CSCI 3022

intro to data science with probability & statistics

Sept 17, 2018

HW2 Posted! Due in 11 days.

1. Bayes' Rule
2. Random variables and probability distributions

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Public Service Announcements

- Don't miss 10 days in all your classes and shave 0.5-1.0 off your semester GPA!   (Or: don't put others at risk for the flu!)
- **Free Flu Shots for CU**: https://www.colorado.edu/healthcenter/flu !

- Register to vote, and vote—one of the most beautiful parts of Adulting.
- **Online Reg**: colorado.edu/registrar/students/registration/mycuinfo/register-vote
- Call your parents.

Previously, on CSCI 3022

Conditional probability: The probability that A occurs **given** that C has occurred is

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

Product rule: $P(A \cap C) = P(A \mid C)P(C)$

Independence: events A and B are independent if and only if:

1. $P(A \mid B) = P(A)$
2. $P(B \mid A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$

Law of total probability (LTP): If $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \ldots \cup C_m = \Omega$ then the probability of an arbitrary event A can be written as

$$P(A) = P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \ldots + P(A \mid C_m)P(C_m)$$

Bayes' Rule

- Recall the probability that two events intersect: $P(A \cap B) = P(A|B)P(B)$ ✓

$$\|$$

- But we can write it the other way, too: $P(A \cap B) = P(B|A)P(A)$ ✓

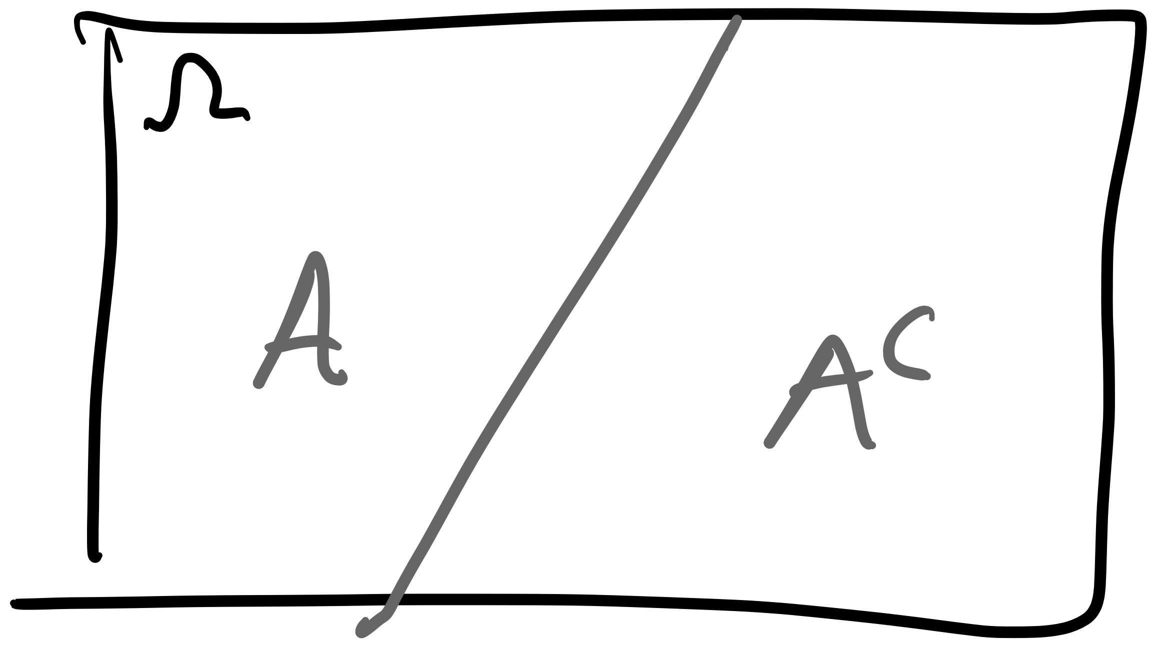
- And then… $P(A|B)P(B) = P(B|A)P(A)$ Assume $P(B) > 0$

- And this, is Bayes' Rule: $$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

- [Bonus: suppose A and B are independent…]

$P(A|B) = P(A)$ AND $P(B|A) = P(B)$ subs… $P(A) = \frac{\cancel{P(B)}P(A)}{\cancel{P(B)}}$

Bayes' Rule + Law of T. P.



