QM

Assignment - 2

Q1. $\hat{H} = \alpha(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)$

a) $\hat{A}$ is hermitian $|\phi_1\rangle$ & $|\phi_2\rangle$ are eigen states of $\hat{A}$

$\therefore$ $|\phi_1\rangle$ & $|\phi_2\rangle$ are orthogonal i.e.

~~$|\phi_1\rangle$~~ $\langle\phi_1|\phi_2\rangle = 0$

$$\hat{H}^2 = \alpha^2(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)$$

$$= \alpha^2(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_2|)$$

$$\neq \hat{H}$$

$\Rightarrow \hat{H}$ is not a projection operator

$$a^{-2}\hat{H}^2 = (|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)$$

$$= |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$$

$$= a^{-2}\hat{H}^2$$

$a^{-2}\hat{H}^2$ is a projection operator

b) $\hat{H}|\phi_1\rangle = a|\phi_1\rangle\langle\phi_2|\phi_1\rangle + a|\phi_2\rangle\langle\phi_1|\phi_1\rangle$

$= a|\phi_2\rangle$

since → $\langle\phi_2|\phi_1\rangle = 0$

also $\hat{H}|\phi_2\rangle = a|\phi_1\rangle$

∴ $|\phi_1\rangle$ & $|\phi_2\rangle$ are not eigenstate of $\hat{H}$.

c) $[\hat{H}, |\phi_1\rangle\langle\phi_1|]$ & $[\hat{H}, |\phi_2\rangle\langle\phi_2|]$

$\hat{H}|\phi_1\rangle = a|\phi_2\rangle$ & $\hat{H}|\phi_2\rangle = a|\phi_1\rangle$

∴ $[\hat{H}, |\phi_1\rangle\langle\phi_1|] = a[|\phi_2\rangle\langle\phi_1| - |\phi_1\rangle\langle\phi_2|]$

$[\hat{H}, |\phi_2\rangle\langle\phi_2|] = a[|\phi_1\rangle\langle\phi_2| - |\phi_2\rangle\langle\phi_1|]$

$[\hat{H}, |\phi_1\rangle\langle\phi_1|] = -[\hat{H}, |\phi_2\rangle\langle\phi_2|]$

d) consider $|\psi\rangle = \lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle$

$\hat{H}|\psi\rangle = a(|\phi_1\rangle\langle\phi_2| + |\phi_2\rangle\langle\phi_1|)(\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle)$

$= a(\lambda_2|\phi_1\rangle + \lambda_1|\phi_2\rangle)$

$\langle\psi|\psi\rangle = |\lambda_1|^2 + |\lambda_2|^2 = 1$

(∵ $|\psi\rangle$ is normalized)

∴ $|\lambda_1| = |\lambda_2| = 1/\sqrt{2}$

$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle \pm |\phi_2\rangle)$

Eigen values are $\pm a$ →

$\hat{H}|\psi_\pm\rangle = \pm a|\psi_\pm\rangle$

e) $\langle \phi_1 | \phi_2 \rangle = \langle \phi_2 | \phi_1 \rangle = 0$

$$\langle \phi_1 | \phi_1 \rangle = \langle \phi_2 | \phi_2 \rangle = 1$$

$$H_{11} = \langle \phi_1 | \hat{H} | \phi_1 \rangle = 0$$
$$H_{22} = \langle \phi_2 | \hat{H} | \phi_2 \rangle = 0$$

$$H_{12} = \langle \phi_1 | \hat{H} | \phi_2 \rangle = \alpha$$
$$H_{21} = \langle \phi_2 | \hat{H} | \phi_1 \rangle = \alpha$$

$$H = \alpha \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

eigen values $= \pm e$

eigen vectore $= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$

Q2. a) The eigenvalue problem for $-\frac{d^2}{dx^2}$ consists of solving the differential equation →

$$-\frac{d^2 \psi(x)}{dx^2} = a \psi(x)$$

& finding the eigenvalues a & eigenfunction $\psi(x)$. The most general solution to this equation is –

$$\psi(x) = Ae^{ibx} + Be^{-ibx}$$

with $\alpha = +b^2$. Using the boundary conditions of $\psi(x)$ at $x=0$ & $x=a$, we have →

$$\psi(0) = A + B = 0 \Rightarrow B = -A$$
$$\psi(a) = Ae^{iba} + Be^{-iba} = 0 \quad -(2)$$

A substitution of $B=-A$ into the 2nd eqn gives $A(e^{iba} - e^{-iba}) = 0$ or $e^{iba} = e^{-iba}$ which leads to $e^{2iba} = 1$. Thus, we have soln $2ba = 0$ & $\cos 2ba = 1$, so $ba = n\pi$

The eigenvalue are then given by $a_n = \frac{n^2\pi^2}{a^2}$ & the corresponding eigenvectors by $\psi_n(x) = A(e^{in\pi x/a} - e^{-in\pi x/a})$, i.e

$$a_n = \frac{n^2\pi^2}{a^2}, \quad \psi_n(x) = c_n \sin\left(\frac{\pi n x}{a}\right)$$

So, the eigenvalue spectrum of the operator $\hat{A} = -\frac{d^2}{dx^2}$ is discrete, because the eigenvalue & eigenfunctions depend on a discrete number n.

b) The normalization of $\psi_n(x)$ →

$$1 = c_n^2 \int_0^a \sin^2\left(\frac{\pi n x}{a}\right) dx = \frac{c_n^2}{2}\int_0^a \left[1 - \cos\left(\frac{2\pi n x}{a}\right)\right] dx$$

$$= \frac{c_n^2}{2} a$$

$$\Rightarrow c_n = \sqrt{\frac{2}{a}}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

The probability in the region $0 < x < a/2$ is $\rightarrow$

$$\frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{1}{a}\int_0^{a/2} \left[1-\cos\left(\frac{2\pi n x}{a}\right)\right] dx$$

$$= 1/2$$

This is expected since the total probability is 1.

$$\int_0^a |\psi_n(x)|^2 dx = 1$$

Q3. $f(x) = 3\sin\pi x$ $\rightarrow$ represents a physically acceptable wave function, since $f(x)$ & its derivative are finite, continuous, single valued everywhere & integrable.

