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Computational Physics
Exam 3

1. Solving an ODE like it's transcontinental railroad

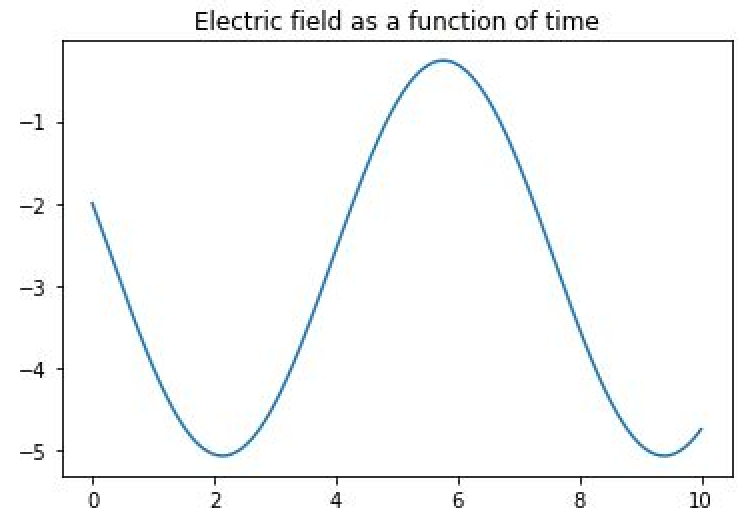
   To solve for the wave function for a particle bound in a finite potential well, we solve the second-order linear differential equation from different directions. With other problems that require using ODE's, only one direction was necessary to integrate. Here it is vitally important to integrate from two different directions because you are really working with two different functions. The function will act differently once it is inside the potential well (V is not zero) and it is important to note the boundary conditions at each side of the well. SInce the particle is confined, it can be normalizeable and we can determine how the function behaves in special cases such as positive and negative infinity. This function also has to satisfy the boundary conditions.

   With the special cases and the boundary conditions, this is an eigenvalue problem, not just a regular ODE problem. Therefore, it has special eigenvalues for certain energies. So in order to solve for this eigenfunction/ wave function, we must combine the ordinary way to solve for the function while simultaneously guessing a wave function that satisfies the boundary conditions. Remembering that the function has to be continuous at the meeting point, both the wave function and the derivatives have to be continuous. Going from each side, you can check how well the two wave functions agree which will provide the change in energy. When that change is smaller than the tolerance, you can conclude the functions are equal.

   The advantage of this approach is that it allows one to solve for the wave function and determine the ground state energy. The approach uses multiple methods to solve for the function. One part is the shooting method, the bi-section method. The good thing about this method is that it will always converge, always find a solution. The variance in the guess is used to update the new guess. But a disadvantage to this method is that the time to convergence is dependent upon the initial guesses for the parameters. If the guesses are far off, it will take a very long time to converge. The closer the initial guess, the quicker it will converge.

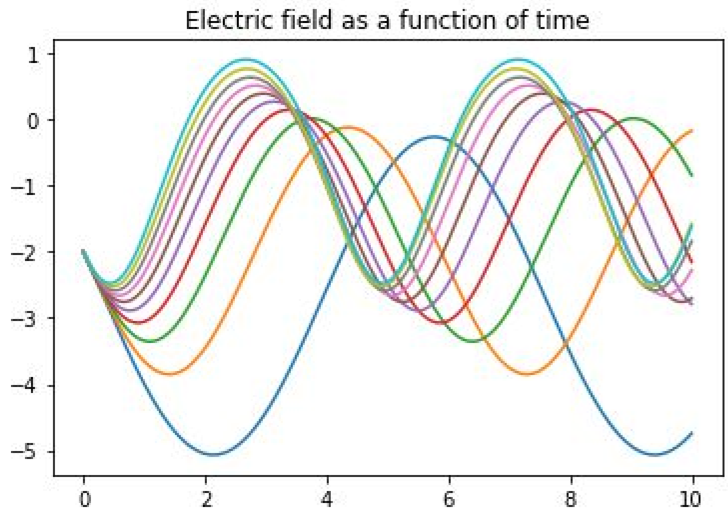
   Another method to solve this problem is to use an RK2 or RK4 method instead of an Eulerian scheme like I used. The precision is better. Even more the book suggests using Numerov method which can speed up the method due to higher precision. This is actually a commonly used method to solve for the wave equation. It is a linear multistep method to the fourth order, increasing the precision.

2. Van der Pol's Equation & the Electric Field of a Laser
   I chose to approach this problem with the Euler Method. This is a more straightforward method. The error may be larger than RK methods, but it should suffice.
   Beginning with the initial constants at:
   g = 2
   g_t= 0
   tau = 2
   omega = 0.5
   h = 0.0001

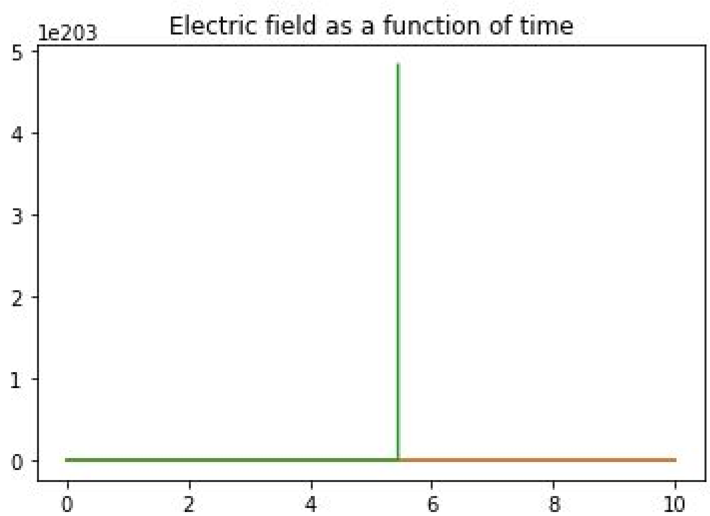


   Here we can see that E(t) oscillates if E(0) does not equal 0 and the other constants above. This is what we would expect for a laser that oscillates in a single mode. If it oscillates more, we would expect more variability, more periodic oscillations. This is better seen in frequency space. There, the different oscillations would show as individual peaks.

