CSCI 3022

intro to data science with probability & statistics

September 12, 2018

1. HW 1 due this Fri. 9/14/18 ← 5 P.M. Moodle.
2. Mathematical probability

OH today till 5:30

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Last Time

- **Sample Space** Ω: set of all possible outcomes of an experiment.
- **Event**: a *set* of one or more outcomes.
- **Probability Function P**: assigns value in [0,1] to each outcome or event.
  - Two requirements:
    1. Probability of the sample space is 1. $P(\Omega) = 1$
    2. $P(A \cup B) = P(A) + P(B)$ only if A and B are disjoint.
  - If A and B are not disjoint then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - If results of two trials don't affect each other, we say they are *independent*.

Warming up

- Suppose you draw one card from a standard 52-card deck

Question: What is the probability that the card is **A**♦?

$$\frac{1}{52}$$

Question: What is the probability that the card is an **A** *or* a ♦?

$$\frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \boxed{\frac{16}{52}}$$

$$P(\diamondsuit) + P(A) - P(A \cap \diamondsuit)$$

A rigorous way to compute probabilities

to be clear: disjoint

- Suppose we know $P(\omega)$ for each outcome ω in Ω.

- We can compute the probability of an event A (1 or more outcomes) as the sum of the probabilities of the outcomes in A

$A = \{w_1, w_2, w_3\}$ then $P(A) = P(w_1) + P(w_2) + P(w_3)$

Question: suppose we flip a *biased* coin P({H,T}) = {p,1-p} exactly 3 times. What is the probability that we get two or more T?

T T H, w_1 $P(w_1) = (1-p)^2 p$

T H T, w_2 $P(w_2) = (1-p)^2 p$

H T T, w_3 $P(w_3) = (1-p)^2 p$

T T T, w_4 $P(w_4) = (1-p)^3$

$P(A) = 3(1-p)^2 p + (1-p)^3$

To Infinity and Beyond!

- Suppose you flip a biased coin until a H comes up. Prove that the probability that you flip a H eventually is 1.

discrete.

$|\Omega| = \infty$

Question 1: what is the sample space for this experiment?

$\Omega = \{H, TH, TTH, TTTH, TTTTH, \ldots\}$

Question 2: what is the probability that you flip a heads eventually?

$$P(E) = P(H) + P(TH) + P(TTH) + P(TTTH) + \ldots$$

$$= p + (1-p)p + (1-p)^2 p + (1-p)^3 p + \ldots$$

$$= \sum_{i=0}^{\infty} p(1-p)^i = p\sum_{i=0}^{\infty}(1-p)^i = \frac{p}{1-(1-p)} = \frac{p}{1-1+p} = \frac{\cancel{p}}{p} = 1$$

geom series

Conditional probability

Question: Stop a random person on the street and ask them in what month they were born. What is the probability that they were born in a 31-day month?

1 2 3 4 5 6 7

$E = \{\text{Jan}, \text{Mar}, \text{May}, \text{June}, \text{Aug}, \text{Oct}, \text{Dec}\}$ $P(E) = \frac{7}{12}$

Question: What is the probability that they were born in a month with an *r* in the name?

1 2 3 4 5 6 7 8

$R = \{\text{Jan}, \text{Feb}, \text{Mar}, \text{Apr}, \text{Sept}, \text{Oct}, \text{Nov}, \text{Dec}\}$

$P(R) = \frac{8}{12}$

Conditional probability

Question: Suppose that the person tells you that they were born in a 31-day month. *Now* what is the probability that they were born in a month with an *r* in it?

$P(R) = \frac{8}{12}$

$P(E_{31}) = \frac{7}{12}$

we know this event E happened.

E is true.

mid

"given"

