Hamiltonian → $H(\{\theta\}) = \frac{J}{2}\int d^2\vec{r}\,(\nabla\theta)^2$

Euler eq^n / saddle pt eq^n

↳ $$\frac{\delta H\{\theta\}}{\delta\theta(r)} = 0 \Rightarrow \nabla^2\theta(\vec{r}) = 0$$

↳ Its soln

① trivial soln

$\theta(r) = \text{constant}$

② Non-trivial soln:

↳ no NTS for scalar pot. (since no charges)

(didn't get??)
The vectors we're talking about, are phase $\angle$les. Although the form looks same to us.

Because of periodicity we have NTS

↳ NTS may be found out by the b.c

$$\oint(\vec{\nabla}\theta \cdot d\vec{l}) = 2\pi n$$

↳ so, what kind of θ satisfies this

& $\nabla^2\theta(r)$ ultimately this.

$$|\nabla\theta| = \frac{n}{r}, \quad \theta(r) = n\tan^{-1}(y/x)$$

$$\nabla\theta(r) = n\left[\frac{\hat{i}(-y/x^2)}{1+y^2/x^2} + \frac{\hat{j}(1/x)}{1+y^2/x^2}\right] = \frac{n}{r^2}[-\hat{i}y + \hat{j}x]$$

In radial coordinate →

$$\nabla\theta = \frac{n}{r}\left[-\hat{i}\sin\theta + \hat{j}\cos\theta\right]$$

$y = r\sin\theta$
$x = r\cos\theta$

$\boxed{\frac{n}{r}(-\sin\theta\cos\theta + \sin\theta\cos\theta) = 0}$
↳ Is this correct??

$$\oint_C (\vec{\nabla}\theta \cdot d\vec{l}) = \int_S (\nabla\times\nabla\theta)\cdot d\vec{S}$$

↳ from Navier-Stokes Thm.

~~$\nabla\times\nabla\theta = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & 0 \end{vmatrix}$~~

$$\nabla\times\nabla\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{y}{r^2} & \frac{x}{r^2} & 0 \end{vmatrix}$$

$$= \vec{i}\left(\overset{0}{\cancel{\partial_x \frac{x}{r^2}}} \;\; \frac{x}{r^2}\partial_y\right)$$

$$-\vec{j}\left(0 + \frac{y}{r^2}\partial_z\right)$$

$$+\vec{k}\left(\partial_x \frac{x}{r^2} + \frac{y}{r^2}\partial_y\right)$$

~~$\nabla\times\nabla\theta =$~~ $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix}$

$$= \vec{i}\left(\partial_y z - y\partial_z\right)$$

$$-\vec{j}\left(\partial_y z - x\partial_z\right)$$

$$+\vec{k}\left(\partial_x y - x\partial_y\right)$$

$$= \vec{i}(-x\partial_y)$$

$$+\vec{j}\, x\partial_z$$

$$+\vec{k}(-x\partial_y)$$

$$= -y\partial_z\vec{i}$$

$$+x\partial_z\vec{j}$$

$$-x\partial_y\vec{k}$$

~~$= x(-\partial_y\vec{i} + \partial_z\vec{j} - \partial_y\vec{k})$~~

$$\nabla\times\nabla\theta = \delta(\vec{r})\hat{k}$$

(from electrodynamics)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{y}{r^2} & \frac{x}{r^2} & 0 \end{vmatrix} = \nabla \times \nabla \theta$$

$$\partial_x\left(\frac{x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{1}{r^2} - \frac{x \cdot 2x}{(x^2+y^2)^2} + \frac{1}{r^2} - \frac{y \cdot 2y}{(x^2+y^2)^2}$$

$$\theta(\vec{r}) = n \tan^{-1}\left(\frac{y}{x}\right)$$

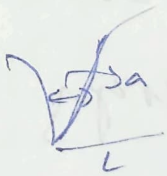
$$\nabla \times \nabla\theta = \delta(\vec{r})\hat{k}$$

$$|\nabla\theta| = \frac{n}{r} \rightarrow \text{vortex soln}$$

$$\vec{r} = (r, \varphi)$$

$$\theta(\vec{r}) = n\varphi$$

$$E = \frac{J}{2}\int d^2r \left(\frac{n}{r}\right)^2$$



This is a vortex soln

↳ for trivial soln: $= 0$

→ didn't get it

$L, a \rightarrow$ far from the vortex

$a \rightarrow$ arbitrarily small

$a << \xi$

$$E_{vortex}(L) = \frac{Jn^2}{2}\int_a^L dr \left(r/r^2\right)$$

L → system size

where did it go?

why r??

$$= \pi J n^2 \ln\left(\frac{L}{a}\right) \rightarrow a << \xi$$

how??

↳ divergence when we're close to vortex & also away from it.

What kind of excitations we need high energy??!

Single vortex energy is not favorable energetically

→ Vortex energy is divergent.
Single vortex is not possible.