The other functions are not wave funct's $\rightarrow$

$g(x) = 4-|x|$ $\rightarrow$ not continuous
not finite & not square integrable

$h^2(x) = 5x$ $\rightarrow$ not finite & not square integrable

$e(x) = x^2$ $\rightarrow$ not finite & not square integrable

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Q4. $|\psi\rangle = \frac{\sqrt{3}}{3}|\phi_1\rangle + \frac{2}{3}|\phi_2\rangle + \frac{\sqrt{2}}{3}|\phi_3\rangle$

a) $\langle\psi|\psi\rangle = \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2 = \frac{9}{9} = 1$

$\Rightarrow |\psi\rangle$ is normalized

$P_1 = \langle\phi_1|\psi\rangle^2$

$= \Big[\frac{\sqrt{3}}{3}\langle\phi_1|\phi_1\rangle + \frac{2}{3}\langle\phi_1|\phi_2\rangle$

$+ \frac{\sqrt{3}}{3}\langle\phi_1|\phi_3\rangle\Big]^2 = \frac{1}{3}$

IIIly, $P_2 = \frac{4}{9}$, $P_3 = \frac{2}{9}$

b) Required number $\rightarrow n_1 = 810 P_1$
(state 1)

$n_1 = \frac{810}{3} = 270$

$n_2 = 810 \times P_2 = \frac{810 \times 4}{9} = 360$

$n_3 = 810 P_3$

$= \frac{810 \times 2}{9} = 180$

$N_1, N_2, N_3 \rightarrow$ no. of systems that will be found in states $|\phi_1\rangle, |\phi_2\rangle$ & $|\phi_3\rangle$ respectively.

Q5. $|\psi\rangle \rightarrow$ not normalized

$\langle\psi|\psi\rangle = \sum_{n=1}^{5} a_n^2 \langle\phi_n|\phi_n\rangle$

$= \sum_{n=1}^{5} a_n^2 = \frac{1}{19} + \frac{4}{19} + \frac{2}{19} + \frac{3}{19} + \frac{5}{19}$

$= \frac{15}{19}$

a) Since, $E_n = \langle \phi_n | \hat{H} | \phi_n \rangle = n\epsilon_0$
$[n = 1, 2, 3, 4, 5]$

The various measurements of the energy of the system yield the values $E_1 = \epsilon_0$, $E_2 = 2\epsilon_0$, $E_3 = 3\epsilon_0$, $E_4 = 4\epsilon_0$ & $E_5 = 5\epsilon_0$ & their probabilities $\Rightarrow$

$$P_1(E_1) = \frac{|\langle \phi_1 | \psi \rangle|^2}{\langle \psi | \psi \rangle} = \left| \frac{1}{\sqrt{19}} \langle \phi_1 | \phi_1 \rangle \right|^2 \frac{19}{15}$$
$$= \frac{1}{15}$$

$$P_2(E_2) = \frac{|\langle \phi_2 | \psi \rangle|^2}{\langle \psi | \psi \rangle}$$
$$= \left| \frac{2}{\sqrt{19}} \langle \phi_2 | \phi_2 \rangle \right|^2 \times \frac{19}{15} = \frac{4}{15}$$

$|||^{ly} \Rightarrow P_3(E_3) = 2/15$
$P_4(E_4) = 3/15$

$P_5(E_5) = 5/15$

b) The average entropy of a system $\Rightarrow$

$$E = \sum_{j=1}^{5} P_j E_j = \left(\frac{1}{15}\epsilon_0\right) + \left(\frac{4}{15}(2\epsilon_0)\right) + \left(\frac{2}{15} \times 3\epsilon_0\right)$$
$$+ \left(\frac{3}{15} \times 4\epsilon_0\right) + \left(\frac{5}{15} \times 5\epsilon_0\right)$$

$$= \frac{52}{15} \epsilon_0$$

[other way is from the ~~exped~~ expectation value of the hamiltonian)

Q6. a) General ~~Solution~~ Relation is $\rightarrow$

$$\{A, B\} = \sum_j \left(\frac{\partial A}{\partial x_j} \frac{\partial B}{\partial P_j} - \frac{\partial A}{\partial P_j} \frac{\partial B}{\partial x_j} \right)$$

$\hookrightarrow$ Applying this to $\{x, P\}$

$$\{x, P\} = \frac{\partial x}{\partial x} \frac{\partial P}{\partial P} - \frac{\partial x}{\partial P} \frac{\partial P}{\partial x}$$

$$= 1$$

b) $[\hat{x}, \hat{p}] = i\hbar \Rightarrow \frac{1}{i\hbar} [\hat{x}, \hat{p}] = 1$

which is equal to Poisson bracket $\rightarrow$

$$\frac{1}{i\hbar} [\hat{x}, \hat{p}] = \{x, P\}_{classical} = 1$$

Q7. Since the funct^{n}s $\rightarrow$

$$\phi_n(r) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$ are othornormal,

$$\langle \phi_n | \phi_m \rangle = \int_0^a \phi_n^*(x) \phi_m(x)\, dx$$

$$= \frac{2}{a} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$= \delta_{mn}$$

$$\Psi(x, 0) = \frac{A}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) + \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi m}{a}\right)$$

$$= \frac{A}{\sqrt{2}} \phi_1(x) + \sqrt{\frac{3}{10}} \phi_3(x) + \frac{1}{\sqrt{10}} \phi_5(x)$$

a) Since $\langle \phi_n | \phi_m \rangle = \delta_{nm}$, the normalization at $\psi(x,0)$ gives →

$$1 = \langle \psi | \psi \rangle = \frac{A^2}{2} + \frac{3}{10} + \frac{1}{10}$$

$$A = \sqrt{\frac{6}{5}}$$

$$\psi(x,0) = \sqrt{\frac{3}{5}} \phi_1(x) + \sqrt{\frac{3}{10}} \phi_3(x) + \frac{1}{\sqrt{10}} \phi_5(x)$$

b) Since the 2nd derivative of $\phi_n(x)$ is given by → $\frac{d^2}{dx^2}\{\phi_n(x)\} = -\left(\frac{n^2\pi^2}{a^2}\right)\phi_n(x)$

& since the hamiltonian of a ~~free~~ free particle is $\hat{H} = \frac{-\hbar^2}{2m}\frac{d}{dx^2}$, the expectation value of $\hat{H}$ with respect to $\phi_n(x)$ is →

$$E_n = \langle \phi_n | \hat{H} | \phi_n \rangle = \frac{-\hbar^2}{2m}\int_0^a \phi_n^*(x) \frac{d^2}{dx^2}\phi_n(x)\,dx$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$ → has probability

