Course: Qunatum Information Theory (Assignment 4)

Q.1 Consider a generic state of a qubit (spin 1/2 particle) given by the density matrix $\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Find the following:

(i) is the state pure or mixed?

(ii) find the average spin components $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$.

(iii) Calculate the Von Neumann entropy of the state.

Q.2 Consider a generic state of a qubit (spin 1/2 particle) given by the density matrix $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find the following:

(i) is the state pure or mixed?

(ii) find the average spin components $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$.

(iii) Calculate the Von Neumann entropy of the state.

Q.3 Consider the three local Hilbert spaces $H_i, i = 1; 2; 3$ representing three subsystems of a system described in the Hilbert space $H = H_1 \otimes H_2 \otimes H_3$. Let $|0\rangle, |1\rangle, |3\rangle$ be an orthonormal basis in a single three dimensional Hilbert space. Then a basis of the Hilbert space for H is given by $|ijk\rangle$, where $i, j, k \in \{0, 1, 2\}$. Consider the states $|\psi\rangle = |000\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$

(i) Compute the reduced density matrix for the system with Hilbert space $H_1 \otimes H_2$ for each case.

(ii) Compute the entanglement entropy in each case for part (i).

Q.4 Consider a system with total angular momentum 1. We choose a basis corresponding to the three eigenvectors of the z-component of the angular momentum, J_z, with eigenvalues +1, 0,1, respectively. The state of the system is described by a density matrix

$$\rho = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{1}$$

(a) Is ρ a permissible density matrix? Justify your answer.

(b) Does it describe a pure or mixed state? Justify your answer

Q.5 Prove the von Neumann mixing theorem which states that, given two distinct density matrices ρ_1 and ρ_2, and a mixed state $\rho = \theta\ \rho_1 + (1-\theta)\ \rho_2$, the VN entropy of the state ρ is given as $S(\rho) > \theta\ S(\rho_1) + (1-\theta)\ S(\rho_2)$.

Q.6 An attempt to perform a Bell-state measurement on two photons produces a mixed state, one in which the two photons are in the entangled state $|\phi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$ with probability p and in each of the states $|++\rangle$ and $|--\rangle$ with probability $(1-p)/2$. Here $|+\rangle$ and $|-\rangle$ represent photons linearly polarized at angle $\pi/4$ and $-\pi/4$ respectively. Determine the density matrix for this ensemble using the linear (horizontal and vertical) polarization states of the photons as basis states.

Q.7 A machine tries to produce qubits in the state $|0\rangle$. But it is not very good so it only produces $|0\rangle$ with probability q. And, with probability $1-q$, it produces a state $|\psi\rangle = \cos(\theta/2)\ |0\rangle + \sin(\theta/2)\ |1\rangle$ where θ may be some small angle. Obtain the density matrix for this system.