Q1.

We know

$$F = -\mu_z \frac{\partial B_z}{\partial z}$$

$$= \mp 9.27 \times 10^{-24}\ J/T \times 10^3\ T/m$$

$$= \mp 9.27 \times 10^{-21}\ J/m$$

So, acceleration $= a = F/m$

$$= \mp \frac{9.27 \times 10^{-21} \times 1000\ N_A}{107.8}\ m/s^2$$

$$= \mp 5.178 \times 10^4\ m/s^2$$

Inside the magnet →

$$s_z = \tfrac{1}{2} a t^2$$

$$s_x = v_x t$$

$$t = \frac{s_x}{v_x} = \frac{s_x \sqrt{m}}{\sqrt{3KT}}$$

$$t = \frac{0.1 \sqrt{107.8}}{\sqrt{1000\ N_A}\ \sqrt{3(1.38)(10^{-23} \times 600)}}\ \text{seconds}$$

$$t = 0.268 \times 10^{-3}\ \text{sec}$$

$$\delta_z = \frac{1}{2} \times 5.178 \times 10^4 \left(0.268 \times 10^{-3}\right)^2$$

$$= 0.186 \times 10^{-2} \, m$$

z-direction velocity →

$$V_z = at$$

$$= 5.178 \times 10^4 \times 0.268 \times 10^{-3}$$

$$= 13.88 \, m/s$$

Outside the pole piece →

$$t = \frac{1}{V_{RMS}} = 2.68 \times 10^{-3} \, sec$$

So → $\delta_z = V_z t$

$$= (13.88 \times 2.68 \times 10^{-3}) \, m$$

$$= 0.0372$$

In one half, i.e, P_1P_2 or P_3P_1

$$= (0.00186 + 0.0372) \, m$$

$$= (0.03906) \, m$$

Total deflection $= P_3P_1 + P_1P_2 = P_2P_3$

$$= 2 \times 0.03906 = 0.07812 \, m$$

Q2.

Option - D : split vertically into 2 beams

↳ since that vertical ∴ two beam will represent two state, spin up & spin down of electron. The spin of proton is ignored because $\mu_e >> \mu_p$, so deflection of proton would be very small

Q3.

a) $f(x) = 4$, $g(x) = x^2$, $h(x) = e^{2x}$

Linear combination $\Rightarrow$

$$4c_1 + x^2 c_2 + e^{2x} c_3 = 0$$

$\hookrightarrow$ This is possible only when $c_1 = c_2 = c_3 = 0$ for any value of x. So, they're linearly independent.

b) $f(x) = x$, $g(x) = x^2$, $h(x) = x^3$

linear combination of them $\Rightarrow$

$$c_1 x + c_2 x^2 + c_3 x^3 = 0$$

$\hookrightarrow$ possible when $c_1 = c_2 = c_3 = 0$

$\therefore$ they are linearly independent

c) $f(x) = x$, $g(x) = 5x$, $h(x) = x^2$

Clearly, $g(x) = 5f(x) + 0 \times h(x)$

$\hookrightarrow$ $g(x)$ is linear combination of $f(x)$ & $h(x)$

$\therefore$ they are linearly dependent

d) $f(x) = 2 + x^2$, $h(x) = 2x + 3x^2 - 8x^3$

$g(x) = 3 - x + 4x^3$

Clearly $\rightarrow$ $3f(x) = 2g(x) + h(x)$

$$\Rightarrow f(x) = \frac{2}{3} g(x) + \frac{1}{3} h(x)$$

$\Rightarrow f(x)$ is a linear combination of $g(x)$ & $h(x)$

Q4.

a) $\vec{A} = (3,0,0)$, $\vec{B}(0,-2,0)$,
$\vec{C} = (0,0,-1)$

So $\rightarrow$ $c_1\vec{A} + c_2\vec{B} + c_3\vec{C} = 0$

$3c_1\hat{i} - 2c_2\hat{j} - c_3\hat{k} = 0$

clearly, this is possible only if
$c_1 = c_2 = c_3 = 0$

$\therefore$ $\vec{A}, \vec{B}, \vec{C}$ are linearly independent

b) $\vec{A} = (6,-9,0)$, $\vec{B} = (-2,3,0)$

$c_1\vec{A} + c_2\vec{B} = 0$

$\Rightarrow 6c_1\hat{i} - 9c_1\hat{j} - 2c_2\hat{i} + 3c_2\hat{j} = 0$

$(6c_1 - 2c_2)\hat{i} + (3c_2 - 9c_1)\hat{j} = 0$

$6c_1 = 2c_2 \qquad\qquad c_2 = 3c_1$
$\Rightarrow c_2 = 3c_1$

$\therefore$ They are not linearly independent.

c) $\vec{A} = (2,3,-1)$, $(0,0,-5) = \vec{C}$
$\vec{B} = (0,1,2)$

$c_1\vec{A} + c_2\vec{B} + c_3\vec{C} = 0$

$\Rightarrow 2c_1\hat{i} + (3c_1 + c_2)\hat{j} + (-c_1 + 2c_2 - 5c_3)\hat{k} = 0$

$c_1 = 0,\ c_3 = 0$
$c_2 = -3c_1 = 0 \Rightarrow c_2 = c_1 = c_3 = 0$

$\therefore$ $\vec{A}, \vec{B}, \vec{C} \rightarrow$ linearly independent

d) $\vec{A} = (1, -2, 3)$.

$\vec{B} = (-4, 1, 7)$

$\vec{C} = (0, 10, 11)$, $\vec{D} = (14, 3, -4)$

$$c_1\vec{A} + c_2\vec{B} + c_3\vec{C} + c_4\vec{D} = 0.$$

$$(c_1 - 4c_2 + 14c_4)\vec{i} + (-2c_1 + c_2 + 10c_3 + 3c_4)\vec{j} + (3c_1 + 7c_2 + 11c_3 - 4c_4)\vec{k} = 0$$

$$c_1 - 4c_2 + 14c_4 = 0 \quad -(1)$$

$$-22c_1 + 11c_2 + 110c_3 + 33c_4 = 0 \quad -(2)$$

$$30c_1 + 70c_2 + 110c_3 - 40c_4 = 0 \quad -(3)$$

$(3) - (2) \rightarrow 52c_1 + 59c_2 - 73c_4 = 0 \quad -(4)$

$(1)\times 52 - (4) \rightarrow 52c_1 - 4\times 52c_2 + 14(52c_4) = 0$

$$52c_1 + 59c_2 - 73c_4 = 0$$

$$(-4\times 52 - 59)c_2 + (14\times 52 + 73)c_4 = 0$$

$\Rightarrow (14\times 52 + 73)c_4 = (4\times 52 + 59)c_2 \quad -(5)$

$\Rightarrow c_1, c_2, c_3, c_4 \rightarrow$ not linearly independent

They are related by $\rightarrow$ $\vec{D} = 2\vec{A} - 3\vec{B} + \vec{C}$ $\vec{A}, \vec{B}, \vec{C}, \vec{D}$

Q5. a) $\hat{A} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, $\hat{A}^{+} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ $\neq \hat{A}$

$\hat{A} \rightarrow$ not Hermitian

$\hat{B} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$, $\hat{B}^{+} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} = \hat{B} \rightarrow$ Hermitian

b) $\hat{A}\hat{B} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ -2 & 0 & 4 \end{pmatrix}$

Q6.

a) $|\phi\rangle = \begin{pmatrix} 2 \\ -i \\ 2-3i \end{pmatrix}$, $\langle\phi| = (2 \quad i \quad 2+3i)$

b) $\langle\phi|\psi\rangle = (2 \quad i \quad 2+3i) \begin{pmatrix} -3i \\ 2+i \\ 4 \end{pmatrix}$

$= -6i + 2i - 1 + 8 + 12i$

$= 7 + 8i$

c) $|\psi\rangle \rightarrow 3\times 1$ matrix, $|\phi\rangle \rightarrow 3\times 1$ matrix

$|\psi\rangle|\phi\rangle \rightarrow (3\times 1) \times (3\times 1)$

$\hookrightarrow$ This kind of multiplication is not allowed in matrix.

Ill^y for same reason, $\langle\phi|\langle\phi|$ is not possible. These objects do make any physical sense.

