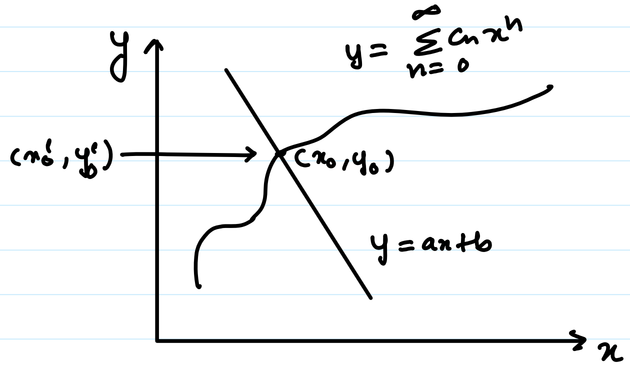
Lecture 01: Basics of computation+Error

Tuesday, August 16, 2022 8:07 AM

Numerical error:-

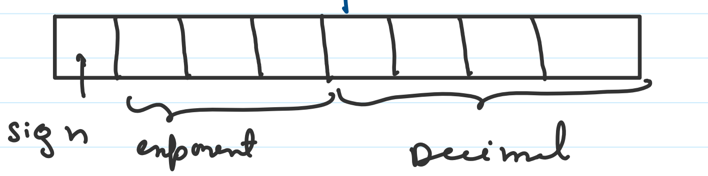
$$e_x = \frac{|x_0' - x_0|}{x_0}$$

$$e_y = \frac{|y_0' - y_0|}{y_0}$$



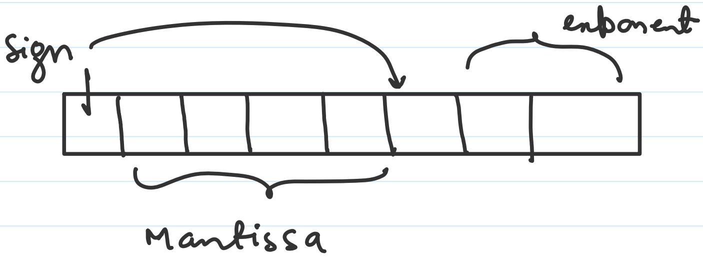
Truncation or Round off error :- for non-integer numbers.

→ 8-bit system :-
Hypothetical computer on base of 10 :-



$999.9999 \rightarrow \sim 10^3$

$000.0001 \rightarrow 10^{-4}$



	Mantissa	Exponent		
$80.9 =$	0.809	$\times 10^2$	$\rightarrow$	$+\ 8090\ +\ 02$
$0.33 =$	0.33	$\times 10^0$	$\rightarrow$	$+\ 3300\ +\ 00$
$-0.000019 =$	-0.19	$\times 10^{-4}$	$\rightarrow$	$-\ 1900\ -\ 04$

Highest no. we can stored = 9999 E+99

Lowest " " " " = 1000 E−99

→ Mathematical Rules:-

1) Addition :- exponents has to be made equal

0.3780 E 16 → keep the larger one unchanged

0.1953 E 14 → 0.3780 E 16

$\underbrace{53}$ least digit → 0.0019 E 16

0.1953 E 14 → 0.3780 E 16

lost digit which is truncated

0.0019 E 16

0.3799 E 16

0.4546 E03 → 0.0000 E07

0.5433 E07 → 0.5433 E07

0.5433 E07

$0.4546 \times 10^3 = 454.6$

$0.5433 \times 10^7 = 5433000.0$

+

5433454.6 → 0.5433 E07

0.6389 E99 →

0.4117 E99

1.0506 E99 →

0.1050 E100 → out of range a.k.a $\boxed{\text{overflow error.}}$

0.1512 E−12 → 0.1512 E−12

− 0.3573 E−13 − 0.0357 E−12

0.1155 E−12

0.5452 E31

− 0.5424 E31

0.0028 E31 → 0.2800 E29

0.5452 E−99

− 0.5424 E−99

0.0028 E−99 → 0.2800 E−101 → $\boxed{\text{underflow error}}$

$0.0028 \; E-99 \longrightarrow 0.2800 \; E-101 \rightarrow$ $\boxed{\text{underflow error}}$

2) <u>Multiplication</u>:- Multiply mantissa, add exponents

$$\begin{array}{ll} 0.7500 & E-32 \\ 0.9248 & E-33 \\ \hline 0.6936 & E-65 \end{array}$$

$$\begin{array}{ll} 0.1111 & E10 \\ 0.1231 & E15 \\ \hline 0.01367 & E25 \end{array} \longrightarrow 0.1367 \; E24$$

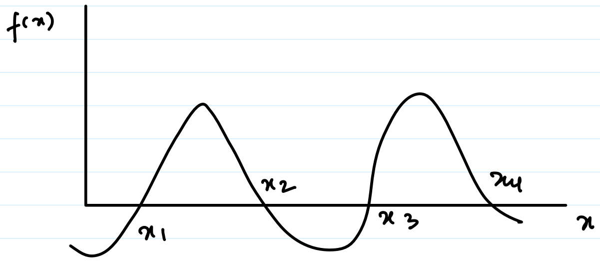
3) Division :- Divide the mantissa, subtract the exponent of Denominator

$$\begin{array}{lll} & 0.1111 & E10 \\ \div & 0.1231 & E03 \\ \hline & 0.9025 & E07 \end{array}$$

→ Log table :-

		Characteristics	Mantissa	$\log_{10}(x)$
$27.262 \rightarrow$	$2.7262\times10^{1} \rightarrow$	1	$2726 \Rightarrow 4346+9 = 4355$	1.4355
$553.0 \rightarrow$	$5.53\times10^{2} \rightarrow$	2	$5530 \Rightarrow 7427$	2.7427
$0.0058939 \rightarrow$	$5.8939\times10^{-3} \rightarrow$	-3	$5894 \Rightarrow 7701+3 = 7704$	$-3+0.7704 = -2.2296$

→ Root finding :-



$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$$

→ User defined error limit

$d^i < e$, then stop $\quad d^i$ = width of the i^{th} iteration

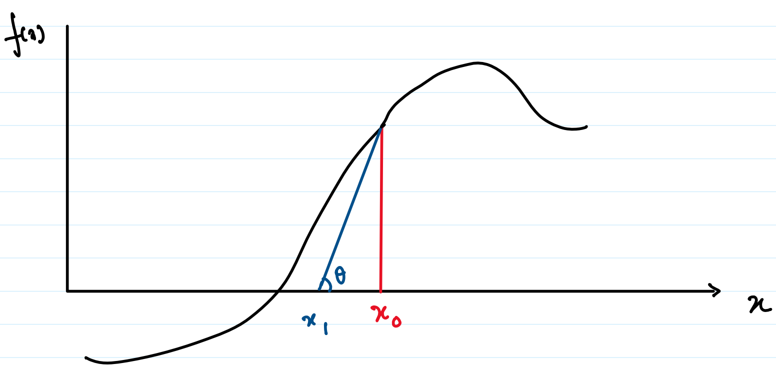
interval d^i after i^{th} iteration.

Q. Try the bisection method for different values of ϵ -

$\epsilon \equiv e =$ 0.1 $\quad$ 0.01 $\quad$ 0.001 $\quad$ 0.0001 $\quad$ - - - -

no. of iteration, $N =$ N_1 $\quad$ N_2 $\quad$ N_3 $\quad$ N_4 $\quad$ - - -

→ Newton - Raphson method :-



$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \quad \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

→ Discussion on Newton - Raphson :-

1) Issue with convergence

2) Error propagation :-

let us assume that the root is found in $(i+1)$ iteration

$$f(x_{i+1}) = 0 = f(x_i + h)$$

$$\therefore \quad x_{i+1} = x_i + h \qquad (h \rightarrow \text{increment in } x)$$

Now, $$f(x_i+h) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \frac{h^3}{3!} f'''(x_i) + \ldots$$

$$0 \simeq f(x_i) + hf'(x_i)$$

$$\Rightarrow \quad 0 \simeq f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

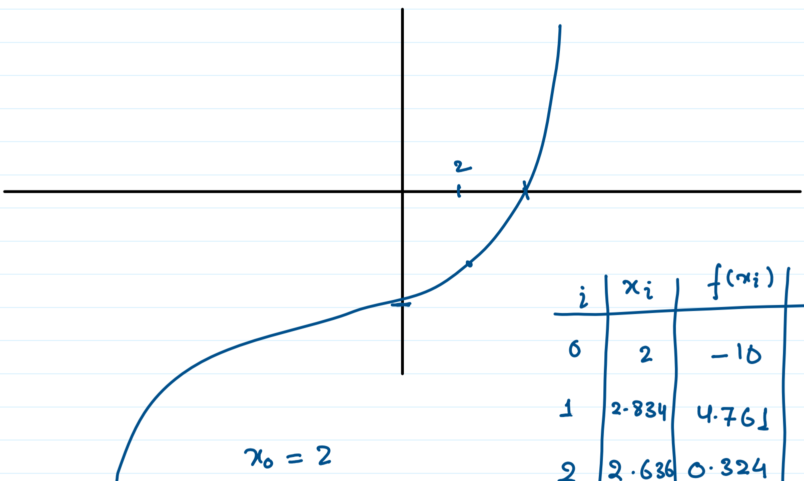
$$\Rightarrow \quad x_{i+1} - x_i = -\frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \Rightarrow 1^{st} \text{ order Convergence}$$

3) Presence of differential.

Ex:- $\sqrt[3]{18}$ using Newton- Raphson :-

$f(x): x^3 - 18 = 0$



$x_0 = 2$

i	x_i	$f(x_i)$	$f'(x_i)$
0	2	−10	12
1	2.834	4.761	24.094
2	2.636	0.324	20.851

1	2.834	4.761	24.094
2	2.636	0.324	20.851
3	2.620	−0.005	20.600

$x_0 = 2$

$f'(x) = 3x^2$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_1 = 2 + \frac{10}{12} = \frac{34}{12} = \frac{17}{6} = 2.834$$

$$\Rightarrow x_2 = 2.834 - \frac{4.761}{24.094} = 2.636$$

$$x_3 = 2.636 - \frac{0.324}{20.851} = 2.620$$

$$\sqrt[3]{18} = 2.620$$

now, for $x_0 = 10$;

i	x_i	$f(x_i)$	$f'(x_i)$
0	10	982	300
1	6.726	286.368	135.744
2	4.616	80.379	63.933
3	3.358	19.894	33.845

$$x_1 = 10 - \frac{982}{300} = 6.726$$

$$x_2 = 6.726 - \frac{286.368}{135.744} = 4.616$$

$$x_3 = 4.616 - \frac{80.374}{63.933} = 3.358$$

→ Gauss - Siedel method :-

System of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$$

→ Write in matrix form :-

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n$

→ Write in matrix form:-

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$AX = C$$

$$\Rightarrow \boxed{X = A^{-1}C}$$

$A \longrightarrow$ has to be diagonally dominant

Lecture 03: Convergence of bisection and N-R + solution of simultaneous linear eq.