But there's a possibility →

$\left.\begin{array}{l} (+1) \\ (-1) \end{array}\right\}$ diff top. charges

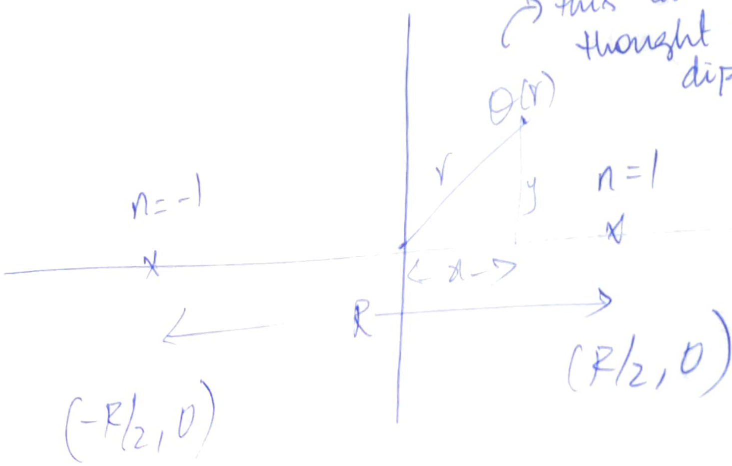
(+1)
$n = \cancel{+1} -1$
↳ is the $2\pi n$ from $\oint_C (\vec{\nabla}\theta \cdot d\vec{l})$

(−1)
$n = +1$

↳ In such a case, the total energy of a vortex

→ this can be thought of dipoles

so, they get cancelled.



We know phase for ~~n=1~~ these 2

$(0 \text{ to } 2\pi) \leftarrow \underbrace{\theta(r)}_{\text{vortex solns}} = \tan^{-1}\left(\dfrac{Ry}{\left(\frac{R}{2}\right)^2 - (x^2+y^2)}\right)$

If we consider a loop around $n=1,-1$ → we get solns ↗ & $E = 0$ → because for $n=-1$, phase $-2\pi n$ & $n=1$, phase $2\pi n$

$$E = \frac{J}{2}\int (d\vec{r}) |\nabla\theta|^2$$

2 for 2 vortex

$$E_{2D} = E_1 (J) \ln(R/a) + 2E_c$$

What's this?

core energy = nonuniversal part.

Each vortex has some finite core energy

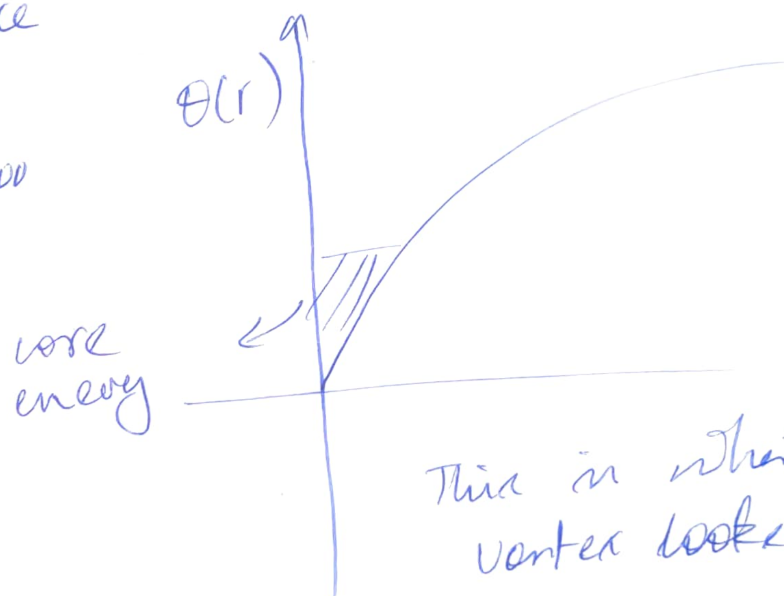
$-ve \to n=-1 \to$ Antivortex

$a \to$ min distance

$R \to$ max dist betn the two vortices.

$R < L$

since its the system's size



This is what vortex looks like

Free energy, $F = E - TS$

$$F = (\pi J - 2k_B T)\ln\left(\frac{L}{a}\right)$$

changes sign when we ↑ temp for $T = \frac{\pi J}{2k_B}$

$$\boxed{T_{KT} = \frac{\pi J}{2k_B}}$$

~~for $T < T_K$ $\Rightarrow$ free energy is +ve $F > 0 \Rightarrow$ we get vortex & anti-vortex pair~~ ~~They in~~

for $T < T_{KT} \rightarrow F < 0$

since $\ln(R/a)$ is -ve for $R < a$

↳ vortex-antivortex binding

for $T > T_{KT}$, $F > 0$

↳ proliferation of vortex and anti-vortex

⇒ They are not bounded anymore.

↓

(V+AV) They are widely spread.

And have diff phases

→ This gives resistive state.

This PT (KT PT)

(2D system)

(not possible for 3D ↓ since there's d^2r)

In these kind of PT, no order param in involved

We have non-zero n value for $T > T_{KT}$ & $R > a$ for certain loops

for $T < T_{KT}$

↳ $n = 0$

This is KT model valid for even dim.

They have roles for higher dim.

8th March 2024

$T \to 0$, $H(\theta) = \frac{J}{2}\int d^2\vec{r}\ (\nabla\theta)^2$

$K = \beta J \to \infty$

Bound state of a vortex and an anti-vortex

$T > T_{KT} = \frac{\pi J}{2K_B}$; unbinding of vortices happens

This is like the mean field theory kind of temp.

(This is PT)

↳ This is topological (no local order param is involved)

since what changes is the topological no.

from TN = 0 in B.S & TN ≠ 0 in unbinding case

$T \to \infty$, $K \to 0$

$C(\vec{r}) \sim e^{-r/\xi(K)}$

$\xi = \frac{1}{\ln(K/2)}$

↳ This is short range correlations

for large range correlations, this goes to zero.

