Zero-temperature properties of Ferromagnets

Ground state of the Heisenberg Ferromagnet

Consider a set of magnetic ions at Bravais lattice sites $\vec{R}$,

$$\mathcal{H} = -\frac{1}{2}\sum_{\vec{R},\vec{R}'} \hat{S}(\vec{R})\cdot\hat{S}(\vec{R}')\,J(\vec{R}-\vec{R}') - g\mu_B B \sum_{\vec{R}} \hat{S}_z(\vec{R}),$$

where $\quad J(\vec{R}-\vec{R}') = J(\vec{R}'-\vec{R}) \geq 0.$

Note that the positive exchange interaction $(J>0)$ favours parallel spin alignment.

$\vec{B}$ is the local magnetic field acting on each magnetic ion.

Before proceeding further, let us consider spins as classical vectors for the time being. The lowest energy state is obtained when all the spins are aligned along the z-axis, parallel to the applied magnetic field and parallel to each other.

Let us now build a quantum picture of the ground state motivated by the above classical picture of spins.

It seems a plausible quantum mechanical ground state $|0\rangle$ is one that is an eigenstate of $\hat{S}_z(\vec{R})$ for every $\vec{R}$ with the maximum eigenvalue, S:

$$|0\rangle = \prod_{\vec{R}} |S\rangle_{\vec{R}},$$

where $\quad \hat{S}_z(\vec{R})\,|S\rangle_{\vec{R}} = S\,|S\rangle_{\vec{R}}.$

Let us now show that $|0\rangle$ is indeed an eigenstate of $\mathcal{H}$.
To do so, we first rewrite the Hamiltonian in terms of the (spin ladder) operators:

$$\hat{s}_{\pm}(\vec{R}) = \hat{s}_x(\vec{R}) \pm i\, \hat{s}_y(\vec{R}),$$

where
$$\hat{s}_{\pm}(\vec{R})|s_z\rangle_{\vec{R}} = \sqrt{(s \mp s_z)(s+1 \pm s_z)}\; |s_z \pm 1\rangle_{\vec{R}} .$$

$$\begin{aligned}
\hat{s}(\vec{R})\cdot\hat{s}(\vec{R}') &= \hat{s}_x(\vec{R})\hat{s}_x(\vec{R}') + \hat{s}_y(\vec{R})\hat{s}_y(\vec{R}') + \hat{s}_z(\vec{R})\hat{s}_z(\vec{R}') \\
&= \frac{1}{4}\left(\hat{s}_+(\vec{R}) + \hat{s}_-(\vec{R})\right)\left(\hat{s}_+(\vec{R}') + \hat{s}_-(\vec{R}')\right) \\
&\quad - \frac{1}{4}\left(\hat{s}_+(\vec{R}) - \hat{s}_-(\vec{R})\right)\left(\hat{s}_+(\vec{R}') - \hat{s}_-(\vec{R}')\right) \\
&\quad + \hat{s}_z(\vec{R})\hat{s}_z(\vec{R}') . \\
&= \frac{1}{2}\left\{\hat{s}_+(\vec{R})\hat{s}_-(\vec{R}') + \hat{s}_-(\vec{R})\hat{s}_+(\vec{R}')\right\} + \hat{s}_z(\vec{R})\hat{s}_z(\vec{R}') \\
&= \hat{s}_-(\vec{R}')\hat{s}_+(\vec{R}) + \hat{s}_z(\vec{R})\hat{s}_z(\vec{R}') .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{H} = &- \frac{1}{2}\sum_{\vec{R},\vec{R}'}{}' J(\vec{R}-\vec{R}')\, \hat{s}_z(\vec{R})\hat{s}_z(\vec{R}') - g\mu_B B \sum_{\vec{R}} \hat{s}_z(\vec{R}) \\
&- \frac{1}{2}\sum_{\vec{R},\vec{R}'}{}' J(\vec{R}-\vec{R}')\, \hat{s}_-(\vec{R}')\hat{s}_+(\vec{R}) .
\end{aligned}$$

Observe that the application of spin increase operator to a function with maximum spin must lead to zero:

$$\hat{s}_+(\vec{R})\,|s\rangle_{\vec{R}} = 0, \quad \text{when } s_z = s .$$

Thus, when $\mathcal{H}$ acts on $|0\rangle$, only terms in $\hat{s}_z$ contribute

Moreover, by construction $|0\rangle$ is an eigenstate of each $\hat{S}_z(\vec{R})$ with eigenvalue S, this gives

$$\mathcal{H}|0\rangle = E_0 |S\rangle,$$

where

$$E_0 = -\frac{1}{2} S^2 \sum_{\vec{R},\vec{R}'} J(\vec{R}-\vec{R}') - Ng\mu_B B S.$$

This shows that $|0\rangle$ is indeed an eigenstate of $\mathcal{H}$.

