Electronic Properties of Metals: Drude Theory

Drude model is based on the following assumptions –

(i) The electrons in a solid do not interact with each other at all

(There is no Coulomb interaction and, as opposed to a classical gas model, they do not collide with each other. This is known as independent electron approximation) -

Somehow surprisingly good

(ii) The positive charge is located on immobile ion cores. The electrons can collide with the ion cores. These collisions instantaneously change their velocity. However, in between collisions, the electrons do not interact with the ions either. This is known as the free electron approximation.

This approximation is not good. Some scattering mechanism exists

Assumptions of Drude Theory - Contd.

(iii) The electrons reach thermal equilibrium with the lattice through the collisions with the ions. According to equipartition theorem, their mean kinetic energy is -

$$\frac{1}{2}m_e v_t^2 = \frac{3}{2}k_B T$$

(At room temperature this results in average speed $v_t \sim 10^5$ m/s, We shall see later that mean velocity is 100 times larger)

Immediately after each collision, an electron is taken to emerge with a velocity that is not related with its velocity just before the collision, but randomly directed and with a speed appropriate to the temperature prevailing at the place where the collision occurred

(iv) In between collisions, the electrons move freely. The mean length of this free movement is called the mean free path λ. Given the average speed v_t , the mean free path also corresponds to a mean time between the collisions, given by τ, called the relaxation time, i.e. $\lambda = v_t \tau$

DC electrical conductivity – Drude model

$$\vec{j} = \frac{ne^2\tau}{m_e}\vec{\mathcal{E}}$$

So, the current density is in the direction of the electric field and proportional to the field strength. $\longrightarrow$ Ohm's law
(Qualitative explanation)

$$\sigma = \frac{ne^2\tau}{m_e} = \frac{1}{\rho} \qquad \left[\begin{array}{l}\sigma \to \text{conductivity} \\ \rho \to \text{resistivity}\end{array}\right].$$

Mobility of the electrons $\mu = \dfrac{e\tau}{m_e} = \dfrac{|\bar{v}|}{|\mathcal{E}|}$

Room temperature resistivities are typically $\sim$ μohm-cm

$$\tau = \frac{m}{\rho ne^2} \sim 10^{-14} \text{ to } 10^{-15} \text{ seconds.}$$

DC electrical conductivity – Drude model

- Given that the total momentum per electron is $p(t)$ at time 't'
- An electron at time 't' will have a collision before time '$t + dt$' with probability $\dfrac{dt}{\tau}$
- An electron will survive till time '$t + dt$' without suffering a collision with probability $(1 - \dfrac{dt}{\tau})$.

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- An electron will survive till time '$t + dt'$ without suffering a collision with probability $(1 - \dfrac{dt}{\tau})$. If it suffers no collision, it simply evolves under the influence of $f(t)$ [$f(t)$ is due to spatially uniform electric or magnetic field]

$$p(t+dt) = \left(1 - \frac{dt}{\tau}\right)\left\{p(t) + f(t)\,dt\right\} + O(dt)^2$$

Correction to the above equation from electrons that had a collision between time 't' and '$t + dt$' is of order (dt^2) as shown

DC electrical conductivity – Drude model

$$p(t+dt) - p(t) = (-)\left(\frac{dt}{\tau}\right)p(t) + f(t)dt + O(dt)^2$$

In the limit $dt \to 0$

$$\boxed{\frac{dp(t)}{dt} = (-)\frac{p(t)}{\tau} + f(t)}$$

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- Simply states that the effect of individual electron collisions is to introduce a damping term into the equation of motion for the momentum per electron
- The average electronic velocity is $\boxed{v(t) = \frac{p(t)}{m}}$
- Current density: $j = -\frac{ne\,p(t)}{m}$

Hall conductivity: Partial success of Drude model

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- **Electrons in electric field-**

$$\frac{dp}{dt} = (-)\frac{p}{\tau} - eE$$

In the steady state $\frac{dp}{dt} = 0$

and $mv = p = (-)eE\tau$

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- **Electrons in electric field and magnetic field ($B\,\hat{z}$)-**

$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e(\vec{v} \times \vec{B})$$