Q7.

a) $|\psi + \chi\rangle = |\psi\rangle + |\chi\rangle$

$= 3i|\phi_1\rangle - 7i|\phi_2\rangle - |\phi_1\rangle + 2i|\phi_2\rangle$

$= (3i-1)|\phi_1\rangle - 5i|\phi_2\rangle$

$\langle \Psi + \chi| = (-1-3i)\langle \phi_1| + 5i \langle \phi_2|$

b) $\langle \Psi | \chi \rangle = (-3i\langle \phi_1| + 7\langle \phi_2|)$
$(-|\phi_1\rangle + 2i|\phi_2\rangle)$

$= -14 + 3i$

$\langle \chi | \Psi \rangle = (-\langle \phi_1| - 2i\langle \phi_2|)$
$(3i|\phi_1\rangle - 7i|\phi_2\rangle)$

$= -14 - 3i$ $\quad \langle \Psi | \chi \rangle \neq \langle \chi | \Psi \rangle$

c) Schwarz Inequality $\Rightarrow$

$$|\langle \Psi | \chi \rangle|^2 < \langle \Psi | \Psi \rangle \langle \chi | \chi \rangle$$

$\langle \Psi | \Psi \rangle = 58$, $\langle \chi | \chi \rangle = 5$

$\langle \Psi | \Psi \rangle \langle \chi | \chi \rangle$

$= 290$

$|\langle \Psi | \chi \rangle|^2 = 14^2 + 3^2 = 205$

$\Rightarrow 205 < 240$ (proved)

d) Triangle Inequality -

$$\sqrt{\langle \Psi + \chi | \Psi + \chi \rangle} < \sqrt{\langle \Psi | \Psi \rangle} + \sqrt{\langle \chi | \chi \rangle}$$

$\langle \Psi + \chi | \Psi + \chi \rangle = 35$

$\sqrt{35} < \sqrt{58} + \sqrt{5}$ (proved)

Q8. If $|\psi_1\rangle$ & $|\psi_2\rangle$ are orthogonal

then $\rightarrow$ $\langle\psi_2|\psi_1\rangle = 0$

$\langle\psi_1| = -2i\langle\phi_1| + \langle\phi_2| - a\langle\phi_3|$

$\langle\psi_2| : 3\langle\phi_1| + i\langle\phi_2| + 5\langle\phi_3| - \langle\phi_4|$

$\Rightarrow \langle\psi_2|\psi_1\rangle$

$6i + i - 5a - 4 = 0$

$\Rightarrow a = \dfrac{7i - 4}{5}$

Q9.

a) $(\hat{A} + \hat{A}^{+})$

$\hookrightarrow$ To get its Hermitian

$\therefore (\hat{A} + \hat{A}^{+})^{+}$

$= \hat{A}^{+} + (\hat{A}^{+})^{+} = \hat{A}^{+} + \hat{A}$

$= \hat{A} + \hat{A}^{+}$

So $\Rightarrow (\hat{A} + \hat{A}^{+})^{+} = \hat{A} + \hat{A}^{+}$

$\Rightarrow$ Its a hermitian operator

For $[i(\hat{A} + \hat{A}^{+})]^{+} = (\hat{A} + \hat{A}^{+})^{+}(i)^{+}$

$= (\hat{A} + \hat{A}^{+})(-i)$

$\Rightarrow$ Its not a hermitian operator.

For $[i(\hat{A}-\hat{A}^{+})]^{+} = (i)^{+}(\hat{A}-\hat{A}^{+})^{+}$

$= (-i)[\hat{A}^{+} - (\hat{A}^{+})^{+}] = -i(\hat{A}^{+}-\hat{A})$

$= i(\hat{A}-\hat{A}^{+})$

$\Rightarrow$ Its a hermitian operator.

b) $f(\hat{A}) = (1+i\hat{A}+3\hat{A}^2)(1-2i\hat{A}-9\hat{A}^2)/(5+7\hat{A})$

$\therefore [f(\hat{A})]^{+} = \dfrac{(1-2i\hat{A}-9\hat{A}^2)^{+}(1+i\hat{A}+3\hat{A}^2)^{+}}{(5+7\hat{A})^{+}}$

$= \dfrac{(1+2i\hat{A}^{+}-9(\hat{A}^{+})^2)(1-i\hat{A}+3(\hat{A}^{+})^2)}{(5+7\hat{A}^{+})}$

$= f(\hat{A}^{+})$

c) Let $\hat{A}$ be an hermitian operator & $|\psi\rangle$ be a normalized eigenvector

$\hat{A}|\psi\rangle = \lambda|\psi\rangle$ where λ is eigenvalue of $\hat{A}$

$\langle\psi|A|\psi\rangle = \lambda\langle\psi|\psi\rangle = \lambda$

Now $\rightarrow$ $\langle\psi|A\psi\rangle^{*} = \langle\psi|A^{*}|\psi\rangle$

$= \langle\psi|A|\psi\rangle$

& $\langle\psi|A^{*} = \langle\psi|\lambda^{*} \Rightarrow \langle\psi|A^{*}|\psi\rangle = \lambda^{*}$

$\therefore \lambda^{*} = \lambda \Rightarrow \lambda$ is a real no.

Property of anti-hermitian operator $\rightarrow$

$$\hat{A}^{\dagger} = -\hat{A}$$

so $\rightarrow$ $\hat{A}|\varphi\rangle = \lambda|\varphi\rangle$

$$\langle\varphi|\hat{A}^{*} = \langle\varphi|\lambda^{*}$$

$$\langle\varphi|\hat{A}|\varphi\rangle = \lambda\langle\varphi|\varphi\rangle = \lambda$$

$$\langle\varphi|\hat{A}^{\dagger}|\varphi\rangle = \langle\varphi|A|\varphi\rangle^{*}$$

$$= -\langle\varphi|\hat{A}|\varphi\rangle$$

$$\Rightarrow \langle\varphi|\hat{A}|\varphi\rangle = -\langle\varphi|A|\varphi\rangle^{*}$$

$$\lambda = -\lambda^{*}$$

$$\therefore \lambda + \lambda^{*} = 0$$

$\lambda^{*} \rightarrow$ is the complex conjugate of λ

so $\lambda + \lambda^{*} = 0$ means λ is a purely imaginary quantity.

Q10. $(|\varphi\rangle\langle\varphi|)^{\dagger} = |\varphi\rangle\langle\varphi|$

so, $|\varphi\rangle\langle\varphi|$ is a hermitian operator

$$(|\varphi\rangle\langle\varphi|)^2 = (|\varphi\rangle\langle\varphi|)(|\varphi\rangle\langle\varphi|)$$

$$= |\varphi\rangle\langle\varphi|\varphi\rangle\langle\varphi|$$

If $|\psi\rangle$ is normalized, we get →

$$\langle\psi|\psi\rangle = 1$$

$$(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|$$

If $|\psi\rangle$ is normalized, the product of ket $[|\psi\rangle]$ with the bra $[\langle\psi|]$ is a projection operator.

Q11.

a) Let $\hat{A}, \hat{B}$ → 2 hermitian operators.

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

so, $$[\hat{A},\hat{B}]^{+} = (\hat{A}\hat{B} - \hat{B}\hat{A})^{+}$$

$$= \hat{B}^{+}\hat{A}^{+} - \hat{A}^{+}\hat{B}^{+}$$

Since $\hat{A} = \hat{A}^{+}$, $\hat{B} = \hat{B}^{+}$

$$[\hat{A},\hat{B}]^{+} = \hat{B}\hat{A} - \hat{A}\hat{B} = -[\hat{A},\hat{B}]$$

∴ The commutator of 2 hermitian operator is anti-hermitian.

b) $[\hat{A}, [\hat{B},\hat{C}]\hat{D}]$

$$= [\hat{A},[\hat{B},\hat{C}]]\hat{D} + [\hat{B},\hat{C}][\hat{A},\hat{D}]$$

$$= [\hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) - (\hat{B}\hat{C}-\hat{C}\hat{B})\hat{A}]\hat{D}$$

$$+ (\hat{B}\hat{C}-\hat{C}\hat{B})(\hat{A}\hat{D}-\hat{D}\hat{A})$$

$$= \hat{A}\hat{B}\hat{C}\hat{D} - \hat{A}\hat{C}\hat{B}\hat{D} - \hat{B}\hat{C}\hat{A}\hat{D} + \hat{C}\hat{B}\hat{A}\hat{D}$$

$$+ \hat{B}\hat{C}\hat{A}\hat{D} - \hat{B}\hat{C}\hat{D}\hat{A} - \hat{C}\hat{B}\hat{A}\hat{D} + \hat{C}\hat{B}\hat{D}\hat{A}$$

$= \hat{A}\hat{B}\hat{C}\hat{D} - \hat{A}\hat{C}\hat{B}\hat{D} - \hat{B}\hat{C}\hat{D}\hat{A} + \hat{C}\hat{B}\hat{D}\hat{A}$

Q12.

