

A Preferential Attachment Model for the Initial Mass Function

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Abstract

Accurate specification of a likelihood function is becoming increasingly difficult in many inference problems in astronomy. As the sample sizes resulting from astronomical surveys continue to grow, deficiencies in the likelihood function lead to larger biases in key parameter estimates. These deficiencies result from the oversimplification of the physical processes that generated the data, and from the failure to account for observational limitations. Unfortunately, realistic models often do not yield an analytical form for the likelihood. The estimation of a stellar initial mass function (IMF) is an important example. The stellar IMF is the mass distribution of stars initially formed in a given cluster of stars, a population which is not directly observable due to stellar evolution and other disruptions of the cluster. There are several difficulties with specifying a likelihood in this setting, since the physical processes and observational challenges result in measurable masses that cannot legitimately be considered independent draws from an IMF. This work improves inference of the

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IMF by using an approximate Bayesian computation approach that both accounts for observational and astrophysical effects and incorporates a physically-motivated model for star cluster formation. The methodology is illustrated via a simulation study, demonstrating that the proposed approach can recover the true posterior in realistic situations, and applied to observations from the Orion Nebula Cluster.

Keywords: Approximate Bayesian computation, astrostatistics, computational statistics, dependent data

1 Introduction

The Milky Way is home to billions of stars (McMillan 2016), many of which are members of *stellar clusters* - gravitationally bound collections of stars. Stellar clusters are formed from low temperature and high density clouds of gas and dust called *molecular clouds*, though there is uncertainty as to how the stars in a cluster form (Beccari et al. 2017). Each theory of star formation yields a different prediction for the distribution of the masses of stars that initially formed in a cluster. Hence, it is of fundamental interest to estimate this distribution, referred to as the *stellar initial mass function (IMF)*, and assess the validity of these competing theories. In fact, research advances in many areas of stellar, galactic, and extragalactic astronomy are at least somewhat reliant upon accurate understanding of the IMF (Bastian et al. 2010). For example, the IMF is a key component of galaxy and stellar evolution and planet formation (Bally & Reipurth 2005, Bastian et al. 2010, Shetty & Cappellari 2014), along with chemical enrichment and abundance of core-collapse supernovae (Weisz et al. 2013).

There is also ongoing discussion surrounding the *universality* of the IMF, i.e., if a single IMF describes the generative distribution of stellar masses for all star clusters (Bastian et al. 2010). Even though the consensus of the astronomical community is that there is a universal IMF, most of the observations had been consistent with universality. [\[\[Jessi: Do we mean the consensus is that not a universal imf?\]\]](#) With further research and growing sample sizes, however, there is increased theoretical (Dib et al. 2010, Bonnell et al. 2006) and observational (Treu et al. 2010, Spiniello et al. 2014, Geha et al. 2013, Dib et al. 2017) support for an IMF that varies from cluster to cluster.

Salpeter (1955) studied the evolutionary properties of certain populations of stars, and in the process defined the first IMF (which he called the “original mass function”). This work put forth the now-classic model for the IMF, a power law with a finite upper bound equal to the physical maximum mass of a star that could form in a cluster (Salpeter 1955). More recent studies continue to use this power law form for the IMF, especially for stars of mass greater than half that of our sun (e.g., Massey 2003, Bastian et al. 2010, Da Rio et al. 2012, Lim et al. 2013, Weisz et al. 2013, 2015, Jose et al. 2017). Similar models have been proposed and used in the astronomical literature for inference of the stellar IMF; these will be discussed in the next section. By relying on the noted power law form for the IMF, the estimation of the parameters of these proposed models relies on the unphysical assumption that the observed stars in a stellar cluster form independently; more specifically, the assumption that the masses of the individual stars form independently. [[Jessi: Chad: please read the previous sentence and see if that makes sense. Rather than citing all the same papers again, I try to reference back to the power law model assumption]] The proposed model in this work loosens the assumption of independence in order to explore one of several possible physical formation mechanisms of cluster formation.

Despite this seemingly simple form of the power law model, the statistical challenges of estimating the IMF using *observed* stars from a cluster are significant. Many of the limitations are related to observational issues and to the adequate modeling of the evolution of a star cluster after the initial formation. For example, since stars of greater mass die more rapidly, the upper tail of the IMF is not observed in a cluster of sufficient age. Also, the death of massive stars can trigger additional star formation, contaminating the lower end of the IMF with new stars (Woosley & Heger 2015). There are also issues related to missing lower-mass stars due to the sensitivity of the instruments. The observational astronomers will often estimate the *completeness function* of an observed cluster, which is the probability of observing a star of a particular mass. The completeness function is discussed in more detail below.

The observational limitations and the challenge of modeling cluster evolution make approximate Bayesian computation (ABC) appealing for estimation of the IMF, as ABC

allows for relatively easy incorporation of such effects. The difficulty of addressing these limitations implied by the fact that observational effects are often ignored or accounted for in an ad-hoc or unspecified manner (e.g., Da Rio et al. 2012, Ashworth et al. 2017, Jose et al. 2017, Kalari et al. 2018), though Weisz et al. (2013) discuss how some observational limitations can be incorporated into their proposed Bayesian model. A primary appeal of ABC for this application is the ability to incorporate more complex models for cluster formation. Standard IMF models do not specify the process by which a large mass of gas (the molecular cloud) transforms into a gravitationally bound collection of stars. ABC is based on a simple rejection-sampling approach, in which draws of model parameters from a prior distribution are fed through a simulation model to generate a sample of data. If the generated sample is “close” (based on an appropriately chosen metric) to the observed data, the prior draw that produced that generated sample is retained. The collection of accepted parameter values comprise draws from an approximation to the posterior. The simulations (the *forward model*) can include any of the complex processes that make it challenging to derive a likelihood function for the observable data.

This situation is typical of inference challenges that arise in astronomy. See Schafer & Freeman (2012), Akeret et al. (2015), Ishida et al. (2015) for reviews. Recent years have seen a rapid increase in the use of ABC methods for estimation in this field, including specific application to Milky way properties (Robin et al. 2014), strong lensing of galaxies (Killedar et al. 2018, Birrer et al. 2017), large scale structure of the Universe (Hahn, Vakili, Walsh, Hearin, Hogg & Campbell 2017), estimating the redshift distribution (Herbel et al. 2017), galaxy evolution (Hahn, Tinker & Wetzel 2017), weak lensing (Peel et al. 2017, Lin & Kilbinger 2015), exoplanets (Parker 2015), galaxy morphology (Cameron & Pettitt 2012), and supernovae (Weyant et al. 2013).

This paper is organized as follows. In Section 2 we will present background on the IMF along with inference challenges, and introduce ABC. We propose a new stochastic model for stellar formation in Section 3 and discuss our ABC procedure. Section ?? applies the proposed methodology to the estimation of the IMF of a realistic astrophysical simulation (Bate 2012). Finally, Section 6 provides a discussion.

