CHAPTER

2

Fuzzy Sets and Logic

Syllabus Topic : Basic Concepts of Fuzzy Logic, Fuzzy Set Theory

2.1 Basic Concepts of Fuzzy Logic

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- Fuzzy logic was introduced by Prof. Lofti A. Zadeh in 1965.
- The word fuzzy means "Vagueness".
- Fuzziness occurs when a boundary of a piece of information is not clear.
- He proposed a mathematical way of looking at the vagueness of the human natural language.

Why fuzzy logic is required?

- Most of our traditional tools for formal modelling, reasoning and computing are crisp, deterministic and precise.
- While designing the system using classical set, we assume that the structures and parameters of the model are definitely known and there are no doubts about their values or their occurrence.
- But in real world there exists much fuzzy knowledge; knowledge that is vague, imprecise, uncertain, ambiguous, inexact or probabilistic in nature.
- There are two facts;
  1. Real situations are very often not crisp and deterministic and they cannot be described precisely.
  2. The complete description of a real system often would require more detailed data than a human being could ever recognize simultaneously, process and understand.

- Because of these facts, modeling the real system using classical sets often do not reflect the nature of human concepts and thoughts which are abstract, imprecise and ambiguous.
- The classical (crisp) sets are unable to cope with such unreliable and incomplete information.
- We want our systems should also be able to cope with unreliable and incomplete information and give expert opinion.
- Fuzzy set theory has been introduced to deal with such unreliable, incomplete, vague and imprecise information.
- Fuzzy set theory is an extension to classical set theory where element have degree of membership.
- Fuzzy logic uses the whole interval between 0 (false) and 1 (true) to describe human reasoning.

Syllabus Topic : Fuzzy Sets and Crisp Sets

2.2 Fuzzy Sets and Crisp Sets

2.2.1 Crisp Sets

2.2.1.1 Basics

- A classical set (or conventional or crisp set) is a set with a crisp boundary.
- For example, a classical set A of real numbers greater than 6 can be expressed as

$$A = \{x \mid x > 6\}$$

Where there is a clear, unambiguous boundary '6' such that if x is greater than this number, then x belongs to the set A, otherwise x does not belong to the set.

- Although classical sets are suitable for various approximations and have proven to be an important tool for mathematics and computer science, they do not reflect the nature of human concepts and thoughts, which are abstract, imprecise and ambiguous.
- For example, mathematically we can express the set of all tall persons as a collection of persons whose height is more than 6 ft.

$$A = \{x \mid x > 6\}$$

Where A = "tall person" and x = "height".

The problem with the classical set is that it would classify a person 6.001 ft. tall as a tall person, but a person 5.999 ft. tall as "not tall". This distinction is intuitively unreasonable.

- The flaw comes from the sharp transition between inclusion and exclusion in a set. (Fig. 2.2.1)

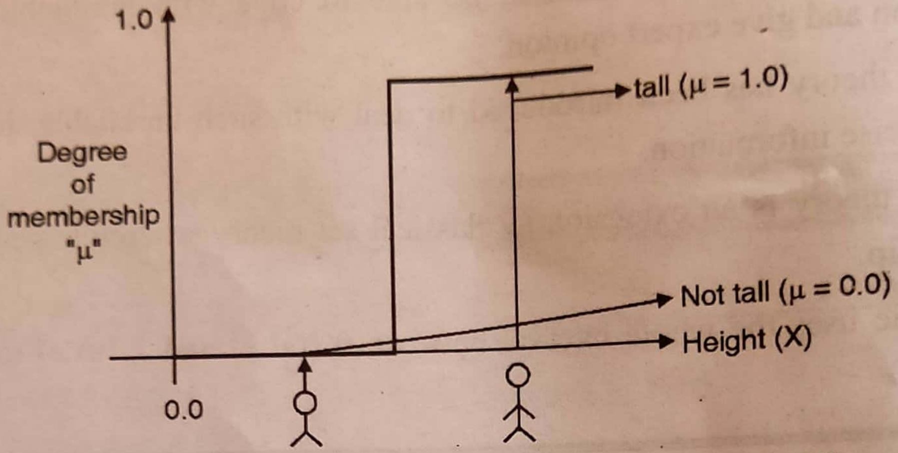


Fig. 2.2.1 : Sharp edged membership function for TALL

2.2.1.2 Operations on Classical / Crisp Sets

1. Union

The union of two classical sets A and B is given by the set of all elements which belong to either set A or set B and is denoted by $A \cup B$.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

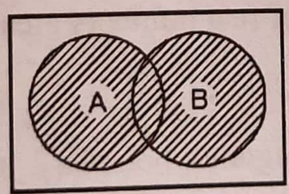


Fig. 2.2.2 : Union of A and B

2. Intersection

The intersection of two classical sets A and B is represented by the set of all the elements which belong to both A and B and is denoted by $A \cap B$.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

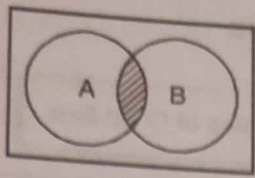


Fig. 2.2.3 : Intersection of A and B

3. **Complement**

The complement of set A is defined as the collection of all elements that do not belong to A and is denoted by $\bar{A}$.

$$\bar{A} = \{x \mid x \notin A,\ x \in X\}$$

Where X is the universal set and A is a given set formed from universe X.

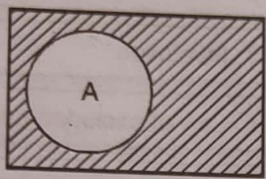


Fig. 2.2.4 : Complement of set A

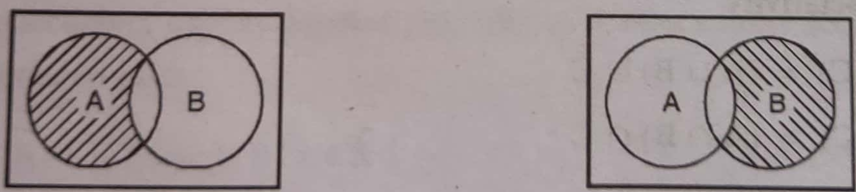
4. **Difference (Subtraction)**

The difference of set A with respect to set B is the collection of all elements in the universe which belong to A but do not belong to B and is denoted by A|B or A – B.

$$A \mid B \text{ or } A - B = \{x \mid x \in A \text{ and } x \notin B\} = A - (A \cap B)$$

Similarly, the difference of set B w.r.t. set A is given by,

$$B \mid A \text{ or } B - A = \{x \mid x \in B \text{ and } x \notin A\} = B - (B \cap A)$$



A|B B|A

(a) (b)

Fig. 2.2.5 : Difference operation

2.2.1.3 Properties of Crisp Sets

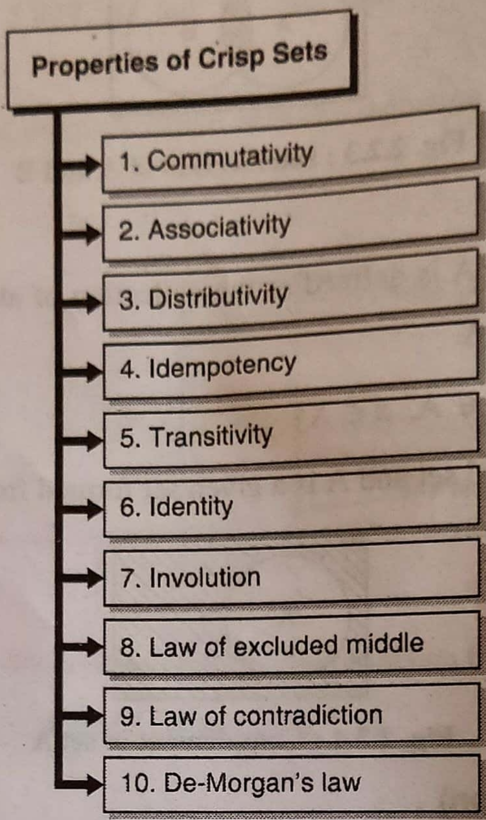


Fig. 2.2.6

→ 1) **Commutativity**

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

→ 2) **Associativity**

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

→ 3) **Distributivity**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

→ 4) **Idempotency**

$$A \cup A = A$$

$$A \cap A = A$$

5) Transitivity

If $A \subseteq B \subseteq C$, then $A \subseteq C$

6) Identity

$A \cup \phi = A, \quad A \cap \phi = \phi$

$A \cup X = X, \quad A \cap X = A$

7) Involution

$\bar{\bar{A}} = A$

8) Law of excluded middle

$A \cup \bar{A} = X$

9) Law of contradiction

$A \cap \bar{A} = \phi$

10) De-Morgan's law

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

2.2.2 Fuzzy Sets

2.2.2.1 Fuzzy Set : Definition

- If X is a collection of objects denoted generally by x, then a fuzzy set $\tilde{A}$ in X is defined as a set of ordered pairs :

$$\tilde{A} = \{ (x, \mu_{\tilde{A}}(x)) \mid x \in X \}$$

where $\mu_{\tilde{A}}(x)$ is called the membership function (MF for short) for the fuzzy set $\tilde{A}$:

- The MF maps each element of X to a membership grade between 0 and 1.
- If the value of membership function $\mu_{\tilde{A}}(x)$ is restricted to either 0 or 1, then $\tilde{A}$ is reduced to a classical set.

Note : Classical sets are also called ordinary sets, crisp sets, non fuzzy sets or just sets.

- Here, X is referred to as the **Universe of discourse** or simply the **Universe** and it may consist of discrete objects or continuous space.

2.2.2.2 Types of Universe of Discourse

Types of Universe of Discourse

- 1. Fuzzy sets with a discrete non ordered universe
- 2. Fuzzy sets with a discrete ordered universe
- 3. Fuzzy sets with a continuous universe

Fig. 2.2.7

→ 1. Fuzzy sets with a discrete non ordered universe

- Universe of discourse may contain discrete non-ordered objects.

  For example,

  Let X = {San Francisco, Boston, Los Angeles} be the set of cities one may choose to live in.

  The fuzzy set "desirable city to live in" may be described as follows :

  $\tilde{A}$ = {(San Francisco, 0.9), (Boston, 0.8), (Los Angeles, 0.6)}

- Here, the universe of discourse X is discrete and it contains non-ordered objects, in this case three big cities in United States.

→ 2. Fuzzy sets with a discrete ordered universe

- Let X = {0, 1, 2, 3, 4, 5, 6} be the set of number of children a family may choose to have.
- Then, a fuzzy set "Sensible number of children in family" may be described as ,

$$\tilde{A} = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}$$

Here we have a discrete ordered universe X.

The MF is shown in Fig. 2.2.8.

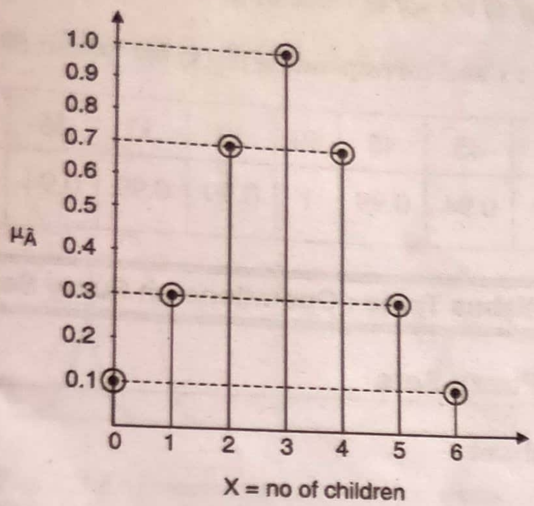


Fig. 2.2.8 : MF on a discrete universe

> **Note :** Membership grades of the fuzzy set are subjective measures. (For example, the height 5'5" may be considered tall in Japan, but in Australia, it may be considered medium).

➜ 3. Fuzzy sets with a continuous universe

Let $X = R^+$ be the set of possible ages for human beings. (Real numbers - continuous). Then the fuzzy set B = "about 50 years old" may be expressed as,

$$\tilde{B} = \{ (x, \mu_{\tilde{B}}(x)) \mid x \in X\}$$

Where, $$\mu_{\tilde{B}}(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

This is illustrated in Fig. 2.2.9.

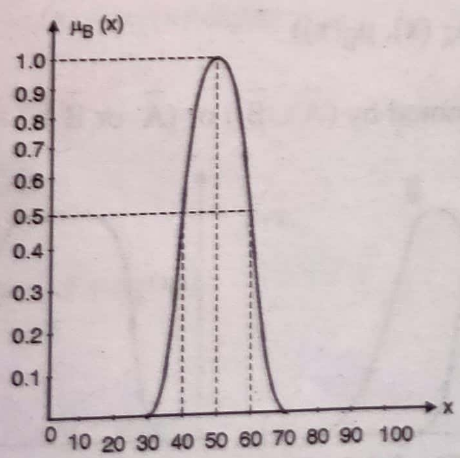


Fig. 2.2.9 : MF for "about 50 years old"

Table 2.2.1 shows $\mu_{\tilde{B}}(x)$ for some value of x.

Table 2.2.1 : x and corresponding $\mu_{\tilde{B}}(x)$ for "about 50 years old"

x	40	42	45	48	50	52	53	55	56	58	60
$\mu_{\tilde{B}}(x)$	0.5	0.71	0.94	0.99	1	0.99	0.99	0.94	0.89	0.71	0.5

Syllabus Topic : Operations on Fuzzy Sets

2.3 Operations on Fuzzy Sets

1. Containment or Subset

Fuzzy set $\tilde{A}$ is contained in fuzzy set $\tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ for all x.

$$\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$$

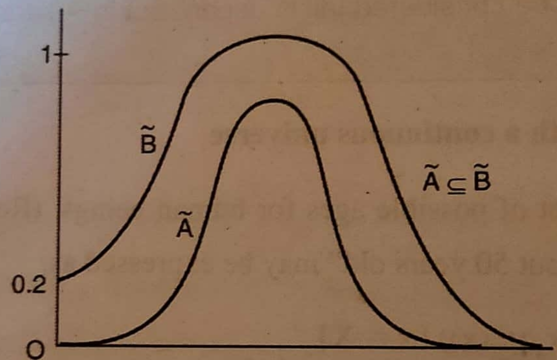


Fig. 2.3.1 : Containment or subset

2. Union (disjunction)

A union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, such that whose MF is,

$$\mu_{\tilde{C}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

Union of $\tilde{A}$ and $\tilde{B}$ is denoted by $(\tilde{A} \cup \tilde{B})$ or $(\tilde{A}$ or $\tilde{B})$

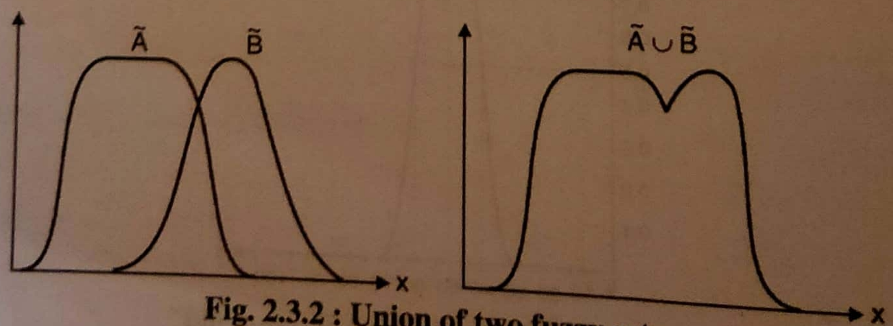


Fig. 2.3.2 : Union of two fuzzy sets

3. Intersection (conjunction)

The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set $\tilde{C}$, such that whose MF is defined as,

$$\mu_{\tilde{C}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

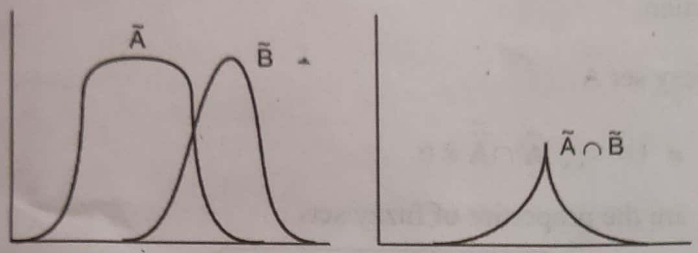


Fig. 2.3.3 : Intersection of two fuzzy sets

4. Complement (negation)

The complement of a fuzzy set $\tilde{A}$, denoted by $\bar{\tilde{A}}$ is defined as,

$$\mu_{\bar{\tilde{A}}}(x) = 1 - \mu_{\tilde{A}}(x)$$

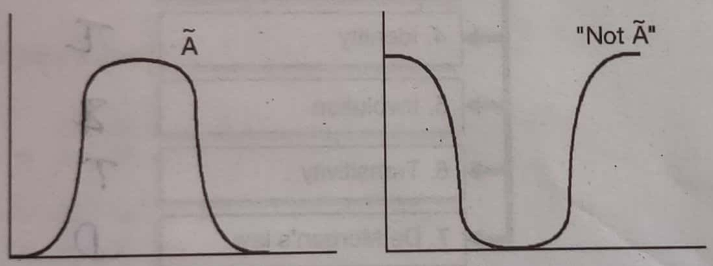


Fig. 2.3.4 : Complement of a fuzzy set

More operations of fuzzy sets

1. **Algebraic sum**

$$\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) . \mu_{\tilde{B}}(x)$$

2. **Algebraic product**

$$\mu_{\tilde{A}\cdot\tilde{B}}(x) = \mu_{\tilde{A}}(x) . \mu_{\tilde{B}}(x)$$

3. **Bounded sum** min!

$$\mu_{\tilde{A}\oplus\tilde{B}}(x) = \min[1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)]$$

4. **Bounded difference**

$$\mu_{\tilde{A}\odot\tilde{B}}(x) = \max[0, \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)]$$

Syllabus Topic : Properties of Fuzzy Sets

2.4 Properties of Fuzzy Sets

Fuzzy sets follow the same properties as crisp set except for the law of excluded middle and law of contradiction.

That is, for fuzzy set $\tilde{A}$

$$\tilde{A} \cup \overline{\tilde{A}} \neq U \quad ; \quad \tilde{A} \cap \overline{\tilde{A}} \neq \phi$$

The following are the properties of fuzzy sets :

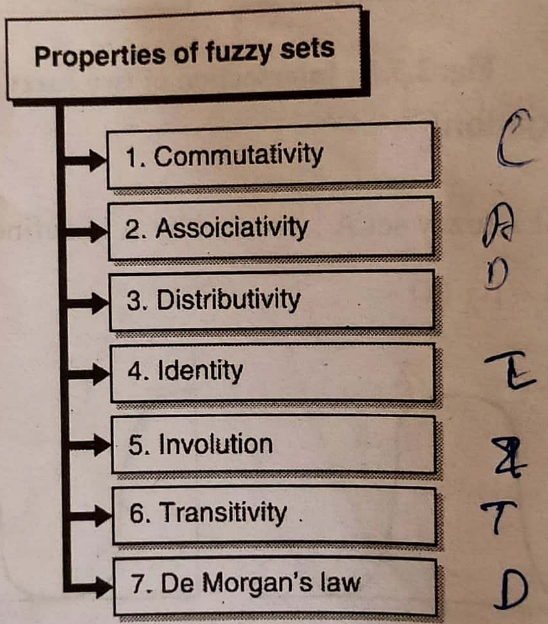


Fig. 2.4.1

➔ 1. Commutativity

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A} \quad \text{and}$$

$$\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$$

➔ 2. Assoiciativity

$$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$$

$$\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}$$

3. Distributivity

$$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$$

$$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$$

4. Identity

$$\tilde{A} \cup \phi = \tilde{A} \quad ; \quad \tilde{A} \cup U = U$$

$$\tilde{A} \cap \phi = \phi \quad ; \quad \tilde{A} \cap U = \tilde{A}$$

5. Involution

$$\overline{\overline{\tilde{A}}} = \tilde{A}$$

6. Transitivity

if $\tilde{A} \subset \tilde{B} \subset \tilde{C}$, then $\tilde{A} \subset \tilde{C}$

7. De Morgan's law

$$\overline{\tilde{A} \cup \tilde{B}} = \overline{\tilde{A}} \cap \overline{\tilde{B}}$$

$$\overline{\tilde{A} \cap \tilde{B}} = \overline{\tilde{A}} \cup \overline{\tilde{B}}$$

Syllabus Topic : Fuzzy and Crisp Relations

2.5 Fuzzy and Crisp Relations

2.5.1 Crisp (Classical) Relation

- An n-ary relation over $M_1, M_2, M_3, \ldots M_n$ is a subset of the Cartesian product $M_1 \times M_2 \times \ldots M_n$. Where n = 2, the relation is a subset of the Cartesian product $M_1 \times M_2$. This is called a binary relation from M_1 to M_2.

  Let X and Y be two universe and $X \times Y$ be their Cartesian product.

  Then $X \times Y$ can be defined as,

  $$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

  Every element in X is related to every element in Y.