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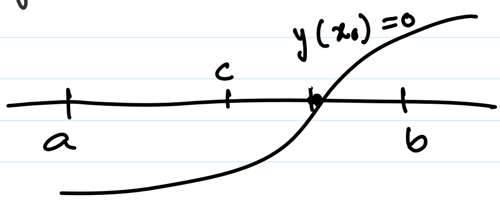
→ Convergence of bisection method:-



0^{th} $\quad \epsilon^0_{max} \approx |b-a| = \Delta$

1^{st} $\quad \epsilon^1_{max} \approx |c-b| = \frac{\Delta}{2}$

$\vdots \qquad \vdots \qquad \vdots$

n^{th} $\quad \epsilon^n_{max} \approx \quad = \frac{\Delta}{2^n}$

→ Brut-force method (the series always converges).

→ General formula for error propagation:-

$$\epsilon^{k+1} = C(\epsilon^k)^m$$

m → order of convergence

C → constant

for method of bisection:-

$$\epsilon_b^{k+1} = \frac{\epsilon_b^k}{2} \qquad m = 1$$

So, the order of convergence of error propagation is 1.

→ for a user defined error ϵ_{user}:-

the iteration terminate at

$$\frac{\Delta}{2^n} \leq \epsilon_{user}$$

Ex: $\Delta = 0.3$, $\quad \epsilon = 0.00001$

the iteration terminate at

$$\frac{\Delta}{2^n} = \epsilon \Rightarrow n\log 2 = \log\Delta - \log\epsilon$$

$$n = \frac{\log\left(\frac{\Delta}{\epsilon}\right)}{\log(2)} = 16.28$$

$$\Rightarrow \boxed{(n)_{actual} = \text{Round}(n) + 1}$$

→ Order of Convergence for N-R :-

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Actual root $= x_0$

Error in $(i+1)^{th}$ iteration $= \epsilon_{i+1}$

" " i^{th} " $= \epsilon_i$

$$\Rightarrow (x_0 + \epsilon_{i+1}) = x_0 + \epsilon_i - \frac{f(x_0+\epsilon_i)}{f'(x_0+\epsilon_i)}$$

taylor expansion of the function ⇒

$$\epsilon_{i+1} = \epsilon_i - \frac{\cancel{f(x_0)}^{0} + \epsilon_i f'(x_0) + \frac{\epsilon_i^2}{2} f''(x_0)}{f'(x_0) + \epsilon_i f''(x_0) + \cancel{O(\epsilon_i^2)} \to 0}$$

$$= \epsilon_i - \frac{\epsilon_i\left[1 + \frac{\epsilon_i}{2}\frac{f''(x_0)}{f'(x_0)}\right]}{\left[1 + \epsilon_i \frac{f''(x_0)}{f'(x_0)}\right]}$$

$$= \epsilon_i - \epsilon_i\left[\left(1 + \frac{\epsilon_i}{2}\frac{f''(x_0)}{f'(x_0)}\right)\left(1 - \epsilon_i \frac{f''(x_0)}{f'(x_0)}\right)\right]$$

$$\Rightarrow \epsilon_{i+1} = \epsilon_i - \epsilon_i\left[1 + \frac{\epsilon_i}{2}\frac{f''(x_0)}{f'(x_0)} - \epsilon_i \frac{f''(x_0)}{f'(x_0)} - \frac{\epsilon_i^2}{2}\left\{\frac{f''(x_0)}{f'(x_0)}\right\}^2\right]$$

$$\Rightarrow \epsilon_{i+1} = \epsilon_i - \epsilon_i \left[1 + \frac{\epsilon_i}{2} \frac{f''(x_0)}{f'(x_0)} - \epsilon_i \frac{f''(x_0)}{f'(x_0)} - \frac{\epsilon_i^2}{2} \left\{ \frac{f''(x_0)}{f'(x_0)} \right\} \right]$$

$$= \epsilon_i - \epsilon_i - \frac{\epsilon_i^2}{2} \frac{f''(x_0)}{f'(x_0)} + \epsilon_i^2 \frac{f''(x_0)}{f'(x_0)} - \frac{\epsilon^3}{2} (\quad)$$

$$\boxed{\epsilon_{i+1} \simeq \epsilon_i^2 \left[\frac{1}{2} \frac{f''(x_0)}{f'(x_0)} \right]}$$

$\Rightarrow m = 2$

$\Rightarrow$ 2nd order convergence.

→ Solution of simultaneous linear equations :-

$$A_{n\times n} \bar{X}_{n\times 1} = \bar{b}_{n\times 1}$$

Let us take a system of two linear eqn:-

$$\times 10 \left[x_1 + x_2 = 2 \right.$$

$$3x_1 - 10x_2 = 3$$

$$13x_1 = 23$$

$$x_1 = 23/13$$

$$\cancel{+}10\!\!\!/\; x_2 = 3 - \frac{3 \cdot 23}{13}$$

$$= \frac{39 - 69}{13}$$

$$= \cancel{+} \frac{3\cancel{0}}{13}$$

$$\Rightarrow x_1 = \frac{23}{13} = 1.769$$

$$x_2 = \frac{3}{13} = 0.231$$

Now,

$x_1 + x_2 = 2 \Rightarrow x_1 = 2 - x_2$ — (A)

$3x_1 - 10x_2 = 3 \Rightarrow x_2 = \frac{-3 + 3x_1}{10}$ — (B)

Iteration	x_1	x_2	$\lvert\epsilon_{x_1}\rvert$	$\lvert\epsilon_{x_2}\rvert$
0	0	0 $\xrightarrow{\text{to}}$ (A)	1.769	0.231
1	2	3/10 → (A)		

2	$17/10$	$21/100$
3.	$179/100$	$237/1000$
4.	$\frac{1763}{1000}$	$\frac{2289}{10000}$

Now, (Switching the order)

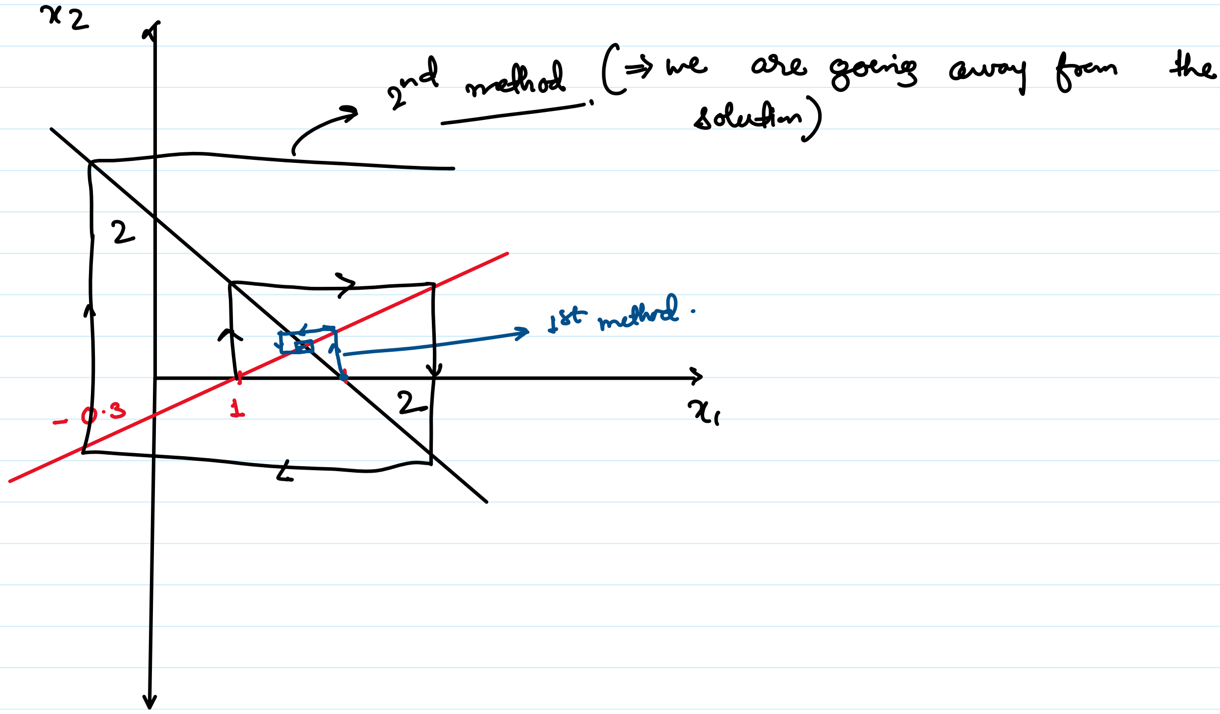
$3x_1 - 10x_2 = 3 \Rightarrow x_1 = \frac{3+10x_2}{3}$ — (A)

$x_1 + x_2 = 2 \Rightarrow x_2 = 2 - x_1$ — (B)

Method – 2

Iteration	x_1	x_2
0	0	0
1	1	1
2	$13/3$	$-7/3$
3	$-61/9$	$79/9$
4.	$817/27$	$-763/27$

→ Won't come to the solution



$$\underbrace{\begin{pmatrix} 3 & -10 \\ 1 & 1 \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\bar{X}} = \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{\bar{b}} \quad \rightarrow \text{Method 2}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 3 & -10 \end{pmatrix}}_{A_2} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\bar{X}} = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\bar{b}} \quad \rightarrow \text{Method 1}$$

for diagonal dominance:-

$$|A_{ii}| \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |A_{ij}|$$

(If this criteria is satisfied then convergence is guaranteed)

for,
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$$

$$A_{n \times n} \bar{X}_{n \times 1} = \bar{b}_{n \times 1}$$

Ex:-

$$9x_1 + 2x_2 + 4x_3 = 20$$

$$x_1 + 10x_2 + 4x_3 = 6$$

$$2x_1 - 4x_2 + 10x_3 = -15$$

$$\begin{pmatrix} 9 & 2 & 4 \\ 1 & 10 & 4 \\ 2 & -4 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 6 \\ -15 \end{pmatrix}$$

$\Downarrow$

diagonally dominant $\Rightarrow$ convergence.

$$x_1 = \frac{20 - 2x_2 - 4x_3}{9}$$

$$x_2 = \frac{6 - x_1 - 4x_3}{10}$$

$$x_3 = \frac{-15 - 2x_1 + 4x_2}{10}$$

Iteration	x_1 (2.7367)	x_2 (0.9872)	x_3 (−1.6525)
0	0	0	0
1	$20/9$	$17/45$	$-\frac{269}{150}$

Lecture 04: Linear Interpolation + Numerical Differentiation & Integration(Trapezoid, simpson & mid-Point)

Tuesday, September 6, 2022 8:17 AM

x	$f(x)$
x_1 $\updownarrow h_1$	$f(x_1) \rightarrow f_1$
x_2 $\updownarrow h_2$	$f(x_2) \rightarrow f_2$
x_3 $\updownarrow h_3$	$f(x_3)$
$\vdots$	$\vdots$
x_n $\updownarrow h_{n-1}$	$f(x_n)$

Max^m $(n-1)^{th}$ order of polynomial

$\Rightarrow$ linear interpolation

- $f(x) = a + bx$ (linear)

$$f(x) = c_1(x-x_2) + c_2(x-x_1)$$

$$\Rightarrow \quad c_1 = \frac{f_1}{(x_1-x_2)} \quad , \quad c_2 = \frac{f_2}{(x_2-x_1)}$$

$$\Rightarrow \boxed{f(x) = \frac{f_1(x-x_2)}{(x_1-x_2)} + \frac{f_2(x-x_1)}{(x_2-x_1)}}$$

$\rightarrow$ lagrange representation.

