1st March 2024 (1hr Class late) | Class is from 2PM on Fridays

Coherent State Path Integrals for Spins

(Take notes from Prasant)

Spin-S

$|S, +S\rangle \to S_z$ basis

$\hookrightarrow$ highest projected case

$[S_\alpha, S_\beta] = i\epsilon_{\alpha\beta\gamma} S_\gamma$ $\quad \hbar = 1$ $\to$ commutation relation for spins

$\hookrightarrow$ comp are related by this relation.

$\langle S, +S | \vec{S} | S, +S \rangle = S\hat{n}_0$, $\hat{n}_0 = (0, 0, 1)$

$\downarrow$ direction along z-direction

$|S, +S\rangle \equiv |\hat{n}_0, S\rangle$

Now define $\to$ what is coherent state?

If we have a vector in 3D, apply Euler rotations for??

Euler Rotations $\to R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$

$\hookrightarrow$ coherent state

$|\hat{n}, S\rangle = e^{-i\alpha S_z} e^{-i\beta S_y} e^{-i\gamma S_z} |\hat{n}_0, S\rangle$

$\downarrow$ different angle but same spin S

$\equiv e^{-i\alpha S_z} e^{-i\beta S_y} |\hat{n}_0, S\rangle$

Coherent State $\to$ what is the defn of coherent state?

$(\vec{S}\cdot\hat{n})|\hat{n}, S\rangle = S|\hat{n}, S\rangle$

$|\hat{n}, s\rangle = ?$

Euler rot of a vector

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

$$\langle \hat{n}, s | \vec{S}.\vec{n} | \hat{n}, s\rangle = S$$

Identity: $\underbrace{e^{i\theta S_a} S_b e^{-i\theta S_a}}_{\text{expand in power series}} = S_b \cos\theta - \epsilon_{abc} S_c \sin\theta$

$a \neq b$

for $a=b$, this identity does not work

↳ expand in power series

↳ reassemble terms of powers of θ → that will give commutation relation $[S_a, S_b]$

Prove $\langle \hat{n}, s | (\vec{S}.\hat{n}) | \hat{n}, s\rangle = S$

$$LHS = \langle \hat{n}_0, s | e^{i\beta S_y} \underbrace{e^{i\alpha S_z} (\vec{S}.\hat{n}) e^{-i\alpha S_z}}_{} e^{-i\beta S_y} |\hat{n}_0, s\rangle$$

$(S_x n_x + S_y n_y + S_z n_z)$

$\epsilon_{213} S_3$

ϵ_{231} $e^{i\beta S_y} \{ n_x \{ S_x \cos\alpha - S_y \sin\alpha \}$
$+ n_y \{ S_y \cos\alpha + S_x \sin\alpha \} + n_z S_z \} e^{-i\beta S_y}$

$e^{i\beta S_y} [S_x (n_x \cos\alpha + n_y \sin\alpha) + S_y (n_y \cos\alpha - n_x \sin\alpha)$
$+ S_z n_z] e^{-i\beta S_y}$

$(n_x \cos\alpha + n_y \sin\alpha)[S_x \cos\beta + S_z \sin\beta]$

$+ (n_y \cos\alpha - n_x \sin\alpha) S_y$

$+ (n_z \{ S_z \cos\beta - S_x \sin\beta \} = S_z$

$n_z = \cos\beta$

$n_x = \sin\beta \cos\alpha$

$n_y = \sin\beta \sin\alpha$

$\langle \hat{n}_0, S | S_z | \hat{n}_0, S \rangle = S$

Spin-coherent state $|\hat{n}, S\rangle = e^{-i\alpha S_z} e^{-i\beta S_y} |\hat{n}_0, S\rangle$

$\downarrow$

$|+S, S\rangle$

spin 1/2 system

$|\hat{n}, 1/2\rangle = e^{-i\frac{\alpha}{2}\sigma_z} e^{-i\frac{\beta}{2}\sigma_y} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hookrightarrow$ highest projected state

$\hookrightarrow |\hat{n}_0, 1/2\rangle$

$|\hat{n}, 1/2\rangle = \left(\cos\frac{\alpha}{2} - i\sigma_z \sin\frac{\alpha}{2}\right)$

$\left(\cos\beta/2 - i\sigma_y \cos\beta/2\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\alpha/2 - i\sin\alpha/2 & 0 \\ 0 & \cos\alpha/2 + i\sin\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \left(\quad \right) \times \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} = \begin{pmatrix} \cos\beta/2 \, e^{-i\alpha/2} \\ \sin\beta/2 \, e^{i\alpha/2} \end{pmatrix}$