- We can define characteristic function f that gives the strength of the relationship between the each element of X and Y.

$$f_{X\times Y}(x, y) = \begin{cases} 1 , & (x, y) \in X \times Y \\ 0 , & (x, y) \notin X \times Y \end{cases}$$

- We can represent the relation in the form of matrix.
- An n-dimensional relation matrix represents on n-ary relation.
- So, binary relation is represented by 2 dimensional matrices.

Example : Consider the following two universe,

$X = \{a, b, c\}, \quad Y = \{1, 2, 3\}$

The Cartesian product - $X \times Y$ is,

$X \times Y = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$

From the above set, we may select a subset R, such that

$R = \{(a, 1), (b, 2), (b, 3), (c, 1), (c, 3)\}$

Then R can be represented in matrix form as,

R	1	2	3
a	1	0	0
b	0	1	1
c	1	0	1

The relation between set X and Y can also be represented as coordinate diagram as shown in Fig. 2.5.1.

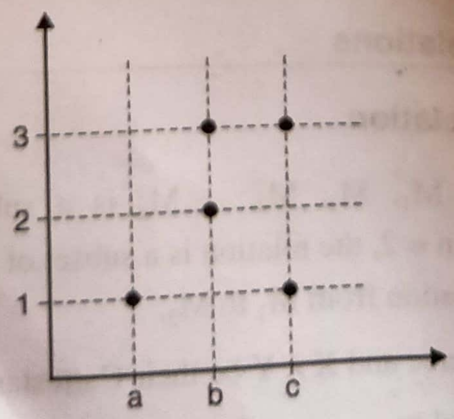


Fig. 2.5.1 : Co-ordinate diagram of a relation

The relation R can also be expressed by mapping representation as shown in Fig. 2.5.2.

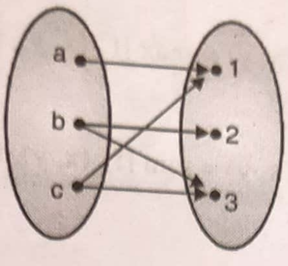


Fig. 2.5.2 : Mapping representation of a relation

- A characteristic function is used to assign values of relationship in the mapping of $X \times Y$ to the binary values and is given by,

$$f_{R(x,y)} = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

2.5.1.1 Cardinality of Classical Relation

Let X and Y be two universe and n elements of X are related to m elements of y.

Let the Cardinality of X is η_X and cardinality of Y is η_Y, then the cardinality of relation R between X and Y is,

$$\eta_{X \times Y} = \eta_X \times \eta_Y$$

The Cardinality of the power set P $(X \times Y)$ is given as,

$$\eta_{P(X \times Y)} = 2^{(\eta_X \eta_Y)}$$

2.5.1.2 Operations on Classical Relations

Let A and B be two separate relations defined on the Cartesian universe $X \times Y$.

Then the null relation defined as,

$$\phi_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And complete relation is defined as,

$$E_A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The following operations can be performed on two relations A and B.

1. **Union**

$A \cup B \rightarrow f_{A \cup B}(x, y) : f_{A \cup B}(x, y) = \max [f_A(x, y), f_B(x, y)]$

2. **Intersection**

$A \cap B \rightarrow f_{A \cap B}(x, y) : f_{A \cap B}(x, y) = \min [f_A(x, y), f_B(x, y)]$

3. **Complement**

$\bar{A} \rightarrow f_{\bar{A}}(x, y) : f_{\bar{A}}(x, y) = 1 - f_{\bar{A}}(x, y)$

4. **Containment**

$A \subset B \rightarrow f_A(x, y) : f_B(x, y) \leq f_B(x, y)$

5. **Identity**

$\phi \rightarrow \phi_A$ and $X \rightarrow E_A$

2.5.1.3 Properties of Crisp (Classical) Relations

- The properties of classical set such as commutativity, associativity, involution, distributivity and idempotency hold good for classical relation also.
- Also De Morgan's law and excluded middle laws hold good for crisp relations.

2.5.1.4 Composition of Crisp Relations

Composition is a process of combining two compatible binary relations to get a single relation.

Let A be a relation that maps elements from universe X to universe Y.

Let B be a relation that maps elements from universe Y to universe Z.

The two binary relations A and B are said to be compatible if,

$A \subseteq X \times Y$ and $B \subseteq Y \times Z$

The composition between the two relations A and B can be denoted as A o B.

Example :

$$\text{Let } X = \{a_1, a_2, a_3\}$$
$$Y = \{b_1, b_2, b_3\}$$
$$Z = \{c_1, c_2, c_3\}$$

Let the relation A and B as,

$$A = X \times Y = \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\}$$
$$B = Y \times Z = \{(b_1, c_1), (b_2, c_3), (b_3, c_2)\}$$

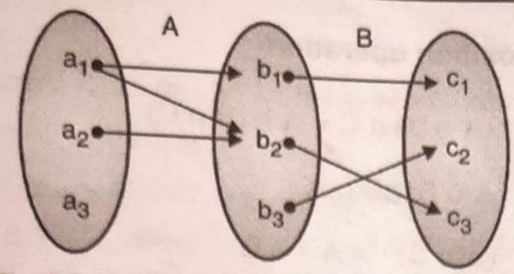


Fig. 2.5.3 : Illustration of relations A and B

Then A o B can be written as,

$$A \circ B = \{(a_1, c_1), (a_2, c_3), (a_2, c_3)\}$$

The representation of A and B in matrix form is given as,

$$A = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \; ; \quad B = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Then, composition A o B is represented as,

$$A \circ B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Types of composition operations

There are two types of composition operations:

1. Max-min composition
2. Max-product composition

The max-min composition is defined as,

$$T = A \circ B$$

$$f_T(x, z) = \bigvee_{y \in Y} [f_A(x, y) \wedge f_B(y, z)]$$

The max-product composition is defined as,

$$T = A \circ B$$

$$f_T(x, z) = \bigvee_{y \in Y} [f_A(x, y) \cdot f_B(y, z)]$$

Note : In the above equations $\vee$ represents max operation, $\wedge$ represents min operation and $\cdot$ represents product operation.

Properties of composition operation

1. Associative : $(A \circ B) \circ C = A \circ (B \circ C)$
2. Commutative : $A \circ B \neq B \circ A$
3. Inverse : $(A \circ B)^{-1} = A^{-1} \circ B^{-1}$

2.5.2 Fuzzy Relations

In general, a relation can be considered as a set of tuples, where a tuple is an ordered pair. Similarly, a fuzzy relation is a fuzzy set of tuples i.e. each tuple has a membership degree between 0 and 1.

2.5.2.1 Definition

Let U and V be continuous universe, and $\mu_R : U \times V\ [0, 1]$ then,

$$R = \int_{U \times V} \mu_R(u,v)/(u,v)$$

Is a binary fuzzy relation on $U \times V$.

If U and V are discrete universe then ,

$$R = \sum_{U \times V} \mu_R(u, v)/(u,v)$$

We can express fuzzy relation $R = U \times V$ in matrix form as,

$$R = \begin{bmatrix} \mu_R(u_1, v_1) & \mu_R(u_1, v_2) & \dots & \mu_R(u_1, v_n) \\ \mu_R(u_2, v_1) & \mu_R(u_2, v_2) & \dots & \mu_R(u_2, v_n) \\ \vdots & & & \\ \mu_R(u_m, v_1) & \mu_R(u_m, v_2) & \dots & \mu_R(u_m, v_n) \end{bmatrix}$$

Where $U = \{u_1, u_2, u_3, \dots, u_m\}$ and

$V = \{v_1, v_2, v_3, \dots v_n\}$ are universe of discourse.

Ex. 2.5.1 : *Given universe of discourse*

$U = \{1,2,3\}$ form a relation R where "x is approximately equal to y"

Then Relation R can be defined as,

$$R = \left\{\frac{1}{(1,1)} + \frac{1}{(2,2)} + \frac{1}{(3,3)} + \frac{0.8}{(1,2)} + \frac{0.3}{(1,3)} + \frac{0.8}{(2,1)} + \frac{0.8}{(2,3)} + \frac{0.3}{(3,1)} + \frac{0.8}{(3,2)}\right\}$$

Soln. :

The membership function μ_R of this relation can be described as,

$$\mu_R(x, y) = \begin{cases} 1 & \text{when} \quad x = y \\ 0.8 & \text{when} \quad |x - y| = 1 \\ 0.3 & \text{when} \quad |x - y| = 2 \end{cases}$$

The matrix notation is,

X \ Y	1	2	3
1	1	0.8	0.3
2	0.8	1	0.8
3	0.3	0.8	1

N - ary fuzzy relation

It is possible to define n-ary fuzzy relations as fuzzy set of n-tuples.

In general it is a relation of pairs.

$\mu_R(x_1, \ldots, x_n)/(x_1, \ldots, x_n);$

2.5.2.2 Operations on Fuzzy Relation

- Fuzzy relations are very important in fuzzy controller because they can describe interaction between variables.
- Four types of operations can be performed on fuzzy relation.

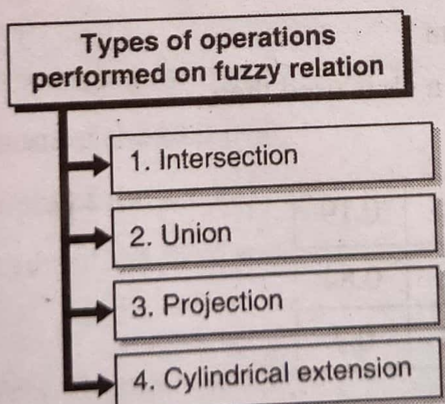


Fig. 2.5.4

1. Intersection

- Let R and S be binary relations defined on $X \times Y$. The intersection of R and S is defined by,
  $\forall (x, y) \in X \times Y : \mu_{R \cap S}(x, y) = \min(\mu_R(x, y), \mu_s(x, y))$
- Instead of the minimum, any T - Norm can be used.

2. Union

- The union of R and S is defined as,
  $\forall (x, y) \in X \times Y : \mu_{R \cup S}(x, y) = \max(\mu_R(x, y), \mu_s(x, y))$.
- Instead of maximum, any S – norm can be used.

Given two relations R and S

R =

	y_1	y_2	y_3
x_1	0.3	0.2	0.1
x_2	0.4	0.6	0.1
x_3	0.2	0.3	0.5

S =

	y_1	y_2	y_3
x_1	0.4	0	0.1
x_2	1	0.2	0.8
x_3	0.3	0.2	0.4

Then using max operation,

$R \cup S$ =

0.4	0.2	0.1
1	0.6	0.8
0.3	0.3	0.5

Suppose a simple s - norm

$S(a, b) = a + b - a \cdot b$ is used then,

$R \cup S$ =

0.58	0.2	0.19
1	0.68	0.84
0.44	0.44	0.7

This operation is more optimistic than the max operation. All the membership degrees are at least as high as in the max operation.

Now,

Using min operation,

$R \cap S$ =

0.3	0	0.1
0.4	0.2	0.1
0.2	0.2	0.4

Suppose a simple T norm $T(a, b) = \frac{a \cdot b}{a + b - a \cdot b}$ is used then,

$R \cap S$ =

0.20	0	0.1
0.4	0.17	0.10
0.13	0.13	0.28

The above operation is more optimistic than the min operation. All the membership degrees are less than in the min operation.

➜ 3. Projection

The projection relation brings a ternary relation back to a binary relation, or a binary relation to a fuzzy set, or a fuzzy set to a single crisp value.

Ex. Consider the relation R as given below.

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

Then the projection on X means that

- x_1 is assigned the maximum of the first row.
- x_2 is assigned the maximum of the second row.
- x_3 is assigned the maximum of the third row.

Thus,

$$\text{Proj. R on X} = \frac{1}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3}$$

Similarly,

$$\text{Proj. R on Y} = \frac{0.9}{y_1} + \frac{1}{y_2} + \frac{0.7}{y_3} + \frac{0.8}{y_4}$$

4. Cylindrical Extension

- The projection operation is almost always used in combination with cylindrical extension.
- Cylindrical extension is more or less opposite of projection.
- It converts fuzzy set to a relation.

e.g. consider a fuzzy set,

$$A = \text{proj. of } R \text{ on } X = 1/x_1 + 0.8/x_2 + 1/x_3.$$

Its cylindrical extension on the domain $X \times Y$ is

ce (A) =

	y_1	y_2	y_3	y_4
x_1	1	1	1	1
x_2	0.8	0.8	0.8	0.8
x_3	1	1	1	1

Consider the fuzzy set

$$B = \text{proj. of } R \text{ on } X = \frac{0.9}{y_1} + \frac{0.8}{y_2} + \frac{0.7}{y_3} + \frac{0.8}{y_4}$$

ce (B) =

	y_1	y_2	y_3	y_4
x_1	0.9	0.8	0.7	0.8
x_2	0.9	0.8	0.7	0.8
x_3	0.9	0.8	0.7	0.8

2.5.2.3 Properties of Fuzzy Relations

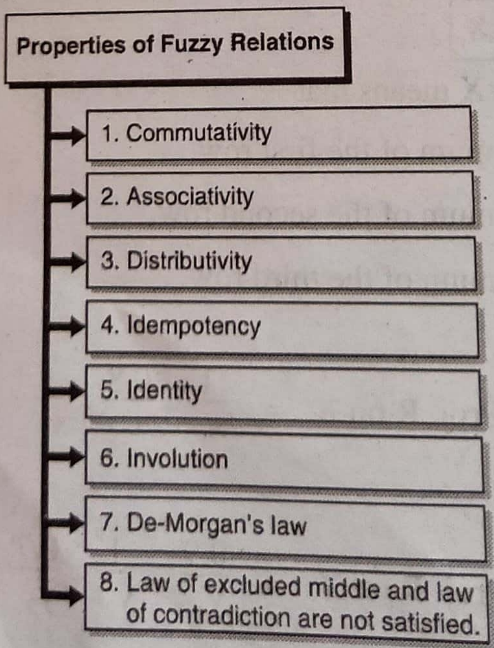


Fig. 2.5.5

Let R, S and T be fuzzy relations defined on the universe X × Y. Then, the properties of fuzzy relations are stated below :

→ **1. Commutativity**

$$R \cup S = S \cup R$$

$$R \cap S = S \cap R$$

→ **2. Associativity**

$$R \cup (S \cup T) = (R \cup S) \cup T$$

$$R \cap (S \cap T) = (R \cap S) \cap T$$

→ **3. Distributivity**

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$$

→ **4. Idempotency**

$$R \cup R = R$$

$$R \cap R = R$$

→ **5. Identity**

$$R \cup \phi_R = R, \quad R \cap \phi_R = \phi_R$$

$$R \cup E_R = E_R, \quad R \cap E_R = R$$

Where ϕ_R and E_R are null relation (null matrix) and complete relation (unit matrix of all 1s) respectively.

→ **6. Involution**

$$\overline{\overline{R}} = R$$

→ **7. De-Morgan's law**

$$\overline{R \cap S} = \overline{R} \cup \overline{S}$$

$$\overline{R \cup S} = \overline{R} \cap \overline{S}$$

→ 8. **Law of excluded middle and law of contradiction are not satisfied.**

i.e. $R \cup \bar{R} \neq E_R$

and $R \cap \bar{R} \neq \phi_R$

2.5.3 Fuzzy Composition

Composition operation can be used to combine two fuzzy relations in different product spaces.

There are two compositions that are used commonly.

1. Max - min composition
2. Max - product composition

2.5.3.1 Max - Min Composition

- Let R_1 be a fuzzy relation defined on $X \times Y$.
  And R_2 be a fuzzy relation defined on $Y \times Z$.
- Then the max - min composition of two fuzzy relations R_1 and R_2 is denoted by $R_1 \circ R_2$ and defined as,

$$R_1 \circ R_2 = \{[(x, z), \max_{y \in Y} (\min (\mu_{R1}(x, y), \mu_{R2}(y, z))] \mid x \in X, y \in Y, z \in Z\}$$

OR

$$\mu_{R1 \circ R2}(x, z) = \max_{y \in Y} \{\min (\mu_{R1}(x, y), \mu_{R2}(y, z))\}$$

2.5.3.2 Max - Product Composition

The max - product composition is defined as,

$$R_1 \circ R_2 = \{[(x, z), \max_{y \in Y} (\mu_{R1}(x, y) \cdot \mu_{R2}(y, z))] \mid x \in X, y \in Y, z \in Z\}$$

OR

$$\mu_{R1 \circ R2}(x, z) = \max_{y \in Y} \{\mu_{R1}(x, y) \cdot \mu_{R2}(y, z)\}$$

The following are the properties of fuzzy composition. Assuming R, S and T are binary relations defined on $X \times Y$, $Y \times Z$ and $Z \times W$ respectively.

1. Associativity $\rightarrow R \circ (S \circ T) \Rightarrow (R \circ S) \circ T$

2\.
3\.
4\.
Ex. 2.5
Soln.
1\.
Z =

2. Monotonicity $\rightarrow S \subseteq T \Rightarrow R \circ S \subseteq R \circ T$
3. Distributivity $\rightarrow R \circ (S \cup T) \Rightarrow (R \circ S) \cup (R \circ T)$
4. Inverse $\rightarrow (R \circ S)^{-1} = S^{-1} \circ R^{-1}$

Ex. 2.5.2 : Given two fuzzy relations R_1 and R_2 defined on $X \times Y$ and $Y \times Z$ respectively, where $X = \{1, 2, 3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$ and $z = \{a, b\}$. Find :

1. Max - min composition.
2. Max - product composition.

$$R_1 = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.5 \\ 0.4 & 0.3 & 0.7 & 0.9 \\ 0.6 & 0.1 & 0.8 & 0.2 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0.1 & 0.9 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}$$

Soln. :

1. Max - min composition

Here R_1 is defined on $X \times Y$ where $X = \{1, 2, 3\}$ and $Y = (\alpha, \beta, \gamma, \delta)$

$$R_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.5 \\ 0.4 & 0.3 & 0.7 & 0.9 \\ 0.6 & 0.1 & 0.8 & 0.2 \end{bmatrix} \end{array} \quad 3 \times 4$$

Similarly R_2 is defined on $Y \times Z$ where $Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$

So,

$$R_2 = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.1 & 0.9 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} \end{array} \quad 4 \times 2$$

So composition of R_1 and R_2 will be defined on $X \times Z$ where $X = \{1, 2, 3\}$ and $Z = \{a, b\}$.

Here $R_1 \circ R_2$ is 3×2 matrix 3×2

$$\mathbf{R_1 \circ R_2} = \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{a} & \mathbf{b} \end{array} \\ \begin{bmatrix} \mathbf{0.5} & \mathbf{0.3} \\ \mathbf{0.7} & \mathbf{0.6} \\ \mathbf{0.5} & \mathbf{0.6} \end{bmatrix} \end{array}$$

...Ans.

We compute each element of $R_1 o R_2$ as follows :

$\mu_{R1 o R2}(1, a)$ = max (min (0.1,0.1), min (0.2,0.2), min(0.3,0.5), min (0.5,0.7))
= max (0.1, 0.2, 0.3, 0.5)
= 0.5

$\mu_{R1 o R2}(1, b)$ = max (min (0.1, 0.9), min (0.2, 0.3), min (0.3, 0.6), min (0.5, 0.3))
= max (0.1, 0.2, 0.3, 0.3)
= 0.3

$\mu_{R1 o R2}(2, a)$ = max (min (0.4, 0.1), min (0.3, 0.2), min (0.7, 0.5), min (0.9, 0.7))
= Max (0.1, 0.2, 0.5, 0.7)
= 0.7

$\mu_{R1 o R2}(2, b)$ = max (min (0.4, 0.9), min (0.3, 0.3), min (0.7, 0.6), min (0.9, 0.3))
= max (0.4, 0.3, 0.6, 0.3)
= 0.6

$\mu_{R1 o R2}(3, a)$ = max (min (0.6, 0.1), min (0.1, 0.2), min (0.8, 0.5), min (0.2, 0.7))
= max (0.1, 0.1, 0.5, 0.2)
= 0.5

$\mu_{R1 o R2}(3, b)$ = max (min (0.6, 0.9), min (0.1, 0.3), min (0.8, 0.6), min (0.2, 0.3))
= max (0.6, 0.1, 0.6, 0.2)
= 0.6

2. Max - product composition

$$R_1 o R_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.35 & 0.18 \\ 0.63 & 0.42 \\ 0.40 & 0.54 \end{bmatrix} \end{array}$$

...Ans.

$\mu_{R1 o R2}(1, a)$ = max (0.1 × 0.1, 0.2 × 0.2, 0.3 × 0.5, 0.5 × 0.7)
= max (0.01, 0.04, 0.15, 0.35)
= 0.35

$\mu_{R1 o R2}(1, b)$ = max (0.1 × 0.9, 0.2 × 0.3, 0.3 × 0.6, 0.5 × 0.3)
= max (0.09, 0.06, 0.18, 0.15)
= 0.18

$\mu_{R1 o R2}(2, a)$ = max (0.4 × 0.1, 0.3 × 0.2, 0.7 × 0.5, 0.9 × 0.7)
= max (0.04, 0.06, 0.35, 0.63)
= 0.63

$\mu_{R1 o R2}(2, b)$ = max (0.4 ...

