Radioactivity

Lecture 1

Introduction

The disintegration of nuclei into two or more particles are called radioactivity.
The radioactivity can be of two types
Natural radioactivity ; example : U and Th series, K
Artificial radioactivity ; example : Nuclear reaction

Decay law

It has been observed that the decay rate is decreasing with time and the decay is statistical in nature, i.e. it is difficult to predict when a nucleus will decay. if there are N radioactive nuclei are present at time t and no nuclei are introduced into the sample then the number dN decaying in time dt is proportional to N, or

$$\frac{dN}{dt} \propto N, \text{ or, } \frac{dN}{dt} = -\lambda N$$

Solving the differential equation we have

$$N = N_0 e^{-\lambda t}$$

where N_0 is the number of nuclei present at time $t = 0$. The λ is called as disintegration constant. Higher the value of λ faster will be the decay process.

The activity A of the sample is defined as

$$A = \lambda N \;=\; \lambda N_0 e^{-\lambda t} \;=\; A_0 e^{-\lambda t}$$

Here, A_0 is the activity of the sample at time $t = 0$. The unit of activity is

$$1Ci = 3.7 \times 10^{10} \text{disintegration/sec.}$$

The SI unit of activity is Bq, where one Bq is equal to one disintegration per second.

If at time $t = t_{1/2}$ the number of nuclei present in the sample is half of the initial number, i.e. $N = \frac{N_0}{2}$ then we can write

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}}$$

or

$$t_{1/2} = \frac{ln2}{\lambda}$$

From the relation $A = A_0 e^{-\lambda t}$ it is evident that the disintegration constant can be obtained from the slope of the plot lnA vs time t.

From the equation $N = N_0 e^{-\lambda t}$ we can say that $e^{-\lambda t}$ is the survival probability per nuclei at time t and when it is multiplied by the number of nuclei present at time $t = 0$, it gives number of nuclei present at the time t. Therefore, $(1 - e^{-\lambda t})$ is the probability that a nucleus will disintegrate.

The mean lifetime τ can then be defined as

$$\tau = \frac{\int_0^\infty t e^{-\lambda t} dt}{\int_0^\infty e^{-\lambda t} dt} \;=\; \frac{1}{\lambda}$$

Now let us consider two nuclei of same sample.
The probability that both will survive in time t is $e^{-\lambda t} e^{-\lambda t} = e^{-2\lambda t}$
The probability that both will decay in time t is $(1 - e^{-\lambda t})^2$
The probability that one will survive and other will decay in time t is $(1 - e^{-\lambda t})e^{-\lambda t} + e^{-\lambda t}(1 - e^{-\lambda t})$

If we have N nuclei then probability that n nuclei will decay is

$$P(n) =^{N} C_n e^{-(N-n)\lambda t}(1-e^{-\lambda t})^n$$

We now make an assumption that N is very large compared to n and $\lambda t << 1$ or $t << t_{1/2}$. Therefore

$$(1-e^{-\lambda t})^n = e^{-n\lambda t}(e^{\lambda t}-1)^n \simeq e^{-n\lambda t}(1+\lambda t-1)^n = e^{-n\lambda t}(\lambda t)^n$$

Therefore,

$$P(n) = \frac{N(N-1)(N-2)...(N-n)!}{n!(N-n)!}e^{-N\lambda t}e^{n\lambda t}e^{-n\lambda t}(\lambda t)^n$$

As N is very large, each factor in $N(N-1)(N-2)...$ can be approximated to N. Therefore,

$$P(n) = \frac{N^n}{n!}e^{-N\lambda t}(\lambda t)^n \;=\; \frac{(N\lambda t)^n e^{-N\lambda t}}{n!}$$

If we define $\mu = N\lambda t$ we can write

$$P(n) = \frac{\mu^n e^{-\mu}}{n!}$$

This is a Poisson distribution function. That means radioactivity decays can be described by Poisson distribution function.

Total probability for all possible values of n is

$$\sum_0^\infty \frac{\mu^n e^{-\mu}}{n!} \;=\; e^{-\mu}\sum_0^\infty \frac{\mu^n}{n!} \;=\; 1$$

This shows that the total probability is normalised to one. The average value of n can be obtained for the distribution function as

$$\bar{n} = \sum_0^\infty n\frac{\mu^n e^{-\mu}}{n!} \;=\; e^{-\mu}\sum_1^\infty \mu\frac{\mu^{(n-1)}}{(n-1)!} \;=\; \mu e^{-\mu}e^{\mu} \;=\; \mu$$

Therefore, mean of the distribution is described by the parameter μ. Similarly we can calculate mean of n^2 as

$$\begin{aligned}\bar{n^2} &= \sum_0^\infty n^2 \frac{\mu^n e^{-\mu}}{n!} \;=\; e^{-\mu}\sum_1^\infty n\frac{\mu^n}{(n-1)!} \;=\; e^{-\mu}\sum_1^\infty [(n-1)+1]\frac{\mu^n}{(n-1)!} \\ &= e^{-\mu}\left[\sum_2^\infty \mu^2\frac{\mu^{n-2}}{(n-2)!} + \sum_1^\infty \mu\frac{\mu^{n-1}}{(n-1)!}\right] \\ &= \mu^2+\mu\end{aligned}$$

The variance σ^2 of the distribution function is then

$$\sigma^2 = \bar{n^2} - (\bar{n})^2 \;=\; \mu^2+\mu-mu^2 \;=\; \mu$$

Therefore, $\sigma = \sqrt{\mu}$

If the number of decay is relatively large the distribution function becomes symmetric and can be described by the Gaussian distribution function

$$G(n) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[\frac{-(x-\mu)^2}{\sigma^2}\right]$$

Calculation of activity

If there is a mixture of two radioactive samples with disintegration constant λ_1 and λ_2 such that $\lambda_1 >> \lambda_2$ then the total activity A of the sample is sum of the activities of both the samples, i.e. $A = A_1 + A_2$. From the measurement of activity of the combined sample one can determine disintegration constants or the half-lives of the individual sample, as shown in Fig. **??**.

Let us take some examples of the decay processes and calculation of its activity.

Production of a radioactive sample and its decay

Let us consider a sample is being produced at a rate Q per second. Its decay constant is λ. We need to calculate activity of the sample produced at any

time t. The decay equation of the sample can be written as

$$\frac{dN}{dt} = Q - \lambda N$$

$$\text{or, } \frac{dN}{Q - \lambda N} = dt$$

$$\text{or, } \frac{dN}{N - \frac{Q}{\lambda}} = -\lambda dt$$

Integrating we have

$$ln\left(N - \frac{Q}{\lambda}\right) = -\lambda t + C_1$$

where C_1 is an integration constant whose value can be determined from initial considitions.

Considering $N = N_0$ at at time $t = 0$, we have

$$C_1 = ln\left(N_0 - \frac{Q}{\lambda}\right)$$

Therefore, using the value of C_1 we have

$$\frac{N - \frac{Q}{\lambda}}{N_0 - \frac{Q}{\lambda}} = e^{-\lambda t}$$

$$\text{or, } N - \frac{Q}{\lambda} = \left(N_0 - \frac{Q}{\lambda}\right) e^{-\lambda t}$$

$$\text{or, } N = N_0 e^{-\lambda t} + \frac{Q}{\lambda}\left(1 - e^{-\lambda t}\right)$$

If we assume that $N_0 = 0$ at time $t = 0$ we have

$$N = \frac{Q}{\lambda}\left(1 - e^{-\lambda t}\right)$$

This is an increasing funtion with time t and is shown in Fig **??**. The plot shows saturation after sufficient longer time. The activity can be calculated as

$$A = \lambda N \;=\; Q\left(1 - e^{-\lambda t}\right)$$

and as $t \to \infty, A \to Q$.

Sample with radioactive daughter nucleus

Let us consider a rdiactive nucleus A decays to B with decay constant λ_A. The daughter nucleus B is also radioactive and it decays to nucleus C with a decay constant λ_B. The nucleus C is stable. Let us also assume that at time $t = 0$ the number of nuclei $N_A = N_0$ and $N_B = N_C = 0$. The decay equations are the following

$$\frac{dN_A}{dt} = -\lambda_A N_A \tag{1}$$

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B \tag{2}$$

$$\frac{dN_C}{dt} = \lambda_B N_B \tag{3}$$

The solution of equation (1) can be written as

$$N_A = N_0 e^{\lambda_A t} \tag{4}$$

The equation (2) can be written as

$$\frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_A$$

$$\text{or, } e^{\lambda_B t}\left(\frac{dN_B}{dt} + \lambda_B N_B\right) = \lambda_A N_0 e^{(\lambda_B - \lambda_A)t}$$

$$\text{or, } \frac{d}{dt}\left(e^{\lambda_B t} N_B\right) = \lambda_A N_0 e^{(\lambda_B - \lambda_A)t}$$

Integrating we have

$$e^{\lambda_B t} N_B = \frac{\lambda_A N_0}{(\lambda_B - \lambda_A)} e^{(\lambda_B - \lambda_A)t} + C_1$$

The constant C_1 can be determined by using the initial condition $N_B = 0$ at $t = 0$. This gives

$$C_1 = -\frac{\lambda_A N_0}{(\lambda_B - \lambda_A)}$$

Therefore,

$$N_B = \frac{\lambda_A N_0}{(\lambda_B - \lambda_A)}\left[e^{-\lambda_A t} - e^{-\lambda_B t}\right] \tag{5}$$

Using the expression of N_B we can solve the equation (3) in straight forward way

$$\frac{dN_C}{dt} = \frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)} \left[e^{-\lambda_A t} - e^{-\lambda_B t}\right]$$

Integrating we have

$$N_C = \frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)} \left[-\frac{1}{\lambda_A} e^{-\lambda_A t} + \frac{1}{\lambda_B} e^{-\lambda_B t}\right] + C_2$$

With the initial condition as $t = 0$ $N_C = 0$

$$0 = \frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)} \left[-\frac{1}{\lambda_A} + \frac{1}{\lambda_B}\right] + C_2$$

This gives $C_2 = N_0$ and then N_C is equal to

$$N_C = N_0 \left[1 - \frac{\lambda_B}{\lambda_B - \lambda_A} e^{-\lambda_A t} + \frac{\lambda_A}{\lambda_B - \lambda_A} e^{-\lambda_B t}\right] \tag{6}$$

small time behaviour

Assuming t to very small such that $t << \tau_A$ and $t << \tau_B$, where τ_A and τ_B are the half-lives of A and B, respectively. We can then expand the exponential of equation (5)

$$N_B = \frac{\lambda_A N_0}{(\lambda_B - \lambda_A)} \left[1 - \lambda_A t - 1 + \lambda_B t)\right] \;=\; N_0 \lambda_A t$$

This shows that the N_B is increasing linearly in time t.

