–don't compute).

a)因为 $X_1, X_2, \ldots, X_n, \ldots$独立同分布，$P(X = \alpha) = 0$ 对于所有 α成立，以及 $P(X_1 = X_2) = 0$，所以我们有：$P(X_1 > X_2)$与$P(X_1 < X_2)$概率相等，所以$P(X_1 > X_2) = P(X_1 < X_2) = 1/2$。
b)X_1, X_2, X_3三个随机变量独立同分布，任意一个为三个随机变量中取值最小的概率相等，所以 $P(X_1 \leqslant X_2; X_1 \leqslant X_3) = 1/3$。