~~$C(\vec{r}) =$~~

for $T < T_{KT} \to C(\vec{r}) = \langle \cos(\theta_r - \theta_0) \rangle$

$$C(\vec{r}) = \mathrm{Re}\,\frac{\int (D\theta)\, e^{-\beta H}\, e^{i(\theta_r - \theta_0)}}{\int (D\theta)\, e^{-\beta H}}$$

Convert Hamiltonian to fourier space

↳ $H = \frac{J}{2}\int d\vec{r}\, (\nabla\theta)^2$, $\theta(\vec{r}) = \int \frac{d\vec{k}}{(2\pi)^2}\, \theta_k\, e^{i\vec{k}\cdot\vec{r}}$

2 Dim so 2 square

$$H = \frac{J}{2}\int d\vec{r} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\vec{k}'}{(2\pi)^2}\, \theta_k \theta_{k'} (\nabla e^{i k\cdot \vec{r}})(\nabla)(e^{i\vec{k}'\cdot\vec{r}})$$

$$H = \frac{J}{2} \int d\vec{r} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\vec{k}'}{(2\pi)^2} \, \theta_k \theta_{k'} (\nabla e^{i k \cdot r}) \cdot (\nabla e^{i k' \cdot r})$$

$$\downarrow \qquad \downarrow$$

$$(i\vec{k} e^{i k \cdot r}) \cdot ((i\vec{k}') e^{i \vec{k}' \cdot \vec{r}})$$

$$= \frac{J}{2} \int \frac{d\vec{k}}{(2\pi)^2} \int \frac{d\vec{k}'}{(2\pi)^2} (2\pi)^2 \delta(k+k') (-\vec{k} \cdot \vec{k}') \, \theta_k \theta_{k'}$$

F.T

$$\int d\vec{r} \, e^{i\vec{k} \cdot \vec{r}} = (2\pi)^2 \delta(\vec{k})$$

$$\int \frac{d\vec{k}}{(2\pi)^2} e^{-i\vec{k} \cdot \vec{r}} = \delta(\vec{r})$$

↳ This $(2\pi)^2$ comes from the way we define k in QM

$k = \frac{2\pi}{L}$. By convention, we get, $(2\pi)^2$ to momentum

$$\frac{1}{V} \sum_{\vec{k}} = \int \frac{d\vec{k}}{(2\pi)^d}$$

$$= \frac{J}{2} \int \frac{d\vec{k}}{(2\pi)^2} \theta_{-\vec{k}} k^2 \theta_{\vec{k}}$$

$$C(\vec{r}) = \mathrm{Re} \int (D\theta_k) \, e^{-\beta \frac{J}{2} \int \frac{d\vec{k}}{(2\pi)^2} \theta_{-k} k^2 \theta_k} \, e^{i \int \frac{d\vec{k}}{(2\pi)^2} \theta_k (e^{ik \cdot r} - 1)}$$

$$\int (D\theta_k) \, e^{-\beta \frac{J}{2} \int \frac{d\vec{k}}{(2\pi)^2} \theta_{-k} k^2 \theta_k}$$

$\langle \theta_{-k} \theta_k \rangle_0 = ?$

$H = -\frac{J}{2} \int d\vec{r}\, \theta\, \nabla^2 \theta$ (how??)

$\langle \theta(r)\, \theta(r') \rangle =$ some C-F functn
$= f(r - r')$

↳ should satisfy →

$-\frac{J}{2} \nabla^2 f(r-r') = \delta^{(2)}(r-r')$ → Remember this

here ~~the~~ the operators are non-local → gradient

$\langle \theta_{-k}\, \theta_k \rangle_0 = \frac{1}{\left(\frac{\beta J}{2}\right) k^2}$ (how??) → from the 2-part in denominator

↓

here its local, we do not ~~need~~ need to know pts/values at some other coordinates to find K. Because we already have value of θ in K.

~~$e^{i \int \frac{d\vec{k}}{(2\pi)^2} \theta_k \left(e^{ik\cdot r} - e^{ik\cdot(0)}\right)}$~~

$e^{i \int \frac{d\vec{k}}{(2\pi)^2} \theta_k \left(e^{ik\cdot r} - 1\right)}$ → Ask??

↓

linear

While $e^{-\frac{\beta J}{2} \cdots}$

↳ is quadratic in k??

$$C(\vec{r}) = \langle \cos(\theta_r - \theta_0) \rangle$$

$$= \frac{\frac{1}{2}\int (D\theta) e^{-\beta H}\left[e^{i(\theta r - \theta_0)} + e^{-i(\theta r - \theta_0)}\right]}{\int D\theta \, e^{-\beta H}}$$

$$C(\vec{r}) = \mathrm{Re}\,\frac{1}{2} \frac{\int (D\theta_k)\left[e^{-\frac{\beta J}{2}\int \frac{d\vec{k}}{(2\pi)^2}\theta_{-k} k^2 \theta_k}\right]\left(e^{i\int \frac{d\vec{k}}{(2\pi)^2}\theta_k (e^{i\vec{k}\cdot r} - 1)} + e^{-i\int \frac{dk}{(2\pi)^2}\theta_k (e^{ik\cdot r} - 1)}\right)}{\int (D\theta_k) e^{-\frac{\beta J}{2}\int \frac{d\vec{k}}{(2\pi)^2}\theta_{-k} k^2 \theta_k}}$$

