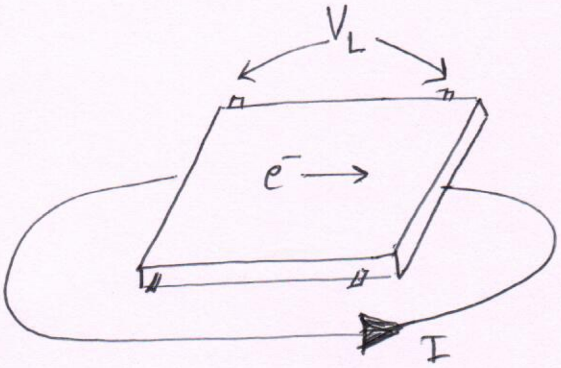
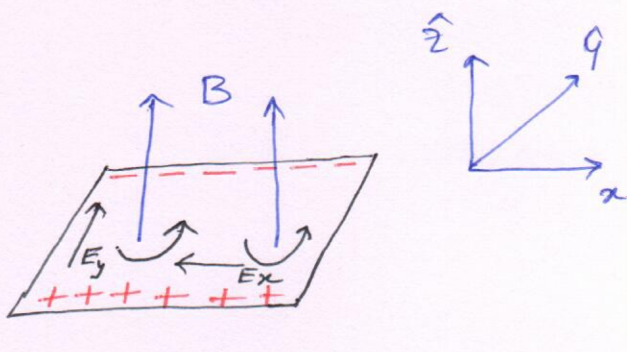
Hall Effect

→ Consider a thin conducting metal slab in which a current I is directed along the $\hat{x}$-direction.





→ The current produces a voltage drop V_L and an electric field E_x and current density j_x along the x-axis.

→ Let us assume that the current density and the electric field are linearly related (Ohm's law).

$$j_x = \sigma_0 E_x ,$$

where
$$\sigma_0 = \frac{n_e e^2 \tau}{m} .$$

→ Apply a magnetic field that is oriented upward along the $\hat{z}$-axis.

- The electrons will be deflected in the ~~negative~~ positive y-direction by the Lorentz force.

- If the sample is infinite in the $\hat{y}$-direction, the current will no longer be directed solely along the $\hat{x}$-axis. The electrons will move at some angle relative to the x-axis. This angle defines the Hall angle.

- However, if the sample is constrained in the y-direction, electrons deflected by the magnetic field will ultimately run into the edges of the sample.

- Electrons accumulating at the edges will produce an electric field E_y which will point in the positive y-direction.

- When the electric field E_y becomes sufficiently large it will cancel the Lorentz forces and the further accumulation of electrons at the edges will be stopped.

- Cancellation of the Lorentz force in equilibrium signifies that electrons will transport purely along the x-axis.

- The field E_y is referred to as the Hall field. In a uniform system at equilibrium, the Hall field is perpendicular to the current.

→ Let us now examine the Drude model for this system. The equation of motion is

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - e\,\vec{v}\times\vec{B} - \frac{m\vec{v}}{\tau},$$

where τ denotes the average time between collisions, the scattering time.

→ In equilibrium, $d\vec{v}/dt = 0$,

$$\vec{v} + \frac{e\tau}{m}\,\vec{v}\times\vec{B} = -\frac{e\tau}{m}\vec{E}.$$

Define the current density as $\vec{J} = -ne\vec{v}$, where n_e is the density of electrons.

→ The above equation can be written as (in matrix notation)

$$\boxed{\begin{bmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{bmatrix}\vec{J} = \frac{e^2 n_e \tau}{m}\vec{E}.}$$

$$\omega_B \equiv \frac{eB}{m}$$

→ The matrix can be inverted to write $\vec{J} = \sigma\vec{E}$.

→ In the presence of magnetic field, σ is not a single number, it is a matrix. It is often referred to as the conductivity tensor.

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix}.$$

→ Drude model gives.

$$\sigma = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}$$

with $\sigma_{DC} = \frac{n_e e^2 \tau}{m}$, the DC conductivity in the absence of a magnetic field.

→ Observation: The off-diagonal terms in the matrix are responsible for the Hall effect.
In equilibrium, a current in the x-direction requires an electric field with a component in the y-direction.

→ Resistivity

$$\rho = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}.$$

$$= \frac{1}{\sigma_{DC}}\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}.$$

Observations:

(i) The off-diagonal terms $\rho_{xy} = \frac{\omega_B\tau}{\sigma_{DC}}$ are independent of the scattering time τ.