$P_n(E_n) = |\langle \phi_n | \psi \rangle|^2$

Since the initial wave function $\psi(x,0)$ contains only three eigenstates of $\hat{H}$, $\phi_1(x)$, $\phi_3(x)$, $\phi_5(x)$ the results of the energy measurement along with the corresponding probabilities are →

$$E_1 = \langle \phi_1 | A | \phi_1 \rangle = \frac{\pi^2 \hbar^2}{2ma^2} \rightarrow P(E_1) = |\langle \phi_1 | \psi \rangle|^2 = \frac{3}{5}$$

$$E_3 = \langle \phi_3 | \hat{H} | \phi_3 \rangle = \frac{9\pi^2 \hbar^2}{2ma^2}, \cancel{P(E_3) = \langle 4 \phi}$$

$$\hookrightarrow P_3(E_3) = |\langle \phi_3 | \psi \rangle|^2 = \frac{3}{10}$$

$$E_5 = \langle \phi_5 | \hat{H} | \phi_5 \rangle = \frac{25 \pi^2 \hbar^2}{2ma^2}$$

$$\hookrightarrow P_5(E_5) = |\langle \phi_5 | \psi \rangle|^2 = \frac{1}{10}$$

The average energy is →

$$E = \sum_n P_n E_n = \frac{3}{5} E_1 + \frac{3}{10} E_3 + \frac{1}{10} E_5$$

$$= \frac{29 \pi^2 \hbar^2}{10ma^2}$$

c) $A = \sqrt{\frac{6}{5}}$

$$\psi(x,0) = \sqrt{\frac{3}{5}} \phi_1(x) + \sqrt{\frac{3}{10}} \phi_3(x) + \frac{1}{\sqrt{10}} \phi_5(x)$$

$$\Rightarrow \psi(x,t) = \sqrt{\frac{3}{5}} \phi_1(x) e^{-iE_1 t/\hbar} + \sqrt{\frac{3}{10}} \phi_3(x) e^{-iE_3 t/\hbar}$$

$$+ \frac{1}{\sqrt{10}} \phi_5(x) e^{-iE_5 t/\hbar}$$

d) $\psi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{5\pi x}{a}\right) e^{-iE_5 t/\hbar}$

$= \phi_5(x) e^{-iE_5 t/\hbar}$

Probability of finding the system at a time t in the state $\phi(x,t)$ is $\rightarrow$

$$P = |\langle \phi | \psi \rangle|^2 = \left| \int_0^a \psi^*(x,t) \psi(x,t) dx \right|^2$$

$$= \frac{1}{10} \left| \int_0^a \phi_5^*(x) \phi_5(x) dx \right|^2 = \frac{1}{10}$$

as $\langle \psi | \phi_1 \rangle = \langle \psi | \phi_3 \rangle = 0$
& $\langle \psi | \phi_5 \rangle = e^{iE_5 t/\hbar}$

$|||^{ly}$, $\chi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) e^{-iE_2 t/\hbar}$

$= \phi_2(x) e^{-iE_2 t/5}$

Probability of finding the system in the state $\chi(x,t)$ is zero $\rightarrow$

$$P = |\langle \chi | \psi \rangle|^2$$

$$= \left| \int_0^a \chi^*(x,t) \psi(x,t) dx \right|^2 = 0$$

as $\langle \chi | \phi_1 \rangle = \langle \chi | \phi_3 \rangle = \langle \chi | \phi_5 \rangle = 0$

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Q8. a) Since $\psi(x,0)$ can be expressed in terms of $\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right)$

as $\to$ $\psi(x,0) = \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) + \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi x}{a}\right)$

$$= \sqrt{\frac{3}{10}}\, \phi_3(x) + \frac{1}{\sqrt{10}}\, \phi_5(x)$$

$\Rightarrow \psi(x,t) = \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) e^{-iE_3 t/\hbar}$

$$+ \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi x}{a}\right) e^{-iE_5 t/\hbar}$$

$$= \frac{\sqrt{3}}{10}\, \phi_3(x) e^{-iE_3 t/\hbar} + \frac{1}{\sqrt{10}}\, \phi_5(x)\, e^{-iE_5 t/\hbar}$$

b) Since $\rho(x,t) = \psi^*(x,t)\psi(x,t)$

$$\rho(x,t) = \frac{3}{10}\phi_3^2(x) + \frac{\sqrt{3}}{10}\phi_3(x)\phi_5(x)\left[e^{i(E_3-E_5)t/\hbar} + e^{-i(E_3-E_5)t/\hbar}\right]$$

$$+ \frac{1}{10}\phi_5^2(x)$$

$$E_3 - E_5 = 9E_1 - 25E_1 = -16E_1 = \frac{-8\pi^2\hbar^2}{(ma^2)}$$

$$\rho(x,t) = \frac{3}{10}\phi_3^2(x) + \frac{\sqrt{3}}{10}\phi_3(x)\phi_5(x)\cos\left(\frac{16E_1 t}{\hbar}\right) + \frac{1}{10}\phi_5^2(x)$$

$$= \frac{3}{5a}\sin^2\left(\frac{3\pi x}{a}\right) + \frac{2\sqrt{3}}{5a}\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{16E_1 t}{\hbar}\right)$$

$$+ \frac{1}{5a}\sin^2\left(\frac{5\pi x}{a}\right)$$

Since the system is in 1-D space, the action of gradient operator on $\psi(x,t)$ & $\psi^*(x,t)$ is given by →

$$\vec{\nabla}\psi(x,t) = \left(\frac{d\psi(x,t)}{dx}\right)\vec{i}$$

$$\& \ \vec{\nabla}\psi^*(x,t) = \frac{d\psi^*}{dx}(x,t)\,\vec{i}$$

We can thus write the current density

$$\vec{J}(x,t) = \frac{i\hbar}{2m}\left(\psi(x,t)\frac{d\psi^*(x,t)}{dx} - \psi^*(x,t)\frac{d\psi(x,t)}{dx}\right)\vec{i}$$

$$\frac{d\psi(x,t)}{dx} = \frac{3\pi}{a}\sqrt{\frac{3}{5a}}\cos\left(\frac{3\pi x}{a}\right)e^{-iE_3t/\hbar}$$

$$+\frac{5\pi}{a}\frac{1}{\sqrt{5a}}\cos\left(\frac{5\pi x}{a}\right)e^{-iE_5t/\hbar}$$

$$\frac{d\psi^*(x,t)}{dx} = \frac{3\pi}{a}\sqrt{\frac{3}{5a}}\cos\left(\frac{3\pi x}{a}\right)e^{iE_3t/\hbar}$$

$$+\frac{5\pi}{a}\frac{1}{\sqrt{5a}}\cos\left(\frac{5\pi x}{a}\right)e^{iE_5t/\hbar}$$

$$\psi\frac{d\psi^*}{dx} - \psi^*\frac{d\psi}{dx} = -\frac{2i\pi\sqrt{3}}{5a^2}\left[5\sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right) - 3\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)\right]\sin\left(\frac{(E_3-E_5)t}{\hbar}\right)$$

put this in $\vec{J}(x,t)$ & using $E_3 - E_5 = -16E_1$

$$\vec{J}(x,t) = -\frac{\hbar}{m}\frac{\sqrt{3}}{5a^2}\left[5\sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right) - 3\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)\right]$$

$$\times \sin\left(\frac{16E_1 t}{\hbar}\right)\hat{i} \quad - (1)$$

c) $$\frac{\partial \rho}{\partial t} = -\frac{32\sqrt{3}E_1}{5a\hbar}\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{16E_1 t}{\hbar}\right)$$

$$= -\frac{16}{5ma^3}\pi^2\hbar\sqrt{3}\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{16E_1 t}{\hbar}\right)$$

$$\vec{\nabla}\cdot\vec{J}(x,t) = \frac{dJ(x,t)}{dx} = \frac{16\pi^2\hbar\sqrt{3}}{5ma^3}\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{16E_1 t}{\hbar}\right)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}(x,t) = 0$$