Similarly we can write N_C as

$$N_C = N_0 \left[1 - \frac{\lambda_B}{\lambda_B - \lambda_A} \left(1 - \lambda_A t + \frac{1}{2}(\lambda_A t)^2\right) + \frac{\lambda_A}{\lambda_B - \lambda_A} \left(1 - \lambda_B t + \frac{1}{2}(\lambda_B t)^2\right)\right]$$

$$N_C = \frac{1}{2} \lambda_A \lambda_B N_0 t^2$$

i.e., N_c is proportional to t^2 The plots of N_A, N_B and N_C are shown in Fig. **??**.

We have seen that N_B increases as t increases from zero. One can also see from equation (5) that $N_B \to 0$ as $t \to \infty$. Therefore, N_B must have a maximum at some time $t = t_m$. The t_m can be obtained by maximising the expression of N_B, or by equating $\frac{dN_B}{dt}$ to zero.

$$\frac{dN_B}{dt} = \frac{\lambda_A N_0}{(\lambda_B - \lambda_A)}\left[-\lambda_A e^{-\lambda_A t_m} + \lambda_B e^{-\lambda_B t_m}\right] = 0$$

$$\text{or, } e^{(\lambda_A - \lambda_B)t_m} = \frac{\lambda_A}{\lambda_B}$$

$$\text{or, } t_m = \frac{1}{\lambda_A - \lambda_B} ln\left(\frac{\lambda_A}{\lambda_B}\right)$$

The activities of the nuclei A and B at any time t are

$$A_A = \lambda_A N_A \;=\; \lambda_A N_0 e^{-\lambda_A t}$$

$$A_B = \lambda_B N_B \;=\; \frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)}\left[e^{-\lambda_A t} - e^{-\lambda_B t}\right]$$

If $\lambda_A > \lambda_B$, i.e. parent is short-lived compared to daughter or $\tau_A << \tau_B$. If $t >> \tau_A$ we have $\lambda_A t >> 1$ and $e^{-\lambda_A t} \to 0$ Therfore,

$$A_B = -\frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)} e^{-\lambda_B t} \;=\; \frac{\lambda_A \lambda_B N_0}{(\lambda_A - \lambda_B)} e^{-\lambda_B t}$$

This shows that the decrease in activity of B is decided by the disintegration constant λ_B of B, as shown in Fig. **??**.

If $\lambda_B > \lambda_A$, i.e. the daughter is short-lived compared to parent nucleus or $\tau_B << \tau_A$. If $t >> \tau_B$ we have $\lambda_B t >> 1$ and $e^{-\lambda_B t} \to 0$ Therefore,

$$A_B = -\frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)} e^{-\lambda_A t} \;=\; \frac{\lambda_B}{(\lambda_B - \lambda_A)} A_A \quad [\text{As } A_A = \lambda_A N_0 e^{-\lambda_A t}]$$

$$\text{or, } \frac{A_B}{A_A} = \frac{\lambda_B}{(\lambda_B - \lambda_A)} \;=\; \text{constant}$$

, i.e. the ratio of the activities of daughter and parent is constant and does not change with time. This is called as transient equilibrium.

If $\lambda_A \simeq 0$ and very small compared to λ_B. If $t << \tau_A$ then $e^{-\lambda_A t} \simeq 1$

$$A_B = \frac{\lambda_A \lambda_B N_0}{(\lambda_B - \lambda_A)} \left[1 - e^{-\lambda_B t}\right]$$

$$= \lambda_A N_0 \left[1 - e^{-\lambda_B t}\right] \;=\; A_A(1 - e^{-\lambda_B t})$$

If $t >> \tau_B$ then $e^{-\lambda_B t} \simeq 0$. Therefore

$$A_B = A_A$$

and the parent and daughter nuclei are in secular equilibrium. If there is a chain of nuclei in the decay seies and if they are in secular equilibrium, we can write

$$\lambda_A N_A = \lambda_B N_B = \lambda_C N_C = ...$$

$$\text{or,} \frac{N_A}{\tau_A} = \frac{N_B}{\tau_B} = \frac{N_C}{\tau_C} = ...$$

Alpha Decay

Lecture 1

Kinematics

Alpha decay is due to instability in nuclei and is observed spontaneously in heavier nuclei. In this process mass number of the daughter nucleus is reduced by four and atomic number by two and is represented by the decay equation

$$^{A}_{Z}X \rightarrow ^{A-4}_{Z-2}Y + \alpha + Q$$

Here, Q is the energy released in this process. The process goes spontaneously if the $Q > 0$. The Q value can be calculated using the masses of the participating nuclei

$$Q = (M_X - M_Y - m_\alpha)\, c^2.$$

We can also write Q value expresssion in terms of binding energies of the nuclei

$$Q = (B_Y + B_\alpha - B_X)$$

The decay process is a two-body process in which daughter nucleus and alpha particle appear in the final state. Therfore, momentum is conserved in the CM frame of the decay process, i.e. $|P_\alpha| = |P_Y|$. The energy of the alpha particle (around 5-10 MeV) is much smaller compared to its rest mass, we can write kinetic energies of alpha particle and daughter nucleus as

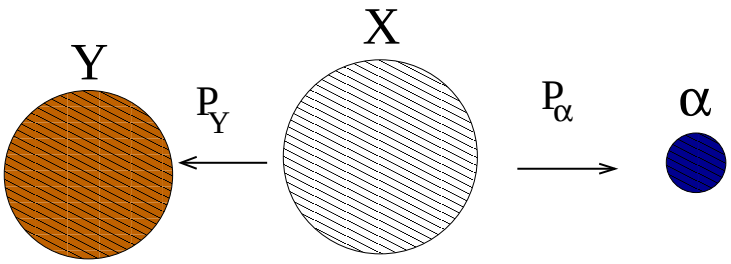

Figure 1: Schematic diagram of alpha decay process. It is two-body decay process and in CM frame linear momentum is conserved.

$$T_\alpha = \frac{P_\alpha^2}{2m_\alpha}$$

and

$$T_Y = \frac{P_Y^2}{2M_Y}$$

since $|P_\alpha| = |P_Y|$ we cn write

$$T_Y = \frac{P_\alpha^2}{2M_Y} \;=\; T_\alpha \frac{m_\alpha}{M_Y}$$

Since

$$Q = T_\alpha + T_Y \;=\; T_\alpha \left(1 + \frac{m_\alpha}{M_Y}\right)$$

$$\text{or } T_\alpha = \left(\frac{M_Y}{m_\alpha + M_Y}\right) Q$$

We can approximate $M_Y + m_\alpha \simeq A$ and $M_Y \simeq (A-4)$ then

$$T_\alpha = \left(1 - \frac{4}{A}\right) Q$$

Similarly, we can write

$$T_Y = \left(\frac{4}{A}\right) Q$$

For A=200, $T_\alpha = 0.98Q$ and $T_Y = 0.02Q$. This shows that 98% of the Q is shared by the α particle and the daughter nucleus take only 2% of the available energy.

Calssically α decay is difficult to understand as it has to cross the Coulomb barrier which is higher than the energy of the alpha particle inside the nucleus. The maximum barrier height can be calculated by assuming spherical shape of the nuclei as

$$B = \frac{2Z_D e^2}{4\pi\epsilon_0 R} \tag{7}$$

The distance R is the sum of the radii of daughter nucleus and alpha particle. As an example, the barrier height for alpha decay process from $^{238}_{92}$U is

$$B = \frac{2 \times 90 \times 1.44 MeV fm}{1.2 \times [(234)^{1/3} + (4)^{1/3}]} \simeq 28 MeV.$$

The alpha particle energy is between 5-10 MeV and it was difficult to explain how alpha particle comes out of the nucleus.

Another observation was made by Geiger and Nuttall that there is strong correlation between the Q value of the decay process and hilf-life of the decaying state. Looking at the nuclei across the periodic table it was observed that for nuclei with same atomic number

$$log_{t_1/2} \propto \frac{1}{\sqrt{Q}}.$$

For example,
(1) $Q_\alpha = 4.08$ MeV for alpha decay from $^{232}_{90}$Th with half-life $t_{1/2} = 5 \times 10^{17}$ s
(2) $Q_\alpha = 9.85$ MeV for alpha decay from $^{218}_{90}$Th with half-life $t_{1/2} = 10^{-7}$ s

We can see that as Q value becomes little more than double, the half-life changes by an order of 10^{24}. It has been observed that for isotopes, Q value decreases by 0.3-0.4 MeV per addition of neutron.

