classmate
Date
Page

Assignment - 1

Roll NO: 20PH20014

Mathematical Methods I
Curvilinear coordinates and Tensors

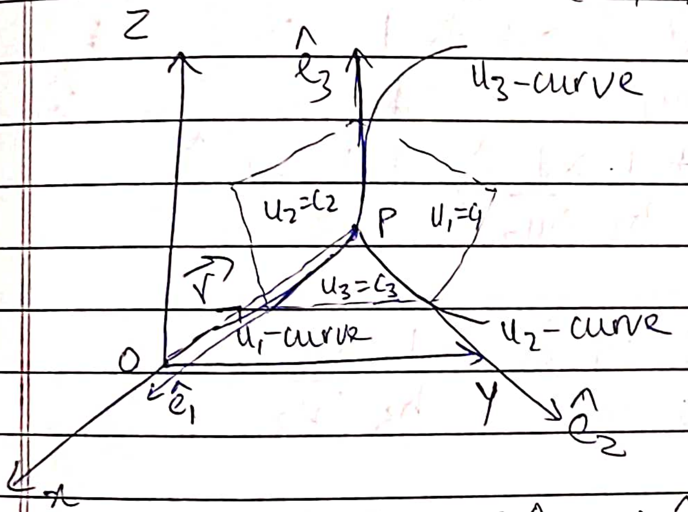
Q1. Given $\rightarrow$ $ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$

a) Gradient

$\nabla\phi$ where ϕ is a scalar funct^n $\rightarrow \phi = \phi(u_1, u_2, u_3)$

We know $d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$ — (1)

as $\vec{r} = \vec{r}(u_1, u_2, u_3)$ where $\vec{r}$ is position vector of point P which is the intersection of the three coordinate surfaces which are orthogonal curvilinear coordinates.



let $\nabla\phi = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$ — (2)

~~$\nabla\phi \cdot d\vec{r}$~~

Also, $d\vec{r} = h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$ — (3)

$d\phi = \nabla\phi \cdot d\vec{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3$ — (4)

Since $\phi = \phi(u_1, u_2, u_3) \rightarrow d\phi = \frac{\partial\phi}{\partial u_1} du_1 + \frac{\partial\phi}{\partial u_2} du_2 + \frac{\partial\phi}{\partial u_3} du_3$ — (5)

From (4) & (5)

$h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3$
$= \frac{\partial\phi}{\partial u_1} du_1 + \frac{\partial\phi}{\partial u_2} du_2 + \frac{\partial\phi}{\partial u_3} du_3$

$$\Rightarrow h_1 f_1 = \frac{\partial \phi}{\partial u_1}$$

$$\Rightarrow f_1 = \frac{\partial \phi}{\partial u_1} \cdot \frac{1}{h_1}$$

Similarly $\rightarrow$ $f_2 = \frac{1}{h_2} \frac{\partial \phi}{\partial u_2}$

$$f_3 = \frac{1}{h_3} \frac{\partial \phi}{\partial u_3}$$

$\therefore$, $\nabla \phi = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3$

$$\cancel{\nabla \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3}}$$

$$\nabla \phi = \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial u_3} \hat{e}_3$$

$$\therefore, \boxed{\begin{aligned} \nabla \phi &= \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial u_3} \hat{e}_3 \\ \nabla &= \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} \end{aligned}}$$

$\hookrightarrow$ Gradient

Divergence

Let $\vec{F}(u_1, u_2, u_3)$ be a vector pt funct^n

$$\vec{F} = F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3$$

$$\nabla \cdot \vec{F} = \nabla \cdot (F_1 \hat{e}_1) + \nabla \cdot (F_2 \hat{e}_2) + \nabla \cdot (F_3 \hat{e}_3) \quad (1)$$

$$\nabla \cdot (F_1 \hat{e}_1) = \nabla \cdot (\vec{F_1}\; h_2 h_3 \nabla u_2 \times \nabla u_3)$$