$= |\hat{n}, 1/2\rangle$

Spin 1

$$|\hat{n},1\rangle = e^{-i\alpha\Sigma_z} e^{-i\beta\Sigma_y} \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\Sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}, \quad \Sigma_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$

$$\Sigma_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

$$|\hat{n},1\rangle = \begin{pmatrix} e^{-i\alpha}\cos^2\beta/2 \\ \frac{1}{\sqrt{2}}\sin\beta \\ e^{i\alpha}\sin^2\beta/2 \end{pmatrix} \rightarrow \text{spin 1 coherent state}$$

$$\langle\hat{n},1|\vec{\Sigma}\cdot\hat{n}|\hat{n},1\rangle = 1 \quad \rightarrow \text{since spin is 1}$$

$$\langle\hat{n},\tfrac{1}{2}|\tfrac{1}{2}(\vec{\sigma}\cdot\vec{n})|\hat{n},\tfrac{1}{2}\rangle = \tfrac{1}{2}$$

for PI → find coherent state
identity operator
Then write PI

Identity operator using coherent state

$$|\hat{n},s\rangle\langle\hat{n},s| \rightarrow \text{only funct}^{n} \text{ of } (\alpha,\beta)$$

$$\frac{1}{4\pi}\int_0^{\pi}\sin\beta\, d\beta \int_0^{2\pi} d\alpha\; |\hat{n},s\rangle\langle\hat{n},s| = \text{diag}(1,1,1,1) = \mathbb{1}_{(2s+1, 2s+1)}$$

$$\frac{1}{4\pi}\int_0^{\pi} \sin\beta \, d\beta \int_0^{2\pi} d\alpha \, |\hat{n}, s\rangle\langle \hat{n}, s| = \mathbb{1}_{(2s+1, 2s+1)}$$

↳ normalization constant
(he's not sure of it → check online)

$H(\vec{S})$

$$\langle \hat{n}_f, t_f | e^{-iHt} | \hat{n}_i, t_i\rangle$$

↳ Insert coherent state, after doing this, we generalise it to this →

$$\langle \hat{n}_{j+1}, t_{j+1} | e^{-iH(\Delta t)} | \hat{n}_j, t_j\rangle$$

$$\simeq \langle \hat{n}_{j+1}, t_{j+1} | (1 - iH\Delta t) | \hat{n}_j, t_j\rangle$$

$$|\hat{n}_{j+1}, t_{j+1}\rangle \simeq | n_j, t_j\rangle + (\Delta t) | \partial_t \hat{n}_j, t_j\rangle$$

$$\langle n_j, t_j | + (\Delta t)\langle \partial_t n_j, t_j | \Big) (1 - iH\Delta t) | n_j, t_j\rangle$$

$$= 1 + (\Delta t)\left[\langle \partial_t n_j, t_j | n_j, t_j\rangle - i\langle n_j, t_j | H | n_j, t_j\rangle\right.$$

$$\langle \hat{n}_{j+1}, t_{j+1} | e^{-iH\Delta t} | \hat{n}_j, t_j\rangle \cancel{\simeq 1 + (\Delta t)\left[\langle \partial_t n_j, t_j | n_j, t_j\rangle\right.}$$

$$\simeq 1 + (\Delta t)\left[\langle \partial_t n_j, t_j | n_j, t_j\rangle - i\langle n_j, t_j | H | n_j, t_j\rangle\right]$$

$\hbar = 1$

$$Z \equiv \int (Dn(x)) \, e^{+i\int_{t_i}^{t_f} dt \, [i\langle \partial_t \hat{n} | \hat{n}\rangle - \langle \hat{n}, t | H | \hat{n}, t\rangle]}$$

$$\equiv \int Dn(x) \, e^{i\int_{t_i}^{t_f} [i\langle \hat{n} | \partial_t \hat{n}\rangle - \underbrace{H(\hat{n}_i)}_{\hookrightarrow \text{ how??}}]}$$

$H \to$ spin operator $\vec{S}.\hat{n}$

eigenstate $|\hat{n}, t\rangle$

Ex $\to$ spin 1/2 & spin 1 (atmost)
General case $\to$ difficult

~~Calculate~~

Prove: $\langle n | \partial_t n\rangle = i S(1 - \cos\beta)\partial_t \alpha \to$ Approach??
$\hookrightarrow$ General for any spin

Check for $S = 1/2$ and 1

$$\boxed{\text{Spin-lagrangian} = S(\cos\beta - 1)\partial_t \alpha - H}$$

Ask??