Q9. a) $$\langle \hat{P} \rangle (t) = \frac{P_0}{m} t^2 + x_0$$

Ehrenfest eqn $\rightarrow$
$$\frac{d\langle \hat{p} \rangle}{dt} = \langle [\hat{P}, \hat{V}(x,t)] \rangle / i\hbar$$

For a free particle, $\hat{V}(x,t) = 0$
$$\Rightarrow \frac{d\langle \hat{p} \rangle}{dt} = 0$$

$$\langle \hat{P} \rangle (t) = P_0$$ (linear momentum of a free particle is conserved)

$$\frac{d\langle \hat{x} \rangle}{dt} = \frac{1}{m} P_0$$

With initial condition, $\langle \hat{x} \rangle (0) = x_0$

$$\langle \hat{x} \rangle (t) = \frac{P_0 t}{m} + x_0$$

b) $[\hat{P}^2, \hat{H}] = [\hat{P}^2, \hat{P}/2m] = 0$

$\& \ \frac{\partial \hat{P}^2}{\partial t} = 0$

$$\frac{d}{dt}\langle \hat{P}^2 \rangle = \frac{1}{i\hbar} \langle [\hat{P}^2, \hat{H}] \rangle + \left\langle \frac{\partial \hat{P}^2}{\partial t} \right\rangle$$

$$d\langle \hat{x}^2 \rangle / dt = \frac{1}{i\hbar} \langle [\hat{x}^2, \hat{H}] \rangle$$

$$= \frac{1}{2im\hbar} \langle [\hat{x}^2, \hat{P}^2] \rangle$$

$\& \ \frac{\partial \hat{x}^2}{\partial t} = 0$

$$[\hat{x}, \hat{P}] = i\hbar$$

$$[\hat{x}^2, \hat{P}^2] = \hat{P}[\hat{x}^2, \hat{P}] + [\hat{x}^2, \hat{P}]\hat{P}$$

$$= \hat{P}\hat{X}[\hat{X},\hat{P}] + \hat{P}[\hat{X},\hat{P}]\hat{X}$$

$$+ \hat{X}[\hat{X},\hat{P}]\hat{P} + [\hat{X},\hat{P}]\hat{X}\hat{P}$$

$$= 2i\hbar(\hat{P}\hat{X} + \hat{X}\hat{P})$$

$$= 2i\hbar(2\hat{P}\hat{X} + i\hbar)$$

$$\frac{d}{dt}(\langle\hat{X}^2\rangle) = \frac{2}{m}\langle\hat{P}\hat{X}\rangle + \frac{i\hbar}{m}$$

c) $(\Delta x)^2 = \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$

$$\frac{d}{dt}(\Delta x)^2 = \frac{d}{dt}(\langle\hat{X}^2\rangle - 2\langle\hat{X}\rangle\frac{d}{dt}\langle\hat{X}\rangle$$

$$= \frac{2}{m}\langle\hat{P}.\hat{X}\rangle + \frac{i\hbar}{m} - \frac{2}{m}\langle\hat{X}\rangle\langle\hat{P}\rangle$$

since $\dfrac{d\langle\hat{X}\rangle}{dt} = \dfrac{\langle\hat{P}\rangle}{m}$

since $\dfrac{d}{dt}(\langle\hat{X}\rangle\langle\hat{P}\rangle) = \langle\hat{P}\rangle\dfrac{d\langle\hat{X}\rangle}{dt}$

$$= \frac{\langle\hat{P}\rangle^2}{m}$$

$$\frac{d}{dt}\langle\hat{P}\hat{X}\rangle = \frac{1}{i\hbar}\langle[\hat{P}\hat{X},\hat{H}]\rangle$$

$$= \frac{1}{2im\hbar}\langle[\hat{P}\hat{X},\hat{P}^2]\rangle = \frac{1}{m}\langle\hat{P}^2\rangle$$

$$\frac{d^2}{dt^2}(\Delta x)^2 = \frac{2}{m}\left(\frac{d}{dt}\langle \hat{p}\hat{x}\rangle - \frac{d}{dt}(\langle \hat{x}\rangle\langle \hat{p}\rangle)\right)$$

$$= \frac{2}{m}\left(\langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2\right) = \frac{2}{m^2}(\Delta p)_0^2$$

$$(\Delta p)_0^2 = \langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2$$

$$= \langle \hat{p}^2\rangle_0 - \langle \hat{p}\rangle_0^2$$

The momentum of the free particle is a constant of the motion
∴, solution is →

$$(\Delta x)^2 = \frac{1}{m^2}(\Delta p)_0^2 t^2 + (\Delta x)_0^2$$

Q10. E.F → $E_0 \cos\omega t$

$$\hat{H} = \frac{p^2}{2m} + \frac{k\hat{x}^2}{2} + qE_0\hat{x}\cos\omega t$$

a) $\frac{\partial \hat{x}}{\partial t} = 0$

$$\frac{d}{dt}\langle \hat{x}\rangle = \frac{1}{i\hbar}\langle[\hat{x}, \hat{H}]\rangle$$

$$= \frac{1}{i\hbar}\langle[\hat{x}, \frac{p^2}{2m}]\rangle$$

$$= \frac{\langle \hat{p}\rangle}{m}$$

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Since $[\hat{P}, \hat{X}] = -i\hbar [\hat{P}, \hat{X}^2]$

$$= -2i\hbar \hat{X}$$

$$\frac{\partial \hat{P}}{\partial t} = 0$$

$$\frac{d}{dt} \langle p \rangle = \frac{1}{i\hbar} \langle [\hat{P}, \hat{H}] \rangle$$

$$= \frac{1}{i\hbar} \langle [\hat{P}, \frac{1}{2} k \hat{X}^2 + qE_0 \hat{X} \cos\omega t] \rangle$$

$$= -k \langle \hat{X} \rangle - qE_0 \cos\omega t$$

$$\frac{d}{dt} \langle \hat{H} \rangle = \frac{1}{i\hbar} \langle [\hat{H}, \hat{H}] \rangle + \langle \frac{\partial \hat{H}}{\partial t} \rangle$$

$$= \langle \frac{\partial \hat{H}}{\partial t} \rangle = -qE_0\omega \langle \hat{X} \rangle \sin\omega t$$

b) $\frac{d^2}{dt^2} \langle \hat{X} \rangle = \frac{1}{m} \frac{d}{dt} \langle \hat{P} \rangle = -\frac{k}{m} \langle \hat{X} \rangle$

$$- \frac{qE_0 \cos\omega t}{m}$$

$$\langle \hat{X} \rangle(t) = \langle \hat{X} \rangle(0) \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$- \frac{q}{m} E_0 \sin\omega t + A$$