- Quadratic : for three points x_1, x_2, x_3

$$f(x) = a + bx + cx^2$$

$$f(x) = c_1(x-x_2)(x-x_3) + c_2(x-x_1)(x-x_3) + c_3(x-x_1)(x-x_2)$$

$$c_1 = \frac{f_1}{(x_1-x_2)(x_1-x_3)} \; ; \quad c_2 = \frac{f_2}{(x_2-x_1)(x_2-x_3)} \; ;$$

$$c_3 = \frac{f_3}{(x_3-x_1)(x_3-x_2)}$$

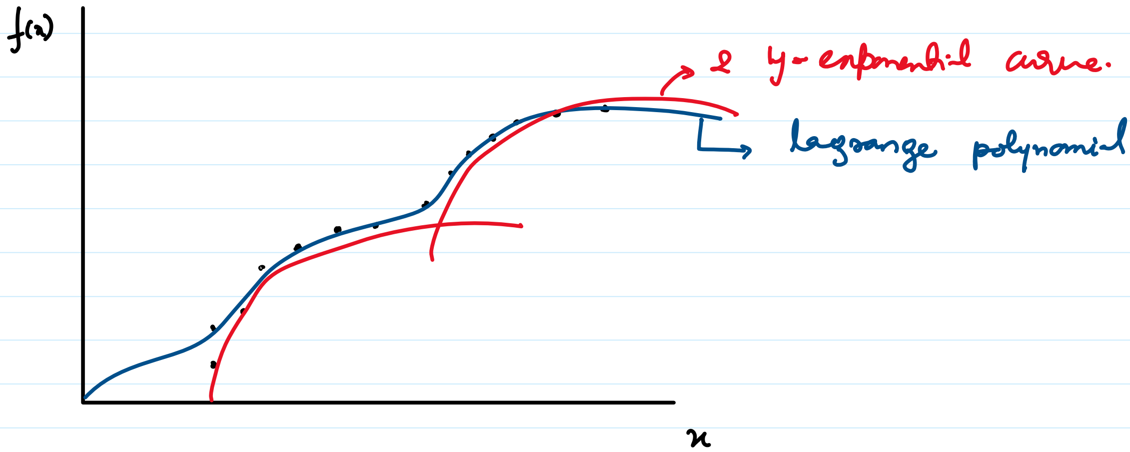
$$c_3 = \frac{f_3}{(x_3 - x_1)(x_3 - x_2)}$$

$$\Rightarrow f(x) = \frac{f_1 (x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f_2 (x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + \frac{f_3 (x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Generalization :-

$$\boxed{f(x) = \sum_{i=1}^{N} \prod_{\substack{j=1 \\ i \neq j}}^{N} \frac{f_i (x - x_j)}{(x_i - x_j)}}$$

$$\equiv \sum_{i=1}^{N} f_i \prod_{j \neq 1} \frac{(x - x_j)}{(x_i - x_j)}$$



→ Differentiation and integration :-

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f_{n+1} - f_n}{h_n}$$

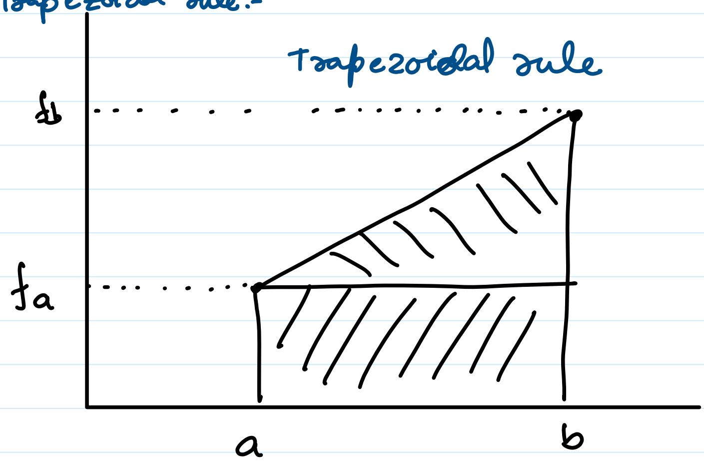
$$\left.\frac{df}{dx}\right|_{\text{in } n^{th} \text{ place}} \approx \frac{f_{n+1} - f_n}{h_n}$$

$$\approx \frac{f_n - f_{n-1}}{h_{n-1}}$$

$$\approx \frac{\cdots \; f_{n-1}}{h_{n-1}}$$

→ Integration :-

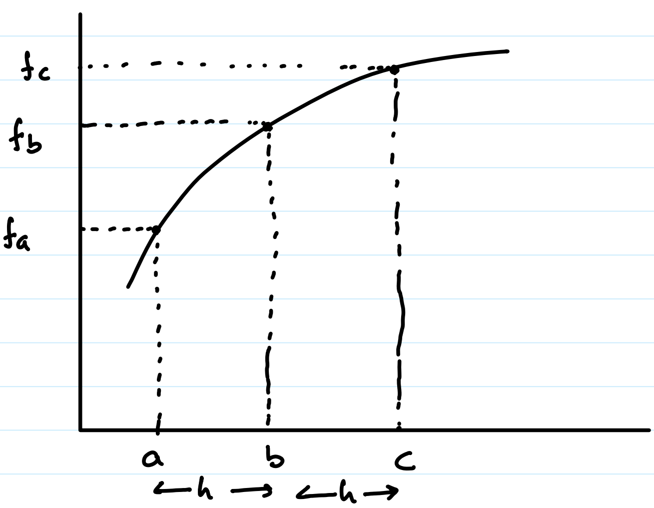
1. Trapezoidal rule:-

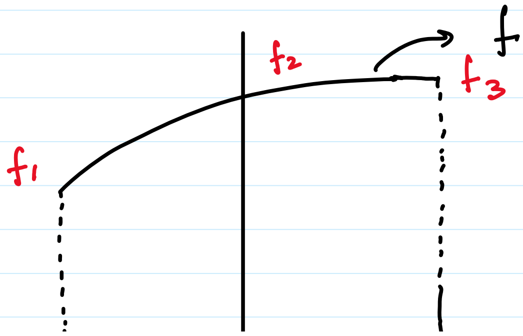


$$\text{Area } A = f_a(b-a) + \frac{1}{2}(b-a)(f_b - f_a)$$

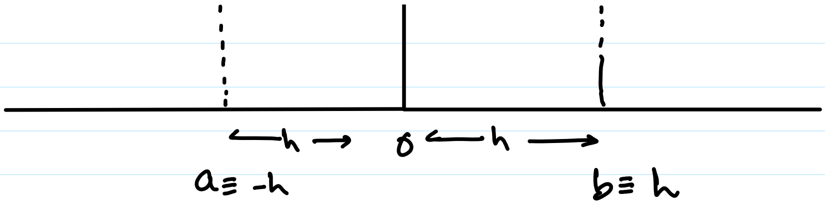
$$= \frac{(b-a)}{2}(f_b + f_a) = \frac{h}{2}(f_b + f_a)$$

2. Simpson's $\frac{1}{3}$rd rule





$$f(x) = c_1 + c_2 x + c_3 x^2$$



$$\int_{-h}^{h} f(x) = \int_{-h}^{h} (c_1 + c_2 x + c_3 x^2)\, dx$$

$$= c_1 x \Big|_{-h}^{h} + c_2 \frac{x^2}{2} \Big|_{-h}^{h} + c_3 \frac{x^3}{3} \Big|_{-h}^{h}$$

$$= 2 c_1 h + \frac{2 c_3}{3} h^3$$

Now,

$$f(-h) = c_1 - c_2 h + c_3 h^2 = f_1$$

$$f(0) = c_1 = f_2 \Rightarrow \boxed{c_1 = f_2}$$

$$f(h) = c_1 + c_2 h + c_3 h^2 = f_3$$

$$\Rightarrow 3c_1 + 2c_3 h^2 = f_1 + f_2 + f_3$$

$$\Rightarrow \boxed{c_3 = \frac{f_1 - 2f_2 + f_3}{2h^2}}$$

$$\boxed{c_2 = \frac{f_3 - f_1}{2h}}$$

$$\Rightarrow \int_{-h}^{h} f(x)\, dx = 2c_1 h + \frac{2c_3}{3} h^3$$

$$= 2 f_2 h + \frac{\cancel{2}}{3} \frac{f_1 + f_3 - 2f_2}{\cancel{2}\cancel{h^2}} h^3$$

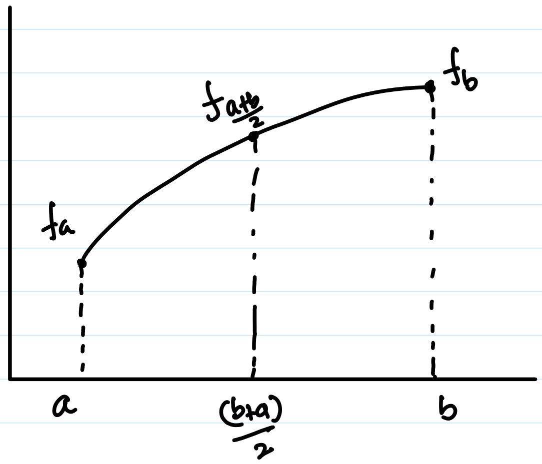
$$= \frac{h}{3} \left[6f_2 + f_1 + f_3 - 2f_2 \right]$$

$$\boxed{\int_{-h}^{h} f(x)\,dx = \frac{h}{3}\left[f_1 + 4f_2 + f_3 \right]} \rightarrow \text{simpson's formula.}$$

3. Midpoint rule

$$(b-a)\, f\left(\frac{a+b}{2}\right)$$

→ Numerical Integration summary :-



$$T \approx \left(\frac{b-a}{2}\right)\left(f_b + f_a\right)$$

$$M \approx (b-a)\, f_{\frac{a+b}{2}}$$

$$S \approx \frac{(b-a)}{6}\left(f_a + 4 f_{\frac{a+b}{2}} + f_b\right)$$

$$\Rightarrow S = \frac{b-a}{6}\left[\frac{2T}{b-a} + 4\frac{M}{b-a}\right]$$

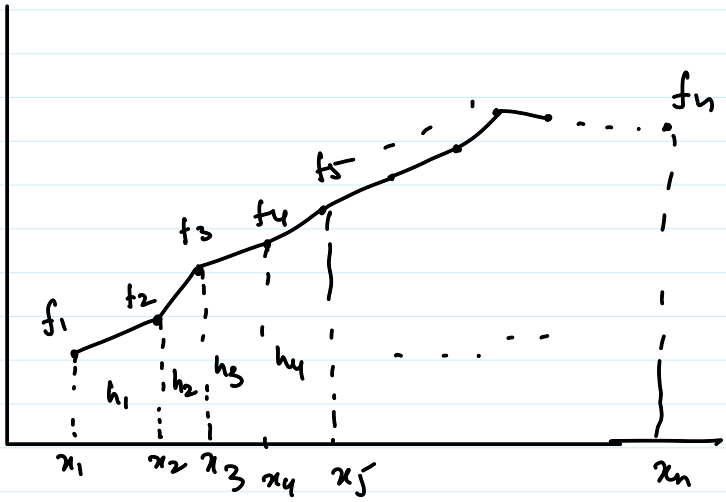
$$= \left(\frac{T}{3} + \frac{\cancel{4}^{2} M}{\cancel{6}_{3}}\right)$$

$$\Rightarrow \boxed{S = \frac{2M+T}{3}}$$

→ Numerical Integration :-

Composite formulae

x	$f(x)$
x_1	f_1
x_2	f_2
x_3	f_3
$\vdots$	$\vdots$



$$\rightarrow \quad I \approx T = \frac{h_1}{2}(f_1+f_2) + \frac{h_2}{2}(f_2+f_3) + \frac{h_3}{2}(f_3+f_4) + \cdots$$

