The Ginzburg-Landau ~~freq~~ free energy density of the superconductor

$$f_s(T) = f_n(T) + \frac{1}{2m^*}\left|(-i\hbar\nabla - qA)\psi\right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4$$

where $-i\hbar\nabla \longrightarrow -i\hbar\nabla - qA$.

The total free energy is obtained by integrating this over the whole system and by including the electromagnetic field energy of the field $\vec{B}(\vec{r}) = \vec{\nabla}\times A(\vec{r})$ at each point $\vec{r}$.

Thus,

$$F_s(T) = F_n(T) + \int d^3r \left[\frac{1}{2m^*}\left|(-i\hbar\nabla - qA)\psi\right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4\right]$$

$$+ \frac{1}{\mu_0}\int B^2(\vec{r})\, d^3r.$$

(The first integral is carried out over points inside the samples, while the second is perfored over all space).

The condition for the minimum free energy state is obtained by performing a functional differentiation to minimize $F_s(T)$ with respect to $\psi(\vec{r})$ & $\psi^*(\vec{r})$.

It yields

$$\boxed{-\frac{1}{2m^*}(-i\hbar - qA)^2\psi(\vec{r}) + a\psi(\vec{r}) + b|\psi|^2\psi(\vec{r}) = 0}$$

~~where a and b depend~~ The supercurrent due to the magnetic field is given by

$$\vec{J}_s = -\frac{\partial F_s[\vec{A}]}{\partial \vec{A}(\vec{r})}, \text{ giving}$$

$$\boxed{\vec{J}_s = \frac{\hbar^2}{2m^*}\frac{q}{i\hbar}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{m^*}|\psi|^2\vec{A}}$$

The vector potential must be obtained from the magnetic field arising from both the supercurrents and any other currents, such as the external currents, $\vec{J}_{ext}$, in the solenoid coils

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_{ext} + \vec{J}_e \right)$$

as given by Maxwell's equations.

Gauge symmetry and symmetry breaking

The GL order parameter for a superconducting state can be written as

$$\psi(\vec{r}) = |\psi(\vec{r})| \, e^{i\theta(\vec{r})} .$$

Something very interesting happens when we consider the gauge invariance for this system.

If we make a gauge transformation of the magnetic vector potential $A(\vec{r}) \to A(\vec{r}) + \nabla\chi(\vec{r})$, then we must make a corresponding change in the phase of the order parameter θ.

Let's focus on the term containing $\vec{A}$ in the GL free energy density. If we change the phase of the order parameter by

$$\psi(\vec{r}) \longrightarrow \psi(\vec{r})\, e^{i\alpha(\vec{r})},$$

then we have

$$(-i\hbar\nabla - qA)\,\psi(\vec{r})\,e^{i\alpha(\vec{r})} = e^{i\alpha(\vec{r})}\left[-i\hbar\nabla - qA\right]\psi(\vec{r}) + \psi(\vec{r})e^{i\alpha(\vec{r})}\,\hbar\nabla\alpha(\vec{r}) .$$

$$= e^{i\alpha(\vec{r})}\left[-i\hbar\nabla - q\left(A - \frac{\hbar}{q}\nabla\alpha(\vec{r})\right)\right]\psi(\vec{r}) .$$

From this it follows that the free energy remains unchanged if we simultaneously change $\psi(\vec{r})$ to $\psi(\vec{r})\,e^{i\alpha(\vec{r})}$ and the vector potential according to

$$A(\vec{r}) \longrightarrow A(\vec{r}) - \frac{\hbar}{q}\nabla\alpha(\vec{r}) .$$

Thus demonstrating that ~~our~~ GL theory satisfies a local gauge invariance.

Both the phase of the order parameter and the vector potential depend on the choice of gauge, but all physical observables (free energy, magnetic field $\vec{B}$ etc.) are gauge invariant.

In the bulk region the superconductor has a ground state with a constant order parameter ψ.

Therefore it must have the same phase θ everywhere.

There must be a phase-stiffness, or an energy cost associated with changing θ from one part of the solid to another.

Now consider a superconducting state wherein the order parameter has a constant amplitude $|\psi|$ and a phase $\theta(\vec{r})$, which varies only slowly with position $\vec{r}$, then we can write the GL free energy as

$$F_s = F_s^0 + \rho_s \int d^3r \left[\vec{\nabla}\theta - \frac{q}{\hbar}\vec{A}\right]^2 .$$

Here the superfluid stiffness is defined by

$$\rho_s = \frac{\hbar^2}{2m^*} |\psi|^2$$

and F_s^0 is the total free energy in the ground state ($\theta = \text{constant}, \vec{A} = 0$).

Now if we fix a gauge for $\vec{A}(\vec{r})$ (say the Landau gauge $\vec{\nabla}\cdot\vec{A}=0$), then within this fixed gauge there is now a free energy cost associated with further gradients in $\theta(\vec{r})$.

To minimize the gradient energy, we must minimize the gradients, by making $\theta(\vec{r})$ as constant as possible throughout the system.

If $\vec{B}_{ext}=0$, then we can choose $\vec{A}=0$, and $\theta(\vec{r})$ will be constant everywhere in the system. Since the system effectively chooses an (arbitrary) constant order parameter everywhere in the system, we can say that the system ~~order parameter phase~~ exhibits long range order in the order parameter phase. Because the long range order is in the phase variable, we say that the system has spontaneously broken global gauge symmetry.

The point is that global gauge symmetry refers to change $\theta(\vec{r})$ by a constant amount everywhere in the whole solid (which does not require any change in $\vec{A}$). This is in contrast to local gauge symmetry in which $\theta(\vec{r})$ and $\vec{A}(\vec{r})$ are change simultaneously.