However, we still have to demonstrate that $|0\rangle$ has the lowest eigenvalue, i.e., E_0 is lower bound of the eigen values.

Consider another eigenstate $|0'\rangle$ of $\mathcal{H}$ with eigenvalue E_0';

$$E_0' = \langle 0'|\mathcal{H}|0'\rangle.$$

When all the $J(\vec{R}-\vec{R}')$ are positive, E_0' has the lower bound

$$-\frac{1}{2}\sum_{\vec{R}\vec{R}'} J(\vec{R}-\vec{R}') \max\langle \hat{S}(\vec{R})\hat{S}(\vec{R}')\rangle - g\mu_B B \max\langle \hat{S}_z\rangle,$$

where $\max\langle X\rangle$ is the largest diagonal matrix element that operator X can assume (in any state whatsoever).

$$\hat{S}(\vec{R}) + \hat{S}(\vec{R}') \longrightarrow 2S$$

$$\max.\ \left[\hat{S}(\vec{R}) + \hat{S}(\vec{R}')\right]^2 \to 2S(2S+1) = S_{\vec{R}}^2 + S_{\vec{R}'}^2 + 2S_{\vec{R}}\cdot S_{\vec{R}'}$$

$$= S(S+1) + S(S+1) + 2\vec{S}_{\vec{R}}\cdot\vec{S}_{\vec{R}'}$$

$$\Rightarrow \max\langle \hat{S}(\vec{R})\hat{S}(\vec{R}')\rangle = S^2 \quad \vec{R}, \vec{R}'$$

Note that for $\vec{R} = \vec{R}'$, $\langle \hat{S}(\vec{R}) \rangle = S(S+1)$.

Also, $\max \langle \hat{S}_z(\vec{R}) \rangle = S$.

So, show that

$$\langle \hat{S}(\vec{R}) \cdot \hat{S}(\vec{R}') \rangle \leq S^2 \qquad \vec{R} \neq \vec{R}'$$

and

$$\langle \hat{S}_z(\vec{R}) \rangle \leq S.$$

Now use the above inequalities in the bond for E_0'. A comparison with E_0 shows that E_0' can not be less than E_0 and therefore, E_0 must be the energy of the ground state.

Hence, the state $|0\rangle$ is the ground state.

Remark: It is great to see that an intuitive classical ground state is also the quantum mechanical ground state.

Low-temperature behavior of the Heisenberg Ferromagnet: Spin Waves

A ferromagnet is perfectly ordered at $T=0$ (ignore the ~~quan~~ zero-point fluctuations) but at non-zero temperature the order is disrupted by spin waves, quantized as magnons.

In what follows we will resort to a semiclassical treatment. ~~Moreover~~

Assume that $J(\vec{R}-\vec{R}') = J(\vec{R}'-\vec{R}) = J \quad \forall \ \vec{R}, \vec{R}'$

and we will work with a one-dimensional chain in order to illustrate the spin wave excitations.

Here, each spin has two neighbors, so the Hamiltonian simplifies to

$$\hat{H} = -J \sum_i \hat{S}_i \cdot \hat{S}_{i+1},$$

where we have ignored the interaction with the external magnetic field.

The temporal evolution of the expectation value of any operator is given by

$$\frac{d\langle \hat{A} \rangle}{dt} = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle,$$

where $\hat{A}$ itself is not time dependent.

Time derivative of expectation values

Compute the expectation value of an operator $\hat{A}$ in the state ψ.

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \left\langle\frac{d\psi}{dt}\right|A|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle + \langle\psi|A\left|\frac{d\psi}{dt}\right\rangle$$

$$= \frac{-1}{i\hbar}\langle H\psi|A|\psi\rangle + \frac{1}{i\hbar}\langle\psi|A|H\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle$$

$$\boxed{\frac{d}{dt}\langle\psi|A|\psi\rangle = \frac{1}{i\hbar}\langle\psi|[A,H]|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle}.$$

If the operator does not explicity depend on time,

$$\frac{d\langle\psi|A|\psi\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|[A,H]|\psi\rangle.$$

So, expectation values of operators that commute with the Hamiltonian are constants of the motion.