In special case when $h_1 = h_2 = h_3 \ldots = h$

$$\Rightarrow \quad T = \frac{h}{2}\left[f_1 + f_2 + f_2 + f_3 + f_3 + f_4 \cdots + f_{n-1} + f_{n-1} + f_n\right]$$

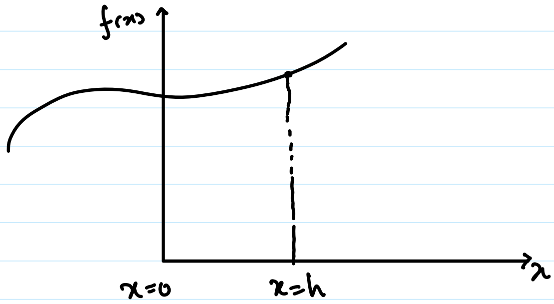
$$\boxed{T = \frac{h}{2}\left[f_1 + 2f_2 + 2f_3 + \cdots + 2f_{n-1} + f_n\right]}$$

$$\rightarrow \quad I \approx S = \frac{h}{3}\left[(f_1 + 4f_2 + f_3) + (f_3 + 4f_4 + f_5) + (f_5 + 4f_6 + f_7) + \cdots\right]$$

$$S = \frac{h}{3}\left[f_1 + 4f_2 + 2f_3 + 4f_4 + 2f_5 + 4f_6 + 2f_7 + \cdots + 4f_{n-1} + f_n\right]$$

$$\rightarrow \quad I \approx M = 2h\left[f_2 + f_4 + f_6 + \cdots + f_{n-1}\right]$$

→ Error in numerical integrations:-



Actual :- $E = \int_0^h f(x)\,dx$

$I \approx T = \frac{h}{2}\left(f(0) + f(h)\right)$

In order to calculate the error $e_T = |E - T|$ we need to expand $f(x)$ in taylor expansion.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\Rightarrow E(x) = h f(0) + \frac{h^2}{2} f'(0) + \frac{h^3}{6} f''(0) + \frac{h^4}{24} f'''(0) + \frac{h^5}{120} f^{IV}(0)$$

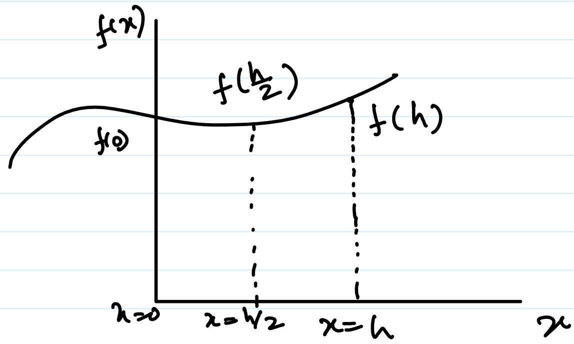
and
$$T = \frac{h}{2}\left[f(0) + f(0) + h f'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) + \cdots \right.$$

$$= h f(0) + \frac{h^2}{2} f'(0) + \frac{h^3}{4} f''(0) + \frac{h^4}{12} f'''(0) + \frac{h^5}{48} f^{IV}(0)$$

$$\Rightarrow e_T = |E - T| = h^3\left(\frac{1}{4} - \frac{1}{6}\right) f''(0) + h^4\left(\frac{1}{12} - \frac{1}{24}\right) f'''(0)$$
$$+ h^5\left(\frac{1}{48} - \frac{1}{120}\right) f^{IV}(0)$$

$$\therefore \boxed{e_T = \frac{h^3}{12} f''(0) + \frac{h^4}{24} f'''(0) + \frac{h^5}{80} f^{IV}(0)}$$

- Error for mid-point rule:-



$$I \approx M = h f\left(\frac{h}{2}\right)$$

$$= h\left[f(0) + \frac{h}{2} f'(0) + \frac{h^2}{8} f''(0) + \frac{h^3}{48} f'''(0) + \frac{h^4}{384} f^{iv}(0)\right].$$

$$\therefore \quad \mathcal{E}_M = E - M$$

$$= h^3\left(\frac{1}{6} - \frac{1}{8}\right) f''(0) + h^4\left(\frac{1}{24} - \frac{1}{48}\right) f'''(0)$$

$$+ h^5\left(\frac{1}{120} - \frac{1}{389}\right) f^{iv}(0)$$

$$\boxed{\mathcal{E}_M = \frac{h^3}{24} f''(0) + \frac{h^4}{24} f'''(0) + \frac{11 \times h^5}{1920} f^{iv}(0)}$$

- For Simpson's rule :-

$$\mathcal{E}_S = \frac{2\mathcal{E}_M + \mathcal{E}_T}{3} \qquad \left(\because S = \frac{2M+T}{3}\right)$$

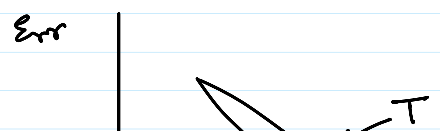
$$= \frac{1}{3}\left[\cancel{h^3 f''(0)\left(\frac{1}{12} - \frac{1}{12}\right)} + \cancel{h^4 f'''(0)\left(\frac{1}{24} - \frac{1}{24}\right)} + h^5 f^{iv}(0)\left(\frac{11}{960} - \frac{1}{80}\right) + \dots\right]$$

$$= -\frac{1}{3} h^5 \frac{1}{960} f^{iv}(0)$$

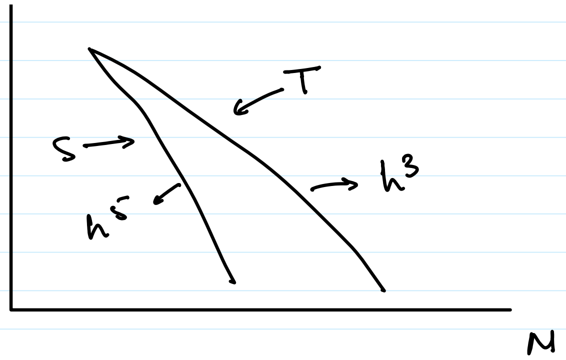
$$\therefore \quad \mathcal{E}_S = \frac{-1}{2880} h^5 f^{iv}(0)$$

for $h = 2h$:-

$$\boxed{\mathcal{E}_S = -\frac{h^5}{90} f^{iv}(0)}$$



$f(x) = 6x^5 \qquad I = 1$



$f(x) = 6x^5 \qquad I = 1$

$\varepsilon_r^T = \left| \frac{T-1}{1} \right|$

$\varepsilon_r^S = \left| \frac{S-1}{1} \right|$

→ <u>Gauss quadrature</u> :- (Gauss - Legendre formula)

Let say we have the integral :- $\int_{-1}^{1} f(x)\,dx$

so,
$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} \omega_i f(x_i)$$

↓ weight

↳ nodes (nodes of n^{th} order legendre polynomial)

$$\omega_i = \frac{2}{(1-x_i^2)\,[P_n'(x_i)]^2}$$

- This formula is exact upto $(2n-1)^{th}$ order.

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(y)\,dy$$

$$y = px + q$$

$$= \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} y + \frac{b+a}{2}\right) dy$$

no. of points	Points (x_i)	ω_i
1	0	2
2	$\pm\sqrt{1/3}$	1
3	$0, \pm\sqrt{3/5}$	$8/9$, $5/9$

4	± 0.339981	0.652145
	± 0.861136	0.347853

Ex:- $I = \int_5^9 (7x^3 - 9y)\,dx = 10,136$

$b = 9, \quad a = 5 \qquad \frac{b-a}{2} = 2, \quad \frac{b+a}{2} = 7$

$\Rightarrow I = 2\int_{-1}^{1} f(2y+7)\,dx$ $\qquad f(2y+7) = 7(2y+7)^3 - 9(2y+7)$

$= 7(8y^3 + 343 + \underline{3.7.4y^2} + 3.7^2.2y) - 18y - 63$

$= 56y^3 + 588y^2 + 2338 + 2040y$

$= 2\int_{-1}^{1} (56y^3 + 588y^2 + 2040y + 2338)\,dy$

I(for one point) $\approx 2[2f(0)] = 4 \times 2338 = 9354$

I (for two point Gauss quadrature) $= 2\left[1.f\left(\frac{1}{\sqrt{3}}\right) + 1.f\left(-\frac{1}{\sqrt{3}}\right)\right]$

$= 2\left[(588.\frac{1}{3} + 2338) \times 2\right]$

$= 10,136$

I(for 3 point) $= 2\left[\frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) + \frac{5}{4}f\left(-\sqrt{\frac{3}{5}}\right)\right]$

$= 2\left[\frac{8}{9} \times 2338 + \frac{10}{4}(2338 + 588 \times \frac{3}{5})\right]$

Example $I = \int_2^3 \frac{dx}{1+x}$, 5 point Simpson

$h = \frac{b-a}{n-1} = 0.25$

$= \ln(1+x)\Big|_2^3$

$$= \ln\frac{4}{3} = 0.287682.$$

$$f(2) = \frac{1}{1+2} = 0.333333$$

$$f(2.25) = \frac{1}{1+2.25} = 0.307692$$

$$f(2.5) = \frac{1}{1+2.5} = 0.285714$$

$$f(2.75) = 0.266666$$

$$f(3) = \frac{1}{4} = 0.25$$

$$S = \frac{h}{3}\Big[0.33333 + 4\times(0.307692 + 0.266666) + 2\times(0.285714) + 0.25\Big]$$

$$= 0.287682.$$

→ with 2 point Gauss - quadrature.

$$I = \int_2^3 \frac{dx}{1+x}$$

$$= \int_{-1}^{1} f\left(\frac{5}{2} + \frac{1}{2}y\right) dy$$

$$f(y) = \frac{1}{2}\,\frac{2}{2+\frac{5}{2}+\frac{1}{2}y} = \frac{1}{7+y}$$

$$\Rightarrow G^1 = 2\times\frac{1}{7} = 0.285714$$

$$G^2 = 1\left(\frac{1}{7+\sqrt{1/3}} + \frac{1}{7-1/\sqrt{3}}\right)$$

$$= 0.287671$$

$$G^3 = \frac{8}{9} \times \frac{1}{7} + \frac{5}{9}\left(\frac{1}{7+\sqrt{3/5}} + \frac{1}{7-\sqrt{3/5}}\right)$$

$$= 0.126984 + 0.160697$$

$$= 0.287682$$

→ **Solution of ODE :-**

I.C :-

$y(x_0) = y_0 \qquad \dfrac{dy}{dx} = f(x, y) \longrightarrow y = y(x)$

$y(x_0) = y_0 \qquad \dfrac{d^2y}{dx^2} = g(x, y, y') \longrightarrow y = y(x)$

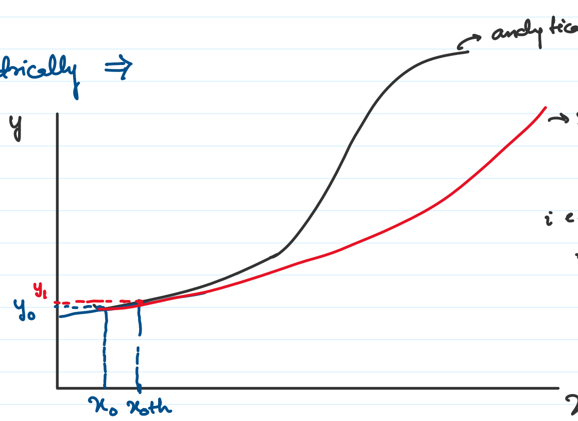
$y'(x_0) = y'_0$

Taylor expansion ⇒

Euler's first order approximation

$$y(x_0+h) = \underbrace{y(x_0) + \frac{h}{1!}y'(x_0)}_{\text{Euler's first order approximation}} + \frac{h^2}{2!}y''(x_0) + \mathcal{O}(h^3)$$