Since $\langle \hat{X} \rangle(0) = x_0$, $A = 0$

$$\langle \hat{x} \rangle (t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right) - \frac{qE_0}{m\omega} \sin \omega t$$

Q11. a)

Poisson bracket between two dynamical variables A & B is defined in terms of the generalized co-ordinates q_i & the momenta P_i of the system →

$$\{A, B\} = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial B}{\partial P_j} - \frac{\partial A}{\partial P_j} \frac{\partial B}{\partial q_j} \right)$$

$$\{I_x, I_y\} = \sum_{j=1}^{3} \left(\frac{\partial I_x}{\partial q_j} \frac{\partial I_y}{\partial P_j} - \frac{\partial I_x}{\partial p_j} \frac{\partial I_y}{\partial q_j} \right)$$

$$q_1 = x, \quad q_2 = y, \quad q_3 = z$$

$$P_1 = P_x, \quad P_2 = P_y, \quad P_3 = P_z$$

As $I_x = yP_z - zP_y$

$I_y = zp_x - xp_z$

$I_z = xp_y - yp_x$

The only partial derivatives that survive are →

$$\frac{\partial I_x}{\partial z} = -P_y, \quad \frac{\partial I_y}{\partial P_z} = -x$$

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$$\frac{\partial I_x}{\partial P_z} = y \quad \& \quad \frac{\partial I_y}{\partial z} = P_x$$

$$\{ I_x, I_y \} = \frac{\partial I_x}{\partial z}\frac{\partial I_y}{\partial P_z} - \frac{\partial I_x}{\partial P_z}\frac{\partial I_y}{\partial z}$$

$$= x P_y - y P_x = I_z$$

b) $\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y$

$\hat{L}_y = \hat{Z}\hat{P}_x - \hat{X}\hat{P}_x$

$\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$

$\hat{X}, \hat{Y}, \hat{Z}$ mutually commute
$\hat{P}_x, \hat{P}_y, \hat{P}_z$ also mutually commute

$$[\hat{L}_x, \hat{L}_y] = [\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \hat{Z}\hat{P}_x - \hat{X}P_z]$$

$$= [\hat{Y}\hat{P}_z, \hat{Z}\hat{P}_x] - [\hat{Y}\hat{P}_z, \hat{X}\hat{P}_z]$$

$$- [\hat{Z}\hat{P}_y, \hat{Z}\hat{P}_x]$$

$$+ [\hat{Z}\hat{P}_y, \hat{X}\hat{P}_z]$$

$$= \hat{Y}[\hat{P}_z, \hat{Z}]\hat{P}_x + \hat{X}[\hat{Z}, \hat{P}_z]\hat{P}_y$$

$$= i\hbar(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x)$$

$$= i\hbar\hat{L}_z$$

c) $\{I_x, I_y\} = I_z$

$$\Rightarrow [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

Q12. $|\psi(0)\rangle = \frac{1}{5}\begin{bmatrix}3\\0\\4\end{bmatrix}$

$$H = \begin{bmatrix}3 & 0 & 0\\0 & 0 & 5\\0 & 5 & 0\end{bmatrix}$$

a) Energy measurement gives the values

$E_1 = -5$, $E_2 = 3$, $E_3 = 5$. eigenvectors are →

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$|\phi_2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

The corresponding probabilities are →

$$P(E_1) = |\langle\phi_1|\psi(0)\rangle|^2$$

$$= \left|\frac{1}{5\sqrt{2}}(0 \;\; -1 \;\; 1)\begin{pmatrix}3\\0\\4\end{pmatrix}\right|^2$$

$$= 8/25$$

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$$P(E_2) = |\langle \phi_2 | \psi(0) \rangle|^2$$

$$= \left| \frac{1}{5} [1\ 0\ 0] \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right|^2 = \frac{9}{25}$$

$$P(E_3) = |\langle \phi_3 | \psi(0) \rangle|^2$$

$$= \left| \frac{1}{5\sqrt{2}} (0\ 1\ 1) \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right|^2 = 8/25$$

b) $$|\psi(0)\rangle = \frac{1}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \frac{2\sqrt{2}}{5} |\phi_1\rangle + \frac{3}{5} |\phi_2\rangle + \frac{2\sqrt{2}}{5} |\phi_3\rangle$$

$$|\psi(t)\rangle = \frac{2\sqrt{2}}{5} e^{-iE_1 t} |\phi_1\rangle + \frac{3}{5} e^{-iE_2 t} |\phi_2\rangle + \frac{2\sqrt{2}}{5} e^{-iE_3 t} |\phi_3\rangle$$

$$= \frac{1}{5} \begin{bmatrix} 3e^{-3it} \\ -4i\sin 5t \\ 4\cos 5t \end{bmatrix}$$

c) As $\langle \psi(0) | \psi(0) \rangle = 1$

$\langle \phi_n | \phi_m \rangle = \delta_{nm}$

$A|\phi_n\rangle = E_n |\phi_n\rangle$

I method (bra-ket)

$$E(0) = \langle \psi(0) | \hat{A} | \psi(0) \rangle$$

$$= \frac{8}{25} \langle \phi_1 | \hat{H} | \phi_1 \rangle + \frac{9}{25} \langle \phi_2 | \hat{H} | \phi_2 \rangle + \frac{8}{25} \langle \phi_3 | \hat{H} | \phi_3 \rangle$$

$$= \frac{8}{25}(-5) + \frac{9}{25}(3) + \frac{8}{25}(5) = \frac{27}{25}$$

$= \frac{8}{25}$

II method (Matrix algebra)

$$E(0) = \langle \psi(0) | \hat{H} | \psi(0) \rangle$$

$$= \frac{1}{25} \begin{bmatrix} 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$= \frac{27}{25}$$