Consider the position and momentum operators.

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\langle[x,H]\rangle = \frac{1}{i\hbar}\left\langle\left[x,\frac{p^2}{2m}\right]\right\rangle = \left\langle\frac{p}{m}\right\rangle.$$

(Ehrenfest theorem).

$$\frac{d\langle p\rangle}{dt} = \frac{1}{i\hbar}\langle[p,H]\rangle = \frac{1}{i\hbar}\left\langle\left[-i\hbar\frac{d}{dx}, V(x)\right]\right\rangle = -\left\langle\frac{dV(x)}{dx}\right\rangle.$$

Newton's law.

Therefore,

$$\frac{d\langle\hat{S}_j\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{S}_j, \hat{\mathcal{H}}]\rangle$$

$$= -\frac{J}{i\hbar}\langle[\hat{S}_j, \ldots + \hat{S}_{j-1}\cdot\hat{S}_j + \hat{S}_j\cdot\hat{S}_{j+1} + \ldots]\rangle$$

$$= -\frac{J}{i\hbar}\langle[\hat{S}_j, \hat{S}_{j-1}\cdot\hat{S}_j] + [\hat{S}_j, \hat{S}_j\cdot\hat{S}_{j+1}]\rangle.$$

<u>Digression</u>

Pauli spin matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$$

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, then $\vec{\sigma}\cdot\vec{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$.

Also, $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$.

and $(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2$.

$\hat{S} = \frac{1}{2}\hat{\sigma}$ (note that it is in units of $\hbar$)

or written explicitly

$$\hat{S} = \frac{\hbar}{2}\hat{\sigma}.$$

consider two operators $\vec{X}$ and $\vec{Y}$ ($\hat{S}\cdot\vec{X}$ is a scalar operator and $\hat{S}$ is a vector operator).

$$[\hat{S}\cdot\vec{X}, \hat{S}\cdot\vec{Y}] = \frac{1}{4}[(\hat{\sigma}\cdot X)(\hat{\sigma}\cdot Y) - (\hat{\sigma}\cdot Y)(\hat{\sigma}\cdot X)] = \frac{i}{4}\{\hat{\sigma}\cdot(\vec{X}\times\vec{Y}) - \hat{\sigma}\cdot(\vec{Y}\times\vec{X})\}$$

So,

$$\boxed{[\vec{S}\cdot\vec{X}, \hat{S}] = i\,\hat{S}\times\vec{X}}$$

Therefore,

$$\frac{d\langle\hat{S}_j\rangle}{dt} = \frac{J}{\hbar}\left\langle \hat{S}_j \times (\hat{S}_{j-1} + \hat{S}_{j+1})\right\rangle.$$

Now treat spins at each site as classical vectors.

The ground state has all its spins aligned, say along the z-axis.

So, in the ground state $S_j^z = S$, $S_j^x = 0$, $S_j^y = 0$.

Let us now consider a state which is a small departure from the ground state with $S_j^z \approx S$, $S_j^x, S_j^y \ll S$.

(Do the cross-product explicitly)

$$\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ S_j^x & S_j^y & S_j^z \\ S_{j-1}^x + S_{j+1}^x & S_{j-1}^y + S_{j+1}^y & S_{j-1}^z + S_{j+1}^z \end{vmatrix}$$

$$\Rightarrow \frac{\hbar}{J}\frac{d\langle S_j^x\rangle}{dt} \simeq S_j^y\left(S_{j-1}^z + S_{j+1}^z\right) - S_j^z\left(S_{j-1}^y + S_{j+1}^y\right)$$

$$= 2S S_j^y - S\left(S_{j-1}^y + S_{j+1}^y\right)$$

$$= S\left(2S_j^y - S_{j-1}^y + S_{j+1}^y\right)$$

$$\frac{\hbar}{J}\frac{d\langle S_j^y\rangle}{dt} \simeq -S_j^x\left(S_{j-1}^z + S_{j+1}^z\right) + S_j^z\left(S_{j-1}^x + S_{j+1}^x\right)$$

$$= -2S S_j^x + S S_{j-1}^x + S S_{j+1}^x$$

$$= S\left(-2S_j^x + S_{j-1}^x + S_{j+1}^x\right).$$

$$\frac{\hbar}{J}\frac{d\langle S_j^z\rangle}{dt} \simeq 0.$$

Consider normal mode solutions.