2 Background

2.1 Stellar Initial Mass Function

As noted above, Salpeter (1955) introduced the power law model for the shape of the IMF for masses larger than $0.5M_{\odot}$, where M_{\odot} is the mass of the Sun. Kroupa (2001) extended the range of the IMF by proposing a three-part broken power law model over the range $0.01M_{\odot} < m < M_{\max}$, where M_{\max} is the mass of the largest star that could form with nonzero probability. This model postulates different forms for the IMF for stars of masses $0.01M_{\odot} < m < 0.08M_{\odot}$, $0.08M_{\odot} < m < 0.5$ and $m > 0.5M_{\odot}$. Focusing on the upper part, and defining $\theta = (\alpha, M_{\max})$, the probability density function for mass x in the upper tail of the stellar IMF is assumed to be given by

$$f_M(m | \theta) = cm^{-\alpha}, \quad m \in [M_{\min}, M_{\max}], \quad (1)$$

where the constant c is chosen such that f_M is a valid probability density. Alternative models have been proposed that include log-normal distributions, joint power law and log-normal parts, and truncated exponential distributions (Chabrier 2005, Corebello et al. 2005, Bastian et al. 2010, Offner et al. 2014). The Kroupa (2001) and Chabrier (2005) models are displayed in Figure 1 along with observational challenges discussed §2.1.1. Power law distributions and log-normal distributions are closely related and may be the result of subtle differences in the underlying formation mechanism (Mitzenmacher 2004). The IMF model we propose will exploit one possible underlying formation mechanism that we show results in power law tails. [[Jessi: previous statement may have to be modified]]

2.1.1 Observational Challenges

Observing all stars comprising an IMF is not feasible, as the most massive stars ($m > 10M_{\odot}$) have lifetimes of only a few million years. The lifetime of a star (the time it takes for the star to burn through its hydrogen) depends strongly on its mass: the most massive stars have shorter lives due to the hotter temperatures they must maintain to avoid collapse from the strong gravitational forces. In particular, stellar life is approximately proportional

to $m^{-\rho}$ where $\rho \approx 3$ (Hansen et al. 2004, p. 30, Chaisson & McMillan 2011, p. 439). Hence, the mass of the largest star observed in a given cluster is depends on the cluster age.

Furthermore, the IMF will be estimated using a noisy, incomplete view of that cluster. Whether or not a star is observed is dependent on several factors including its mass, its location in the cluster, and its neighbors. Some of these factors are described by a data set's *completeness function*, which quantifies a given star's probability of being observed. This depends on its luminosity (i.e. intrinsic brightness) since it needs to be sufficiently bright to be observable; in particular, completeness depends on stellar flux in comparison with the flux limits of the observations. There are also issues with *mass segregation*: stars with lower mass tend to be on the edge of the cluster, while the most massive stars are often found in the center (Weisz et al. 2013). Due to *stellar crowding* in the center, stars in this region can be more difficult to observe. Additionally, binary stars (star systems consisting of a pair of stars) are difficult to distinguish from a single star, creating the potential for overstating the mass of an object and understating the number of stars in the cluster.

There are additional uncertainties involved in translating the actual observables (e.g. photometric magnitudes) into a mass measurement; that is, the mass values for observable stars are only estimates. The *Hertzsprung-Russell (H-R) Diagram* is a classic visual summary of the distribution of the luminosity and temperature of a collection of stars. A typical H-R Diagram includes a *main sequence* of stars that trace a line from bright and hot stars to dim and cool stars. Stellar mass also evolves along this one-dimensional feature, and since luminosity and temperature are estimable, mass can thus also be estimated. The mass of binary stars can be determined via Kepler's Laws, and hence a *mass-luminosity relationship* can be fit to binaries and then extended to other stars on the main sequence. Unfortunately, luminosity and temperature are nontrivial to estimate, as corrections for effects such as *accretion* and *extinction* are required, along with an accurate estimate of the distance to the stars (Da Rio et al. 2010). The process is further complicated by the dependence of how these transformations are made on the *spectral type* of the star. Careful budgeting of the errors that accumulate is required in order to produce a reasonable error bars on mass estimates; Da Rio et al. (2010) utilize a Monte Carlo approach in which the

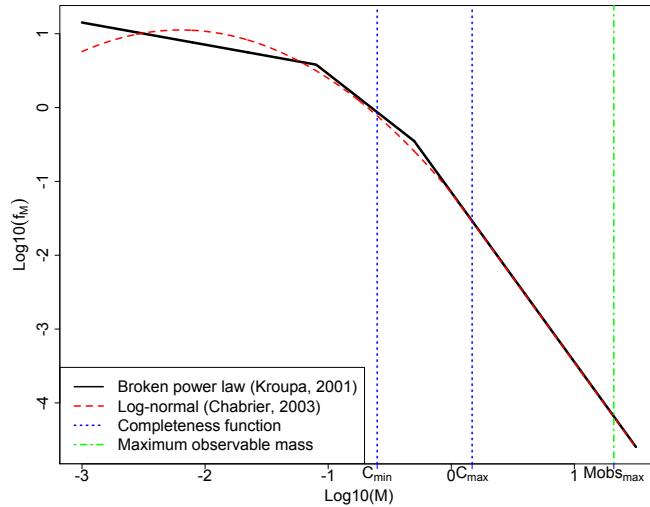


Figure 1: The broken power law model of Kroupa (2001) (black, line) and the lognormal model of Chabrier (2005) (red, dashed) are displayed along with vertical lines representing several observational challenges. The blue vertical dotted lines indicate the range of values (C_{\min}, C_{\max}) on which the completeness function may be defined, and the vertical green dotted and dashed line indicates the maximum observable mass (M_{obs}_{\max}) due to the aging cutoff. The observational challenges are discussed further in §3.2.

errors in magnitudes are propagated forward through to uncertainties in the spectral type, the accretion and reddening corrections, and finally to an uncertainty on the mass.

2.2 Approximate Bayesian Computation

Standard approaches to Bayesian inference, either analytical or built on MCMC, require the specification of a likelihood function, $f(m | \boldsymbol{\theta})$, with data $m_{\text{obs}} \in \mathcal{D}$, and parameter(s) $\boldsymbol{\theta} \in \Theta$. In many modern scientific inference problems, such as for some emerging models for the stellar IMF, the likelihood is too complicated to be derived or otherwise specified. As noted previously, ABC provides an approximation to the posterior without specifying a likelihood function, and instead relies on forward simulation of the data generating process.

The basic algorithm for sampling from the ABC posterior is attributed to Tavaré et al. (1997) and Pritchard et al. (1999), used for applications to population genetics. The algorithm has three main steps which are repeated until a sufficiently large sample is generated: *Step 1*, Sample $\boldsymbol{\theta}^*$ from the prior; *Step 2*, Generate m_{sim} from forward process assuming truth $\boldsymbol{\theta}^*$; *Step 3*, Accept $\boldsymbol{\theta}^*$ if some distance function $\rho(m_{\text{obs}}, m_{\text{sim}}) \leq \epsilon$, where ϵ is a tuning parameter that should be close to 0. This last step typically consists of comparing low-dimensional summary statistics generated for the observed and simulated datasets. Adequate statistical and computational performance of ABC algorithms depends greatly on the selection of such summary statistics (Joyce & Marjoram 2008, Blum & François 2010, Blum 2010, Fearnhead & Prangle 2012, Blum et al. 2013).

The basic ABC algorithm can be inefficient in cases where the parameter space is of moderate or high dimension. Hence, important adaptations of the basic ABC algorithm incorporate ideas of sequential Monte Carlo (SMC) in order to improve the sampling efficiency (Marjoram et al. 2003, Sisson et al. 2007, Beaumont et al. 2009, Del Moral et al. 2011). A nice overview of ABC can be found in Marin et al. (2012). Here, we use a sequence of decreasing tolerances $\epsilon_{1:T} = (\epsilon_1, \dots, \epsilon_T)$ with the tolerance ϵ_t shrinking until further reductions do not significantly affect the resulting ABC posterior. The improvement in efficiency is due to the modification that happens after the first time step: instead of sampling from the prior distribution, the proposed $\boldsymbol{\theta}$ are drawn from the previous time

step’s ABC posterior. Using this adaptive proposal distribution can help to improve the sampling efficiency. The resulting draws, however, are not targeting the correct posterior, and so importance weights, W_t , are used to correct this discrepancy.