We know →

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

For position & momentum, lets calculate first for x-component

So → $\Delta x \Delta P_x \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{P}_x] \rangle \right|$

We know →

$$[\hat{x}, \hat{P}_x] = i\hbar$$

So, we get → $\Delta x \Delta P_x \geq \hbar/2$

III^ly, $[\hat{y}, \hat{P}_y] = i\hbar = [\hat{z}, \hat{P}_z]$

So, we get →

$$\Delta y \Delta P_y \geq \hbar/2$$

$$\Delta z \Delta P_z \geq \hbar/2$$

Q13.

Condition to be an unitary operator is →

$$\hat{U}^{+} = \hat{U}^{-1}$$

$$\hat{U}^{+}\hat{U} = \hat{U}\hat{U}^{+} = \hat{I}$$

Here, $\hat{U} = e^{i\varepsilon\hat{G}}$

So, $\hat{U}^{+} = e^{-i\hat{G}^{+}\epsilon^{+}}$

$\therefore e^{i(-\hat{G}^{+}\epsilon^{+} + \epsilon\hat{G})} = 1 = e^{i(\epsilon\hat{G} - \hat{G}^{+}\epsilon^{+})}$

$$\Rightarrow \hat{G}^{+}\epsilon^{+} = \epsilon\hat{G}.$$

If ϵ is a real number, then $\rightarrow$

$$\epsilon^{+} = \epsilon$$

& if $\hat{G}$ is a hermitian operator then $\rightarrow$

$$\hat{G}^{+} = G$$

This is the condition such that the operator $\hat{U}$ is unitary.

Q14. We know $\Rightarrow$

$$\hat{A}^{-1}\hat{A} = \hat{I}$$

So, $\hat{A}^{-1}\hat{A}|\varphi\rangle = \hat{I}|\varphi\rangle \equiv |\varphi\rangle$ — ①

Also, $\hat{A}^{-1}\hat{A}|\varphi\rangle = \lambda(\hat{A}^{-1}|\varphi\rangle)$ — ②

From ① & ②, we get $\rightarrow$

$$\lambda(\hat{A}^{-1}|\varphi\rangle) = |\varphi\rangle$$

$$\Rightarrow \hat{A}^{-1}|\varphi\rangle = \frac{1}{\lambda}|\varphi\rangle.$$

So, λ is the eigenvalue of $\hat{A}$ & $\frac{1}{\lambda}$ is the eigenvalue of $\hat{A}^{-1}$

Q15.

a) $|\psi\rangle = 9i|\phi_1\rangle + 2|\phi_2\rangle$

$\langle\chi| = \frac{i}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{2}}\langle\phi_2|$

$|\psi\rangle\langle\chi| = -\frac{9}{\sqrt{2}}|\phi_1\rangle\langle\phi_1| + \frac{9i}{\sqrt{2}}|\phi_1\rangle\langle\phi_2|$

$+ \frac{2i}{\sqrt{2}}|\phi_2\rangle\langle\phi_1| + \frac{2}{\sqrt{2}}|\phi_2\rangle\langle\phi_2|$

$|\chi\rangle = -\frac{i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$

$\langle\psi| = -9i\langle\phi_1| + 2\langle\phi_2|$

$|\chi\rangle\langle\psi| = -\frac{9}{\sqrt{2}}|\phi_1\rangle\langle\phi_1| - \frac{2i}{\sqrt{2}}|\phi_1\rangle\langle\phi_2|$

$-\frac{9}{\sqrt{2}}|\phi_2\rangle\langle\phi_1| + \frac{2}{\sqrt{2}}|\phi_2\rangle\langle\phi_2|$

So, $|\psi\rangle\langle\chi| \neq |\psi\rangle\langle\varphi|$

b) $(|\psi\rangle)^\dagger = \langle\varphi|$

$= -9i\langle\phi_1| + 2\langle\phi_2|$

$(|\psi\rangle\langle\chi|)^\dagger = |\chi\rangle\langle\varphi|$

$= -\frac{9}{\sqrt{2}}|\phi_1\rangle\langle\phi_1| - \frac{2i}{\sqrt{2}}|\phi_1\rangle\langle\phi_2| - \frac{9i}{\sqrt{2}}|\phi_2\rangle\langle\phi_1|$

$+ \frac{2}{\sqrt{2}}|\phi_2\rangle\langle\phi_2|$

$$(|\chi\rangle\langle\varphi|)^{\dagger} = |\varphi\rangle\langle\chi|$$

$$= \frac{-9}{\sqrt{2}}|\phi_1\rangle\langle\phi_1| + \frac{9i}{\sqrt{2}}|\phi_1\rangle\langle\phi_2| + \frac{2i}{\sqrt{2}}|\phi_2\rangle\langle\phi_1|$$

$$+ \frac{2}{\sqrt{2}}|\phi_2\rangle\langle\phi_2|$$

c.) We know →

$$\mathrm{Tr}(|\psi\rangle\langle\chi|) = \langle\chi|\psi\rangle$$

$$= \left(\frac{i}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{2}}\langle\phi_2|\right)\left(9i|\phi_1\rangle + 2|\phi_2\rangle\right)$$

$$= -\frac{9}{\sqrt{2}} + \frac{2}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$$

$$\mathrm{Tr}(|\chi\rangle\langle\varphi|) = \langle\psi|\chi\rangle$$

$$= (-9i\langle\phi_1| + 2\langle\phi_2|)\left(\frac{-i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle\right)$$

$$= \frac{-9}{\sqrt{2}} + \frac{2}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$$

$$= \mathrm{Tr}(|\psi\rangle\langle\chi|) = \mathrm{Tr}(|\chi\rangle\langle\varphi|)$$

They are equal because result is a real number.
If result is a complex no, then we know–

$$\langle\chi|\varphi\rangle = \langle\varphi|\chi\rangle^{*}$$

d) $|\psi\rangle\langle\psi|$

$= 81|\phi_1\rangle\langle\phi_1| + 18i\,|\phi_1\rangle\langle\phi_2| - 18i\,|\phi_2\rangle\langle\phi_1|$
$\quad + 4|\phi_2\rangle\langle\phi_2|$

$|\chi\rangle\langle\chi|$
$= \frac{1}{2}|\phi_1\rangle\langle\phi_1| - \frac{i}{2}|\phi_1\rangle\langle\phi_2| + \frac{i}{2}|\phi_2\rangle\langle\phi_1|$
$\quad + \frac{1}{2}|\phi_2\rangle\langle\phi_2|$

$Tr(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle$
$= 81 + 4 = 85$

$Tr(|\chi\rangle\langle\chi|) = \langle\chi|\chi\rangle = \frac{1}{2} + \frac{1}{2} = 1$

As we can see $|\chi\rangle$ is normalized but $|\psi\rangle$ is not so $|\chi\rangle\langle\chi|$ is a projection operator but $|\psi\rangle\langle\psi|$ is not a projection operator.

Q16. We know →

a) $\cos(n\theta) = \dfrac{e^{in\theta} + e^{-in\theta}}{2}$

$$\psi(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N} a_n e^{in\theta} + \sum_{n=-N}^{0} a_{-n} e^{-in\theta}\right)$$

$$= \sum_{n=-N}^{N} c_n e^{in\theta}$$

where $c_n = a_n/2$ for $n>0$

and $c_n = a_{-n}/2$ for $n<0$

& $c_0 = a_0$ at $n=0$

As we know trigonometric function of this form $\psi(\theta) = \sum_{n=0}^{N} a_n \cos(in\theta)$

can be expressed in terms of functions $\alpha_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$, we can try to make

the set $\alpha_n(\theta)$ as a basis now.

$$\langle \alpha_m | \alpha_n \rangle = \int_{-\pi}^{\pi} \alpha_m^*(\theta)\, \alpha_n(\theta)\, d\theta$$

$$\langle \alpha_m | \alpha_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta = \delta_{mn}$$

If $n=m$, $\langle \alpha_n | \alpha_n \rangle$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta = 1$$

If $n \neq m$, $\langle \alpha_m | \alpha_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta$

$$= \frac{1}{2\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right] \cdot \frac{1}{i(n-m)}$$

$$= \sum_{n=-N}^{N} c_n e^{in\theta}$$

where $c_n = a_n/2$ for $n>0$

and $c_n = a_{-n}/2$ for $n<0$

& $c_0 = a_0$ at $n=0$

As we know trigonometric function of this form $\psi(\theta) = \sum_{n=0}^{N} a_n \cos(in\theta)$ can be expressed in terms of functions $\alpha_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$, we can try to make the set $\alpha_n(\theta)$ as a basis now.