Not part of class

$$H_{1st} = -\frac{\hbar^2}{2m}\sum_i (\nabla_i^2 + V) + \sum_{i<j} V(r_i - r_j)$$

$$H_{2nd} = \int dr \, \psi^\dagger(r)\left[\frac{-\hbar^2}{2m}\nabla_r^2 + V(r)\right]\psi(r)$$

$$+ \int d\vec{r} \int d\vec{r}\,' \, \psi^\dagger(r)\psi^\dagger(r') V(r - r') \cancel{\psi(r)} \, \psi(r')\psi(r)$$

$$\hat{n} = \psi^\dagger(r)\psi(r)$$

then apply FT on this

$$H_{2nd} \equiv \sum_n \epsilon_n \underbrace{c_n^\dagger c_n}_{\substack{\text{density}\\ \text{operators}}}$$

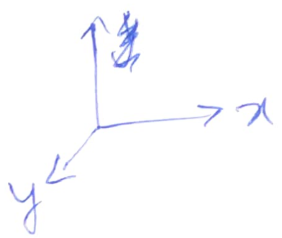
$$\psi(r) = \sum_n \psi_n(r)\, c_n$$

1st March Class Notes from Prasenjit

coherent state PI for spins:

spin $1/2$ $\quad |\uparrow\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$

Rotate about y-axis by an angle θ



$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \qquad ; \quad e^{-i\left(\frac{\hbar}{2}\sigma_y\right)\theta}\begin{pmatrix}1\\0\end{pmatrix}$$

↳ $e^{-\frac{i}{\hbar}L_z\theta}$ (how??)

~~$e^{-i\left(\frac{\hbar}{2}\sigma_y\right)\theta}\begin{pmatrix}1\\0\end{pmatrix}$~~

$$e^{-i\left(\frac{\hbar}{2}\sigma_y\right)\theta}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= e^{-i\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\theta}\begin{pmatrix}1\\0\end{pmatrix}$$

didn't get this < $\quad = e^{-\hbar/2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\theta}\begin{pmatrix}1\\0\end{pmatrix}$ what happened to $\hbar$?

$$= \left(\cos\theta/2 - i\sigma_y \sin\frac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

"Spin-s" $\quad |s, +s\rangle$

↳ eigenstate of S_z operator

$-s, -s+1, \dots, +s \to (2s+1)$ multiplets

$\langle s, +s | \vec{S} | s, +s \rangle = S\hat{n}_0$ why write that? Isn't it evident? $\hat{n}_0 = (0,0,1)$

z-direction

$[S_\alpha, S_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} S_\gamma$

$[S_x, S_y] = i\hbar S_z$

$|s, +s\rangle \equiv |\hat{n}_0, s\rangle$

what's this??

coherent state

$(\vec{S}\cdot\hat{n})|\hat{n}, s\rangle = \hbar s |\hat{n}, s\rangle$ → coherent state

~~chosen~~ arbitrary direction

$\langle \hat{n}, s | (\vec{S}\cdot\hat{n}) | \hat{n}, s\rangle = \hbar s$

Now, what is $|\hat{n}, s\rangle = ?$

Euler rotation of vectors:

$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$

Euler rotation matrix for spin S. $S_z = \frac{1}{2}\sigma_z$

coherent state

$|\hat{n}, s\rangle = e^{-i\alpha S_z} e^{-i\beta S_y} e^{-i\gamma S_z} |\hat{n}_0, s\rangle$

eigenstate of (what??) $|\hat{n}_0, s\rangle$

prove that this state is coherent state

$|\hat{n}, s\rangle = e^{-i\alpha S_z} e^{-i\beta S_y} |\hat{n}_0, s\rangle$

Identity $e^{i\theta S_a} S_b e^{-i\theta S_a} = S_b \cos\theta - \epsilon_{abc} S_c \sin\theta$

$$\left(1 + i\theta S_a + \frac{1}{2!}(i\theta S_a)^2 + \frac{1}{3!}(i\theta S_a)^3 + \dots\right) S_b \left(1 - i\theta S_a + \frac{1}{2!}(-i\theta S_a)^2 + \frac{1}{3!}(-i\theta S_a)^3 + \dots\right)$$

Term without S_a (Ask??)

$$S_b + i\theta\{S_a S_b - S_b S_a\} + \frac{(i\theta)^2}{2!}\{S_a^2 S_b + S_b S_a^2 - 2 S_a S_b S_a\}$$

$$+\frac{(i\theta)^3}{3!}\{S_a^3 S_b - S_b S_a^3 + 3 S_a S_b S_a^2 - 3 S_a^2 S_b S_a\}$$