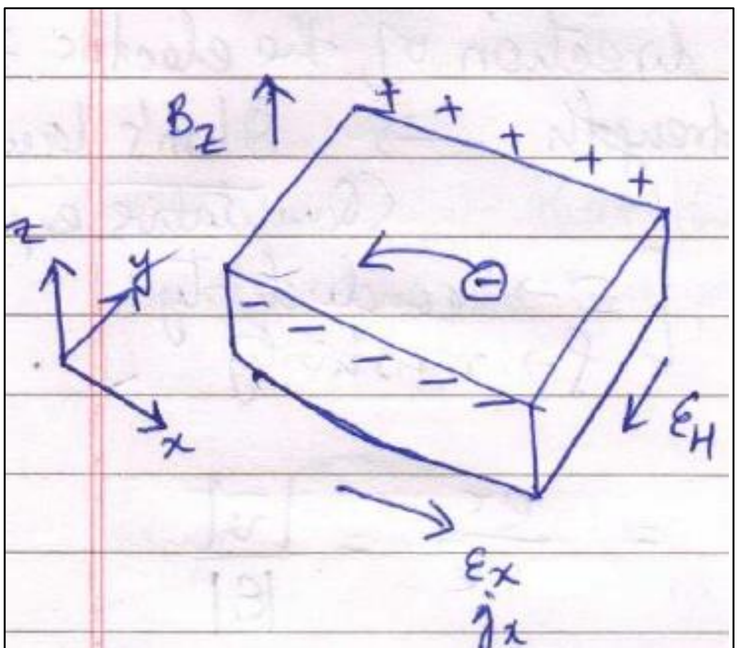
Hall conductivity: Partial success of Drude model

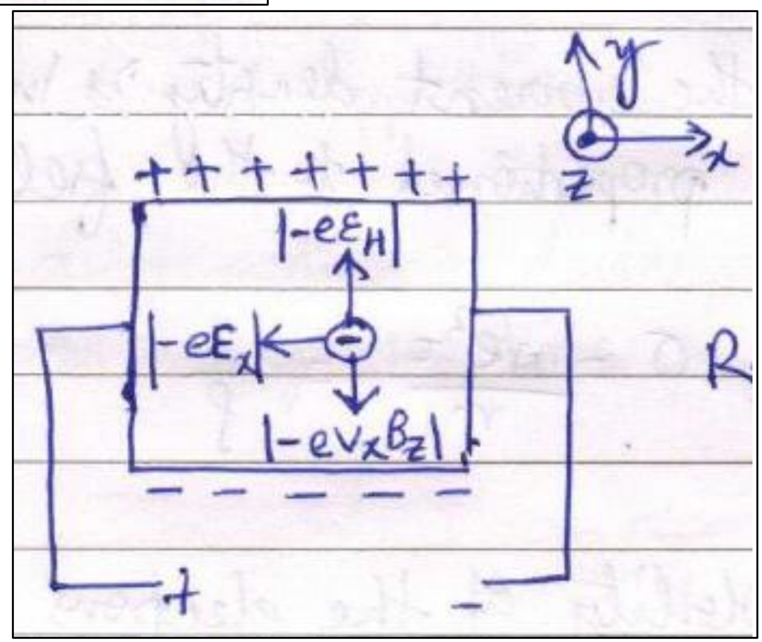
Again setting the L.H.S. to zero in the steady state, and using $\vec{p} = m\vec{v}$ and $\vec{j} = -ne\vec{v}$

$$0 = (-)\,eE_x - \omega_c p_y - \frac{p_x}{\tau}$$

$$0 = (-)\,eE_y + \omega_c p_x - \frac{p_y}{\tau}$$

$$\omega_c = \frac{eB}{m}$$





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Multiplying by $(-)\dfrac{ne\tau}{m}$ and $\vec{j} = (-)\,ne\vec{v} = (-)\,\dfrac{ne\vec{p}}{m}$

$$\sigma_0 E_x = \omega_c \tau j_y + j_x$$

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$\sigma_0 \rightarrow$ DC conductivity in absence of magnetic field.

Hall conductivity: Partial success of Drude model

The Hall field is determined by the requirement that there be no transverse current j_y

$$E_y = (-)\left(\frac{\omega_c \tau}{\sigma_0}\right) j_x = (-)\frac{\cancel{e}B}{\cancel{m}} \times \frac{\cancel{\tau}\cancel{m}}{ne^{\cancel{2}}\cancel{\tau}} j_x$$

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$$\boxed{R_H = \frac{E_y}{j_x B} = (-)\frac{1}{ne}}$$

Alkali metal agree well with the above formula

For some metals, like Be, Mg and Al, the measured R_H is positive and not negative. It appears, thus, in these metals the current is carried by positive charges, which does not make sense in the Drude model