3 Forward Model for the IMF

Due to their simple interpretations, mathematical ease, and demonstrated consistency with observations, power law IMFs (or similar variants) have been widely adopted in the astronomy literature (Kroupa et al. 2012); however, open questions remain about stellar formation processes. The proposed forward model is a way to link a possible stellar formation process with the realized mass function (MF). One known underlying mechanism for producing data with power-law tails is based on *preferential attachment* (PA). The earliest PA model, the Yule-Simon process, was popularized by Simon (1955), and was originally used to model biological genera and word frequencies[[Jessi: add more citations]]. [[Jessi: make connection to IMF]]

3.1 Preferential attachment for the IMF

describe formation theory we are going to try to capture. Rather than assuming that stellar masses in a cluster arise independently of each other, our PA model proposes a resource-limited mass accretion process between stellar cores whose ability to accumulate additional mass is a function of their existing masses. This dependence feature is particularly important for statistical inference, as models that assume independent observations of stellar masses are vulnerable to incorrect and misleading inference. Additionally, the mass of the largest star to form in a cluster is limited by the total cluster mass.

Our proposed stochastic model for stellar formation is as follows: we first fix a total available cluster mass M_{tot} . This quantity can be physically interpreted as the total mass available for stellar formation in a molecular cloud. At each time step $t = 1, 2, \dots$, a random quantity of mass $m_t \sim \text{Exponential}(\lambda)$ enters the collection of stars; $M_{1,1} = m_1$ becomes the mass of the first star. Subsequent masses entering the system form a new star with

probability π_t or join existing star $k = 1, \dots, n_t$ with probability π_{kt} . These probabilities are specified as

$$\pi_t = \min(1, \alpha) \quad \text{and} \quad \pi_{kt} \propto M_{k,t}^\gamma. \quad (2)$$

The generating process is complete when the total mass of formed stars reaches M_{tot} . The possible ranges of the three parameters are $\lambda > 0$, $\alpha \in [0, 1]$, and $\gamma > 0$.

The parameter α defines the probability that entering mass forms a new star in a cluster. For the growth component, the model allows for linear ($\gamma = 1$), sublinear ($\gamma < 1$), and superlinear ($\gamma > 1$) behavior; the limiting case of $\gamma \rightarrow 0$ gives a uniform attachment model. Finally, the parameter λ acts as a scaling factor which controls the average ‘coarseness’ of masses joining the forming stellar cores.

The proposed PA mass generation model offers considerable flexibility to approximate existing IMF models in the literature. To illustrate the generality of the proposed model, IMF realizations were drawn assuming the Kroupa (2001) broken power-law model as the true model and the Chabrier (2005) lognormal model as the true model. Our proposed PA ABC procedure was then used for inference and Figure 2 displays the resulting posterior predictive IMFs. The proposed model captures the general shape of the true model. Figures 3a - 3c display ABC marginal posteriors for the broken power-law model and Figures 3d - 3f display the ones for the lognormal model. Both the broken power-law and lognormal models have similar ABC posterior means for α (0.293 and 0.304, respectively). However, the ABC posterior means for γ are notably different. The broken power-law model has an ABC posterior mean of 0.889 while the estimate for the lognormal model is 1.050. [[Jessi: perhaps expand on the previous points to emphasize why this may be scientifically interesting]]

3.1.1 Connection to power law tails?

In the stellar formation context, linear PA models generate power law IMFs; sublinear variants produce slight deviations from power laws, and exponential cutoffs can also be induced. [[Jessi: I'm not sure about the previous statement - it may have been written in

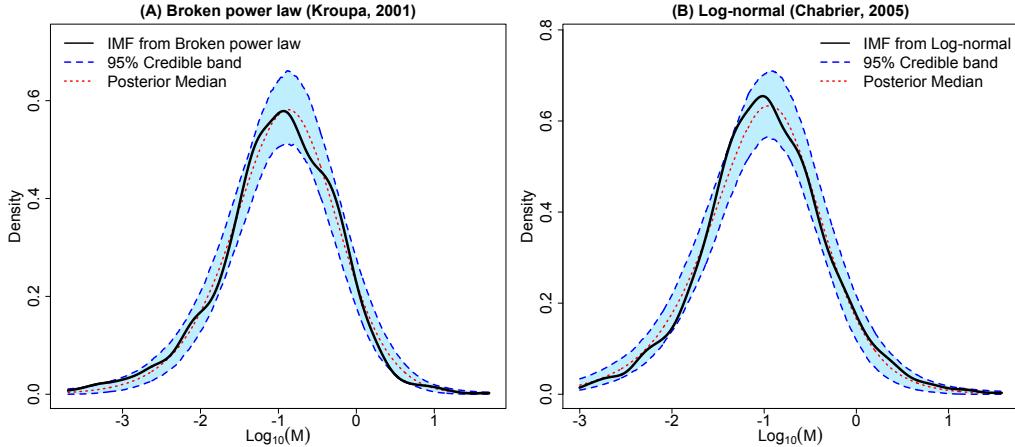


Figure 2: The solid black lines display the IMF with 1,000 stars simulated from (A) the broken power-law model of Kroupa (2001) and (B) the log-normal model of Chabrier (2005). The proposed PA ABC model was used with $N = 1,000$ particles, and 95% point-wise credible bands are displayed (blue, dashed lines) along with the posterior median (red, dotted) for each data set. The preferential attachment model provides flexibility to approximate both existing models. Both models have similar ABC posterior means for α (0.293 and 0.304, respectively). However, the ABC posterior means for γ are notably different. The broken power-law model has an ABC posterior mean of 0.889 while the estimate for the lognormal model is 1.050.