$$S_j^x = A\exp\left(i\left[qja - \omega t\right]\right)$$

$$S_j^y = B\exp\left(i\left[qja - \omega t\right]\right),$$

where q is a wave vector,

A simple substitution and a little manipulation will yield the following:

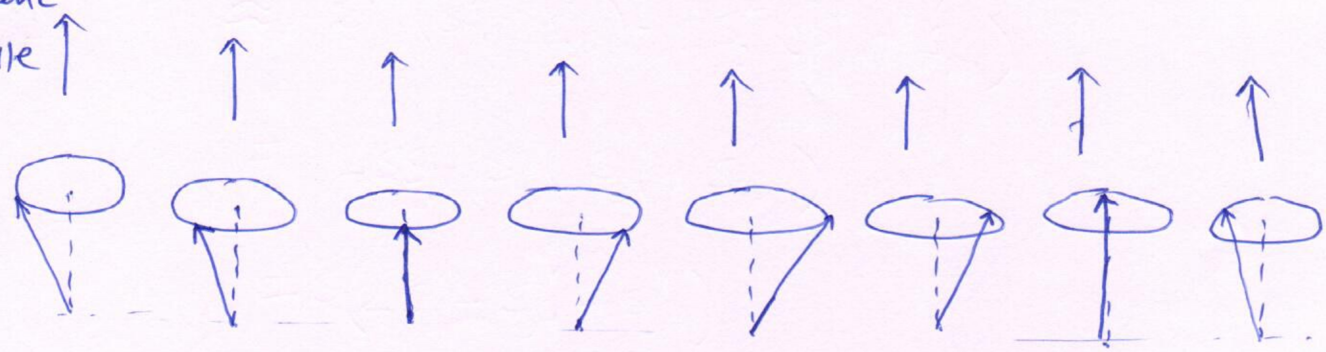
$$A = iB$$

and

Schematic representations of the orientations in a row of spins in:

Ferromagnetic ground state

Spin wave state.



Thermodynamic properties at the onset of magnetic ordering

→ The critical temperature T_c above which magnetic ordering vanishes: in

Ferromagnets/Ferrimagnets → Curie temperature

antiferromagnets → Néel temperature T_N

→ The spontaneous magnetization (or, in antiferromagnets, the sublattice magnetization) <u>drops continuously</u> to zero as the critical temperature is approached from below.

→ The observed magnetization just below T_c is well described by a power law

$$M(T) \sim (T_c - T)^{\beta},$$

where β is typically between 0.33 and 0.37.

→ In a ferromagnet, the susceptibility is observed to diverge as T drops to T_c, following the power law:

$$\chi(T) \sim (T - T_c)^{-\gamma},$$

where γ is typically between 1.3 and 1.4.

→ A singularity in the zero-field specific heat at a critical point

$$c(T) \sim (T - T_c)^{-\alpha},$$

where α is of order 0.1 or less.

Singular behavior near a continuous phase transition is characterized by a set of critical exponents
$\{\alpha, \beta, \gamma, \delta, \mu, \nu, \eta, \dots\}$.

One can try exact results, mean field theory, etc. to obtain these exponents. It is well known that the prediction of these based on the mean field theory are unreliable due to the role of fluctuations.

Question: Are these exponents independent?
Hint: Since the various thermodynamic quantities are related, these exponents can not be independent of each other.

Aim: Find the relationships between exponents, and to determine the minimum number of independent exponents needed to describe the critical point.

Summary: Only two are independent and the remaining can be obtained from the knowledge of these two through the so-called "scaling laws".

Examples: $\alpha + 2\beta + \gamma = 2$ Rushbrooke scaling law

$$\beta\delta = \beta + \gamma$$

these are examples of thermodynamic scaling laws. They relate the exponents describing thermodynamic quantities.

