$$V_{eff}^{RPA} = \frac{V_{eff}(q, \nu_m)}{1 + \Pi_0(q,\nu_m)\, V_{eff}(q, \nu_m)}$$ → m ↔ n? → its m

$$V_{eff}(q, \nu_m) = V_c(q)\, \frac{(i\nu_m)^2}{(i\nu_m)^2 - \Omega_{q\lambda}^2}$$

$$V_{eff}^{RPA} = \frac{V_c(q)\left(\frac{(i\nu_m)^2}{(i\nu_m)^2 - \Omega_{q\lambda}^2}\right)}{1 + V_c(q)\frac{(i\nu_m)^2}{(i\nu_m)^2 - \Omega_{q\lambda}^2}\, \Pi_0(q, i\nu_m)}$$

$$= \underbrace{\frac{V_c(q)}{1 + V_c(q)\Pi_0(q, i\nu_m)}} \cdot \frac{(i\nu_m)^2}{(i\nu_m)^2 - \left(\frac{\Omega_{q\lambda}^2}{1 + V_c \Pi_0}\right)}$$

This can be considered as V_c^{RPA} for coulomb part (how?? → ~~physically~~?
↳ screens the coulomb pot.

→ this is renormalized due to $\frac{1}{1+V_c \Pi_0}$ factor

$$= V_c^{RPA} \frac{(i\nu_m)^2}{(i\nu_m)^2 - \omega_{q\lambda}^2(i\nu_m)} \quad \omega_{q\lambda}(\nu_m) = \frac{\Omega_{q\lambda}}{\sqrt{1 + V_c \Pi_0(q, \nu_m)}}$$

For small q →

$\Pi_0 \simeq \rho(\epsilon_F)$, DOS at fermi energy

$$\omega_{q\lambda} \simeq \frac{\Omega_\lambda}{\sqrt{1 + \frac{4\pi e^2}{q^2}\rho(\epsilon_F)}}$$

independent of q (Even if we do approx on Ω → without considering q, the q is introduced in denominator)

$\bar{e}$ at FS ↳ conduct electricity

used here

↳ screened ⇒ value is since the denominator is a +ve quantity and "q" dependence is introduced in the denominator despite making an assumption of Ω being q-independent.

In $V_{eff}^{RPA} = V_C^{RPA} \times \frac{(i\nu_m)^2}{(i\nu_m)^2 - \omega_{q\lambda}^2(i\nu_m)}$

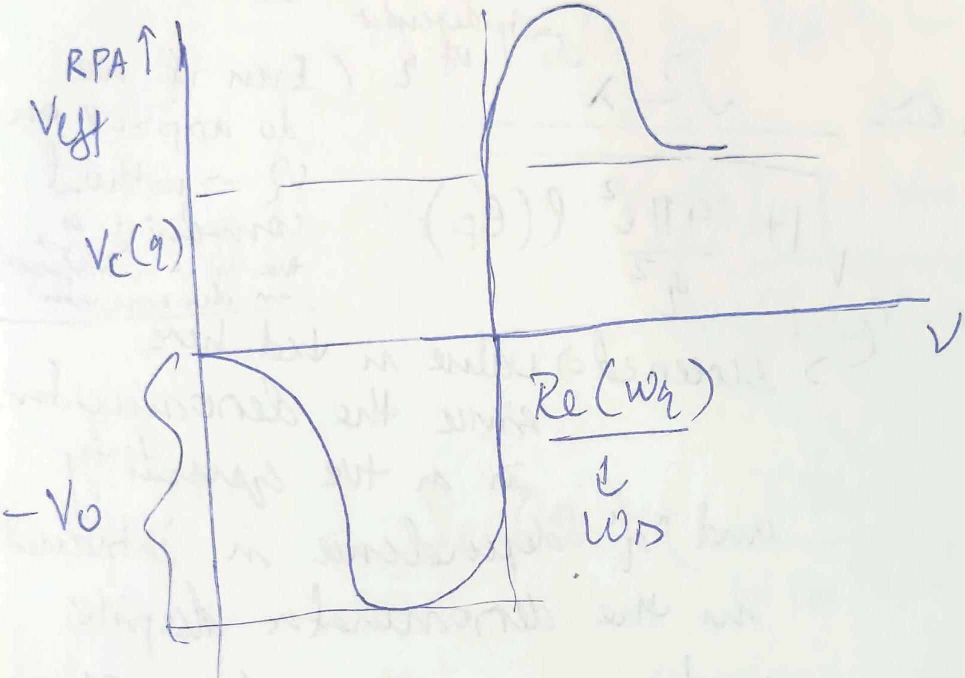
When we take the real part of V_{eff}^{RPA}

its ~~real~~ imag part also contributes at slightly above the small q regime → This results in smearing ~~the~~ out the divergence

a sizable amt of imag is converted as real in the w_{tin} form.

Debye freq → is relatively higher (ω_D)

In the slightly above the small q regime, we have attractive potential.



To understand the actual SC, you do not need to bother about freq.

In very crud approx →

$$V_{eff}^{RPA} = \begin{cases} -V_0 & \text{for } |\omega_n|, |\omega_n| < \omega_D \\ 0 & \text{otherwise} \end{cases}$$

~~beyond~~ ~~($|\omega_n| < \omega_D$)~~ ??

$\omega_D \to$ higher than the ?? in SC

the higher energy cut-off is from the ω_D

BCS theory: weak coupling S.C

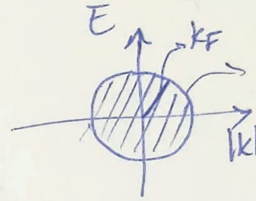
→ Show that → there's some sort of instability in C repulsion (tomorrow's discussion)

15th March 2024

cooper instability

$$V_{eff}^{RPA} \simeq \begin{cases} -V_0 & \text{for } |\omega_n| \ \& \ |\omega_n'| < \omega_D \\ 0 & \text{otherwise} \end{cases}$$

Without phonon interactions ⇒ electrons have fermi surface

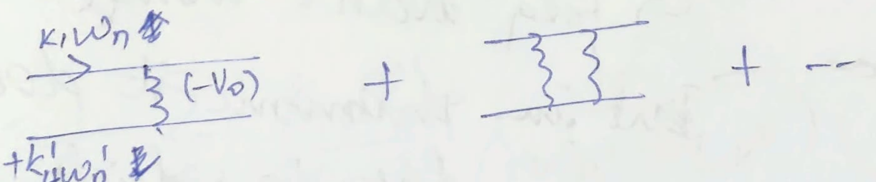


→ beyond this, electrons are not there

→ In this condition, fermi surface gets destroyed due to attractive interaction regime.