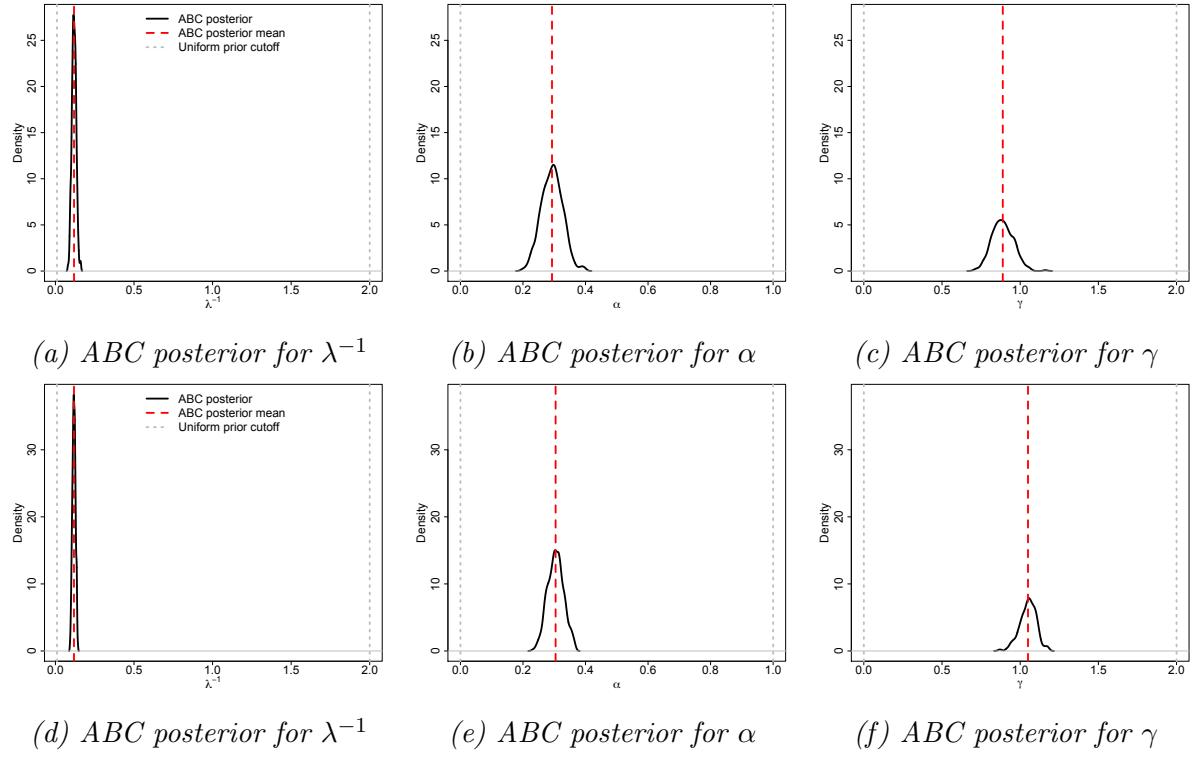


Figure 3: (top row) Marginal ABC posteriors for data generated from the broken power-law model of Kroupa (2001) and for data generated from the lognormal model of Chabrier (2005). The vertical dashed red lines indicate the ABC posterior mean, and the dotted gray lines indicate the range of the uniform prior for the parameter.

a much earlier draft. It can/should be revised based on what we decide to include in the discussion about the power law tails]]

3.2 Initial Mass Function to the Observed Mass Function

The PA model describes the formation of a star cluster at initial formation. However, we are not generally able to observe the star cluster after initial formation due to observational uncertainties, measurement uncertainties, and aging and dynamical evolution of the cluster. A cluster’s present-day observed mass function (MF) is the observed distribution of the stellar masses of a particular cluster.

Observation limitations can be easily incorporated into the ABC framework. For simplicity, we adopt a “linear ramp” completeness function describing the probability of observing a star of mass m :

$$\text{Pr}(\text{observing a star} \mid m) = \begin{cases} 0, & m < C_{\min} \\ \frac{m - C_{\min}}{C_{\max} - C_{\min}}, & m \in [C_{\min}, C_{\max}] \\ 1, & m > C_{\max}. \end{cases} \quad (3)$$

We assume that the values C_{\min} and C_{\max} are known, though we note that selecting an appropriate completeness function is a difficult process which requires quantification from the observational astronomers for each set of data. Different models for the completeness function could also be considered, including those which allow for spatially-varying observational completeness. A benefit of ABC is the ease at which a new completeness function can be incorporated – it amounts to a simple change in the forward model.

Due to measurement error and practical limitations in translating luminosities into masses, the masses of stars are not perfectly known. This uncertainty can be incorporated in different ways; following Weisz et al. (2013), we assume that the inferred mass of a star m_i is related to its true mass M_i via

$$\log m_i = \log M_i + \sigma_i \eta_i, \quad (4)$$

where η_i is a standard normal random variable, and σ_i is known measurement error. The

model for mass uncertainties in (4) is simple and could be extended to account for other sources of uncertainty (e.g. redshift).

As noted previously, the lifecycle of a star depends on certain characteristics such as mass. In the proposed algorithm, stars generated in a cluster are aged using a simple truncation of the largest masses. That is, the distribution of stellar masses for a star cluster of age τ Myr is given by

$$f_M(m \mid \theta, \tau) = f_M(m \mid \theta) \mathbb{I}\{M \leq \tau^{-1/3} \times 10^{4/3}\}, \quad (5)$$

corresponding to stellar lifetimes of $10^4 M^{-3}$ Myr, where M is the mass of the star (Hansen et al. 2004, Chaisson & McMillan 2011). More sophisticated models that account for effects such as binary stars and stellar wind mass loss can be inserted into this framework.

4 Methods

We propose an ABC framework to make inferences on the IMF given a cluster’s present-day observed MF. The proposed ABC algorithm is displayed in Algorithm (1), where N is the desired particle sample size to approximate the posterior distribution, and is motivated by the adaptive ABC algorithm of Beaumont et al. (2009). The forward model, F , in Algorithm (1) is where observational limitations and uncertainties, stellar evolution, and other astrophysical elements can be incorporated as outlined in Section 3.

4.1 Proposed ABC algorithm

Algorithm (1) is initialized using the basic ABC rejection algorithm at time step $t = 1$ using a distance function $\rho(m_{\text{sim}}, m_{\text{obs}})$ to measure the distance between the simulated and observed datasets, m_{sim} and m_{obs} , respectively. The first tolerance, ϵ_1 , is adaptively selected by drawing kN particles for some $k > 0$. Then the N particles that have the smallest distance are retained, and ϵ_1 is defined as the largest of those N distances retained. For subsequent time steps ($t > 1$), rather than proposing a draw, θ^* , from the prior, $\pi(\theta)$, the proposed θ^* is selected from the previous time step’s ($t - 1$) ABC posterior samples.

The selected θ^* is then moved according to some kernel, $K(\theta^*, \cdot)$, to ameliorate degeneracy as the sampler evolves. In order to ensure the true posterior (which requires sampling from the prior) is targeted, the retained draws are weighted according to the appropriate importance weights, W_t – this incorporates the proposal distribution’s kernel.

Data: Observed stellar masses, (m_{obs})

Result: ABC posterior sample of θ

At iteration $t = 1$:

for $j = 1, \dots, kN$ **do**

 Propose $\theta_t^{(j)}$ by drawing $\theta_t^* \sim \pi(\theta)$

 Generate cluster stellar masses m_{sim} from $F(m | \theta_t^*)$

 Calculate distance $\rho_t^{(j)} \leftarrow \rho(m_{\text{sim}}, m_{\text{obs}})$

end

$\theta_t^{(j)} \leftarrow \theta_t^{(l)}, l = \text{index of } N \text{ smallest } \rho_t^{(q)}, q = 1, \dots, kN$

$W_t^{(j)} \leftarrow 1/N, j = 1, \dots, N$

At iterations $t = 2, \dots, T$:

for $j = 1, \dots, N$ **do**

while $\rho(m_{\text{sim}}, m_{\text{obs}}) > \epsilon_t$ **do**

 Select $\theta^{(j)}$ by drawing from the $\theta_{t-1}^{(i)}$ with probabilities $W_{t-1}^{(i)}, i = 1, \dots, N$