$= \max (0.36, 0.09, 0.42, 0.27)$

$= 0.42$

$\mu_{R1 \circ R2} (3, a) = \max (0.6 \times 0.1, 0.1 \times 0.2, 0.8 \times 0.5, 0.2 \times 0.7)$

$= \max (0.06, 0.02, 0.40, 0.14)$

$= 0.40$

$\mu_{R1 \circ R2} (3, b) = \max (0.6 \times 0.9, 0.1 \times 0.3, 0.8 \times 0.6 \times 0.2 \times 0.3)$

$= \max (0.54, 0.03, 0.48, 0.06) = 0.54$

Ex. 2.5.3 : Let R be the relation that specifies the relationship between the 'color of a fruit' and 'grade of maturity'. Relation S specifies the relationship between 'grade of maturity' and 'taste of a fruit', where color, grade and taste of a fruit are characterized by crisp sets x, y, z respectively as follows.

X = {green, yellow, red}

Y = {verdant, half mature, mature}

Z = {sour, tasteless, sweet}

Consider following relations R and S and find the relationship between 'color and taste' of a fruit using

1. Max - min composition
2. Max - product composition

R	Verdant	Half mature	Mature
Green	1	0.5	0
Yellow	0.3	1	0.4
Red	0	0.2	1

S	Sour	Tasteless	Sweet
Verdant	1	0.2	0
Half mature	0.7	1	0.3
Mature	0	0.7	1

Soln. :

1. Max - min composition

T	Sour	Tasteless	Sweet
Green	1	0.5	0.3
Yellow	0.7	1	0.4
Red	0.2	0.7	1

...Ans.

T (green, sour) $= \max (\min (1, 1), \min (0.5, 0.7), \min (0, 0))$

$= \max (1, 0.5, 0)$

$= 1$

T (green, tasteless) $= \max (\min (1, 0.2), \min (0.5, 1), \min (0, 0.7))$

$= \max (0.2, 0.5, 0)$

$= 0.5$

$$
\begin{aligned}
T\,(\text{green, sweet}) &= \max\,(\min\,(1, 0), \min\,(0.5, 0.3), \min\,(0, 1)) \\
&= \max\,(0, 0.3, 0) \\
&= 0.3 \\
T\,(\text{yellow, sour}) &= \max\,(\min\,(0.3, 1), \min\,(1, 0.7), \min\,(0.4, 0.7)) \\
&= \max\,(0.3, 0.7, 0.4) \\
&= 0.7 \\
T\,(\text{yellow, tasteless}) &= \max\,(\min\,(0.3, 0.2), \min\,(1, 1), \min\,(0.4, 0.7)) \\
&= \max\,(0.2, 1, 0.4) = 1 \\
T\,(\text{yellow, sweet}) &= \max\,(\min\,(0.3, 0), \min\,(1, 0.3), \min\,(0.4, 1)) \\
&= \max\,(0, 0.3, 0.4) \\
&= 0.4 \\
T\,(\text{red, sour}) &= \max\,(\min\,(0, 1), \min\,(0.2, 0.7), \min\,(1, 0)) \\
&= \max\,(0, 0.2, 0) = 0.2 \\
T\,(\text{red, tasteless}) &= \max\,(\min\,(0, 0.2), \min\,(0.2, 1), \min\,(1, 0.7)) \\
&= \max\,(0, 0.2, 0.7) \\
&= 0.7 \\
T\,(\text{red, sweet}) &= \max\,(\min\,(0, 0), \min\,(0.2, 0.3), \min\,(1, 1)) \\
&= \max\,(0, 0.2, 1) \\
&= 1
\end{aligned}
$$

2. Max - product composition

T	Sour	Tasteless	Sweet
Green	1	0.5	0.15
Yellow	0.7	1	0.4
Red	0.14	0.7	1

...Ans.

$$
\begin{aligned}
T\,(\text{green, sour}) &= \max\,(1 \times 1, 0.5 \times 0.7, 0 \times 0) \\
&= \max\,(1, 0.35, 0) \\
&= 1 \\
T\,(\text{green, tasteless}) &= \max\,(1 \times 0.2, 0.5 \times 1, 0 \times 0.7) \\
&= \max\,(0.2, 0.5, 0) \\
&= 0.5 \\
T\,(\text{green, sweet}) &= \max\,(1 \times 0, 0.5 \times 0.3, 0 \times 1) \\
&= \max\,(0, 0.15, 0)
\end{aligned}
$$

$$= 0.15$$

$$T(\text{yellow, sour}) = \max(0.3 \times 1, 1 \times 0.7, 0.4 \times 0.7)$$
$$= \max(0.3, 0.7, 0.28)$$
$$= 0.7$$

$$T(\text{yellow, tasteless}) = \max(0.3 \times 0.2, 1 \times 1, 0.4 \times 0.7)$$
$$= \max(0.06, 1, 0.28)$$
$$= 1$$

$$T(\text{yellow, sweet}) = \max(0.3 \times 0, 1 \times 0.3, 0.4 \times 1)$$
$$= \max(0, 0.3, 0.4)$$
$$= 0.4$$

$$T(\text{red, sour}) = \max(0 \times 1, 0.2 \times 0.7, 1 \times 0)$$
$$= \max(0, 0.14, 0)$$
$$= 0.14$$

$$T(\text{red, tasteless}) = \max(0 \times 0.2, 0.2 \times 1, 1 \times 0.7)$$
$$= \max(0, 0.2, 0.7)$$
$$= 0.7$$

$$T(\text{red, sweet}) = \max(0 \times 0, 0.2 \times 0.3, 1 \times 1)$$
$$= \max(0, 0.06, 1)$$
$$= 1$$

Syllabus Topic : Fuzzy to Crisp Conversion

2.6 Fuzzy to Crisp Conversion

Ans. Fuzzy to crisp conversion involves various defuzzification techniques. Defuzzification is the process of converting fuzzy values in to crisp values. Defuzzification has been discussed in Section 2.11.2.

Syllabus Topic : Membership Function

2.7 Membership Function

2.7.1 Types of Membership Functions

- One way to represent a fuzzy set is by stating its membership function (MF). MFs can be represented using any mathematical equation as per requirement or using one of the standard MFs available.

- There are several different standard MFs available.

Types of Membership Functions

1. Increasing MFs (T Function)
2. Decreasing MF (L function)
3. Triangular MF ($\wedge$ function)
4. Trapezoidal MFs (π function)
5. Gaussian MFs
6. Generalized bell MF / Cauchy MF
7. Sigmoidal MFs

Fig. 2.7.1

→ 1. Increasing MFs (T Function)

An increasing MF is specified by two parameters (a, b) as follows:

$$T(x ; a, b) = \begin{cases} 0 & ; \ x \leq a \\ (x-a)/(b-a) & ; \ a \leq x \leq b \\ 1 & ; \ x \geq b \end{cases}$$

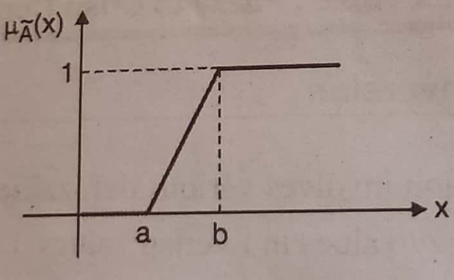


Fig. 2.7.2 : Increasing MF

→ 2. Decreasing MF (L function)

A decreasing MF is specified by two parameters (a, b) as follows :

$$L(x ; a, b) = \begin{cases} 1 & ; \ x \leq a \\ (b-x)/(b-a) & ; \ a \leq x \leq b \\ 0 & ; \ x \geq b \end{cases}$$

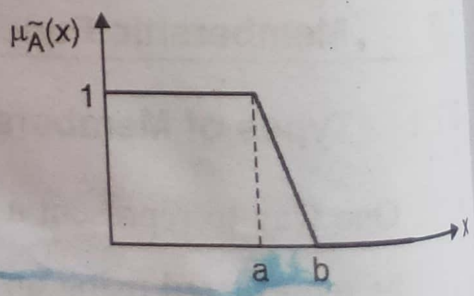


Fig. 2.7.3 : Decreasing MF

3. Triangular MF ($\wedge$ function)

A triangular MF is specified by three parameters (a, b, c) as follows :

$$\wedge(x ; a, b, c) = \begin{cases} 0 & ; x \le a \\ (x-a)/(b-a) & ; a \le x \le b \\ (c-x)/(c-b) & ; b \le x \le c \\ 0 & ; x \ge c \end{cases}$$

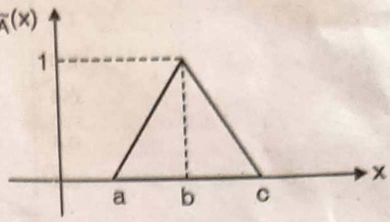


Fig. 2.7.4 : Triangular MF

4. Trapezoidal MFs (π function)

A trapezoidal MF is specified by four parameters (a, b, c, d) as follows :

trapezoid (x ; a, b, c, d) =

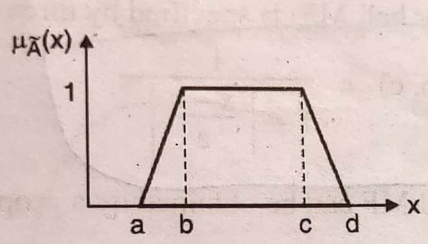


Fig. 2.7.5 : Trapezoid MF

An alternative expression using min and max can be given as,

$$\text{trapezoid}(x ; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

The parameters {a, b, c, d} (with $a < b < c < d$) determine the x coordinates of the four corners of the trapezoidal MF.

Note that if $b = c$ then trapezoidal MF reduces to triangle MF.

Since two MFs triangular and trapezoidal are composed of straight line segment, they are not smooth at the corner points specified by the parameters. However due to the simple formulae and computational efficiency, they are used extensively.

Some **smooth** and **non-linear** MFs are discussed below :

5. Gaussian MFs

A Gaussian MF is specified by two parameters $\{c, \sigma\}$.

$$\text{Gaussian}(x ; c, \sigma) = e^{-1/2\left(\frac{x-c}{\sigma}\right)^2}$$

c represents MFs center and

σ determines MFs width.

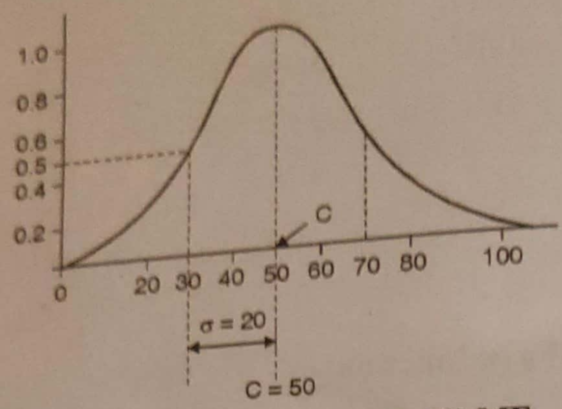


Fig. 2.7.6 : Gaussian (x; 50, 20) MF

6. **Generalized bell MF / Cauchy MF**

A generalized bell MF (or bell MF) is specified by three parameters (a, b, c).

$$\text{bell}(x\,;\,a, b, c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

A desired generalized bell MF can be obtained by a proper selection of the parameters a, b, c.

c specifies the center of a bell MF

a specifies the width of a bell MF

and b determines the slope at the crossover points.

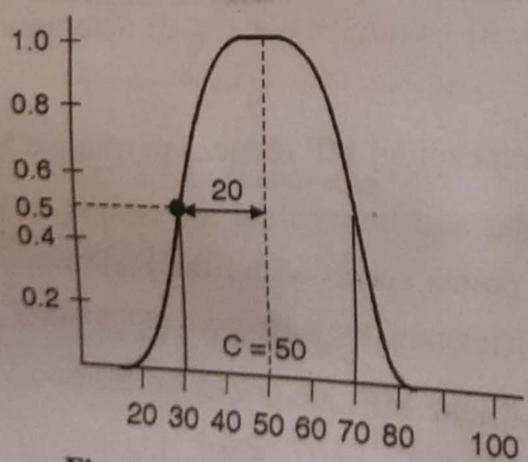


Fig. 2.7.7 : Bell (x ; 20, 4, 50)

Fig. 2.7.8 illustrates the effect of changing these parameters on the shape of the curve.

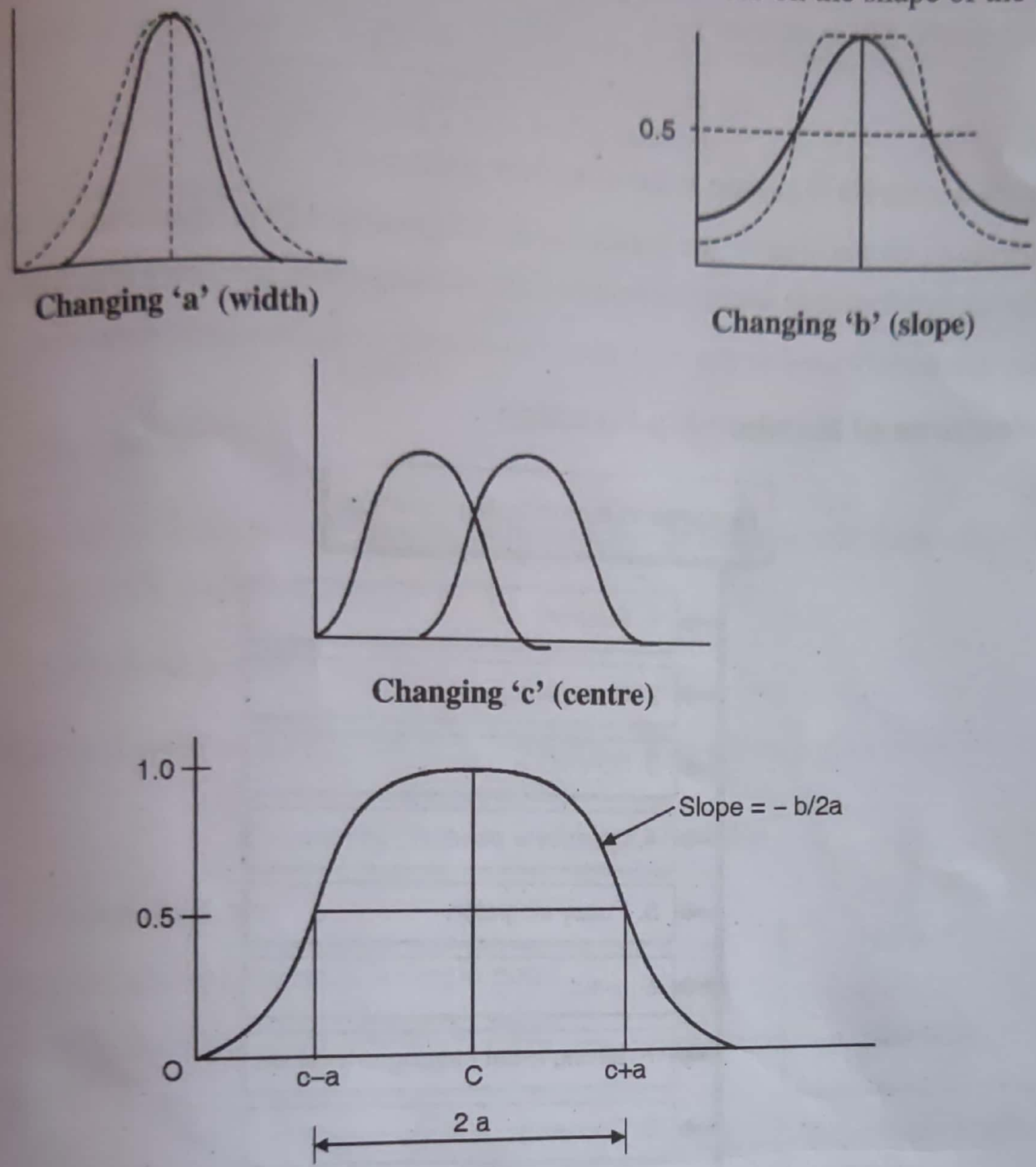


Fig. 2.7.8 : Effect of change of different parameters in Bell MF

The bell MF is direct generalization of Cauchy distribution used in probability theory; so it is also referred to as the Cauchy MF. The bell MF has more parameter than Gaussian MF, so it has more degree of freedom to adjust the steepness at the crossover point.

Although Gaussian and bell MFs achieves smoothness, they are unable to specify asymmetric MFs.

Asymmetric MFs

Asymmetric and close MFs can be achieved by using either the absolute difference or the product of two sigmoidal functions.

Sigmoidal MFs are discussed below.

7. Sigmoidal MFs

A sigmoidal MF is defined by,

$$\text{sig}(x\,;\,a,\,c) = \frac{1}{1 + \exp[-a(x - c)]}$$

where, *a* controls the slope at the crossover point $x = c$.

Depending on the sign of the parameter *a*, a sigmoidal MF is open right or open left and thus is appropriate for representing concepts such as "very large" or "very negative".

They are widely used as the activation function in artificial neural networks.

2.7.2 Features of Membership Function

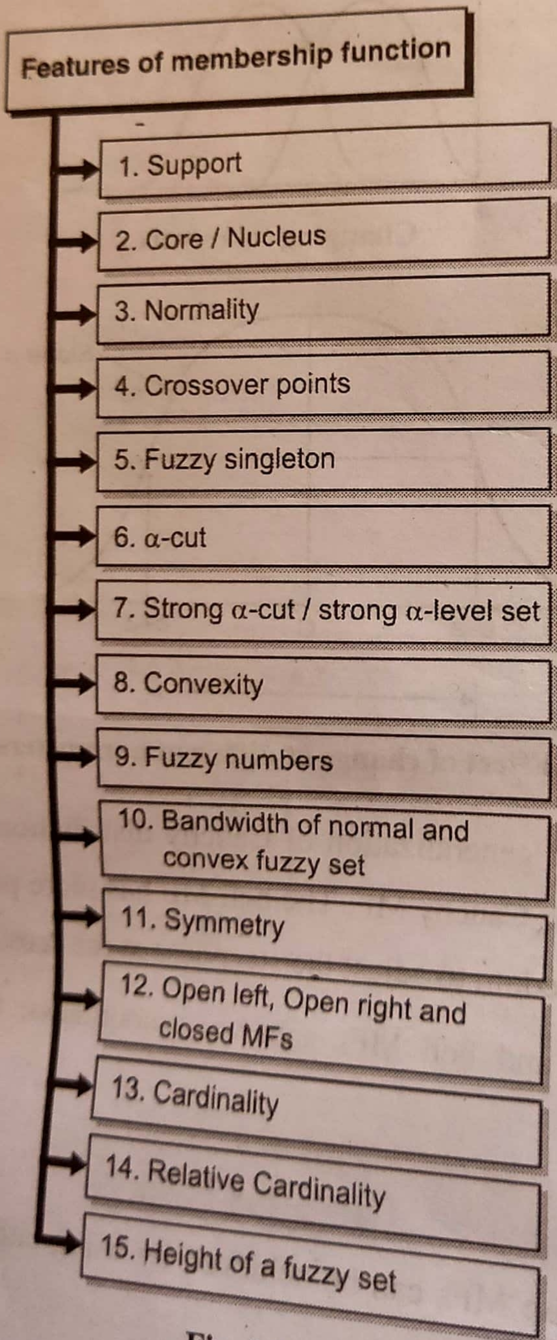


Fig. 2.7.9

1. Support

A support of a fuzzy set $\tilde{A}$ is the set of all points x in X such that, $\mu_{\tilde{A}}(x) > 0$.

$$\text{Support}(\tilde{A}) = \{x \mid \mu_{\tilde{A}}(x) > 0\}$$

2. Core / Nucleus

The core of a fuzzy set $\tilde{A}$ is the set of all points x in X such that $\mu_{\tilde{A}}(x) = 1$.

$$\text{Core } \tilde{A} = \{x \mid \mu_{\tilde{A}}(x) = 1\}$$

3. Normality

A fuzzy set $\tilde{A}$ is normal if its core is nonempty. In other words there must be at least one point $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

4. Crossover points

A cross over point of a fuzzy set $\tilde{A}$ is a point $x \in X$ at which $\mu_{\tilde{A}}(x) = 0.5$.

$$\text{Crossover}(\tilde{A}) = \{x \mid \mu_{\tilde{A}}(x) = 0.5\}$$

5. Fuzzy singleton

A fuzzy set whose support is a single point in X with $\mu_{\tilde{A}}(x) = 1$ is called a fuzzy singleton.

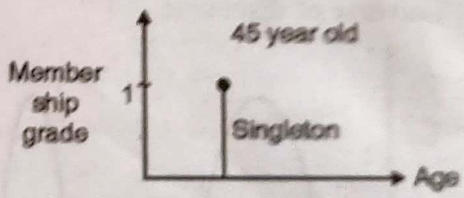


Fig. 2.7.10 : A fuzzy Singleton

Fig. 2.7.11 shows three parameters (core, support and crossover points) of a fuzzy set.