(Can S_a, S_b commute?) They don't

$$= S_b + i\theta[S_a, S_b] - \frac{\theta^2}{2!}[S_a, [S_a, S_b]]$$

$$+ \frac{(i\theta)^3}{3!}[[S_a, [S_a, [S_a, S_b]]]]$$

$$[S_a, [S_a, [S_a, S_b]]]$$

$$= [S_a, [S_a, [S_a S_b - S_b S_a]]]$$

$$= [S_a S_a^2 S_b - S_a S_b S_a + \dots$$

$$S_b + i\theta(i\hbar\epsilon_{abc} S_c) - \frac{\theta^2}{2!}(i\hbar)\epsilon_{abc}[S_a, S_c]$$

$$-\frac{\theta^2}{2!} i\hbar \epsilon_{ab}\epsilon_{acd} S_d$$

$$+\frac{\theta^2}{2!}(i)^2 \epsilon_{acb}\epsilon_{acd} S_d$$

$S_b + i\theta\,(i\hbar\,\epsilon_{abc}\,S_c) - \frac{\theta^2}{2!}(i\hbar)\,\epsilon_{abc}\,[S_a, S_c]$

$\downarrow$

$-\frac{\theta^2}{2!}\, i\hbar\,\epsilon_{abc}\,\epsilon_{acd}\,S_d$

$\downarrow$

$+\frac{\theta^2}{2!}\,(i)^2\,\epsilon_{acb}\,\epsilon_{acd}\,S_d$

$\downarrow$

$-\frac{\theta^2}{2!}\,S_b$

$= S_b\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right] - \epsilon_{abc}\,S_c\left[\theta - \frac{\theta^3}{3!} + \cdots\right]$

$= S_b \cos\theta - \sin\theta\;\epsilon_{abc}\,S_c$ $\qquad \epsilon_{321}\,S_1$

Prove that $\langle \hat{n}, s | (\vec{S}\cdot\hat{n}) | \hat{n}\, s\rangle = s$

$\downarrow$

LHS $\qquad$ When $a \neq b$

$\langle \hat{n}_0, s | e^{i\beta S_y}\, \underbrace{e^{i\alpha S_z}\,(\vec{S}\cdot\hat{n})\,e^{-i\alpha S_z}}_{(S_x n_x + S_y n_y + S_z n_z)}\, e^{-i\beta S_y} | \hat{n}_0, s\rangle$

$\langle \hat{n}_0, s \,|\, e^{i\beta S_y} \{ n_x \{ S_x \cos\alpha - S_y \sin\alpha \}$ $\qquad$ how??

$+ n_y \{ S_y \cos\alpha + S_x \sin\alpha \}$

$+ n_z S_z \}\; e^{-i\beta S_y} \,|\, \hat{n}_0, s\rangle$

$= e^{i\beta S_y} [S_x (n_x \cos\alpha + n_y \sin\alpha)$
$+ S_y (n_y \cos\alpha - n_x \sin\alpha) + S_z n_z] e^{-i\beta S_y}$

$= (n_x \cos\alpha + n_y \sin\alpha) [S_x \cos\beta + S_z \sin\beta] +$
$(n_y \cos\alpha - n_x \sin\alpha) [S_y] + n_z \{S_z \cos\beta - S_x \sin\beta\}$

(What is going on??)

$= S_z$

$n_z = \cos\beta$
$n_x = \sin\beta \cos\alpha$
$n_y = \sin\beta \sin\alpha$

$\langle \hat{n}_0 . S | S_z | \hat{n}_0 . S \rangle = S$ → how?? why not S_z??

spin coherent state $|\hat{n}, S\rangle$
$e^{-i\alpha S_z} e^{-i\beta S_y} |\hat{n}_0, S\rangle$

$|\hat{n}, S\rangle = e^{-i\alpha S_z} e^{-i\beta S_y} \underset{\downarrow}{|\hat{n}_0, S\rangle}$
$|+S, S\rangle$

Spin 1/2 System

$|\hat{n}, 1/2\rangle = e^{-i\alpha/2\, \sigma_z} e^{-i\beta/2\, \sigma_y} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ → $|n_0, 1/2\rangle$

