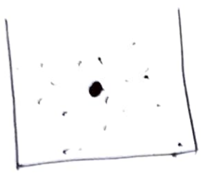
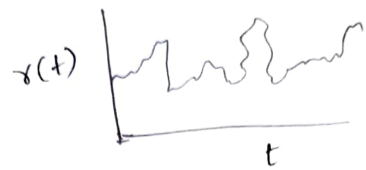
Brownian motion

The random motion of a small particle immersed in a fluid is called Brownian motion





start with a simple picture

Lets assume that particle could only feel the drag force (viscous drag). ~~the and of force~~

Eq. of motion

$$m\frac{d\bar{v}}{dt} = \bar{F}_{Total}.$$

$$m\frac{d\bar{v}}{dt} = -\xi\bar{v}$$

$\bar{v} \to \bar{v}(t)$

$\xi \to$ assumed as constant

It is a linear 1^{st} order differential equation,

$$\frac{d\bar{v}}{dt} = -\frac{\xi}{m}\bar{v}$$

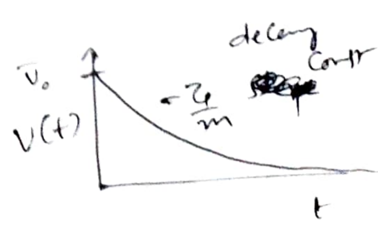
$$\frac{d\bar{v}}{\bar{v}} = -\frac{\xi}{m}dt$$

$$\log\bar{v} = -\frac{\xi}{m}t + C.$$

at $t=0$ $\bar{v} = \bar{v}_0$

$$\Rightarrow \log\bar{v} = -\frac{\xi}{m}t + \log\bar{v}_0$$

$$\Rightarrow \boxed{\bar{v} = \bar{v}_0\, e^{-\frac{\xi t}{m}}}$$



However this is not true in general.

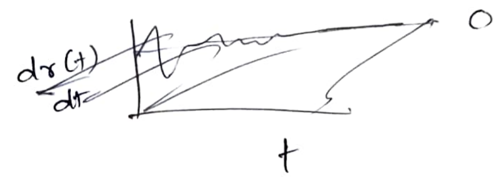
The instantaneous velocity of the Brownian particle may not be zero as the surrounding medium may exert ^the fluid molecules in a random force on the particle. Lets call it as $F(t)$

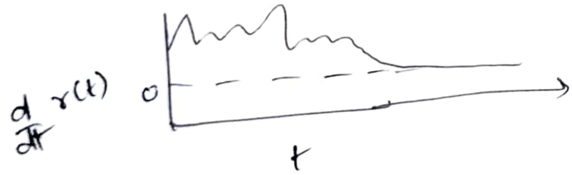
$$\boxed{m\frac{d\bar{v}}{dt} = -\zeta\bar{v} + \bar{F}(t)}$$

Friction force $\rightarrow$ systamatic force
$F(t)$ $\rightarrow$ random force

} Both are arrising from the surrounding medium. So they are related

properties of the random force





The mean velocity of the Brownian particle is zero unless there is an external force acts on the particle !!

$\Rightarrow \langle F(t)\rangle = 0$
$\hookrightarrow$ time averaged

properties of the random force:

$$\langle F(t)F(t')\rangle = 2B\,\delta(t-t')$$

⊛ Where 'B' measures the strength of the fluctuating force.

⊛ Fluctuating force has a Gaussian distribution

Solve $\quad m \frac{dv}{dt} = -\eta v + F(t)$

general eq: $\quad \frac{d x(t)}{dt} = a\, x(t) + b(t)$

say $\quad x(t) = e^{at} y(t)$

$$e^{at} \frac{d y(t)}{dt} + \cancel{\underbrace{y(t) e^{at}}_{x(t)} a} = \cancel{a x(t)} + b(t)$$

$$\Rightarrow \quad \frac{d y(t)}{dt} = e^{-at} b(t)$$

$$y(t) = y(0) + \int_0^t ds\, e^{-as} b(s)$$

Where $y(0) = x(0)$

$$\Rightarrow \quad x(t) = x(0) e^{+at} + \int_0^t ds\, e^{a(t-s)} b(s)$$

$$\Rightarrow \quad v(t) = v_0 e^{-\frac{\eta}{m} t} + \int_0^t dt'\, e^{-\frac{\eta(t-t')}{m}} \frac{F(t')}{m}$$

Now, Calculate $v(t)^2$

$$v^2(t) = v_0^2 e^{-\frac{2\eta t}{m}} + 2 v_0 e^{-\frac{\eta t}{m}} \int_0^t e^{-\frac{\eta(t-t')}{m}} \frac{F(t')}{m} + \int_0^t dt'\, e^{-\frac{\eta}{m}(t-t')} \frac{F(t')}{m} \int_0^t dt''\, e^{-\frac{\eta}{m}(t-t'')} \frac{F(t'')}{m}$$

$$\langle v^2(t) \rangle = v_0^2 e^{-\frac{2\eta}{m} t} + 0 + \int_0^t dt'\, e^{-\frac{\eta}{m}(t-t')} \int_0^t dt''\, e^{-\frac{\eta}{m}(t-t'')} \frac{1}{m^2} \underbrace{\langle F(t') F(t'') \rangle}_{2B\,\delta(t'-t'')}$$

$$= v_0^2 e^{-\frac{2\eta}{m} t} + \frac{2B}{m^2} \int_0^t dt'\, e^{-\frac{\eta}{m}(t-t')} \cdot e^{-\frac{\eta}{m}(t-t')}$$

Using the property of the delta function

$$\langle v^2(t)\rangle = v_0^2\, e^{-\frac{2\gamma t}{m}} + \frac{\cancel{2}B}{m^2}\,\frac{\cancel{m}}{\cancel{2}\gamma}\, e^{-\frac{2\gamma}{m}(t-t')}\Big|_0^t$$

$$\langle v^2(t)\rangle = v_0^2\, e^{-\frac{2\gamma t}{m}} + \frac{B}{\gamma m}\left(1 - e^{-\frac{2\gamma t}{m}}\right)$$

As $t \to \infty$

$$\boxed{\langle v^2(t)\rangle = \frac{B}{\gamma m}}$$

However the mean squared velocity must approach its equilibrium value $\frac{KT}{m}$.

$$\Rightarrow \frac{B}{\gamma m} = \frac{KT}{m} \Rightarrow \boxed{B = \gamma KT}$$ Fluctuation-dissipation theorem.

Where $B \to$ strength of the random force

$\gamma \to$ magnitude of the friction or dissipation

It expresses the balance between friction, which tends to drive any system to a completely "dead state" and noise, which tends to keep the system "alive". This balance is required to have a thermal equilibrium state at long times.