III method (probabilities)

$$E(0) = \sum_{n=1}^{2} p(E_n) E_n$$

$$= \frac{8}{25}(-5) + \frac{9}{25}(3) + \frac{8}{25}(5)$$

$$= \frac{27}{25}$$

The energy at a time t is $\rightarrow$

$$E(t) = \langle \psi(t) | \hat{H} | \psi(t) \rangle$$

$$= \frac{8}{25} e^{iE_1 t} e^{-iE_1 t} \langle \phi_1 | \hat{H} | \phi_1 \rangle$$

$$+ \frac{9}{25} e^{iE_2 t} e^{-iE_2 t} \langle \phi_2 | \hat{H} | \phi_2 \rangle$$

$$+ \frac{8}{25} e^{iE_3 t} e^{-iE_3 t} \langle \phi_3 | \hat{H} | \phi_3 \rangle$$

$$= \frac{8}{25}(-5) + \frac{9}{25}(3) + \frac{8}{25}(5)$$

$$= \frac{27}{25} = E(0)$$

As expected $E(t) = E(0)$

since $\frac{d\langle \hat{H} \rangle}{dt} = 0$

d) None of the eigenvalues of $\hat{H}$ is degenerate. So, the eigenvectors $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ form a complete (orthonormal) basis, i.e., $\{\hat{H}\}$ forms a complete set of commuting operators.

Q13. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

a) The measurements of A, B, C & D give

$a_1 = -1, \; a_2 = 3, \; a_3 = 5$

$b_1 = -3, \; b_2 = 1, \; b_3 = 3$

$c_1 = -1/\sqrt{2}, \; c_2 = 0, \; c_3 = 1/\sqrt{2}$

$d_1 = -1, \; d_2 = d_3 = 1$

The corresponding ~~eigenstat~~ eigen vectors are,

$$|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$|a_3\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$|b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad |b_2\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |b_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$|c_1\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix}3\\-\sqrt{13}\\2\end{pmatrix}, \quad |c_2\rangle = \frac{1}{\sqrt{13}}\begin{pmatrix}2\\0\\-3\end{pmatrix}$$

$$|c_3\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix}3\\\sqrt{13}\\2\end{pmatrix}$$

$$|d_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\1\end{pmatrix}, \quad d_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$|d_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\1\end{pmatrix}$$

b) It can be seen that, only A & B are compertible, since the matrices A & B commute.

$$|a_1, b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix} \quad\Big|\quad |a_3, b_2\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$|a_2, b_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

form a common complete basis for A & B,
since $\hat{A} | a_n, b_m \rangle = a_n | a_n, b_m \rangle$
& $\hat{B} | a_n, b_m \rangle = b_m | a_n, b_m \rangle$

c) Eigenvalues of the operators $\{\hat{A}\}$, $\{\hat{B}\}$ & $\{\hat{C}\}$ are all non-degenerate & each one forms a separate CSCO. Additionally, since two eigenvalues of $\{\hat{D}\}$ are degenerate ($d_2 = d_3 = 1$), the operator $\{\hat{D}\}$ does not form a CSCO. Among the combinations, $\{\hat{A}, \hat{B}\}$, $\{\hat{A}, \hat{C}\}$, $\{\hat{B}, \hat{C}\}$, $\{\hat{A}, \hat{D}\}$ & $\{\hat{A}, \hat{B}, \hat{C}\}$ only.

$\{\hat{A}, \hat{B}\}$ forms a CSCO, because $\{\hat{A}\}$ & $\{\hat{B}\}$ are the only operators that commute the set of their joint eigenvectors are given by

$$|a_1, b_1\rangle, \; |a_2, b_3\rangle, \; |a_3, b_2\rangle$$

Q14. $|\psi(t)\rangle = \frac{1}{6}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

a) eigen values of A are $a_1 = 0$, $a_2 = 2$, $a_3 = 2$ (doubly degenerate). eigen states are $\rightarrow$

$|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$, $|a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

$$|a_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The probability that a measurement of A gives $\rightarrow a_1 = 0$ is given by $\rightarrow$

$$P(a_1) = \frac{|\langle a_1 | \psi(t)\rangle|^2}{\langle \psi(t) | \psi(t)\rangle}$$

$$= \frac{36}{17} \left| \frac{1}{\sqrt{2}} \frac{1}{6} \begin{pmatrix} 0 & i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right|^2 = \frac{8}{17}$$

$$\langle \psi(t) | \psi(t)\rangle$$

$$= \frac{1}{36} \begin{pmatrix} 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

$$= \frac{17}{36}$$

Since the ~~eigen~~ system was initially in ~~the~~ ~~old~~ the state $|\psi(t)\rangle$ after a measurement of A gives $a_1 = 0$, the system is left

$$|\phi\rangle = |a_1\rangle\langle a_1 | \psi(t)\rangle$$

$$= \frac{1}{2} \times \frac{1}{6} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \begin{pmatrix} 0 & i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

eigenvalues of B are $\rightarrow b_1 = -1,\ b_2 = b_3 = 1$

eigenvectors are $\rightarrow$

$$|b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ i\\ 1\end{pmatrix}$$

$$|b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ -i\\ 1\end{pmatrix},\quad |b_3\rangle = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

The probability of obtaining the value $b_2 = b_3 = 1$ for B is $\rightarrow$

$$P(b_2) = \frac{|\langle b_2|\phi\rangle|^2}{\langle\phi|\phi\rangle} + \frac{|\langle b_3|\phi\rangle|^2}{\langle\phi|\phi\rangle}$$

$$= \frac{1}{2}\left|\frac{1}{\sqrt{2}}\begin{pmatrix}0 & i & 1\end{pmatrix}\begin{pmatrix}0\\ -i\\ 1\end{pmatrix}\right|^2$$

$$+ \frac{1}{2}\left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ i\\ 1\end{pmatrix}\right|^2$$

$$= 1$$

The reason $P(b_2) = 1$ is due to the new state $|\phi\rangle$ being an eigenstate of B

