Quantum Mechanics

Assignment-1

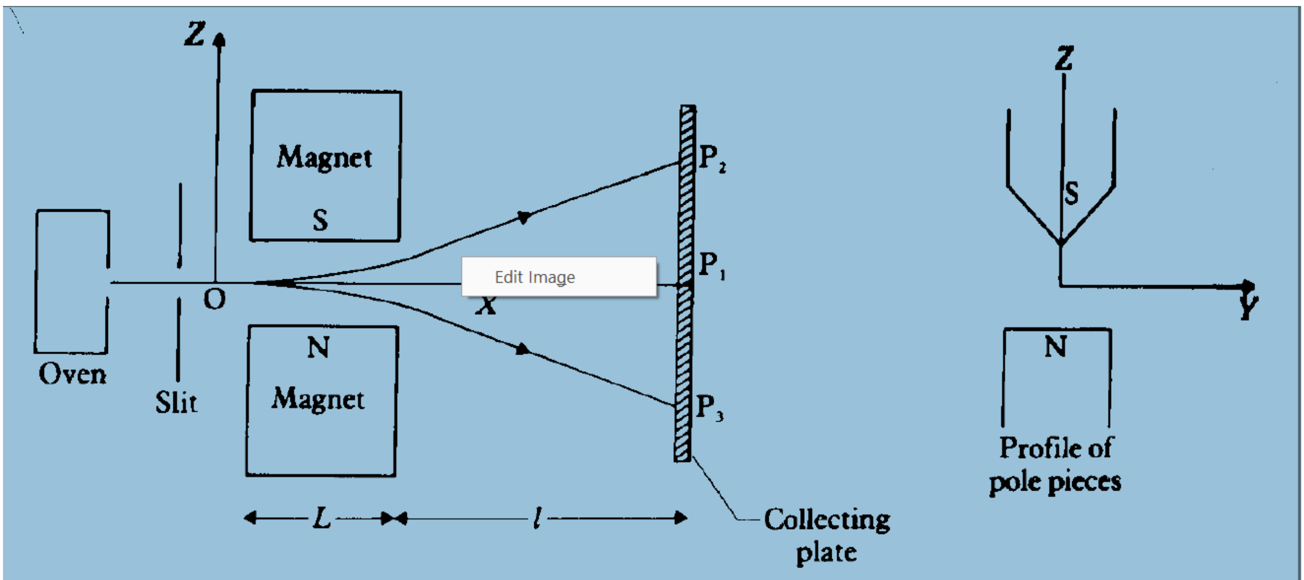
1.

With reference to Fig. 1.22 showing a Stern–Gerlach apparatus, calculate the distance P_2P_3 from the following data:

Field gradient: $\dfrac{\partial \mathcal{B}_z}{\partial z} = 10^3\ T/\text{m}$

Length of pole piece: $L = 0.1$ m
Distance to screen: $l = 1$ m
Atomic beam composed of silver atoms, for which $\mathcal{M}_z = \pm\mu_B$
Temperature of oven: 600 K

Assume that the velocity of the silver atoms is equal to the root mean square velocity of $(3kT/M)^{1/2}$, where k is Boltzmann's constant and M is the mass of a silver atom.



1.22 The Stern–Gerlach apparatus.

2. Find the correct option with justification:

A beam of neutral hydrogen atoms in their ground state is moving into the plane of this page and passes through a region of a strong inhomogeneous magnetic field that is directed upward in the plane of the page. After the beam passes through this field, a detector would find that it has been

A. deflected upward
B. deflected to the right
C. undeviated
D. split vertically into two beams
E. split horizontally into three beams.

3.

Check whether the following sets of functions are linearly independent or dependent on the real x-axis.

(a) $f(x) = 4$, $g(x) = x^2$, $h(x) = e^{2x}$
(b) $f(x) = x$, $g(x) = x^2$, $h(x) = x^3$
(c) $f(x) = x$, $g(x) = 5x$, $h(x) = x^2$
(d) $f(x) = 2 + x^2$, $g(x) = 3 - x + 4x^3$, $h(x) = 2x + 3x^2 - 8x^3$

4.

Are the following sets of vectors (in the three-dimensional Euclidean space) linearly independent or dependent?

(a) $\vec{A} = (3, 0, 0)$, $\vec{B} = (0, -2, 0)$, $\vec{C} = (0, 0, -1)$
(b) $\vec{A} = (6, -9, 0)$, $\vec{B} = (-2, 3, 0)$
(c) $\vec{A} = (2, 3, -1)$, $\vec{B} = (0, 1, 2)$, $\vec{C} = (0, 0, -5)$
(d) $\vec{A} = (1, -2, 3)$, $\vec{B} = (-4, 1, 7)$, $\vec{C} = (0, 10, 11)$, and $\vec{D} = (14, 3, -4)$

5.