Since $\hat{e}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3$

$\cancel{\nabla \cdot \vec{F}}$ $\nabla \cdot (F_1 \hat{e}_1) = \nabla (F_1 h_2 h_3) \cdot (\nabla u_2 \times \nabla u_3)$

and $\hat{e}_2 = \dfrac{\nabla u_2}{|\nabla u_2|}$

$$\nabla \cdot (F_1 \hat{e}_1) = \nabla (F_1 h_2 h_3) \cdot \left(\frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} \right)$$

and $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$

$$\nabla \cdot (F_1 \hat{e}_1) = \nabla (F_1 h_2 h_3) \cdot \frac{\hat{e}_1}{h_2 h_3}$$

$$= \left(\left[\frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} \right] (F_1 h_2 h_3) \right) \cdot \frac{\hat{e}_1}{h_3 h_2}$$

$$\nabla \cdot (F_1 \hat{e}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (F_1 h_2 h_3)$$

$\therefore$, $\nabla \cdot (F_2 \hat{e}_2) = \dfrac{1}{h_1 h_2 h_3} \dfrac{\partial}{\partial u_2} (F_2 h_1 h_3)$

$$\nabla \cdot (F_3 \hat{e}_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (F_3 h_1 h_2)$$

$$\boxed{\therefore, \nabla \cdot \vec{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (F_2 h_1 h_3) + \frac{\partial}{\partial u_3} (F_3 h_1 h_2) \right]}$$

↳ Divergence

Curl

Let $\vec{F}(u_1, u_2, u_3)$ be a vector pt funct^n

$$\vec{F} = F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3$$

$$\nabla \times \vec{F} = \nabla \times (F_1 \hat{e}_1) + \nabla \times (F_2 \hat{e}_2) + \nabla \times (F_3 \hat{e}_3) \quad -(1)$$

classmate
Date
Page

$$\nabla \times (F_1 \hat{e}_1) = \nabla \times (F_1 h_1 \nabla u_1)$$
$$= \nabla(F_1 h_1) \times \nabla u_1 + F_1 h_1 (\nabla \times \nabla u_1)$$
$$= \nabla(F_1 h_1) \times \frac{\hat{e}_1}{h_1} + 0$$
$$= \left(\frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1}(F_1 h_1) + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2}(F_1 h_1) + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3}(F_1 h_1) \right] \times \frac{\hat{e}_1}{h_1}$$
$$= \frac{\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_3}(F_1 h_1) - \frac{\hat{e}_3}{h_1 h_2} \frac{\partial}{\partial u_2}(F_1 h_1) \quad -(2)$$

Similarly →

$$\nabla \times (F_2 \hat{e}_2) = \left[\frac{\hat{e}_3}{h_1 h_2} \frac{\partial}{\partial u_1}(F_2 h_2) - \frac{\hat{e}_1}{h_2 h_3} \frac{\partial}{\partial u_3}(F_2 h_2) \right) \quad -(3)$$

$$\nabla \times (F_3 \hat{e}_3) = \left[\frac{\hat{e}_1}{h_1 h_3} \frac{\partial}{\partial u_2}(F_3 h_3) - \frac{\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_1}(F_3 h_3) \right] \quad -(4)$$