Exercise : Check the above statement using semi-empirical mass formula for ^{226}Th. [Hint: calculate $\left(\frac{\partial Q}{\partial A}\right)_Z$ and show that it is $\simeq -0.17$]

Semi-classical theory of Alpha decay

In this model we assume that alpha particle exists inside the nucleus and is trapped by Coulomb barrier but has finite probability to come out through quantum tunnelling. The potential energy of the alpha particle is shown as function of distance r in Fig. 2. It is clear that Q value must be positive otherwise α-particle will be bound and can not escape from the nucleus. Let R be the distance between centers of daughter nucleus and alpha paricle and is given by

$$R = R_0 \left[(A-4)^{1/3} + 4^{1/3}\right]$$

where $R_0 = 1.2$ fm. For simplicity, let us neglect the recoil energy of daughter nucleus and, therefore, $Q \simeq T_\alpha$. We can also mark a distance $r = b$ at which the barrier height is equal to Q_α. This will define barrier width.

In Fig. 2, there are three regions, $r <$, $R < r < b$ and $r > b$ For $r <$, potential energy is $-V_0$. Classically the α particle can move in this region with kinetic energy $Q + V_0$, as total energy $Q = T + (-V_0)$, but can not escape from it.

For the region $R < r < b$, potential energy is more than the available energy Q and classically α particle is not allowed to enter the region. The $r > b$ region is clasically permitted region where particle energy is more than the barrier height.

if Q is the kinetic energy of the α particle , we can write the barrier height at $r = b$ as Q

$$Q = \frac{2Z_D e^2}{4\pi\epsilon_0 b}$$

$$\text{or } b = \frac{2Z_D e^2}{4\pi\epsilon_0 Q} \tag{8}$$

Using equations 7 and 8 we can write

$$\frac{B}{Q} = \frac{b}{R} \tag{9}$$

As can be seen in Fig. 2 that the barrier does not have fixed height with r. It varies from $B - Q$ at $r = R$ to zero and $r = b$. let us make a very

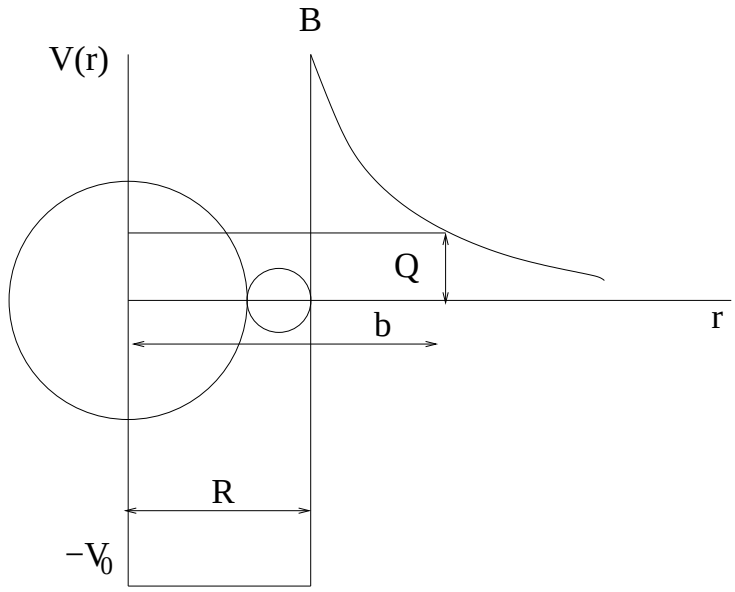

Figure 2: Schematic diagram of potential energy between daughter nucleus and alpha particle. The depth of the attractive potential in V_0. Here, R is the sum of the radii of daughter nucleus and alpha particle at which Coulomb barrier is B. The barrier height is Q at $r = b$.

crude approximation and replace the barrier with a rectangular barrier with following height and width.

$$W = \frac{1}{2}(b - R)$$

and

$$H = \frac{1}{2}(B - Q)$$

We can calculate transmission probability of the particle using quantum mechanical approach of rectangular barrier problem of fixed height and fixed with, as shown in Fig. 3. Clearly, there are three regions, as marked in Fig. 3.

In region I, $V_0 = 0$ and the wave function of the particle is

$$\psi_I \sim e^{ik_1 r}, \text{where } k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

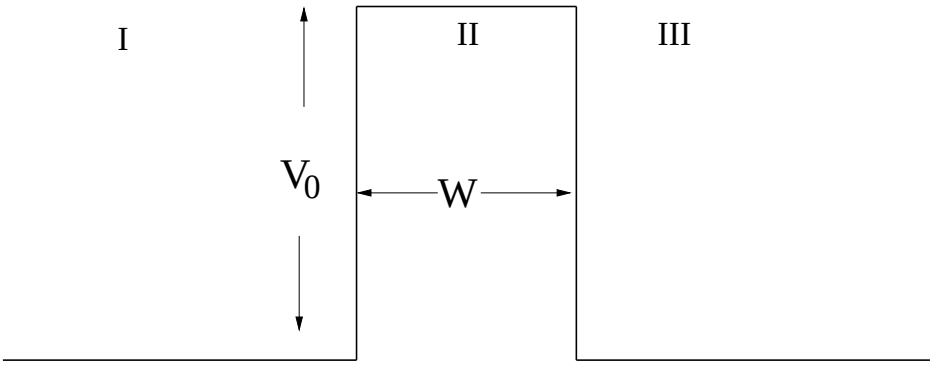

Figure 3: Schematic diagram of rectangular potential barrier of height V_0 and width W. The three regions are marked as I, II and III.

and

$$\psi_{II} \sim e^{-k_2 r}, \text{where } k_2 = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

The transmission probability is then given by

$$T = \frac{16e^{-2k_2 W}}{\left[1 + \left(\frac{k_1}{k_2}\right)^2\right]\left[1 + \left(\frac{k_2}{k_1}\right)^2\right]}.$$

We can drop the factor in front of exponential which can be of the order one for the present calculation and write

$$T \sim e^{-2k_2 W}$$

Replacing $E = Q$ and $(V_0 - E) = H$ we can write

$$k_1 = \sqrt{\frac{2m}{\hbar^2}Q}$$

$$k_2 = \sqrt{\frac{2m}{\hbar^2}H}$$

Let us take the example of the alpha decays listed below

(1) $Q_\alpha = 4.08$ MeV for alpha decay from $^{232}_{90}$Th with half-life $t_{1/2} = 5 \times 10^{17}$ s
(2) $Q_\alpha = 9.85$ MeV for alpha decay from $^{218}_{90}$Th with half-life $t_{1/2} = 10^{-7}$ s

(1) $^{232}_{90}\text{Th} \rightarrow {}^{228}_{88}\text{Ra} + \alpha$

The distance R can be calculated as

$$1.2 \times \left[(228)^{1/3} + (4)^{1/3}\right] \;=\; 9.24 \text{ fm}$$

using equation 8 the value of $b = 62.12$ fm and, hence, barrier width $W = \frac{1}{2}(b - R) \;=\; 26.44$ fm.

The coulomb barrier can be calculated using $\frac{B}{Q} = \frac{b}{R} \;=\; 27.4MeV$ and then the barrier height $H = \frac{1}{2}(B - Q) \;=\; 11.68MeV$

With all these parameters,
$k_2 = \sqrt{\frac{2m}{\hbar^2}H} \;=\; \sqrt{\frac{2mc^2}{\hbar^2c^2}H} \;=\; \sqrt{\frac{2 \times 3727}{197^2} \times 11.68} \;=\; 1.49fm^{-1}$ Therefore for decay (1), $T_1 = e^{-2k_2W} \;=\; 5.52 \times 10^{-35}$

For the decay $^{218}_{90}\text{Th} \rightarrow {}^{214}_{88}\text{Ra} + \alpha$

we can calculate the same and the values are listed below $Q_\alpha = 9.85MeV$
$R = 9.08fm$
$b = 25.73fm$
$W = 8.324fm$
$B = 27.9MeV$
$H = 9.03MeV$
$k_2 = 1.31fm^{-1}$
$T_2 = 3.3 \times 10^{-10}$

Inside the spherical potential, α particle moves with a speed v hitting the potential wall frequently. If f is the frequency with which α particle is hitting the barrier then

$$f = \frac{v}{2R}\ s^{-1} \text{ as } t = \frac{2R}{v}s$$

Inside the nucleus the kinetic energy of the α particle is $Q + V_0$, where V_0 is the depth of the attractive potential which can be of the order of 40 MeV.

Using the equation $v = \sqrt{\frac{2E}{m}}$, the velocity of the alpha particle is $v = 0.154c$ and the $f = 2.5 \times 10^{21} s^{-1}$

The transition probability is then given by $\lambda = fT$

Therefore, for decay equation (1)

$$\lambda_1 = 1.38 \times 10^{-13} s^{-1}$$

and the half-life is

$$t_{1/2} = \frac{0.693}{\lambda_1} = 5.0 \times 10^{12} s$$

The experimental value is $t_{1/2} = 5 \times 10^{17} s$.

For the decay equation (2), $v = 0.163c$ and the $f = 3.86 \times 10^{21} s^{-1}$ and

$$\lambda_2 = fT_2 \ = \ 1.27 \times 10^{12} s^{-1}$$

and the half-life is

$$t_{1/2} = \frac{0.693}{\lambda_2} = 5.4 \times 10^{-13} s$$

compared to the experimental value $t_{1/2} = 1 \times 10^{-7} s$

We can see that the calculated value of half-life is very different from the experimental values. This is expected as we have made a crude approximation on the shape and size of the barrier. However, we can see that the ratio of the lifetimes, 10^{24}, compares well with ratio of experimental observed values.

Lecture 2

Gamow theory of alpha decay

As we have seen that replacing the Coulonb barrier by a rectangular barrier does not reproduce the half-life of the decay. The better way to deal with the barrier is to divide the whole barrier into several very thin strip of rectangular

barrier of fixed width dr, as shown in Fig. **??**. We can then calculate transmission probability for each barrier as given in previous lecture and then multiply all the probabilities to get total transmission probability of alpha particle.