$e^{i\int d\vec{k}}$

$\hookrightarrow e^{\int_{-\alpha}^{\alpha} d\vec{k}}$ $\quad k' = -k$

$dk' = -dk$

$$-\int_{\alpha}^{-\alpha} dk' = \int_{-\alpha}^{\alpha} dk'$$

change k to $-k$

We have to add something to these 2 terms to get it of the form $\theta_{-k} k^2 \theta_k$

$$e^{-\frac{\beta J}{2}\int d\vec{k}\left\{\theta_{-k} + \frac{i(e^{ik\cdot r} - 1)}{k^2(-\frac{\beta J}{2})}\right\} k^2 \left\{\theta_k + \frac{i(e^{ik\cdot r} - 1)}{k^2(-\frac{\beta J}{2})}\right\}}$$

$$e^{-\frac{\beta J}{2}\int d\vec{k}\left\{\theta_{-k}+\frac{i(e^{ik\cdot r}-1)}{k^2(-\frac{\beta J}{2})}\right\}k^2\left\{\theta_k+\frac{i(e^{-ik\cdot r}-1)}{k^2(-\frac{\beta J}{2})}\right\}}$$

$$\times\, e^{-\frac{\beta J}{2}\int d\vec{k}\frac{(e^{i\vec{k}\cdot\vec{r}}-1)(e^{-i\vec{k}\cdot\vec{r}}-1)}{k^2}}$$

It's a gaussian integral shifted by some constant → so numerator & denom get cancelled.

$$c(\vec{r})=\frac{1}{2}e^{-\frac{\beta J}{2}\int\frac{d\vec{k}}{(2\pi)^2}\frac{(e^{i\vec{k}\cdot\vec{r}}-1)(e^{-i\vec{k}\cdot r}-1)}{k^2}}$$

$$=\frac{1}{2}e^{g(r)},\quad g(r)=\frac{-\beta J}{2}\int\frac{d\vec{k}}{(2\pi)^2}\left[\frac{2-2\cos(k\cdot r)}{k^2}\right]$$

$$g(r)=-\beta J\int\frac{d\vec{k}}{(2\pi)^2}\left(\frac{1-\cos(k\cdot r)}{k^2}\right)$$

This integral is not trivial

If we have a L size system, it → 0 to L but for k → 0 to $2\pi/L$ (not ∞)

because of this

$$\sim\frac{-\beta J}{2\pi}\ln\left(\frac{r}{L}\right)$$

$$1\quad \cancel{e^{ik\cdot r-ik\cdot r}}-e^{ik\cdot r}$$
$$-e^{-ik\cdot r}+1$$
$$=(1-e^{-ik\cdot r})+(1-e^{ik\cdot r})$$
$$=1-\cos(k\cdot r)+1-\cos(k\cdot r)$$
$$=2-2\cos(k\cdot r)$$

$$C(r) \sim e^{-\frac{\beta J}{2\pi}\ln(r/L)}$$

$$\sim (r/L)^{-\eta} \qquad \eta = \frac{\beta J}{2\pi}$$

So what we understand here?

↳ $T \to 0, K \to \infty$

$$C(\vec{r}) \sim (r/L)^{-\eta}, \quad \eta = \frac{\beta J}{2\pi} = \frac{K}{2\pi}$$

For $T \to \infty, K \to 0$

$$C(\vec{r}) \sim e^{-r/\xi(K)}, \quad \xi = \frac{1}{\ln(\frac{K}{2})}, \quad K = \beta J$$

If we are in low temp phase, the decay funct decays algebraically. So for small Temp, below T_{KT} → for $T < T_{KT}$,

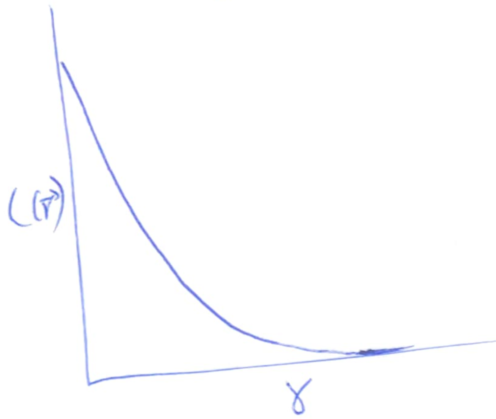
$$C(\vec{r}) \sim (r/L)^{-\eta}, \text{ decays}$$

in power law.

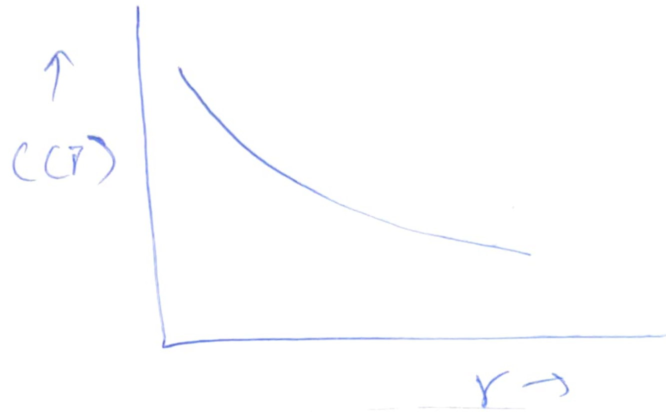
For $T > T_{KT}$, $C(\vec{r}) \sim e^{-r/\xi}$ decays exponentially

so there is much faster decay at high temp ⟵ & changes slowly for low temp.