$= (\cos\alpha/2 - i\sigma_z \sin\alpha/2)(\cos\beta/2 - i\sigma_y \sin\beta/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e^{-i\alpha/2\, \sigma_z}$
$= \begin{bmatrix} \cos\alpha/2 - i\sin\alpha/2 & 0 \\ 0 & \cos\alpha/2 + i\sin\alpha/2 \end{bmatrix} \begin{bmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\hat{n}, 1/2\rangle = \begin{pmatrix} \cos\beta/2 \; e^{-i\alpha/2} \\ \sin\beta/2 \; e^{i\alpha/2} \end{pmatrix}$$

What is coherent state for spin-1?

$$|\hat{n}, 1\rangle = e^{-i\alpha \Sigma_z} e^{-i\beta \Sigma_y} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Spin 1 matrix,

$$\Sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Sigma_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

not getting this exp??

$$|\hat{n}, 1\rangle = \begin{pmatrix} e^{-i\alpha} \cos^2 \beta/2 \\ \frac{1}{\sqrt{2}} \sin\beta \\ e^{i\alpha} \sin^2 \beta/2 \end{pmatrix} \qquad \langle \hat{n}, 1 | (\vec{\Sigma} \cdot \hat{n}) | \hat{n}, 1\rangle = 1$$

Why $(\frac{1}{2})$??

$$\langle \hat{n}, 1/2 | \frac{1}{2} (\vec{\sigma} \cdot \hat{n}) | \hat{n}, +\tfrac{1}{2}\rangle = 1/2$$

Identity operation using coherent state

$$|\hat{n}, s\rangle\langle \hat{n}, s|$$

function of (α, β)

(Remaining parts - I noted down earlier → Check)

7th March 2024

XY Model

Kosterlitz Thouless

(Topological Phase Transition occurs)

in a continuum model.

There is no order parameter involved in this kind of PT.

Top. no. changes

More generalized model : (nonlinear σ-model)

Generality functional

$$Z = \int (D n(r))\, e^{\beta J \sum_{\langle r, r' \rangle} \hat{n}(r) \cdot \hat{n}(r')}$$

$$H = -J \sum_{\langle r, r' \rangle} \hat{n}(r) \cdot \hat{n}(r')$$ $|\hat{r}| = 1$

(Ferromagnetic interaction) $K = \beta J$

$\downarrow$ XY model

$$\hat{n}(r) = \hat{i} \underbrace{\cos\theta(\vec{r})}_{} + \hat{j} \sin\theta(\vec{r})$$

angular no for $\hat{n}(r)$ vector. Not the normal θ.

2-pt correlation function : $\langle e^{i\alpha\theta(r)}\, e^{i\beta\theta(r')} \rangle$

$$= C_{\alpha\beta}(\vec{r} - \vec{r}')$$

$C_{\alpha\beta}(\vec{r}-\vec{r}') = \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$

$= \int (D\Theta(r)) \, e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} e^{K\sum_{\langle r,r'\rangle} \cos(\Theta(r)-\Theta(r'))}$

Z

$\Theta \rightarrow$ its the angular variable

$\Rightarrow$ should be invariant under $\Theta \rightarrow \Theta + \underbrace{2\pi}_{\text{any multiple of } 2\pi}$

$\hookrightarrow$ To keep them as a single-valued functn

$\Rightarrow \alpha, \beta$ are integers due to invariance

In 3rd exp $\rightarrow$ this transformation doesn't matter if we take the difference.

What about 1st & 2nd exponents?

$\hookrightarrow$ why any integers?

Action is invariant under $\Theta(r) \rightarrow \Theta(r) + \Theta_0$

$\equiv S(\Theta(r))$

If $C_{\alpha\beta}$ is also invariant under this transformation, $S \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$

$= \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$

$= e^{i(\alpha+\beta)\Theta_0} \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$

$= \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$

$\Rightarrow \alpha + \beta = 0$

$$C_\alpha(\vec{r}-\vec{r}') = \langle e^{i\alpha\theta(r)} e^{-i\alpha\theta(r')} \rangle$$

$$C(\vec{r}-\vec{r}') = \langle e^{i\theta(\vec{r})} e^{-i\theta(r')} \rangle$$

where did the α go? Take $\alpha=1$

We will take only real part $\rightarrow \stackrel{L}{=} \langle \cos(\theta(r)-\theta(r')) \rangle$

$$= \underbrace{C(\vec{r}-\vec{r}')}$$

$\downarrow$ relative quantity $\vec{r}$

$$C(\vec{r}) = \langle \cos(\theta(0)-\theta(\vec{r})) \rangle$$

$$= \frac{\int (D\theta(r)) \cos(\theta(0)-\theta(\vec{r}))\, e^{K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r'))}}{\int (D\theta(r))\, e^{K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r'))}}$$