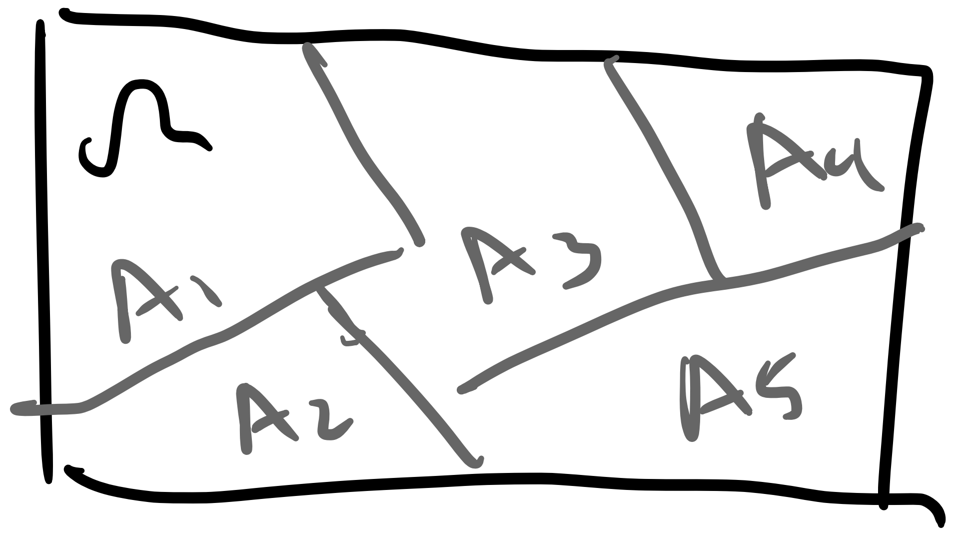
- Use Law of Total Probability to rewrite the denominator:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}} \overset{\text{LTP}}{=} \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

- Or, if B can be broken into K disjoint events:

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} \overset{\text{LTP}}{=} \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_m)P(A_m)}$$

$$= \frac{P(B|A_1)P(A_1)}{\sum_{i=1}^{m} P(B|A_i)P(A_i)}$$



Bayes' Rule + Law of T. P.

- Use Law of Total Probability to rewrite the denominator:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)}$$

- Or, if B can be broken into K disjoint events:

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots}$$

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{\sum_{k=1}^{K} P(B|A_k)P(A_k)}$$

Drugs, Prostates, TSA

- One of the most common applications of Bayes' rule is when we develop a test for something, but **the test is not always accurate**.
- Imagine that 1% of CS professors are using a drug *Bayes Salts*. There is a test that detects Bayes Salts on the breath of professors 98% of the time when a prof is using, and incorrectly calls BS only 1% of the time when a prof *is not* using.
- Suppose we test Professor Charles Xavier… he's **positive** for Bayes Salts! What is the probability that Professor Xavier is **actually** on Bayes Salts?

Drugs, Prostates, TSA

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- 1% of ~~math~~ CS professors *Bayes Salts* users. ⟵
- If on *Bayes Salts*, test says **+** 98% of the time
- If not on *Bayes Salts*, test says **+** only 1% of the time.
- What is probability that Prof. X is a user if he tests positive for *Bayes Salts*?

is user $= A$

not user $= A^c$

$$P(\text{user} \mid \text{test}\oplus) \overset{\text{Bayes}}{=} \frac{P(\text{test}\oplus \mid \text{user})P(\text{user})}{P(\text{test}\oplus)} \overset{\text{LTP}}{=} \frac{P(\text{test}\oplus \mid \text{user})P(\text{user})}{P(\text{test}\oplus \mid \text{user})P(\text{user}) + P(\text{test}\oplus \mid \text{not user})P(\text{not user})}$$

$$= \frac{0.98 \cdot \cancel{0.01}}{0.98 \cdot \cancel{0.01} + \cancel{0.01} \cdot 0.99} = \frac{0.98}{0.98 + 0.99} \approx \boxed{0.497}$$

