INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Department of Physics, IIT Kharagpur

Practice Problem Set 1, Autumn 2023-24

No submission is required.

Below is a collection of problems that have been directly taken from the book "Superconductivity, Superfluids and Condensates" by James F. Annett. Solutions to these exercises are given at the end of the same book.

The purpose of suggesting these exercises is to help in consolidating what has been covered in the lectures.

Additional practice problems (may be slightly non-trivial) would be provided by the 6th September 2023.

Q1.

(1) Using the London equation show that

$$\nabla \times (\nabla \times \mathbf{B}) = -\frac{1}{\lambda^2}\mathbf{B}$$

in a superconductor.

(2) Suppose the surface of the superconductor lies in the y-z plane. A magnetic field is applied in the z direction parallel to the surface, $\mathbf{B} = (0, 0, B_0)$. Given that inside the superconductor the magnetic field is a function of x only, $\mathbf{B} = (0, 0, B_z(x))$ show that

$$\frac{d^2 B_z(x)}{dx^2} = \frac{1}{\lambda^2} B_z(x).$$

(3) Solving the ordinary differential equation in (2) show that the magnetic field near a surface of a superconductor has the form

$$B = B_0 \exp(-x/\lambda).$$

Sketch it.

Q2. Consider a thin superconducting slab, of thickness $2L$. If an external parallel magnetic field, B_0, is applied to the slab surface, show that inside the slab the magnetic field becomes

$$B_z(x) = B_0 \frac{\cosh(x\lambda)}{\cosh(L/\lambda)}.$$

Q3.

(1) A vortex in a superconductor can be modeled as having a cylindrical core of normal metal of radius ξ_0. Use $\nabla \times (\nabla \times \mathbf{B}) = -\mathbf{B}/\lambda^2$ and the expression for curl in cylindrical polar coordinates to show that the magnetic field $B_z(r)$ outside of the core obeys the Bessel equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_z}{dr}\right) = \frac{B_z}{\lambda^2}.$$

(2) For small r, obeying $\xi_0 < r \ll \lambda$, the right hand side of the Bessel equation in (1) can be approximated by zero. Show that this approximation leads to

$$B_z(r) = a\ln(r) + b,$$

where a and b are unknown constants.

(3) Show that the current corresponding to the field $B_z(r)$ found in (2) is equal to

$$\mathbf{j} = -\frac{a}{\mu_0 r}\mathbf{e}_\phi.$$

Hence find the vector potential $\mathbf{A}$ and find a as a function of the magnetic flux enclosed by the vortex core, Φ.

(4) For larger values of r ($r \sim \lambda$ and above) assume that we can approximate the Bessel equation from (1) by:

$$\frac{d}{dr}\left(\frac{dB_z}{dr}\right) = \frac{B_z}{\lambda^2}.$$

Hence show that $B_z(r) \sim e^{-r/\lambda}$ for large r.

(5) The large r solution given in (4) is not exactly the correct asymptotic form of the solution. For large values of r, assume that

$$B_z(r) \sim r^p e^{-r/\lambda}$$

and hence show that the correct exponent is $p = -1/2$.

Q4.

(1) For type I superconductor $H_c(T)$ is the boundary between normal metal and superconductor in the H, T phase diagram. Everywhere on this boundary thermal equilibrium equires

$$G_s(T, H) = G_n(T, H).$$

Apply this equation and $dG = -SdT - \mu_0 VMdH$ to two points on the $H - T$ phase boundary (T, H) and $(T + \delta T, H + \delta H)$. Hence show that:

$$-S_s\delta T - \mu_0 VM_s\delta H = -S_n\delta T - \mu_0 VM_n\delta H,$$

where δT and δH are small, and where $S_{s/n}, M_{s/n}$ are the superconducting and normal state entropy and magnetization, respectively.

(2) Using (1), and $M_n = 0$, and $M_s = -H$ show that the latent heat per unit volume for the phase transition, $L = T(s_n - s_s)$, is given by

$$L = -\mu_0 TH_c\frac{dH_c(T)}{dT}$$

where the phase boundary curve is $H_c(T)$.

Q5. Find $|\psi|^2$, the free energy $F_s - F_n$, the netropy and the heat capacity of a superconductor near T_c, using the bulk Ginzburg-Landau free energy. Sketch their variations with temperature assuming that $a = \dot{a} \times (T - T_c)$ and $\dot{a}$ and b are constant near T_c.

Q6.

(1) Show that for one-dimensional problems, the Ginzburg-Landau equations for $\psi(x)$ can be rewritten as:

$$-\frac{d^2}{dy^2}f(y) - f(y) + f(y)^3 = 0,$$

where $\psi(x) = \psi_0 f(x/\xi)$, $y = x/\xi$ and $\psi_0 = \sqrt{|a|/b}$.

(2) Verify that

$$f(y) = \tanh(y/\sqrt{2})$$

is a solution to the equation in (2) corresponding to the boundary condition $\psi(0) = 0$. Hence sketch $\psi(x)$ near the surface of a superconductor.

(3) Often the surface boundary condition of a superconductor is not $\psi(x) = 0$, but $\psi(x) = C$ where C is a numerical constant. Show that if $C < \psi_0$ we can just translate the solution from the problem **Q5.** sideways to find a valid solution for any value of C in the range $0 \le C < \psi_0$.

(4) In the proximity effect a metal (in the half-space $x > 0$) is in contact with a superconductor (occupying region $x < 0$). Assuming that the normal metal can be described by a Ginzburg-Landau model but with $a > 0$, show that the order parameter $\psi(x)$ included in the metal by the contact with the superconductor

is approximately

$$\psi(x) = \psi(0)e^{-x/\xi(T)},$$

where $\hbar^2/2m^*\xi(T)^2 = a > 0$, and $\psi(0)$ is the order parameter of the superconductor at the interface.