At any distance r, the Coulomb barrier width is dr and the height is

$$V(r) = \frac{2Z_D e^2}{4\pi\epsilon_0 r}$$

The wave function at each barrier is of the form $e^{-k(r)r}$, where

$$k(r) = \sqrt{\frac{2m}{\hbar^2}\left(V(r) - E\right)} \;=\; \sqrt{\frac{2m}{\hbar^2}\left(V(r) - Q\right)}$$

The probability to penetrate each barrier is then

$$dT = \exp\left(-2\sqrt{\frac{2m}{\hbar^2}\left(V(r) - Q\right)}dr\right)$$

and the total probability is

$$T = \exp\left(-2\int_R^b \sqrt{\frac{2m}{\hbar^2}\left(V(r) - Q\right)}dr\right) \;=\; \exp\left(-2G\right)$$

Where Gamow factor

$$G = \int_R^b \sqrt{\frac{2m}{\hbar^2}\left(V(r) - Q\right)}dr$$

Using the expressions of $V(r)$ and Q we can write

$$G = \int_R^b \sqrt{\frac{2m}{\hbar^2}\left[\frac{2Z_D e^2}{4\pi\epsilon_0 r} - \frac{2Z_D e^2}{4\pi\epsilon_0 b}\right]}dr$$

$$= \sqrt{\frac{2m}{\hbar^2}}\sqrt{\frac{2Z_D e^2}{4\pi\epsilon_0}}\int_R^b \left(\frac{1}{r} - \frac{1}{b}\right)^{1/2} dr$$

In order to calculate the integral,

$$\int \left(\frac{1}{r} - \frac{1}{b}\right)^{1/2} dr$$

let us take $r = bcos^2\theta$ and therefore $dr = -2bcos\theta sin\theta d\theta$. Therefore,

$$\int \left(\frac{1}{r} - \frac{1}{b}\right)^{1/2} dr \;=\; \sqrt{b}\int (cos2\theta - 1)\, d\theta \;=\; \sqrt{b}(sin\theta cos\theta - \theta)$$

Since $r = bcos^2\theta$, we have

$$cos\theta = \sqrt{\frac{r}{b}}$$

and

$$sin\theta = \sqrt{1 - \frac{r}{b}}$$

Therefore the above integral with the limits is

$$\sqrt{b}\left[\sqrt{1 - \frac{r}{b}}\sqrt{\frac{r}{b}} - cos^{-1}\left(\sqrt{\frac{r}{b}}\right)\right]_R^b$$

The above integral becomes zero at the upper limit $r = b$ and therefore,its value is non-zero only at the lower limit. We can then write Gamow factor G as

$$G = \sqrt{\frac{2m}{\hbar^2}}\sqrt{\frac{2Z_D e^2 b}{4\pi\epsilon_0}}\left[cos^{-1}\left(\sqrt{\frac{R}{b}}\right) - \sqrt{1 - \frac{R}{b}}\sqrt{\frac{R}{b}}\right]$$

Using the relation

$$\frac{B}{Q} = \frac{b}{R}$$

and

$$b = \frac{2Z_D e^2}{4\pi\epsilon_0 Q}$$

we can write the Gamow factor as

$$G = 2Z_D\alpha\sqrt{\frac{2mc^2}{Q}}\left[cos^{-1}\left(\sqrt{\frac{Q}{B}}\right) - \sqrt{1 - \frac{Q}{B}}\sqrt{\frac{Q}{B}}\right] \tag{10}$$

where fine structure constant α is

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

The transmission probability $T = e^{-2G}$ can be calculated by calculation the Gamow factor G using the equation 10.

We can again take the examples of alpha decay in Th isotopes.

(1) $Q_\alpha = 4.08$ MeV for alpha decay from $^{232}_{90}$Th with half-life $t_{1/2} = 5 \times 10^{17}$ s
(2) $Q_\alpha = 9.85$ MeV for alpha decay from $^{218}_{90}$Th with half-life $t_{1/2} = 10^{-7}$ s

For ^{232}Th decay, we have $Q = 4.08MeV$, $B = 27.4MeV$ and the Gmow factor is $G = 44.98$ The transmisson probability $T_1 = e^{-2G} = 8.5 \times 10^{-40}$. With the frequency $f = 2.5 \times 10^{21}$, the transition rate λ is

$$\lambda_1 = 2.13 \times 10^{-18}$$

and the half life is

$$t_{1/2} = 3.2 \times 10^{17} s$$

For ^{218}Th decay, we have $Q = 9.85MeV$, $B = 27.9MeV$ and the Gmow factor is $G = 16.14$ The transmisson probability $T_2 = e^{-2G} = 9.6 \times 10^{-15}$. With the frequency $f = 3.86 \times 10^{21}$, the transition rate λ is

$$\lambda_2 = 3.7 \times 10^{7} s^{-1}$$

and the half life is

$$t_{1/2} = 1.9 \times 10^{-8} s$$

You can see that the calculated lifetimes for the decay processes compare with those of experimental results.

Geiger-Nuttall Law

We have seen that barrier is of the order of 28 MeV and Q is in the range 5-10 MeV. Under the approximation of

$$\sqrt{\frac{Q}{B}} < 1$$

we can write

$$cos^{-1}\left(\sqrt{\frac{Q}{B}}\right) = \frac{\pi}{2} - \frac{Q}{B}$$

and then the Gamow factor G is

$$G = 2Z_D\alpha\sqrt{\frac{2mc^2}{Q}}\left[\frac{\pi}{2} - \sqrt{\frac{Q}{B}} - \sqrt{\frac{Q}{B}}\right]$$

Here we have neglected $(\frac{Q}{B})^2$ term.

$$\text{or}G = \pi Z_D\alpha\sqrt{2mc^2}\frac{1}{\sqrt{Q}} - 4Z_D\alpha\sqrt{\frac{2mc^2}{B}}$$

$$= \frac{C_1}{\sqrt{Q}} + C_2$$

Here, C_1 and C_2 are constants.

The transmission probability $T = e^{-2G}$ and $\lambda = fT$. Therefore, $lnt_{1/2} \propto G = \frac{C_1}{\sqrt{Q}} + C_2$ This is Geiger-Nuttall Law.

We can try to understand the dependence of half-life on Q using Geiger-Nuttall Law. If $Q_2 > Q_1$ for alpha decay processes, both the barrier height and the width are less for the decay with Q_2 and, hence its transmission probability is higher and smaller half-life.

We also know that the Coulomb barrier $B \propto Z_D$. Therefore, higher the Z_D higher the barrier and less will be the transmission proability and hence, longer half-life.

Lecture 3

Alpha spectroscopy

If we place a alpha source in front of a detector, the spectrum will show several peaks, as shown in the following Fig. 4

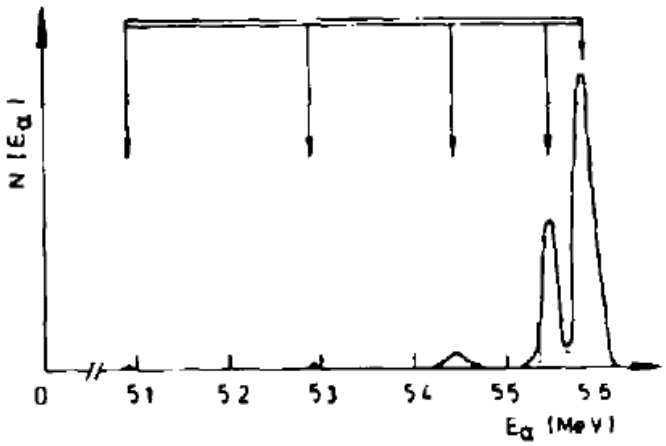

Figure 4: Alpha spectrum of ^{238}Pu. The Q value of the decay is 5.59 MeV

Observations:

1. There are several peaks in the spectrum, called as "fine structure" of alpha spectrum. Please note that this is nothing to do with fine structure of atomic spectrum.

2. The intensity (number of alpha particles) is decreasing with decreasing alpha energy. The highest energy peak is the most intense peak.

How do we understand this?

The peaks in the spectrum are due to decay of the parent nucleus to ground state as well as to the excited states of daughter nucleus. For example, the decay of ^{238}Pu is shown in the Fig. 5. The Q-value of the decay process is

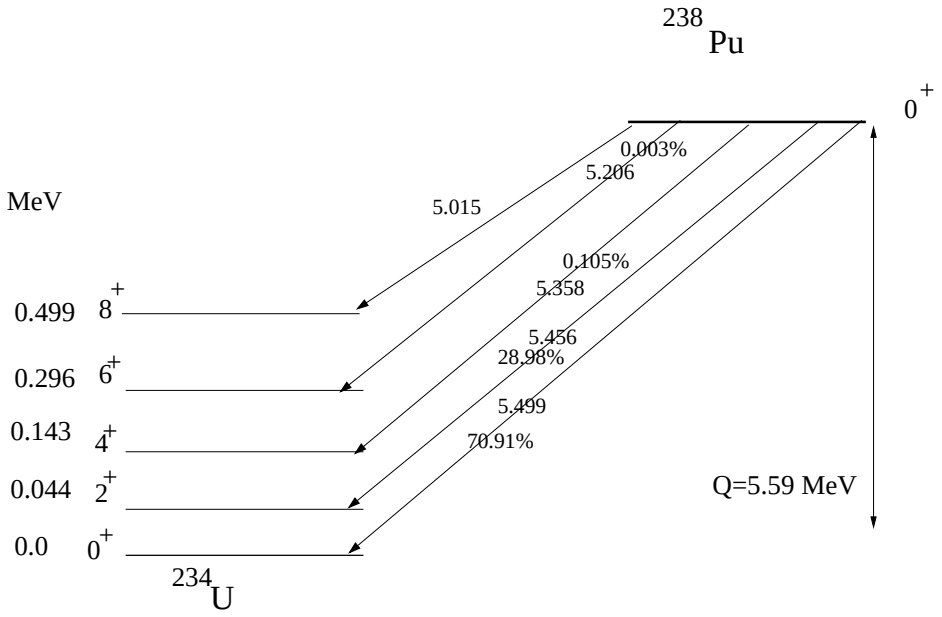

Figure 5: Decay scheme of ^{238}Pu. The Q value of the decay is 5.59 MeV

5.59 MeV. The energy of the alpha particles and their branching ratio are also shown in the figure. Note that for $0^+ \rightarrow 0^+$ decay $E_\alpha \simeq 0.98Q$

It is clear from Fig. 5 that for decays to higher spin states, the energy of alpha particle is less, i.e. the Q in the expression of Gamow factor G is smaller. This will increase factor G and, hence, less transmission probability and less intensity.