$$\langle \alpha_m | \alpha_n \rangle = \int_{-\pi}^{\pi} \alpha_m^*(\theta)\, \alpha_n(\theta)\, d\theta$$

$$\langle \alpha_m | \alpha_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)\theta}\, d\theta = \delta_{mn}$$

If $n=m$, $\langle \alpha_n | \alpha_n \rangle$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta = 1$$

If $n \neq m$, $\langle \alpha_m | \alpha_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)\theta}\, d\theta$

$$= \frac{1}{2\pi} \left[e^{i(n-m)\pi} - e^{-i(n-m)\pi} \right] \cdot \frac{1}{i(n-m)}$$

$$= \frac{2i \sin[(n-m)\pi]}{2i\pi(n-m)} = 0$$

Since $\sin[(n-m)\pi] = 0$, $\alpha_n(\theta)$ is a complete and orthonormal set. We can see the $\varphi(\theta)$ has $(2N+1)$ number of function of $\alpha_m(\theta)$. The dimension of the function space is $(2N+1)$.

b) If $N=5$, dimension $= 2N+1 = 11$

Basis has 11 vectors.

$$\alpha_{-5}(\theta) = \frac{e^{-5i\theta}}{\sqrt{2\pi}}, \quad \alpha_5(\theta) = \frac{e^{5i\theta}}{\sqrt{2\pi}}$$

$$\alpha_{-4}(\theta) = \frac{e^{-4i\theta}}{\sqrt{2\pi}}, \quad \alpha_4(\theta) = \frac{e^{4i\theta}}{\sqrt{2\pi}}$$

$$\alpha_{-3}(\theta) = \frac{e^{-3i\theta}}{\sqrt{2\pi}}, \quad \alpha_3(\theta) = \frac{e^{3i\theta}}{\sqrt{2\pi}}$$

$$\alpha_{-2}(\theta) = \frac{e^{-2i\theta}}{\sqrt{2\pi}}, \quad \alpha_2(\theta) = \frac{e^{2i\theta}}{\sqrt{2\pi}}$$

$$\alpha_{-1}(\theta) = \frac{e^{-i\theta}}{\sqrt{2\pi}}, \quad \alpha_1(\theta) = \frac{e^{i\theta}}{\sqrt{2\pi}}$$

$$\alpha_0(\theta) = \frac{1}{\sqrt{2\pi}}$$

Q17.

a) Let $\hat{P}_1$ & $\hat{P}_2$ be 2 projective operators

$$(\hat{P}_1 + \hat{P}_2)^+ = \hat{P}_1^+ + \hat{P}_2^+ = \hat{P}_1 + \hat{P}_2$$

$$(\hat{P}_1 + \hat{P}_2)^2 = (\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1)$$

$$= \hat{P}_1 + \hat{P}_2 + (\hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1)$$

If $\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1$, then we can say that the sum of 2 projection operator is a projection operator if & only if $\hat{P}_1\hat{P}_2 = 0$, when they are not projection operator.

b) $(\hat{P}_1 \hat{P}_2)^+ = \hat{P}_2^+ \hat{P}_1^+$

$$= \hat{P}_2 \hat{P}_1$$

$$(\hat{P}_1\hat{P}_2)^2 = (\hat{P}_1\hat{P}_2)(\hat{P}_1\hat{P}_2)$$

Now if we take $\hat{P}_1$ & $\hat{P}_2$ to be →

$$[\hat{P}_1, \hat{P}_2] = 0$$

then → $\hat{P}_1\hat{P}_2 = \hat{P}_2\hat{P}_1$

$(\hat{P}_1\hat{P}_2)^+ = \hat{P}_1\hat{P}_2$, $(\hat{P}_1\hat{P}_2)^2 = (\hat{P}_1\hat{P}_1)(\hat{P}_2\hat{P}_2)$

$$= \hat{P}_1^2\hat{P}_2^2 = \hat{P}_1\hat{P}_2$$

So, $\hat{P}_1 \hat{P}_2$ is a projection operator if $\hat{P}_1$ & $\hat{P}_2$ commute, otherwise $\hat{P}_1 \hat{P}_2$ will not be a projection operator.

Q18.

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$$

$|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ are orthonormal eigenstates

$$\text{Expectation value of } \hat{B} = \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle}$$

Given $\rightarrow \hat{B}|\phi_n\rangle = n^2|\phi_n\rangle$

$$\therefore \langle\psi| = \frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|$$

$$\langle\psi|\psi\rangle = 1/2 + 1/5 + 1/10 = 8/10$$

$$\langle\psi|\hat{B}|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)$$
$$\left(\frac{1}{\sqrt{2}}{}^2|\phi_1\rangle + \frac{2^2}{\sqrt{5}}|\phi_2\rangle + \frac{3^2}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{9}{10} = \frac{22}{10}$$

$$\langle\hat{B}\rangle = \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{22}{8} = \frac{11}{4}$$

Q19.

a) If $\hat{x}^* = \hat{x}$, then eigen value of $\hat{x}$ will be real

$$\langle \psi | \hat{x} | \psi \rangle$$

$$= \int_{-\infty}^{\infty} \psi^*(x) \left[x \psi(x) \right] dx$$

$$= \int_{-\infty}^{\infty} \left[x \psi(x) \right]^* \psi(x) \, dx$$

$$= \langle \hat{x} \psi | \psi \rangle$$

$$\hat{x}^\dagger = \hat{x} \quad \& \quad \hat{x}^* = \hat{x}$$

$$\langle \varphi | \frac{d}{dx} \psi \rangle = \int_{-\infty}^{\infty} \varphi^*(x) \left[\frac{d}{dx} \psi(x) \right] dx$$

$$= \underbrace{\left[\varphi^*(x) \psi(x) \right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} \frac{d\varphi^*(x)}{dt} \psi(x) \, dx$$

$$= \int_{-\infty}^{\infty} \left[-\frac{d}{dt} \varphi^*(x) \right] \psi(x) \, dx = \langle -\frac{d}{dx} \varphi | \psi \rangle$$

$$\left(\frac{d}{dx} \right)^\dagger = -\frac{d}{dx} \quad , \quad \left(\frac{d}{dx} \right)^* = \frac{d}{dx}$$

$$\left(i \frac{d}{dx} \right)^\dagger = (i)^\dagger \left(\frac{d}{dx} \right)^\dagger = i \frac{d}{dx}$$

$$\left(i\frac{d}{dx}\right)^* = -i\frac{d}{dx}$$

Hermitian conjugate of position & momentum →

$$\hat{x}^\dagger = \hat{x} \quad , \quad \hat{p}^\dagger = \hat{p}$$

Complex conjugate of position & momentum

$$\hat{x}^* = \hat{x} \quad , \quad \hat{p}^* = \left(-i\hbar\frac{d}{dx}\right)^*$$

$$= i\hbar\frac{d}{dx} = -\hat{p}$$

$$\hat{x} = \hat{x}^*$$

$$\hat{p} = -\hat{p}^*$$

b) $(e^{\hat{x}})^\dagger = e^{\hat{x}^\dagger} = e^{\hat{x}}$ → Hermitian operator

$$\left(e^{\frac{d}{dx}}\right)^\dagger = e^{\left(\frac{d}{dx}\right)^\dagger} = e^{-d/dx}$$

$$\left(e^{id/dx}\right)^\dagger = e^{id/dx}$$ → Hermitian

c) $\hat{x}^\dagger = \hat{x}$ → Hermitian

$$\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$ → anti-Hermitian

$$\left(\hat{x}\frac{d}{dx}\right)^\dagger = \left(\frac{d}{dx}\right)^\dagger (\hat{x})^\dagger = -\frac{d}{dx}\hat{x}$$

Now →

$$\frac{d}{dx}(\hat{x}\varphi(x))$$

$$= \psi(x) + x\frac{d}{dx}[\varphi(x)]$$

$$= \left(1 + x\frac{d}{dx}\right)\varphi(x)$$

$$\left(\hat{x}\frac{d}{dx}\right)^{+} = -1 - \hat{x}\frac{d}{dx}$$

d) $\hat{L}_x = -i\hbar\left(\hat{y}\frac{\partial}{\partial z} - \hat{z}\frac{\partial}{\partial y}\right)$

$$\hat{L}_x^{+} = i\hbar\left[\left(\hat{y}\frac{\partial}{\partial z}\right)^{+} - \left(\hat{z}\frac{\partial}{\partial y}\right)^{+}\right]$$

$$= -i\hbar\left(\hat{y}\frac{\partial}{\partial z} - \hat{z}\frac{\partial}{\partial y}\right)$$

$= \hat{L}_x \rightarrow$ Hermitian Operator

|||ly, $\hat{L}_y^{+} = \hat{L}_y$ & $\hat{L}_z^{+} = \hat{L}_z \rightarrow$ Hermitian operator.

