Assignment 1: (Course: Quantum Information)

Topic: Quantum Gates/ Circuits

1. Express the Hadamard gate H as a product of R_x and R_z rotations and $e^{i\phi}$ for some ϕ.

2. If $\mathbf{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions, a rotation by θ about the $\mathbf{n}$ axis can be defined by the equation

$$R_n\hat{}(\theta). \equiv. \exp(-i\theta\, \mathbf{n}.\boldsymbol{\sigma}/2) = \cos(\theta/2)\, I - I \sin(\theta/2)\, (n_X X + n_Y Y + n_Z Z)$$

where $\boldsymbol{\sigma}$ denotes the three component vector (X, Y, Z) of Pauli matrices.

3. Show that X Y X = −Y and use this to prove that $X\, R_y(\theta)\, X = R_y(-\theta)$.

4. (a) Show that an arbitrary single qubit unitary operator can be written in the form

$$U = \exp(i\alpha)\, R_{\mathbf{n}}\, (\theta)$$

for some real numbers α and θ, and a real three-dimensional unit vector $\mathbf{n}$ **.**

(b) Find values for α, θ, and $\mathbf{n}$ giving the Hadamard gate H

(c) Find values for α, θ, and $\mathbf{n}$ giving the phase gate. S= $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

(5) (a) Suppose U is a unitary operation on a single qubit.
Then there exist real numbers α, β, γ and δ such that

$$U = e^{i\alpha}\, R_Z(\beta)\, R_Y(\gamma).\, R_Z(\delta).$$

(b) Find real numbers α, β, γ and δ using R_x instead of R_z.

(6) Suppose U is a unitary gate on a single qubit. Then there exist unitary operators A, B, C on a single qubit such that $A B C = I$ and $U = e^{i\alpha} A X B X C$, where α is some overall phase factor.

(7) Give A, B, C, and α for the Hadamard gate.

(8) Prove the following three identities: HXH=Z; HYH=−Y; HZH=X.

show that $HTH = R_x(\pi/4)$, up to a global phase. (H, X, Y, Z: standard Gate notations)

(9). What is the 4×4 unitary matrix for the circuit in the computational basis?

x___________H_______

y___________________-

(10) What is the unitary matrix for the circuit

x___________________

y_______H__________

(11) Construct a CNOT gate from one controlled-Z gate, that is, the gate whose action in the computational basis is specified by the unitary matrix U where

$$U =. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and two Hadamard gates, specifying the control and target qubits.

(12). Show that CNOT gate is a simple permutation whose action on a density matrix ϱ is to rearrange the elements in the matrix. Write out this action explicitly in the computational basis.

(13) Prove that a $C^2(U)$ gate (for any single qubit unitary U) can be constructed using at most eight one-qubit gates, and six controlled-NOTs.

(14) Construct a $C^1(U)$ gate for $U = R_x(\theta)$ and $U = R_y(\theta)$, using only and single qubit gates. Can you reduce the number of single qubit gates needed in the construction from three to two?

(15) Let subscripts denote which qubit an operator acts on, and let C be a CNOT with qubit 1 as the control qubit and qubit 2 as the target qubit. Prove the following identities:

$CX_1C = X_1X_2$

$CY_1C = Y_1X_2$

$CZ_1C = Z_1$

$CX_2C = X_2$

$CY_2C = Z_1Y_2$

$CZ_2C = Z_1Z_2$

$R_{z,1}(\theta)C = CR_{z,1}(\theta)$

$R_{x,2}(\theta)C = CR_{x,2}(\theta)$.