putting (2), (3), (4) in (1) →

$$\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}$$

→ curl

classmate
Date
Page

Laplacian

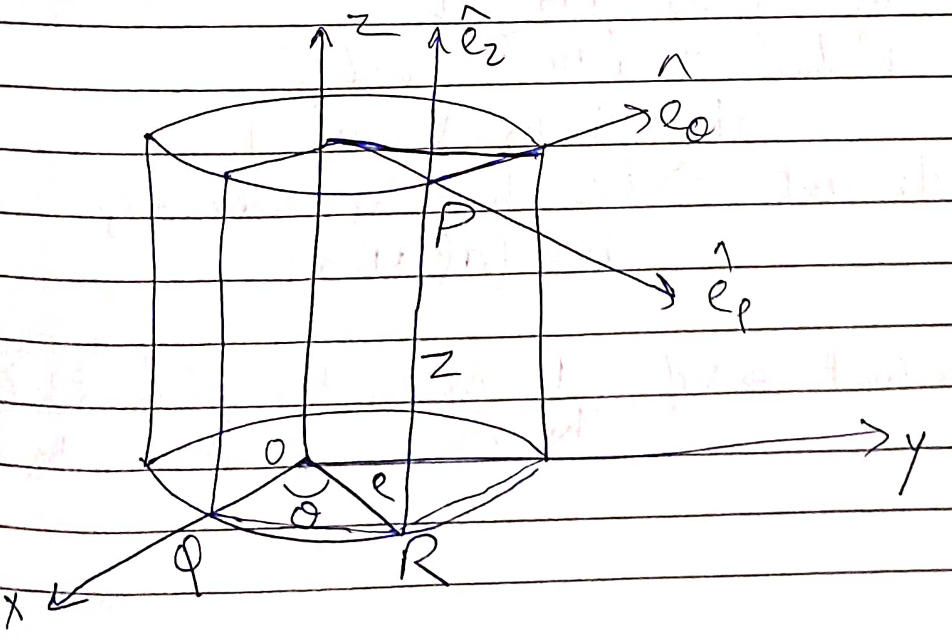
$$\nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$\nabla^2\phi = \left[\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right] \cdot \left[\frac{\hat{e}_1}{h_1}\frac{\partial \phi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial \phi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial \phi}{\partial u_3}\right]$$

$$\boxed{\begin{aligned}\nabla^2\phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right] \\ \nabla^2 &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_2h_1}{h_3}\frac{\partial}{\partial u_3}\right)\right]\end{aligned}}$$

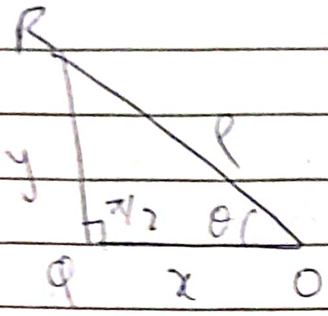
↳ Laplacian

b) for cylindrical coordinates



$u_1 = \rho, \; u_1 = \theta, \; u_3 = z$

$OR = \rho, \; \angle XOR = \theta, \; PR = z$



$\cos\theta = \frac{x}{\rho} \Rightarrow x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = PR = z$

Transformation eq^n^s

$\rho \geq 0$

$0 \leq \theta \leq 2\pi$

$-\alpha \leq z \leq \alpha$

$x = x(\rho, \theta, z)$

$y = y(\rho, \theta, z)$

$z = z(z)$

$$dx = \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial z} dz$$

$$= \cos\theta \, d\rho - \rho\sin\theta \, d\theta + 0$$

$$dy = \sin\theta \, d\rho + \rho\cos\theta \, d\theta$$

$$dz = dz$$

Arc length $= ds^2 = dx^2 + dy^2 + dz^2$

$$ds^2 = (\cos\theta \, d\rho - \rho\sin\theta \, d\theta)^2 + (\sin\theta \, d\rho + \rho\cos\theta \, d\theta)^2 + dz^2$$

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 \quad \text{(1)}$$

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

$$\Rightarrow h_1 = 1, \; h_2 = \rho, \; h_3 = 1$$

vol element $\rightarrow dV = h_1 h_2 h_3 \, du_1 du_2 du_3$

$$dV = \rho \, d\rho \, d\theta \, dz$$

Gradient $\Rightarrow \nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\hat{e}_3$

$$\boxed{\nabla\phi = \frac{\partial\phi}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z}$$

↳ gradient in cylindrical coordinates

Divergence

↳ $\nabla\cdot\vec{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{\partial}{\partial\theta}(F_\theta) + \frac{\partial}{\partial z}(\rho F_z)\right]$