Temperature Driven Phase Transition

High-T limit: $K \to 0$

$$e^{K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r'))} \simeq 1 + K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r')) + \dots$$

$$Z \sim \int \left[\prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij\rangle} (1+K\cos(\theta_i-\theta_j))\right]$$

$$C_{\alpha\beta}(\vec{r}-\vec{r}') = \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$$

$$= \int (D\Theta(r))\, e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} e^{K\sum_{\langle r,r'\rangle} \cos(\Theta(r)-\Theta(r'))}$$

$$Z$$

$\Theta \to$ its the angular variable

$\Rightarrow$ should be invariant under $\Theta \to \Theta + \underbrace{2\pi}$

$\hookrightarrow$ To keep them as a single-valued functn — any multiple of 2π

$\Rightarrow \alpha, \beta$ are integers due to invariance

In 3rd exp $\to$ this transformation doesn't matter if we take the difference.

What about 1st & 2nd exponents?

$\hookrightarrow$ why any integers?

Action is invariant under $\Theta(r) \to \Theta(r) + \Theta_0$

$$\equiv S(\Theta(r))$$

If $C_{\alpha\beta}$ is also invariant under this transformation, $S\langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')}\rangle$

$$= \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')}\rangle$$

$$= e^{i(\alpha+\beta)\Theta_0} \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')}\rangle$$

$$= \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')}\rangle$$

$$\Rightarrow \alpha + \beta = 0$$

$$C_\alpha(\vec{r}-\vec{r}') = \langle e^{i\alpha\theta(r)} e^{-i\alpha\theta(r')} \rangle$$

$$C(\vec{r}-\vec{r}') = \langle e^{i\theta(\vec{r})} e^{-i\theta(r')} \rangle$$

where did the α go? Take $\alpha = 1$

We will take only real part $\rightarrow$ $\langle \cos(\theta(r)-\theta(r')) \rangle$

$= C(\underbrace{\vec{r}-\vec{r}'}_{\substack{\text{relative}\\\text{quantity}\\ \vec{r}}})$

$$C(\vec{r}) = \langle \cos(\theta(0) - \theta(\vec{r})) \rangle$$

$$= \frac{\int (D\theta(r)) \cos(\theta(0) - \theta(\vec{r}))\, e^{K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r'))}}{\int (D\theta(r))\, e^{K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r'))}}$$

Temperature Driven Phase Transition

High-T limit: $K \to 0$

$$e^{K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r'))} \simeq 1 + K\sum_{\langle r,r'\rangle} \cos(\theta(r)-\theta(r')) + \ldots$$

$$Z \sim \int \left[\prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij\rangle} (1 + K\cos(\theta_i - \theta_j))\right]$$

$$C_{\alpha\beta}(\vec{r}-\vec{r}') = \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$$

$$= \int (D\Theta(r))\, e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} e^{K\sum_{\langle r,r'\rangle}\cos(\Theta(r)-\Theta(r'))}$$

Z

$\Theta \rightarrow$ its the angular variable

$\Rightarrow$ should be invariant under $\Theta \rightarrow \Theta + \underbrace{2\pi}_{\text{any multiple of } 2\pi}$

↳ To keep them as a single-valued functn

$\Rightarrow \alpha, \beta$ are integers due to invariance

In 3rd exp $\rightarrow$ this transformation doesn't matter if we take the difference.

What about 1st & 2nd exponents?

↳ why any integers?

Action is invariant under $\Theta(r) \rightarrow \Theta(\vec{r}) + \Theta_0$

$\equiv S(\Theta(r))$

If $C_{\alpha\beta}$ is also invariant under this transformation, $S\langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$

$$= \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$$

$$= e^{i(\alpha+\beta)\Theta_0} \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$$

$$= \langle e^{i\alpha\Theta(r)} e^{i\beta\Theta(r')} \rangle$$

$$\Rightarrow \alpha + \beta = 0$$

$$C_\alpha(\vec{r}-\vec{r}') = \langle e^{i\alpha\,\theta(r)}\, e^{-i\alpha\,\theta(r')}\rangle$$

$$C(\vec{r}-\vec{r}') = \langle e^{i\theta(\vec{r})}\, e^{-i\theta(r')}\rangle$$

where did the α go? Take $\alpha = 1$

We will take only real part $\rightarrow$ $\underline{L}$ $\langle \cos(\theta(r)-\theta(r'))\rangle$