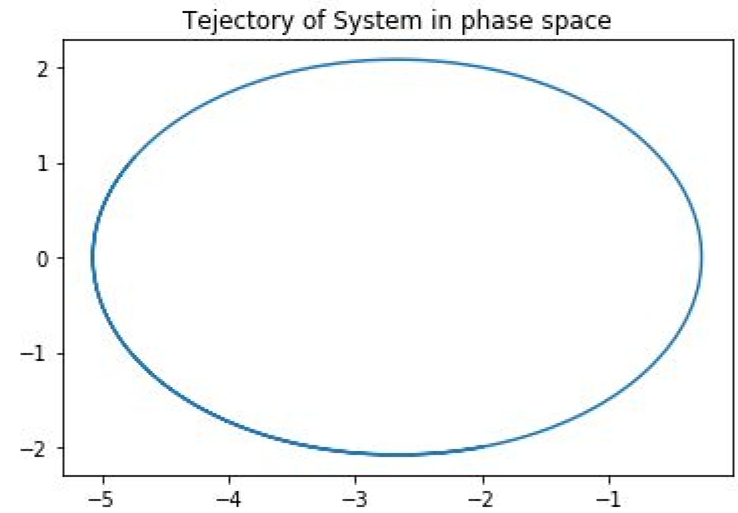
   Now to vary the value of g_t (saturation):
   gt= (0,1, 0.1)

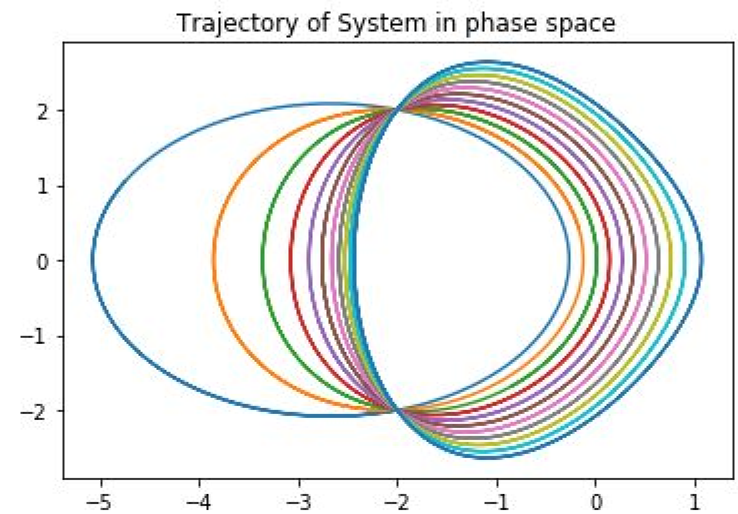


Once g_t increases past 1.1, the values become extremely large and non-variable. THis means that the amplitude of the oscillations will decrease and the value of the electric field will reach its maximum.



Now looking into the phase space, below are the two figures representing g_t = 0 and g_t = (1-1.1) respectively.





Looking into the trajectory of the system in phase space, we can see that initially the system is at its maximum negative value for the electric field. This is where the system is behaving like a harmonic oscillator with an elliptical orbit. As the E increases (becomes less negative) the system behaves more like an anharmonic oscillator and is distorted. It is not quite ellipse shaped or symmetric.

*Below are the derivations for the relevant equations needed to solve for E(t) in the provided problem.

$$\ddot{E} = -\omega_0^2 E - \frac{1}{\tau}\dot{E} + (g - \bar{g}E^2)\dot{E}$$

ω_0 - natural freq.
τ = loses
g = gain
$\bar{g}$ = saturation

Euler Method

$$y^{(0)} = E$$

$$y^{(1)} = \dot{E} = \frac{dE}{h} = \frac{\Delta E}{h}$$

$$y^{(2)} = \ddot{E}$$

$$\frac{\Delta E}{h} = \frac{E(t_{n+1}) - E(t_n)}{h}$$

$$\therefore y^{(1)} = E(t_{n+1}) = h(\Delta E(t_n)) + E(t_n)$$

$$\ddot{E} = \frac{d\dot{E}}{h} = \frac{\Delta\dot{E}}{h} = \frac{\dot{E}(t_{n+1}) - \dot{E}(t_n)}{h}$$

$$\therefore y^{(2)} = \dot{E}(t_{n+1}) = h\left[-\omega_0^2 E(t_n) - \left[\frac{1}{\tau}\dot{E}(t_n) + (g - \bar{g}E^2(t_n))\right]\right] + \dot{E}(t_n)$$

$$y^{(0)} = E$$

$$y^{(1)} = E(t_{n+1}) = h(\dot{E}(t_n)) + E(t_n)$$

$$y^{(2)} = \dot{E}(t_{n+1}) = h\left[-\omega_0^2 E(t_n) - \left[\frac{1}{\tau}\dot{E}(t_n) + (g - \bar{g}E^2(t_n))\right]\right] + \dot{E}(t_n)$$