assume to be true

Want to know: $P(R \mid E)$

Conditional probability

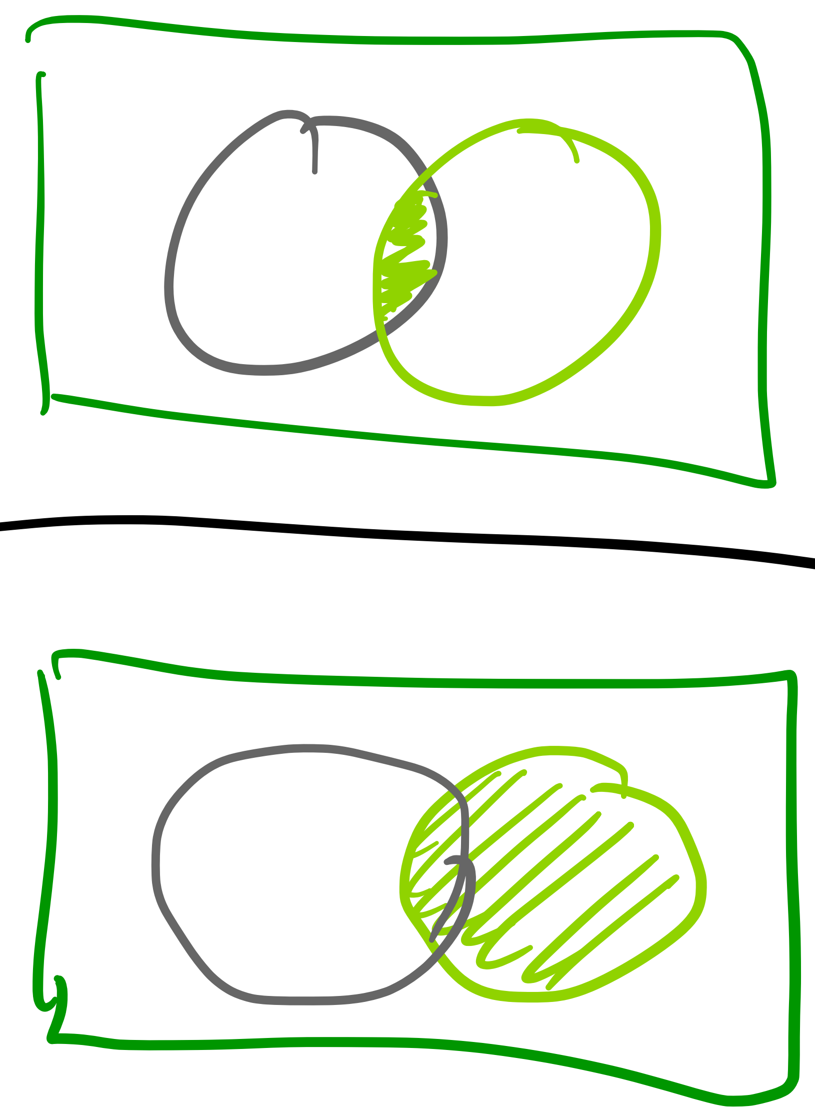
1111 1
1110 2
1101 3
1100 4
1011 5
1010
1001
1000
0111 6
0110 7
0101
0100
0011 8
0010
0001
0000

Definition: The conditional probability of A given C is

$$P(A|C) = \frac{P(A \cap C)}{P(C)}$$

provided that

$\frac{5}{16}$



$\frac{1}{2}$

Example: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it contains *at least* two consecutive 1s given that the first bit is a 1?

8 contain 2 consec. ones

8 start with ones.

5 start w/ one, have two consec 1s.

$P(\text{start w/ } 1 \cap \text{2 in a row}) = \frac{5}{16}$

$P(\text{start w/ } 1) = \frac{1}{2}$

$\frac{5}{16} / \frac{1}{2} = \frac{10}{16} = \frac{5}{8}$

Conditional probability

Example2: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it **does not** contain two consecutive 1s given that the first bit is a 1?

$$P(E \mid C) = 1 - P(\text{not } E \mid C) = 1 - \frac{5}{8} = \frac{3}{8}$$

Prev slide

NB: the conditional probability $P(\,\cdot\,|C)$ is a valid probability function.

$$P(A \mid C) + P(\text{not } A \mid C) = 1$$

i.e. $P(\Omega) = 1$

The product rule of probability

- The definition of conditional probability can be manipulated!

- In this form, we call this the **product rule**:

$$P(A \cap C) = P(A|C)P(C)$$

- The product rule is useful when the conditional probability is easy to compute, but the probability of intersections of events is difficult.

Example: you draw 2 cards from a deck. What is the probability that they are both black?

$$P(1^{st} \text{ card black} \cap 2^{nd} \text{ card black}) = P(2^{nd} \text{ black card} \mid 1^{st} \text{ card black})\, P(1^{st} \text{ card black})$$

$$= \frac{25}{51} \times \frac{1}{2} = \frac{25}{102}$$

Independent events - intuition

Example: you draw 2 cards from a deck. What is the probability that they are both black?