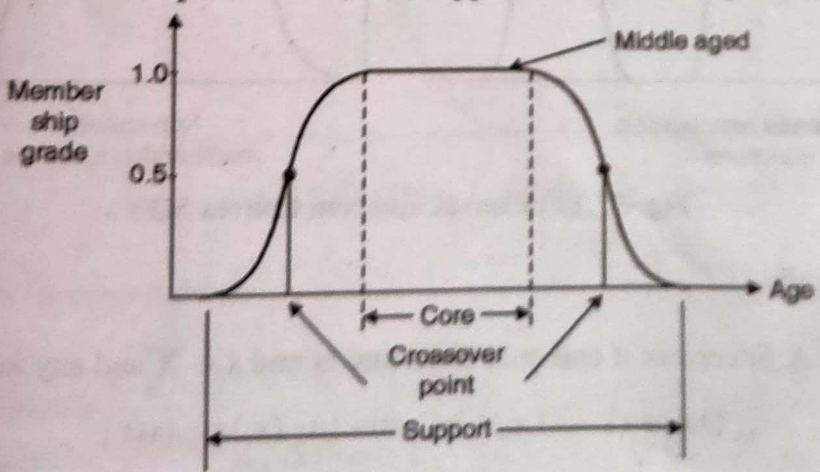


Fig. 2.7.11 : Core, Support and Crossover points of a fuzzy set

6. α-cut

The α-cut or α-level set of a fuzzy set $\tilde{A}$ is a crisp set defined by,

$$A_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$$

7. Strong α-cut / strong α-level set

Strong α-cut is defined by

$$A'_\alpha = \{x \mid \mu_{\tilde{A}}(x) > \alpha\}$$

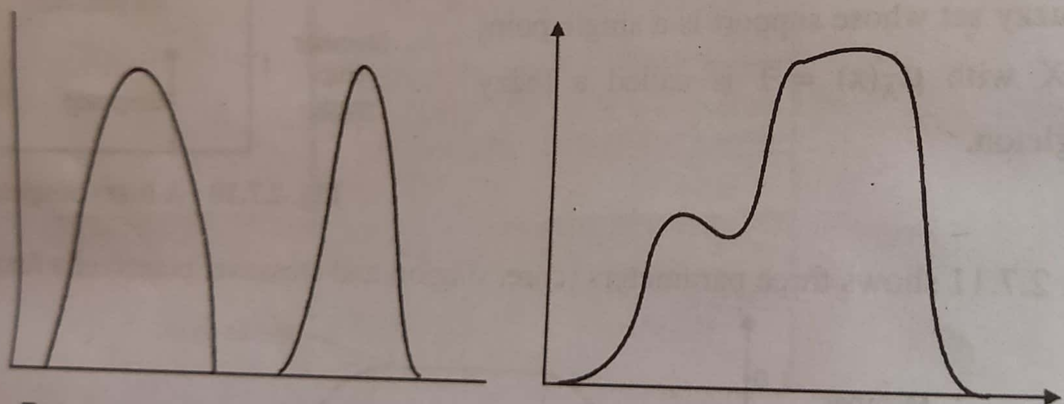
Using the above notations, we can express support and core of a fuzzy set A as,

$$\text{Support}(\tilde{A}) = A'_0 \qquad \text{Here } \alpha = 0$$

$$\text{Core}(\tilde{A}) = A_1 \qquad \text{Here } \alpha = 1$$

8. Convexity

A convex fuzzy set has membership function whose membership values are strictly monotonically increasing or strictly monotonically decreasing or strictly monotonically increasing than strictly monotonically decreasing with increasing values for elements in universe of discourse.



Two convex membership functions

Non convex membership functions

Fig. 2.7.12 : Convex and Non Convex MFs

Mathematically,

A fuzzy set $\tilde{A}$ is convex if and only if for any x_1 and $x_2 \in X$ and any $\lambda \in [0, 1]$.

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$$

or

$\tilde{A}$ is convex if all its α-level sets are convex.

9. Fuzzy numbers

A fuzzy number $\tilde{A}$ is a fuzzy set in the real line (R) that satisfies the conditions for normality and convexity.

10. Bandwidth of normal and convex fuzzy set

For a normal and convex fuzzy set, the bandwidth or width is defined as the distance between two unique crossover points.

$$\text{Width}(\tilde{A}) = |x_2 - x_1|$$

Where $\mu_{\tilde{A}}(x_1) = \mu_{\tilde{A}}(x_2) = 0.5$

11. Symmetry

A fuzzy set $\tilde{A}$ is symmetric if its MF is symmetric around a certain point $x = c$,

$$\mu_{\tilde{A}}(c + x) = \mu_{\tilde{A}}(c - x) \quad \text{for all } x \in X$$

12. Open left, Open right and closed MFs

A fuzzy set $\tilde{A}$ is open left if,

$$\lim_{x \to -\infty} \mu_{\tilde{A}}(x) = 1 \text{ and}$$

$$\lim_{x \to +\infty} \mu_{\tilde{A}}(x) = 0$$

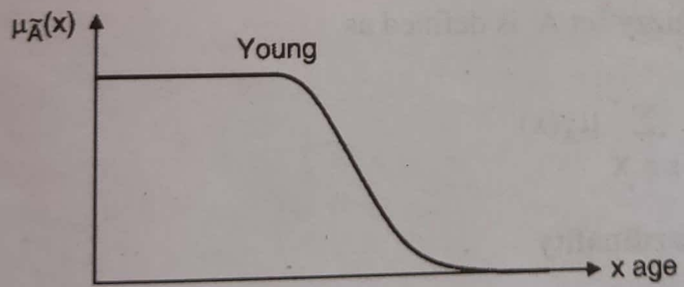


Fig. 2.7.13 : Open Left MF

A fuzzy set $\tilde{A}$ is open right if,

$$\lim_{x \to -\infty} \mu_{\tilde{A}}(x) = 0 \quad \text{and}$$

$$\lim_{x \to +\infty} \mu_{\tilde{A}}(x) = 1$$

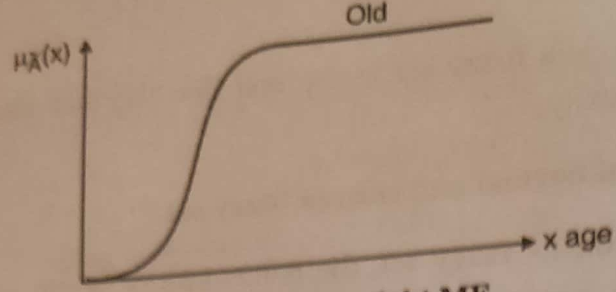


Fig. 2.7.14 : Open right MF

A fuzzy set $\tilde{A}$ is closed if,

$$\lim_{x \to +\infty} \mu_{\tilde{A}}(x) = 0 \quad \text{and}$$

$$\lim_{x \to -\infty} \mu_{\tilde{A}}(x) = 0$$

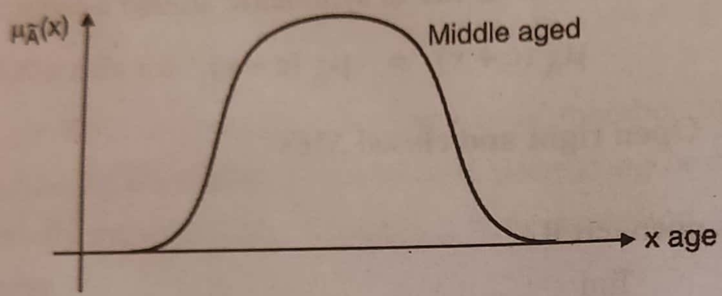


Fig. 2.7.15 : Closed MF

→ 13. Cardinality

Cardinality of a fuzzy set $\tilde{A}$ is defined as

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$$

→ 14. Relative Cardinality

Relative Cardinality of a fuzzy set $\tilde{A}$ is defined as,

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{|X|}$$

→ 15. Height of a fuzzy set

The height of a fuzzy set $\tilde{A}$ in X, is equal to the largest membership degree μ_m

$$hgt(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$$

if hgt $(\tilde{A}) = 1$ then, $\tilde{A}$ is normal

if hgt$(\tilde{A}) < 1$ then, $\tilde{A}$ is subnormal

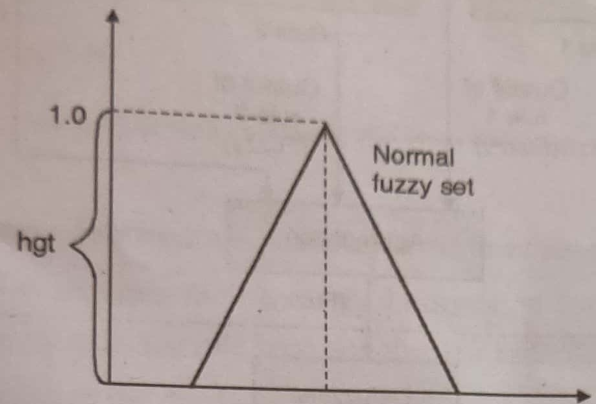


Fig. 2.7.16 : Height of a fuzzy set

Syllabus Topic : Inference in Fuzzy Logic

2.8 Inference in Fuzzy Logic

- Fuzzy Inference System is the key unit of a fuzzy logic system. Fuzzy inference (reasoning) is the actual process of mapping from a given input to an output using fuzzy logic.
- It uses the "IF…THEN" rules along with connectors "OR" or "AND" for drawing essential decision rules.

Fig. 2.8.1 (a) shows the block diagram of general fuzzy inference system.

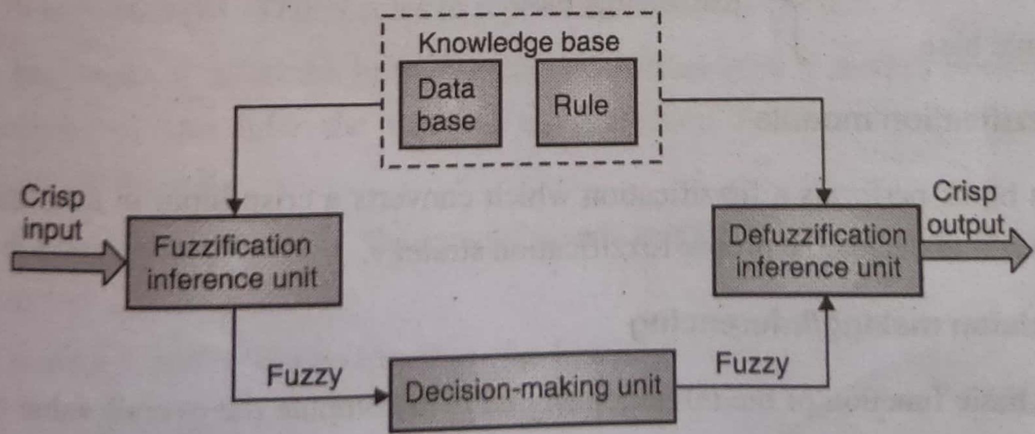


Fig 2.8.1(a) : Block diagram : Fuzzy inference system

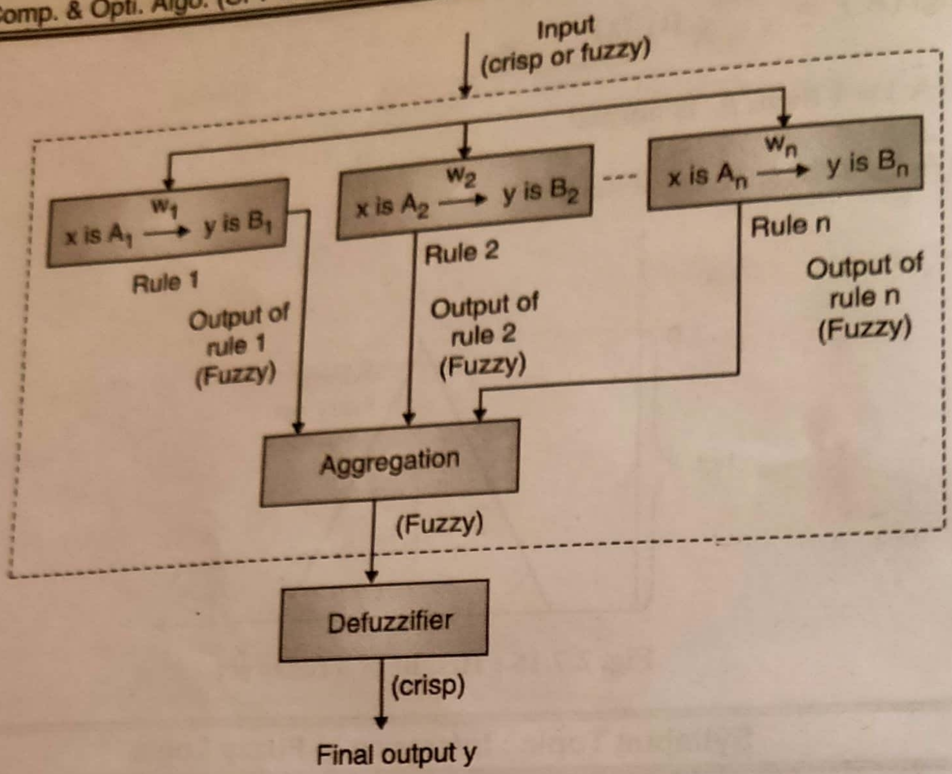


Fig 2.8.1 (b) : Fuzzy Inference using If-Then rules

- As shown in Fig. 2.8.1 (a) ,FIS involves three important modules.
  1. Fuzzification module (FM)
  2. Decision making or Inferencing module
  3. Defuzzification module (DM)
- In addition to this, it uses two more components
  - Data base and
  - Rule base

  } Knowledge base

Fuzzification module

This block performs a fuzzification which converts a crisp input in to a fuzzy set. Here we need to decide the proper fuzzification strategy.

Decision making/Inferencing

- The basic function of the inference engine is to compute the overall value of the control output variable based on the individual contribution of each rule in the rule base.
- The output of the fuzzification module representing the crisp input is matched to each rule-antecedent.

- The degree of match of each rule is established. Based on this degree of match, the value of the control output variable in the rule-consequent is modified. The result is, we get the "clipped" fuzzy set representing the control output variable.
- The set of all clipped control output values of the matched rules represent the overall fuzzy value of control output.

Defuzzification module

It performs defuzzification which converts the overall control output into a single crisp value.

The **rule base** and the **database** are jointly referred to as the **knowledge base.**

- The **database** provides the necessary information for proper functioning of the fuzzification module, the rule base and the defuzzification module.
- The information in the database includes :
  - Fuzzy MFs for the input and output control variables.
  - The physical domains of the actual problems and their normalized values along with the scaling factors.

Syllabus Topic : Fuzzy If-Then Rules

2.9 Fuzzy If-Then Rules

- Fuzzy inference is the process of obtaining a new knowledge from an existing knowledge.
- To perform inference, knowledge must be represented in some form.
- Fuzzy logic uses IF –THEN rules to represent its knowledgebase.
- The basic rule of inference in traditional two-valued logic is **modus ponens**, according to which, we can infer the truth of a proposition B from the truth of A and the implication $A \rightarrow B$.

  e.g. if A is identified with - "the tomato is red" and B with "the tomato is ripe" then if it is true that

  "the tomato is red" it is also true that "the tomato is ripe".

i.e.		
	Premise 1 (fact)	X is A
	Premise 2 (rule)	if X is A then Y is B
	Consequence (conclusion) :	Y is B

- However, in most of the human reasoning, modus ponens is employed in an approximate manner.

  For e.g. : if we have the same implication rule, "if the tomato is red, then it is ripe" and we know that,

  "the tomato is more or less red" then we may infer that

  "the tomato is more or less ripe"

Premise 1 (fact)	X is A′
Premise 2 (rule)	if X is A then Y is B
Conclusion	Y is B′

Where A′ is close to A and

B′ is close to B

- When A, B, A′ and B′ are fuzzy sets of appropriate universes, the inference procedure is called "**approximate reasoning**" or **fuzzy reasoning**, it is also called Generalized Modus Ponens (GMP).

Definition : Approximate reasoning / fuzzy reasoning

Let A, A′ and B be fuzzy sets of X, x and Y respectively. Assume that the fuzzy implication $A \rightarrow B$ is expressed as a fuzzy relation R on $X \times Y$. Then the fuzzy set B induced by "x is A" (fact) and the fuzzy rule "if x is A then y is B" is defined by,

$$\mu_{B'}(y) = \max \min [\mu_{A'}(x), \mu_R(x, y)]$$
$$= V_x [\mu_{A'}(x) \wedge \mu_R(x, y)]$$

or

$$\beta' = A' \circ (A \rightarrow B)$$

if x is A, y is B

Single rule with single antecedent

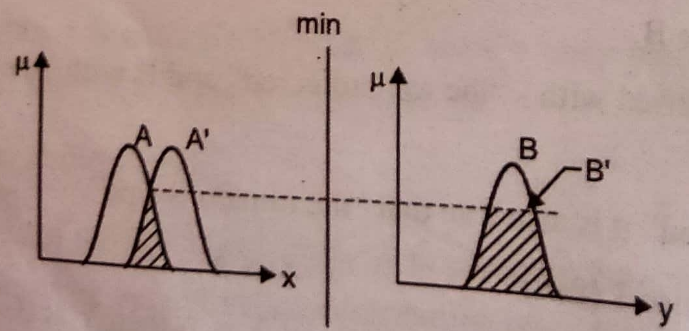


Fig. 2.9.1 : Graphic interpretation of GMP using Mamdani's fuzzy implication and max-min composition

Here $\mu_{B'}$ can be defined as ;

$$\mu_{B'}(y) = [V_x(\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y)$$
$$= w \wedge \mu_B(y)$$

Thus, we first find the degree of match w which is the maximum of $\mu_{A'}(x) \wedge \mu_A(x)$.

Then the MF of resulting B′ is equal to the MF of B clipped by w.

Single rule with multiple antecedents

A fuzzy if then rule with two antecedents can be written as,

"if x is A and y is B then z is C"

Premise 1 (fact)	:	x is A′ and y is B′
Premise 2 (rule)	:	If x is A and y is B then z is C
Consequence		z is C′

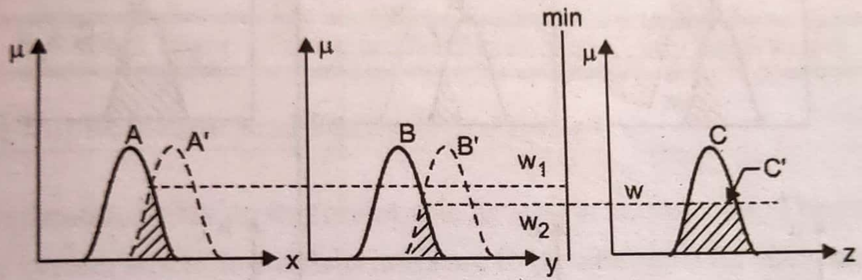


Fig. 2.9.2 : Approximate reasoning for multiple antecedents

$$\mu_{C'}(z) = \underbrace{\{V_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{V_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z)$$

$$= \underbrace{(w_1 \wedge w_2)}_{\text{firing strength}} \wedge \mu_C(z)$$

- When w_1 and w_2 are the maxima of the MFs of $A \cap A'$ and $B \cap B'$ respectively. Thus, w_1 denotes the degree of compatibility between A and A′, similarly for w_2.
- Since the antecedent parts of the fuzzy rule is constructed using **and** connective, $w_1 \wedge w_2$ is called **firing strength or degree of fulfilment of the fuzzy rule.**

- The firing strength represents the degree to which the antecedent part of the rule is satisfied.
- The MF of the resulting C′ is equal to the MF of clipped by the firing strength w (when $w = w_1 \wedge w_2$)

Multiple Rules with Multiple antecedents

- The GMP problem for multiple rules with multiple antecedents can be written as,

Premise 1 (fact)	:	x is A′ and y is B′
Premise 2 (rule 1)	:	If x is A_1 and y is B_1 then z is C_1
Premise 3 (rule 2)	:	If x is A_2 and y is B_2 then z is C_2
Consequence	:	z is C′

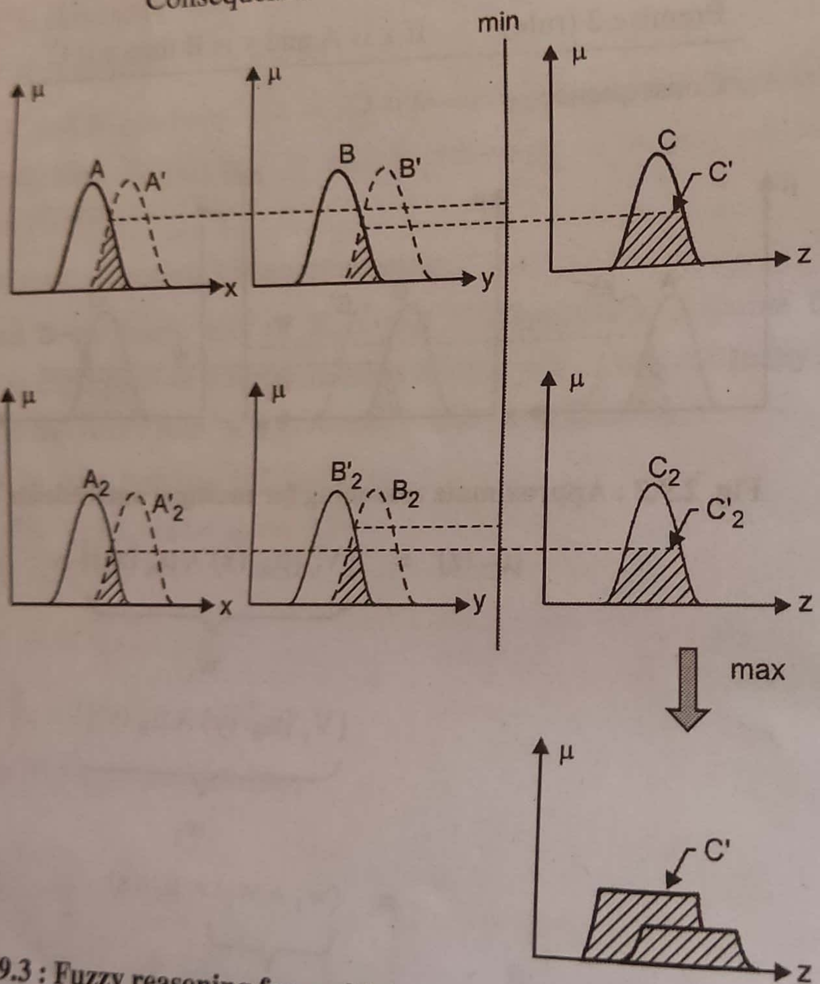


Fig. 2.9.3 : Fuzzy reasoning for multiple rules with multiple antecedents

- Here C_1' and C_2' are the inferred fuzzy sets for rule 1 and rule 2 respectively.
- When a given fuzzy rule assumes the form "if x is A or y is B" then firing strength is given as the maximum of degree of match on the antecedent part for a given condition.

e.g.

if x is A_1 or y is B_1 then z is C_1.