 Generate $\theta^{*(j)}$ from transition kernel $K(\theta^{(j)}, \cdot)$

 Generate cluster stellar masses m_{sim} from $F(m | \theta^{*(j)})$

end

$\theta_t^{(j)} \leftarrow \theta^{*(j)}$

$W_t^{(j)} \leftarrow \frac{\pi(\theta_t^{(j)})}{\sum_{i=1}^N W_{t-1}^{(i)} K(\theta_{t-1}^{(i)}, \theta_t^{(j)})}$

end

$W_t^{(j)} \leftarrow \frac{W_t^{(j)}}{\sum_{l=1}^N W_t^{(l)}}, j = 1, \dots, N$

Algorithm 1: Stellar IMF ABC algorithm with sequential sampling

A key step in the implementation of an ABC algorithm is to quantify the distance between the simulated and observed stellar masses. We define a bivariate summary statistic

and distance function that captures the shape of the present-day MF and the random number of stars observed, displayed in Equations (6) and (7), respectively. For the shape of the present-day MF, we use a kernel density estimate of the \log_{10} masses (due to the heavy-tailed distribution of the initial masses), and an L_2 distance between the simulated and observed \log_{10} mass function estimates. The number of stars observed is the other summary statistic, with the distance being the absolute value of the difference in the ratio of the counts from 1. The bivariate summary statistic is defined as

$$\rho_1(m_{\text{sim}}, m_{\text{obs}}) = \left[\int \left\{ \hat{f}_{\log m_{\text{sim}}}(x) - \hat{f}_{\log m_{\text{obs}}}(x) \right\}^2 dx \right]^{1/2} \quad (6)$$

$$\rho_2(m_{\text{sim}}, m_{\text{obs}}) = \max \left\{ \left| 1 - n_{\text{sim}}/n_{\text{obs}} \right|, \left| 1 - n_{\text{obs}}/n_{\text{sim}} \right| \right\}, \quad (7)$$

where the \hat{f} are kernel density estimates, and n_{sim} and n_{obs} are the number of stars comprising the observed MF. These summary statistics were selected based on performance of a simulation study using the high-mass section of the broken power-law model because the true posterior is known in this setting. Results and additional discussion of the simulation study can be found in Appendix A.

With the bivariate summary statistic, we use a bivariate tolerance sequence, $(\epsilon_{1t}, \epsilon_{2t})$, for $t = 1, \dots, T$ is such that $\epsilon_{i1} \geq \epsilon_{i2} \geq \dots \geq \epsilon_{iT}$ for $i = 1, 2$. At time step t , the tolerances are determined based on the empirical distribution of the retained distances from time step $t - 1$ (e.g. the 25th percentile). As noted previously, the tolerance sequence is initialized adaptively by selecting kN proposals from the prior distributions, then the N proposals that result in the N smallest distances were selected.¹

In practice, M_{tot} is an unknown quantity of interest. A prior can be assigned to M_{tot} and an additional summary statistic and tolerance sequence can be used. The summary statistics we select in this case is

$$\rho_3(m_{\text{sim}}, m_{\text{obs}}) = \left| \sum_{i=1}^{n_{\text{sim}}} m_{\text{sim},i} - \sum_{j=1}^{n_{\text{obs}}} m_{\text{obs},j} \right|, \quad (8)$$

where $m_{\text{sim},i}$ and $m_{\text{obs},j}$ are the masses of the individual simulated and observed stars, respectively.

¹The kN sampled distances were scaled, squared, and then added together; the N smallest of these combined distances were retained.

4.2 Simulation study

A short simulation study was carried out to illustrate the performance of the proposed model in the case where the solution is known, using the summary statistics in Equations (6)-(8). Values for the parameters $(\lambda^{-1}, \alpha, \gamma, M_{\text{tot}})$ are displayed in Table 1 along with the number of stars produced, n_{obs} , which are used in the simulation as the observations. The generated IMF's used as the observations for each setting are displayed in Figure 4; number counts rather than densities are displayed in order to emphasize the different number of stars produced under each setting.

Number	λ^{-1}	α	γ	M_{tot}	n_{obs}
1	0.25	0.3	0.5	1000	1166
2	0.25	0.3	1.0	1000	1142
3	0.25	0.3	1.5	1000	1148
4	0.25	0.7	0.5	1000	2801
5	0.25	0.7	1.0	1000	2698
6	0.25	0.7	1.5	1000	2710

Table 1: Parameter values for the simulation study and the number of stars in the data set, n_{obs} .

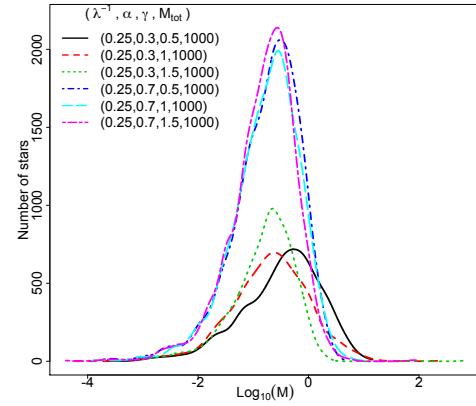


Figure 4: The simulated IMF's used as the observations in the simulation study displayed as number of stars by Log_{10} mass.

One of the more notable differences in the simulated IMFs displayed in Figure 4 used as the observations in the simulation study is the number of stars generated. The α parameter controls the numbers stars that form and so there appears to be two groups of IMFs: one with $\alpha = 0.3$ and one with $\alpha = 0.7$. The resulting marginal ABC posteriors for λ^{-1} , α , γ , and M_{tot} are displayed in Figure 5. For λ^{-1} and M_{tot} (Figure 4(A) and (D), respectively) cover the input values well with differences in the width of the ABC posteriors. In general, the marginal ABC posteriors for α and γ (Figure 4(B) and (C), respectively) cover the input values of the parameters, but their shapes and widths vary. For example, the width of the marginal posteriors for α appear to depend on the corresponding value of γ , where

$\gamma = 0.5$ results in the broadest marginal ABC posteriors for a fixed α , and $\gamma = 1.5$ results in the narrowest marginal ABC posteriors for a fixed α . [[Jessi: I'm not really sure why the previous statement happens]]

The pattern for γ is a bit more complicated. For a $\gamma \leq 1$, a smaller α corresponds with a narrower posterior. This makes sense because a smaller α results in fewer new stars, which means the entering piece of mass has to join an already formed star, at a rate controlled by the value of γ . Hence, more pieces that have to be assigned to existing stars seems to result in more information about the rate of growth of the stars. However, for a $\gamma > 1$, a larger α (producing more new stars) appears to lead to a narrower marginal ABC posterior for γ than when α is larger (0.7). The marginal ABC posterior for γ corresponding to the smaller value for α (0.3) is wider and pushes against the boundary of the prior. Interestingly, it turns out that this behavior seems to occur because when γ is greater than 1 and α is smaller, all the mass that has to be distributed to the already existing stars (rather than forming a new star) tends to be assigned to the same, most massive star. There ends up a being a runaway effect in the scenario with $\gamma > 1$ and smaller α where almost all of the incoming mass is assigned to the same star, resulting in one extremely massive star. This means there is essentially only one star for providing information on the γ , which may be why the ABC posterior is wider and pushes against the upper boundary of the prior on γ .