In fact $|\phi = \frac{\sqrt{2}}{3}|b_2\rangle$

The probability of finding a value of θ be A t for B is given by $\rightarrow$

$$P(a_1, b_2) = P(a_1)\, P(b_2) = 8/17$$

b) Now we measure B first, then A.
The probability of measuring 1 for B →

$$p'(b_2) = \frac{|\langle b_2|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} + \frac{|\langle b_3|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle}$$

$$= \frac{36}{17}\cdot\frac{1}{36}\left[\left|\frac{1}{\sqrt{2}}\begin{pmatrix}0 & i & 1\end{pmatrix}\begin{pmatrix}1\\0\\4\end{pmatrix}\right|^2 + \left|\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\4\end{pmatrix}\right|^2\right]$$

$$= \frac{9}{17}$$

Now, we will measure A. The state of the system immediately after measuring B is given by →

a projection of $|\psi(t)\rangle$ onto $|b_2\rangle$ & $|b_3\rangle$ →

$$|\chi\rangle = |b_2\rangle\langle b_2|\psi(t)\rangle + |b_3\rangle\langle\psi_3|\psi(t)\rangle$$

$$= \frac{1}{12}\begin{pmatrix}0\\-i\\1\end{pmatrix}\begin{pmatrix}0 & i & 1\end{pmatrix}\begin{pmatrix}1\\0\\4\end{pmatrix} + \frac{1}{6}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\4\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1\\-2i\\2i\end{pmatrix}$$

Probability of finding a value of $a_1 = 0$ when measuring A is given by →

$$P'(a_1) = \frac{|\langle a_1 | X \rangle|^2}{\langle X | X \rangle} = \frac{36}{9} \left| \frac{1}{6\sqrt{2}} \begin{pmatrix} 0 & i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2i \\ 2i \end{pmatrix} \right|^2$$

$$= \frac{8}{9}$$

$$\langle X | X \rangle = \frac{9}{36}$$

∴, when measuring B, then A, the probability of finding a value of 1 for B & 0 for A is given by the product of the probabilities →

$$P(b_2, a_3) = P'(b_2)\, P'(a_1) = \frac{9}{17} \cdot \frac{8}{9} = \frac{8}{17}$$

c) $P(a_1, b_2) = P(b_2, a_1)$ → this is expected since A & B commute. The result of the successive measurements of A & B does not depend on the order in which they are carried out.

d) Neither $\{\hat{A}\}$ nor $\{\hat{B}\}$ form a CSCO since their eigenvalues are degenerates the set $\{\hat{A}, \hat{B}\}$, however does form a CSCO since the operator $\{\hat{A}\}$ & $\{\hat{B}\}$ commute. The set of eigenstates that are common to $\{\hat{A}, \hat{B}\}$ are given by →

$$|a_2, b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

$$|a_1, b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

$$|a_3, b_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Q15. $|\psi(t)\rangle = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a) eigenvalues of A are $a_1 = -1, a_2 = 0, a_3 = 1$.
The eigenvectors are →

$$|a_1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ -\sqrt{2} \\ -1 \end{pmatrix}, \quad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$|a_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Probability of getting $a_1 = -1$ is →

$$P(-1) = \frac{|\langle a_1|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{6}\left|\frac{1}{2}\begin{pmatrix} -1 & \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right|^2$$

$$= 1/3$$

as $\langle \psi(t) | \psi(t) \rangle$

$$= \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

b) eigenvalues of B are $b_1 = -1$, $b_2 = 0$, $b_3 = 1$

eigenvectors are →

$$|b_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|b_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Probability of obtaining $b_2 = 0$ for B is →

$$P(0) = \frac{|\langle b_2 | \psi(t) \rangle|^2}{\langle \psi(t) | \psi(t) \rangle}$$

$$= \frac{1}{6} \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right|^2 = 2/3$$

After measuring B, the system is left in a state $|\phi\rangle$

$$|\phi\rangle = |b_2\rangle \langle b_2 | \psi(t) \rangle$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

The probability of finding 1 when we measure A is given by →

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$$P(a_3) = \frac{|\langle a_3|\phi\rangle|^2}{\langle\phi|\phi\rangle}$$

$$= \frac{1}{4}\left|\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}0\\2\\0\end{pmatrix}\right|^2 = 1/2$$

$\langle\phi|\phi\rangle = 4$

$$P(b_2, a_3) = P(b_2)\,P(a_3)$$
$$= \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

c) Now, we measure A first, then B. Since the system is in state $|\psi(t)\rangle$, the probability of measuring $a_3 = 1$ for A is given by →

$$P(a_3) = \frac{|\langle a_3|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle}$$

$$= \frac{1}{6}\left|\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix}\right|^2 = \frac{1}{3}$$

Now, measure B, the state of the system just after measuring A (with a value of $a_3 = 1$) is given by a projection of $|\psi(t)\rangle$ onto $|a_3\rangle$

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$$|\chi\rangle = |a_3\rangle\langle a_3|\psi(t)\rangle$$

$$= \frac{1}{4}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}$$

$$= \frac{\sqrt{2}}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$$

So, the probability of finding a value of $b_2 = 0$ when measuring B is given by →

$$P'(b_2) = \frac{|\langle b_2|\chi\rangle|^2}{\langle\chi|\chi\rangle}$$

$$= \frac{1}{2}\left|\frac{\sqrt{2}}{2}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}\right|^2 = 1/2$$

as $(\langle\chi|\chi\rangle = 2)$

So, when measuring A, then B, the probability of finding a value for A & 0 for B is given by the product of the probabilities

$$P(a_3, b_2) = P'(a_3)\,P'(b_2)$$

$$= \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

d) $P(b_2, a_3)$ & $P(a_3, b_2)$ are different. This is expected since A & B do not commute. The result of the successive measurements of A & B, ∴, depends on the order in which they are carried out. The probability of obtaining 0 for B, then 1 for A is equal to 1/3.

On the other hand, the probability of obtaining 1 for A, then 0 for B is equal to 1/6. However, if the observables A & B commute, the result of the measurements will not depend on the order in which they are carried out.

e) Any operator with non-degenerate eigen values constitutes all by itself, a CSCO. Hence, each of $\{\hat{A}\}$ & $\{\hat{B}\}$

form a CSCO, since their eigenvalues are not degenerate. However, the set

$\{\hat{A}, \hat{B}\}$ does not form a CSCO since

the operators $\{\hat{A}\}$ & $\{\hat{B}\}$ do not commute.