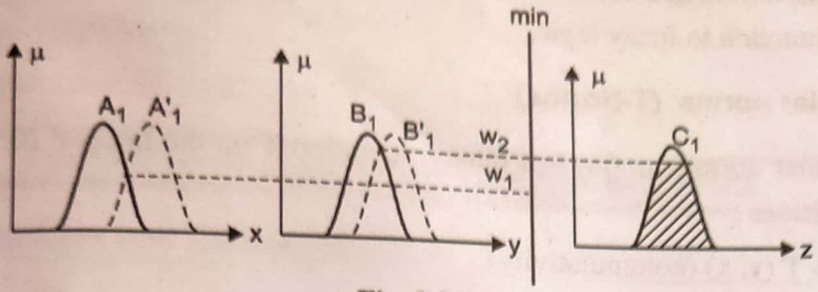


Fig. 2.9.4

- In the above example, because two antecedents are connected using **or**, we take maximum of w_1 and w_2 as a firing strength.
- Since $w_2 > w_1$, we take w_2 as a firing strength and then we apply **min** implication operator on the output MF C_1.

Syllabus Topic : Fuzzy Implications and Fuzzy Algorithms

2.10 Fuzzy Implications and Fuzzy Algorithms

- Fuzzy implications play an important role as logical connectives. They are used in fuzzy inference system where multiple criteria are to be combined for decision making.
- The most common logical connectives used in propositional logic (classical logic) are negation (NOT), conjunction (AND), disjunction (OR).
- In fuzzy logic theory, we can consider the t-norm and t-conorm operators as an abstraction of classical conjunction and disjunction. We can say that, fuzzy implication is a generalization of classical implications.

☞ Classical implications

An implication in classical logic is a function: $l : \{0, 1\} \times \{0, 1\} \rightarrow \{0, 1\}$, which satisfies the following boundary conditions.

(i) $l(0, 0) = 1$

(ii) $l(0, 1) = 1$

(iii) $l(1, 0) = 0$

(iv) $l(1, 1) = 1$

Fuzzy implications

Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic.

Triangular norms (T-Norms)

A triangular norm (t-norm) is a binary operation T on the interval [0, 1], satisfying the following conditions :

- $T(x, y) = T(y, x)$ (commutativity)
- $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)
- $y \leq z \Rightarrow T(x, y) \leq T(x, z)$ (monotonicity)
- $T(x, 1) = x$ (neutral element 1)

Example of T-norms

- $T_M(x, y) = \min(x, y)$ (minimum or Godel t-norm)
- $T_P(x, y) = x \cdot y$ (product t-norm)
- $T_L(x, y) = \max(x + y - 1, 0)$ (Lukasiewict t-norm)

Triangular co-norm (also called S-norm)

- Its neutral element is 0 instead of 1, all other conditions remained same.
- $S(x, y) = S(y, x)$ (commutativity)
- $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)
- $y \leq z \Rightarrow S(x, y) \leq S(x, z)$ (monotonicity)
- $S(x, 0) = x$ (neutral element 0)

Examples of t-conorm

- $S_M(x, y) = \max(x, y)$ (maximum or Godel t-conorm)
- $S_P(x, y) = x + y - x \cdot y$ (product t-conorm)
- $S_L(x, y) = \min(x + y, 1)$ (Lukasiewict t-conorm , bounded sum)

Other implications

1. Kleene-Wienes implication

   $R(x, y) = \max(1 - x, y)$

2. The zadeh implication

   $R(x, y) = \max\{1 - x, \min(x, y)\}$

3. The Reichenbach implication

   $R(x, y) - 1 - x + xy$

4. The Willmott implication

   $R(x, y) = \min\{\max\{1 - xy\}, \max\{1 - x, y\}, \max\{1 - y, y\}\}$

Syllabus Topic : Fuzzifications and Defuzzifications

2.11 Fuzzifications and Defuzzifications

2.11.1 Fuzzification

- Fuzzification is the process of converting a crisp set into a fuzzy set.
- Here the crisp value is transformed into linguistic variables.
- In real word problems, many a times the input values are not very precise and accurate rather they are uncertain, imprecise and unknown.
- The uncertainty may arise due to the vagueness and incompleteness of data. In such cases, variable may be represented as fuzzy and can be represented as fuzzy membership function.

Methods of membership value of assignment

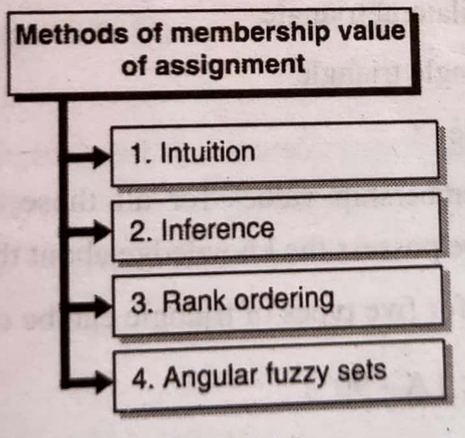


Fig. 2.11.1

1. Intuition

As the name suggest, this method is based upon the common intelligence of human. The human develops membership functions based on their own understanding capability.

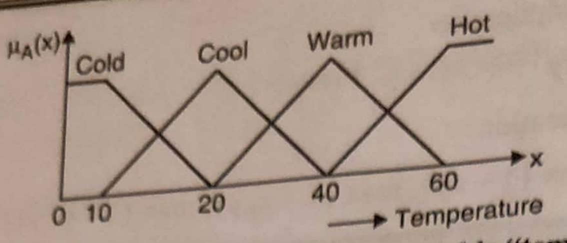


Fig. 2.11.2 : Membership functions for fuzzy variable "temperature"

As shown in Fig. 2.11.2 each triangular curve is a membership function corresponding to various fuzzy (linguistic) variables such as cold, coal, warm etc.

→ 2. Inference

In inference method we use knowledge to perform deductive reasoning. To deduce or infer a conclusion, we use the facts and knowledge on that particular problem. Let us consider the example of Geometric shapes for the identification of a triangle.

Let A, B, C be the interior angles of a triangle such that,

$A \geq B \geq C > 0°$ and $A + B + C = 180°$

For this purpose we define five types of triangles.

1. R = Approximately Right-angle triangle
2. I = Approximately Isosceles triangle
3. E = Approximately Equilateral triangle
4. I · R = Isosceles Right-angle triangle
5. T = Other type of triangle

Now, we can infer membership values for all those types of triangles through the method of inference because we possess the knowledge about the geometry of their shapes.

The membership values for five types of triangle can be defined as,

$$\mu_R(A, B, C) = 1 - \frac{1}{90°} \mid A - 90° \mid$$

$$\mu_I(A, B, C) = 1 - \frac{1}{60°} \min\{(A - B), (B - C)\}$$

$$\mu_E(A, B, C) = 1 - \frac{1}{80°} \mid A - C \mid$$

$$\mu_{I \cdot R}(A, B, C) = \mu_{I \cap R}(A, B, C)$$

$$= \min\{\mu_I(A, B, C), \mu_R(A, B, C)\}$$

$$\mu_T(A, B, C) = \overline{(R \cup I \cup E)} = \bar{R} \cap \bar{I} \cap \bar{E}$$

Example :

$$\mu\,(A, B, C) = \{80, 65, 35\}$$

$$\mu_R\,(A, B, C) = 1-\frac{1}{90}\,|\,80-90\,| = \frac{8}{9}$$

$$\mu_I\,(A, B, C) = 1-\frac{1}{60}\,\min\{\,15, 45\,\} = \frac{3}{4}$$

$$\mu_E\,(A, B, C) = 1-\frac{1}{180}\,|\,45\,| = \frac{3}{4}$$

$$\mu_{IR}\,(A, B, C) = \min\{\mu_I, \mu_R\}\;\frac{3}{4}$$

$$\mu_T = R^C \cap I^C \cap E^{\varepsilon}$$

$$= \min\left\{\frac{1}{9}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$= \frac{1}{4}$$

➜ 3. Rank ordering

- In rank ordering method, preferences are assigned by a single individual, committee, a poll and other opinion methods can be used to assign membership values to fuzzy variables.
- Here the preferences are determined by pair wise comparisons and they are used to determine ordering of the membership.

Example :

Let's suppose 1000 people respond to a questionnaire and their pair wise preferences among the colors red, orange, yellow and blue is given as below.

	Red	Orange	Yellow	Green	Blue
Red	- -	517	525	545	661
Orange	483	-	891	477	576
Yellow	474	159	-	534	614
Green	455	523	466	-	643
Blue	339	424	386	357	-

➜ 4. Angular fuzzy sets

- Angular fuzzy sets differ from normal fuzzy sets only in their coordinate description.
- Angular fuzzy sets are defined on a universe of angles; hence they are of repeating shapes for every 2π cycles.

- Angular fuzzy sets are used in the quantitative description of the linguistic variables, which are known as "truth values".

Example :

Let's consider that pH values of water samples are taken from a contaminated pond. We know that,

- If P_n value is 7 means it's a neutral solution.
- Levels of P_n between 14 and 7 are labelled as Absolute Basic (AB), Very Basic (VB), Basic (B), Fairly Basic (FB), Neutral (N) drawn from $\theta = \frac{\pi}{2}$ to $\theta = -\frac{\pi}{2}$
- Levels of P_n between 7 to 0 are called neutral, Fairly Acidic (FA), Acidic (A), Very Acidic (VA), Absolutely Acidic (AA), are drawn from $\theta = 0$ to $\theta = -\frac{\pi}{2}$.
- Linguistic values vary with θ and their membership values are given by equation,

$$E_\theta = t \tan \theta$$

Here 't' is the horizontal projection of the radial vector.

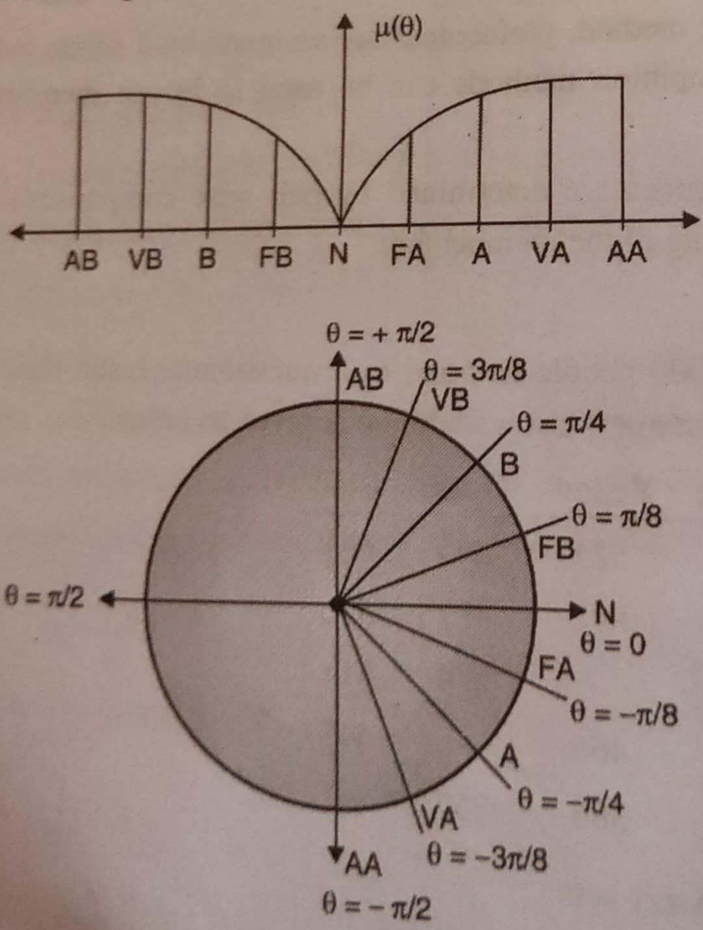


Fig. 2.11.3 : Model of angular fuzzy set

2.11.2 Defuzzification

- Defuzzification is the process of converting a fuzzy set into a crisp value.
- The output of a fuzzy process may be union of two or more fuzzy membership functions. In that case we need to find crisp value as a representative of the entire fuzzy MF.
- Different methods of defuzzification are listed below :

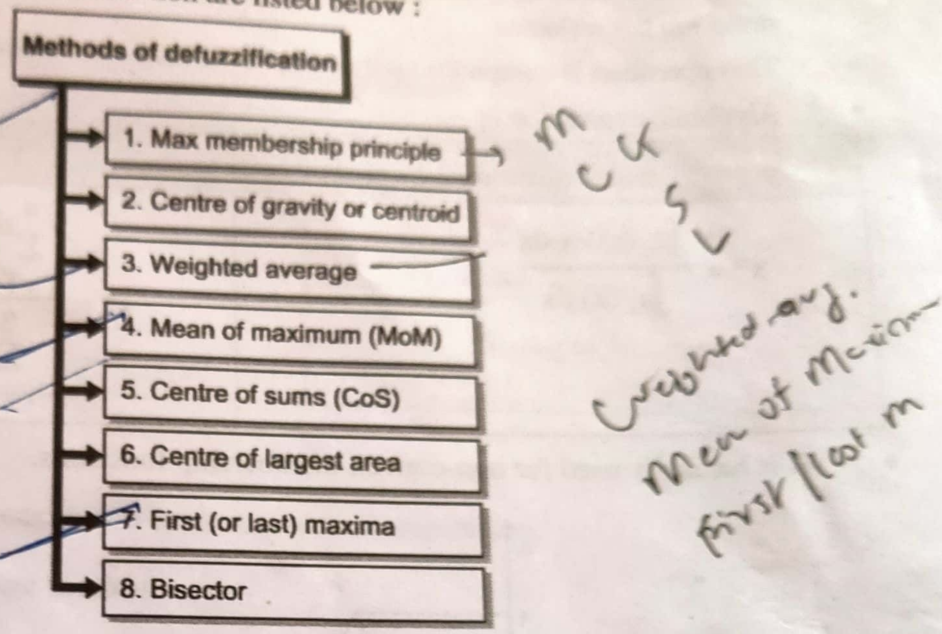


Fig. 2.11.4

1. Max-membership principle / Height method

This method is limited to peak output functions. It uses the individual clipped or scaled central outputs.

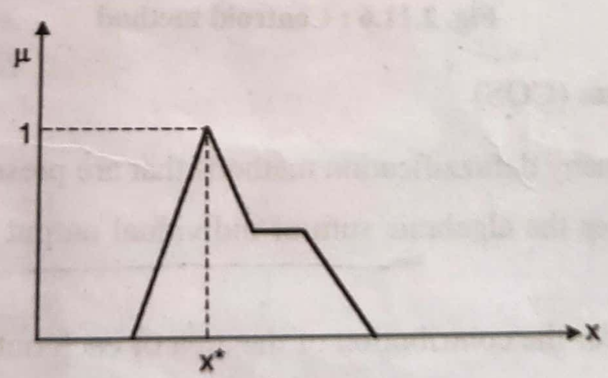


Fig. 2.11.5 : Max-membership

The algebraic expression is :

$$\mu_{\tilde{c}}(x^*) \geq \mu_{\tilde{c}}(x) \text{ for all } x \in X$$

→ 2. Centre of Area / Gravity (Centroid) Method

- This method is the most preferred and physically appealing of all the defuzzification methods.
- This method determines the centre of the area below the **combined membership function.** (i.e. it takes union of all output fuzzy sets).
- So, if there exist an overlapping area, will be considered only once. Thus overlapping areas are not reflected.
- This operation is computationally complex and therefore results in slow inference cycle.
- Algebraic expression is

For Continuous	For Discrete
$x^* = \dfrac{\int \mu_{\tilde{c}}(x) \cdot x\, dx}{\int \mu_{\tilde{c}}(x)\, dx}$	$x^* = \dfrac{\sum_{i=1}^{n} \mu_{\tilde{c}}(x_i) \cdot x_i}{\sum_{i=1}^{n} \mu_{\tilde{c}}(x_i)}$

- It is basically used for non-convex membership functions.

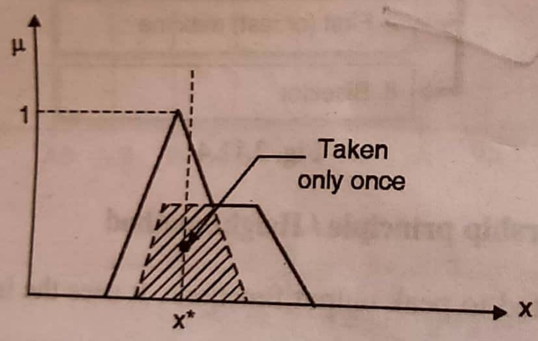


Fig. 2.11.6 : Centroid method

→ 3. Centre of sum (COS)

- This is faster than many defuzzification methods that are presently in use.
- This method involves the algebraic sum of individual output fuzzy sets, instead of their union.

The idea is to consider the contribution of the area of each output membership curve.

In contrast, the centre of area/gravity method considers the union of all output fuzzy sets.

In COS method, we take overlapping areas. If such overlapping areas exist, they are reflected more than once.

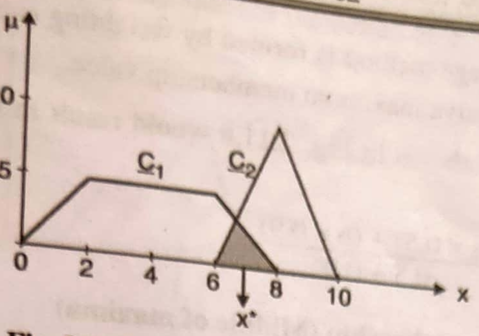


Fig. 2.11.7 : Centre of Sum method

For Continuous	For Discrete
$x^* = \dfrac{\int x \sum_{k=1}^{N} \mu_{\bar{C}_k}(x)\, dx}{\int \sum_{k=1}^{N} \mu_{\bar{C}_k}(x)\, dx}$	$x^* = \dfrac{\sum_{i=1}^{N} x_i \sum_{k=1}^{n} \mu_{\bar{C}_k}(x_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{\bar{C}_k}(x_i)}$

Advantage

It can be implemented easily and leads to a faster computation.

4. Weighted average method

This method is only valid for **symmetrical** output membership functions.

Algebraic expression is :

$$x^* = \frac{\sum_{i=1}^{n} \mu_{\bar{c}}(x_i) \cdot xi}{\sum_{i=1}^{n} \mu_{\bar{c}}(x_i)}$$

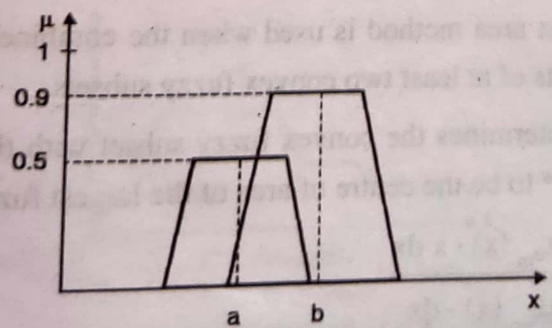


Fig. 2.11.8 : Weighted average method

- The weighted average method is formed by weighting each membership function in the output by its respective maximum membership value.
- The two functions shown in Fig. 2.11.8 would result in the following general form of defuzzification.

$$x^* = \frac{(a \times 0.5) + (b \times 0.9)}{0.5 + 0.9}$$

→ 5. Mean-max membership (Middle of maxima)

- This method is closely related to the max-membership principle (height defuzzification) method; except that the locations of the maximum membership can be non-unique (can be more than one).
- In that case we take the average of the elements having maximum membership value of Maximizing MF.