4.2.1 Simulated data with observational effects

5 Astrophysical simulation data

Next we consider a star cluster generated from a radiation hydrodynamical simulation presented in Bate (2012). This simulation results in 183 stars and brown dwarfs formed from a $500 M_{\odot}$ molecular cloud of uniform density, along with the growth of the stars and brown dwarfs across the duration of the simulation. Figure 7 displays the growth curves of the 183 objects and the IMF. The total mass of the resulting objects is about $88.68 M_{\odot}$. The details of the simulation can be found in Bate (2012). Validation of the simulated cluster was carried out by comparing the simulated cluster with real observations using

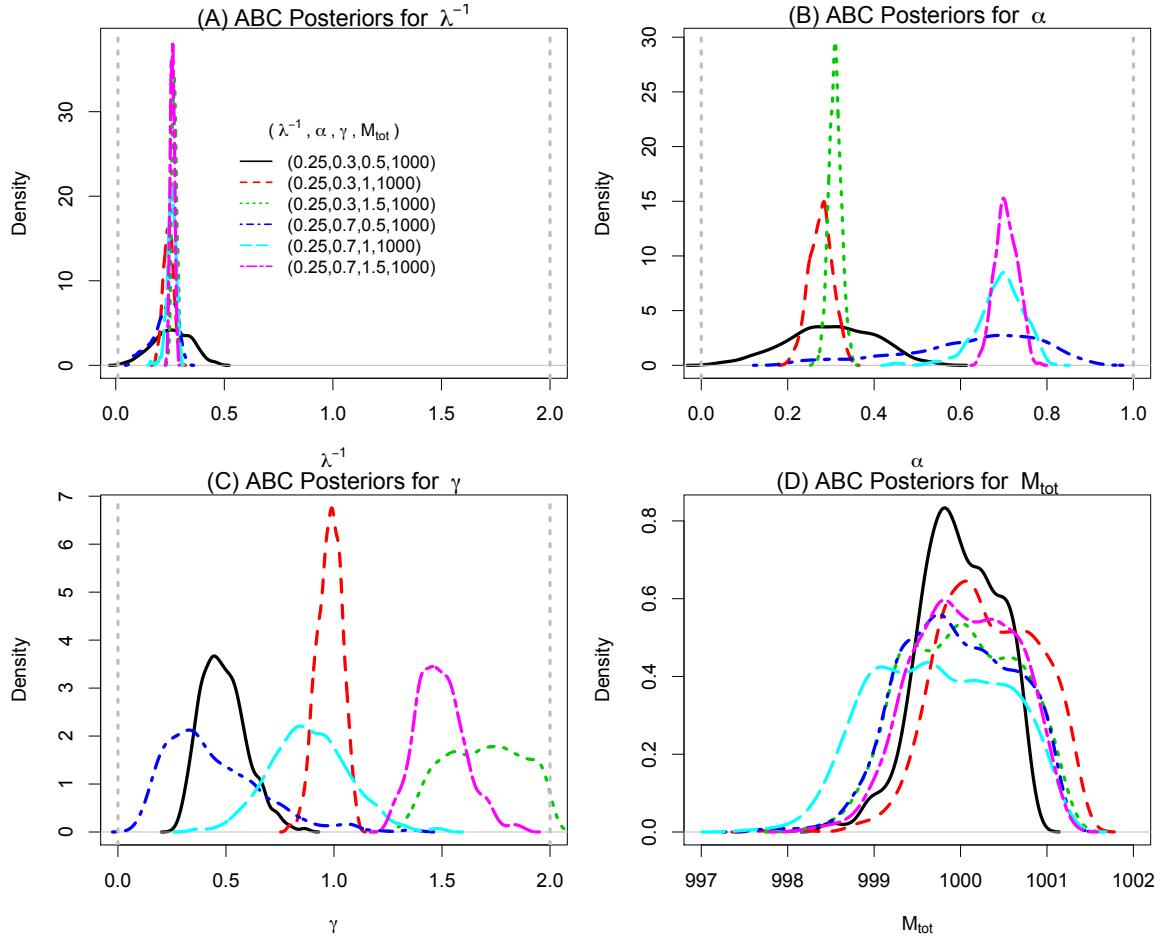


Figure 5: Marginal ABC posteriors for the simulation settings of Table 1. The different color and types of lines indicate the input parameter values corresponding to the weighted kernel density estimates of the marginal ABC posteriors for (A) λ^{-1} , (B) α , (C) γ , and (D) M_{tot} . The vertical dotted gray lines in plots (A) - (C) indicate the range of the uniform prior for the parameter. The prior used for M_{tot} was a Normal distribution with a prior mean of 1200 and a prior standard deviation of 600.

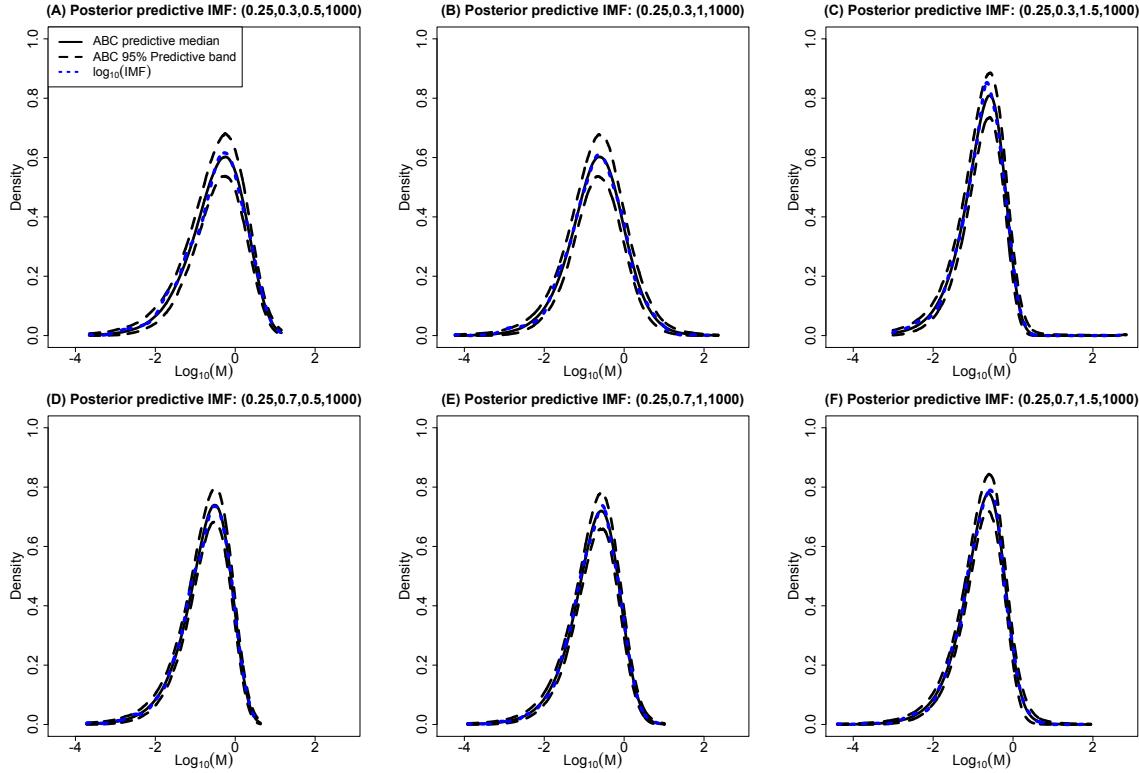


Figure 6: ABC posterior predictive IMFs for the simulation settings of Table 1. The blue dotted lines display the true IMF used in each simulation setting, the solid black line is the median ABC posterior predictive IMF, and the dashed black lines indicate a point-wise 95% credible band. The posterior predictive IMFs and band were derived from taking 1000 draws from the final ABC posteriors in each setting, and then simulating an IMF using the sampled values. The input parameter values, $(\lambda^{-1}, \alpha, \gamma, M_{tot})$, are displayed in each sub-plot title.