The second factor responsible for decreasing intensity is due to presence of centrifugal barrier for decays to higher spin states. Since angular momentum is conserved in the decay process, the alpha particle must be carrying difference of angular momentum between initial and final states. That means they also carry rotational energy.

$$V_{rot} = \frac{\mathrm{L}^2}{2\mathrm{m}_\alpha \mathrm{r}^2}$$

Therefore, total potential energy will be $V_{coul} + V_{rot}$ resulting barrier height

to be higher. The increase in barrier height will be higher for the decays to higher spin states.

Among the other reasons, the difference in the structure of the initial and final states can also influence transmission probability T. In all the calculation, we have assumed spherical shape of the nucleus. Departure from spherical shape may also produce hindrance.

At r=R, the new barrier height is

$$B' = \frac{2Z_D e^2}{4\pi\epsilon_0 R} + \frac{l(l+1)\hbar^2}{2m_\alpha R^2}$$

$$= \frac{2Z_D e^2}{4\pi\epsilon_0 R}\left[1 + \frac{l(l+1)\hbar^2}{2m_\alpha R^2} \times \frac{4\pi\epsilon_0 R}{2Z_D e^2}\right]$$
$$= B\left(1 + \beta l(l+1)\right) = B(1+\sigma)$$

where,

$$\beta = \frac{\hbar^2 4\pi\epsilon_0}{4Z_D e^2 m_\alpha R}.$$

The value of β is equal to 0.002. The correction to the Coulomb barrier is $B\sigma$.

The new barrier B' will replace B in the expression of Gamow factor G

$$\sqrt{\frac{Q}{B'}} = \sqrt{\frac{Q}{B}}\left(1 - \frac{1}{2}\sigma\right)$$

and the approximate expression of Gamow factor will be

$$G' = G + 2Z_D\alpha\sqrt{\frac{2m_\alpha c^2}{B}}\sigma$$

One can calculate transmission probability using modified Gamow factor G'.

0.0.1 Selection rules

The angular part of the wavefunction of alpha particle is represented by spherical harmonics $Y_l^m(\theta, \phi)$ and parity of the wavefunction will be decided by the factor (-1)$^{l_\alpha}$. Since parity is conserved in strong interaction, we can write

$$\pi_i = (-1)^{l_\alpha} \pi_f$$

Here, i is the initial state, f is representing final state and l_α is orbital angular momentum of alpha particle. Alpha particle is an even-even nucleus and its ground state spin J is zero. Since nucleons in alpha particle are occupying l=0 state (ground state), total intrinsic spin S=0. Therefore, $J_\alpha = l_\alpha$. The total angular momentum carried away by the alpha particle is J_α which can have all the values between $|J_i - J_f|$ and $(J_i + J_f)$.

Consider following decays

(i) $0^+ \rightarrow 2^+$

Here, $J_i = 0$, $J_f = 2$, $J_\alpha = 2 = l_\alpha$. Therefore, $\pi_i = (-1)^2 \pi_f$ is satisfied and the decay is allowed.

(ii) $1^+ \rightarrow 3^-$

Here, $J_i = 1$, $J_f = 3$, $J_\alpha = l_\alpha = 2, 3,$ and 4. Therefore, conservation of parity $\pi_i = (-1)^3 \pi_f$ demands $l_\alpha = 3$ $l_\alpha = 2$ and 4 are not allowed.

(iii) $1^+ \rightarrow 3^+$

Here, $J_i = 1$, $J_f = 3$, $J_\alpha = l_\alpha = 2, 3,$ and 4. In this case, conservation of parity $\pi_i = (-1)^{2 \text{ or } 4} \pi_f$ suggests $l_\alpha = 2$ or 4 but not 3.

(iv) $0^+ \rightarrow 4^-$

Here, $J_i = 0$, $J_f = 4$, $J_\alpha = 4 = l_\alpha$. Therefore, $\pi_i = (-1)^4 \pi_f$ is NOT satisfied and the decay is NOT allowed.

(v) $0^+ \rightarrow 3^+$

Here, $J_i = 0$, $J_f = 3$, $J_\alpha = 3 = l_\alpha$. Therefore, $\pi_i = (-1)^3 \pi_f$ is NOT satisfied and the decay is also NOT allowed.

Therefore, we can say that decay from 0^+ state to odd-parity even-spin state is forbidden. Similarly, decay from 0^+ state to even-parity odd-spin state is also forbidden.

Beta Decay

Lecture 1

Introduction

The beta decay process is due to weak interaction. Following decay processes can happen inside the nucleus.

$$n \rightarrow p + e^- + \bar{\nu} \tag{11}$$

$$p \rightarrow n + e^+ + \nu \tag{12}$$

$$e + p \rightarrow n + \nu \tag{13}$$

The first equation is for β^- decay in which anti-neutrino comes out along with β^-. This is to conserve lepton number. We do not have any lepton on the left hand side of the decay equation. Therefore, total lepton number should be zero on the right hand side of it. Since electron is classified as a particle, an anti-particle of neutrino comes out. The decay process is possible in free space, as mass of neutron is higher than those of proton, electron and anti-neutrino. The half life of neutron in free space is around 10.3 min.

The second equation is for β^+ decay. The β^+ is anti-particle of electron and, therefore, neutrino comes out along with β^+. The third decay equation is for electron capture process. Both of these decay processes are not possible in free space due to energy conservation. In the electron capture process, an electrons from K or higher shells are captured by the nucleus. The probability of capture process is higher for the nearest shell, i.e. K shell. This process

competes with β^+ decay as you can see that the end product is same in the both the decays. These two processes are linked by cross symmetry. If a particle from r.h.s of the decay equation is brought to its l.h.s, it has to be replaced by its anti-particle. Same is true if a particle is transferred from l.h.s to r.h.s, it has to be replaced by its anti-particle. In the third equation e^+ is transferred to l.h.s and is replaced by e^-

The β decay processes are three-body process unlike alpha decay and, hence, total energy is shared by both β and ν. Therefore, the beta spectrum has continuous energy distribution.

For equation 1, the $Q-$value of the decay process is
$Q = (m_n - m_p - m_e - m_{\bar{\nu}})c^2$.

For the decay of neutron at rest, we can also write
$Q = (T_p + T_e + T_{\bar{\nu}})$.

If we neglect recoil energy of the proton, which is around 0.3 keV, then
$Q \simeq (T_e + T_{\bar{\nu}})$.

Therefore, in case neutrino carries nearly zero energy, we can write $Q \simeq (T_e)_{\max}$

The measured value of maximum energy of the electrons from neutron decay is 0.782 ± 0.013 MeV. Using measured values of masses of neutron, proton and electron, the computed $Q-$value is
$Q = 0.782 - m_\nu c^2$

Within the precession of the measurement the neutrino mass comes out to be nearly zero. In the standard model, neutrino is considered to be massless and, hence, its velocity is c. However, a limit on its mass, which is of the order of eV, has been established in recent experiments.

Calculation of Q-values

β^- decay

$$^A_ZX \rightarrow ^A_{Z+1}Y + \beta^- + \bar{\nu} \tag{14}$$

$$Q(\beta^-) = \left[m_N(^A_ZX) - m_N(^A_{Z+1}Y) - m_e\right] c^2 \tag{15}$$

Here, m_N represents nuclear masses and m_e is the mass of electron. The equation 15 can also be expressed in terms of atomic masses m_A by replacing nuclear mass

$$m_N(^A_ZX)c^2 = \left[m_A(^A_ZX) + Zm_e\right] c^2 - \Sigma B_e.$$

Here, B_e is the binding energy of the electrons. Therefore, equation 15 becomes

$$Q(\beta^-) = \left[m_A(^A_ZX) - m_A(^A_{Z+1}Y)\right] c^2$$

The difference between electronic binding energies of parent and daughter atoms is small and is neglected.

β^+ decay

$$^A_ZX \rightarrow ^A_{Z-1}Y + \beta^+ + \nu \tag{16}$$

Following the same as for the β^- decay, the Q value in terms of atomic masses of parent and daughter atoms is (Check this)

$$Q(\beta^+) = \left[m_A(^A_ZX) - m_A(^A_{Z-1}Y) - 2m_e\right] c^2$$

Electron capture

$$^A_ZX + e^- \rightarrow ^A_{Z-1}Y + \beta^- + \nu \tag{17}$$

In this process the daughter atom is produced in excited state because of vacancy of electrons in atomic shell and one should take care of this in the calculation of Q-value.

$$Q(\varepsilon) = \left[m_A(^A_ZX) - m_A(^A_{Z-1}Y)\right] c^2 - B_e$$

Here, B_e is the binding energy of the captured electron.

All the above decay processes will proceed spontaneously if its Q-value is positive. For β^+, the atomic mass difference between parent and daughter must be greater than twice the mass of electron, i.e. 1.022 MeV. If the difference is less than 1.022 MeV, the decay process will be through electron capture, as both the processes competes. If the Q-value is more than 1.022 MeV, both the processes will follow. For electron capture process, the difference of masses of parent and daughter should be greater than the binding energy of electron, which is of the order of a few keV.