$T > T_{KT}$



for $T > T_{KT}$



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normal vector rot symm $O(3)$ → this is broken for spin 1/2

$\downarrow$ spin-1/2

$SU(2)$ (3D spin)

$\hat{n} = \hat{i}\cos\theta + \hat{j}\sin\theta$

$\downarrow$ If we consider symm in plane

$U(1)$

XY model

because only by a phase we can describe

we can break this further

Z_2 symm — Ising Mode

↳ 2 possibilities for spins +1, -1

Prelude of Superconductivity

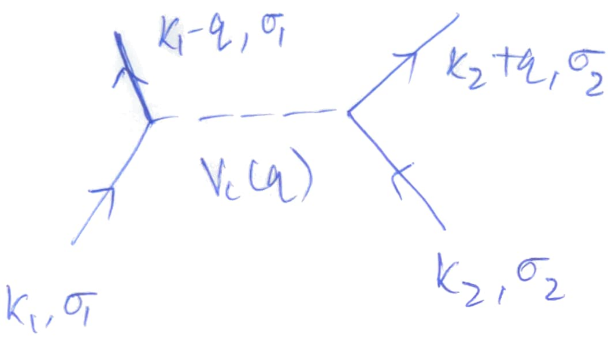
In which regime e^- have attractive ~~att~~ interact

coulomb interaction → changes only momentum but energy remains same

→ same for spins

↳ $V_{col} \neq$ [illegible]

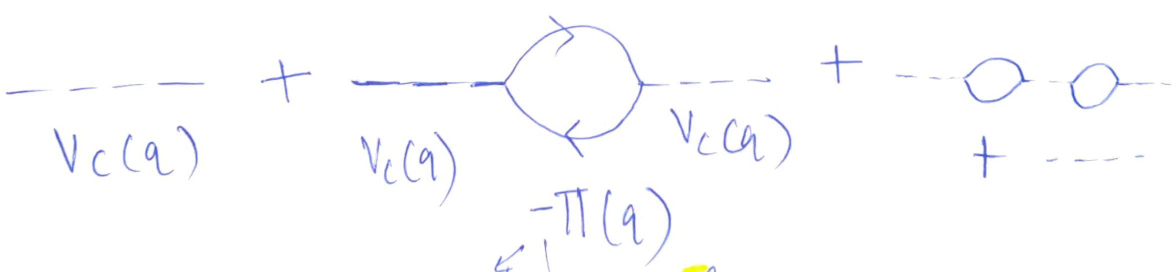
$$V_{col} = \frac{1}{2V} \sum_{k_1, k_2, q} \sum_{\sigma_1, \sigma_2} V_c(q)\, c^{\dagger}_{k_1-q,\sigma_1} c^{\dagger}_{k_2+q,\sigma_2} c_{k_2\sigma_2} c_{k_1\sigma_1}$$



$$V_c(q) = \frac{e^2}{\epsilon_0 q^2}$$

$$V_c(r) = \frac{e^2}{4\pi\epsilon_0 r}$$

$\Pi(q) \to$ fermioni grp



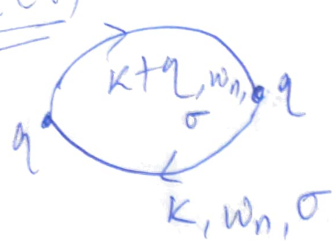
$\tilde{V}_c(q)$ = Summing over infinite series

$$= V_c(q) + V_c(q)\left(-\Pi(q)\right)V_c(q) + V_c(q)\left\{-\Pi(q)\right\}V_c(q)\left\{-\Pi(q)\right\}V_c(q) + \cdots$$

$$= \frac{V_c(q)}{1+\Pi(q)V_c(q)}$$

(Random Page Approx) → Check on google
???

$\Pi(q) \rightarrow$



fermions have freq. but it won't be affected in coulomb interaction.
Spin & freq are conserved.
$k \rightarrow$ can be anything ~~but~~

$$\sum_{k,\sigma} \sum_{\omega_n} G_\sigma^e(\omega_n, k)\, G_\sigma^e(\omega_n, k+q)$$

$\hookrightarrow$ electronic green's functn

$$\Pi(q)\ \frac{1}{\beta} \frac{1}{V} \sum_{k,\sigma} \sum_{\omega_n} G_\sigma^e(\omega_n, k)\, G_\sigma^e(\omega_n, k+q)$$

$\frac{1}{\beta} \rightarrow$ N.C for ω_n

$G_\sigma^e(\omega_n, k) \rightarrow \dfrac{1}{i\omega_n - \epsilon_k}$, $\quad G_\sigma^e(\omega_n, k+q) \rightarrow \dfrac{1}{i\omega_n - \epsilon_{k+q}}$

$\hookrightarrow$ independent of spins

$\Pi(q\to 0) \sim \rho(\epsilon_F) \rightarrow$ independent of q

$\hookrightarrow$ DOS at fermi energy of e^-

$$\tilde{V}_c = \frac{e^2}{\epsilon_0 (q^2 + k_s^2)}, \quad k_s = \sqrt{4\pi e^2 N(\epsilon_F)}$$

$\hookrightarrow$ when we do FT $\rightarrow$ the Colomb Int $\rightarrow$ have no longer $\frac{1}{r}$ depende.