Geometrically ⇒



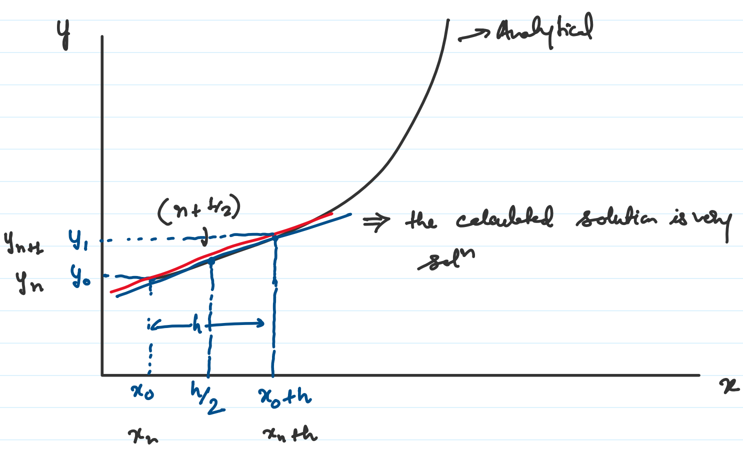
i.e., if the slope change
⇒ there will be huge deviation in the Euler's method.
∵ it is a first order approximation.

Euler Method :- $\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$

- Improvement in the method ⇒ second order approximation.

$$y(x_0+h) = \underbrace{y(x_0) + h\,y'(x_0) + \frac{h^2}{2!}y''(x_0)}_{\text{Runge - Kutta 2nd order}} + \mathcal{O}(h^3)$$

Runge - Kutta 2nd order



⇒ the calculated solution is very close to Analytical sol^n

- Runge-Kutta 2nd order method / Midpoint method / modified Euler method

let us assume that we know $y_{n+1/2}$ analytically

• Runge-Kutta [illegible] order method / Midpoint method / Modified Euler method

let us assume that we know $y_{n+1/2}$ analytically

$$y_n \equiv y_{n+1/2}\left(x - \frac{h}{2}\right) = y_{n+1/2} - \frac{h}{2} y'_{n+1/2} + \frac{h^2}{8} y''_{n+1/2} + O(h^3)$$

$$\Rightarrow \quad y_{n+1} \equiv y_{n+1/2}\left(x + \frac{h}{2}\right) = y_{n+1/2} + \frac{h}{2} y'_{n+1/2} + \frac{h^2}{8} y''_{n+1/2} + O(h^3)$$

$$\Rightarrow \quad y_{n+1} - y_n = h y'_{n+1/2}$$

$$\Rightarrow \quad \boxed{y_{n+1} = y_n + h y'_{n+1/2} + O(h^3)}$$

where $y'_{n+1/2}$ can be calculated using Euler's Method:-

Now, $y'_{n+1/2} = f\left(x_{n+1/2},\ y_{n+1/2}\right)$

$$\Rightarrow \quad \boxed{y'_{n+1/2} = f\left(x_{n+1/2},\ \underbrace{y_n + \frac{h}{2} f(x_n, y_n)}_{\text{Euler method}}\right)}$$

(∴ It is called modified Euler's method)

let
$$\begin{cases} S_1 = f(x_n, y_n) \longrightarrow \text{Euler} \\ S_2 = f\left(x_{n+1/2},\ y_n + \frac{h}{2} S_1\right) \end{cases}$$

$$\Rightarrow \quad y'_{n+1/2} \equiv S_2$$

$$\Rightarrow \quad \boxed{y_{n+1} = y_n + h S_2} \longrightarrow \text{final recursion relation}$$

where S_1 and S_2 are defined.

Now, from taylor expansion, we have

$$y_{n+1} \approx y_n + h y'_n + \frac{h^2}{2} y''_n$$

also, $$y''_n = \frac{y'_{n+1} - y'_n}{h}$$

$$\Rightarrow \quad y_{n+1} \approx y_n + \frac{h}{2}\left(y'_n + y'_{n+1}\right)$$

$$= y_n + \frac{h}{2}\left(f(x_n, y_n) , f(x_{n+1}, y_{n+1})\right)$$

$$\Rightarrow \quad \boxed{y_{n+1} = y_n + \frac{h}{2}\Big(\underbrace{f(x_n,y_n)}_{S_1} + \underbrace{f(x_n+h,\ y_n + h f(x_n,y_n))}_{S_2' \text{ (Euler)}}\Big)}$$

$$\Rightarrow \quad y_{n+1} = y_n + \frac{h}{2}\left(S_1 + S_2'\right)$$

$$S_1 = f(x_n, y_n)$$

$$S_2' = f(x_n + h,\ y_n + h f(x_n, y_n))$$

• R-K 4th order approximation:-

k_i = slope

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(x_n + h,\ y_n + h k_3\right)$$

$$\text{Error} \sim O(h^5)$$

Ex- $\dfrac{dy}{dx} = -xy$, $y(0) = 1$ | $\ln y = -\dfrac{x^2}{2}$

$\Rightarrow y = e^{-x^2/2}$

$f(x,y) = -xy$, $h = 0.1$ $\quad \Rightarrow y(0.1) = 0.995012$

Mid-point method

x	$x_{1/2}$	y	$S_1 = f(x,y)$	$S_2 = f(x_{n+1/2}, y_n + h\frac{S_1}{2})$	$y_{n+1} = y_n + hS_2$	y_{n+1} Analytical value
0	0.05	1	0	$-(0.05)(1+0) = -0.05$	0.995	0.995012
0.1	0.15	0.995	-0.0995	$-(0.15)(0.995 - 0.05 \times 0.0995)$ $= -0.1485$	0.98015	0.98019
0.2	0.25	0.98015	-0.19603	$-(0.25)(0.98015 - 0.05 \times 0.19603)$	0.9559	0.95599

$0.2 \quad 0.25 \quad 0.98015 \quad -0.19603 \quad -(0.25)(0.98015 - 0.05 \times 0.19603) \quad 0.9559 \quad 0.95599$

$= -0.2425$

→ R-K 2nd order:-

$$\frac{dy}{dx} = f(x,y)$$

$S_1 = f(x_n, y_n)$

$S_2 = f(x_n + dh, \; y_n + dhS_1)$

weight factor

$y_{n+1} = y_n + h \cdot (w_1 S_1 + w_2 S)$

$w_1 + w_2 = 1$ and $d = \frac{1}{2}$

- for mid-point method:-
  $w_1 = 0$, $w_2 = 1$ and $d = 1/2$
- for Heun's method:-
  $w_1 = w_2 = 1/2$ and $d = 1/2$

→ Coupled ODE:-

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y,z) \\ \frac{dz}{dx} &= g(x,y,z) \end{aligned}\right\}$$

I.C.:-
at $x = x_0$, $y = y_0$, $z = z_0$

$S = f(x_n, y_n, z_n)$

$p = g(x_n, y_n, z_n)$

— Using Euler's method the solution for 1st eqn using S is given by:-

$$y_{n+1} = y_n + hS$$

Similarly for the 2nd eqn using P:-

Similarly for the 2^{nd} eq^n using p :-

$$z_{n+1} = z_n + hp$$

and, $$x_{n+1} = x_n + h$$

- Using RK2 (Heun's method) :-

$$S_1 = f(x_n, y_n, z_n)$$

$$P_1 = g(x_n, y_n, z_n)$$

$$S_2 = f(x_n+h, y_n + hS_1, z_n + hP_1)$$

$$P_2 = g(x_n+h, y_n + hS_1, z_n + hP_1)$$

$$y_{n+1} = y_n + \frac{h}{2}(S_1 + S_2)$$

$$z_{n+1} = z_n + \frac{h}{2}(P_1 + P_2)$$

$$x_{n+1} = x_n + h$$

- For 2^{nd} order ODE :-

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = d$$ → we solve by breaking it into two coupled ODE.

$$\frac{dy}{dx} = z = f(x,y,z) \quad \text{—— (1)}$$

$$\Rightarrow \quad a\frac{dz}{dx} + bz + cy = d$$

$$\Rightarrow \quad \frac{dz}{dz} = \frac{d - cy - bz}{a} = g(x,y,z) \quad \text{—— (2)}$$

Eg: $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 5, \quad y(0) = 0 = y'(0)$$

Analytical solⁿ: $y_a = \frac{5}{4} - \frac{5}{3}e^{-x} + \frac{5}{12}e^{-4x}$

$$\frac{dy}{dx} = z \quad \text{— (1)} \qquad f(x,y,z) = S$$

$$\frac{dz}{dx} + 5z + 4y = 5$$

$$\Rightarrow \frac{dz}{dx} = 5 - 4y - 5z \quad \text{— (2)} \qquad g(x,y,z) = P$$