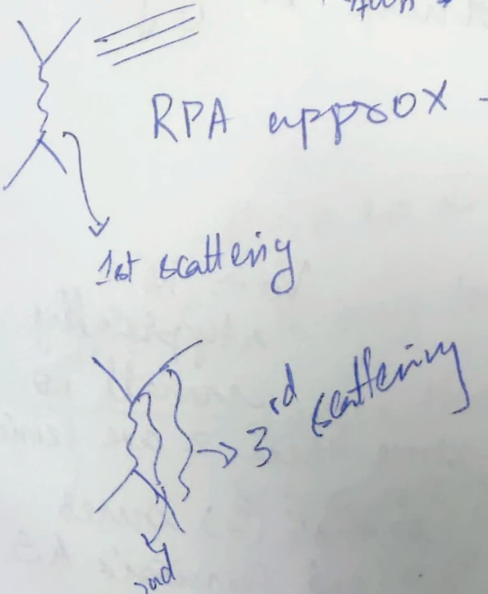
Consider scattering betn 2 e^- →

(ladder diagrams)

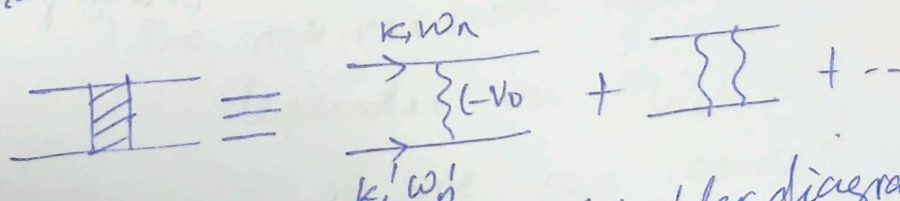


≡

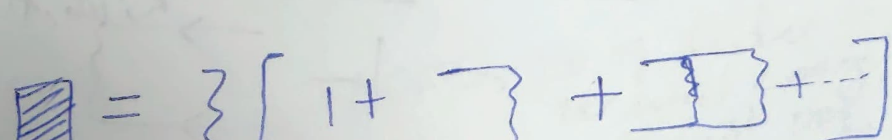
1st scattering



RPA approx → involve several fermion loops



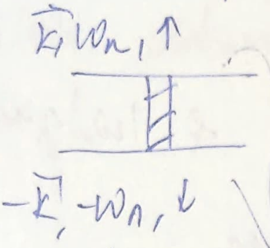
(ladder diagrams)



(vertex correction in effective directions)

given by

Now we're specifying things →

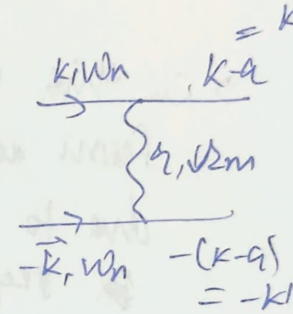


↓

(can [illegible])

→ special case ≡ for this case only, instability occurs for the $2e^-$ we're considering have opp momentum & spin

→ spins remain same.
k & ω → change and become opp for up & down
→ whatever gained by one will be lost by another.



In coulomb interaction
↳ freq doesn't change

~~[illegible]~~

But in phononic it does change due to which energy also change.

$$\boxed{\text{▨}} = \cfrac{-V_0}{1 - \underset{-k', -\omega_n', \downarrow}{\overset{k', \omega_n, \uparrow}{\bigcirc}}}$$

→ since these 2 are fermions so this (−) comes and fermion's A.S

V_0 → is atypically small no.

$$\Lambda = \frac{-V_0}{1 - \frac{(-1)}{\beta} \sum_{\omega_n'} \int \frac{dk'}{(2\pi)^3} G^{\uparrow}_{k'}(\omega_n') G^{\downarrow}_{-k'}(-\omega_n') (-V_0)}$$

Instability occurs $(\Lambda \to \infty)$

$$\frac{1}{\beta}\sum_{\omega_{n'}}\int\frac{d\vec{k}}{(2\pi)^3}\, G^{\uparrow}_{k'}(i\omega_{n'})\, G^{\downarrow}_{-k'}(-i\omega_{n'}) = +\frac{1}{V_0}$$

$\Rightarrow$ Fermi surface is unstable

$V_0 \to -ve$ then there's a regime where FS is stable?

$\hookrightarrow$ Proving this

$$\frac{1}{\beta}\sum_{\omega_{n'}}\int\frac{dk'}{(2\pi)^3}\frac{1}{i\omega_{n'}-\xi_{k'}}\cdot\frac{1}{-i\omega_{n'}-\xi_{-k'}}$$

$$\xi_{k'} = \frac{\hbar^2 k'^2}{2m} - \omega_F = \xi_{-k'}$$

$\hookrightarrow$ why opp sign $\to k^2$??

$$= -\frac{1}{\beta}\sum_{\omega_{n'}}\int\frac{d\vec{k}'}{(2\pi)^3}\frac{1}{(i\omega_{n'}-\xi_{k'})(i\omega_{n'}+\xi_{k'})}$$

$\hookrightarrow$ The momentum integral depends on $\xi_{k'}$

$\Rightarrow$ so we can convert it to $\xi_{k'}$

$k' \to 0$ to ∞

valid?? $\swarrow$ ω_F is very large

$\to$ larger than Debye energy

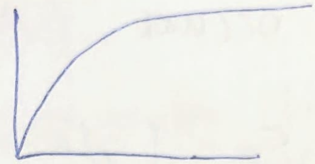
DOF $\to$ no. of states per unit energy per unit vol.