Q20.

a) $\hat{A} = i(\hat{x}^2+1)\frac{d}{dx} + i\hat{x}$

$$(\hat{A})^{+} = \left[i(\hat{x}^2+1)\frac{d}{dx}\right]^{+} + (i\hat{x})^{+}$$

$$= \left(\frac{d}{dx}\right)^{+}(\hat{x}^2+1)^{+}(i)^{+} - \hat{x}^{+}$$

$$= \left(-\frac{d}{dx}\right)(\hat{x}^2+1)(-i) - i\hat{x}$$

$$= \frac{d}{dx}(\hat{x}^2+1)\, i - i\hat{x}$$

$$= i\frac{d}{dx} - i\hat{x} + i\frac{d}{dx}(\hat{x}^2)$$

$$(\hat{A})^{+} = i\frac{d}{dx} - i\hat{x} + i\frac{d}{dx}(\hat{x}^2) - i(\hat{x}^2)\frac{d}{dx}$$
$$+ i(\hat{x}^2)\frac{d}{dx}$$

$$= i\hat{x}^2\frac{d}{dx} + i\frac{d}{dx} - i\hat{x} + i\left[\frac{d}{dx}, \hat{x}^2\right]$$

$$\left[\frac{d}{dx}, \hat{x}^2\right]$$
$$= \hat{x}\left[\frac{d}{dx}, \hat{x}\right] + \left[\frac{d}{dx}, \hat{x}\right]\hat{x}$$

$$= 2\hat{x}$$

$$(\hat{A})^{+} = i(\hat{x}^2+1)\frac{d}{dx} - i\hat{x} + 2i\hat{x}$$

$$= i(\hat{x}^2+1)\frac{d}{dx} + i\hat{x} = \hat{A} \rightarrow \text{Hermitian Operator}$$

b) $\hat{A}\psi(x) = 0$

$$i(\hat{x}^2+1)\frac{d\psi(x)}{dx} + i\hat{x}\,\psi(x) = 0$$

$$\frac{d\psi(x)}{dx} = -\frac{x}{x^2+1}\psi(x)$$

$$\int \frac{d\psi(x)}{\psi(x)} = -\int \frac{x}{x^2+1}\,dx$$

$$\ln|\psi(x)| = -\frac{1}{2}\ln|x^2+1| + \ln c$$

$$\therefore \psi(x) = \frac{c}{\sqrt{x^2+1}}$$

Normalization $\rightarrow$

$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1$$

$$\Rightarrow c^2\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 1$$

$$\Rightarrow c^2\pi = 1$$

$$\Rightarrow c = \frac{1}{\sqrt{\pi}}$$

$$\psi(x) = \frac{1}{\sqrt{\pi(x^2+1)}}$$

c) Probability $= \int_{-1}^{+1} |\psi(x)|^2\,dx$

$$= \frac{1}{\pi}\int_{-1}^{+1} \frac{dx}{(x^2+1)}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi}{2}\cdot\frac{1}{\pi} = \frac{1}{2}$$

q21.

a) $\left(\frac{1+i\hat{A}}{1-i\hat{A}}\right)^{+}$

$$= \left(\frac{1-i\hat{A}^{+}}{1+i\hat{A}^{+}}\right)$$

$$\left(\frac{1+i\hat{A}}{1-i\hat{A}}\right)^{+}\left(\frac{1+i\hat{A}}{1-i\hat{A}}\right)$$

$$= \frac{(1-i\hat{A}^{+})(1+i\hat{A})}{(1+i\hat{A}^{+})(1-i\hat{A})}$$

$$= \frac{1+i\hat{A}-i\hat{A}^{+}+\hat{A}^{+}\hat{A}}{1-i\hat{A}+i\hat{A}^{+}+\hat{A}^{+}\hat{A}}$$

If $\hat{A}^{+} = \hat{A}$

$$= \left(\frac{1+i\hat{A}}{1-i\hat{A}}\right)^{+}\left(\frac{1+i\hat{A}}{1-i\hat{A}}\right)$$

$$= \frac{1+\hat{A}\hat{A}}{1+\hat{A}\hat{A}} = \hat{I}$$

So, condition for $\left(\frac{1+i\hat{A}}{1-i\hat{A}}\right)$ to be unitary is $\hat{A}^{+} = \hat{A}$, i.e, $\hat{A}$ is a hermitian operator.

b) $\left[\frac{(\hat{A}+i\hat{B})}{\sqrt{\hat{A}^2+B^2}}\right]^{+}$

$$= \frac{\hat{A}^{+}-i\hat{B}^{+}}{\sqrt{(\hat{A}^{+})^2+(\hat{B}^{+})^2}}$$

$$\left(\frac{\hat{A}+i\hat{B}}{\sqrt{\hat{A}^2+\hat{B}^2}}\right)^{+}\left(\frac{\hat{A}+i\hat{B}}{\sqrt{\hat{A}^2+\hat{B}^2}}\right)$$

If $\hat{A}^{+}=\hat{A}$ & $\hat{B}^{+}=\hat{B}$

$$\therefore \left(\frac{\hat{A}+i\hat{B}}{\sqrt{\hat{A}^2+\hat{B}^2}}\right)^{+}\left(\frac{\hat{A}+i\hat{B}}{\sqrt{\hat{A}^2+\hat{B}^2}}\right)$$

$$=\frac{(\hat{A}^{+}-i\hat{B}^{+})(\hat{A}+i\hat{B})}{(\hat{A}^2+\hat{B}^2)}$$

$$=\frac{(\hat{A}^{+}-i\hat{B}^{+})(\hat{A}+i\hat{B})}{\hat{A}^2+\hat{B}^2}$$

$$=\frac{(\hat{A}-i\hat{B})(\hat{A}+i\hat{B})}{\hat{A}^2+\hat{B}^2}$$

$$=\frac{\hat{A}^2+i\hat{A}\hat{B}-i\hat{B}\hat{A}+\hat{B}^2}{\hat{A}^2+\hat{B}^2}$$

$$=\frac{\hat{A}^2+\hat{B}^2+i[\hat{A}\hat{B}-\hat{B}\hat{A}]}{(\hat{A}^2+\hat{B}^2)}$$

If $[\hat{A},\hat{B}]=0$, then this becomes unitary

conditions are $\rightarrow$

(1) $\hat{A}^{+}=\hat{A}$ & $\hat{B}^{+}=\hat{B}$

(2) $[\hat{A},\hat{B}]=0$

Q22.

a) lets use induction method, assume

$[\hat{x}^m, \hat{P}] = im\hbar \hat{x}^{m-1}$ is true for $m=c$

$$[\hat{x}^c, \hat{P}] = ic\hbar\hat{x}^{c-1}$$

Now if $c+1 = m$, then →

$$[\hat{x}^{c+1}, \hat{P}]$$

$$= \hat{x}^c[\hat{x}, \hat{P}] + [\hat{x}^c, P]\hat{x}$$

$$= i\hbar\hat{x}^c + ic\hbar\hat{x}^{c-1}\hat{x}$$

$$= i(c+1)\hbar\hat{x}^c$$

So, this relation is valid for any c, for $c = m-1$

$$[\hat{x}^m, \hat{P}] = im\hbar\hat{x}^{m-1} \text{ (hence proved)}$$

Also, $[\hat{x}^2, \hat{P}] = \hat{x}[\hat{x}, \hat{P}] + [\hat{x}, \hat{P}]\hat{x}$

$$= 2i\hbar\hat{x};$$

$$[\hat{x}^3, \hat{P}] = \hat{x}^2[\hat{x}, \hat{P}] + [\hat{x}^2, P]\hat{x}$$

$$= \hat{x}^2 i\hbar + 2i\hbar\hat{x}^2$$

$$= 3i\hbar\hat{x}^2$$

By this, we can say →

$$[\hat{x}^m, \hat{P}] = im\hbar\hat{x}^{m-1}$$

Direct Method

$$[\hat{x}^m, \hat{p}_x]\psi(x)$$

$$= (\hat{x}^m \hat{p}_x - \hat{p}_x \hat{x}^m)\psi(x)$$

$$= \hat{x}^m \left[-i\hbar \frac{d}{dx}\psi(x)\right] + i\hbar \frac{d}{dx}\left[x^m \psi(x)\right]$$

$$= x^m\left[-i\hbar \frac{d\psi(x)}{dx}\right] + mi\hbar x^{m-1}\psi(x) + x^m i\hbar \frac{d\psi(x)}{dx}$$

$$= mi\hbar x^{m-1}\psi(x)$$

So, $[\hat{x}^m, \hat{p}] = im\hbar \hat{x}^{m-1}$

b) Taylor expansion of $F(\hat{x}) \rightarrow$

$$F(\hat{x}) = \sum_n a_n \hat{x}^n$$

$$[F(\hat{x}), \hat{p}] = \sum_n a_n [\hat{x}^n, \hat{p}]$$

$$= \sum_n a_n i\hbar n \hat{x}^{n-1} = i\hbar \sum_n n a_n \hat{x}^{n-1}$$

$$= i\hbar \frac{d}{d\hat{x}}\left(\sum_n a_n \hat{x}^n\right) = i\hbar \frac{dF(\hat{x})}{d\hat{x}}$$