Instead → $\tilde{V}_c(r) \approx \frac{1}{r} e^{-k_s r}$

⇒ suppression of CI at larger distances

↳ At smaller interactions, exp term can be neglected.

Phonon interaction

→ phonon dispersio

$$H_{phonon} = \sum_q \Omega_q \left(b^\dagger_{q\lambda} b_{q\lambda} + \frac{1}{2} \right)$$

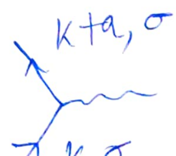
freq for some q

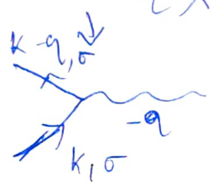
Vibrations can have different kinds of polarizations

λ → polarization phononic

$$H_{el-ph} = \sum_{k,\sigma} \sum_{q,\lambda} g_{q\lambda}\, c^\dagger_{k+q,\sigma}\, c_{k,\sigma} \left(b_{q\lambda} + b^\dagger_{-q\lambda} \right) \quad *$$

(absorbed)





(phonon emitted)

Bare ~~&~~ Phononic Green's function

↳ from $H_{ph} = \sum_{q,\lambda} \Omega_{q\lambda} \left(b^{\dagger}_{q\lambda} b_{q\lambda} + \frac{1}{2} \right)$

comparing it with

$H_e = \sum_{k,\sigma} \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma}$ → Its g.f ↳ is $\frac{1}{i\omega_n - \epsilon_k}$

to bring in the analogy

|||ly G.F for Hph → $\frac{1}{i\nu_n - \Omega_{q\lambda}}$

but considering Hel-ph → we see a similarity betn $b_{q\lambda}$ & $b^{\dagger}_{-q\lambda}$

↓

considering this ← $b_{q\lambda} + b^{\dagger}_{-q\lambda}$ term

the $\frac{1}{2}$ have a significance & it contributes to g.f

2?

$~~[b_q, b^{\dagger}_q]~~$

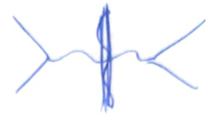
(Ex) ~~Use~~ commutation relations to find the 2nd term in $D^0_{q\lambda}(i\nu_n)$

∴, the G.F for Hph

$$D^0_{q\lambda}(i\nu_n) = \left[\frac{1}{i\nu_n - \Omega_{q\lambda}} - \frac{1}{i\nu_n + \Omega_{q\lambda}} \right]$$

$$= 2\Omega_{q\lambda} / ((i\nu_n)^2 - \Omega^2_{q\lambda})$$

$e^- - e^-$ comes from col & phononic too



→ phonon

vibrations & e^- hits this e-e forms

⟹ This results in 2nd order phononic interaction.

Together with coulomb & phonon → result in effectively attractive force.

<u>14th March 2024</u>

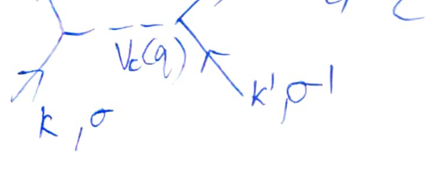
<u>Electron & Phonon Hamiltonian</u>

→ are bosons → Discrete freq = even multiple of $\frac{\pi}{\beta}$

$$H = H_{el} + H_{ph} + H_{el-ph}$$

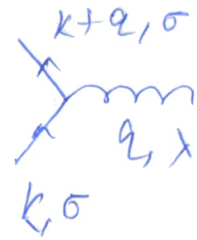
$$(b^\dagger b + 1/2)$$
$$= (b^\dagger b + b b^\dagger)$$
$$= 2b^\dagger b + 1$$
$$= 2(b^\dagger b + 1/2)$$

$$H_{el} = \sum_{k,\sigma} \epsilon_k C^\dagger_{k,\sigma} C_{k\sigma} + \frac{1}{2V} \sum_{\sigma,\sigma'} \sum_{k,k',q} V_c(q) C^\dagger_{k-q,\sigma} C^\dagger_{k'+q,\sigma'} C_{k',\sigma'} C_{k\sigma}$$



$$H_{ph} = \sum_{q,\lambda} \Omega_{q\lambda} \left(b^\dagger_{q\lambda} b_{q\lambda} + \frac{1}{2} \right)$$

$$H_{el-ph} = \frac{1}{V} \sum_{k\sigma} \sum_{q\lambda} g_{q\lambda} \, c^\dagger_{k+q,\sigma} \, c_{k\sigma} \left(b_{q\lambda} + b^\dagger_{-q,\lambda} \right)$$



Free-electron Green's functⁿ →

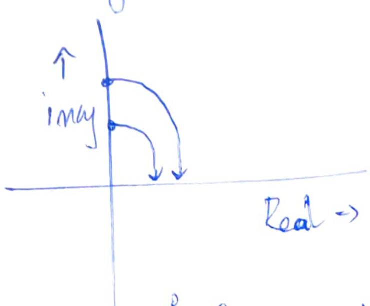
$$\omega_n = \frac{\pi}{\beta}(2n+1) \qquad G^0_k(\omega_n) = \frac{1}{i\omega_n - \xi_k}$$