$= C(\underbrace{\vec{r}-\vec{r}'}_{\text{relative quantity } \vec{r}})$

$$c(\vec{r}) = \langle \cos(\theta(0) - \theta(\vec{r}))\rangle$$

$$= \frac{\int (D\theta(r)) \cos(\theta(0) - \theta(\vec{r}))\, e^{K\sum_{\langle r,r'\rangle}\cos(\theta(r)-\theta(r'))}}{\int (D\theta(r))\, e^{K\sum_{\langle r,r'\rangle}\cos(\theta(r)-\theta(r'))}}$$

Temperature Driven Phase Transition

High-T limit: $K \to 0$

$$e^{K\sum_{\langle r,r'\rangle}\cos(\theta(r)-\theta(r'))} \simeq 1 + K\sum_{\langle r,r'\rangle}\cos(\theta(r)-\theta(r')) + \dots$$

$$Z \sim \int \prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij\rangle} \left[1 + k\cos(\theta_i - \theta_j)\right]$$

$$\int_0^{2\pi} \frac{d\theta_i}{2\pi} = 1$$

$\underset{r}{|}\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,\underset{r'}{|}$ → break them into infinitely many terms & insert $\int_0^{2\pi} \frac{d\theta_i}{2\pi} = 1$ → bet^n them

↳ (n-1) of them.

$$Z \sim \int \prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} \left[1 + k\cos(\theta_i - \theta_j)\right]$$

$$\langle \cos(\theta(0) - \theta(r)) \rangle \sim \int \prod_{i=1}^{r-1} \frac{d\theta_i}{2\pi} \cos(\theta(r) - \theta(r))$$

$$\times \left[1 + k\cos(\theta_0 - \theta_1)\right]\left[1 + k\cos(\theta_1 - \theta_2)\right]$$

$$+ \cdots \left[1 + k\cos(\theta_{r-1} - \theta_r)\right])$$

θ_0 & θ_r → are fixed

so i starts from 1 & goes till (r-1)

→ Numerator

Denominator → $\int \prod_{i=1}^{r-1} \frac{d\theta_i}{2\pi} \left[1 + k\cos(\theta_0 - \theta_1)\right]$

$\cdots \left[1 + k\cos(\theta_{r-1} - \theta_r)\right])$

$$\int_0^{2\pi} \frac{d\theta_j}{2\pi} \cos(\theta_i - \theta_j) = 0$$

↳ take it to be θ

$d\theta_j = d\theta$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \cos(\theta) = 0$$

$$\int_0^{2\pi} \frac{d\theta_j}{2\pi} \cos(\theta_i - \theta_j) \cos(\theta_j - \theta_k)$$

$$= \int_0^{2\pi} \frac{d\theta_i}{2\pi} (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j] \\ (\cos\theta_j \cos\theta_k + \sin\theta_j \sin\theta_k)$$

$$\cos a \cos b = \frac{\cos(a-b) + \cos(a+b)}{2}$$

$$= \frac{\cos(\theta_i - \theta_k)}{2}$$ ~~(HOW??)~~

Denominator →

$$\{ (1 + k\cos(\theta_0 - \theta_1)) (1 + k\cos(\theta_1 - \theta_2)) (1 + k\cos(\theta_2 - \theta_3)) \text{ - - -}$$

↳ Do ~~$\cos\theta_i$~~ $d\theta_1$ integration

we get → $$\left(1 + \frac{k^2 \cos(\theta_0 - \theta_2)}{2}\right)$$

New $d\theta \rightarrow \underbrace{\left(1+\frac{k^2\cos(\theta_0-\theta_2)}{2}\right)\left(1+k\cos(\theta_2-\theta_3)\right)}$

$\downarrow$

gives $1+\frac{k^3\cos(\theta_0-\theta_3)}{2^2}$

$\swarrow$ generalizing from here

$$\text{Deno} \equiv 1+\frac{k^r\cos(\theta_0-\theta_r)}{2^{r-1}}$$

for numerator $\rightarrow$

$$\cos(\theta_0-\cancel{\ }\theta_r)\left(1+k\cos(\theta_0-\theta_1)\right)$$
$$\left(1+k\cos(\theta_1-\theta_2)\right)----$$

$$=\cos(\theta_0-\theta_r)\left(1+\frac{k^r\cos(\theta_0-\theta_r)}{2^{r-1}}\right)$$

$$\langle\cos(\theta(0)-\theta(r))\rangle=?$$

For denominator, since k is small, the k term can be neglected.