Q23.

a) $A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix}$, $A^\dagger = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix}$

$A = A^{\dagger} \rightarrow$ Hermitian

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2i & 0 \\ i & 0 & -5i \end{pmatrix}, \quad B^{\dagger} = \begin{pmatrix} 1 & 0 & -i \\ 0 & -2i & 0 \\ 3 & 0 & 5i \end{pmatrix}$$

$B \neq B^{\dagger} \rightarrow$ Not Hermitian

$$AB = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2i & 0 \\ i & 0 & -5i \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 21 \\ 1 & 2i & -5 \\ -i & -2 & 5i \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2i & 0 \\ i & 0 & -5i \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3i & -3 \\ 0 & 2i & 2 \\ 7i & 5 & 5i \end{pmatrix}$$

$Tr(AB) = 7 + 2i + 5i = 7 + 7i$

$Tr(BA) = 7 + 2i + 5i = 7 + 7i$

$Tr(AB) = Tr(BA)$

$[A, B] = AB - BA$

$$= \begin{pmatrix} 7 & 0 & 21 \\ 1 & 2i & -5 \\ -i & -2 & 5i \end{pmatrix} - \begin{pmatrix} 7 & 3i & -3 \\ 0 & 2i & 2 \\ 7i & 5 & 5i \end{pmatrix}$$

$$[A,B] = \begin{pmatrix} 0 & -3i & 24 \\ i & 0 & -7 \\ -8i & -7 & 0 \end{pmatrix}$$

$$Tr([A,B]) = 0+0+0 = 0$$

b) $\det(A-\lambda I) = 0$

$$\begin{vmatrix} 7-\lambda & 0 & 0 \\ 0 & 1-\lambda & -i \\ 0 & i & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left(-(1-\lambda)(1+\lambda)+i^2\right) = 0$$

$$\lambda\,(7-\lambda)(-1+\lambda^2-1) = 0$$

$$(\lambda^2-2)(\lambda-7) = 0$$

$$\lambda = 7, \pm\sqrt{2}$$

eigenvalues of A are $\rightarrow$

$$\lambda_1 = -\sqrt{2},\ \lambda_2 = \sqrt{2},\ \lambda_3 = 7$$

eigenvector for $\lambda_1 = -\sqrt{2} \rightarrow$

$$\begin{pmatrix} 7+\sqrt{2} & 0 & 0 \\ 0 & 1+\sqrt{2} & -i \\ 0 & i & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(7+\sqrt{2})c_1 = 0$$

$$(1+\sqrt{2})c_2 - ic_3 = 0$$

$$ic_2 + (\sqrt{2}-1)c_3 = 0$$

$$\Rightarrow c_1 = 0$$

$$c_2 = \frac{i}{(1+\sqrt{2})}c_3$$

$$c_3 = \frac{1+\sqrt{2}}{i} c_2 = -i(1+\sqrt{2})c_2$$

$$|\lambda_1\rangle = \begin{pmatrix} 0 \\ c_2 \\ -i(1+\sqrt{2})c_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ -i(1+\sqrt{2}) \end{pmatrix} c_2$$

<u>Normalization</u>

$$\langle \lambda_1 | \lambda_1 \rangle = 1$$

$$|c_2|^2 + |-i(1+\sqrt{2})c_2|^2 = 1$$

$$c_2^2 + (1+\sqrt{2})^2 c_2^2 = 1$$

$$c_2^2 (1 + 1 + 2\sqrt{2} + 2) = 1$$

$$\Rightarrow c_2 = \frac{1}{\sqrt{4+2\sqrt{2}}}$$

$$|\lambda_1\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \\ \frac{-i(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}$$

For $\lambda_2 = \sqrt{2}$, eigenvector $\rightarrow$

$$\begin{pmatrix} 7-\sqrt{2} & 0 & 0 \\ 0 & 1-\sqrt{2} & -i \\ 0 & i & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(7-\sqrt{2})c_1 = 0 \Rightarrow c_1 = 0$$

$(1-\sqrt{2})c_2 - ic_3 = 0$

$$c_3 = \frac{(1-\sqrt{2})}{i} c_2$$

$$= i(\sqrt{2}-1)c_2$$

eigenvector →

$$|\lambda_2\rangle = \begin{pmatrix} 0 \\ c_2 \\ i(\sqrt{2}-1)c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ i(\sqrt{2}-1) \end{pmatrix} c_2$$

<u>Normalization</u> →

$$\langle \lambda_2 | \lambda_2 \rangle = 1$$

$$c_2^2 (1 + 1 - 2\sqrt{2} + 2) = 1$$

$$\Rightarrow c_2 = \frac{1}{\sqrt{4-2\sqrt{2}}}$$

$$|\lambda_2\rangle = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{4-2\sqrt{2}}} \\ \dfrac{i(\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}$$

For $\lambda_3 = 7$ →

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & -i \\ 0 & i & -8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

c_1 = free variable, i.e, we can choose any value for c_1

$$\left.\begin{array}{l} -6c_2 - ic_3 = 0 \\ ic_2 - 8c_3 = 0 \end{array}\right\} \Rightarrow c_2 = c_3 = 0$$

$$|\lambda_3\rangle = \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Normalization →

$$\langle \lambda_3 | \lambda_3 \rangle = 1$$

$$\Rightarrow c_1 = 1$$

$$|\lambda_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

∴, Sum of eigenvalue of A →

$$= \lambda_1 + \lambda_2 + \lambda_3 = \sqrt{2} - \sqrt{2} + 7 = 7$$

∴, Sum of eigenvalue of $A = Tr(A)$

We can see that eigenvectors $|\lambda_1\rangle, |\lambda_2\rangle, |\lambda_3\rangle$ are mutually orthonormal.

They are complete →

$$\sum_{i=1}^{3} |\lambda_i\rangle\langle \lambda_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since they're orthonormal, they form a complete 4 orthonormal basis.

c) $$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{i(\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}} & \frac{-i(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}$$

$$U^{\dagger}AU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{-i(\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{i(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} AU$$

$$U^{\dagger}AU = \begin{pmatrix} 7 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

$$U^{\dagger}U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{-i(\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{i(1+\sqrt{2})}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \\ 0 & \frac{i(\sqrt{2}-1)}{\sqrt{4-2\sqrt{2}}} & \frac{-i(\sqrt{2}+1)}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So $\rightarrow$ $U^{-1} = U^{\dagger}$

d) $A' = U^{\dagger}AU = \begin{pmatrix} 7 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$

$$A^{-1} = \frac{1}{-14} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -7\sqrt{2} & 0 \\ 0 & 0 & 7\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/7 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} \end{pmatrix}$$

Q24.

a)

$$H = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H^{\dagger} = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$H = H^{\dagger} \rightarrow$ Hermitian

If $|\lambda\rangle$ is a eigenstate of H, then we will get $c.|\lambda\rangle$ where c is a constant if we operate H on $|\lambda\rangle$

$$H|\lambda\rangle = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 7i \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -7+2i \\ -1+7i \\ 7i \end{pmatrix} \neq c|\lambda\rangle$$

$|\lambda\rangle$ is not an eigen state of H.