AC electrical conductivity of a metal

Time dependent electric field-

$$\vec{E}(t) = \text{Re}\left(\vec{E}(\omega)e^{-i\omega t}\right)$$

Now,

$$\frac{d\vec{p}}{dt} = (-)\frac{\vec{p}}{\tau} - e\vec{E}$$

We seek a steady state solution of the form-

$$p(t) = \text{Re}\left(\vec{p}(\omega)e^{-i\omega t}\right)$$

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AC electrical conductivity of a metal

$$(-)i\omega \vec{p}(\omega) = (-)\frac{\vec{p}(\omega)}{\tau} - e\vec{E}(\omega)$$

Since, $\vec{j} = (-)\frac{ne\vec{p}}{m}$

$$j(t) = \mathrm{Re}\left(j(\omega)\, e^{-i\omega t}\right)$$

where, $j(\omega) = (-)\frac{ne\,p(\omega)}{m} = \frac{(ne^2/m)\vec{E}(\omega)}{\left(\frac{1}{\tau} - i\omega\right)}$

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$$\boxed{\vec{j}(\omega) = \sigma(\omega)\vec{E}(\omega)}$$

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}}$$

$$\boxed{\sigma_0 = \frac{ne^2\tau}{m}}$$

Thermal conductivity of a metal

- The most impressive success of the Drude model at the time it was proposed was the explanation of the empirical law of Wiedemann and Franz (1853)

$$\frac{K}{\sigma} \propto T$$

with the proportionality constant being the same for all metals

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- For small temperature gradients, the thermal current is found to be proportional to $\vec{\nabla} T$ (Fourier's law)

$$\vec{j}^{q} = (-) K \vec{\nabla} T$$

The proportionality constant 'k' is called thermal conductivity

Thermal conductivity of a metal

- Considering a one-dimensional model, in which electrons can move along the x-axis, so that at a point 'x', half of the electrons come from high-temperature side of 'x', and half from the low-temperature side

- $\varepsilon(T)$ is the thermal energy per electron in a metal in equilibrium at temperature T

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Their total contribution to the thermal current density at 'x'

$$= \left(\frac{n}{2}\right) v\, \varepsilon\left(T\,[x - v\tau]\right)$$

Electrons arriving from low-temperature side of 'x', on the other hand, will contribute $= \left(\frac{n}{2}\right)(-v)\left[\varepsilon\left(T[x+v\tau]\right)\right]$

Thermal conductivity of a metal

$$j^q = \frac{1}{2} n v \left[\mathcal{E}(T[x - v\tau]) - \mathcal{E}(T[x + v\tau]) \right]$$

Provided that the variation over a mean free path ($l = v\tau$) is small,

$$j^q = n v^2 \tau \frac{d\mathcal{E}}{dT} \left(-\frac{dT}{dx} \right)$$

To go to three dimensions, replacing v by v_x of the electronic velocity $\vec{v}$ and average over all directions.

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{v^2}{3}$$

and, $n \frac{d\mathcal{E}}{dT} = \left(\frac{N}{V} \right) \frac{d\mathcal{E}}{dT} = \frac{1}{V} \frac{dE}{dT} = c_v$ (Electronic specific heat)

Thermal conductivity of a metal

$$j^q = n v^2 \tau \frac{d\varepsilon}{dT}\left(-\frac{dT}{dx}\right)$$

In three dimensions -

$$j^q = \frac{1}{3} v^2 \tau c_v \left(-\vec{\nabla} T\right)$$

So, $K = \frac{1}{3} v^2 \tau c_v$

The ratio of the thermal and electrical conductivities -

$$\frac{K}{\sigma} = \frac{\frac{1}{3} c_v m v^2}{n e^2}$$

Drude naturally took -

$$c_v = \frac{3}{2} n k_B \quad \text{and} \quad \frac{1}{2} m v^2 = \frac{3}{2} k_B T$$

Wiedemann and Franz law from Drude model

$$\frac{\kappa}{\sigma} = \underbrace{\frac{3}{2}\left(\frac{k_B}{e}\right)^2}_{\text{Universal constants}} T$$

$$\frac{\kappa}{\sigma T} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2 = 1.11 \times 10^{-8} \text{ watt-ohm}/K^2$$

experimentally $\frac{\kappa}{\sigma T} = 2.22 \times 10^{-8}$ watt-ohm$/K^2$

Actually this is extremely fortuitous, Drude's impressive success is a consequence of two errors that cancel.

At room temperature (in hindsight) –

- Actual contribution of electronic specific heat is 100 times smaller
- Mean square electronic speed is 100 times larger