(1) Euler's method $\qquad x_0 = y_0 = z_0 = 0$

$$S = z = f(x,y,z)$$

$$P = 5 - 4y - 5z = g(x,y,z)$$

$$h = 0.1$$

$$x_1 = 0 + 0.1 = 0.1$$

$$S_0 = 0$$
$$P_0 = 5$$

$$y_1 = y_0 + hS_0 = 0$$

$$z_1 = z_0 + hP_0 = 0.1 \times 5 = 0.5$$

$$x_2 = 0.1 + 0.1 = 0.2$$

$$S_1 = 0.5$$
$$P_1 = 5 - 4y_1 - 5z_1$$
$$= 5 - 5(0.5) = 2.5$$

$$y_2 = y_1 + hS_1 = 0 + 0.1 \cdot 0.5 = 0.05$$

$$z_2 = z_1 + hP_1 = 0.5 + 0.1 \times 2.5 = 0.75$$

$$x_3 = 0.2 + 0.1 = 0.3$$

$$S_2 = 0.75$$

$$P_2 = 5 - 4 \times 0.05 - 5 \times 0.75$$

$$= 5 - 0.2 - 3.75$$

$$= 5 - 3.95 = 1.05$$

$$y_3 = y_2 + hS_2 = 0.05 + 0.1 \times 0.75$$
$$= 0.125$$

$$z_3 = z_2 + hP_2 = 0.75 + 0.1 \times 1.05$$
$$= 0.855$$

i	0	1	2	3
x	0	0.1	0.2	0.3
y	0	0	0.05	0.125
z	0	0.5	0.75	0.855

(2) RK2 method :- $x_0 = y_0 = z_0 = 0$

$$S_1 = f(x, y, z) = z$$

$$P_1 = g(x, y, z) = 5 - 4y - 5z$$

$$S_2 = f(x_n + h, y_n + hS_1, z_n + hP_1) = z + hP_1$$

$$P_2 = g(x_n + h, y_n + hS_1, z_n + hP_1)$$

$$= 5 - 4(y_n + hS_1) - 5(z_n + hP_1)$$

$$= 5 - 4y_n - 5z_n - h(4S_1 + 5P_1)$$

$$y_{n+1} = y_n + \frac{h}{2}(S_1 + S_2)$$

$$z_{n+1} = z_n + \frac{h}{2}(P_1 + P_2)$$

$$x_{n+1} = x_n + h$$

$x_1 = 0.1$

$S_1 = 0$

$P_1 = 5$

$S_2 = 0 + (0.1) \times 5 = 0.5$

$P_2 = 5 - 0 - 5(0 + 0.1 \times 5)$

$= 5 - 2.5 \quad = 2.5$

$y_1 = 0 + \frac{0.1}{2} \times (0.5) = \frac{0.05}{2} = 0.025$

$z_2 = 0 + \frac{0.1}{2} \times (5 + 2.5) = \frac{0.75}{2} = 0.375$

$x_2 = 0.2$

$S_1 = 0.375$

$P_1 = 5 - 4(0.025) - 5(0.375)$

$= 5 - 0.1 - 1.875$

$= +3.025$

$S_2 = 0.375 + 0.1 \times 3.025$

$= 0.6775$

$P_2 = 5 - 4 \times (0.025) - 5 \times (0.325) - 0.1(4(0.325) + 5(3.025))$

$= 5 - 0.1 - 1.825 - 1.6625$

$= 1.3625$

$$y_2 = 0.025 + \frac{0.1}{2}(0.375 + 0.6775)$$

$$= 0.077625$$

$$z_2 = 0.375 + \frac{0.1}{2}(3.025 + 1.3625)$$

$$= 0.594375$$

$$x_3 = 0.3$$

S_1

i	0	1	2	3
x	0	0.1	0.2	0.3
y	0	0.025	0.077625	0.14565062 5
z	0	0.375	0.594375	0.711309375
y_a	0	0.021	0.074	0.14

RK2 - method is more accurate than Euler's method.

→ Finite difference method:-

① In Euler/RK method a differential eqn is converted into a recursion relation.

However, in finite difference method we convert the differential eqn to a set of linear/non-linear eqn.

② In Euler/RK - I.C. are given. → $y'' + \omega_0^2 y = 0 \mid y(0) = a \,;\, y'(0) = 0$

But FD method deals with I.C. and Boundary values (B.V.).

↓

$$y'' + \omega_0^2 y = 0$$

$$y'' + \omega_0^2 y = 0$$

$$\boxed{y(0) = a \quad , \quad y(\pi) = -a}$$

$\rightarrow$ Euler method cannot work with this boundary condition.

③ FD method can be applicable for PDEs.

$\rightarrow$ Finite difference Method:-

$$f(x) \longrightarrow \begin{cases} f(x+h) = f(x) + h f'(x) + \dfrac{h^2}{2!} f''(x) + O(h^3) & \text{— ①} \\ f(x-h) = f(x) - h f'(x) + \dfrac{h^2}{2!} f''(x) - O(h^3) & \text{— ②} \end{cases}$$

from -①, truncating the series to order 2

Forward difference

$$\Rightarrow f'(x) \simeq \frac{f(x+h) - f(x)}{h} + O(h) \quad \text{— ③}$$

Similarly from -②

Backward difference

$$f'(x) \simeq \frac{-f(x-h) + f(x)}{h} + O(h) \quad \text{— ④}$$

Now,

$$f(x+h) - f(x-h) = 2h f'(x) + O(h^3)$$

Central difference

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \quad \text{— ⑤}$$

Also

$$f(x+h) + f(x-h) = 2f(x) + \cancel{2}\frac{h^2}{\cancel{2}} f''(x) + O(h^4)$$

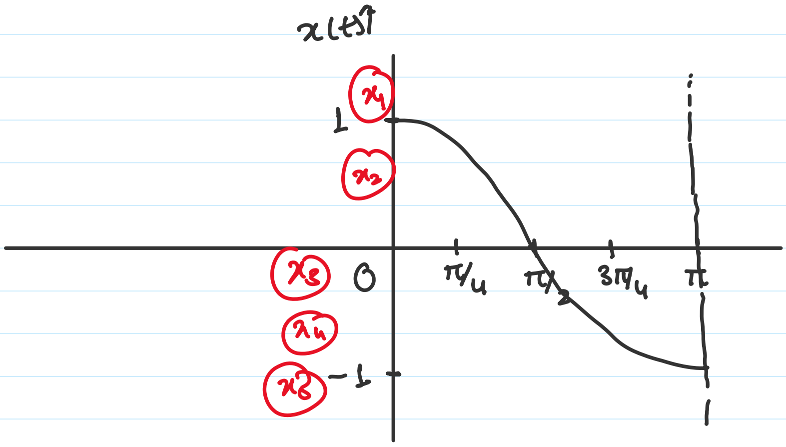
$$\Rightarrow f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2) \quad \text{— ⑧}$$

Central difference method

Central difference method for 2nd derivative " — (8)

Ex: $\ddot{x} + \omega_0^2 x = 0$, $\omega_0 = 1$

$\Rightarrow \ddot{x} + x = 0$, $x(0) = 1$, $x(\pi) = 1$, $\Delta t = \pi/4$



$x_a(t) = \cos t$

& $\Delta t = \frac{\pi}{4} \Rightarrow \Delta t^2 = \frac{\pi}{16}$

using — (8) we can have the expression for $\ddot{x}$

Hence, our differential eq^n becomes -

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} + x_n = 0$$

$$x_{n+1} + x_n(\Delta t^2 - 2) + x_{n-1} = 0$$

for $n=2$:- $x_3 + x_2\underbrace{\left(\frac{\pi^2}{16} - 2\right)}_{= -1.383} + x_1 = 0$

$\Rightarrow x_3 - 1.383\,x_2 + 1 = 0$ — (1)

$n=3$:- $x_4 - 1.383\,x_3 + x_2 = 0$ — (2)

$n=4$:- $x_5 - 1.383\,x_4 + x_3 = 0$

or $-1 - 1.383\,x_4 + x_3 = 0$ — (3)

$$x_2 = \frac{1 + x_3}{1.383}$$

$$1.383 \left(\cancel{x_4} - 1.383\, x_3 + \cancel{\frac{1}{1.383}} + \cancel{\frac{x_3}{1.383}} = 0 \right)$$

$$-1 - \cancel{1.383\, x_4} + x_3 = 0$$

$$\Rightarrow x_3 = 0$$

$$\Rightarrow x_2 = \frac{1}{1.383} = 0.7230658$$

$$x_4 = 1.383\, x_3 - x_2 = -0.7230658.$$

Now for n eqns :-

$$x_1 = a \quad , \quad x_N = -a$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ \times & \times & \times & 0 & 0 & \cdots & 0 \\ 0 & \times & \times & \times & 0 & 0 \cdots & 0 \\ 0 & 0\ 0 & \times & \times & \times & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} a \\ c_2 \\ \vdots \\ \vdots \\ -a \end{pmatrix}$$

Spur matrix.

Lecture 08: ODE (finite difference method) + PDE (heat eq and laplace eq)

Wednesday, October 19, 2022 6:01 PM

→ Finite difference method :- (FD) method

$$y' = \frac{y_{n+1} - y_n}{h} + O(h)$$

$$= \frac{-y_{n-1} + y_n}{h} + O(h)$$

$$= \frac{y_{n+1} - y_{n-1}}{2h} + O(h^2)$$

$$y'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + O(h^2)$$

- SHO :-

$$\ddot{x} + \omega_0^2 x = 0 \quad , \quad \begin{cases} x(\theta_0) = a_0 \\ x(\theta_n) = a_n \\ \text{Dirichlet Boundary Condition} \end{cases}$$

or we can have

$$\begin{cases} x(\theta_0) = a_0 \\ \dot{x}(\theta_0) = b_0 \\ \text{Neumann B.C.} \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \times & \times & \times & & 0 \\ 0 & \times & \times & \times & \\ \vdots & 0 & \times & \times \; \times & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ \vdots \\ a_n \end{pmatrix}$$

Sparse matrix

- damped Harmonic Oscillator :-

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta t)^2} + 2\beta \frac{x_{n+1} - x_{n-1}}{2\Delta t} + \omega_0^2 x_n = 0$$

$$\Rightarrow x_{n+1}(1+\beta\Delta t) + x_n(\omega_0^2 \Delta t^2 - 2) + x_{n-1}(1 - \beta\Delta t) = 0$$

$$\boxed{x_{n+1} = -a_1 x_n - a_2 x_{n-1}} \quad \text{—— (1)}$$

where $a_1 = \dfrac{\omega_0^2 \Delta t^2 - 2}{1+\beta\Delta t}$, $a_2 = \dfrac{1-\beta\Delta t}{1+\beta\Delta t}$

from Neumann condition:-

$x(\theta_0) = a_0 \Rightarrow x_0 = a_0$

$\dot{x}(\theta_0) = b_0 \Rightarrow$ $\boxed{\dfrac{x_1 - x_0}{h} = b_0 \Rightarrow x_1 = h b_0 + a_0}$

– determination of Δt:-

$x_{n+1} = z x_n$ where z is the propagator

$$\left.\begin{aligned} &\text{and} \quad x_n = z x_{n-1} \\ &\Rightarrow \quad x_{n+1} = z^2 x_{n-1} \end{aligned}\right\} \text{—— (2)}$$

$\Rightarrow$ from —(1)

$$(z^2 + a_1 z + a_2) x_{n-1} = 0$$

$$\Rightarrow z = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$\Rightarrow$ from —(2)

$$z = \frac{1}{2}\left[-a_1 \pm i D\right]$$

$$D = \sqrt{4a_2 - a_1^2}$$

for decaying solution:- $|z| < 1$

$$\Rightarrow \frac{1}{4}(a_1^2 + D^2) < 1$$

$$\Rightarrow (\cancel{a_1^2} + \cancel{4}a_2 - \cancel{a_1^2}) < \cancel{4}$$

$$\Rightarrow a_2 < 1$$

$$\Rightarrow \frac{1-\beta\Delta t}{1+\beta\Delta t} < 1 \;, \quad \beta\Delta t > 0 \Rightarrow \beta > 0 .$$

2nd condition:- $\text{Im}(z) \neq 0$

$$\Rightarrow D \neq 0$$

$$\Rightarrow 4a_2 - a_1^2 > 0$$

$$\Rightarrow 4a_2 > a_1^2$$

$$\Rightarrow \frac{4(1-\beta\Delta t)}{1+\beta\Delta t} > \frac{(\omega_0^2\Delta t^2 - 2)^2}{(1+\beta\Delta t)^2}$$

$$\Rightarrow \boxed{\Delta t < \frac{2\sqrt{\omega_0^2 - \beta^2}}{\omega_0^2}} \rightarrow \text{Stability Criteria.}$$