b) $\det(H - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & i & 0 \\ -i & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(\lambda(\lambda-1)-1) + i(-\lambda i) = 0$$

$$(2-\lambda)(\lambda^2 - \lambda - 1) + \lambda = 0$$

$$2\lambda^2 - 2\lambda - 2 - \lambda^3 + \lambda^2 + \lambda + \lambda = 0$$

$$\lambda^3 - 3\lambda^2 + 2 = 0$$

$$(\lambda - 1)(\lambda^2 - 2\lambda - 2) = 0$$

$$\lambda_1 = 1 \, , \quad \lambda_2 = \frac{2 + \sqrt{4+8}}{2} = 1+\sqrt{3}$$

$$\lambda_3 = \frac{2 - \sqrt{4+8}}{2} = 1-\sqrt{3}$$

For $\lambda_1 = 1 \rightarrow$

$$\begin{pmatrix} 1 & i & 0 \\ -i & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 + ic_3 = 0$$

$$-ic_1 + c_3 = 0$$

$$c_2 - c_3 = 0$$

$$\Rightarrow c_2 = c_3 , \quad c_3 = ic_1 = c_2$$

$$|\lambda_1\rangle = \begin{pmatrix} c_1 \\ ic_1 \\ ic_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$$

Normalization $\rightarrow$

$$\langle \lambda_1 | \lambda_1 \rangle = 1$$

$$\Rightarrow c_1^2 + c_1^2 + c_1^2 = 1 \Rightarrow c_1 = 1/\sqrt{3}$$

$$|\lambda_1\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$$

For $\lambda_2 = 1+\sqrt{3} \rightarrow$

$$\begin{pmatrix} 1-\sqrt{3} & i & 0 \\ -i & -\sqrt{3} & 1 \\ 0 & 1 & -1-\sqrt{3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(1-\sqrt{3})c_1 + ic_2 = 0 \Rightarrow c_2 = -\frac{(1-\sqrt{3})}{i} c_1$$

$$-ic_1 - \sqrt{3}c_2 + c_3 = 0$$

$$c_2 = i(1-\sqrt{3})c_1$$

$$c_2 - (1+\sqrt{3})c_3 = 0$$

$$\Rightarrow c_2 = (1+\sqrt{3})c_3$$

$$c_3 = \frac{1}{1+\sqrt{3}} \cdot c_2$$

$$|\lambda_2\rangle = \begin{pmatrix} c_1 \\ i(1-\sqrt{3})c_1 \\ \frac{i(1-\sqrt{3})}{1+\sqrt{3}} c_1 \end{pmatrix}$$

<u>Normalization</u>

$$\langle \lambda_2 | \lambda_2 \rangle = 1.$$

$$c_1^2 + (1-\sqrt{3})^2 c_1^2 + \frac{(1-\sqrt{3})^2}{(1+\sqrt{3})^2} c_1^2 = 1$$

$$c_1^2 \left(1 + 1 - 2\sqrt{3} + 3 + \left(\frac{1-2\sqrt{3}+3}{1-3} \right)^2 \right) = 1$$

$$c_1^2 \left(5 - 2\sqrt{3} + \frac{(4-2\sqrt{3})^2}{2^2} \right) = 1$$

$$a^2(5-2\sqrt{3}+4-4\sqrt{3}+3)=1$$

$$a=\frac{1}{\sqrt{12-6\sqrt{3}}}$$

$$|\lambda_2\rangle=\frac{1}{\sqrt{12-6\sqrt{3}}}\begin{pmatrix}1\\ i(1-\sqrt{3})\\ \frac{i(1-\sqrt{3})}{1+\sqrt{3}}\end{pmatrix}$$

III) for $\lambda_3=1-\sqrt{3}$,

$$|\lambda_3\rangle=\frac{1}{\sqrt{12+6\sqrt{3}}}\begin{pmatrix}1\\ i(1+\sqrt{3})\\ \frac{i(1+\sqrt{3})}{1-\sqrt{3}}\end{pmatrix}$$

c) $P=|\lambda_1\rangle\langle\lambda_1|$

$$=\frac{1}{3}\begin{pmatrix}1\\ i\\ i\end{pmatrix}\begin{pmatrix}1 & -i & -i\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}$$

$$P^2=\frac{1}{9}\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}=P$$

So, P is a projection operator

$$H|\lambda_1\rangle = |\lambda_1\rangle$$

$$\langle\lambda_1| H = \langle\lambda_1|$$

$$[P, H] = PH - HP$$

$$= |\lambda_1\rangle\langle\lambda_1| H - H|\lambda_1\rangle\langle\lambda_1|$$

$$= |\lambda_1\rangle\langle\lambda_1| - |\lambda_1\rangle\langle\lambda_1|$$

$$= 0$$

Q25.

a) $A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$

$$A^+ = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$A = A^+ \rightarrow$ Hermitian

$B = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix}$, $B^+ = \begin{pmatrix} 2 & 3 & 0 \\ -i & 1 & i \\ 0 & 5 & -2 \end{pmatrix}$

$B \neq B^+ \rightarrow$ Not hermitian

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 & i \\ 0 & 1-\lambda & 0 \\ -i & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda[-\lambda(1-\lambda)] + i^2(1-\lambda) = 0$$

$$\lambda^2(1-\lambda) - (1-\lambda) = 0$$

$$\lambda = 1, 1, -1$$

eigen vectors for $\lambda_1 = 1$

$$\begin{pmatrix} -1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$c_1 = ic_3$

$c_2 =$ free variable

$c_3 = -ic_1$

$$|\lambda_1\rangle = c_1 \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Normalization

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = 1$$

So $\rightarrow$ $|\lambda_1\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$

$$|\lambda_1\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 1 \rightarrow |\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

b)

$$AB = \begin{pmatrix} 0 & 1 & -2i \\ 3 & 1 & 5 \\ -2i & 1 & 0 \end{pmatrix}$$

$$Tr(AB) = 0+1+0 = 1$$

$$BA = \begin{pmatrix} 0 & i & 2i \\ -5i & 1 & 3i \\ 2i & -i & 0 \end{pmatrix}$$

$$Tr(BA) = 0+1+0 = 1$$

$$Tr(AB) = Tr(BA)$$

$$det(AB) = 4-16i$$

$$det(A) = i \cdot i = -1, \quad det(B) = -4+16i$$

$$det(A)\,det(B) = 4-16i = det(AB)$$

$$det(B^{+}) = -4-16i = (det(B))^{*}$$

c) $[A,B] = (AB - BA)$

$$= \begin{pmatrix} 0 & 1 & -2i \\ 3 & 1 & 5 \\ -2i & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & i & 2i \\ -5i & 1 & 3i \\ 2i & -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1-i & -4i \\ 3+5i & 0 & 5-3i \\ -4i & 1+i & 0 \end{pmatrix}$$

$$\{A,B\} = AB + BA = \begin{pmatrix} 0 & 1+i & 0 \\ 3-5i & 2 & 5+3i \\ 0 & 1-i & 0 \end{pmatrix}$$

d)

$$A^{-1} = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$B^{-1} = \frac{1}{68} \begin{pmatrix} 22+3i & 8-2i & 20-5i \\ -6-24i & 4+16i & 10+40i \\ -12+3i & 8-2i & -14-5i \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{68} \begin{pmatrix} -5-20i & 8-2i & -3+22i \\ 40-10i & 4+16i & 24-6i \\ -5+14i & 8-2i & -3-12i \end{pmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{68} \begin{pmatrix} -5-20i & 8-2i & -3+22i \\ 40-10i & 4+16i & 24-6i \\ -5+14i & 8-2i & -3-12i \end{pmatrix}$$

$$= (AB)^{-1}$$

e) $$A^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A^3 = A^2 . A, \quad A^4 = A^2 A^2 = I$$

$$A^{2n} = (A^2)^n = I$$

$$A^{2n+1} = A^{2n} . A = A$$

$$e^{xA} = \sum_{n=0}^{\infty} \frac{x^n A^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n} A^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1} A^{2n+1}}{(2n+1)!}$$

$$= I \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} + A \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

The relation →

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} = \cosh x$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$$

$$e^{xA} = I\cosh x + A\sinh x$$

$$= \begin{pmatrix} \cosh x & 0 & i\sinh x \\ 0 & \cosh x + \sinh x & 0 \\ -i\sinh x & 0 & \cosh x \end{pmatrix}$$

Q26.