(1) (Approx)

$$= -\frac{1}{\beta}\sum_{\omega_{n'}}\int_{-\infty}^{\infty} d\xi_{k'}\, D(\omega_F)\,\frac{1}{\xi_{k'}^2+\omega_{n'}^2}$$

$\hookrightarrow$ DOS at ω_F since there's no angular dependency so no angular integral

(why??) then ang integral is also considered $\leftarrow$ If there had be ang. depen

In principle → (i) DOS →  where we consider all DOS but here we're doing approx.

$$= \cancel{\tfrac{1}{\beta}} \int \frac{d\omega_{k'}}{\omega_{k'}^2 + \omega_{n'}^2} = \cancel{\tan} \; \frac{\pi}{|\omega_{n'}|}$$

→ mod since $\omega_{n'}$ can be +ve or -ve so we're putting mod

$$= \frac{1}{\beta} \sum_{n'} \int_{-\infty}^{\infty} d\varepsilon_{k'} \, D(\omega_F) \frac{1}{\omega_{k'}^2 + \omega_{n'}^2}$$

DOS at ω_F

$$\Rightarrow D(\omega_F) \frac{\pi}{\beta} \sum_{\omega_{n'}} \frac{1}{|\omega_{n'}|}$$

⇒ Instability to occur

$$\pi V_0 D(\omega_F) \frac{1}{\beta} \sum_{\omega_{n'}} \frac{1}{|\omega_{n'}|} = 1$$

ω_n → freq of fermions $= \frac{\pi}{\beta}(2n+1)$

= fermionic discrete freq

$$\pi V_0 D(\omega_F) \frac{1}{\beta} \sum_{n'} \frac{1}{\frac{2\pi}{\beta} |n' + \frac{1}{2}|} = 1$$

$|n' + \frac{1}{2}| \to$ symmetric & $\omega_{n'} \to$ can be from ∞ to $-\infty$ but due to cut-off freq, i.e, $|\omega_n| < \omega_D$ $\to$ we get cut-off freq.

$$\to V_0 D(\omega_F) \sum_{n' \geq 0}^{\beta\omega_D/2\pi} \frac{1}{n + 1/2} = 1$$

$\downarrow$ arbitrarily small no.

$$V_0 D(\omega_F) \sum_{n' \geq 0}^{\frac{\beta\omega_D}{2\pi}} \frac{1}{n + 1/2} = 1$$

$$\omega_D = \frac{\pi}{\beta} (2n_{max} + \underbrace{1}_{\downarrow})$$

neglected since $2n_{max} >> 1$

$$\omega_D = \frac{\pi}{\beta} 2n_{max}$$

$\hookrightarrow$ final result:

$$V_0 D(\omega_F) \left[\underbrace{\gamma}_{\hookrightarrow \text{Euler constant}} + \underbrace{\ln \left(4 \frac{\beta\omega_D}{2\pi} \right)}_{\hookrightarrow \text{correction term}} \right] = 1$$

correction term since we're not going upto ∞.

$$V_0 D(\omega_F) \ln \left(4 \frac{e^{\gamma} \beta\omega_D}{2\pi} \right) = 1$$

$$V_0 D(\omega_F) \ln \left(\frac{2e^{\gamma} \beta\omega_D}{\pi} \right) = 1 \quad \Big| \quad \beta = \frac{1}{k_B T}$$

$\hookrightarrow$ put $\beta_c = \frac{1}{k_B T_c}$ inorder to see up to which temp this holds.

$$\ln \left(\frac{2e^{\gamma} \omega_D}{\pi k_B T_c} \right) = \frac{1}{V_0 D(\omega_F)}$$

$$\boxed{k_B T_c = \frac{2 \cancel{\gamma} e^{\gamma}}{\pi} \omega_D \, e^{-1/V_0 D(\epsilon_F)}}$$

i

Vo shows this means T_c changes with V_0???

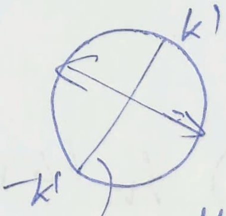
↳ so what we have found is a temp which is consistent with +ve value of V_0 (and however small it can be)

With V_0 becoming smaller & smaller, the temp T_c also becomes smaller.

So, then we find a Temp (T_c) where the FS becomes unstable for the case where e-e attraction inter. occurs/regime.

Upto a few kelvins, the FS is unstable.

Lastly cooper showed FS is unstable, so now what happens →

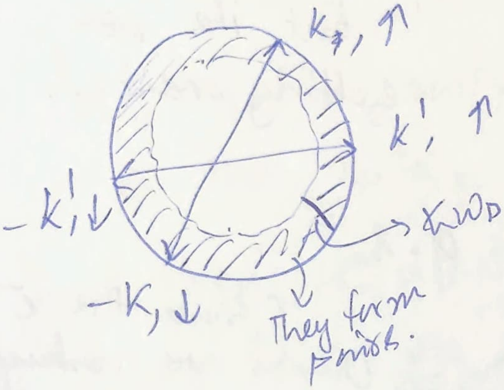


exactly opp. momen ⇒ so this is done very coherently. (Ask??)

Next class: PI formulation of this theory

This class includes discussion of BCS theory → physical intuition.

FS is unstable below certain temp if the effective interaction is attractive (at least up to a certain energy of e^-)



$\Rightarrow$ Since there's opp. mom. this implies there's a pair formation since this mom. pair is intact.

And also, all e^- can form their respective pairs but only with particular e^- which have opp mom. ~~e^- on F.S cannot form pairs.~~

Only e^- in the range of ω_D can form pairs. Deep inside FS cannot form pairs

e^- that lie near the FS conduct e^- since these are the e^- that can be excited quickly.

e^- in the scale of $\omega_D \ll \omega_F$

** Condition: e^- with opp. spin & momenta form pairs when ~~they are close~~ their energies are close to ω_F within the window of $\hbar\omega_D$

Ground state in case of FS $\rightarrow$ $\langle c_k^\dagger c_k \rangle \neq 0$

 $\rightarrow$ FS g.s $\nearrow$ (since for every mom. there's an electron)

$\langle C_K C_{-K} \rangle_{FS} = 0$ $\quad$ $n_{K\uparrow}$ = no. op for $K\uparrow$ $\leftarrow$ $\langle C^\dagger_{K\uparrow} C_{K\uparrow} \rangle_{FS} \neq 0$

$\langle C_{K,\uparrow} C_{-K\downarrow} \rangle_{FS} = 0 \rightarrow$ why?