[ADD CLUSTER COMPARISONS MADE].

marginal ABC posteriors in Figure 8

Posterior predictive IMF in Figure 9

1000 particles, quantile = 0.25, 5 iterations

final tolerance for MF shape: 0.009360724

final tolerance for star count: 0.005464481

λ^{-1} posterior mean = 0.260

α posterior mean = 0.537

γ posterior mean = 1.091

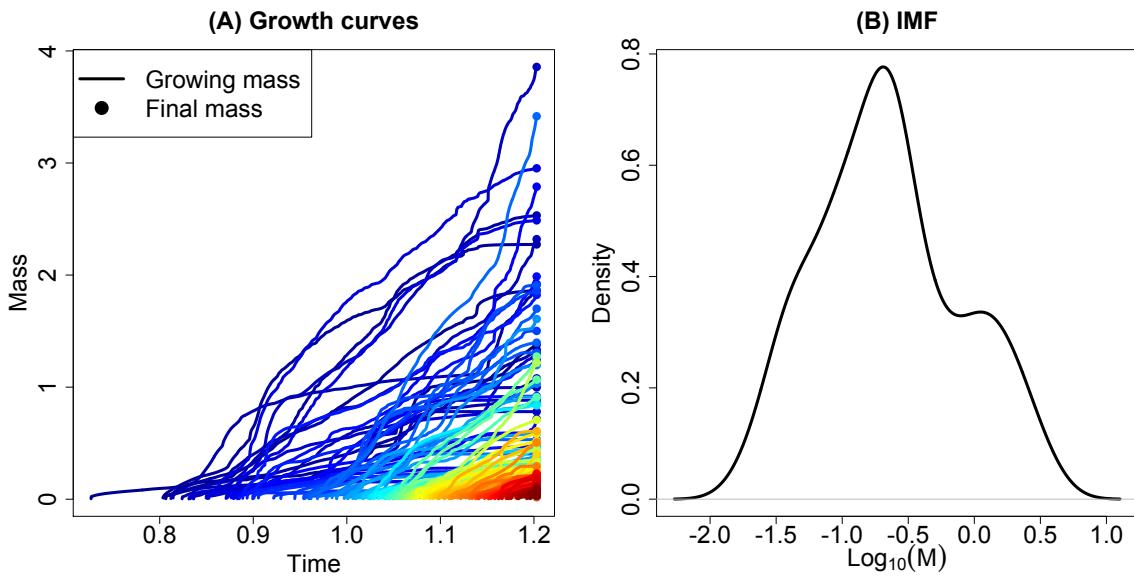


Figure 7: Astrophysical simulation data. (A) The growth of stars and brown dwarfs across simulation time from Bate (2012), and (B) the resulting IMF. The colors in (A) distinguish the 183 stars and brown dwarfs.

- Section 2.1 of Bate on sink particles seems to describe the accretion process
- In Bate - should not have significant super-linear attachment. Check to see if he has examples where there is super-linear attachment.

“Sink particles are permitted to merge in the calculation if they passed within 0.01 au of each other. However, no mergers occurred during the calculation.”

- 2.3.2 has discussion about the accretion radius
- Changes in the number of stars and mean masses with vs. without radiative feedback
- “In this paper, we refer to the mass function as an IMF because we compare it to the observed IMF since the PMF cannot yet be determined observationally. However, it should be noted that how a PMF transforms into the IMF depends on the accretion history of the protostars and how the star formation process is terminated.”

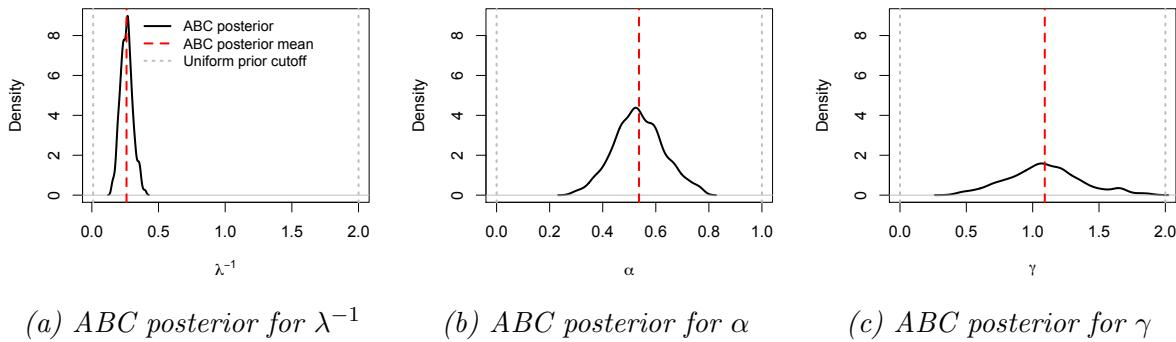


Figure 8: Marginal ABC posteriors for astrophysical simulation data from Bate (2012). The vertical dashed red lines indicate the ABC posterior mean, and the dotted gray lines indicate the range of the uniform prior for the parameter.

6 Discussion

Accounting for complex dependencies in observations, such as the initial masses of stars forming from a molecular cloud, is a challenging statistical problem. A possible, but unsatisfactory, resolution is simply to proceed as though the dependency is weak enough to assume independence. Instead, we draw from ideas in the preferential attachment literature proposing a new forward model to be used in an ABC algorithm. Though the new generative model was motivated by inference on stellar IMFs, it is generalizable to other applications that operate on a similar principle of dependent data. First we verified that the ABC approach works under the simplifying assumption of independence to ensure the

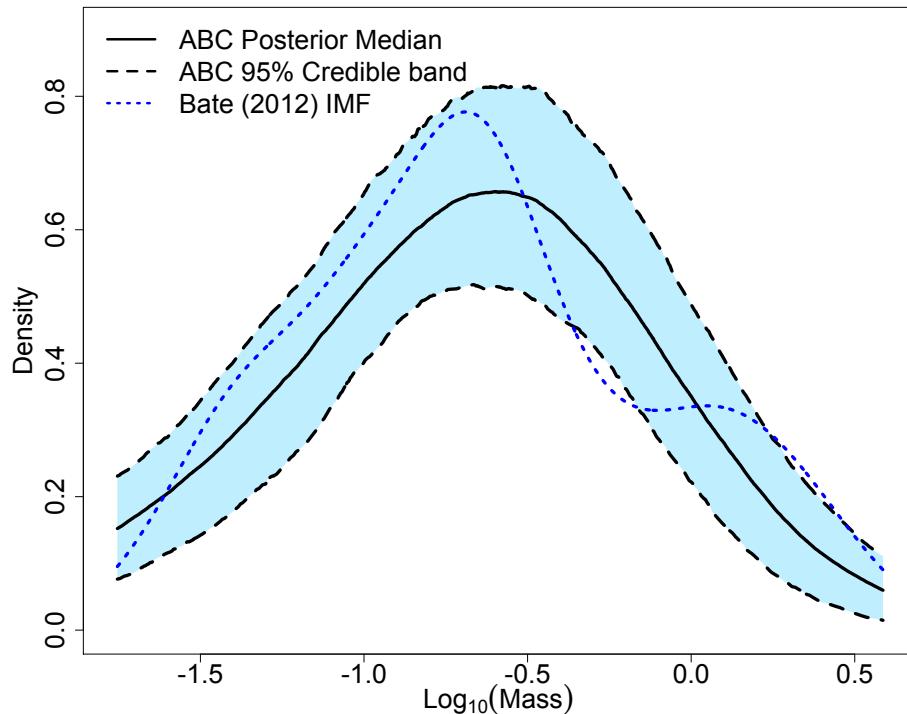


Figure 9: Posterior predictive IMF for astrophysical simulation data from Bate (2012). The median of the posterior predictive IMF (solid black) with a corresponding 95% point-wise predictive band (dashed black) compared to the true IMF (blue dotted). For the posterior predictive IMF, 1000 draws were made from the ABC posteriors of Figure 8 and then 1000 cluster samples were drawn from the proposed forward model.

true posterior was recoverable using ABC with the various observational effects. Then we introduced the new generative model, which starts with the total mass and stochastically builds individual stars of particular mass. The new model is general in the sense that it captures the dependencies among the masses of the stars, and also can accommodate standard models proposed in the astronomical literature.