Recover London eqn. and Meissner-Ochsenfeld effect

$$F_s = F_s^0 + \rho_s \int d^3r \left(\vec{\nabla}\theta - \frac{q}{\hbar}\vec{A}\right)^2$$

$$\vec{j}_s = -\frac{\partial F_s[\vec{A}]}{\partial \vec{A}(\vec{r})} = -2\frac{q}{\hbar}\rho_s\left(\nabla\theta + \frac{q}{\hbar}\vec{A}\right).$$

Consider a ground state where θ is constant, we find that with a small constant external vector potential, the current is

$$\boxed{\vec{j}_s = -2\rho_s \frac{q^2}{\hbar^2}\vec{A}}.$$

This is exactly same as the London equation.
The (superfluid) density ρ_s is associated with the London superfluid density n_s.

Recall the London eqn. $\vec{j}_s = -\frac{n_s e^2}{m_e}\vec{A}$.

$$\vec{j}_s = -2|\psi|^2\frac{\hbar^2}{2m^*}\cdot\frac{(2e)^2}{\hbar^2}\vec{A} = -\frac{(2|\psi|^2)\,e^2}{(m^*/2)}\vec{A},$$

(convention)
London superfluid density $n_s = 2|\psi|^2$ and the GL effective mass is $m^* = 2m_e$ (where m_e is the bare electron mass).
Also, with this choice the eqn. can be interpreted physically as implying that $|\psi|^2$ is the density of pair of electrons in the ground state.

Therefore in comparison with the BCS theory of supercond. we can interpret $|\psi|^2$ with density of cooper pairs in the ground state and n_s as the density of electrons belonging to the cooper pairs.

The normal fraction, $n_n = n - n_s$ corresponds to the density of unpaired electrons.

The GL parameter m^* is the mass of the cooper pair which is naturally twice the original electron mass

So,
$$n_s = 2|\psi|^2 = \frac{2\dot{a}(T-T_c)}{b}.$$

Therefore, the London penetration depth $\lambda(T)$ is given by

$$\lambda(T) = \left[\frac{m_e b}{2\mu_0 e^2 \dot{a}(T_c - T)}\right]^{1/2}.$$

Observe that this will diverge at the critical temperature as $(T_c - T)^{-1/2}$.

The GL coherence length $\xi(T)$ also diverges with the same power of $(T_c - T)$ and so the dimensionless ratio
$$\kappa = \frac{\lambda(T)}{\xi(T)},$$
is independent of temperature within the GL theory.

Flux quantization

Consider a superconducting ring.
Choose cylindrical polar coordinates $\vec{r} = (r, \phi, z)$, with 3-axis perpendicular to the plane of the ring.

The order parameter $\psi(\vec{r})$ is periodic in the angle ϕ,

$$\psi(r, \phi, z) = \psi(r, \phi + 2\pi, z).$$

Assumption: The variation of $\psi(\vec{r})$ across the cross-section of the ring are unimportant, thus neglect the r and z dependence.

Therefore, the possible order parameter inside the superconduction is of the form.

$$\psi(\phi) = \psi_0 e^{in\phi},$$

where n is an integer and ψ_0 is a constant.
n can be interpreted as a winding number of the macroscopic wave function.

Assume — a magnetic flux Φ through the ring, then we can choose $\vec{A}$ to be in the tangential direction $\hat{e}_\phi$ with $A_\phi = \Phi/2\pi R$, where R is the radius of the ring.

$$\Phi = \int \vec{B} \cdot d\vec{s} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = \oint \vec{A} \cdot d\vec{r} = 2\pi R A_\phi .$$

The free energy corresponding to this wave function and vector potential is

$$F_s(T) = F_n(T) + \int d^3r \left[\frac{\hbar^2}{2m^*} \left| (i\nabla + \frac{q}{\hbar} i A)\psi \right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right] + E_B.$$

$$= F_s^0(T) + V \left[\frac{\hbar^2}{2m^*} \left| \frac{in}{R} + \frac{q i \Phi}{\hbar\, 2\pi R} \right|^2 |\psi|^2 \right] + \frac{1}{2\mu_0} \int B^2(r)\, d^3r.$$

Note that $\nabla \chi = \frac{\partial \chi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \chi}{\partial \phi} \hat{e}_\phi + \frac{\partial \chi}{\partial z} \hat{e}_z$.

V is the total volume of the superconducting ring and $F_s^0(T)$ is the ground state free energy of the ring in the absence of any currents and magnetic fluxes.

The vacuum magnetic field energy $E_B = \frac{1}{2\mu_0} \int B^2 d^3r$ can be expressed in terms of the inductance L of the ring and the current I.

$$E_B = \frac{1}{2} L I^2 \propto \Phi^2,$$

The free energy of the superconductor contains a term dependent on both the flux Φ and winding number n.

This term is $\frac{V\hbar^2}{2m^*} \cdot \frac{1}{R^2} \frac{q^2}{\hbar^2} \left| \frac{h}{q} n + \Phi \right|^2 = \frac{Vq^2}{2m^* R^2} |\Phi - n\Phi_0|$,

where the flux quanta is $\Phi_0 = \frac{h}{-q} = \frac{h}{2e} = 2.07 \times 10^{-15}$ Wb.

Therefore, the free energy is equal to the bulk free energy plus two additional terms depending only on the winding number n and the flux Φ.

The free energy of the superconducting ring is given by

$$F_s(T) = F_s^{bulk}(T) + const_1 (\Phi - n\Phi_0)^2 + const_2 \Phi^{\prime 2}.$$

The free energy is minimum whenever the flux through the loop obeys $\Phi = n\Phi_0$.

This is the phenomena of flux quantization in superconductors.