Are the events of drawing the first black card and the second black card **independent?**

NO

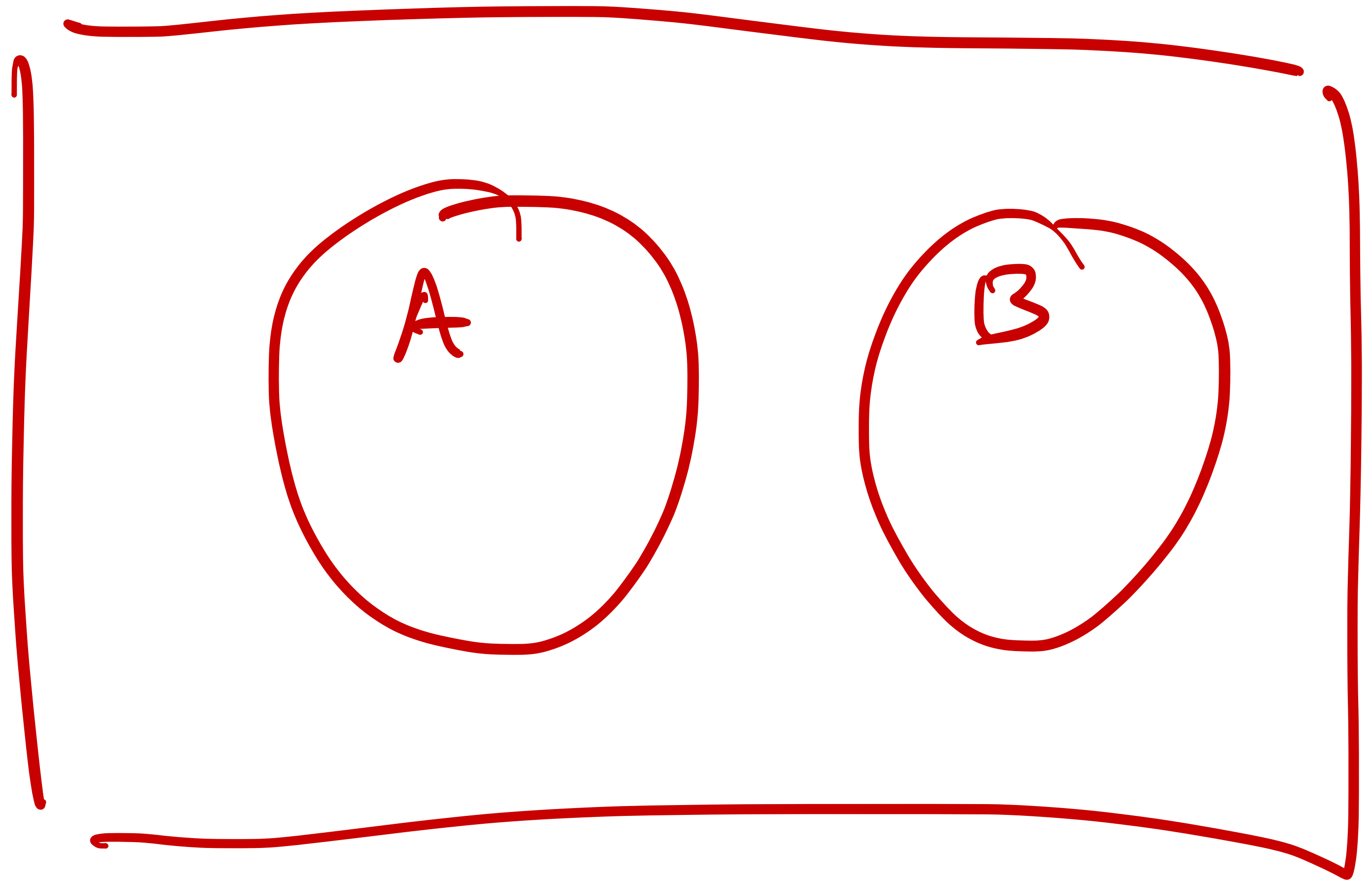
Independent events - math

Jan glasses.

Definition: An event A is said to be independent of event B if $P(A|B) = P(A)$

This definition, combined with the product rule (or the definition of conditional probability) gives us many equivalent definitions (or tests!) for independence:

1. $P(A|B) = P(A)$ $\Longleftrightarrow$ indep.
2. $P(B|A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$



$P(B|A)P(A) = P(A \cap B) = P(A|B)\, P(B)$

Subtleties of Independence

- **Def**: Events $A_1, A_2, \ldots, A_m$ are independent if

$$P(A_1 \cap A_2 \cap \cdots \cap A_m) = P(A_1)P(A_2)\ldots P(A_m)$$

Question: is independence of A, B, and C the same as: A and B are independent; B and C are independent; A and C are independent?

Example: Flip a fair coin twice. Let A be "Heads on flip 1", let B be "Heads on flip 2", let C be "the two flips match.