↳ $C_{-K\downarrow}$ → destroys but its not getting created.

↳ This is opp for Cooper instability

G.S in C.I.S ≡ BCS g.s

This state is called S.C state $\leftarrow$ $\langle C_{K\uparrow} C_{-K\downarrow} \rangle_{BCS} \neq 0$ (since the $\bar{e}$ are making pairs with opp. mom & opp spin)

$\langle C^\dagger_{K\uparrow} C_{K\downarrow} \rangle_{BCS} = 0$

↳ (no. op is not conserved) → v. N. Trivial result.

Within this understanding,
WF^{n}s were written.

WF^n for fermi gas of e^- (with FS)

$$|FS\rangle = \prod_{k,\sigma} C^\dagger_{k\sigma} C_{k\sigma} |0\rangle$$

(in momentum space)

With this analogy → BCS came up with the WF^n for BCS state

(this is not very simple)

$$|BCS\rangle = \prod_K (u_K + v_K C^\dagger_{K\uparrow} C^\dagger_{-K\downarrow}) |0\rangle$$

This is a variational WF^n since there are 2 params u_K & V_K → these are prob amp

$$|u_K|^2 + |V_K|^2 = 1$$

V_K^2 → prob of creating a pair for k-state

u_K^2 → prob of not creating a pair for k-state.

so, what is unknown is the theta$_K$ param

and then what they did is
→ you have the complete hamiltonian & then the exp val of BCS state and minize this exp val to find u_K

$$\langle BCS | H | BCS \rangle = E_{BCS}$$ → minimize it to find u_K
↓
once we find this, we can write the wf^n.

But what is this Hamiltonian? for S.C
↳ the basic form

→ so the minimal Hamiltonian for SC is H

~~H (in momentum space)~~

$$\cancel{H = \int d\vec{r}\, \psi^\dagger(r) \left[\left(\frac{-\hbar^2 \nabla^2}{2m} - e\vec{A} \right)^2 \right.}$$
~~(in coordinate space)~~

for temp=0 → its $\epsilon_F = \mu$

whatever $-V_0$ was, we have g for that (but there's diff since g is in c-space & V_0 in k-space

$$H_{(\text{in c-space})} = \sum_\sigma \int d\vec{r}\, \psi_\sigma^\dagger(r) \left[\frac{(\vec{P} - e\vec{A})^2}{2m} - \mu \right] \psi_\sigma$$
$$+ \int d\vec{r}\, (-g)\, \psi_\uparrow^\dagger(r) \psi_\downarrow^\dagger(r) \psi_\downarrow(r) \psi_\uparrow(r)$$

$-g$ ↓ attractive pot.

μ ↓ chemical Pot since we're doing at finite temp

$$H = \sum_\sigma \int d\vec{r}\, \psi_\sigma^\dagger(r) \left[\frac{(\vec{p} - e\vec{A})^2}{2m} - \mu \right] \psi_\sigma$$

$$+ \int d\vec{r}\, (-g)\, \psi_\uparrow^\dagger(\vec{r}) \psi_\downarrow^\dagger(\vec{r}) \psi_\downarrow(r) \psi_\uparrow(r)$$

$\Rightarrow$ The e^- are at the same pt in physical space but in k-space, they are far apart.

Since the pot. are contact so that's why they're not far apart when their relative diff is zero $\Rightarrow$ so same pt.

5^{th}, 21^{st} $\to$ Revision class

22, 28 $\to$ class

& 4^{th} class

11^{th} holiday, 29^{th}

10^{th} $\to$ project submission

12^{th} $\to$ presentation.

End-sem $\to$ derivations

21st March 2024 (Doubt Clearing Session)

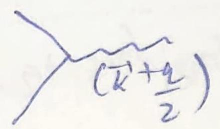
$$H = \sum_k \epsilon_k c_k^\dagger c_k + \frac{e}{2m} \sum \vec{A}_q \left(\vec{k} + \frac{\vec{q}}{2}\right) c_{k+q}^\dagger c_k$$

$$+ \frac{e^2}{2m} \sum_q \vec{A}_q \vec{A}_{-q}$$

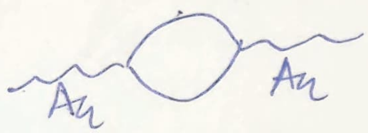
$$G_k^0 = \frac{1}{i\omega - \epsilon_k} = \to$$

conductivity

$$\vec{j} = \sigma \vec{E} = -\sigma \frac{\partial \vec{A}}{\partial t} \quad \Big| \quad \vec{j}_q = i\omega\sigma \vec{A}_q \quad \Big| \quad \vec{j}_q = \frac{\partial H}{\partial A_q}$$



(Ask for these) diagrams from Pascal



$\hookrightarrow$ Hamiltonian is linear

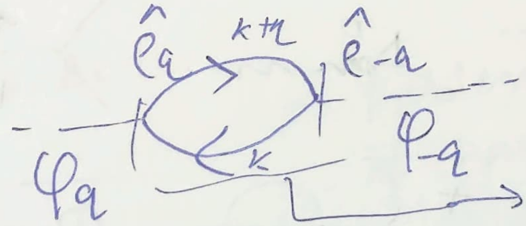
$$H = \sum_k \epsilon_k c_k^+ c_k + \sum_q \varphi_{-q} \hat{\rho}_q , \quad \hat{\rho}_q = \sum_k c_{k+q}^+ c_k$$

$\hookrightarrow$ std density op.