Teach the controversy! 🫢

- Should we test men for prostate cancer?
- Bayes' Rule allows us to write down the probability that someone who tests positive for prostate cancer *actually has* prostate cancer.
- False positives may cause huge amounts of stress, heartache, and even unnecessary surgery!
- On the other hand, if you don't test for cancer, you may not discover it until it's too late.
- Things are slightly more complicated than this: age, PSA cutoffs, etc.

Flipping around a previous problem

- **Suppose I have two bags of marbles.** The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles. Now suppose I put the two bags in a box. If I close my eyes, grab a bag from the box, reach into the bag and pull out a white marble. What is the probability that I picked Bag 1?

$$P(\text{bag } 1 \mid \text{white marble}) = \frac{P(\text{white marble} \mid \text{bag } 1)\, P(\text{bag } 1)}{P(\text{white marble})}$$

Bayes!

$$= \frac{P(\text{white marble} \mid \text{bag } 1)\, P(\text{bag } 1)}{P(\text{white marble} \mid \text{bag } 1)\, P(\text{bag } 1) + P(\text{white marble} \mid \text{bag } 2)\, P(\text{bag } 2)}$$

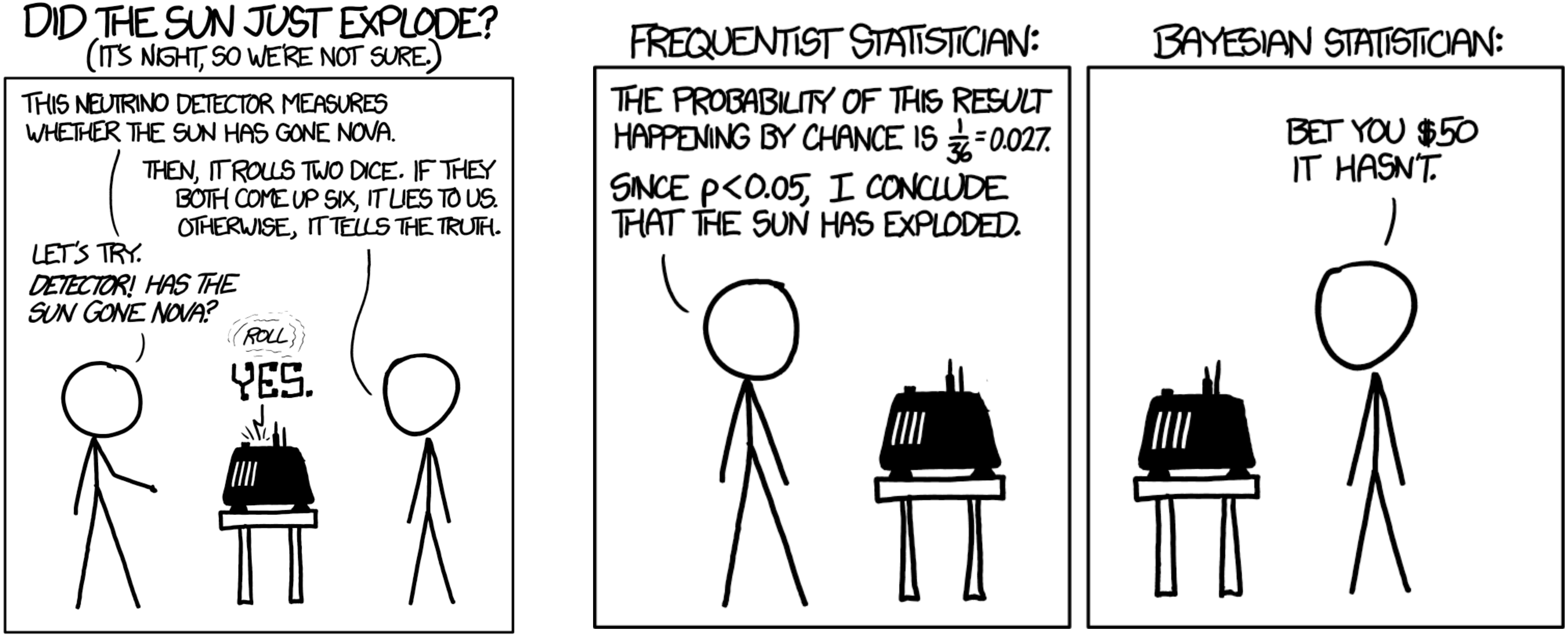
LtP

$$= \frac{\frac{6}{\cancel{10}} \cdot \cancel{\frac{1}{2}}}{\frac{6}{\cancel{10}} \cdot \cancel{\frac{1}{2}} + \frac{3}{\cancel{10}} \cdot \cancel{\frac{1}{2}}} = \frac{6}{6+3} = \frac{6}{9} = \boxed{\frac{2}{3}}$$

Bayes' Rule in the wild ☀️

xkcd
it is funny.

- Bayes' Rule is very helpful because it helps us incorporate our **prior knowledge** about probabilities into our conclusions.



Bayes' Rule in the wild

Likelihood

- Bayes' Rule is very helpful because it helps us incorporate our **prior knowledge** about probabilities into our conclusions.

Prior

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Posterior

evidence

- When we calculated the probability that Professor X was on Bayes Salts, which one of these terms was our prior knowledge of the background rate of Salts use?

1% background rate.

Bayes' Rule in machine learning

- Often, we have a **model with parameters** M and we have **data** D.

- Our goal is to learn the parameters M from the data. Yet we also have some beliefs about the parameters, and no particular beliefs about the data.

$$P(M|D) = \frac{P(D|M)P(M)}{P(D)}$$

Flip coin, $\{9H, 1T\} = D$

M = bias of the coin

I tell you: "I'm pretty sure it's not biased" Prior: $p \sim 0.5$