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

$P(C) = \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{4}$

$P(A) \cdot P(B) \cdot P(C) = \frac{1}{8}$

$\underbrace{P(C|B)}_{\frac{1}{2}} = \underbrace{P(C)}_{\frac{1}{2}}$

Law of Total Probability

6, 4   3, 7

- Suppose I have a bag of marbles. The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles.

- Now suppose I put the two bags in a box. If I close my eyes, grab a bag from the box, and then grab a marble from that bag, what is the probability that it is black?

$$P(\text{marble is black } B) = P(B \mid \text{bag } 1)\, P(\text{bag } 1) + P(B \mid \text{bag } 2)\, P(\text{bag } 2)$$

$$P(B \cap \text{bag } 1) + P(B \cap \text{bag } 2)$$

$$= \frac{4}{10} \cdot \frac{1}{2} + \frac{7}{10} \cdot \frac{1}{2}$$

$$= \frac{11}{20}$$

Law of Total Probability

bag 1   bag 2

$2x$ x

$2x + x = 1 \Rightarrow x = \frac{1}{3}$

- Same as before: The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles. But suppose the first bag is much larger than the second bag—I'm twice as likely to grab the first bag as the second bag. What's the probability that I get a black marble?

$$P(B) = P(B|\text{bag } 1)\,P(\text{bag } 1) + P(B|\text{bag } 2)\,P(\text{bag } 2)$$

$$= \frac{4}{10} \cdot \frac{2}{3} + \frac{7}{10} \cdot \frac{1}{3}$$

$$= \frac{8}{30} + \frac{7}{30} = \frac{1}{2}$$

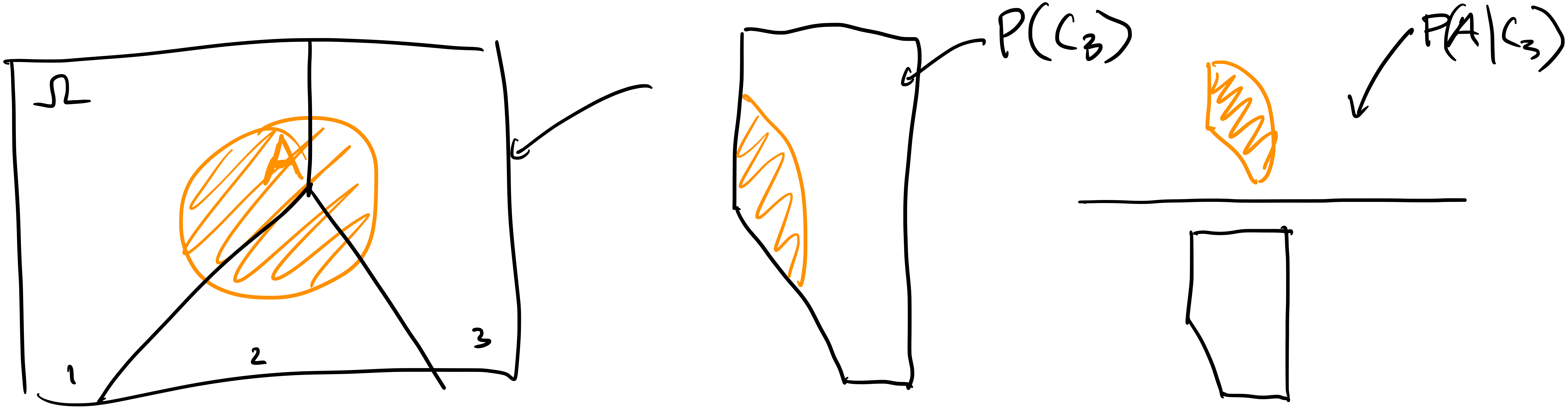
Law of Total Probability

- **Definition**: Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that

$$C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$$

Then the probability of an arbitrary event A can be expressed as:

$$P(A) = P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \cdots + P(A|C_m)P(C_m)$$



Data Party!

- https://tinyurl.com/octopusbirthday

Quizlet on Moodle.

Office Hrs: 4-7 till 5:00! ^_^

The Birthday Paradox

- Say there are two random people in a room. What is the probability that they have different birthdays?

Assume: 1 each day of yr has an equal prob of being a person's bday.

2 Feb 29 D.N.E. :(

$$P(\text{same}) = \frac{1}{365}$$

$$P(\text{diff}) = 1 - \frac{1}{365} = \frac{364}{365}$$

The Birthday Paradox

- Say there are n random people in a room. What is the probability that they all have different birthdays?