Fermi theory of β decay

Fermi was guided by the analogy to electromagnetic γ ray emission. In this process, excited nucleus decay to its ground state creating a photon. In the beta decay process, a neutron is transformed into a proton and an electron and an anti-neutrino are created.

The transition rate can be calculated using Fermi golden rule 2

$$\lambda = \frac{2\pi}{\hbar}|H_{fi}|^2\rho(E_f) \tag{18}$$

Here, the $\rho(E_f)$ is density of final states and can be written as

$$\rho(E_f) = \frac{dn}{dE_f}$$

i.e., the number dn of final states in the energy interval dE_f. It is the density of final states which determines the shape of the β spectrum. The H_{fi} in eqn. 18 is the matrix element of interaction Hamiltonian between initial and final states.

$$H_{fi} = \int \psi_f^* H \psi_i d\tau$$

Here, H is the operator representing the interaction. Because the initial and final states do not refer to same particles, the operator must be able to destroy the initial particle and create the final ones.

let

$$\psi_i = u_i$$

describes the nucleus before decay and

$$\psi_f = u_f \psi_e \psi_{\bar{\nu}u}$$

is the product of the wave function of the particles in the final state. Here, u_f is the wave function of nucleus in the final state, and ψ_e and $\psi_{\bar{\nu}}$ are the wave functions of electron and anti-neutrino in the final state. Therefore, the matrix element can be expressed as

$$H_{fi} = g \int u_f^* \psi_e^* \psi_{\bar{\nu}}^* \hat{O} \psi_i d\tau$$

Here, g is the strength of the interaction and $\hat{O}$ is the operator representing the interaction process.

We shall assume that the wave functions of the electron and the neutrino have the plane wave form, i.e. assumed to be free particle. This may be a good approximation for neutrino as it is a neutral particle. it is a fair approximation for electron as it is subject to electromagnetic interaction with nucleus. A correction factor for this will be introduced later. The wave function of the electron in the free-particle form is

$$\psi_e = \frac{1}{\sqrt{V}} e^{i\mathbf{k_e}.\mathbf{r}}$$

where, V is the volume in which we enclose the system for normalisation purpose and $p_e = \hbar k_e$ is momentum of electron.

similarly for neutrino we can write

$$\psi_\nu = \frac{1}{\sqrt{V}} e^{i\mathbf{k}_\nu.\mathbf{r}}$$

Here, we are neglecting spin of the electron. Let us take an example of a beta particle which comes out with a kinetic energy of 1.5 MeV. There is nothing specific about the above considered value. We can then write kinetic energy of the electron using relativistic energy expression as

$$E_K = \sqrt{p^2c^2 + m_e^2c^4} - m_e c^2$$

For $E_K = 1.5$MeV, one can calculate $pc = 1.945$ or $p = 1.945/c$. Therefore, for a typical value of nuclear radius r=10 fm, we can calculate

$$k_e r = \frac{p}{\hbar} \times r \simeq 0.1$$

Thus over the nuclear volume $k_e r << 1$ and, therefore, we can expand the exponential term in the wave functions of electron and neutrino.

$$e^{i\mathbf{k}_\nu.\mathbf{r}} = 1 + i\mathbf{k}_\nu.\mathbf{r} + ...$$

In the first approximation, we can neglect the second term in the wave function as it is small. This also means that we are replacing the wave functions of electron and neutrino by their values at $r = 0$. This approximation is known as *allowed* approximation. Therefore,

$$\psi_e = \psi_\nu = \frac{1}{\sqrt{V}}$$

Thus the matrix element becomes

$$H_{fi} = g\frac{M_{fi}}{V},$$

where

$$M_{fi} = \int u_f^* \hat{O} u_i \; d\tau$$

and the transition rate is

$$\lambda = \frac{2\pi}{\hbar}\frac{g^2}{V^2}|M_{fi}|^2\frac{dn}{dE_f}$$

In the next lecture, the density of the states $\frac{dn}{dE_f}$ will be calculated.

Lecture 2

Density of final states

For the calculation of density of final states, let us consider a particle confined in a cubical box of side L with volume $V = L^3$. Since particle is confined in the box, we can take the potential to be infinity. The wave function of a particle in $1d$ box is

$\psi = Asin(k_x x) + Bcos(k_x x)$

using the boundary conditions $\psi = 0$ at $x = 0$ and at $x = L$, we have $k_x L = n_x \pi$ or $n_x = \frac{k_x L}{\pi}$. Similarly, we can write $n_y = \frac{k_y L}{\pi}$ and $n_z = \frac{k_z L}{\pi}$.

The energy expression is

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

where, $n^2 = n_x^2 + n_y^2 + n_z^2 = \frac{p^2 L^2}{\hbar^2 \pi^2}$

Also,

$$ndn = \frac{L^2}{\hbar^2 \pi^2} pdp$$

The allowed states of the particle are represented by n_x, n_y and n_z which are positive integers. If the coordinate axes are represented by n_x, n_y and n_z, a state in this coordinate system is represented by a point with position vector n. Therefore, volume of a shell with radius n and $n + dn$ is

$$\frac{4\pi n^2 dn}{8} = \frac{\pi n^2 dn}{2}$$

Note that only $\frac{1}{8}$ of the states are allowed due to restrictions on the values of n_x, n_y and $n_z > 0$ (only positive values). Since volume of a unit cell in

this space is one, as n_x, n_y and n_z changes by 1, number of states within the volume of the shell can be obtained by dividing the volume by volume of a unit shell. Therefore, the number of states is equal to the volume. Using the expression of ndn and n, the number of states with momentum between p and $p + dp$ is

$$dn = \frac{L^3}{2\hbar^3\pi^2}p^2dp = \frac{Vp^2dp}{2\hbar^3\pi^2}$$

In the final state of beta decay process we have two particles, electron and neutrino. For each state of the electron, all possible states of neutrino are allowed to occupy. Therefore, total number of states dn is

$$\begin{aligned} dn &= V^2\frac{p_e^2dp_e}{2\hbar^3\pi^2} \times \frac{p_\nu^2dp_\nu}{2\hbar^3\pi^2} \\ &= \frac{V^2}{(2\hbar^3\pi^2)^2}p_e^2dp_ep_\nu^2dp_\nu \end{aligned}$$

Therefore,

$$\frac{dn}{dE_f} = \frac{V^2}{(2\hbar^3\pi^2)^2}p_e^2dp_ep_\nu^2\frac{dp_\nu}{dE_f}$$

Final energy $E_f = E_e + E_\nu$ with $E_\nu = p_\nu c$. The recoil energy of the daughter nucleus is neglected. If we are looking for beta particles with constant momentum p_e then E_e is constant. Therefore,

$$dE_f = c\ dp_\nu \text{ or } \frac{dp_\nu}{dE_f} = \frac{1}{c}$$

The transition rate is then

$$\lambda(p_e)dp_e = \frac{2\pi}{\hbar}g^2|M_{fi}|^2\frac{1}{(2\hbar^3\pi^2)^2}p_e^2dp_ep_\nu^2\frac{1}{c}$$

If Q is the decay energy then

$$Q = T_\beta + T_\nu$$

where, T_β and T_ν are kinetic energy of β and ν, respectively.

$$= \sqrt{p^2c^2 + m_e^2c^4} - m_ec^2 + p_\nu c$$

or

$$p_\nu = \frac{1}{c}\left[Q - \sqrt{p^2c^2 + m_e^2c^4} + m_ec^2\right]$$

Therefore, the number of electrons per unit time with momentum p_e and $p_e + dp_e$ is

$$\lambda(p_e)dp_e = \frac{2\pi}{\hbar}g^2|M_{fi}|^2\frac{1}{(2\hbar^3\pi^2)^2}\frac{p_e^2dp_e}{c^3}\left[Q - \sqrt{p_e^2c^2 + m_e^2c^4} + m_ec^2\right]^2 \quad (19)$$

The above expression is zero for $p_e = 0$ and for $T_e = Q$, as the term in [..] is zero. We can also replace Q in eqn. 19 by T_β^{max}, maximum kinetic energy of the electron. If the corresponding momentum is p_e^{max} then

$$\lambda(p_e)dp_e = \frac{2\pi}{\hbar}g^2|M_{fi}|^2\frac{1}{(2\hbar^3\pi^2)^2}\frac{p_e^2dp_e}{c^3}\left[\sqrt{p_e^{max2}c^2 + m_e^2c^4} - \sqrt{p_e^2c^2 + m_e^2c^4}\right]^2$$

or in terms of total energy E_β of the beta particle, the transition rate is

$$\lambda(p_e)dp_e = \frac{2\pi}{\hbar}g^2|M_{fi}|^2\frac{1}{(2\hbar^3\pi^2)^2}\frac{p_e^2dp_e}{c^3}\left[E_\beta^{max} - E_\beta\right]^2$$

or

$$\lambda(p_e)dp_e = C_1p_e^2dp_e\left[E_\beta^{max} - E_\beta\right]^2 \quad (20)$$

where,

$$C_1 = \frac{2\pi}{\hbar}g^2|M_{fi}|^2\frac{1}{(2\hbar^3\pi^2)^2c^3}$$

is a constant assuming nuclear matrix element remains constant.