According to the scaling hypothesis in the neighborhood of $T=T_c$ (critical point) the magnetic equation of state

Mean Field Theory

consider the

Heisenberg hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{\vec{R}\vec{R}'} J(\vec{R}-\vec{R}')\, \hat{S}(\vec{R})\hat{S}(\vec{R}') - g\mu_B \mathbf{B} \sum_{\vec{R}}' \hat{S}_z(\vec{R}).$$

Let us focus our attention on a particular site $\vec{R}$ and isolate from $\mathcal{H}$ those terms containing $\hat{S}(\vec{R})$:

$$\boxed{\Delta\mathcal{H} = -\hat{S}(\vec{R})\cdot\left(\sum_{\vec{R}\neq\vec{R}'} J(\vec{R}-\vec{R}')\hat{S}(\vec{R}') + g\mu_B\vec{B}\right)}.$$

The above equation has the form of a spin an <u>effective</u> external field

$$\vec{B}_{eff} = \vec{B} + \frac{1}{g\mu_B}\sum_{\vec{R}'} J(\vec{R}-\vec{R}')\hat{S}(\vec{R}').$$

Note that $\vec{B}_{eff}$ is an operator and <u>depends on the detailed configuration</u> of all <u>the other spins at sites different from $\vec{R}$</u>.

How to get rid of this difficulty? Simplification?

Answer: Replace $\vec{B}_{eff}$ with its thermal equilibrium mean value.

(There are no free lunches! This simplication will cost us and also limits the scope of the mean field approximation.)

consider the case of a ferromagnet. ($J>0$), every spin has the same mean value, which is related to the total magnetization density as

$$\vec{M} = \frac{N}{V} g\mu_B \langle \hat{S}(\vec{R}) \rangle .$$

Now replace each spin in $\vec{B}_{eff}$ by its mean value,

$$\vec{B}_{eff} = \vec{B} + \lambda \vec{M},$$

where

$$\lambda = \frac{V}{N} \frac{J_0}{(g\mu_B)^2}, \quad J_0 = \sum_{\vec{R}} J(\vec{R}).$$

Assumption of the mean field theory (MFT)

The MFT of a ferromagnet assumes that the only effect of interactions is to replace the field each spin experiences by $\vec{B}_{eff}$.

Is this assumption justified? Rarely in practical set ups.

It is asking for too much to give up!

→ It requires that individual spin directions do not deviate too much from their average values.

→ The exchange interaction is of such long range that many spins contribute to $\vec{B}_{eff}$, with individual spin fluctuations about the average canceling among themselves.

To explain experimental findings one has to necessarily

→ The procedure to obtain the magnetization density is similar to what we have used on an earlier occasion in the absence of magnetic interactions.

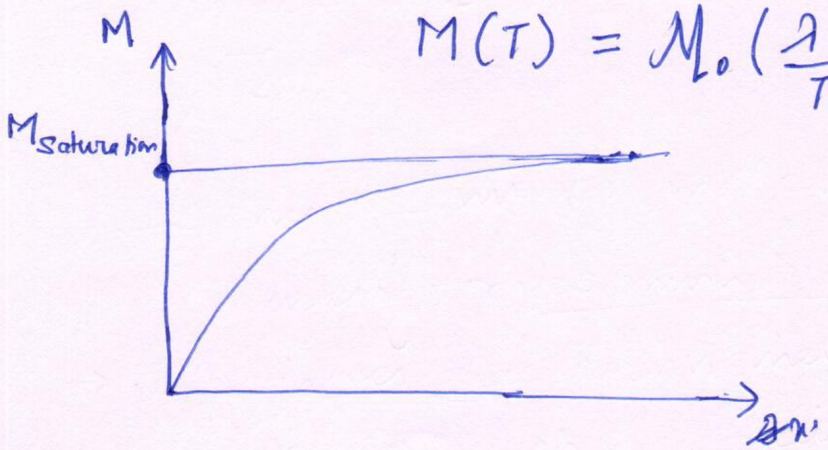
($g\mu_B \vec{S}(\vec{R}) \cdot \vec{B}_{eff}$ } looks familiar?)

$$M = \mathcal{M}_0\left(\frac{B_{eff}}{T}\right), \qquad \left[\mathcal{M}_0 \text{ is a function of } \frac{B_{eff}}{T}\right]$$

where $\mathcal{M}_0$ is the magnetization density ~~calculated~~ in the field $\vec{B}$ at temperature T, calculated in the absence of magnetic interactions.

The existence of a spontaneous magnetization, at a temperature $T < T_c$ is given by a nonzero solution to the above equation. when the applied field vanishes.

$$M(T) = \mathcal{M}_0\left(\frac{\lambda M}{T}\right).$$



Write it as a pair of eqns.

$$M(T) = M_0(x)$$
$$M(T) = \frac{T}{\lambda} x.$$

Soln. given by when $M_0(x)$ intersects $(T/\lambda)\, x$.

$$\chi_0 = \left(\frac{\partial \mathcal{M}_0}{\partial H}\right)_{H=0} \qquad \chi = \frac{\partial M}{\partial H} = \chi_0 (1 + \lambda \chi)$$

$$\Rightarrow \quad \chi = \frac{\chi_0}{1 - \lambda \chi_0}$$

Mean field Theory

- Simplest treatment of an interacting statistical mechanical system.
- Each degree of freedom is assumed to couple to the average of the other degrees of freedom.