(ii) Let the width of the sample be $L_y = W$.
We drop a voltage V_H in the y-direction and measure the resulting current I_x in the x-direction.
The transverse resistance is

$$R_{xy} = \frac{V_H}{I_x} = \frac{WE_y}{WI_x} = \frac{E_y}{J_x} = -\rho_{xy}.$$

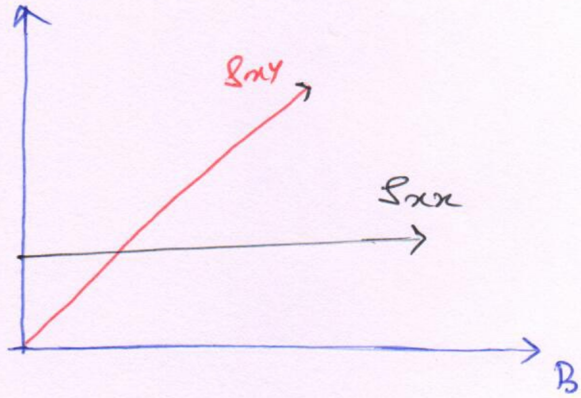
→ For a current I_x flowing in the x-direction, and the associated electric field E_y in the y-direction, the Hall coefficient is defined by

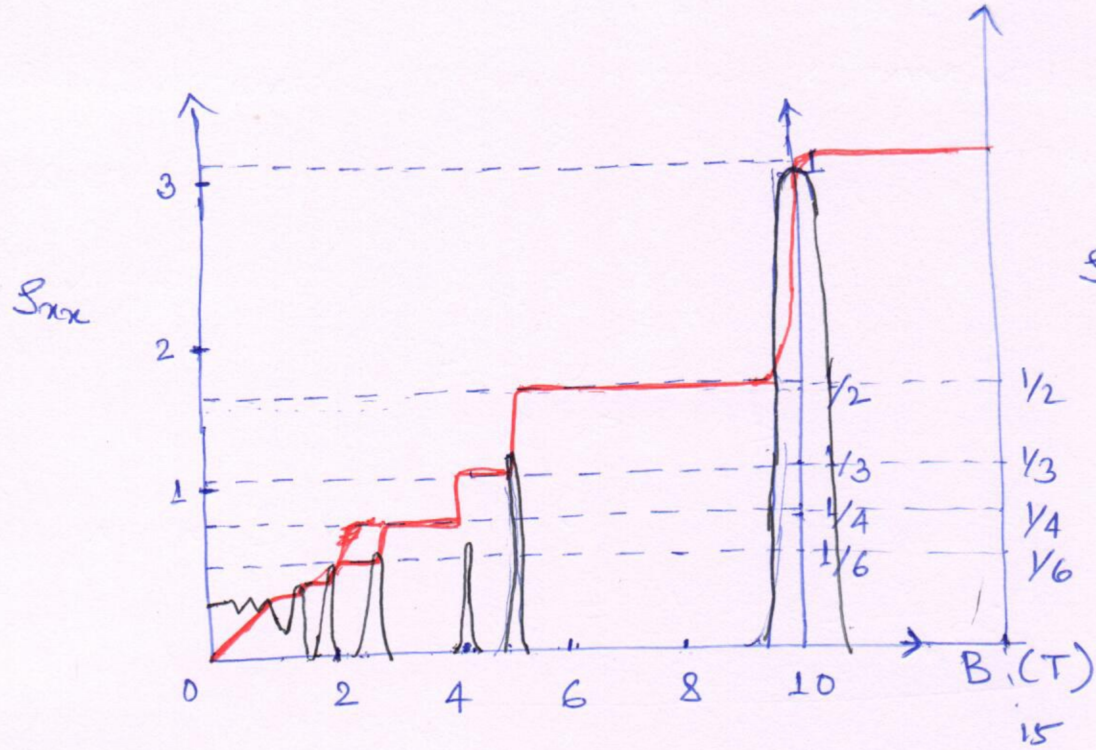
$$R_H = -\frac{E_y}{J_x B} = \frac{\rho_{xy}}{B}.$$

In the Drude model

$$\boxed{R_H = \frac{\omega_B \tau}{B\sigma_{DC}} = \frac{1}{n_e e}.}$$

$$\begin{cases} \rho_{xx} = \dfrac{m}{n_e e^2 \tau} \\ \\ \rho_{xy} = \dfrac{B}{n_e e} \end{cases}$$





In 1980, von Klitzing and collaborators noticed striking deviations from the resistances/resistivity derived based on the Drude's model (see the previous calculation).