Q16. $H = \varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$A = a \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

a) The possible energies are given by the eigenvalues of H. Diagonalization of

It gives three non-degenerate eigen energies. $E_1 = 0$, $E_2 = -\hbar\omega$ & $E_3 = 2\hbar\omega$. The eigenvectors are $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, $|\phi_2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$ which are orthonormal

b) If a measurement of the energy gives $-\hbar\omega$ it means that the system is left in the state $|\phi_2\rangle$. When we measure the next observable, A, the system is in the state $|\phi_2\rangle$. The result we obtain for A is given by any of the eigenvalues of A. Diagonalization of A gives 3 non-degenerate values, $a_1 = \sqrt{17}a$, $a_2 = 0$ & $a_3 = \sqrt{17}\,a$

eigenvectors are →

$$|a_1\rangle = \frac{1}{\sqrt{34}}\begin{pmatrix}4\\-\sqrt{17}\\1\end{pmatrix}$$

$$|a_2\rangle = \frac{1}{\sqrt{17}}\begin{pmatrix}1\\0\\-4\end{pmatrix},\ |a_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}4\\\sqrt{17}\\1\end{pmatrix}$$

Thus, when measuring A on a system which is in the state $|\phi_2\rangle$, the probability of finding $-\sqrt{17}\,a$ is given by →

$$P_1(a_1) = |\langle a_1|\phi_2\rangle|^2 = \left|\frac{1}{\sqrt{34}}(4\ -\sqrt{17}\ 1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{34}$$

$||||^{ly}$, probabilities of measuring 0 &

~~###~~ $\sqrt{17}\,a$ are $\rightarrow$

$$P_2(a_2) = |\langle a_2|\phi_2\rangle|^2$$

$$= \left|\frac{1}{\sqrt{17}}\begin{pmatrix}1 & 0 & -4\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{16}{17}$$

$$P_3(a_3) = |\langle a_3|\phi_2\rangle|^2$$

$$= \left|\frac{1}{\sqrt{34}}\begin{pmatrix}4 & \sqrt{17} & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{34}$$

c) Since the system when measuring A is in the state $|\phi_2\rangle$, the uncertainty ΔA is given by $\rightarrow$

$$\Delta A = \sqrt{\langle\phi_2|A^2|\phi_2\rangle - \langle\phi_2|A|\phi_2\rangle^2}$$

$$\langle\phi_2|A|\phi_2\rangle = a\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 4 & 0\\4 & 0 & 1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= 0$$

$$\langle\phi_2|A^2|\phi_2\rangle = a^2\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 4 & 0\\4 & 0 & 1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 4 & 0\\4 & 0 & 1\\0 & 1 & 0\end{pmatrix}$$

$$\times\begin{pmatrix}0\\0\\1\end{pmatrix} = a^2$$

$\Delta A = a$

Q7.

$$H = \epsilon \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$|\psi_0\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$$

a) A diagonalization of H gives a non-degenerate eigen energy $E_1 = \epsilon$ & a doubly degenerate value $E_2 = E_3 = -\epsilon$ whose respective eigenstates are →

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix}$$

$$|\phi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigen vectors are orthogonal since H is hermitian. $|\psi_0\rangle$ can be written as →

$$|\psi_0\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix} = \sqrt{\frac{2}{5}}\, |\phi_1\rangle + \sqrt{\frac{2}{5}}\, |\phi_2\rangle + \frac{1}{\sqrt{5}}\, |\phi_3\rangle$$

Now, since the eigenvalue is doubly degenerate, $E_2 = E_3 = -\epsilon$, the probability of measuring $-\epsilon$ is →

$$P_2(E_2) = |\langle \phi_2 | \psi_0 \rangle|^2$$

$$+ |\langle \phi_3 | \psi_0 \rangle|^2 = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$$

b) $\langle \hat{H} \rangle = P_1 E_1 + P_2 E_2$

$$= \frac{2}{5}\epsilon - \frac{3}{5}\epsilon = -\frac{1}{5}\epsilon$$

$$\langle \hat{H} \rangle = \langle \psi_0 | \hat{H} | \psi_0 \rangle$$

$$= \frac{\epsilon}{5} \begin{pmatrix} 1+i & 1+i & 1 \end{pmatrix} \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1-i \\ 1-i \\ 1 \end{pmatrix}$$

$$= -\frac{1}{5}\epsilon$$

Q18. $|\psi_0\rangle = \dfrac{\left(\sqrt{2}|\phi_1\rangle + \sqrt{3}|\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle\right)}{\sqrt{7}}$

a) A measurement of energy gives →

$E_n = \langle \phi_n | \hat{H} | \phi_n \rangle = n^2 \epsilon_0$, i.e →

$E_1 = \epsilon_0$, $E_2 = 4\epsilon_0$

$E_3 = 9\epsilon_0$, $E_4 = 16\epsilon_0$

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As $|\psi_0\rangle$ is normalized, $\langle\psi_0|\psi_0\rangle = \frac{2+3+1+1}{7} = 1$

$$P(E_n) = \frac{|\langle\phi_n|\psi_0\rangle|^2}{\langle\psi_0|\psi_0\rangle}$$

$$= |\langle\phi_n|\psi_0\rangle|^2 \qquad \left(\langle\phi_n|\phi_m\rangle = \delta_{nm} \right.$$

$$P(E_1) = \left|\sqrt{\frac{2}{7}}\langle\phi_1|\phi_1\rangle\right|^2 = \frac{2}{7}$$

$$P(E_2) = \left|\sqrt{\frac{3}{7}}\langle\phi_2|\phi_2\rangle\right|^2 = \frac{3}{7}$$

$$P(E_3) = \left|\frac{1}{\sqrt{7}}\langle\phi_3|\phi_3\rangle\right|^2 = \frac{1}{7}$$

$$P(E_4) = \left|\frac{1}{\sqrt{7}}\langle\phi_4|\phi_4\rangle\right|^2 = \frac{1}{7}$$

b) IIIly, a measurement of the observable $\hat{A}$ gives →

$$a_n = \langle\phi_n|\hat{A}|\phi_n\rangle = (n+1)a_0$$

i.e, $a_1 = 2a_0$, $a_2 = 3a_0$

$a_3 = 4a_0$,

$a_4 = 5a_0$

Since $|\psi_0\rangle$ is normalized, the required probabilities are →

$$P(a_n) = \frac{|\langle\phi_n|\psi_0\rangle|^2}{\langle\psi_0|\psi_0\rangle}$$

$$= |\langle\phi_n|\psi_0\rangle|^2$$

$$P(a_1) = \left|\sqrt{\frac{2}{7}}\langle\phi_1|\phi_1\rangle\right|^2 = \frac{2}{7}$$

$$P(a_2) = \left|\sqrt{\frac{3}{7}}\langle\phi_2|\phi_2\rangle\right|^2 = \frac{3}{7}$$

$$P(a_3) = \left|\frac{1}{\sqrt{7}}\langle\phi_3|\phi_3\rangle\right|^2 = \frac{1}{7}$$

$$P(a_4) = \left|\frac{1}{\sqrt{7}}\langle\phi_4|\phi_4\rangle\right|^2 = \frac{1}{7}$$

c) An energy measurement that gives $4\epsilon_0$ implies that the system is left in the state $|\phi_2\rangle$. A measurement of the observable A immediately afterwards leads to $\langle\phi_2|\hat{A}|\phi_2\rangle = 3a_0\langle\phi_2|\phi_2\rangle$

$= 3a_0$