Scalar potential φ_q

$\hookrightarrow$ we have to calculate density-density CF ($\Pi_0(q)$)

$$H_{eff} = \frac{1}{2} \sum_q \varphi_q \, \Pi_0(q) \, \varphi_{-q}$$



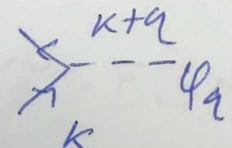
$\rightarrow \Pi_0 = \int d\vec{k} \, G^0_{k+q} \, G^0_k$

$$\Pi_0(q) = \langle \hat{\rho}_q \hat{\rho}_{-q} \rangle = \sum_{k_1,k_2} \langle c_{k_1+q}^+ c_{k_1} c_{k_2-q}^+ c_{k_2} \rangle$$

Linear response theory $\rightarrow$ upto quadratic terms

$\hookrightarrow$ this is for small perturbation.

$$\sum_q \varphi_{-q} \hat{\rho}_q \rightarrow$$



(interaction vertex)

$$(c_{k_1+q}^+ c_{k_1})(c_{k_2-q}^+ c_{k_2})$$

$\langle \rho_q \rangle \langle \rho_{-q} \rangle$ } $\rightarrow$ this is trivial

$\hookrightarrow$ so subtract from $\langle \hat{\rho}_q \hat{\rho}_{-q} \rangle$

$$\langle \hat{\rho}_q \hat{\rho}_{-q} \rangle - \langle \rho_q \rangle \langle \rho_{-q} \rangle = \sum_{k_1,k_2} \langle c_{k_1+q}^+ c_{k_1} c_{k_2-q}^+ c_{k_2} \rangle$$

$$= \sum_{k_1,k_2} G_{k_2}\, \delta_{k_2,k_1+q}\, G_{k_1}\, \delta_{k_1,k_2-q} = \sum_{k_1} G_{k_1} G_{k_1+q}$$

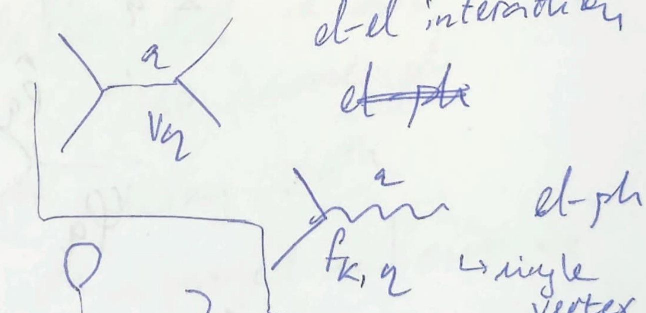
$G_k = \langle (T c_k c_k^+) \rangle \to$ G.F's defn

so $\sum_{k_1,k_2} \langle c^+_{k_1+q} c_{k_1}\, c^+_{k_2-q} c_{k_2} \rangle$ (3 crossings so (-ve) sign)

we also have a -ve in this $\Pi_0 = -\int d^k\, G^0_{k+q} G^0_k$ since its for fermions.

How the phonon interaction changes the el-el interaction

$\to$ 2 vertices



el-el interaction

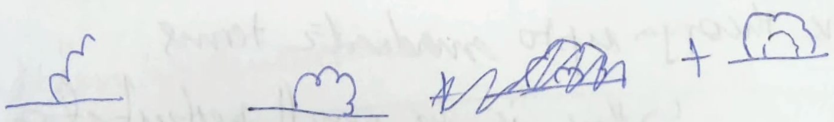
~~el-ph~~

el-ph

$f_{k,q}$ $\hookrightarrow$ single vertex

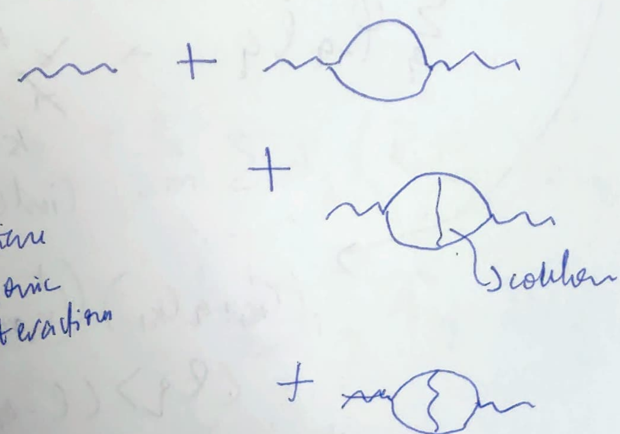
$G^0_k = \rightarrow$ $\quad D^0_{ph}$

$G_k \equiv \rightarrow + \{ \ldots + \ldots \} + -$



$\downarrow$ we can't have this for G_e since it has single vertex.

$\to$ phonon's GF

$D_{ph} = $ 

$\downarrow$ effects of el interaction on phononic interaction

$\hookrightarrow$ scatter

22nd March 2024

The minimal Hamiltonian (k-space) for a S.C

$$H = \sum_{k,\sigma} \varepsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} - \underset{\text{coupling strength}}{g} \sum_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} c_{-k\downarrow} c_{k\uparrow}$$

→ Minimal Hamiltonian to describe a S.C

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} - \mu$$

Formation of Cooper Pairs ⟹ New G.S (of S.C state)

$\hookrightarrow \langle c_{k\uparrow} c_{-k\downarrow} \rangle \neq 0$

$\langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle \neq 0$

$$-g \langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle = \Delta^{*}_k$$

↳ (its physical interpretation → we will come later)

$$-g \langle c_{-k\downarrow} c_{k\uparrow} \rangle = \Delta_k$$

→ Problems arise since it has 4 operators

→ its diagonalized

$$\begin{pmatrix} c_1^{\dagger} & c_2^{\dagger} \end{pmatrix} \begin{bmatrix} \omega_{k_1} & 0 \\ 0 & \omega_{k_2} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Note: If we have the form of $\varepsilon_k c^{\dagger}_{k\sigma} c_{k\sigma}$ then it means its diagonalized

$$AB = \{\langle A \rangle + \Delta A\}\{\langle B \rangle + \Delta B\}$$

$$\simeq \langle A \rangle \langle B \rangle + \langle A \rangle \Delta B$$

$$*AB \simeq \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle$$ (what does it give?)

$$\langle AB \rangle \simeq \langle A \rangle \langle B \rangle + \text{fluctuations}$$

$\rightarrow \langle \Delta B \rangle = 0$

$$\langle AB \rangle = \langle (\bar{A} + \Delta A)(\bar{B} + \Delta B) \rangle = \bar{A}\bar{B} + \bar{A} \times 0 + \bar{B} \times 0 + \langle \Delta A \Delta B \rangle$$

In AB $\simeq$ → we're ignoring fluctuations.