→ $y' = f(x,y)$

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n)$$

$$y_{n+1} = y_n + h f(x_n, y_n) \qquad [\text{Explicit / forward Euler}]$$

Now, using backward method:-

$$\frac{y_n - y_{n-1}}{h} = f(x_n, y_n)$$

$$\Rightarrow \frac{y_{n+1} - y_n}{h} = f(x_{n+1}, y_{n+1})$$

$$\Rightarrow \quad \frac{y_{n+1} - y_n}{h} = f(x_{n+1}, y_{n+1})$$

$$\Rightarrow \quad y_{n+1} = y_n + h f(x_{n+1}, y_{n+1}) \quad [\text{Implicit / Backward Euler}]$$

Ex. $\dfrac{dy}{dx} = \lambda y \qquad y(x=0) = 1$

$$\Rightarrow \quad \ln y = \lambda x + C$$

$\because C = 0 \quad \Rightarrow \quad \boxed{y = e^{\lambda x}}$

Explicit

$$y_{n+1} = y_n + h\lambda y_n$$

or $y_{n+1} = y_n \underbrace{(1 + h\lambda)}_{Z}$

$\lambda < 0 \rightarrow$ decaying

$$\Rightarrow \quad -1 < (1 + h\lambda) < 1$$

$$\Rightarrow \quad h\lambda > -2$$

$$\boxed{h > \frac{2}{|\lambda|}}$$

Stability Criteria

for $\lambda > 0$:-

unconditionally stable

Implicit

$$y_{n+1} = y_n + h\lambda y_{n+1}$$

or $y_{n+1} = \underbrace{\left(\dfrac{1}{1 - h\lambda}\right)}_{Z} y_n$

$\lambda < 0$

$$Z = \frac{1}{1 + h|\lambda|}$$

for $\lambda > 0$:-

$\boxed{h < \dfrac{1}{\lambda}} \rightarrow$ Stability Criteria

Ex. $\dfrac{dy}{dx} = 2y - 3x \qquad y(0) = 1$

explicit :-

$y \quad - y$

implicit :-

$$\frac{y_{n+1} - y_n}{h} = 2y_{n+1} - 3x_{n+1}$$

$$\frac{y_{n+1} - y_n}{h} = 2y_n - 3x_n$$

$$\Rightarrow y_{n+1} = y_n(2h+1) - 3hx_n$$

$$\frac{y_{n+1} - y_n}{h} = 2y_{n+1} - 3x_{n+1}$$

$$\Rightarrow y_{n+1}(1 - 2h) = y_n - 3h\,x_{n+1}$$

$$y_{n+1} = \frac{y_n - 3h(x_n + h)}{1 - 2h}$$

for $h = 0.1$

$y_0 = 1$

$$y_1 = 1(0.2+1) - 3(0.1)\cdot 0$$
$$= 1.2$$

$$y_1 = \frac{1 - 3(0.1)(0+0.1)}{1 - 2(0.1)}$$
$$= \frac{1 - 0.03}{1 - 0.2} = 1.2125$$

$$y_2 = 1.2(2\times0.1+1) - 0.3\times0.1$$
$$= 1.41$$

$$y_2 = \frac{1.2125 - 0.3(0.2)}{0.8}$$
$$= 1.4406$$

$$\frac{dy}{dx} - 2y = -3x$$

$$I.F. = e^{\int -2\,dx} = e^{-2x}$$

$$ye^{-2x} = \int -3x\,e^{-2x}\,dx + c$$

$$= -3\left[x\frac{e^{-2x}}{-2} + \frac{1}{2}\frac{e^{-2x}}{(-2)}\right] + C$$

$$y = \left(\frac{3x}{2} + \frac{3}{4}\right) + C\,e^{2x}$$

$$1 = \frac{3}{4} + C \Rightarrow C = \frac{1}{4}$$

$$\Rightarrow y_{an} = \frac{1}{4}e^{2x} + \frac{3x}{2} + 3/4$$

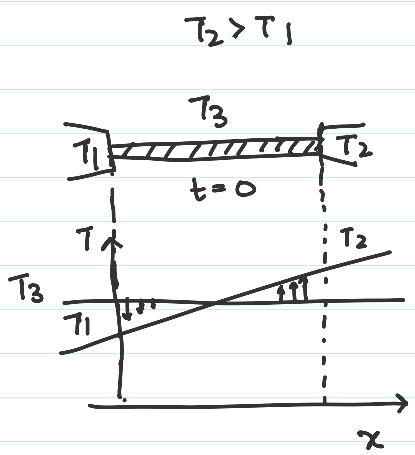
$$y_1(0.1) = 1.20535$$

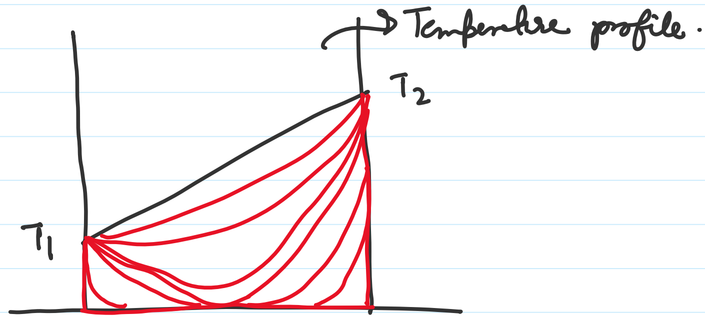
$$y_2(0.2) = 1.42295$$

→ Partial differential equations (PDEs) :-

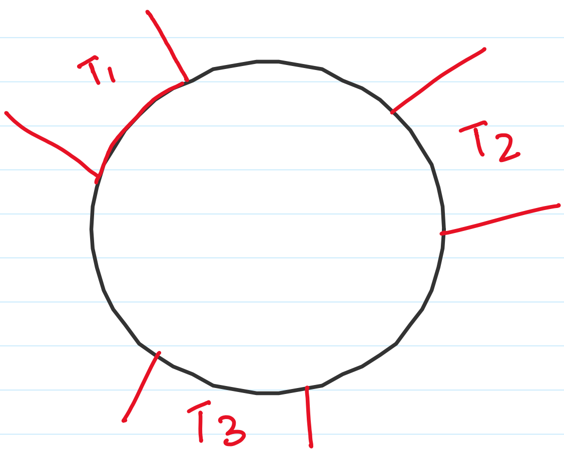
1. 1D - heat eqⁿ :- $\dfrac{\partial u(x,t)}{\partial t} = \alpha \dfrac{\partial^2 u(x,t)}{\partial x^2}$

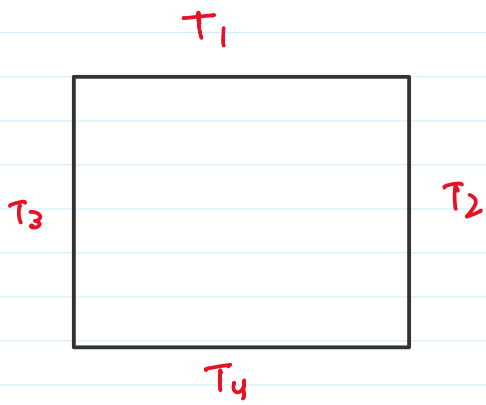
2. Laplace eqⁿ :- $\dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial y^2} = 0$

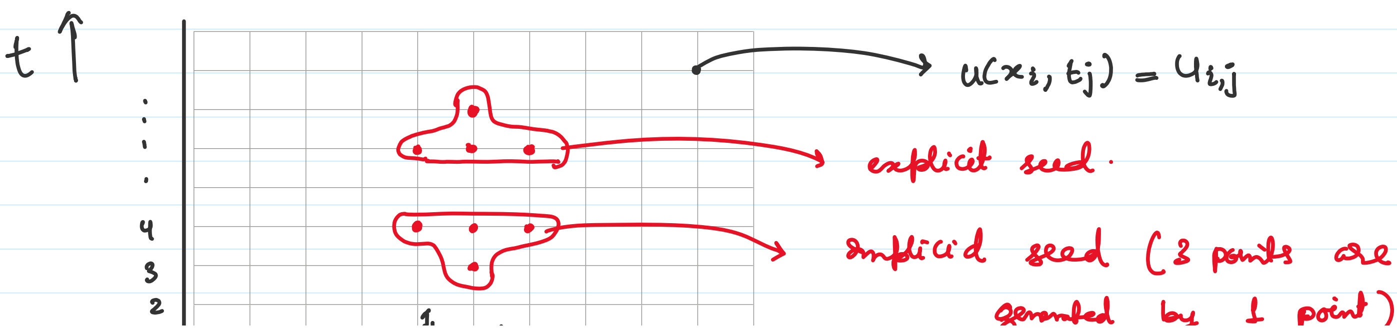


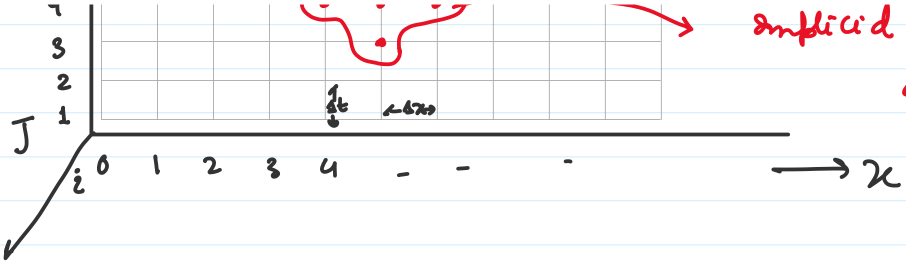


for 2D :-









Implicit seed (3 points are generated by 1 point)

Explicit scheme :-

Forward difference in time

Central " " space

$$\frac{\partial u}{\partial t} = \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta t, t)}{(\Delta x)^2}$$

let $\sigma = \dfrac{\alpha \Delta t}{(\Delta x)^2}$

$$u(x, t+\Delta t) = \sigma u(x+\Delta t, t) - (2\sigma - 1) u(x,t) + \sigma u(x-\Delta x, t)$$

$$\Rightarrow \quad u(x, t+\Delta t) = \underbrace{A^E(\sigma)}_{\text{propagator}} u(x,t)$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & & \cdot \\ u & -(2\sigma-1) & \sigma & 0 & 0 & \cdot & \cdot \\ 0 & u & -(2\sigma-1) & \sigma & 0 & \cdot & \cdot \\ 0 & 0 & & & & & \\ \vdots & \vdots & \cdot & & & & 1 \end{bmatrix}$$

$$u(x,0) = \begin{pmatrix} u_0^0 \\ u_1^0 \\ u_2^0 \\ \vdots \\ u_n^0 \end{pmatrix} \longrightarrow \quad u(x,\Delta t) = \begin{pmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ \vdots \\ u_n^1 \end{pmatrix}$$

→ for N iterations :-

$$\boxed{u(x, N\Delta t) = \left(A^E(\sigma)\right)^N u(x,0)} \rightarrow \text{Conditionally stable}$$

$$\boxed{0 < \sigma < 1/2}$$

Lecture09: FDM (heat equation) + unstability of explicit scheme + Von Neumann stability analysis + Laplace eq (2d)

Tuesday, November 1, 2022 8:19 AM

→ FDM – Heat equation (1-D):-

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

$\alpha \downarrow$ Thermal diffusivity (mm^2/sec)