The ONC is an interesting object of study among astronomers. The proposed preferential attachment model fitted via ABC captures the fluctuations in the observed mass function, as well as accounting for varying uncertainty as a function of available data in different regions of the mass range. While the proposed model is able to account for the dependency among the masses (along with the aging, completeness function, and observation uncertainty effects), it would be scientifically and statistically interesting to generalize the generative model to capture the spatial dependency among the observations. Intuitively, such an approach could account for a mechanism that limits the formation of multiple very massive stars relatively near to each other. Understanding the spatial distribution of masses of stars during formation would help advance our understanding stellar formation and evolution. Additionally, we did not attempt to capture the possible disturbances to the observed mass function as stars die, with the exception of deaths of the most massive stars.

Overall, the proposed generative model and ABC methodology provide a useful framework for dealing with complex physical processes that are otherwise difficult to work with in a statistically rigorous fashion. As increasing computational resources allow for greater model complexity in studies of astronomy and other scientific fields, the proposed and other ABC algorithms may open new opportunities for Bayesian inference in challenging problems.

A ABC summary statistic selection

In order to select effective summary statistics for the proposed model, we first employ the ABC methodology in a simplified study that focuses on the posterior of the power law parameter α from Equation (1). We generate a cluster of $n = 10^3$ stars from an IMF with

slope $\alpha = 2.35$ (Salpeter 1955), $M_{\min} = 2$, and $M_{\max} = 60$, a uniform prior distribution for $\alpha \in (0, 6)$, and with M_{\max} considered known. This simple model was used in order to check the method against the true posterior of α after the observational and aging effects have been incorporated into the forward model. We use the bivariate summary statistic and distance function of Equation (6). Defining the two-dimensional tolerance sequence as $(\epsilon_{1t}, \epsilon_{2t})$ where the subscript t indicates the algorithm time step, and ϵ_{11} and ϵ_{21} were selected using an adaptive started discussed in Section 4 using an initial number of draws of $10 \times N$ with $N = 10^3$. The algorithm ran for $T = 5$ time steps. At steps $t = 2, \dots, T$, ϵ_{1t} and ϵ_{2t} were set equal to the 25th percentile of the distances retained at the previous step from their corresponding distance functions.

The pseudo-data were aged 30 Myrs with the measurement error of $\sigma = 0.25$ and observation completeness defined by the linear-ramp function in (3). Figure 10 displays the ABC posterior resulting from the ABC algorithm along with the true posterior for α . The ABC posterior matches the true posterior, defined as

$$\begin{aligned} \pi_F(\alpha | m_{\text{obs}}, M_{\min}, M_{\max}, n_{\text{obs}}, n_{\text{tot}}, T_{\text{age}}) &\propto & (9) \\ &\left\{ \Pr(M > T_{\text{age}}) + \left(\frac{1-\alpha}{M_{\max}^{1-\alpha} - M_{\min}^{1-\alpha}} \right) \int_2^4 M^{-\alpha} \left(1 - \frac{M-2}{2} \right) dM \right\}^{n_{\text{tot}} - n_{\text{obs}}} \\ &\times \prod_{i=1}^{n_{\text{obs}}} \left\{ \int_2^{T_{\text{age}}} (2\pi\sigma^2)^{-\frac{1}{2}} m_i^{-1} e^{-\frac{1}{2\sigma^2}(\log(m_i) - \log(M))^2} \left(\frac{1-\alpha}{M_{\max}^{1-\alpha} - M_{\min}^{1-\alpha}} \right) M^{-\alpha} \right. \\ &\quad \left. \times \left(I\{M > 4\} + \left(\frac{M-2}{2} \right) I\{2 \leq M \leq 4\} \right) dM \right\} \end{aligned}$$

where $T_{\text{age}} = \tau^{-1/3} \times 10^{4/3}$ is the upper-tail mass cutoff due to aging.

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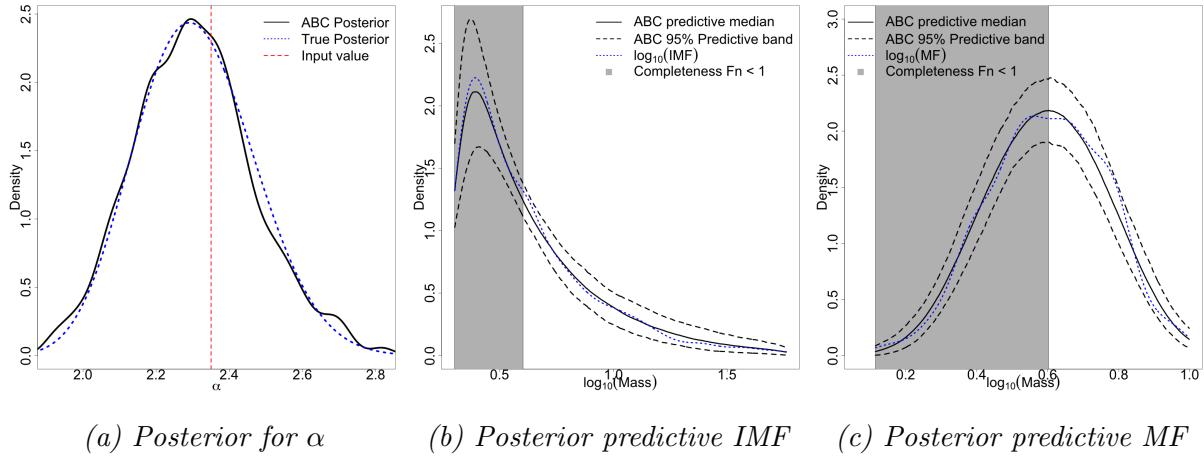


Figure 10: Validation of summary statistics with power law model. (A) The ABC posterior for α (solid black) compared to the true posterior (dotted blue) of Equation (9) using an input value of 2.35 (dashed vertical red). (B) The median of the posterior predictive IMF (solid black) with a corresponding 95% point-wise predictive band (dashed black) compared to the true IMF (blue dotted) which was simulated dataset before aging, completeness, or uncertainty were applied, and the gray shaded region indicates where the completeness function was less than 1. (C) The median of the posterior predictive MF (solid black) with a corresponding 95% point-wise predictive band (dashed black) compared to the observed MF (dotted blue) which was the simulated dataset after aging, completeness, and uncertainty were applied. For the posterior predictive IMF, 1000 draws were made from the ABC posterior of (A) and then 1000 cluster samples were drawn from the power law simulation model. For the posterior predictive MF, the 1000 cluster samples used for (B) were then put through the forward model to apply the aging, completeness, and measurement error effects.

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