$n=1$ $\quad P(\text{unique}) = 1$

$n=2$ $\quad P(\text{unique}) = 1 - \frac{1}{365}$

$n=3$ $\quad P(3^{rd}\text{ bday unique} \mid 2\text{ unique})\, P(2\text{ unique})$

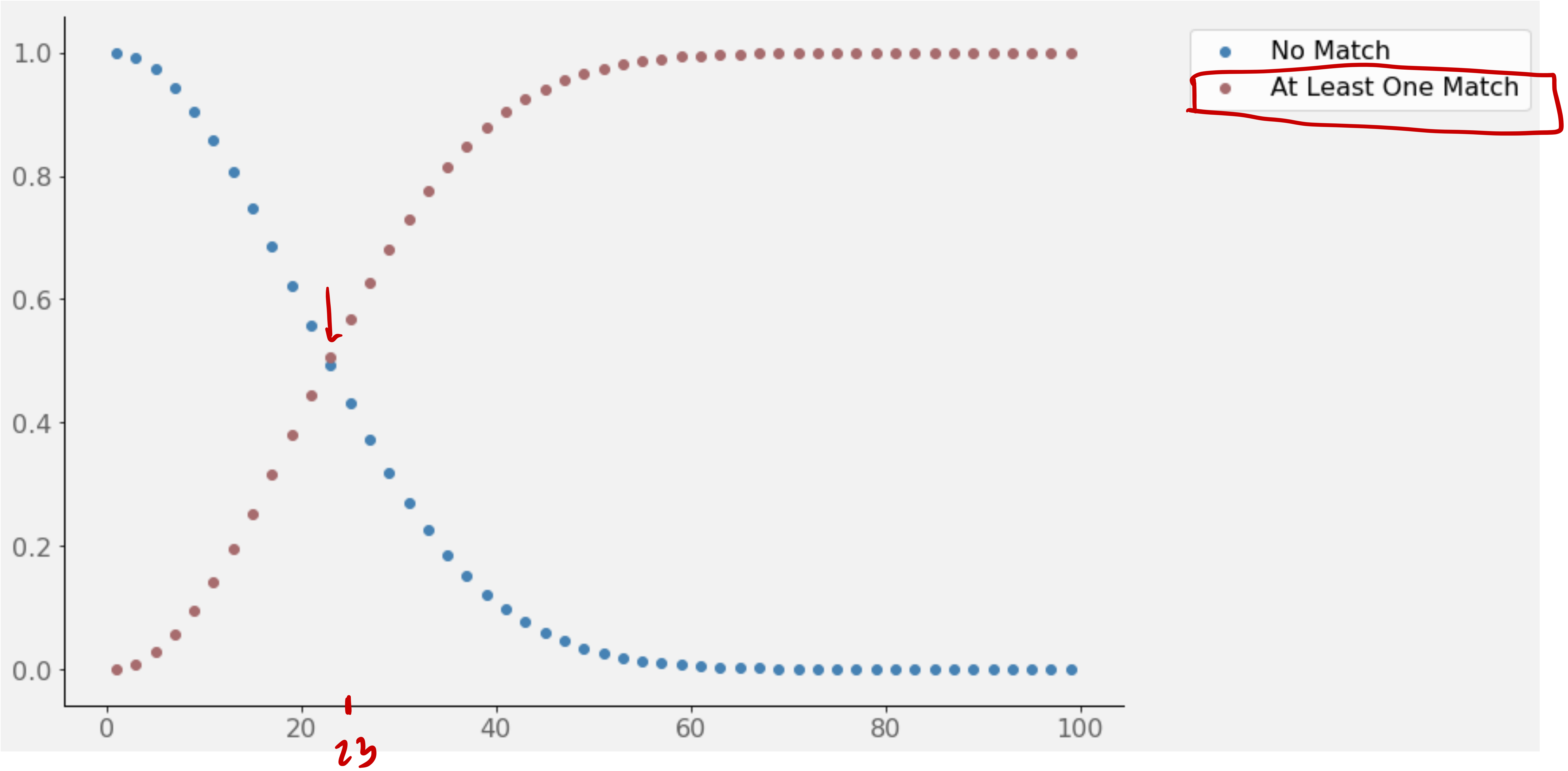
$$= \left(1 - \frac{2}{365}\right)\left(1 - \frac{1}{365}\right)$$

$n=4$ $\quad P(4\text{ unique}) = P(4\text{ unique} \mid 3\text{ unique})\, P(3\text{ unique})$

$$= \left(1 - \frac{3}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{1}{365}\right)$$

The Birthday Paradox

- 23 people in the room: P(shared bday) = 0.5973
- 58 people in the room: P(shared bday) = 0.9917
- 120 people in the room: P(shared bday) = 0.9999999998



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