Consider two operators $\hat{A}$ and $\hat{B}$ whose matrices are

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

(a) Are $\hat{A}$ and $\hat{B}$ Hermitian?
(b) Do $\hat{A}$ and $\hat{B}$ commute?

(c) Find the eigenvalues and eigenvectors of $\hat{A}$ and $\hat{B}$.
(d) Are the eigenvectors of each operator orthonormal?
(e) Verify that $\hat{U}^\dagger \hat{B} \hat{U}$ is diagonal, $\hat{U}$ being the matrix of the normalized eigenvectors of $\hat{B}$.
(f) Verify that $\hat{U}^{-1} = \hat{U}^\dagger$.

6.

Consider the following two kets:

$$\mid \psi \rangle = \begin{pmatrix} -3i \\ 2+i \\ 4 \end{pmatrix}, \qquad \mid \phi \rangle = \begin{pmatrix} 2 \\ -i \\ 2-3i \end{pmatrix}.$$

(a) Find the bra $\langle \phi \mid$.
(b) Evaluate the scalar product $\langle \phi \mid \psi \rangle$.
(c) Examine why the products $\mid \psi \rangle \mid \phi \rangle$ and $\langle \phi \mid \langle \psi \mid$ do not make sense.

7.

Consider the states $\mid \psi \rangle = 3i \mid \phi_1 \rangle - 7i \mid \phi_2 \rangle$ and $\mid \chi \rangle = - \mid \phi_1 \rangle + 2i \mid \phi_2 \rangle$, where $\mid \phi_1 \rangle$ and $\mid \phi_2 \rangle$ are orthonormal.

(a) Calculate $\mid \psi + \chi \rangle$ and $\langle \psi + \chi \mid$.
(b) Calculate the scalar products $\langle \psi \mid \chi \rangle$ and $\langle \chi \mid \psi \rangle$. Are they equal?
(c) Show that the states $\mid \psi \rangle$ and $\mid \chi \rangle$ satisfy the Schwarz inequality.
(d) Show that the states $\mid \psi \rangle$ and $\mid \chi \rangle$ satisfy the triangle inequality.

8.

Consider two states $|\psi_1\rangle = 2i|\phi_1\rangle + |\phi_2\rangle - a|\phi_3\rangle + 4|\phi_4\rangle$ and $|\psi_2\rangle = 3|\phi_1\rangle - i|\phi_2\rangle + 5|\phi_3\rangle - |\phi_4\rangle$, where $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, and $|\phi_4\rangle$ are orthonormal kets, and where a is a constant. Find the value of a so that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal.

9.

(a) Discuss the hermiticity of the operators $(\hat{A} + \hat{A}^\dagger)$, $i(\hat{A} + \hat{A}^\dagger)$, and $i(\hat{A} - \hat{A}^\dagger)$.
(b) Find the Hermitian adjoint of $f(\hat{A}) = (1 + i\hat{A} + 3\hat{A}^2)(1 - 2i\hat{A} - 9\hat{A}^2)/(5 + 7\hat{A})$.
(c) Show that the expectation value of a Hermitian operator is real and that of an anti-Hermitian operator is imaginary.

10.

Show that the operator $|\psi\rangle\langle\psi|$ is a projection operator only when $|\psi\rangle$ is normalized.

11.

(a) Show that the commutator of two Hermitian operators is anti-Hermitian.
(b) Evaluate the commutator $[\hat{A}, [\hat{B}, \hat{C}]\hat{D}]$.

12.

Find the uncertainty relations between the components of the position and the momentum operators.

13.

What conditions must the parameter ε and the operator $\hat{G}$ satisfy so that the operator $\hat{U} = e^{i\varepsilon\hat{G}}$ is unitary?

14.

Show that if $\hat{A}^{-1}$ exists, the eigenvalues of $\hat{A}^{-1}$ are just the inverses of those of $\hat{A}$.

15.

Consider the states $|\psi\rangle = 9i|\phi_1\rangle + 2|\phi_2\rangle$ and $|\chi\rangle = -\frac{i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$, where the two vectors $|\phi_1\rangle$ and $|\phi_2\rangle$ form a complete and orthonormal basis.