$$-g\,(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k\downarrow} c_{k\uparrow} \simeq -g\,(\langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle c_{-k\downarrow} c_{k\uparrow}$$

$$+ c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \langle c_{-k\downarrow} c_{k\uparrow}\rangle$$

$$- \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\rangle \langle c_{-k\downarrow} c_{k\uparrow}\rangle$$

$$= \Delta^\dagger_k c_{-k\downarrow} c_{k\uparrow} + \Delta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \frac{|\Delta_k|^2}{g}$$

↳ Not a fermion operator

~~you can~~ ~~...~~

~~what happened to~~ ~~Δ_k~~

$$H_{eff} = \sum_k \left[\epsilon_k (c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) + \Delta^*_k c_{-k\downarrow} c_{k\uparrow} + \Delta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right]$$

This is not still diagonalize ~~since we dont of the~~ form ~~$c^\dagger_k \dots c$~~ since we do not have the form of $c^\dagger_{k\sigma} c_{k\sigma}$

What about $c^\dagger_{k\downarrow} c_{k\downarrow}\epsilon_k$?? ↳ this gives additionally → $c^\dagger_{k\uparrow}\Delta_k c^\dagger_{-k\downarrow}$ $-\epsilon_k c_{-k\downarrow} c^\dagger_{-k\downarrow}$

$$= \sum_k (c^\dagger_{k\uparrow},\ c_{-k\downarrow}) \underbrace{\begin{bmatrix} \epsilon_k & \Delta_k \\ \Delta^*_k & -\epsilon_k \end{bmatrix}}_{} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}$$

↳ now we digonalized it.

$$\begin{vmatrix} \epsilon_k - \lambda & \Delta_k \\ \Delta^*_k & -(\epsilon_k + \lambda) \end{vmatrix} = 0, \quad \lambda = \pm\sqrt{\epsilon_k^2 + |\Delta_k|^2}$$

$$= \pm E_k$$

$a^2+b^2=1 \rightarrow \begin{pmatrix} \omega_k & \Delta_k \\ \Delta_k^* & -\omega_k \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = E_k \begin{pmatrix} a \\ b \end{pmatrix}$

$(\omega_k a + \Delta_k b = E_k a) \times b$

$(\Delta_k^* a - \omega_k b = E_k b) \times a$

$\omega_k ab + \Delta_k b^2 = E_k ab$

$\Delta_k^* a^2 - \omega_k ba = E_k ab$

$\Delta_k^* a^2 + \Delta_k b^2 = 2E_k ab$

$\Delta_k^* (a^2) + \Delta_k (1-a^2) = 2E_k ab$

$\Delta_k + a^2(\Delta_k^* - \Delta_k) = 2E_k ab$

$\Delta_k + a^2(\Delta_k^* - \Delta_k) = 2E_k (a)(\sqrt{1-a^2})$

$\Delta_k + a^2(\Delta_k^* - \Delta_k) = 2a\sqrt{1-a^2}\,E_k$

$a = \cos\theta_k,\ b = \sin\theta_k$

$\Delta_k + \cos^2\theta_k(\Delta_k^* - \Delta_k)$

$\cancel{= 2\cos\theta_k (\cos^2 k_k}$

$= 2\cos\theta_k \cdot \sin\theta_k E_k$

$(E_k - \omega_k)\cos\theta_k = \Delta_k \sin\theta_k$

$(E_k + \omega_k)\sin\theta_k = \Delta_k^* \cos\theta_k$

lets do this for real (instead of complex $\rightarrow$ $\tan\theta_k = \dfrac{E_k - \omega_k}{\Delta_k}$

$\tan\theta_k = \dfrac{\Delta_k}{E_k + \omega_k}$

$$\sec^2\theta_k = 1 + \frac{\Delta_k^2}{(E_k+\xi_k)^2} = \frac{(E_k+\xi_k)^2+\Delta_k^2}{(E_k+\xi_k)^2} = \frac{2E_k^2+2E_k\xi_k}{(E_k+\xi_k)^2}$$

$$\cos^2\theta_k = \frac{(E_k+\xi_k)^2}{2E_k(E_k+\xi_k)} = \frac{E_k + \xi_k}{2E_k}$$

$$\left\{\begin{array}{l} \overbrace{\cos\theta_k}^{=u_k} = \pm\frac{1}{\sqrt{2}}\sqrt{1+\frac{\xi_k}{E_k}} \\ \\ \sin\theta_k = \pm\frac{1}{\sqrt{2}}\sqrt{1-\frac{\xi_k}{E_k}} \\ \qquad\qquad = v_k \end{array}\right.$$

for $\lambda = E_k$

for $\lambda = -E_k$,

$$= \pm\frac{1}{\sqrt{2}}\sqrt{1-\frac{\xi_k}{E_k}}$$

$$= \pm\frac{1}{\sqrt{2}}\sqrt{1+\frac{\xi_k}{E_k}}$$

We have taken +ve $\tan\theta_k = \frac{\Delta_k}{E_k+\xi_k}$

$$\tan\theta_k = \pm\sqrt{\frac{1-\xi_k/E_k}{1+\xi_k/E_k}}$$

$$= \pm\frac{\sqrt{(1-\xi_k/E_k)^2}}{1-\xi_k^2/E_k^2}$$

$$= \pm\frac{1-\xi_k/E_k}{\Delta_k} \times E_k = \pm\frac{E_k-\xi_k}{\Delta_k}$$

↳ for this, we take $\cos\theta_k$ ~~[illegible]~~ to be +ve, then what should be the sign of $\sin\theta_k$?
↓
have to be +ve

clear from here.