The Weiss theory of ferromagnetism

(an example of a mean field theory)

→ A ferromagnetic substance when subjected to an external magnetic field acquires a net magnetization, which is proportional to the applied field.

→ We will work with Ising model defined in d-dimensions with nearest neighbor interactions

$$\mathcal{H}_\Omega\{S\} = -J\sum_{\langle ij\rangle} S_i S_j - H\sum_i S_i$$

→ Paramagnetic state. $(J=0)$

Partition function

$$Z_\Omega\{0,H\} = \prod_{i=1}^{N}\sum_{S_i} e^{\beta H S_i} = \prod_{i=1}^{N}\left(e^{\beta H}+e^{-\beta H}\right)$$

$$= \left[2\cosh\left(H/k_B T\right)\right]^N$$

Magnetization due to the applied field

$$M = -\frac{1}{N}\frac{\partial F}{\partial H} = k_B T\,\frac{\partial \ln\left[2\cosh\left(H/k_BT\right)\right]}{\partial H}$$

$$\boxed{M = \tanh\left(\frac{H}{k_B T}\right).}$$

→ $J \neq 0$ case. How to proceed?

Mean-field approximation

Any spin on the Lattice experiences a field, which is a combination of the externally applied magnetic field and an effective field H_I due to the spin-spin exchange interaction with other spins. The combination of these two fields then determine the response of this spin i.e. its average magnetic moment.

Note that there is nothing special about this particular spin S_i, the combined effective field should be same for all the spins. Therefore, its average magnetic moment must be the same as the magnetic moment M of the system.

The self-consistent assumption : $H_I \propto M$

We are ultimately interested in the ferromagnetic state which persists even when the external magnetic field is zero. Therefore, the mean field H_I due the spin-spin interactions must be related to the magnetic moment M, which is itself unknown to begin with.

This above discussion becomes evident, if we write the Hamiltonian in the form appropriate to a paramagnetic spin in a site dependent combined effective field:

Therefore, $\boxed{M = \tanh\left(\frac{H + H_I}{k_B T}\right)}$.

→ We have one equation and two unknowns M and H_I.

→ H_I should be a function of M and simplest to assume is $H_I \propto M$.

→ The combined effective field at lattice site i

$$H_e^i = H + H_I^i = H + \underbrace{\sum_j J_{ij} \langle S_j \rangle}_{\text{Mean field}} + \underbrace{\sum_j J_{ij} (S_j - \langle S_j \rangle)}_{\text{Fluctuation}}$$

→ We will ignore the fluctuations.

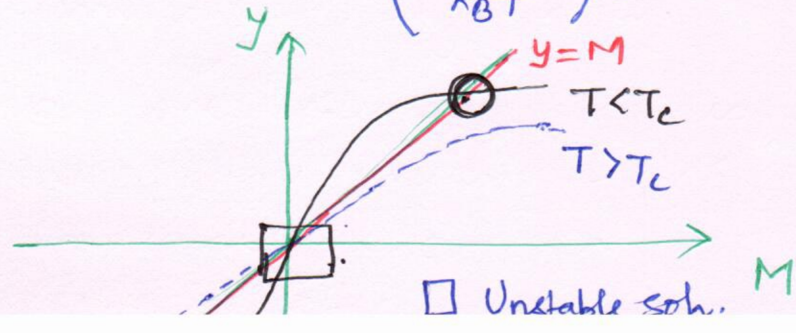
→ For a d-dimensional hypercubic lattice, the coordination number of each site is $2d$.

$$\therefore \quad H_e^i = H + 2dJM.$$

and $\boxed{M = \tanh\left(\frac{H + 2dJM}{k_B T}\right)}$. Let $T_c = \frac{2dJ}{k_B}$.

→ In the absence of H, the spontaneous magnetization

$$M = \tanh\left(\frac{2dJM}{k_B T}\right).$$



→ $T > T_c$, the only intersection is at $M=0$.

→ $T < T_c$, curves intersect at $\pm M_s(T)$.