Free-phonon Green's functⁿ →

$$D^0_{q\lambda}(\nu_m) = \left[\frac{1}{i\nu_m - \Omega_{q\lambda}} - \frac{1}{i\nu_m + \Omega_{q\lambda}} \right]$$

$$\nu_m = \frac{\pi}{\beta}(2m)$$

Analytical continuation



$$D^0_{q\lambda}(\nu) = \frac{1}{\nu - \Omega_{q\lambda} + i\eta} - \frac{1}{\nu + \Omega_{q\lambda} + i\eta}$$

$$= \frac{2\Omega_{q\lambda}}{\nu^2 - \Omega^2_{q\lambda} + i\delta}$$

$$i\omega_n \to \omega + i\eta$$

$$i\nu_m \to \nu + i\eta$$

$$G^0_k(\omega) = \frac{1}{\omega - \xi_k + i\eta}$$

Model for Phonons

Jellium model for ions

In principle, ions are discrete. ~~with some +ve bgd~~

But in Jellium → ions are discrete with some +ve bgd.

$\rho_+(\vec{r},t) \simeq \rho_+^0$ → constant of space & time

this density exactly cancel the e density

Jellium model → everything this model is +ve everywhere, here uniform density

$$\rho_+(\vec{r},t) \simeq \rho_+^0 + \delta\rho_+(\vec{r},t)$$

$$\nabla\cdot\vec{E} = 4\pi\,\delta\rho_+(\vec{r},t)$$

Gauss law

so what is happening is due to the spatial fluctuation & those are causing some E-Fields.

$$\nabla\cdot\vec{E} = 4\pi\,\delta\rho_+(\vec{r},t)$$

force density → $f = \rho_+\vec{E} \approx \rho_+^0\vec{E}$

This is a conservative system, so we have continuity eq^n → $\frac{\partial}{\partial t}\rho_+ + \nabla\cdot(\rho_+\vec{V}) = 0$

↳ $\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{J} = 0$ → current density

$$\frac{\partial}{\partial t}\delta\rho_+ + \rho_+^0\,\nabla\cdot\vec{V} = 0$$

$$\frac{\partial^2}{\partial t^2}\delta\rho_+ \simeq -\rho_+^0 \nabla\cdot \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{acceleration}}$$

$$= -\rho_+^0 \nabla\cdot \frac{f}{\rho_m}$$

$\nabla\cdot(\delta\rho_+)\vec{v} \rightarrow$ not taken because its also very small.

matter density $= \frac{N \times M}{V}$

charge den $= \frac{N \times e}{V}$

$\boxed{M.D = \frac{M}{e} \times C.D}$

$\rho_+ \rightarrow$ charge density

matter density times e = matter mass

$\hookrightarrow$ Ask?? (check online)

$$\frac{\partial^2}{\partial t^2}\delta\rho_+ \simeq -\frac{Ze}{M}\nabla\cdot f$$

~~$\rho_m = \frac{M}{\text{charge}}$~~

$$\frac{\rho_+^0}{\rho_m} \approx \frac{Ze}{M}$$

$$\nabla\cdot f = \rho_+^0 \nabla\cdot\vec{E}$$

$$= 4\pi\rho_+^0\delta\rho_+$$

$$\frac{\partial^2}{\partial t^2}\delta\rho_+ = -\left(\frac{Ze}{M}4\pi\rho_+^0\right)\delta\rho_+$$

$\hookrightarrow$ the density has SHO so the freq of vibration Ω

Vibration freq $\quad \Omega = \sqrt{\frac{4\pi Ze\rho_+^0}{M}}$ $\hookrightarrow$ ionic charge density

$\rho_+^0 = Ze\,n_e$ $\rightarrow$ because e charge is 1 and n_e no. of e^-

$$= \sqrt{\frac{4\pi(Ze)^2 n_e}{M}}$$

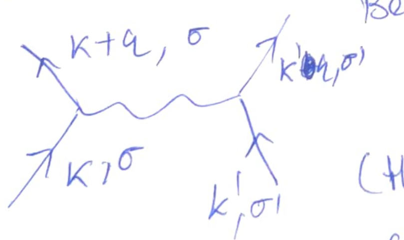
$M \rightarrow$ ionic mass

Ω (is independent of momentum)
↳ so its like optical phonons

$\Omega_{q\lambda}$ → polarizations come into picture since the phonons can vibrate in diff directions.

In general, $\Omega_{q\lambda}$

In el-ph → interaction can induce el-el interaction in 2nd order.



Before the phonon can decay, it reacts with another e to gain momentum to itself.
(this decay process is not considered in our calculations)

|||

this is equivalent to similar kind of coulomb int. (Ask??) but its strength in CI is $V_c(q)$)

$$\rightarrow g_{q\lambda} g_{-q\lambda} D^0_{q\lambda}$$
$$\simeq V^{ph}_{e-e}$$
(e-e int. coming from phonon)

↳ $g_{q\lambda}$ & $g_{-q\lambda}$

→ only exp is written.