One can also express transition rates in terms of kinetic energy T_e of the electron by replacing its momentum (Check!!)

$$\lambda(T_e)dT_e \propto \left(T_e + M_e c^2\right)\left(T_e^2 + 2T_e m_e c^2\right)^{1/2}(Q - T_e)^2 \tag{21}$$

One can see that $\lambda(T_e) \to 0$ if $T_e \to 0$ and $T_e \to Q$

If $N(p_e)dp_e$ is the number of beta particles with momentum p_e and $p_e + dp_e$, one can write $N(p_e)$ using eqn. 19 as

$$N(p_e) = C_1 p_e^2 (Q - T_e)^2$$

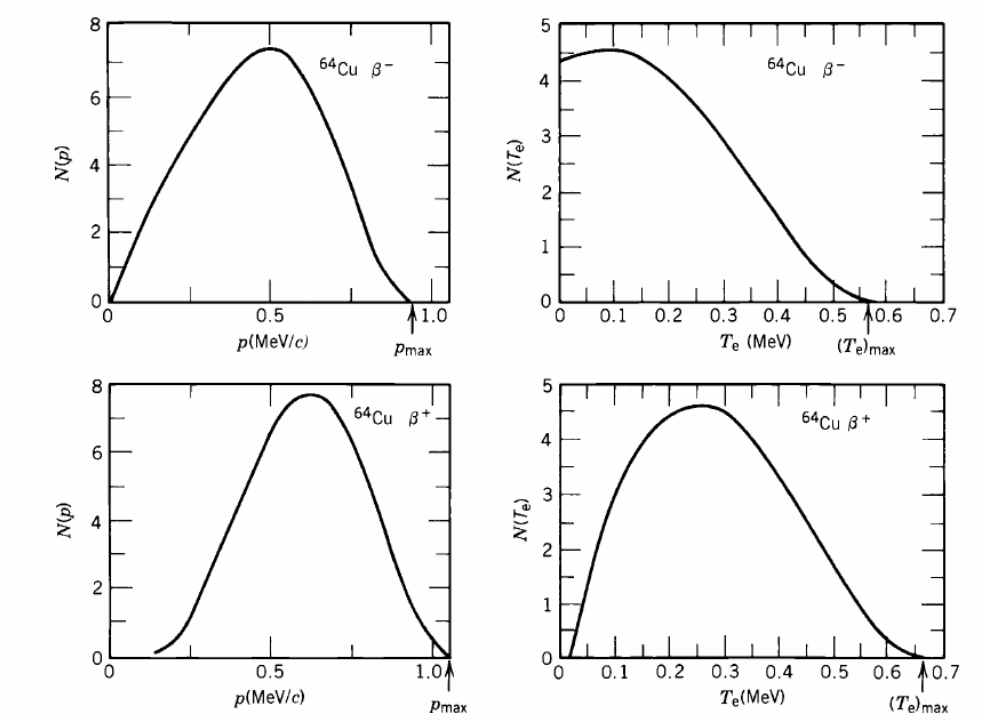

Figure 6: Beta decay spectrum for Cu nuclei. The number of beta particle is plotted against their momentum and kinetic energy.

A typical beta decay spectra in terms of momentum and kinetic energy of beta particles are shown in Fig. 6. Note the difference between spectra of

β^- and β^+ at lower momentum. The plot of β^+ is shifted towards higher momentum. Same is true for the plots with kinetic energy.

It has been observed that the number of low-energy beta particles is more in experiment than that expected from Fermi theory. Also, the same with β^+ is less than the expected number. This is due to the fact that Coulomb repulsion between the nucleus and the emitted β^- or β^+ particles has not been included in Fermi theory. The β^- is attracted by the nucleus and, therefore, its kinetic energy will be less than the expected value from Fermi theory. For β^+, it is repelled and its kinetic energy will be more than the expected value. In order to address this issue, a Fermi function $F(Z', p_e)$ has been included in the expression of transition rate. The Fermi function is defined as

$$F(Z', p_e) = \frac{2\pi\eta}{1 - e^{-2\pi\eta}}$$

Here, Z' is the atomic number of daughter nucleus. The parameter η is defined as

$$\eta = \pm\frac{Z'e^2}{\hbar v_e}$$

where, v_e is the velocity of β^- or β^+ far from the nucleus. For β^-, η is taken positive and it is negative for β^+.

for $\eta << 1$, either v is large or Z' is small. Therefore, for β^-

$$F(Z', p_e) = \frac{2\pi\eta}{1 - (1 - 2\pi\eta)} \simeq 1$$

That means, it does not change the spectrum for large momentum particle.

if $2\pi\eta$ is large, i.e. the velocity of β^- is small (low energy), the exponential term in $F(Z', p_e)$ is negligible. The Fermi function is then

$$F(Z', p_e) = 2\pi\eta \;=\; \frac{2\pi Z'e^2 m_e}{\hbar p_e}$$

.

Therefore,

$$\lambda(p_e)dp_e = C_1 p_e^2 dp_e \frac{2\pi Z'e^2 m_e}{\hbar p_e} (Q - T_e)^2$$

i.e., the transition rate $\lambda(p_e)dp_e \propto p_e dp_e$ (linear in momentum). Note that the transition rate was proportional to p_e^2 before the correction.

The parameter η is negative for β^+. Therefore, for large value of $2\pi\eta$, i.e. low energy β^+ particles, $e^{2\pi\eta} >> 1$ and one can neglect 1 in the denominator of $F(Z', p_e)$. Thus, the Fermi function is

$$F(Z', p_e) = \frac{2\pi\eta}{e^{2\pi\eta}} \propto \frac{1}{p_e} e^{-\frac{C_2}{p_e}}$$

Here, C_2 is a constant. Therefore, the transition rate will be

$$\lambda(p_e)dp_e \propto p_e e^{-\frac{C_2}{p_e}} dp_e$$

The corrected β spectrum including contribution of the Fermi function is shown in Fig. 7. As can be seen that the correction is greater for lower energy β particles. The spectrum is shifted towards lower energy for β^- and, for β^+ it is shifted towards higher energy.

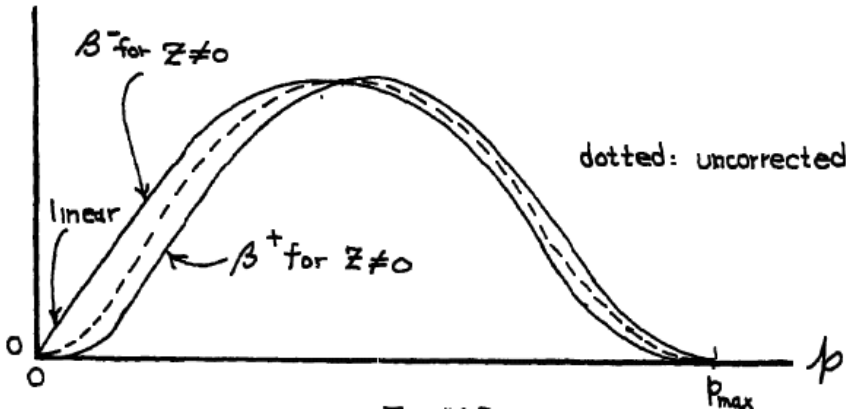

Figure 7: The β^- and β^+ spectra including correction $F(Z', p_e)$. For comparison, the uncorrected spectrum is also shown by the dotted lines.

Lecture 3

Fermi-Kurie Plot

The number of beta particles with momentum between p_e and $p_e + dp_e$ including Coulomb correction is

$$N(p_e) = C_1 p_e^2 \left(Q - T_e\right)^2 F(Z', p_e)$$

Therefore, the function

$$\sqrt{\frac{N(p_e)}{F(Z', p_e) p_e^2}} \propto (Q - T_e)$$

Plotting the function at the left hand side as a function of kinetic energy of the beta particle will produce a straight line with a negative slope. The function is zero if T_e is equal to Q and gives end-point energy of the beta particle. The plot is known as Fermi-Kurie plot and is shown in Fig. 8. The plot deviates from straight line at lower energy due to scattering of beta particle within the source.

Note that, for *allowed* approximation, the nuclear matrix element $|H_{fi}|$ is independent of the electron and neutrino momenta as the first term in the expansion of the wavefunctions $e^{i\mathbf{k}.\mathbf{r}}$ of electron and neutrino is unity. However, this is not true for the second and higher-order terms in the expansion, representing *first forbidden* or *higher-order forbidden* transitions (for details, see next lecture). Therefore, Fermi-Kurie plot for these transitions are not expected to follow a straight line. A shape factor $S(p_e, p_\nu)$, a function of the electron and neutrino momenta, is used to restore the linearity of the plot. For *first forbidden* transition, a function $p_e^2 + p_\nu^2$ can be used.

Neutrino Mass

In the above calculations, mass of neutrino m_ν has been assumed to be zero. If neutrino has mass then the end point energy of the beta particle will be

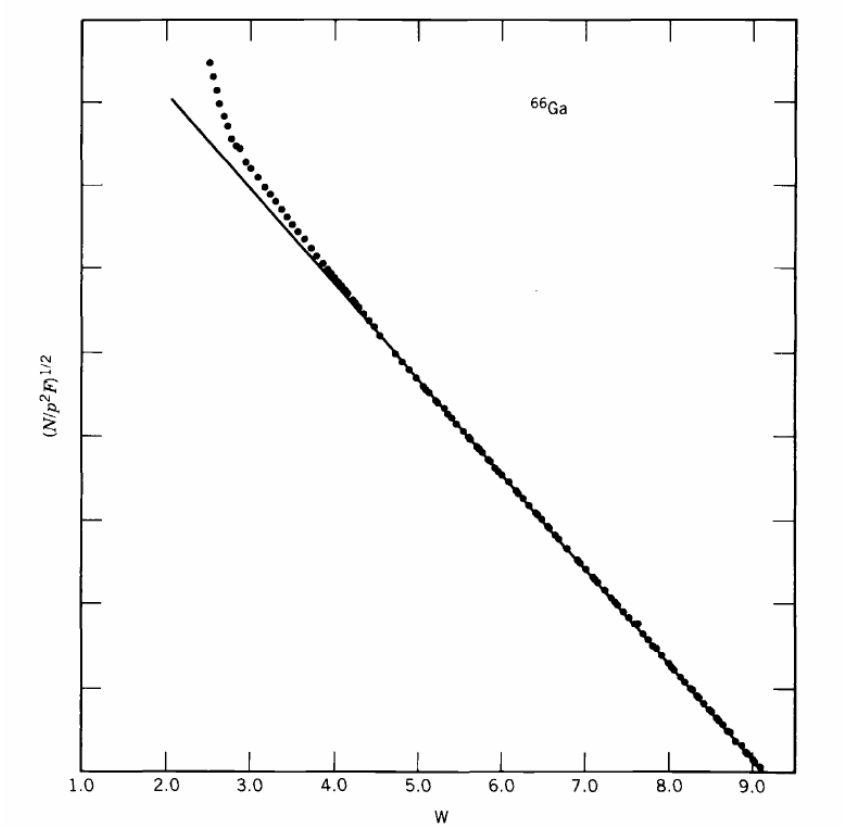

Figure 8: Fermi-Kurie plot of beta decay in ^{66}Ga. The x-axis is the relativistic total energy scaled to mass of electron.

less than the Q value and the difference can be used to calculate the neutrino mass. The problem lies in the uncertainty associated with determination of the end-pont energy. Near the end-point energy the number of beta particles is comparable to the background counts and, therefore, it is difficult to find slope of the function near the end-point energy. let us see what kind of slope is expected if one considers neutrino as massless or having mass.