$$A = \begin{pmatrix} 0 & i & 2 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & i & 0 \\ 3 & +1 & 5 \\ 0 & -i & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 1/2 & -i/2 & 0 \end{pmatrix}$$

$$A^{-1}B = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 1/2 & -i/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -2i \\ 3 & 1 & 5 \\ 1-3i/2 & 0 & -5i/2 \end{pmatrix}$$

$$BA^{-1} = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 1/2 & -i/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i & 2i \\ 5/2 & 1-5i/2 & 3i \\ -1 & 0 & 0 \end{pmatrix}$$

$$A^{-1}B \neq BA^{-1}$$

Q27.

a) $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2-1)+\lambda=0$$

$$\lambda(1-\lambda^2+1)=0$$

$$\lambda=0,\ \lambda=\pm\sqrt{2}$$

So, $a_1=0,\ a_2=\sqrt{2},\ a_3=-\sqrt{2}$

$$|a_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix},\quad |a_2\rangle=\frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}$$

$$|a_3\rangle=\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

For $B \rightarrow$

$b_1=1,\ b_2=0,\ b_3=-1$

$$|b_1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix},\ |b_2\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$|b_3\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

A & B are not degenerate

b) As eigenvectors are normalized -

$\langle a_1|a_1\rangle=1,\ \langle a_2|a_2\rangle=1$

$\langle a_3|a_3\rangle=0$

Also, they are orthonormal since

$\langle a_1|a_2\rangle=0,\quad \langle a_1|a_3\rangle=0$

$\langle a_2 | a_3 \rangle = 0$

$$|a_1\rangle\langle a_1| = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$|a_2\rangle\langle a_2| = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

$$|a_3\rangle\langle a_3| = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$\sum_{i=1}^{3} |a_i\rangle\langle a_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As eigenvectors are normalized $\rightarrow$

$\langle b_1 | b_1 \rangle = 1$, $\langle b_2 | b_2 \rangle = 1$, $\langle b_3 | b_3 \rangle = 1$

Also, they're mutually orthornormal since

$\langle b_1 | b_2 \rangle = 0$, $\langle b_1 | b_3 \rangle = 0$

$\langle b_2 | b_3 \rangle = 0$

$$\sum_{i=1}^{3} |b_i\rangle\langle b_i| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) $a_{ij} = \langle b_i | a_j \rangle$

$$= \begin{pmatrix} \langle b_1|a_1\rangle & \langle b_1|a_2\rangle & \langle b_1|a_3\rangle \\ \langle b_2|a_1\rangle & \langle b_2|a_2\rangle & \langle b_2|a_3\rangle \\ \langle b_3|a_1\rangle & \langle b_3|a_2\rangle & \langle b_3|a_3\rangle \end{pmatrix}$$

So $\rightarrow$

$$U = \frac{1}{2}\begin{pmatrix} -\sqrt{2} & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}$$

$$U^{-1} = \frac{1}{2}\begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix} = U^{+}$$

$$A' = UAU^{+}$$

$$= \frac{1}{4}\begin{pmatrix} -\sqrt{2} & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1-\sqrt{2} & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & 1+\sqrt{2} \end{pmatrix}$$

Q28.

a) $\langle \psi | \psi \rangle = \frac{1}{15} + \frac{1}{3} + \frac{1}{5} = \frac{9}{15}$

$$\text{Norm} = \sqrt{\langle \psi | \psi \rangle} = \frac{3}{\sqrt{15}}$$

b) $\langle \psi | \hat{B} | \psi \rangle$

$\langle \psi | \hat{B} | \psi \rangle$

$$= \left(\frac{1}{\sqrt{15}} \langle \phi_1 | + \frac{1}{\sqrt{3}} \langle \phi_2 | + \frac{1}{\sqrt{5}} \langle \phi_3 |\right)\left(\frac{2}{\sqrt{15}} |\phi_1\rangle + \frac{11}{\sqrt{3}} |\phi_2\rangle + \frac{17}{\sqrt{5}} |\phi_3\rangle\right)$$

$$= \frac{2}{15} + \frac{11}{3} + \frac{17}{5} = \frac{2 + 55 + 51}{15}$$

$$= \frac{108}{15} = \frac{36}{5}$$

Expectation value of $\hat{B} \rightarrow$

$$= \frac{\langle \psi | \hat{B} | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{36}{5} \times \frac{15}{9} = 12$$

c) $B^2 |\phi_n\rangle = (3n^2 - 1)^2 |\phi_n\rangle$

$$\langle \psi | B^2 | \psi \rangle$$

$$= \left(\frac{1}{\sqrt{15}} \langle \phi_1 | + \frac{1}{\sqrt{3}} \langle \phi_2 | + \frac{1}{\sqrt{5}} \langle \phi_3 |\right)\left(\frac{4}{\sqrt{15}} |\phi_1\rangle + \frac{121}{\sqrt{3}} |\phi_2\rangle + \frac{289}{\sqrt{5}} |\phi_3\rangle\right)$$

$$= \frac{4}{15} + \frac{121}{3} + \frac{289}{5} = \frac{4 + 605 + 867}{15}$$

$$= \frac{1476}{15} = \frac{492}{5}$$

Expectation value of $B^2 = \dfrac{\langle \psi | B^2 | \psi \rangle}{\langle \psi | \psi \rangle}$

$$= \frac{492}{5} \cdot \frac{5}{3} = 164$$

Q29.

$x|\phi_1\rangle = |\phi_1\rangle$

$x|\phi_2\rangle = -|\phi_2\rangle$

$|\phi_1\rangle$ & $|\phi_2\rangle \rightarrow$ orthonormal

a) Yes, they form a basis because they're orthogonal states (& hence linearly independent)

b) $\hat{B} = |\phi_1\rangle\langle\phi_2|$

$\hat{B}^+ = |\phi_2\rangle\langle\phi_1| \neq \hat{B}$

So, $\hat{B}$ is not Hermitian

Now, $\hat{B}^2 = |\phi_1\rangle\langle\phi_2|\phi_1\rangle\langle\phi_2|$

$= 0$

c) $\hat{B}\hat{B}^+ = |\phi_1\rangle\langle\phi_2|\phi_2\rangle\langle\phi_1|$

$= |\phi_1\rangle\langle\phi_1| \rightarrow$ Projection Operator

$\hat{B}^+\hat{B} = |\phi_2\rangle\langle\phi_1|\phi_1\rangle\langle\phi_2|$

$= |\phi_2\rangle\langle\phi_2| \rightarrow$ Projection operator

$$\hat{B}^{+}\hat{B} = \hat{B}^{+}\hat{B}$$

d) $|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| = \hat{B}\hat{B}^{+} - \hat{B}^{+}\hat{B} = \hat{C}$

$$\hat{C}^{+} = (|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|$$

$$\hat{C}^{+}C = (|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|)(|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|)$$

$$= |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$$

$= 1$ (by completeness relation)

e) $\hat{C} = \hat{B}\hat{B}^{+} + \hat{B}^{+}\hat{B} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$

(i) $\hat{C}|\phi_1\rangle = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)|\phi_1\rangle$

$$= |\phi_1\rangle$$

(ii) $\hat{C}|\phi_2\rangle = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)|\phi_2\rangle$

$$= |\phi_2\rangle$$

Q.30.

a)

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^{\dagger} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A$$

So, A is hermitian

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda)(1+\lambda) + (1+\lambda) = 0$$

$$(1+\lambda)(-\lambda^2+1) = 0$$

$$(\lambda^2+1)(\lambda+1) = 0$$

$$\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = -1$$

For $\lambda_1 = -1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 + c_3 = 0$$

c_2 = free variable

$$|\lambda_1\rangle = \begin{pmatrix} c_1 \\ 0 \\ -c_1 \end{pmatrix}, \quad |\lambda_3\rangle = \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix}$$

Normalization

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad |\lambda_3\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

For $\lambda_2 = 1$,

$$\begin{pmatrix}-1 & 0 & 1\\0 & -2 & 0\\1 & 0 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$c_1 = c_3$$
$$c_2 = 0$$

$$|\lambda_2\rangle = \begin{pmatrix}c_1\\0\\c_1\end{pmatrix}$$

Normalization

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\langle\lambda_1|\lambda_2\rangle = \frac{1}{2}(1-1) = 0$$

$$\langle\lambda_3|\lambda_2\rangle = 0$$

eigen vectors corresponding to the two non-degenerate eigen values are orthonormal

b) $|\lambda_1\rangle\langle\lambda_2| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\frac{1}{\sqrt{2}}(1\ 0\ 1)$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\-1 & 0 & -1\end{pmatrix} \Rightarrow |\lambda_3\rangle\langle\lambda_2|$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}(1\ 0\ 1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\1 & 0 & 1\\0 & 0 & 0\end{pmatrix}$$