Curl

↳ $\nabla\times\vec{F} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$

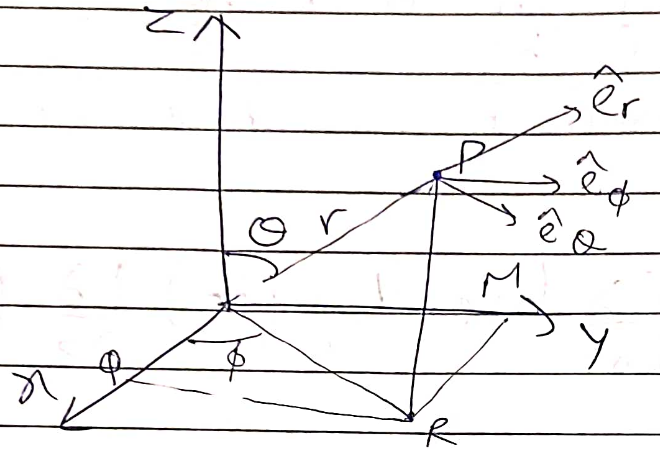
Laplacian

↳ $\cancel{\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}}$

Laplacian

↳ $\nabla^2\phi = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial\phi}{\partial z}\right)\right]$

Spherical polar coordinate system



$u_1 = r$
$u_2 = \theta$
$u_3 = \phi$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$

$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi$

$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$

$dz = \cos\theta\, dr - r\sin\theta\, d\theta$

$ds^2 = dx^2 + dy^2 + dz^2$

$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$

$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$

$ds^2 = du_1^2 + r^2 du_2^2 + r^2\sin^2\theta\, du_3^2$

$\Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ } $\rightarrow$ scale factors

Volume element : $dV = h_1 h_2 h_3\, du_1 du_2 du_3$

$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

Gradient $\rightarrow$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{e}_\phi$$

classmate

Date
Page

Divergence

$$\cancel{\nabla\cdot\vec{F} = \frac{1}{r^2\sin^2\theta}}\Big[$$

$$\nabla\cdot\vec{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(F_r\, r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(F_\theta\, r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(F_\phi\, r\right)\right]$$

Curl

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_\phi\, r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

Laplacian

$$\nabla^2 F = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial F}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial F}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial F}{\partial\phi}\right)\right]$$

Q2. $\nabla\times\left[\vec{v}\times\left(\nabla\times\vec{v}\right)\right]$

$\vec{v}\rightarrow$ fluid velocity, ~~in~~ flowing in z-direction

$\hookrightarrow \vec{V} = v(\rho)\hat{z}$

$$\nabla\times v = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & v(\rho) \end{vmatrix} = -\hat{\phi}\frac{\partial v}{\partial\rho}$$

$$\nabla \times (\nabla \times V) = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & 0 & V \\ 0 & -\frac{\partial V}{\partial \rho} & 0 \end{vmatrix}$$

$$= \hat{\rho}\, V(\rho) \frac{\partial V}{\partial \rho}$$

$$\nabla \times (\nabla \times (\nabla \times V)) = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V\frac{\partial V}{\partial \rho} & 0 & 0 \end{vmatrix}$$

$$= 0$$

Q3. $\mu_0 \vec{J} = \nabla \times \vec{B} = \nabla \times (\nabla \times \vec{e}_\phi A_\phi)$

$\cancel{A\phi = B}$

$\vec{A} = A\phi(r, \theta)$

$$\nabla \times A\phi \hat{e}_\phi = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\, \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & +A\phi\, r\sin\theta \end{vmatrix}$$

classmate

Date ____

Page ____

$$= \frac{1}{r^2\sin\theta}\left[\hat{e}_r\left(\frac{\partial}{\partial\theta}(r\sin\theta A_\phi)\right) - r\hat{e}_\theta\left(\frac{\partial}{\partial r}(r\sin\theta A_\phi)\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{e}_r r\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \hat{e}_\theta r\sin\theta\frac{\partial}{\partial r}(rA_\phi)\right]$$

$$= \hat{e}_r\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right] + \hat{e}_\theta\left(\frac{-1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)$$