Experiments were conducted on a 2d electron gas confined at the interface between SiO_2 and S_i in an S_i MOSFET.

It was found that the Hall voltage, as the magnetic field increased, exhibited distinct flat or plateau regions that were highly reproducible from sample to sample.

From the values of the Hall voltage at the plateau regions, the Hall conductance (the inverse of the resistance) was deduced. It was observed that it must be quantized in units of e^2/h.

$$\boxed{\sigma_H = -\frac{ne^2}{h}}, \text{ where } n \text{ is an integer.}$$

→ Moreover, the Longitudinal resistance vanished exactly at the Hall plateaus, indicating the onset of dissipationless transport.

→ The presence of both quantization and perfect conduction indicates that something fundamental is at the heart of these experiments.

→ At much higher magnetic fields and lower temperatures, it was found that the plateau regions in the Hall voltage were more plentiful than had been thought possible.

→ The plateau regions can occur when the conductance is a fractional multiple of e^2/h, indicating the presence of fractionally charged excitations.

The Hall conductivity (resistivity) is quantized according to

$$\sigma_{xy} = -n\frac{e^2}{h} = -n\frac{e}{\Phi_0} \quad \Big| \quad \left(\rho_{xy} = -\frac{h}{e^2}\frac{1}{n} = -\frac{\Phi_0}{en}\right)$$

Observation: A quick look at the plot shows that ~~the~~ increasing the magnetic field strength still leads to higher Hall resistivity, but it can be seen that the increase in ρ_{xy} are not smoothly linear and occur via discrete jumps.

→ This quantization (IQHE) can be explained by analyzing the interaction of the magnetic field with electrons, while ignoring the electron-electron interactions.

⇒ If the electron-electron interactions are ignored, then we can use the single electron wave function to solve the system.

The Hamiltonian for an electron in a magnetic field is given by

$$\boxed{H = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 + U(\vec{r}),}$$

where $U(\vec{r})$ models the confinement of electrons near the edges of the sample.

→ In what follows, assume that we are far from the edges of the sample/material.

→ Let us choose a gauge. We choose it to be a Landau gauge

$$\vec{A} = Bx\,\hat{y}.$$

→ The Hamiltonian becomes

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}\left(p_y + eBx\right)^2.$$

→ Check that $[p_x, H] \neq 0$, while $[p_y, H] = 0$.

This suggests that we choose the wave function as

$$|\psi_{k_y}\rangle = e^{ik_y y} f(x).$$

→ $H|\psi_{k_y}\rangle = E|\psi_{k_y}\rangle$ depends only on k_y and x.

→ We can write the above eqn. as a Schrödinger equation for a one dimensional quantum harmonic oscillator.

$$\boxed{\frac{1}{2m} p_x^2 f(x) + \frac{e^2B^2}{2m}\left(x + \frac{\hbar}{eB} k_y\right)^2 f(x) = E f(x)} \, .$$

Define $\omega_B \equiv \frac{eB}{m}$. So we have a QHO with the frequency ω_B and centered at $x_0 = -\frac{\hbar}{eB} k_y$.

→ Therefore, we can write $\boxed{f(x) = \phi_n(x - x_0),}$ where

ϕ_n is the nth excited wavefunction of the harmonic oscillator with frequency ω_B.

→ Wave functions have the form

$$\boxed{|\psi_{k_y,n}\rangle = e^{ik_y y}\, \phi_n(x - x_0)}$$

and the energy spectrum

$$\boxed{E_n = \hbar\omega_B \left(n + \tfrac{1}{2}\right).}$$

→ This energy spectrum and the associated wavefunctions are known as Landau Levels.

→ Observation: Each k_y value is degenerate and this degeneracy is of particular importance in understanding the integer quantum Hall effect (IQHE).

Degeneracy

→ The degeneracy of a Landau level is determined by how many allowed values of k_y are there.

We can count these by writing

$$g = \sum_{k_{min}}^{k_{max}} 1 .$$

→ To compute this, we must first examine the boundary conditions of the problem that influence k_y. This depends on $U(\vec{r})$.

$U(\vec{r})$ must satisfy two conditions:

(i) Electrons must be confined, so at the edges of the sample $U(\vec{r})$ must increase rapidly to infinity.