(c_k's in eigenbasis of H_{eff})

$$H_{eff} = \left(\alpha_k^+ ,\ \beta_k\right)\begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix}\begin{pmatrix} \alpha_k \\ \beta_k^+ \end{pmatrix}$$

$$\alpha_k = u_k c_{k\uparrow} + v_k c^+_{-k\downarrow}$$

$\beta_k = ?$

$$\begin{pmatrix} \omega_k & \Delta_k \\ \Delta_k & -\omega_k \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -E_k \begin{pmatrix} a \\ b \end{pmatrix}$$

$\omega_k a + \Delta_k b = -E_k a \Rightarrow a(\omega_k + E_k) + \Delta_k b = 0$

$\Delta_k a - \omega_k b = -E_k b \Rightarrow b(E_k - \omega_k) + \Delta_k a = 0$

let $a = \cos\theta_k$, $b = \sin\theta_k$

$(\omega_k + E_k)\cos\theta_k + \Delta_k \sin\theta_k = 0 \rightarrow \tan\theta_k = -\frac{(\omega_k+E_k)}{\Delta_k}$

$(E_k - \omega_k)\sin\theta_k + \Delta_k \cos\theta_k = 0$

$\hookrightarrow \tan\theta_k = \frac{-\Delta_k}{E_k - \omega_k} = \frac{\Delta_k}{\omega_k - E_k} = \frac{-(\omega_k + E_k)}{\Delta_k}$ — ①

$\cancel{\tan\theta_k \left(\frac{\omega_k + E_k}{E_k - \omega_k}\right) + \frac{1}{\tan\theta_k} = 0}$

$\cancel{\tan^2\theta_k = \frac{\omega_k - E_k}{\omega_k + E_k}}$

$\cancel{\tan\theta_k = \pm\sqrt{\frac{\omega_k - E_k}{\omega_k + E_k}}}$

$\tan\theta_k = \frac{b}{a}$

$$\sec^2\theta_k = 1 + \frac{\Delta_k^2}{(\omega_k - E_k)^2} = \frac{2E_k(E_k - \omega_k)}{(\omega_k - E_k)^2} = \frac{2E_k}{E_k - \omega_k} \Rightarrow \sec\theta_k = \pm\sqrt{\frac{2E_k}{E_k - \omega_k}}$$

$$\cos\theta_k = \pm\frac{1}{\sqrt{2}}\sqrt{1 - \omega_k/E_k}$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}\sqrt{1 + \frac{\omega_k}{E_k}}$$

$\cancel{= \pm\sqrt{1 - \frac{1}{2}\left(1 - \frac{\omega_k}{E_k}\right)}}$

$\cancel{= \pm\frac{1}{\sqrt{2}}\sqrt{1 + \omega_k/E_k}}$

$\tan\theta_k = \mp\sqrt{\frac{1 + \omega_k/E_k}{1 - \omega_k/E_k}}$ ← Which of these signs combination satisfy ①

$= \mp\sqrt{\frac{(1+\omega_k/E_k)^2}{(1 - \omega_k^2/E_k^2)}}$

$= \mp\frac{E_k}{\Delta_k}\left(1 + \frac{\omega_k}{E_k}\right) = \mp\frac{(E_k + \omega_k)}{\Delta_k}$

$\cos\theta_k$ & $\sin\theta_k$ should be of opp. signs

$\therefore \beta_k = -v_k c_{k\uparrow} + u_k c^{\dagger}_{-k\downarrow}$ (how??)

$$H_{eff} = \sum_k E_k \left(\alpha_k^\dagger \alpha_k - \beta_k \beta_k^\dagger \right) = \sum_k E_k \left(\alpha_k^\dagger \alpha_k + \beta_k^\dagger \beta_k \right)$$

here, we ignore "1" from $\beta_k \beta_k^\dagger = 1 - \beta_k^\dagger \beta_k$ since the cont. is independent of fermions → only const energy

(???)

$\{ c_{k\uparrow}, c_{k\uparrow}^\dagger \} = \mathbb{1}$ ↳ since these are fermion op.

$\{ c_{k\uparrow}, c_{-k\downarrow} \} = 0$

↳ what these correspond to α_k & β_k?

fermionic op? since they're new set of op.

$$\{ \alpha_k, \alpha_k^\dagger \} = \{ u_k c_{k\uparrow} + v_k c_{-k\downarrow}^\dagger, \; u_k c_{k\uparrow}^\dagger + v_k c_{-k\downarrow} \}$$

↳ only ~~the two~~ these 2 comb → non-zero

u_k & v_k → real

$\to \{ u_k^2 c_{k\uparrow} c_{k\uparrow}^\dagger + v_k^2 c_{-k\downarrow}^\dagger c_{-k\downarrow} \}$

$= \cancel{u_k^2 - u_k^2 c_{k\uparrow}^\dagger c_{k\uparrow} + v_k^2 c_{-k\downarrow}^\dagger c_{-k\downarrow}}$

??? (how to proceed from here?)

$$= u_k^2 + v_k^2 = \mathbb{1}$$

$$\{ \beta_k, \beta_k^\dagger \} = \{ -v_k c_{k\uparrow} + u_k c_{-k\downarrow}^\dagger, \; -v_k c_{k\uparrow}^\dagger + u_k c_{-k\downarrow} \}$$

$\alpha_k, \beta_k \to$ Bogliubov quasiparticles

$$= u_k^2 + v_k^2 = \mathbb{1}$$

Are α_k & β_k independent? If yes, ~~$\alpha_k \beta_k^\dagger = 0$~~ ↳ $\{ \alpha_k, \beta_k^\dagger \} = 0$

$$\{ \alpha_k, \beta_k^\dagger \} = \{ u_k c_{k\uparrow} + v_k c_{-k\downarrow}^\dagger, \; -v_k c_{k\uparrow}^\dagger + u_k c_{-k\downarrow} \}$$

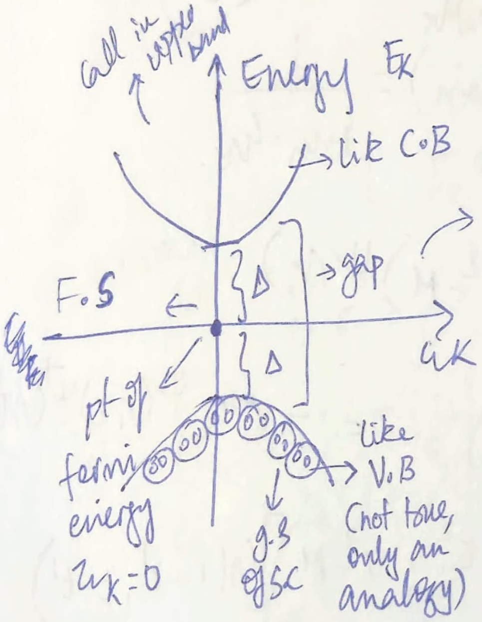
We still have 2 diff. fermions with energy E_k ⇐

$$= -u_k v_k + u_k v_k = 0$$ (they are not independent??)