(a) Calculate the operators $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$. Are they equal?
(b) Find the Hermitian conjugates of $|\psi\rangle$, $|\chi\rangle$, $|\psi\rangle\langle\chi|$, and $|\chi\rangle\langle\psi|$.
(c) Calculate $\mathrm{Tr}(|\psi\rangle\langle\chi|)$ and $\mathrm{Tr}(|\chi\rangle\langle\psi|)$. Are they equal?
(d) Calculate $|\psi\rangle\langle\psi|$ and $|\chi\rangle\langle\chi|$ and the traces $\mathrm{Tr}(|\psi\rangle\langle\psi|)$ and $\mathrm{Tr}(|\chi\rangle\langle\chi|)$. Are they projection operators?

16.

(a) Find a complete and orthonormal basis for a space of the trigonometric functions of the form $\psi(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta)$.
(b) Illustrate the results derived in (a) for the case $N = 5$; find the basis vectors.

17.

(a) Show that the sum of two projection operators cannot be a projection operator unless their product is zero.
(b) Show that the product of two projection operators cannot be a projection operator unless they commute.

18.

Consider a state $|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$ which is given in terms of three orthonormal eigenstates $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$ of an operator $\hat{B}$ such that $\hat{B}|\phi_n\rangle = n^2|\phi_n\rangle$. Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$.

19.

(a) Study the hermiticity of these operators: $\hat{X}$, d/dx, and id/dx. What about the complex conjugate of these operators? Are the Hermitian conjugates of the position and momentum operators equal to their complex conjugates?

(b) Use the results of (a) to discuss the hermiticity of the operators $e^{\hat{X}}$, $e^{d/dx}$, and $e^{id/dx}$.

(c) Find the Hermitian conjugate of the operator $\hat{X}d/dx$.

(d) Use the results of (a) to discuss the hermiticity of the components of the angular momentum operator : $= -i\hbar\left(\hat{Y}\partial/\partial z - \hat{Z}\partial/\partial y\right)$, $\hat{L}_y = -i\hbar\left(\hat{Z}\partial/\partial x - \hat{X}\partial/\partial z\right)$, $\hat{L}_z = -i\hbar\left(\hat{X}\partial/\partial y - \hat{Y}\partial/\partial x\right)$.

20.

(a) Show that the operator $\hat{A} = i(\hat{X}^2 + 1)d/dx + i\hat{X}$ is Hermitian.

(b) Find the state $\psi(x)$ for which $\hat{A}\psi(x) = 0$ and normalize it.

(c) Calculate the probability of finding the particle (represented by $\psi(x)$) in the region: $-1 \leq x \leq 1$.

21.

Discuss the conditions for these operators to be unitary: (a) $(1 + i\hat{A})/(1 - i\hat{A})$,

(b) $(\hat{A} + i\hat{B})/\sqrt{\hat{A}^2 + \hat{B}^2}$.

22.

(a) Using the commutator $[\hat{X}, \hat{p}] = i\hbar$, show that $[\hat{X}^m, \hat{P}] = im\hbar\hat{X}^{m-1}$, with $m > 1$. Can you think of a direct way to get to the same result?

(b) Use the result of (a) to show the general relation $[F(\hat{X}), \hat{P}] = i\hbar dF(\hat{X})/d\hat{X}$, where $F(\hat{X})$ is a differentiable operator function of $\hat{X}$.

23.

Consider the matrices $A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2i & 0 \\ i & 0 & -5i \end{pmatrix}$.

(a) Are A and B Hermitian? Calculate AB and BA and verify that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$; then calculate $[A,\ B]$ and verify that $\mathrm{Tr}([A,\ B]) = 0$.

(b) Find the eigenvalues and the normalized eigenvectors of A. Verify that the sum of the eigenvalues of A is equal to the value of $\mathrm{Tr}(A)$ calculated in (a) and that the three eigenvectors form a basis.

(c) Verify that $U^\dagger AU$ is diagonal and that $U^{-1} = U^\dagger$, where U is the matrix formed by the normalized eigenvectors of A.

(d) Calculate the inverse of $A' = U^\dagger AU$ and verify that A'^{-1} is a diagonal matrix whose eigenvalues are the inverse of those of A'.

24.

Consider a particle whose Hamiltonian matrix is $H = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) Is $|\,\lambda\rangle = \begin{pmatrix} i \\ 7i \\ -2 \end{pmatrix}$ an eigenstate of H? Is H Hermitian?

(b) Find the energy eigenvalues, a_1, a_2, and a_3, and the normalized energy eigenvectors, $|\,a_1\rangle$, $|\,a_2\rangle$, and $|\,a_3\rangle$, of H.

