Mean Squared Displacement

$$\frac{d\,x(t)}{dt} = v(t)$$

$$dx(t) = v(t)\,dt$$

$$\Rightarrow \Delta x(t) = \int_0^t v(t')\,dt'$$

$$dx(t) = v(t)\,dt$$

$$x(t) = \int_0^t v(t')\,dt' + \text{Const}$$

$$t \to 0 \quad x(0) = \text{Const}$$

$$x(t) - x(0) = \int_0^t v(t')\,dt'$$

$$\Delta x = \int_0^t v(t')\,dt'$$

- Now, the aim is to calculate $\langle \Delta x(t)^2 \rangle_{eq}$

For this we use the relation

$$v(t) = e^{-\frac{\zeta}{m}t}\,v(0) + \int_0^t dt'\, e^{-\frac{\zeta}{m}(t-t')}\,\frac{F(t')}{m}$$

Exercise

$$\langle \Delta x(t)^2 \rangle_{eq} = \left\langle \left(\int_0^t v(t')\,dt' \right)^2 \right\rangle_{eq} = 2\,\frac{kT}{\zeta}\left[t - \frac{m}{\zeta} + \frac{m}{\zeta}e^{-\frac{\zeta t}{m}}\right]$$

$$= \frac{2kT}{\zeta}\left[t - \frac{m}{\zeta}\left(1 - e^{-\frac{\zeta t}{m}}\right)\right]$$

- $= \cancel{\int_0^t dt'\, v(t') \int_0^t dt''\, v(t'')}_{eq}$

$= \cancel{\int_0^t dt'\, e^{-\frac{\zeta}{m}(t'-t'')}}$

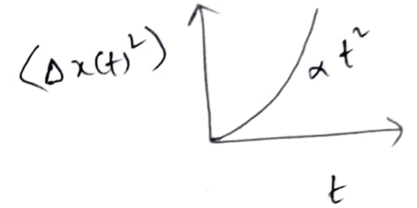
$$\boxed{\langle \Delta x(t)^2 \rangle_{eq} = \frac{2kT}{\zeta}\left[t - \frac{m}{\zeta}\left(1 - e^{-\frac{\zeta t}{m}}\right)\right]}$$

Small t :

$$\langle \Delta x(t)^2 \rangle_{eq} = \frac{2kT}{\zeta}\left[t - \frac{m}{\zeta}\left(\cancel{1} - \cancel{1} + \frac{\zeta t}{m} - \frac{\zeta^2 t^2}{m^2} \right) \right]$$

$$= \frac{2kT}{\zeta}\left[\cancel{t} - \cancel{t} + \frac{\zeta}{m} t^2 \right]$$

$$\boxed{\langle \Delta x(t)^2 \rangle_{eq} = \frac{2kT}{m} t^2}$$

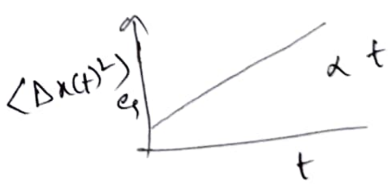


This is the inertial behavior that comes from the initial velocity.

Long t :

$$\langle \Delta x(t)^2 \rangle_{eq} = \frac{2kT}{\zeta}\left[t - \underbrace{\frac{m}{\zeta}}_{\text{Constant}} \right]$$

$$\boxed{\langle \Delta x(t)^2 \rangle_{eq} \to \frac{2kT}{\zeta} t}$$



From Einstein relation, the mean squared displacement of a diffusing particle is $2Dt$, where D is self-diffusion coefficient of the Brownian particle.

$$\cancel{2} D \cancel{t} = \frac{\cancel{2} kT \cancel{t}}{\zeta} \Rightarrow \boxed{D = \frac{kT}{\zeta}}$$

This is the Einstein's expression for the self diffusion coefficient

$$\boxed{D_t = \frac{kT}{6\pi\eta a}}$$ → Stokes - Einstein formula

translational part

$\boxed{\text{when } \zeta = 6\pi\eta a \text{ is called Stoke's law}}$

~~If you include rotations of the Brownian~~

If you take into account the rotational motion of the Brownian particle then the friction $\zeta_r = 8\pi\eta a^3$ and the rotational diffusion constant D_r is

$$\boxed{D_r = \frac{kT}{8\pi\eta a^3}}$$

CTD