(ii) In the low temperature limit, ignore the e-e interactions. In order to be able to do so, $U(\vec{r})$ must vary slowly in the ~~one~~ interior, else the interactions may become significant to ignore.

→ Width of the sample $L_x = W$ and the length $L_x = L$.

For convenience choose the origin on the right (with $x=0$) of the sample.

→ $|\psi_{k_y,n}\rangle \sim e^{ik_y y} H_n(x + k_y l_B^2) \exp\left(-\frac{(x+k_y l_B)^2}{2 l_B^2}\right)$,

where $l_B = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length of the system.

The presence of a Gaussian highly localizes the wavefunction around $x_0 = -k_y l_B^2$, as it falls off exponentially away from it.

→ In y-direction we have plane waves, so this is equivalent to the particle in a box system in this direction. k_y is quantized in units of $\Delta k_y = \frac{2\pi}{L_y} = \frac{2\pi}{W}$.

Also, $0 \leq x_0 \leq |L_x|$.

The constraint on k_y becomes

$$\boxed{-\frac{W}{l_B^2} \leq k_y \leq 0}.$$

→ The number of states per energy level is

$$g = \sum_{k_{min}}^{k_{max}} 1 = \frac{1}{\Delta k_y} \int_{-L_x/\ell_B^2}^{0} dk_y = \frac{eB(L_x L_y)}{h}$$

$$= \frac{eB}{h} A = \frac{\Phi}{\Phi_0},$$

where $\Phi_0 = \frac{h}{e}$ is the quantum of flux and $\Phi = AB$ is the total flux through the sample.

→ The number of states per Landau Level is given by the number of flux quanta that go through the sample.

→ The Landau level filling factor ν is defined as

$$\nu = \frac{n_e}{n_\Phi} = \frac{n_e 2\pi\hbar}{eB},$$ where n_e is the electron density.

→ The sample is finite and has edges.

The eigenstates are localized in one direction, so there should be states that live near the edges of the sample.

It turns out that they have interesting properties that help to understand QHE. These states are called edge states.

→ To explain the resistivity of the IQHE, we must examine the current density.

$$j = I/A$$

$$= -\frac{e}{L_x L_y} \sum_{n=1}^{\nu} \sum_{k_y} \left\langle \psi_{k_y,n} \middle| \frac{d\vec{x}}{dt} \middle| \psi_{k_y,n} \right\rangle,$$

under the assumption that first ν Landau Levels are occupied.

→ $$\vec{J} = -\frac{e}{m L_x L_y} \sum_{n=1}^{\nu} \sum_{k_y} \left\langle \psi_{k_y n} \middle| \vec{p} + e\vec{A} \middle| \psi_{k_y,n} \right\rangle.$$

→ The transverse current density (or the y component).

$$J_y = -\frac{e}{m L_x L_y} \sum_{n=1}^{\nu} \sum_{k_y} \left\langle \psi_{k_y n} \middle| \hbar k_y + eBx \middle| \psi_{k_y n} \right\rangle$$

We know that the expectation value of x for $|\psi_{k_y n}\rangle$ must be x_0.

$$eB \left\langle \psi_{k_y,n} \middle| x \middle| \psi_{k_y n} \right\rangle = eBx_0 = eB\left(-\frac{\hbar k_y}{eB} - \frac{mE_x}{eB^2}\right).$$

$$= -\hbar k_y - \frac{mE_x}{B}.$$

The last term comes into if $-eEx$ term was included in the Hamiltonian.

→ Substitute this in J_y.

$$J_y = e \sum_{n=1}^{\nu}{}' \sum_{k_y}{}' \frac{E_x}{B}.$$

There is no dependence left in our sum over n or k_y.

So the sum is

$$J_y = \frac{e}{L_x L_y} \nu g \frac{E}{B} = e E_x \nu \frac{1}{\Phi_0}.$$

→ $J_x = ?$

$$J_x = -\frac{e}{m L_x L_y} \sum_{n=1}^{\nu}{}' \sum_{k_y}{}' \langle \psi_{k_y n} | p_x | \psi_{k_y n} \rangle = 0. \quad \text{Why?}$$

→ Written in the matrix form

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \rho_{xx} & -\rho_{xy} \\ \rho_{xy} & \rho_{yy} \end{bmatrix} \begin{bmatrix} J_x \\ J_y \end{bmatrix}.$$

$$\Rightarrow \boxed{J_y = -\frac{\Phi_0}{e\nu}.}$$ where ν is an integer.