(Derivation not done) ← $g_{q\lambda} g_{-q\lambda} = |g_{q\lambda}|^2 = \Omega_{q\lambda}(V_c(q))$

$\therefore$, the effective interaction will be sum of
CI & PI $\rightarrow$

$$V_{eff}(q, \nu_m) = \underbrace{V_c(q)}_{\text{CI term}} + \underbrace{V_{ph}(q, \nu_m)}_{\text{old term}}$$

(Ask??)

$$= \cancel{V_c(q)\left[1+\right.}$$

$$= V_c(q)\left[1 + \frac{\Omega^2_{q\lambda}}{V_c(q)\left(1 + \frac{\Omega^2_{q\lambda}}{(i\nu_m)^2 - \Omega^2_{q\lambda}}\right]}\right.$$

(Ask??)

$$V_{eff}(q, \nu_m \rightarrow \nu + \eta)$$

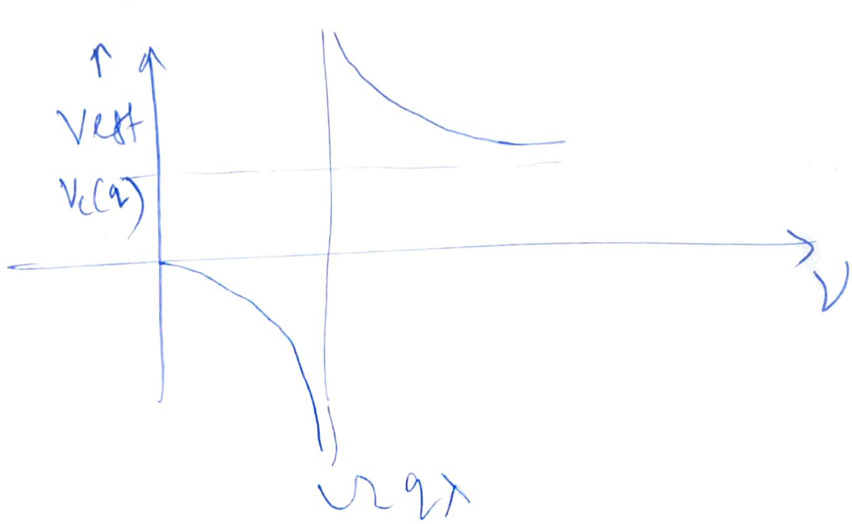
$$= V_c(q)\left[1 + \frac{\Omega^2_{q\lambda}}{\nu^2 - \Omega^2_\lambda + i\delta\,\mathrm{sgn}(\nu)}\right]$$

(Ask??)

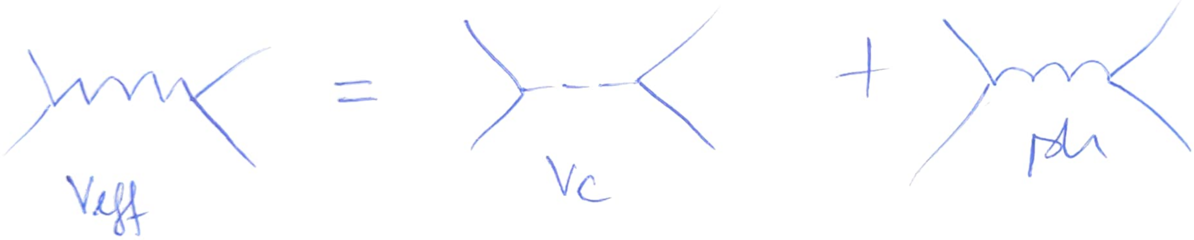
$$V_{eff}(q, \nu_m \rightarrow \nu + i\eta) = V_c(q)\left[1 + \frac{\Omega^2_{q\lambda}}{\nu^2 - \Omega^2_{q\lambda} + i\delta\,\mathrm{sgn}(k)}\right.$$

$$V_{eff}(q, \nu_m \to \nu + i\eta) = V_c(q)\left[1 + \frac{\Omega_{q\lambda}^2}{\nu^2 - \Omega_{q\lambda}^2} \cancel{+ i\delta} + i\epsilon\, \mathrm{sg}(\nu)\right]$$

$$V_{eff}(q, \nu_m \to \nu + i\eta) = V_c(q)\left[1 + \frac{\Omega_{q\lambda}^2}{\nu^2 - \Omega_{q\lambda}^2 + i\delta\, \mathrm{sg}(\nu)}\right]$$

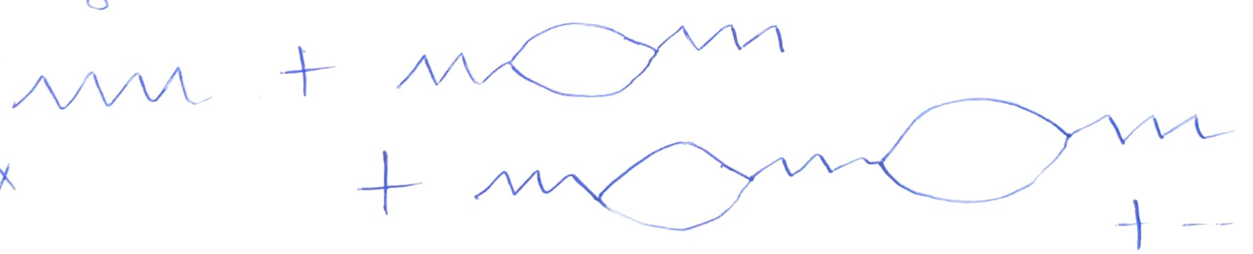


↓ Lowest order



higher-order →

Random Phase Approx



↓

Equivalent to large N Approx

↳ In this case, we can't consider

(why?)

Is it renormalize?

$$V_{eff}^{RPA} = V_{eff}(q, \nu_m) - V_{eff}(q, \nu_m)\,\Pi_0(q, \nu_m)\,V_{eff}(q, \nu_m) + \cdots$$