INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

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Practice Problem Set 2, Autumn 2023-24

No submission is required.

The purpose of suggesting these exercises is to help in consolidating what has been covered in the lectures.

Q1. Use the Bohr-Sommerfeld quantization condition that the orbit have a circumference containing an integral number of de Broglie wavelengths to find the allowed orbits of a 2D electron moving in a uniform magnetic field. Show that each successive orbit encloses precisely one additional quantum of flux in its interior. Hint: It is important to make the distinction between the canonical momentum (which controls the de Broglie wavelength) and the mechanical momentum (which controls the velocity). The calculation is simplified if one uses the symmetric gauge $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$ in which the vector potential is purely azimuthal and independent of the azimuthal angle.

Q2. Consider non-interacting electrons confined to a 2D plane. Use Landau gauge to work out the properties of this system. Write down the Hamiltonian and Schrödinger equation. Obtain the energy levels and eigenfunctions. Use notations and constraints as discussed during the lectures. You may also want to revise and use ladder operator technique to further study the problem, as an alternate way.

Q3. Show for the Landau gauge that even though the Hamiltonian is not invariant for translations in the x direction, the physics is still invariant since the change in the Hamiltonian that occurs under translation is simply equivalent to a gauge change. Prove this for any arbitrary gauge, assuming only that the magnetic field is uniform.

Q4. Using the fact that the energy for the nth harmonic oscillator state is $(n+1/2)\hbar\omega_B$, present a semi-classical argument explaining the result that the width of the support of the wave function scales as $\sqrt{n}\ell$.

Q5. Using the Landau gauge, construct a gaussian wave packet in the lowest Landau level of the form

$$\Psi(x,y) = \int_{-\infty}^{+\infty} a_k e^{iky} e^{-\frac{1}{2\ell^2}(x+k\ell^2)^2}, \qquad (1)$$

choosing a_k in such a way that the wave packet is localized as closely as possible around some point $\mathbf{R}$. What is the smallest size wave packet that can be constructed without mixing in higher Landau levels?

Q6. It is interesting to note that the exact eigenstates in the presence of the electric field can be viewed as displaced oscillator states in the original (zero E field) basis. In this basis the displaced states are linear combinations of all the Landau level excited states of the same k. Use first- order perturbation theory to find the amount by which the $n = 1$ Landau level is mixed into the $n = 0$ state. Compare this with the exact amount of mixing computed using the exact displaced oscillator state. Show that the two results agree to first order in E. Because the displaced state is a linear combination of more than one Landau level, it can carry a finite current. Give an argument, based on perturbation theory why the amount of this current is inversely proportional to the B field, but is independent of the mass of the particle. Hint: how does the mass affect the Landau level energy spacing and the current operator?

Q7. Consider non-interacting electrons confined to a 2D plane. Use symmetric gauge to work out the properties of this system. Write down the Hamiltonian and Schrödinger equation. Obtain the energy levels and eigenfunctions. Use notations and constraints as discussed during the lectures. You may also want to revise and use ladder operator technique to further study the problem, as an alternate way. Compare these with what you obtained for the Landau gauge.

Q8. Show that the potential energy of a two-dimensional classical one-component plasma consisting of a system of particles of charge $\sigma < 0$ over a uniform positive background charge ρ in the region $x^2 + y^2 \leq R^2$, where R

is some macroscopic length scale, is given by

$$\sum_i -\frac{\rho\sigma}{4\epsilon} r_i^2 - \sum_{i>j} \frac{\sigma^2}{2\pi\epsilon} \ln |r_i - r_j|, \tag{2}$$

where ϵ is the dielectric constant.

Q9. Using Laughlin's plasma argument, show that just below the second critical magnetic field, the Abrikosov lattice makes the order parameter of superconductivity as spatially uniform as possible.

Q10. Show that the Vandermonde determinant given below

$$\Psi = \frac{1}{\sqrt{N!}} \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots \\ 1 & z_2 & z_2^2 & \cdots \\ 1 & z_3 & z_3^2 & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{vmatrix} \exp\Big(- \sum_i |z_i^2|^2/4 \Big) \tag{3}$$

can be rewritten such that the wave function can be expressed as

$$\Psi = \frac{1}{\sqrt{N!}} \prod_{i>j} (z_i - z_j) \exp\Big(- \sum_i |z_i^2|^2/4 \Big), \tag{4}$$

where N is the number of electrons.

Q11. Show carefully that the Vandermonde polynomial for N particles is in fact totally antisymmetric.

Q12. Consider a two-electron system in a magnetic field in two-dimensional space. Each electron has a magnetic flux ϕ attached. We neglect the interaction between electrons other than arising from the attached flux.

(1) Write down the Hamiltonian, and separate the center-of-mass motion and the relative motion.
(2) Obtain the eigenstates of the relative angular momentum from the part related to the relative motion.
(3) Obtain the Berry phase that arises when an electron moves around the origin along a circle of radius R.