In BCS → the fermions are ~~the~~ not there but rather the particles are a combination of e^- & hole (both are of opp spins)

$\rightarrow H = \underline{E_c} + H_{eff}$

$\hookrightarrow$ g.s energy for condensate but we're not interested in finding g.s, insted we want to know the energy is affected due to this new set of particles.



$E_k = \pm\sqrt{\xi_k^2 + \Delta_k^2} \equiv \pm\sqrt{\xi_k^2 + \underset{\sim}{\Delta}^2}$

$\hookrightarrow$ is k-independent

ξ_k can be zero (why?) ↓ fermi energy (Ask??)

Even e^- at F.S need Δ to get excited.

What is the min energy of H_{eff} = ?

In FS → all states are occupied.
But here we have gap. (coz FS not at 0??)

So what in the S.C case? In lower band, we need $\underset{\sim}{2\Delta}$ (→ Binding energy) to ~~excite~~ break the pairs. In UB, we get a ~~particle~~ quasi particle & in LB → we get a quasi ~~particle~~ hole

↓ combination of e^- & holes

$\Delta_k = -g\langle C_{k\downarrow} C_{k\uparrow}\rangle$

$(\alpha_k = u_k C_{k\uparrow} + v_k C^{\dagger}_{-k\downarrow})v_k \rightarrow v_k\alpha_k = u_k v_k C_{k\uparrow} + v_k^2 C^{\dagger}_{-k\downarrow}$

$(\beta_k = -v_k C_{k\uparrow} + u_k C^{\dagger}_{-k\downarrow})u_k \rightarrow u_k\beta_k = -u_k v_k C_{k\uparrow} + u_k^2 C^{\dagger}_{-k\downarrow}$

$\underline{v_k\alpha_k + u_k\beta_k = C^{\dagger}_{-k\downarrow}}$

$\cancel{\alpha_k = u_k C_{k\uparrow} + v_k^2\alpha_k + u_k\beta_k v_k}$

$\cancel{\frac{\alpha_k(1+v_k^2) - u_k v_k\beta_k}{u_k} = C_{k\uparrow}}$

$C_{-k\downarrow} = v_k\alpha_k^{\dagger} + u_k\beta_k^{\dagger}$

$C_{k\uparrow} = u_k\alpha_k - v_k\beta_k$

$\Delta_k = -g\langle(v_k\alpha_k^{\dagger} + u_k\beta_k^{\dagger})(u_k\alpha_k - v_k\beta_k)\rangle$

(we will do it in next class)

$$H_{eff} = \sum_k \Big[\xi_k \left(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} \right) + \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k c_{-k\downarrow} c_{k\uparrow} \Big]$$

$$= \sum_k \left(c_{k\uparrow}^\dagger \;\; c_{-k\downarrow} \right) \underbrace{\begin{bmatrix} \xi_k & \Delta_k \\ \Delta_k & -\xi_k \end{bmatrix}}_{\to H_k} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}$$

$$H_0 = \sum_k \xi_k c_k^\dagger c_k \to \quad G_k(\omega_n) = \frac{1}{i\omega_n - \xi_k}$$

$$\to \int d\vec{r}\, \psi^\dagger(r) \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \psi(r)$$

$$\hookrightarrow G(r-r', t-t') = ? = -\langle T_\tau \psi(r,\tau) \psi^\dagger(r',\tau') \rangle$$

$$\left[i\partial_t - \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \right] G(r-r', t-t') = \delta(r-r')\delta(t-t')$$

FT in time

$\hookrightarrow G_k(\tau-\tau') = \langle T_\tau c_k(\tau) c_k^\dagger(\tau') \rangle$

$\hookrightarrow$ In imag time $\to$

$$\left[\partial_\tau - \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \right] G(r-r', \tau-\tau') = \delta(r-r')\delta(\tau-\tau')$$

$$\xrightarrow{F.T} \quad (i\omega_n - \xi_k)\, G(k,\omega) = 1$$

G.F for $H_{eff} \to \left[i\omega_n I_{2\times 2} - H_k \right] G(k, \omega_n) = I_{2\times 2}$

$$\left[i\omega_n \sigma_0 - \xi_k \sigma_3 - \Delta_k \sigma_1 \right] G(k, \omega_n) = \sigma_0$$

$$G(k, \omega_n) = \left[i\omega_n \sigma_0 - \xi_k \sigma_3 - \Delta \sigma_1 \right]^{-1}$$

$G_k(\tau-\tau') = -\langle T_\tau\, c_k(\tau)\, c_k(\tau') \rangle$

$\mathcal{G}_k(\tau-\tau') = -\left\langle T_\tau \begin{bmatrix} c_{k\uparrow}(\tau)\, c^\dagger_{k\uparrow}(\tau') & c_{k\uparrow}\, c_{-k\downarrow} \\ c^\dagger_{k\downarrow} c^\dagger_{k\uparrow} & c^\dagger_{-k\downarrow}\, c_{-k\downarrow} \end{bmatrix} \right\rangle$

↳ why not following T.O here???

in a matrix ↳ write it in τ-space

$\begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{k\downarrow} \end{pmatrix} \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix}$

↳ Anamalous G.F since both c-c involved → non-zero ⇒ which is why we're finding exotic properties of S.C

but ↑ & ↓ same

$= \begin{bmatrix} G_{\uparrow k}(\tau-\tau') & F_k(\tau-\tau') \\ F^\dagger_k(\tau-\tau') & -G_{\downarrow k}(\tau-\tau') \end{bmatrix}$

↳ hole-like G.F

normal G.F = e's G.F

T.O: $\langle T_\tau\, c_k(\tau_1)\, c_k^\dagger(\tau_2) \rangle$

$= c_k(\tau_1)\, c_k^\dagger(\tau_2)\, \Theta(\tau_1-\tau_2)$

$\quad - c_k^\dagger(\tau_2)\, c_k(\tau_1)\, \Theta(\tau_2-\tau_1)$

includes both retarded & advanced G. Function