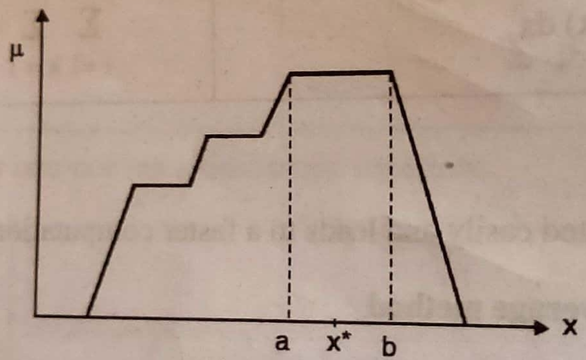


Fig. 2.11.9 : Mean of maximum method

Algebraic expression is,

$$x^* = \frac{a+b}{2}$$

This method only works for convex.

→ 6. Centre of largest area

- The centre of largest area method is used when the combined output fuzzy set is **non-convex** i.e. it consists of at least two convex fuzzy subsets.
- Then the method determines the convex fuzzy subset with the largest area and defines crisp output value x* to be the centre of area of the largest fuzzy subset.

$$x^* = \frac{\int \mu_{\tilde{c}_m}(x) \cdot x \, dx}{\int \mu_{\tilde{c}_m}(x) \cdot dx}$$

- Where, $\tilde{c}_m$ is the convex fuzzy subset that has the largest area.

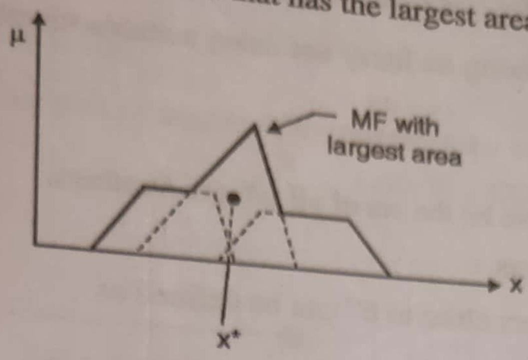


Fig. 2.11.10 : Centre of largest area

→ 7. First (or last) of maxima

- This method uses the overall output (i.e. union of all individual output MF).
- First of maxima is determined by taking the smallest value of the domain with maximized membership degree.
- Last of maxima is determined by taking the greatest value of the domain with maximized membership degree.

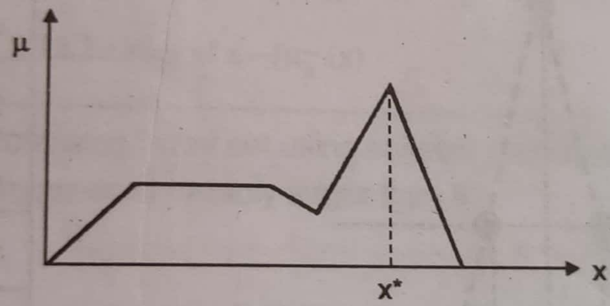


Fig. 2.11.11: First (or last) of maxima

→ 8. Bisector method

- This method uses the vertical line that divides the region into two equal areas as shown in Fig. 2.11.12. This line is called bisector.

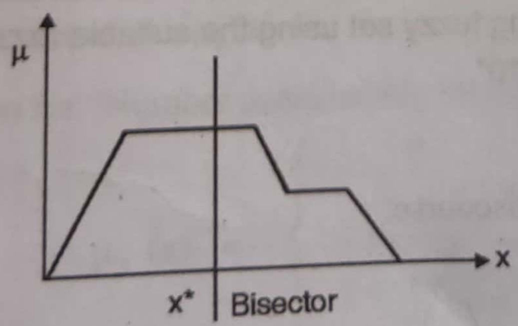


Fig. 2.11.12 : Bisector method of defuzzification

2.12 Solved Problems

Ex. 2.12.1 : Model the following as fuzzy set using suitable membership function. "Numbers close to 6".

Soln. :

Let universe of discourse be the set of all integer numbers.

$$X = \text{Integers}$$

Then fuzzy set "Numbers close to 6" can be defined as

$$\mu_{\tilde{A}}(x) = \frac{1}{1+(x-6)^2} \quad \text{...Ans.}$$

Fig. P. 2.12.1 shows the plot of degree of membership of each element.

Table P. 2.12.1: x and corresponding $\mu_{\tilde{A}}(x)$

x	$\mu_{\tilde{A}}(x)$
2	0.05
3	0.1
4	0.2
5	0.5
6	1
7	0.5
8	0.2
9	0.1
10	0.05

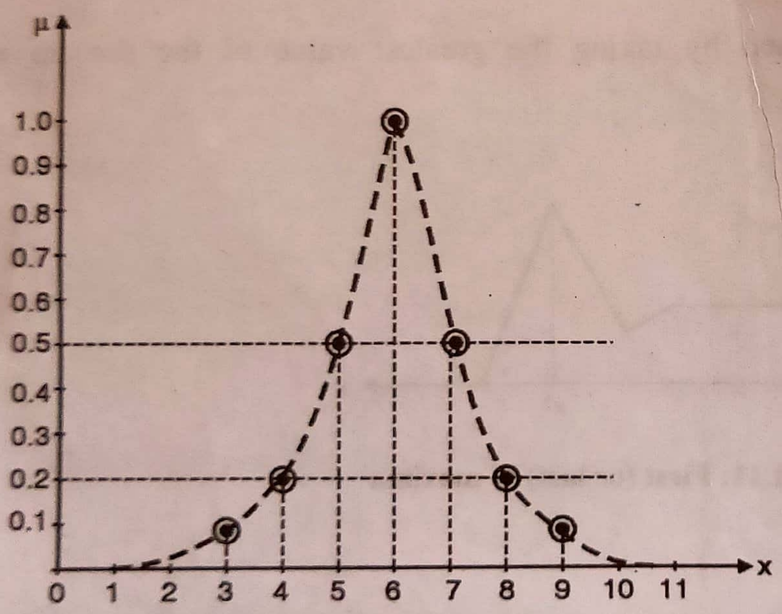


Fig. P. 2.12.1 : Plot of x →$\mu_{\tilde{A}}(x)$

Ex. 2.12.2 : Model the following fuzzy set using the suitable fuzzy membership function Number close to 10".

Soln. :

Let X be the universe of discourse.

$$X = \text{Integers}$$

Then fuzzy set "Number close to 10" can be defined as,

$$\mu_{\tilde{A}}(x) = \frac{1}{1+(x-10)^2}$$

...Ans.

Fig. P. 2.12.2 shows the plot of degree of membership for each element.

Table P. 2.12.2 : x and corresponding $\mu_{\tilde{A}}(x)$

X	$\mu_{\tilde{A}}(x)$
5	0.03
6	0.05
7	0.1
8	0.2
9	0.5
10	1
11	0.5
12	0.2
13	0.1
14	0.05
15	0.03

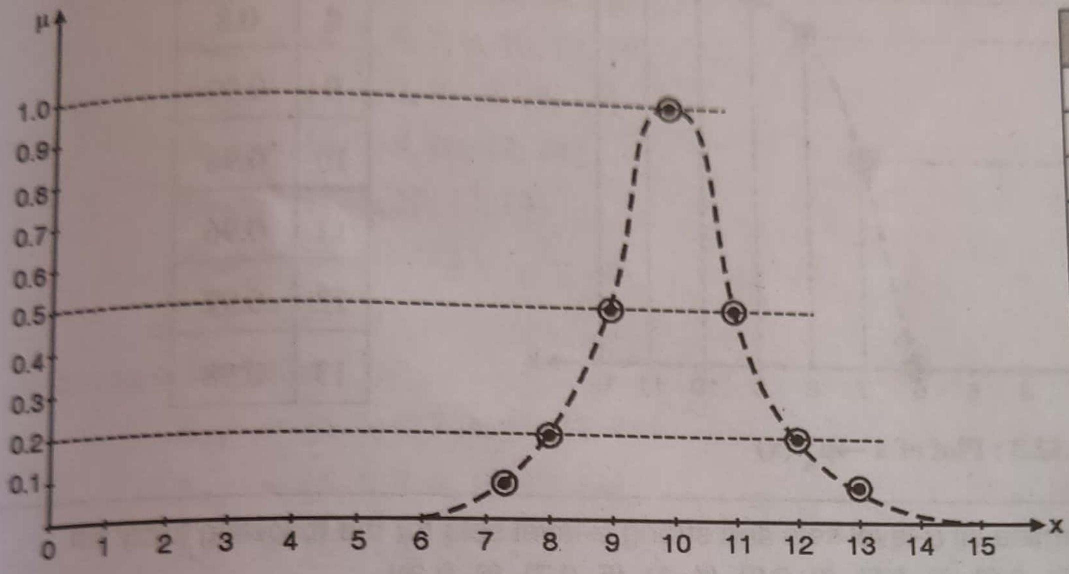


Fig. P. 2.12.2 : Plot of $x \rightarrow \mu_{\tilde{A}}(x)$

Ex. 2.12.3 : Model the following fuzzy set using suitable membership function. "Integer number considerably larger than 6".

Soln. :

Here universe of discourse is set of all integer numbers.

$$X = \text{Integers}$$

Then fuzzy set for "Number considerably larger than 6" can be defined as.

$$\mu_{\tilde{A}}(x) = \frac{1}{1+\frac{1}{(x-6)^2}}$$

Fig. P.2.12.3 shows the plot of $x \rightarrow \mu_A(x)$.

So membership function for "Number considerably larger than 6" is defined as,

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , x \leq 6 \\ \dfrac{1}{1+\dfrac{1}{(x-6)^2}} & , x > 6 \end{cases}$$

...Ans.

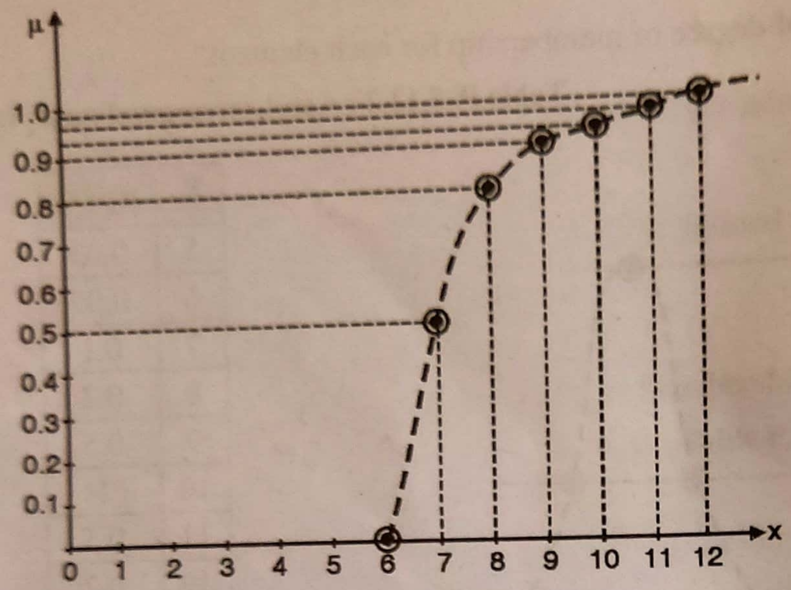


Fig. P. 2.12.3 : Plot of $x \rightarrow \mu_{\tilde{A}}(x)$

Table P. 2.12.3 : x and corresponding $\mu_{\tilde{A}}(x)$

x	$\mu_{\tilde{A}}(x)$
6	0
7	0.5
8	0.8
9	0.90
10	0.94
11	0.96
12	0.97
13	0.98

Ex. 2.12.4 : Determine all α-level sets and strong α-level sets for the following fuzzy set
$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$

Soln. :

The following are α-level sets

$$A_{0.2} = \{1, 2, 3, 4, 5, 6\}$$
$$A_{0.3} = \{2, 3, 4, 5, 6\}$$
$$A_{0.5} = \{2, 3, 4, 5\}$$
$$A_{0.7} = \{3, 4, 5\}$$
$$A_{0.8} = \{3, 4\}$$
$$A_{1} = \{4\}$$

Following are strong α-level sets.

$$A_{0.2}' = \{2, 3, 4, 5, 6\}$$
$$A_{0.3}' = \{2, 3, 4, 5\}$$
$$A_{0.5}' = \{3, 4, 5\}$$
$$A_{0.7}' = \{3, 4\}$$
$$A_{0.8}' = \{4\}$$
$$A_{1}' = \phi$$

Ex. 2.12.5 : Find out all α-level sets and strong α-level sets for the following fuzzy set $\tilde{A} = \{(3, 0.1), (4, 0.2), (5, 0.3), (6, 0.4), (7, 0.6), (8, 0.8), (10, 1), (12, 0.8), (14, 0.6)\}$

Soln. :

- α-level sets

$$A_{0.1} = \{3, 4, 5, 6, 7, 8, 10, 12, 14\}$$
$$A_{0.2} = \{4, 5, 6, 7, 8, 10, 12, 14\}$$
$$A_{0.3} = \{5, 6, 7, 8, 10, 12, 14\}$$
$$A_{0.4} = \{6, 7, 8, 10, 12, 14\}$$
$$A_{0.6} = \{7, 8, 10, 12, 14\}$$
$$A_{0.8} = \{8, 10, 12\}$$
$$A_{1} = \{10\}$$

- Strong α-level sets

$$A_{0.1'} = \{4, 5, 6, 7, 8, 10, 12, 14\}$$
$$A_{0.2'} = \{5, 6, 7, 8, 10, 12, 14\}$$
$$A_{0.3'} = \{6, 7, 8, 10, 12, 14\}$$
$$A_{0.4'} = \{7, 8, 10, 12, 14\}$$
$$A_{0.6'} = \{8, 10, 12\}$$
$$A_{0.8'} = \{10\}$$
$$A_{1'} = \phi$$

Ex. 2.12.6 : A realtor wants to classify the houses he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in them. Let the available types of houses be represented by the following set.

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

The houses in this set are specified by the number of bedrooms in a house. Describe comfortable house for 4-person family" using a fuzzy set.

Soln. :

The fuzzy set for "comfortable type of house for a 4-person family" may be described as,

$$\tilde{A} = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

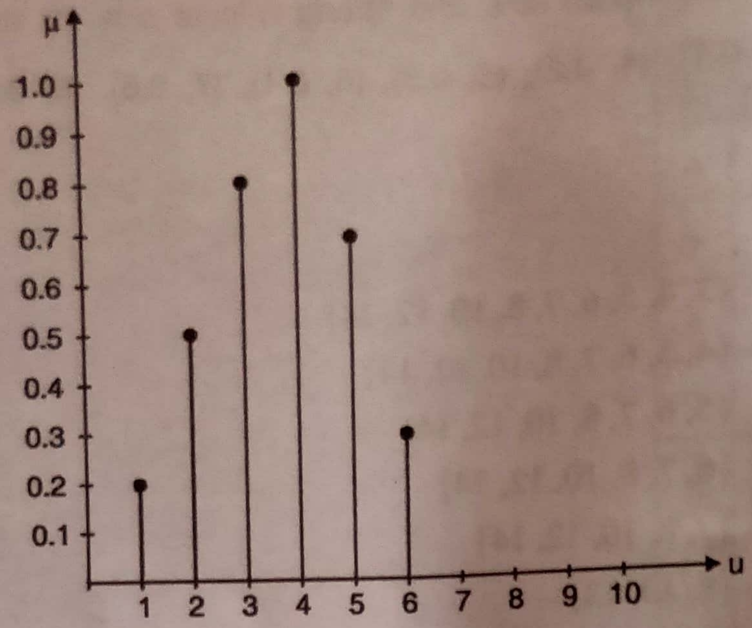


Fig. P. 2.12.6: Plot of u $\rightarrow \mu_A(u)$

Ex. 2.12.7 : Assume $\tilde{A}$ = "x considerably larger than 10" and $\tilde{B}$ = "x approximately 11" characterized by $\tilde{A} = \{x, \mu_{\tilde{A}}(x) \mid x \in X\}$

Draw the plot for both the sets and show

$\tilde{A} \cup \tilde{B}$ and $\tilde{A} \cap \tilde{B}$ in a plot.

Soln. :

Fuzzy set $\tilde{A}$ can be defined as,

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , x \leq 10 \\ \dfrac{1}{1 + \dfrac{1}{(x-10)^2}} & , x > 10 \end{cases}$$

larger

Set $\tilde{B}$ can be defined as,

$$\mu_{\tilde{B}}(x) = \frac{1}{1 + (x-11)^2}$$

approx

Then,

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \begin{cases} \min[(1 + (x-10)^{-2})^{-1}, (1 + (x-11)^2)^{-1}] & , x > 10 \\ 0 & , x \leq 10 \end{cases}$$

That is, intersection operation on fuzzy set $\tilde{A}$ and $\tilde{B}$ represents a new fuzzy set "x considerably larger than 10 and approximately 11".

and

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \{ \max[(1+(x-10)^{-2})^{-1}, (1+(x-11)^{2})^{-1}],\ x \in X$$

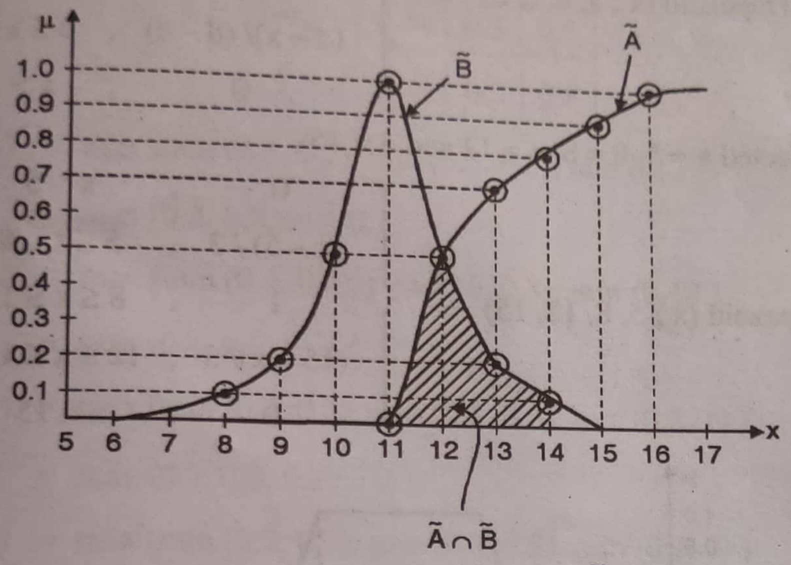


Fig. P. 2.12.7 (a) : Plot of $\tilde{A} \cap \tilde{B}$

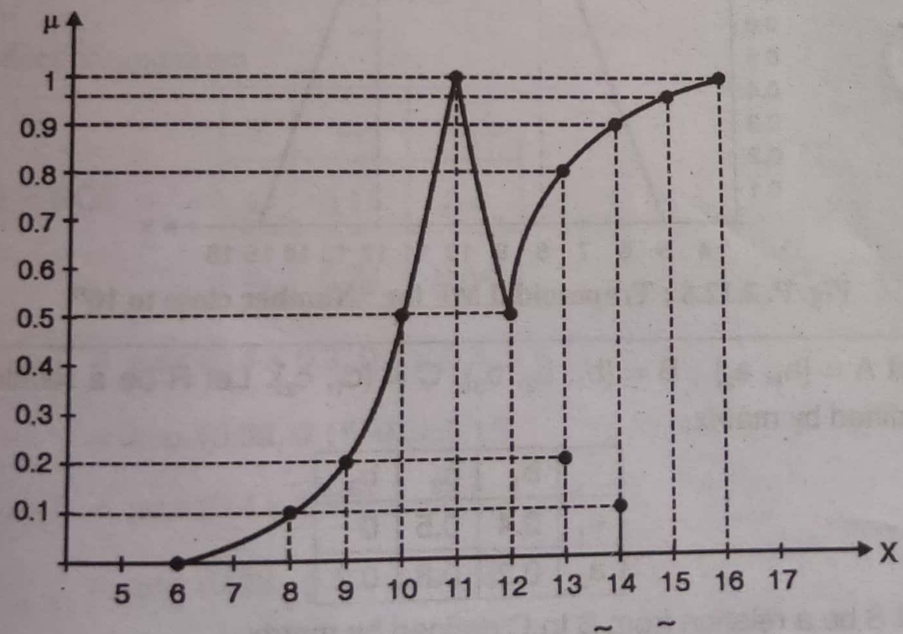


Fig. P. 2.12.7(b) : Plot of $\tilde{A} \cup \tilde{B}$

Ex. 2.12.8 : Model the following as fuzzy set using trapezoidal membership function "Number close to 10".