→ Non-analyticity arises because of the

Study of Critical behavior

$$M = \tanh\left(\frac{H + 2dJM}{k_B T}\right)$$

Let $\quad T_c = \frac{2dJ}{k_B}$, $\quad \tau \equiv \frac{T_c}{T}$ and $t = \frac{T_c - T}{T_c} = 1 - \frac{1}{\tau}$

$$h \equiv \beta H = \frac{H}{k_B T} \qquad\qquad = 1 - \frac{T}{T_c}.$$

$$M = \tanh(h + M\tau) = \frac{\tanh(h) + \tanh(M\tau)}{1 + \tanh(h)\tanh(M\tau)}$$

For small x,

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 \dots$$

$$\Rightarrow \quad M + M\tanh(h)\tanh(M\tau) = \tanh(h) + \tan(M\tau).$$

$$\tanh(h) = \frac{M - \tanh(M\tau)}{1 - M\tanh(M\tau)}$$

$$h \simeq \frac{M - M\tau + \frac{1}{3}(M\tau)^3}{1 - M\left(M\tau - \frac{1}{3}(M\tau)^3\right)}$$

$$\simeq \left[M(1-\tau) + \frac{1}{3}M^3\tau^3\right]\left[1 + M\left(M\tau - \frac{1}{3}(M\tau)^3\right)\right]$$

$$\Rightarrow h \simeq M(1-\tau) + \frac{1}{3}M^3\tau^3 + M(1-\tau)M^2\tau + \frac{1}{3}M^5\tau^4$$
$$- M(1-\tau)\frac{1}{3}(M\tau)^3 M - \frac{1}{9}M^3\tau^3(M\tau)^3$$

$\Rightarrow \quad H$

Extract Critical Exponent

Case : $H = 0, \quad T \to T_c^-$

$$0 = M(1-\tau) + M^3\left(\tau - \tau^2 + \frac{1}{3}\tau^3\right) + \cdots$$

$$= M\left[1 - \frac{T_c}{T}\right] + M^3\left[\frac{T_c}{T} - \frac{T_c^2}{T^2} + \frac{1}{3}\frac{T_c^3}{T^3}\right] + \cdots$$

$$= -M\left(\frac{T_c - T}{T_c}\right)\frac{T_c}{T} + \frac{1}{3}M^3\frac{T_c^3}{T^3}\left[-3\frac{T}{T_c}\left(\frac{T_c-T}{T_c}\right) + 1\right] + \cdots$$

$$= -Mt\frac{T_c}{T} + \frac{1}{3}M^3\frac{T_c^3}{T^3}\left[-3(1-t)t + 1\right] + \cdots$$

$$\Rightarrow \quad 3t = M^2\frac{\left[1 - 3t(1-t)\right]}{(1-t)^2}$$

$$\Rightarrow \quad M^2 = 3t\frac{(1-t)^2}{(1-3t(1-t))} \simeq 3t\left[(1-t)^2(1+3t(1-t))\right]$$

$$M^2 \simeq 3t + \cdots$$

$\Rightarrow \quad M = \sqrt{3}\, t^{1/2}$, (from the leading order term).

$*\quad M \sim t^{\beta} \Rightarrow \beta = \frac{1}{2}.$

(c) The critical isotherm is the curve in the H–M plane corresponding to $T = T_c$.
Its shape near the critical point is characterized by the critical exponent δ:

$$H \sim M^{\delta}$$

At $T = T_c$

$$\frac{H}{k_B T} \simeq M^3$$

$\therefore$ Mean field value of $\delta = 3$.

(e) The isothermal magnetic susceptibility χ also diverges near T_c

$$\chi \equiv \left.\frac{\partial M}{\partial H}\right|_T$$

$$\frac{H}{k_B T} = M(1-\tau) + M^3\left(\tau - \tau^2 + \tfrac{1}{3}\tau^3\right)$$

$$\Rightarrow \quad \frac{1}{k_B T} = \chi_T (1-\tau) + 3M^2 \chi_T \left(\tau - \tau^2 + \tfrac{1}{3}\tau^2\right)$$

For $T > T_c$, $M = 0$.