Random variables

- Say I roll two dice.
- What's the most likely outcome? all same
- What's the most likely sum? 7

Random variables

- Say I roll two dice. What's the sample space? What are the tables of sums, differences, and maxima?

$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Sum

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Random variables

- **The key**: the dice are random, so the sum is random!
- Let's *sidestep* the sample space entirely and just go straight to the thing we care about: the sum.
- We call the sum of the dice a *random variable*.

Discrete random variables

- **Definition**: a discrete random variable is a function that maps the elements of a sample space Ω to a finite number of values $a_1, a_2, \ldots, a_N$ or an infinite number of values $a_1, a_2, \ldots$

- Note: even if there are an infinite number, the values must be discrete.

- **Examples** of discrete random variables:
  - Sum of dice; difference of dice; maximum of dice.
  - Number of flips until we get a heads.

Probability mass functions

- **Definition**: a *probability mass function* is the map between the random variable’s values and the probabilities of those values.

$$f(a) = P(X = a)$$

PMF

- Called a “probability mass” function (PMF) because each of the random variable's values has some probability mass (or weight) attached to it.

- Since the PMF is a probability function, what is the sum of all the masses?

$P(\Omega) = 1$ (def'n of a prob function)

$a \in \Omega$

a \in \Omega

a is in the set Omega.

$$\sum_{a \in \Omega} f(a) = 1$$

Probability mass functions

- **Question**: what is the probability mass function for the number of coin flips until a biased coin comes up heads?

Want: function $f(n)$ that tells me the prob. that I wait n flips to get the first heads.

Cumulative distribution functions

- **Definition**: a *cumulative distribution function* (CDF) is a function whose value at point *a* is the cumulative sum of probability masses up until *a*.

$$F(a) = P(X \leq a)$$

- **Question**: what's the relationship between the PMF and the CDF?

Cumulative distribution functions

- **Example**: What is the probability that I roll two dice and they add up to at most 9?

Cumulative distribution functions

- **Example**: What is the probability that I roll two dice and they add up to between 3 and 6, inclusive?