why these 2 integrals? (not in the defn earlier)

$$C(\vec{r}\,)=\int_0^{2\pi}\frac{d\theta_0}{2\pi}\int_0^{2\pi}\frac{d\theta_r}{2\pi}\times\cos(\theta_0-\theta_r)\left(1+\frac{k^r\cos(\theta_0-\theta_r)}{2^{r-1}}\right)$$

$$=\frac{k^r}{2^r}=\left(\frac{k}{2}\right)^r$$

$$c(\vec{r}) = \left(\frac{k}{2}\right)^r = e^{-r/\xi_4(k)}$$

$$\xi_4(k) = \cancel{1/\ln(2/k)}$$

$$\xi_4(k) = \frac{1}{\ln(2/k)}$$

$$\cancel{e^{-r/\ln(2/k)}}$$

$$\cancel{= \left(e^{-\frac{1}{\ln(2/k)}}\right)^r}$$

$$\cancel{= \left(e^{-1/(\ln 2 - \ln k)}\right)^r}$$

$$= e^{\ln\left(\frac{k}{2}\right)^r}$$

$$= e^{r\ln\left(\frac{k}{2}\right)}$$

$$= e^{-r/\ln(2/k)}$$

Why write them as $e^{-r/\xi_4(k)}$?

→ correlation functn decays exponen, $\xi_4(k)$ → coherence length

T → fixes value of $\xi_4(k)$

This gives the ~~scale~~ length scale for your correlation problem.

In high temp phase, $k \to 0$

$$c(\vec{r}) \sim e^{-r/\xi_4(k)}$$

→ You can't ignore the k term in numerator because r can be anything → large or small but then why not the same for denominator?

Also $0^0 \to$ undefined if γ is also small.

---

for low temp : $K \to \infty$

$$e^{K \sum_{r,r'} \cos(\theta_r - \theta_{r'})} \simeq e^{K \sum_{r,r'} \left\{ 1 - \frac{(\theta_r - \theta_{r'})^2}{2} \right\}}$$

Since $K \to \infty$, you can't do Taylor series on it. So, we do it for $(\theta_r - \theta_{r'})$.

Assumption $\to$ Diff of θ betn 2 nearest pts (neighbor) is very small.

$$e^{K \sum_{r,r'} \cos(\theta_r - \theta_{r'})}$$

$\to$ ~~what about~~

$\to$ here its $\cos\theta$ TY expansion

$$\simeq e^{K \sum_{r,r'} \left\{ 1 - \frac{(\theta_r - \theta_{r'})^2}{2} \right\}}$$

$$\simeq e^{-\frac{a^2 K}{2} \sum_r \left(\frac{(\theta \vec{r} - \theta \vec{r} + \vec{a})}{\vec{a}} \right)^2}$$

$\to$ what happened to 1?

$$\simeq e^{-\frac{K}{2} \int d^2\vec{r} \, (\nabla\theta)^2}$$

$$Z \sim \int (D\theta) \, e^{-\frac{K}{2} \int d\vec{r} \, (\nabla\theta)^2}$$

Hamiltonian → $H(\{\theta\}) = \frac{J}{2}\int d^2\vec{r}\,(\nabla\theta)^2$

Euler eqⁿ / saddle pt eqⁿ

↳ $\frac{\delta H\{\theta\}}{\delta\theta(r)} = 0 \Rightarrow \nabla^2\theta(\vec{r}) = 0$

↳ Its soln

① trivial soln

$\theta(r) = \text{constant}$

② Non-trivial soln:

↳ no NTS for scalar pot. (since no charges)

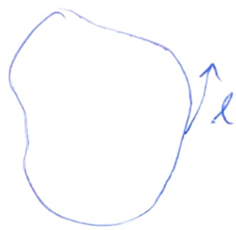
(didn't get??)

The vectors we're talking about, are phase ∠les. Although the form looks same to us.

Because of periodicity we have NTS

↳ NTS may be found out by the b.c

$$\oint(\vec{\nabla}\theta\cdot d\vec{l}) = 2\pi n$$



↳ so, what kind of θ satisfies this & $\nabla^2\theta(r)$ ultimately this.

$|\nabla\theta| = \frac{n}{r}$, $\theta(r) = n\tan^{-1}(y/x)$

$$\nabla\theta(r) = n\left[\frac{\hat{i}(-y/x^2)}{1+y^2/x^2} + \frac{\hat{j}(1/x)}{1+y^2/x^2}\right] = \frac{n}{r^2}\left[-\hat{i}y + \hat{j}x\right]$$