$$\chi_T = \frac{1}{k_B} \cdot \frac{1}{T - T_c}$$

Critical behavior $\chi_T \sim |T - T_c|^{-\gamma}$

For $T < T_c$

$$M = \sqrt{3}\left(\frac{T_c - T}{T_c}\right)^{1/2} = \sqrt{3}\,t^{1/2}.$$

$$\therefore \quad \frac{1}{k_B T} = \chi_T\left(1 - \frac{T_c}{T}\right) + 3 \times \frac{3t}{3}\chi_T\left(3\frac{T^2}{T_c^2} - 3\frac{T}{T_c} + 1\right)\frac{T_c^3}{T^3}$$

$$= -\chi_T \frac{T_c}{T}\left(1 - \frac{T}{T_c}\right) + 3t\chi_T\left(-3(1-t)t + 1\right)\frac{T_c^3}{T^3}$$

$$\Rightarrow \quad \frac{1}{k_B T_c} = -\chi_T t + 3t\,\chi_T\left(1 - 3t(1-t)\right)\frac{1}{(1-t)^2}$$

$$\frac{1}{k_B T_c t} \simeq -\chi_T' + 3\chi_T\left(1 - 3t(1-t)\right)(1-t)^{-2}$$

$$= \chi_T\left[-1 + 3\left((1+t)^2 - 3t(1+t)\right)\right]$$

$$\simeq \chi_T\left[-1 + 3\left(1 + 2t + t^2 - 3t - 3t^2\right)\right]$$

$$\frac{1}{k_B(T_c - T)} \simeq 2\chi_T$$

$$\Rightarrow \quad \chi_T \simeq \frac{1}{2k_B}\frac{1}{T_c - T} \Rightarrow \gamma' = 1.$$

and $\boxed{\gamma' = \gamma = 1}$.

The divergence of the susceptibility below the transition temperature is governed by the critical exponent $\gamma' = \gamma = 1$.

Spatial Correlations

What significance do spatial correlations have within the scope of the MFT, wherein the fluctuations have been averaged out over the entire system?

The spatial correlations are not only the indicator of the spatial extent of the fluctuations of the order parameter, but they also tell us the way in which the order parameter varies in space in response to an "inhomogeneous external field".

For the present purposes, we will make use of thermodynamics and the "static susceptibility sum rule". The latter is an important relation between a thermodynamic quantity – the isothermal susceptibility – and the two-point correlation function.

Defn: Two-point correlation function

$$G(\vec{r}_i - \vec{r}_j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

where $\vec{r}_i$ is the spatial position of the spin s_i.

Partition function

$$Z_N = \mathrm{Tr} \exp \left[\beta J \sum_{\langle ij \rangle}' s_i s_j + H\beta \sum_i s_i \right].$$

$$\sum_i \langle s_i \rangle = \frac{1}{Z_N} \mathrm{Tr} \sum_i s_i e^{-\beta H_N} = \frac{1}{\beta Z_N} \frac{\partial Z_N(H,J)}{\partial H}$$

and $$\sum_{ij} \langle s_i s_j \rangle = \frac{1}{\beta^2 Z_N} \frac{\partial^2 Z_N}{\partial H^2}.$$

General expression for the isothermal susceptibility

$$\chi_T = \frac{\partial M}{\partial H} = \frac{1}{N\beta} \frac{\partial^2 \log Z_N}{\partial H^2}$$

$$= \frac{1}{N} k_B T \left[\frac{1}{Z_N} \frac{\partial^2 Z_N}{\partial H^2} - \frac{1}{Z_N^2} \left(\frac{\partial Z_N}{\partial H} \right)^2 \right]$$

$$= \frac{1}{N} (k_B T)^{-1} \left[\sum_{ij} \langle s_i s_j \rangle - \left(\sum_i \langle s_i \rangle \right)^2 \right]$$

$$= \frac{1}{N} (k_B T)^{-1} \sum_{ij} G(\vec{r}_i - \vec{r}_j)$$

$$\chi_T = (k_B T)^{-1} \sum_i G(\vec{x}_i)$$ (Assumed homogeneity)

Switch to integral

$$\chi_T = (a^d k_B T)^{-1} \int_\Omega d^d \vec{r} \; G(\vec{r}) .$$

This connects the divergence in χ_T with the two-point correlation function G:

G must reflect the divergence of χ_T.

Spatially homogeneous and isotropic

Benefits

1. Homogeneity means that the average values (including correlations) are translationally invariant. This implies that two-point correlations can only depend on the distance between the points and not on their absolute position in space, hence

$$G \equiv G(x_i - x_j).$$

2. Isotropy means that average values are rotationally invariant and hence two-point correlations can not dependen on the direction of the vector connecting the two points. This implies that $G \equiv G(|x_i - x_j|)$.