Soln. :

"Number close to 10" can be represented by,

$$\text{Trapezoid}(x\,;a,b,c,d) = \begin{cases} 0 & , \quad x \le a \\ (x-a)/(b-a) & , \quad a \le x \le b \\ 1 & , \quad b \le x \le c \\ (d-x)/(d-c) & , \quad c \le x \le d \\ 0 & , \quad x > 15 \end{cases}$$

We have selected a = 5, b = 8, c = 12 and d = 15

$$\text{Trapezoid}(x\,;5,8,12,15) = \begin{cases} 0 & , \quad x \le 5 \\ (x-5)/3 & , \quad 5 \le x \le 8 \\ 1 & , \quad 8 \le x \le 12 \\ (15-x)/3 & , \quad 12 \le x \le 15 \\ 0 & , \quad x > 15 \end{cases}$$

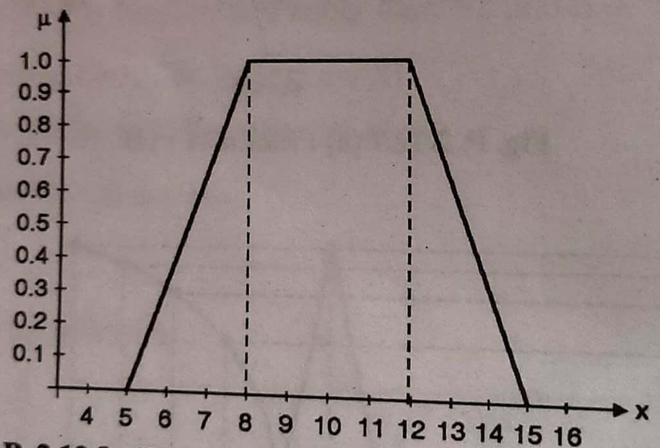


Fig. P. 2.12.8 : Trapezoidal MF for "Number close to 10"

Ex .2.12.9 : Let A = {a_1, a_2} , B = {b_1, b_2, b_3}, C = {c_1, c_2}. Let R be a relation from A to B defined by matrix.

	b_1	b_2	b_3
a_1	0.4	0.5	0
a_2	0.2	0.8	0.2

Let S be a relation from B to C defined by matrix.

	c_1	c_2
b_1	0.2	0.7
b_2	0.3	0.8
b_3	1	0

Find : (1) Max-min composition of R and S.

(2) Max-product composition of R and S.

Soln. :

(1) **Max-min composition**

$T = ROS =$

T	c_1	c_2
a_1	0.3	0.5
a_2	0.3	0.8

...**Ans.**

$$T(a_1, c_1) = \max(\min(0.4, 0.2), \min(0.5, 0.3), \min(0, 1))$$
$$= \max(0.2, 0.3, 0) = 0.3$$
$$T(a_1, c_2) = \max(\min(0.4, 0.7), \min(0.5, 0.8), \min(0, 0))$$
$$= \max(0.4, 0.5, 0) = 0.5$$
$$T(a_2, c_1) = \max(\min(0.2, 0.2), \min(0.8, 0.3), \min(0.2, 1))$$
$$= \max(0.2, 0.3, 0.2) = 0.3$$
$$T(a_2, c_2) = \max(\min(0.2, 0.7), \min(0.8, 0.8), \min(0.2, 0))$$
$$= \max(0.2, 0.8, 0) = 0.8$$

(2) **Max-product composition**

$T = ROS =$

T	c_1	c_2
a_1	0.15	0.4
a_2	0.24	0.64

...**Ans.**

$$T(a_1, c_1) = \max(0.4 \times 0.2, 0.5 \times 0.3, 0 \times 1)$$
$$= \max(0.08, 0.15, 0) = 0.15$$
$$T(a_1, c_2) = \max(0.4 \times 0.7, 0.5 \times 0.8, 0 \times 0)$$
$$= \max(0.28, 0.40, 0) = 0.4$$
$$T(a_2, c_1) = \max(0.2 \times 0.2, 0.8 \times 0.3, 0.2 \times 1)$$
$$= \max(0.04, 0.24, 0.2) = 0.24$$
$$T(a_2, c_2) = \max(0.2 \times 0.7, 0.8 \times 0.8, 0.2 \times 0)$$
$$= \max(0.14, 0.64, 0) = 0.64$$

Ex. 2.12.10 : High speed rail monitoring devices sometimes make use of sensitive sensors to measure the deflection of the earth when a rail car passes. These deflections are measured with respect to some distance from the rail car and, hence are actually very small angles measured in micro-radians. Let a universe of deflection be A = [1, 2, 3, 4] where A is the angle in micro-radians, and let a universe of distance be D = [1, 2, 5, 7] where D is distance in feet, suppose a relation between these two parameters has been determined as follows :

R =

	D_1	D_2	D_3	D_4
A_1	1	0.3	0.1	0
A_2	0.2	1	0.3	0.1
A_3	0	0.7	1	0.2
A_4	0	0.1	0.4	1

Now let a universe of rail car weights be W = [1, 2], where W is the weight in units of 100,000 pounds. Suppose the fuzzy relation of W to A is given by,

S =

	W_1	W_2
A_1	1	0.4
A_2	0.5	1
A_3	0.3	0.1
A_4	0	0

Using these two relations, find the relation $R^T \circ S = T$.

(a) Using max-min composition.

(b) Using max-product composition.

Soln. :

First find R^T,

R^T =

	A_1	A_2	A_3	A_4
D_1	1	0.2	0	0
D_2	0.3	1	0.7	0.1
D_3	0.1	0.3	1	0.4
D_4	0	0.1	0.2	1

and S =

	W_1	W_2
A_1	1	0.4
A_2	0.5	1
A_3	0.3	0.1
A_4	0	0

(1) **Using max-min composition**

T =

	W_1	W_2
D_1	1	0.4
D_2	0.5	1
D_3	0.3	0.3
D_4	0.2	0.1

...Ans.

$T(D_1, W_1) = \max(1, 0.2, 0, 0) = 1$

$T(D_1, W_2) = \max(0.4, 0.2, 0, 0) = 0.4$

$T(D_2, W_1) = \max(0.3, 0.5, 0.3, 0) = 0.5$

$T(D_2, W_2) = \max(0.3, 1, 0.1, 0) = 1$

$T(D_3, W_1) = \max(0.1, 0.3, 0.3, 0) = 0.3$

$T(D_3, W_2) = \max(0.1, 0.3, 0.1, 0) = 0.3$

$T(D_4, W_1) = \max(0, 0.1, 0.2, 0) = 0.2$

$T(D_4, W_2) = \max(0, 0.1, 0.1, 0) = 0.1$

(2) **Using max product composition**

T =

	W_1	W_2
D_1	1	0.4
D_2	0.5	1
D_3	0.3	0.3
D_4	0.06	0.1

...Ans.

$T(D_1, W_1) = \max(1 \times 1, 0.2 \times 0.5, 0 \times 0.3, 0 \times 0)$

$= \max(1, 0.10, 0, 0) = 1$

$T(D_1, W_2) = \max(1 \times 0.4, 0.2 \times 1, 0 \times 0.1, 0 \times 0)$

$= \max(0.4, 0.2, 0, 0) = 0.4$

$T(D_2, W_1) = \max(0.3 \times 1, 1 \times 0.5, 0.7 \times 0.3, 0.1 \times 0)$

$= \max(0.3, 0.5, 0.21, 0) = 0.5$

$T(D_2, W_2) = \max(0.3 \times 0.4, 1 \times 1, 0.7 \times 0.1, 0.1 \times 0)$

$= \max(0.12, 1, 0.07, 0) = 1$

$T(D_3, W_1) = \max(0.1 \times 1, 0.3 \times 0.5, 1 \times 0.3, 0.4 \times 0)$

$= \max(0.1, 0.15, 0.3, 0) = 0.3$

$$T(D_3, W_2) = \max(0.1 \times 0.4, 0.3 \times 1, 1 \times 0.1, 0.4 \times 0)$$
$$= \max(0.04, 0.3, 0.1, 0) = 0.3$$
$$T(D_4, W_1) = \max(0 \times 1, 0.1 \times 0.5, 0.2 \times 0.3, 1 \times 0)$$
$$= \max(0, 0.05, 0.06, 0) = 0.06$$
$$T(D_4, W_2) = \max(0 \times 0.4, 0.1 \times 1, 0.2 \times 0.1, 1 \times 0)$$
$$= \max(0, 0.1, 0.02, 0) = 0.1$$

Ex. 2.12.11 : Model the following fuzzy set using trapezoidal membership function, "Middle age".

Soln. :

Let X be a reasonable age interval of human being.

$$X = \{0, 1, 2, 3, \ldots, 100\}$$

Then a fuzzy set "Middle age" can be represented using Trapezoidal MF as follows.

$$\text{Trapezoid}(x; 30, 40, 60, 70) = \begin{cases} 0 & , \quad x \leq 30 \\ (x-30)/10 & , \quad 30 \leq x \leq 40 \\ 1 & , \quad 40 \leq x \leq 60 \\ (70-x)/10 & , \quad 60 \leq x \leq 70 \\ 0 & , \quad x > 70 \end{cases}$$

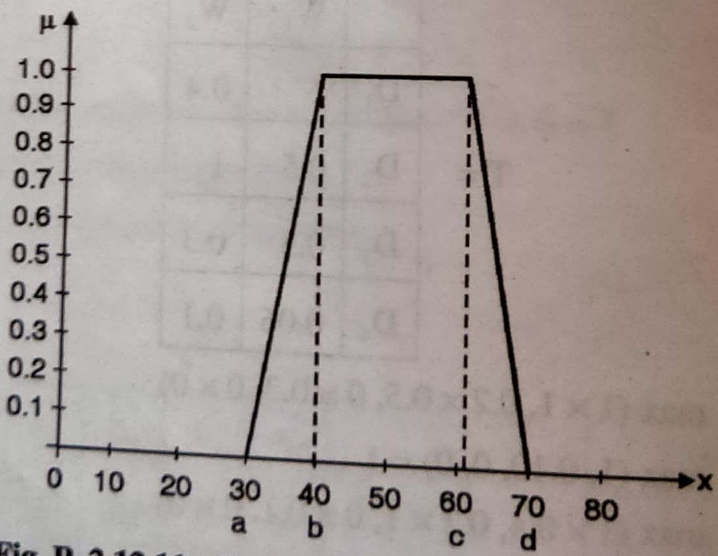


Fig. P. 2.12.11 : Trapezoidal MF for "Middle age"

Ex. 2.12.12 : Represent the set of old people as a fuzzy set using appropriate membership function.

Soln. :

Let X = (0,120) set of all possible ages.

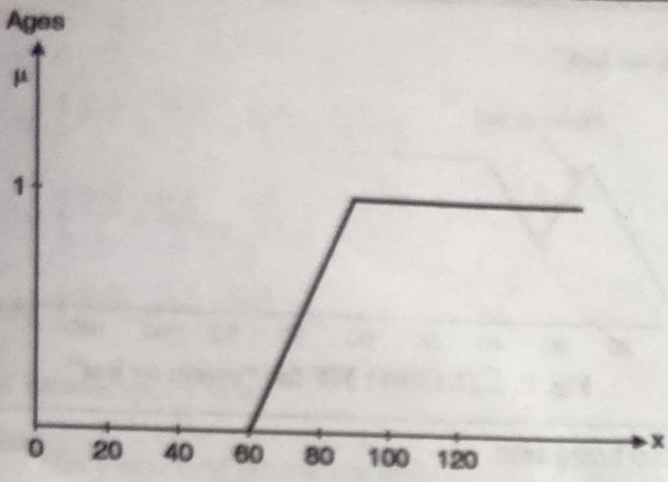


Fig. P. 2.12.12 : Membership function for "old people"

$$\mu_{old}(x) = \begin{cases} 0, & 0 \le x \le 60 \\ (x-60)/20, & 60 \le x \le 80 \\ 1, & x \ge 80 \end{cases}$$

Ex.2.12.13 : Develop graphical representation of membership function to describe linguistic variables "cold", "warm" and "hot". The temperature ranges from O°c to 100°c. Also show plot for "cold and warm" and "warm or hot" temperature.

Soln. :

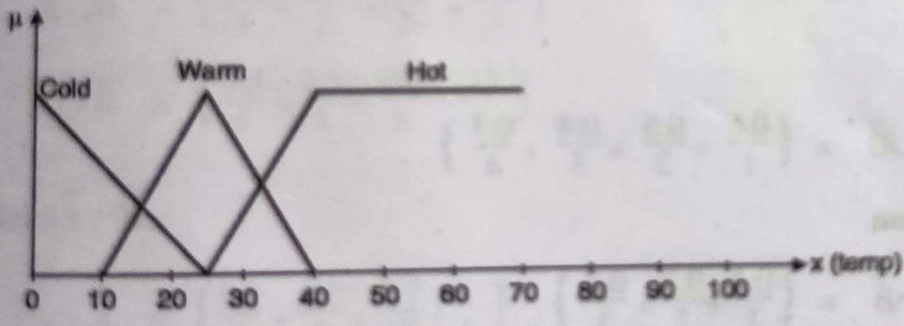


Fig. P. 2.12.13 : MF for cold, warm and hot temperature

Plot for **"cold and warm"**

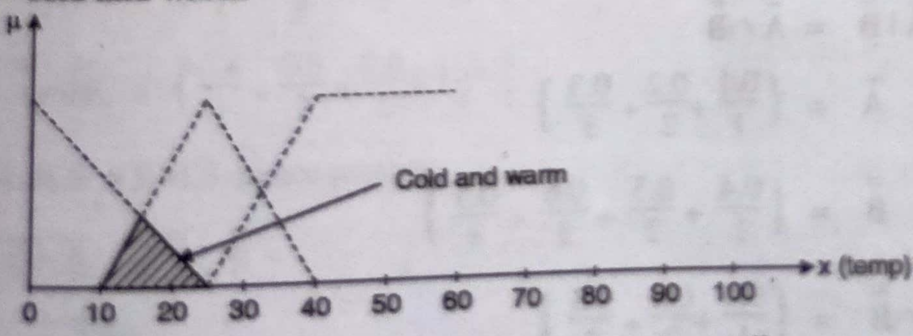


Fig. P. 2.12.13(a) : MF for "cold and warm"

Plot for **"warm or hot"**

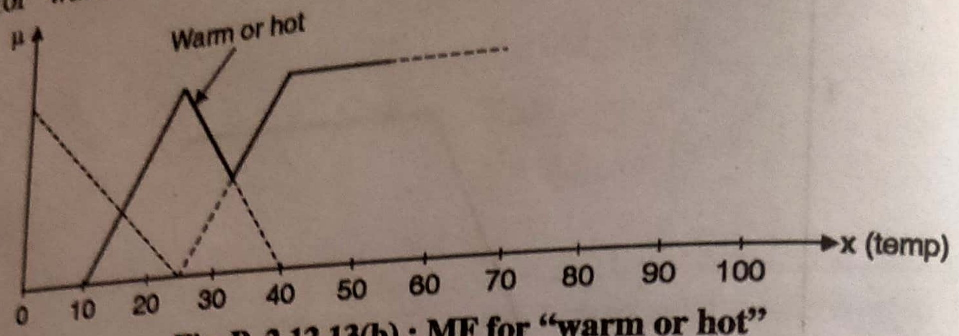


Fig. P. 2.12.13(b) : MF for "warm or hot"

Ex. 2.12.14 : Given two fuzzy sets.

$$\tilde{A} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.3}{3}\right\}$$

$$\tilde{B} = \left\{\frac{0.6}{1} + \frac{0.5}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

Perform following operations on $\tilde{A}$ and $\tilde{B}$

(1) Union

(2) Intersection

(3) Set difference

(4) Verify Demorgan's law.

Soln. :

1. Union

$$\tilde{A} \cup \tilde{B} = \left\{\frac{0.6}{1} + \frac{0.5}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

2. Intersection

$$\tilde{A} \cap \tilde{B} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.3}{3}\right\}$$

3. Set difference

$$\tilde{A} \mid \tilde{B} = \tilde{A} \cap \overline{\tilde{B}}$$

$$\tilde{A} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.3}{3}\right\}$$

$$\overline{\tilde{B}} = \left\{\frac{0.4}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}$$

$$\tilde{A} \cap \overline{\tilde{B}} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.3}{3}\right\}$$

$$\tilde{B} \mid \tilde{A} = \tilde{B} \cap \bar{\tilde{A}}$$

$$\tilde{B} = \left\{\frac{0.6}{1} + \frac{0.5}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

$$\bar{\tilde{A}} = \left\{\frac{0.9}{1} + \frac{0.8}{2} + \frac{0.7}{3}\right\}$$

$$\tilde{B} \cap \bar{\tilde{A}} = \left\{\frac{0.6}{1} + \frac{0.5}{2} + \frac{0.4}{3}\right\}$$

4. **Verification of Demorgan's law**

To verify Demorgan's law first normalize both the sets $\tilde{A}$ and $\tilde{B}$.

$$\tilde{A} = \left\{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.3}{3} + \frac{0}{4}\right\}$$

$$\tilde{B} = \left\{\frac{0.6}{1} + \frac{0.5}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

1. $$\overline{\tilde{A} \cup \tilde{B}} = \bar{\tilde{A}} \cap \bar{\tilde{B}}$$

L.H.S : $\overline{\tilde{A} \cup \tilde{B}}$

$$\tilde{A} \cup \tilde{B} = \left\{\frac{0.6}{1} + \frac{0.5}{2} + \frac{0.4}{3} + \frac{0.5}{4}\right\}$$

$$\overline{\tilde{A} \cup \tilde{B}} = \left\{\frac{0.4}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\} \quad ...(1)$$

R.H.S : $\bar{\tilde{A}} \cap \bar{\tilde{B}}$

$$\bar{\tilde{A}} = \left\{\frac{0.9}{1} + \frac{0.8}{2} + \frac{0.7}{3} + \frac{1}{4}\right\}$$

$$\bar{\tilde{B}} = \left\{\frac{0.4}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\}$$

$$\bar{\tilde{A}} \cap \bar{\tilde{B}} = \left\{\frac{0.4}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.5}{4}\right\} \quad ...(2)$$

Since L.H.S. = R.H.S. hence proved.

2. $$\overline{\tilde{A} \cap \tilde{B}} = \bar{\tilde{A}} \cup \bar{\tilde{B}}$$

L.H.S. : $\overline{\tilde{A} \cap \tilde{B}}$

$$\bar{\tilde{A}} \cap \bar{\tilde{B}} = \left\{\frac{0.1}{1}+\frac{0.2}{2}+\frac{0.3}{3}+\frac{0}{4}\right\}$$

$$\overline{\bar{\tilde{A}} \cap \bar{\tilde{B}}} = \left\{\frac{0.9}{1}+\frac{0.8}{2}+\frac{0.7}{3}+\frac{1}{4}\right\} \quad ...(1)$$

R.H.S. : $\bar{\bar{\tilde{A}}} \cup \bar{\bar{\tilde{B}}}$

$$\bar{\bar{\tilde{A}}} \cup \bar{\bar{\tilde{B}}} = \left\{\frac{0.9}{1}+\frac{0.8}{2}+\frac{0.7}{3}+\frac{1}{4}\right\} \quad ...(2)$$

Since L.H.S. = R.H.S. hence proved.

Ex. 2.12.15 : For the given membership function as shown in Fig. P. 2.12.15, determine the defuzzified output value by any two methods.

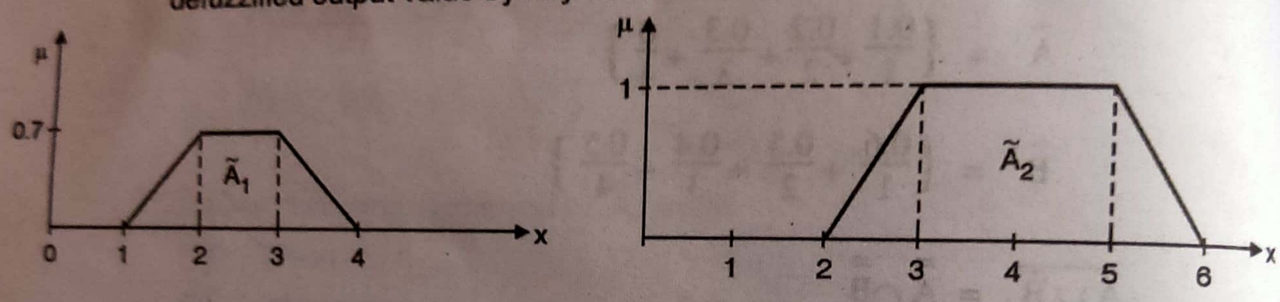


Fig. P. 2.12.15

Soln. :

1) **Weighted Average method**

In weighted average method we first find the centre of each individual fuzzy set .

Centre of $\tilde{A}_1 = 2.5$ Centre of $\tilde{A}_2 = 4.0$

Next we find the membership value at the centre

Membership value of centre of $\tilde{A}_1 = 0.7$

Membership value of centre of $\tilde{A}_2 = 1$

$$x^* = \frac{(0.7 * 2.5) + (1 \times 4.0)}{0.7 + 1} = \frac{1.75 + 4}{1.7} = 3.38$$

...Ans.