(c) Find the matrix corresponding to the operator obtained from the ket-bra product of the first eigenvector $P = |\,a_1\rangle\langle a_1\,|$. Is P a projection operator? Calculate the commutator $[P,\ H]$ firstly by using commutator algebra and then by using matrix products.

25.

Consider the matrices $A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix}$.

(a) Check if A and B are Hermitian and find the eigenvalues and eigenvectors of A. Any degeneracies?

(b) Verify that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, $\det(AB) = \det(A)\det(B)$, and $\det(B^\dagger) = (\det(B))^*$.

(c) Calculate the commutator $[A, B]$ and the anticommutator $\{A, B\}$.

(d) Calculate the inverses A^{-1}, B^{-1}, and $(AB)^{-1}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

(e) Calculate A^2 and infer the expressions of A^{2n} and A^{2n+1}. Use these results to calculate the matrix of e^{xA}.

26.

Consider two matrices: $A = \begin{pmatrix} 0 & i & 2 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix}$. Calculate $A^{-1}\ B$ and $B\ A^{-1}$. Are they equal?

27.

Consider the matrices $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Find the eigenvalues and normalized eigenvectors of A and B. Denote the eigenvectors of A by $\mid a_1\rangle, \mid a_2\rangle, \mid a_3\rangle$ and those of B by $\mid b_1\rangle, \mid b_2\rangle, \mid b_3\rangle$. Are there any degenerate eigenvalues?

(b) Show that each of the sets $\mid a_1\rangle, \mid a_2\rangle, \mid a_3\rangle$ and $\mid b_1\rangle, \mid b_2\rangle, \mid b_3\rangle$ forms an orthonormal and complete basis, i.e., show that $\langle a_j \mid a_k\rangle = \delta_{jk}$ and $\sum_{j=1}^{3} \mid a_j\rangle\langle a_j \mid = I$, where I is the 3×3 unit matrix; then show that the same holds for $\mid b_1\rangle, \mid b_2\rangle, \mid b_3\rangle$.

(c) Find the matrix U of the transformation from the basis $\{\mid a\rangle\}$ to $\{\mid b\rangle\}$. Show that $U^{-1} = U^{\dagger}$. Verify that $U^{\dagger}U = I$. Calculate how the matrix A transforms under U, i.e., calculate $A' = UAU^{\dagger}$.

28.

Consider a state which is given in terms of three orthonormal vectors $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ as follows:

$$|\psi\rangle = \frac{1}{\sqrt{15}}|\phi_1\rangle + \frac{1}{\sqrt{3}}|\phi_2\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle,$$

where $|\phi_n\rangle$ are eigenstates to an operator $\hat{B}$ such that: $\hat{B}|\phi_n\rangle = (3n^2 - 1)|\phi_n\rangle$ with $n = 1, 2, 3$.

(a) Find the norm of the state $|\psi\rangle$.

(b) Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$.

(c) Find the expectation value of $\hat{B}^2$ for the state $|\psi\rangle$.

29.

Consider a two-dimensional space where a Hermitian operator $\hat{A}$ is defined by $\hat{A} \mid \phi_1\rangle = \mid \phi_1\rangle$ and $\hat{A} \mid \phi_2\rangle = - \mid \phi_2\rangle$; $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$ are orthonormal.

(a) Do the states $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$ form a basis?

(b) Consider the operator $\hat{B} = \mid \phi_1\rangle\langle\phi_2 \mid$. Is $\hat{B}$ Hermitian? Show that $\hat{B}^2 = 0$.

(c) Show that the products $\hat{B}\hat{B}^{\dagger}$ and $\hat{B}^{\dagger}\hat{B}$ are projection operators.

(d) Show that the operator $\hat{B}\hat{B}^{\dagger} - \hat{B}^{\dagger}\hat{B}$ is unitary.

(e) Consider $\hat{C} = \hat{B}\hat{B}^{\dagger} + \hat{B}^{\dagger}\hat{B}$. Show that $\hat{C} \mid \phi_1\rangle = \mid \phi_1\rangle$ and $\hat{C} \mid \phi_2\rangle = \mid \phi_2\rangle$.

30.

In a three-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{|\,1\rangle,\ |\,2\rangle,\ |\,3\rangle\}$, is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Is A Hermitian? Calculate its eigenvalues and the corresponding normalized eigenvectors. Verify that the eigenvectors corresponding to the two nondegenerate eigenvalues are orthonormal.

(b) Calculate the matrices representing the projection operators for the two nondegenerate eigenvectors found in part (a).