$$\nabla\times(\nabla\times\hat{e}_\phi A_\phi) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_\phi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) & \frac{-r}{r}\frac{\partial}{\partial r}(rA_\phi) & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[r\sin\theta\hat{e}_\phi\left(\frac{\partial}{\partial r}\left(\frac{-\partial}{\partial r}(rA_\phi)\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right)\right)\right]$$

$$= -\frac{\hat{e}_\phi}{r}\left[\frac{\partial}{\partial r}\left(A_\phi + r\frac{\partial A_\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\left(\cos\theta A_\phi + \sin\theta\frac{\partial A_\phi}{\partial\theta}\right)\right)\right]$$

$$= -\hat{e}_\phi\left[\frac{2}{r}\frac{\partial A_r}{\partial r} + \frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2} + \frac{1}{r^2}\left(\frac{-1}{\sin^2\theta}A_\phi + \cot\theta\frac{\partial A_\phi}{\partial\theta}\right)\right]$$

$$\nabla\times(\nabla\times\hat{e}_\phi A_\phi) = -\hat{e}_\phi\left[\frac{2}{r}\frac{\partial A_\phi}{\partial r} + \frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2} - \frac{1}{r^2\sin^2\theta}A_\phi + \frac{1}{r^2}\cot\theta\frac{\partial A_\phi}{\partial\theta}\right]$$

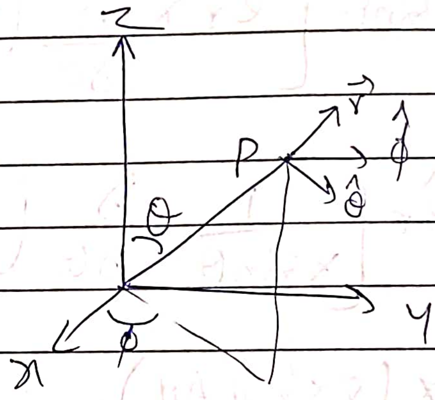
classmate
Date
Page

$$= -\hat{e}_\phi \left[\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A_\phi}{\partial\theta}\right) - \frac{A_\phi}{r^2 \sin^2\theta} \right]$$

Q4. $a_x = \frac{d^2 x}{dt^2}$

$a_y = \frac{d^2 y}{dt^2}$

$a_z = \frac{d^2 z}{dt^2}$



$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\frac{d\hat{r}}{dt} = (\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})\hat{x} + (\cos\theta\sin\phi\,\dot{\theta} + \cos\phi\sin\theta\,\dot{\phi})\hat{y} - \sin\theta\,\dot{\theta}\,\hat{z}$$

$$\frac{d\hat{\theta}}{dt} = (-\sin\theta\,\dot{\theta}\cos\phi - \cos\theta\sin\phi\,\dot{\phi})\hat{x} + (\cos\phi\cos\theta\,\dot{\phi} - \sin\theta\sin\phi\,\dot{\theta})\hat{y} - \cos\theta\,\dot{\theta}\,\hat{z}$$

$$\dot{\hat{\theta}} = \cos\theta\,\dot{\phi}\,\hat{\phi} - \dot{\theta}\,\hat{r}$$

$$\dot{\hat{r}} = \dot{\theta}\,\hat{\theta} + \sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\dot{\hat{\phi}} = -\cos\phi\,\dot{\phi}\,\hat{x} - \sin\phi\,\dot{\phi}\,\hat{y}$$

$$-\sin\theta\,\dot{\phi}\,\hat{r} = -\sin^2\theta\cos\phi\,\dot{\phi}\,\hat{x} - \sin^2\theta\sin\phi\,\dot{\phi}\,\hat{y} - \sin\theta\cos\theta\,\dot{\phi}\,\hat{z}$$

$$-\cos\theta\,\dot\phi\,\hat\theta = -\cos^2\theta\cos\phi\,\dot\phi\,\hat x - \cos^2\theta\sin\phi\,\dot\phi\,\hat y + \sin\theta\cos\theta\,\dot\phi\,\hat z$$

$$\dot{\hat\phi} = -\sin\theta\,\dot\phi\,\hat r - \cos\theta\,\dot\phi\,\hat\theta$$