2) **Centre of sum method**

Find the area of each individual curve

We know that,

$$\text{Area of Trapezoid} = \frac{[\,(\text{Sum of length of parallel lines}) \times (\text{distance between parallel lines})\,]}{2}$$

$$\text{Therefore, Area of } \tilde{A}_1 = \frac{(1+3) * 0.7}{2} = 1.4$$

Similarly, Area of $\tilde{A}_2 = \frac{(2+4) * 1}{2} = \frac{6}{2} = 3$

Next, find center of each curve

Centre of $\tilde{A}_1 = 2.5$

Centre of $\tilde{A}_2 = 4$

$$X^* = \frac{(2.5 \times 1.4) + (4 \times 3)}{1.4 + 3} = \frac{15.5}{4.4} = 3.52$$ ...Ans.

Ex. 2.12.16 : Consider three fuzzy sets $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ given below. Find defuzzified value using :

(1) mean of max

(2) centroid

(3) centre of sum

(4) weighted avg. method.

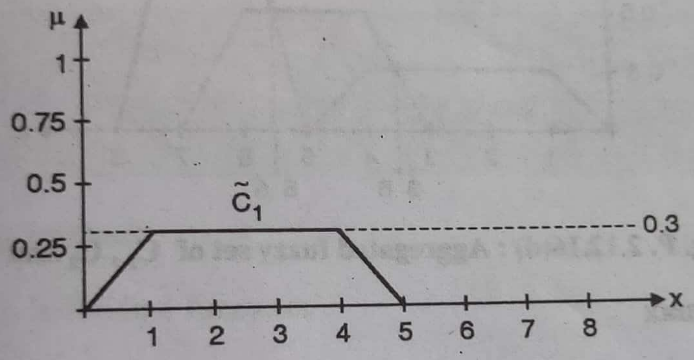


Fig. P. 2.12.16 (a) : fuzzy set $\tilde{C}_1$

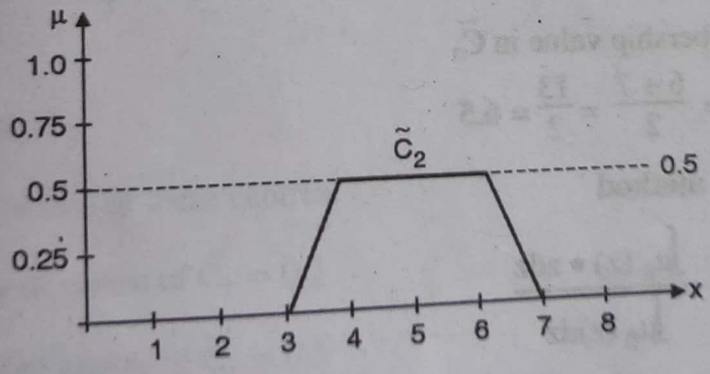


Fig. P. 2.12.16(b) : fuzzy set $\tilde{C}_2$

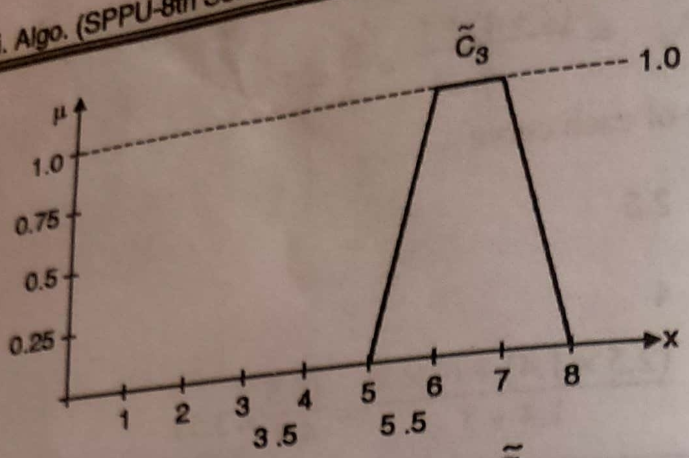


Fig. P. 2.12.16(c) : fuzzy set $\tilde{C}_3$

Soln. :

First find aggregation of all MFs (union)

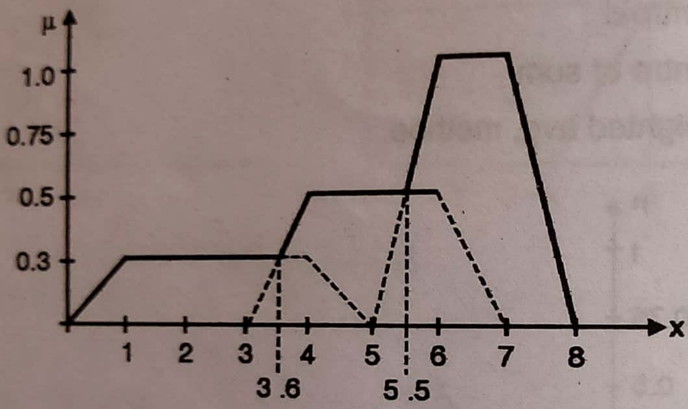


Fig. P. 2.12.16(d) : Aggregated fuzzy set of $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$

1) **Using Mean of max**

Since $\tilde{C}_3$ is the maximizing MF, we take the mean (average) of all the elements having maximum membership value in $\tilde{C}_3$

$$x^* = \frac{6+7}{2} = \frac{13}{2} = 6.5$$

...**Ans.**

2) **Using Centroid method**

$$z^* = \frac{\int \mu_B(z) \bullet z dz}{\int \mu_B(z) dz}$$

$$= \left[\int_0^1 (0.3z)\, z dz + \int_1^{3.6} (0.3z)\, dz + \int_{3.6}^{4} \left(\frac{z-3}{2}\right) z\, dz + \int_4^{5.5} (0.5)\, z\, dz + \int_{5.5}^{6} (z-5)\, z\, dz + \int_6^7 z\, dz + \int_7^8 (8-z)\, z\, dz\right]$$

$$\div \left[\int_0^1 (0.3z)\, dz + \int_1^{3.6} (0.3)\, dz + \int_{3.6}^{4} \left(\frac{z-3}{2}\right) dz + \int_4^{5.5} (0.5)\, dz + \int_{5.5}^{6} (z-5)\, dz + \int_6^7 dz + \int_7^8 (8-z)\, dz\right] = 4.9$$

3) **Using centre of sum method**

First find area of each individual fuzzy set

Area of $\tilde{C}_1 = 1.2$

Area of $\tilde{C}_2 = 1.5$

Area of $\tilde{C}_3 = 2$

Then find centre of each individual fuzzy set

Centre of $\tilde{C}_1 = 2.5$

Centre of $\tilde{C}_2 = 5$

Centre of $\tilde{C}_3 = 6.5$

$$x^* = \frac{(2.5 \times 1.2) + (1.5 \times 5) + (2 \times 6.5)}{1.2 + 1.5 + 2}$$

$$\therefore \quad x^* = 5$$

...Ans.

Note :

Area of Trapezoid = $\frac{\text{[(Sum of length of parallel lines)} \times \text{(distance between parallel lines)]}}{2}$

4) **Weighted average method**

Find centre of each individual fuzzy set

Centre of $\tilde{C}_1 = 2.5$

Centre of $\tilde{C}_2 = 5$

Centre of $\tilde{C}_3 = 6.5$

Find membership values of these centres.

Membership value of centre of $\tilde{C}_1 = 0.3$

Membership value of centre of $\tilde{C}_2 = 0.5$

Membership value of centre of $\tilde{C}_3 = 1$

$$x^* = \frac{(2.5 \times 0.3) + (5 \times 0.5) + (6.5 \times 1)}{0.3 + 0.5 + 1} = \frac{9.75}{1.8}$$

...Ans.

$$\therefore \quad x^* = 5.146$$

Ex. 2.12.17 : Given fuzzy set.

$$\tilde{A} = \left\{ \frac{0.1}{1} + \frac{0.3}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{1}{5} + \frac{0.8}{6} \right\}$$

Find core and support of fuzzy set $\tilde{A}$

Soln. :

Core of $\tilde{A} = \{4, 5\} \rightarrow$ Membership value equal to 1

Support of $\tilde{A} = \{1, 2, 3, 4, 5, 6\} \rightarrow$ Membership value > 0

Ex. 2.12.18 : Consider following two fuzzy sets

$$\tilde{A} = \left\{ \frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4} \right\} \qquad \tilde{B} = \left\{ \frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{1}{4} \right\}$$

Find : 1) Algebraic sum 2) Algebraic product

3) Bounded sum 4) Bounded difference

Soln. :

1) **Algebraic sum**

$$\mu_{\tilde{A}+\tilde{B}}(x) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$$

$$= \left\{ \frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1.5}{4} \right\} - \left\{ \frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4} \right\}$$

$$= \left\{ \frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3} + \frac{1}{4} \right\}$$

2) **Algebraic product**

$$\mu_{\tilde{A}\cdot\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)]$$

$$= \left[\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4} \right]$$

3) **Bounded sum**

$$\mu_{\tilde{A}\oplus\tilde{B}}(x) = \min[1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)]$$

$$= \min\left\{ 1, \left\{ \frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1.5}{4} \right\} \right\} = \left\{ \frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1}{4} \right\}$$

4) **Bounded difference**

$$\mu_{\tilde{A}\odot\tilde{B}}(x) = \max[0, \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)]$$

$$= \max\left\{ 0, \left\{ \frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{-0.5}{4} \right\} \right\} = \left\{ \frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0}{4} \right\}$$

Ex. 2.12.19 : For given two fuzzy sets

$$\tilde{A} = \left\{ \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{1}{x_3} \right\} \text{ and } \tilde{B} = \left\{ \frac{0.4}{y_1} + \frac{0.9}{y_2} \right\}$$

Perform Cartesian product.

Soln. : For Cartesian product we consider min operator (denoted by $\wedge$)

$$\tilde{A} \times \tilde{B} = \left\{ \frac{0.3 \wedge 0.4}{(x_1, y_1)} + \frac{0.3 \wedge 0.9}{(x_1, y_2)} + \frac{0.7 \wedge 0.4}{(x_2, y_1)} + \frac{0.7 \wedge 0.9}{(x_2, y_2)} + \frac{1 \wedge 0.4}{(x_3, y_1)} + \frac{1 \wedge 0.9}{(x_3, y_2)} \right\}$$

$$= \left\{ \frac{0.3}{(x_1, y_1)} + \frac{0.3}{(x_1, y_2)} + \frac{0.4}{(x_2, y_1)} + \frac{0.7}{(x_2, y_2)} + \frac{0.4}{(x_3, y_1)} + \frac{0.9}{(x_3, y_2)} \right\}$$

It can be represented in a matrix form

$\tilde{A} \times \tilde{B} =$

	y_1	y_2
x_1	0.3	0.3
x_2	0.4	0.7
x_3	0.4	0.9

...Ans.

Fig. 2.12.19

Ex.2.12.20 : Consider the following fuzzy sets

$$\text{Low temperature} = \left\{ \frac{1}{131} + \frac{0.8}{132} + \frac{0.6}{133} + \frac{0.4}{134} + \frac{0.2}{135} + \frac{0}{136} \right\}$$

$$\text{High temperature} = \left\{ \frac{0}{134} + \frac{0.2}{135} + \frac{0.4}{136} + \frac{0.6}{137} + \frac{0.8}{138} + \frac{1}{139} \right\}$$

$$\text{High pressure} = \left\{ \frac{0.1}{400} + \frac{0.2}{600} + \frac{0.4}{700} + \frac{0.6}{800} + \frac{0.8}{900} + \frac{1}{1000} \right\}$$

Temperature ranges are 130° F to 140° F and pressure limit is 400 psi to 1000 psi. Find the following membership functions :

1) Temperature not very low.
2) Temperature not very high
3) Pressure slightly high.
4) Pressure very very high.

Soln. :

1) **Temperature not very low**

$$\text{Very low} = \text{low}^2$$

$$= \left\{ \frac{1}{131} + \frac{0.64}{132} + \frac{0.36}{133} + \frac{0.16}{134} + \frac{0.04}{135} + \frac{0}{136} \right\}$$

$$\text{Not very low} = 1 - \text{very low}$$

$$= \left\{ \frac{0}{131} + \frac{0.36}{132} + \frac{0.64}{133} + \frac{0.84}{134} + \frac{0.96}{135} + \frac{1}{136} \right\}$$

...Ans. 1

2) **Temperature not very high**

Very high = high^2

$$\therefore \text{Temp very high} = \left\{\frac{0}{134}+\frac{0.04}{135}+\frac{0.16}{136}+\frac{0.36}{137}+\frac{0.64}{138}+\frac{1}{139}\right\}$$

$$\text{Temp not very high} = 1-\text{very high}$$

$$= \left\{\frac{1}{134}+\frac{0.96}{135}+\frac{0.84}{136}+\frac{0.64}{137}+\frac{0.36}{138}+\frac{0}{139}\right\}$$

...Ans. 2

3) **Pressure slightly high**

Slightly high = dilation (high) = $\sqrt{\text{high}}$

$$\therefore \text{Pressure slightly high} = (\text{high pressure})^{1/2}$$

$$= \left\{\frac{\sqrt{0.1}}{400}+\frac{\sqrt{0.2}}{600}+\frac{\sqrt{0.4}}{700}+\frac{\sqrt{0.6}}{800}+\frac{\sqrt{0.8}}{900}+\frac{\sqrt{1}}{1000}\right\}$$

$$= \left\{\frac{0.31}{400}+\frac{0.44}{600}+\frac{0.63}{700}+\frac{0.77}{800}+\frac{0.89}{900}+\frac{1}{1000}\right\}$$

...Ans. 3

4) **Pressure very very high**

Very very high = $(\text{high})^4$

$$\therefore \text{Pressure very very high} = (\text{high pressure})^4$$

$$= \left\{\frac{0.0001}{400}+\frac{0.0016}{600}+\frac{0.025}{700}+\frac{0.12}{800}+\frac{0.40}{900}+\frac{1}{1000}\right\}$$

Ex. 2.12.21 : Two fuzzy relations are given by

$$R = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Obtain fuzzy relation T as a max-min composition and max-product composition between the fuzzy relations.

Soln. :

1. **Using max-min composition**

$$T(x_1, z_1) = \max(\min(0.6, 1), \min(0.3, 0.8))$$
$$= \max(0.6, 0.3)$$
$$= 0.6$$

$$T(x_1, z_2) = \max(\min(0.6, 0.5), \min(0.3, 0.4))$$
$$= \max(0.5, 0.3)$$
$$= 0.5$$
$$T(x_2, z_3) = \max(\min(0.6, 0.3), \min(0.3, 0.7))$$
$$= \max(0.3, 0.3)$$
$$= 0.3$$
$$T(x_2, z_1) = \max(\min(0.2, 1), \min(0.9, 0.8))$$
$$= \max(0.2, 0.8)$$
$$= 0.8$$
$$T(x_2, z_2) = \max(\min(0.2, 0.5), \min(0.9, 0.4))$$
$$= \max(0.2, 0.4)$$
$$= 0.4$$
$$T(x_2, z_3) = \max(\min(0.2, 0.3), \min(0.9, 0.7))$$
$$= \max(0.2, 0.7)$$
$$= 0.7$$

$$T = R \circ S \quad \begin{array}{c} X_1 \\ X_2 \end{array} \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix}$$ ...Ans.

2. Using max-product

$$T(X_1, Z_1) = \max(0.6 \times 1, 0.3 \times 0.8)$$
$$= \max(0.6, 0.24)$$
$$= 0.6$$
$$T(X_1, Z_2) = \max(0.6 \times 0.5, 0.3 \times 0.4)$$
$$= \max(0.30, 0.12)$$
$$= 0.3$$
$$T(X_1, Z_3) = \max(0.6 \times 0.3, 0.3 \times 0.7)$$
$$= \max(0.18, 0.21)$$
$$= 0.21$$
$$T(X_2, Z_1) = \max(0.2 \times 1, 0.9 \times 0.8)$$
$$= \max(0.2, 0.72)$$
$$= 0.72$$
$$T(X_2, Z_2) = \max(0.2 \times 0.5, 0.9 \times 0.4)$$
$$= \max(0.1, 0.36)$$
$$= 0.36$$
$$T(X_2, Z_3) = \max(0.2 \times 0.3, 0.9 \times 0.7)$$

$= \max(0.06, 0.63)$

$= 0.63$

$$T = R \circ S = \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} Z_1 & Z_2 & Z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.21 \\ 0.72 & 0.36 & 0.63 \end{bmatrix} \end{array}$$

Review Questions

Q. 1 Explain support and core of a fuzzy set with examples. *(Ans. : Refer Section 2.7.2)*

Q. 2 Model the following as fuzzy set using trapezoidal membership function : "Number close to 10". *(Ans. : Refer Ex. 2.12.1)*

Q. 3 Let A = {a_1, a_2}, B = (b_1, b_2, b_3), C = (c_1, c_2)

Let R be a relation from A to B defined by matrix :

	b_1	b_2	b_3
a_1	0.4	0.5	0
a_2	0.2	0.8	0.2

Let S be a relation from B to C defined by matrix :

	c_1	c_2
b_1	0.2	0.7
b_2	0.3	0.8
b_3	1	0

Find (i) Max-min composition of R and S. (ii) Max-products composition of R and S. *(Ans. : Refer Ex. 2.12.9)*

Q. 4 Define Supports, Core, Normality, Crossover paints and α - cut for fuzzy set. *(Ans. : Refer Section 2.7.2)*

Q. 5 High speed rail monitoring devices sometimes make use of sensitive sensors to measure the deflection of the earth when a rail car passes. These deflections are measured with respect to some distance from the rail car and, hence are actually very small angles measured in micro-radians. Let a universe of deflection be A = [1, 2, 3, 4] where A is the angle in micro-radians, and let a universe of distance be D = [1, 2, 5, 7] where D is distance in feet, suppose a relation between these two parameters has been determined as follows :

R =

	D_1	D_2	D_3	D_4
A_1	1	0.3	0.1	0
A_2	0.2	1	0.3	0.1
A_3	0	0.7	1	0.2
A_4	0	0.1	0.4	1

Now let a universe of rail car weights be W = [1, 2], where W is the weight in units of 100,000 pounds. Suppose the fuzzy relation of W to A is given by,

S =

	W_1	W_2
A_1	1	0.4
A_2	0.5	1
A_3	0.3	0.1
A_4	0	0

Using these two relations, find the relation $R^T oS = T$.

(a) Using max-min composition.

(b) Using max-product composition.

(Ans. : Refer Ex. 2.12.10)

Q. 6 Explain standard fuzzy membership functions. *(Ans. : Refer Section 2.7.1)*

Q. 7 Explain any four defuzzification methods with suitable Example. *(Ans. : Refer Section 2.11.2)*

Q. 8 State the different properties of Fuzzy set. *(Ans. : Refer Section 2.4)*

Q. 9 Determine all α - level sets and strong α - level sets for the following fuzzy set.

A = {(1, 0.2) (2, 0.5), (3, 0.8), (4, 1) (5, 0.7), (6, 0.3)}. *(Ans. : Refer Ex. 2.12.4)*

Q. 10 Explain cylindrical extension and projection operations on fuzzy relation with example. *(Ans. : Refer Section 2.5.2.2)*

Q. 11 Model the following as fuzzy set using trapezoidal membership function "Middle age". *(Ans. : Refer Ex. 2.12.11)*

Q. 12 For the given membership function as shown in Fig. Q. 12, determine the defuzzified output value by any 2 methods. *(Refer Ex. 2.12.15)*

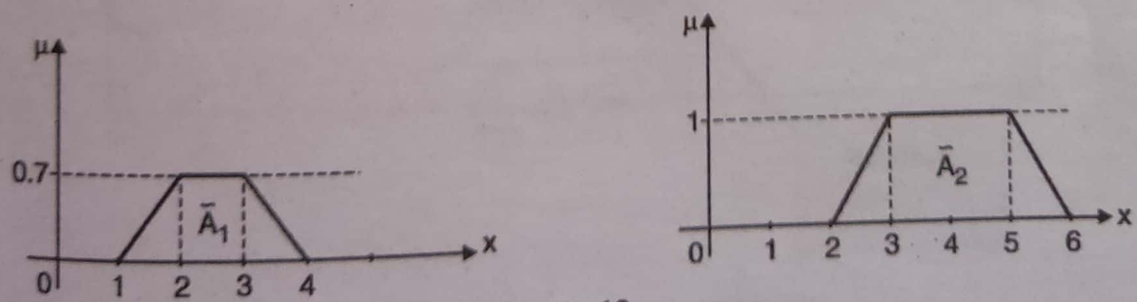


Fig. Q. 12

Q. 13 Discuss fuzzy composition techniques with suitable example. *(Refer Section 2.5.3)*

Q. 14 Two fuzzy relations are given by

$$R = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.9 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Obtain fuzzy relation T as a max-min composition and max-product composition between the fuzzy relations. *(Refer Ex. 2.5.3)*

❑❑❑

Chapter Ends...

$(A \times B \cup \bar{A} \times C)$