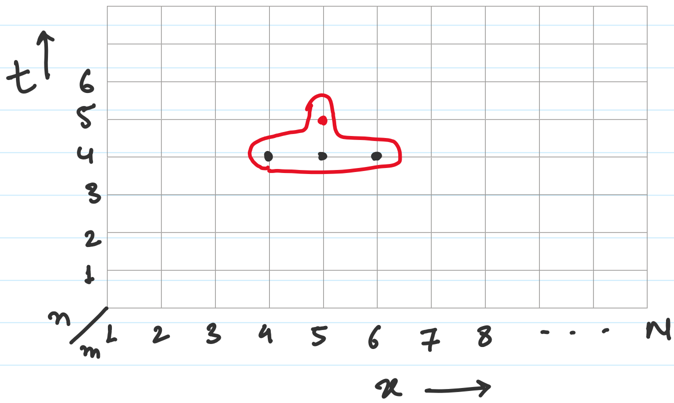
→ Explicit:-

forward difference in time at x

Central " " space at t

$$u(\underset{m}{x}, \underset{n}{t+\Delta t}) = \sigma u(x+\Delta x, t) + (1-2\sigma) u(x,t) + \sigma u(x-\Delta x, t)$$

$$u(m, n+1) = \sigma u(m+1, n) + (1-2\sigma) u(m,n) + \sigma u(m-1, n)$$



$$A^E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & & \\ \sigma & 1-2\sigma & \sigma & 0 & \cdots & & \\ 0 & \sigma & 1-2\sigma & \sigma & 0 & \cdots & \\ \vdots & \vdots & \vdots & & & & \\ & & & & & & 1 \\ 0 & 0 & 0 & \cdots & & & \end{pmatrix}$$

→ Implicit:-

Forward difference in time at x

Central difference in space at $t+\Delta t$

$$u(x,t) = -\sigma u(x+\Delta x, t+\Delta t) + (1+2\sigma) u(x, t+\Delta t) - \sigma u(x-\Delta x, t+\Delta t)$$

$$u(m,n) = -\sigma u(m+1, n+1) + (1+2\sigma) u(m, n+1) - \sigma u(m-1, n+1)$$

$\alpha \rightarrow$ Thermal diffusivity

$$\alpha = \frac{K}{\rho C_p}$$

$K \rightarrow$ Thermal conductivity

$$\sigma = \frac{\alpha \Delta t}{(\Delta x)^2} \quad \text{(dimensionless)}$$

$$A^I = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\sigma & 1+2\sigma & -\sigma & 0 & \cdots \\ 0 & -\sigma & 1+2\sigma & -\sigma & \cdots \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

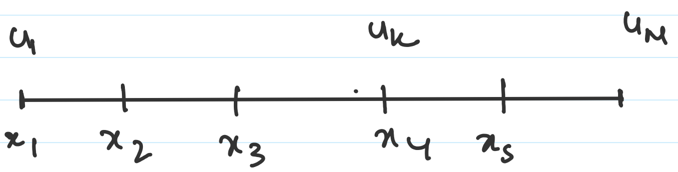
$$\Rightarrow u(x,t) = A^I u(x, t+\Delta t)$$

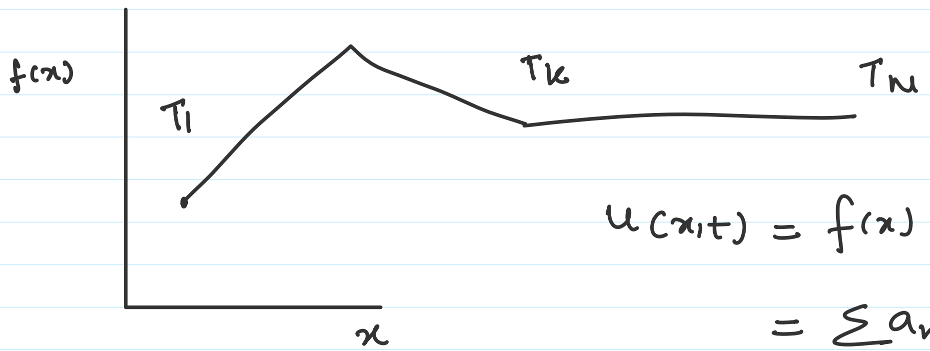
$$\Rightarrow \boxed{u(x, t+\Delta t) = (A^I)^{-1} u(x,t)}$$

or
$$u^k = (A^I)^{-1} u^{k-1}$$
$$= (A^I)^{-k} u^0$$

- Implicit scheme is unconditionally stable.

→ Unstability of Explicit scheme:-





$$u(x,t) = f(x) \rightarrow \text{random function}$$
$$= \sum_n a_n e^{ik_n x}$$

→ Von-Neumann stability analysis:-

PDE with periodic b.c. has general spacial solution

$$u_x = e^{ikx} \quad , \quad k \rightarrow \text{constant}$$

$$u(x,t) = u_x u_t$$

Assume: time evolution to be exponential

$$u_t = e^{at}$$

$$\Rightarrow u(x,t) = e^{ikx} e^{at}$$

$x \rightarrow m\Delta x$

$t \rightarrow n\Delta t$

$$u(m,n) = e^{ikm\Delta x} e^{an\Delta t}$$

$$u(m-1,n) = e^{ik(m-1)\Delta x} e^{an\Delta t}$$

$$u(m+1,n) = e^{k(m+1)\Delta x} e^{an\Delta t}$$

$$u(m,n+1) = e^{ikm\Delta x} e^{a(n+1)\Delta x}$$

Substituting these values in explicit form gives:-

$$u(m,n+1) = \sigma u(m+1,n) + (1-2\sigma) u(m,n) + \sigma u(m-1,n)$$

$$\Rightarrow \underbrace{e^{ikm\Delta x} e^{a(n+1)\Delta t}}_{u(m,n+1)} = \sigma \left[e^{ik(m+1)\Delta x} e^{an\Delta t} \right] +$$

$$(1-2\sigma)\left[e^{ikm\Delta x} e^{an\Delta t} \right] +$$

$u(m, n+1)$

$$= (1-2\sigma)\left[e^{ikm\Delta x} e^{an\Delta t}\right] + \sigma\left[e^{ik(m-1)\Delta x} e^{an\Delta t}\right]$$

$$= \underbrace{e^{an\Delta t} e^{ikm\Delta x}}_{u(m,n)}\left[\sigma\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - (2\sigma - 1)\right]$$

Amplitude factor, $G = \dfrac{u(m, n+1)}{u(m,n)}$

$$\Rightarrow \boxed{G = 1 - 2\sigma + 2\sigma\cos(k\Delta x)}$$

$|G| = 1 - 4\sigma$ for $k\Delta x = n\pi$; $n = \text{odd}$

So, one condition is :- $|G| < 1$ (as we need converging soln)

↳ i.e., the amplitude factor < 1

$$\Rightarrow -1 < (1 - 4\sigma) < 1$$

$$\Rightarrow -2 < -4\sigma < 0$$

$$\Rightarrow \boxed{\tfrac{1}{2} > \sigma > 0}$$

$$\boxed{0 < \sigma < 0.5}$$ $\Rightarrow$ the amplitude factor is not increasing and the soln is converging.

Similarly, for implicit scheme :-

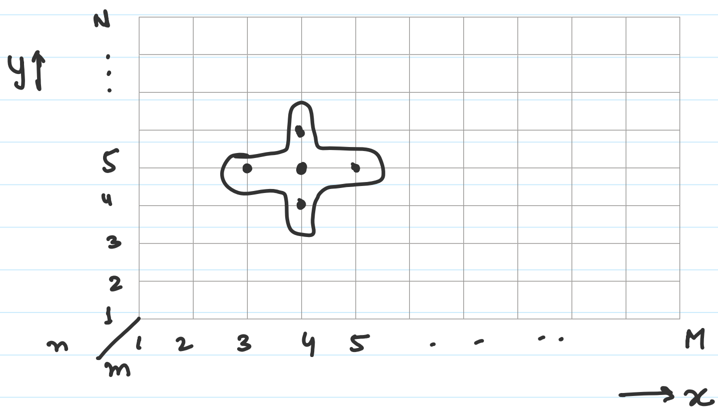
$$u(m,n) = -\sigma\, u(m+1, n+1) + (1+2\sigma)\, u(m, n+1) - \sigma\, u(m-1, n+1)$$

$$G = \frac{u(m, n+1)}{u(n, m)} = \frac{1}{1+2\sigma - 2\sigma \cos k\Delta x}$$

$$= \frac{1}{1+4\sigma} < 1 \Rightarrow$$ Implicit scheme is always stable.

→ Laplace Eqn :- (2D)

$$\frac{\partial^2 u(\overset{\to m}{x}, \overset{\to n}{y})}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$



$$\Rightarrow \frac{u(m+1, n) - 2u(m,n) + u(m-1,n)}{(\Delta x)^2} + \frac{u(m, n+1) - 2u(m,n) + u(m, n-1)}{(\Delta y)^2} = 0$$

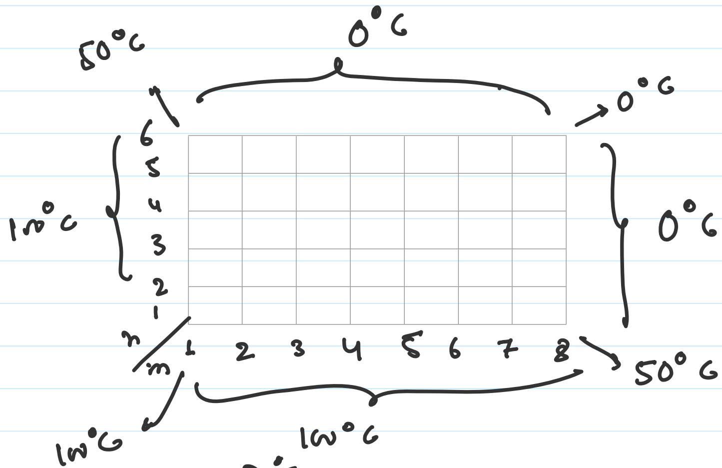
$$\Rightarrow u(m,n) = \frac{u(m-1, n) + u(m+1, n) + \beta^2 \left(u(m, n+1) + u(m, n-1)\right)}{2(1+\beta^2)}$$

$$\beta = \frac{\Delta x}{\Delta y}$$

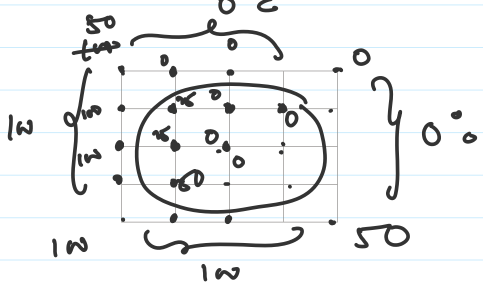
$\beta = \frac{\ }{\Delta y}$

for $\Delta x = \Delta y \quad (\beta = 1)$

$$u(m,n) = \frac{1}{4}\left[u(m+1,n) + u(m-1,n) + u(m,n+1) + u(m,n-1)\right]$$



Average temp. for corner points



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 25 & 0 & 0 \\ 25 & 0 & 0 \\ 50 & 25 & 25 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 31.25 & 6.25 & 0 \\ 43.75 & 12.5 & 6.25 \\ 62.5 & 43.75 & 31.25 \end{pmatrix}$$

difference, $D^{k+1}(m,n) = u^{k+1}(m,n) - u^{k}(m,n)$

user defined precission limit $\rightarrow \in$

If $\max\{|D(m,n)|\} \leq \in$

then stop.