$$u = \dot r = \dot r\hat r + r\dot{\hat r}$$

$$= \dot r\hat r + r(\dot\theta\hat\theta + \sin\theta\,\dot\phi\,\hat\phi)$$

$$u = \dot r\hat r + r\dot\theta\hat\theta + r\sin\theta\,\dot\phi\,\hat\phi$$

$$a = \dot u = \ddot r\hat r + \dot r\dot{\hat r} + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta + r\dot\theta\dot{\hat\theta}$$
$$+ \dot r\sin\theta\,\dot\phi\,\hat\phi + r\cos\theta\,\dot\theta\,\dot\phi\,\hat\phi + r\sin\theta\,\ddot\phi\,\hat\phi$$
$$+ r\sin\theta\,\dot\phi\,\dot{\hat\phi}$$

$$a = \ddot r\hat r + \dot r(\dot\theta\hat\theta + \sin\theta\,\dot\phi\,\hat\phi) + \dot r\dot\theta\hat\theta + r\ddot\theta\hat\theta +$$
$$r\dot\theta(\cos\theta\,\dot\phi\,\hat\phi - \dot\theta\hat r) + \dot r\sin\theta\,\dot\phi\,\hat\phi + r\cos\theta\,\dot\theta\,\dot\phi\,\hat\phi$$
$$+ r\sin\theta\,\ddot\phi\,\hat\phi + r\sin\theta\,\dot\phi(-\sin\theta\,\dot\phi\,\hat r - \cos\theta\,\dot\phi\,\hat\theta)$$

$\therefore$, $$a_\phi = 2\dot r\dot\phi\sin\theta + 2r\dot\theta\dot\phi\cos\theta + r\sin\theta\,\ddot\phi$$
$$a_\theta = r\ddot\theta + 2\dot r\dot\theta - r\sin\theta\cos\theta\,\dot\phi^2$$
$$a_r = \ddot r - r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2$$

Q5. A^{ij}_k and B^p_q are tensors

$$A^{ij}_k B^p_q = \frac{\partial x^i}{\partial \bar x^\alpha}\frac{\partial x^j}{\partial \bar x^\beta}\frac{\partial \bar x^\gamma}{\partial x^k}\frac{\partial x^p}{\partial \bar x^\delta}\frac{\partial \bar x^\epsilon}{\partial x^q}\bar A^{\alpha\beta}_\gamma \bar B^\delta_\epsilon$$

By outer product,
$$A^{ij}_k B^p_q = C^{ijp}_{kq} \text{ is a tensor}$$

if $p = i$
$$\Rightarrow A^{ij}_k B^i_q = \frac{\partial x^i}{\partial \bar x^\alpha}\frac{\partial x^j}{\partial \bar x^\beta}\frac{\partial \bar x^\gamma}{\partial x^k}\frac{\partial x^i}{\partial \bar x^\delta}\frac{\partial \bar x^\epsilon}{\partial x^q}\bar A^{\alpha\beta}_\gamma \bar B^\delta_\epsilon$$

$\Rightarrow$ there are 2 terms of $\frac{\partial x^i}{\partial \bar x^\alpha}$ and $\frac{\partial x^i}{\partial \bar x^\delta}$ here

So, it doesn't follow the tensor transformation

So, $A^{ij}_k B^i_q$ is not a tensor.

Date

Page

Q6.

consider a central force $\vec{F} = F(r)\hat{r}$

for it to be irrotational, $\vec{\nabla} \times \vec{F} = 0$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\sin\theta\hat{\phi} & r\hat{\theta} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F(r) & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \left[\hat{r}(0) - r\sin\theta\hat{\phi} \left[\underbrace{-\frac{\partial}{\partial\theta}(F(r))}_{0} \right] + r\hat{\theta} \left(\underbrace{-\frac{\partial}{\partial\phi}(F(r))}_{0} \right) \right]$$

$$= \frac{1}{r^2 \sin\theta} (0 - 0 + 0) = 0$$

$\therefore$, central forces are irrotational.