Considering neutrino as massless particle ($m_\nu = 0$), we have derived spectrum of a beta particle as

$$N(p_e) = C_1 p_e^2 \left(Q - T_e\right)^2 F(Z', p_e)$$

The slope can be calculated by taking derivative of the function $N(p_e)$ with respect to p_e.

$$N(p_e) = C_1 p_e^2 \left(Q - \sqrt{p_e^2c^2 + m_e^2c^4} + m_e c^2\right)^2 F(Z', p_e)$$

$$\frac{dN(p_e)}{dp_e} = C_1 \left[2p_e \left(Q - \sqrt{p_e^2c^2 + m_e^2c^4} + m_e c^2\right)^2 - 2p_e^2 \left(Q - \sqrt{p_e^2c^2 + m_e^2c^4} + m_e c^2\right) \times \frac{p_e}{\sqrt{p_e^2c^2 + m_e^2c^4}}\right]$$

From the above expression you can see that as $T_e \to Q$, the slope $\frac{dN(p_e)}{dp_e} \to 0$. This because the term in the parenthesis is zero. This means that the slope of beta spectrum will be parallel to the x-axis near the end-point (see Fig. 9).

Now, consider the case of neutrino mass $m_\nu \neq 0$. At very close to the end-point energy, neutrino energy will be very small and the velocity of neutrino v_ν can be treated non-relativistically. we can then go back to the calculation of density of states where neutrino mass was considered zero and repeat the same calculation by considering neutrino to have mass.

The kinetic energy of neutrino in non-relativistic limit is

$$T_\nu = \frac{p_\nu^2}{2m_\nu}$$

Total energy in final state is then

$$E_f = E_e + \frac{p_\nu^2}{2m_\nu} + m_\nu c^2$$

$$dE_f = \frac{p_\nu dp_\nu}{m_\nu}$$

$$\text{or, } \frac{dp_\nu}{dE_f} = \frac{m_\nu}{p_\nu}.$$

The Q value is equal to sum of the kinetic energies of beta and neutrino

$$Q = T_e + \frac{p_\nu^2}{2m_\nu}$$

$$\text{or, } p_\nu^2 = 2m_\nu \left(Q - T_e\right)$$

Therefore, the number of beta particles with momentum between p_e and $p_e + dp_e$ is

$$N(p_e) = C_1 p_e^2 p_\nu^2 \frac{m_\nu}{p_\nu} \;=\; C_1 p_e^2 p_\nu m_\nu$$

Using the expression of p_ν, we can write

$$N(p_e) = C_1 p_e^2 \sqrt{2} m_\nu^{3/2} \left(Q - T_e\right)^{1/2}$$

Therefore, the slope of the spectrum is

$$\frac{dN(p_e)}{dp_e} = \sqrt{2} C_1 \left[2p_e \left(Q - T_e\right)^{1/2} + \frac{p_e^2}{2(Q - T_e)^{1/2}} \times \frac{-p_e}{\sqrt{p_e^2 c^2 + m_e^2 c^4}} \right]$$

In this case, as $T_e \to Q$, $\dfrac{dN(p_e)}{dp_e} \to \infty$, i.e. the plot will become vertical, as shown in Fig. 9. Therefore, we can study the limit of neutrino mass by looking at the slope of the beta spectrum near the end point.

Total transition rate

The transition rate of beta particles with momentum between p_e and $p_e + dp_e$, $\lambda(p_e)dp_e$, has been calculated in the above sections. The total transition rate can be obtained by integrating $\lambda(p_e)dp_e$ for all possible momentum values of β, i.e.

$$\lambda = \int_0^{p_e^{max}} \lambda(p_e) dp_e \;=\; \int_0^{p_e^{max}} C_1 p_e^2 dp_e \left(Q - T_e\right)^2 F(Z', p_e)$$

Let us take $F(Z', p_e) = 1$ for simplicity. This assumes involvement of low-Z nuclei or high-energy beta particles. Using the expression of Q and T_e, we

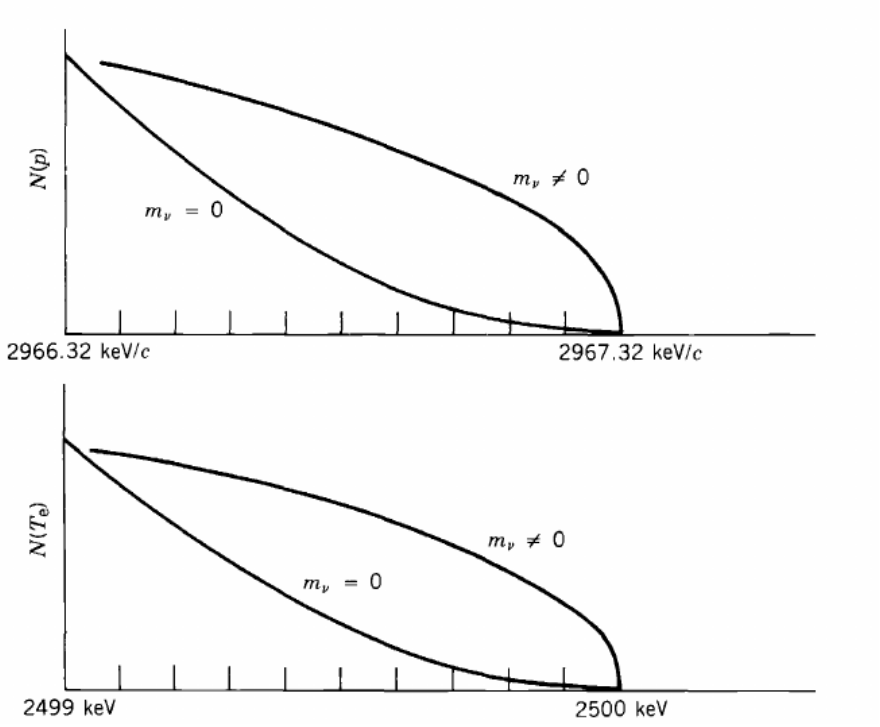

Figure 9: Slope of beta spectrum near the end-point energy considering neutrino to be massless and having mass.

can write

$$\lambda = \int_0^{p_e^{max}} C_1 p_e^2 dp_e \left[\sqrt{p_e^{max2}c^2 + m_e^2c^4} - \sqrt{p_e^2c^2 + m_e^2c^4}\right]^2 \qquad (22)$$

Let us define

$$\epsilon = \frac{p_e}{m_e c}$$

and

$$\epsilon_0 = \frac{p_e^{max}}{m_e c}$$

such that, as $p_e \to 0$, $\epsilon \to 0$ and, as $p_e \to p_e^{max}$, $\epsilon \to \epsilon_0$

Therefore, replacing p_e by ϵ and p_e^{max} by ϵ_0 in equation 22 we have

$$\lambda = C_1 m_e^5 c^7 \int_0^{\epsilon_0} \epsilon^2 d\epsilon \left[(\epsilon_0^2 + 1) + (\epsilon^2 + 1) - 2\sqrt{(\epsilon_0^2 + 1)}\sqrt{(\epsilon^2 + 1)} \right]$$
$$= C_1 m_e^5 c^7 f(\epsilon_0),$$

where, the Fermi integral $f(\epsilon_0)$ can be obtained by evaluating the integral as

$$f(\epsilon_0) = -\frac{1}{4}\epsilon_0 - \frac{1}{12}\epsilon_0^3 + \frac{1}{30}\epsilon_0^5 + \frac{1}{4}\sqrt{1+\epsilon_0^2} ln(\epsilon_0 + \sqrt{1+\epsilon_0^2})$$

Note that for $\epsilon_0 >> 5$, $f(\epsilon_0) \simeq \frac{1}{30}\epsilon_0^5$

using the value of constant C_1, we can write

$$\lambda = \frac{g^2 |M_{fi}|^2 m_e^5 c^4}{2\pi^3 \hbar^7} f(\epsilon_0) \;=\; \frac{0.693}{t_{1/2}}$$

Usually, a comparative half-life $f(\epsilon_0)t_{1/2}$ or in short ft is used in beta decay instead of half-life $t_{1/2}$

$$ft = \frac{0.693 \times 2\pi^3 \hbar^7}{g^2 |M_{fi}|^2 m_e^5 c^4}$$

The ft values are large and, therefore, $\log ft$ is used for beta decay transitions. Note that the matrix element $|M_{fi}|$ in the above expression is still unknown and its calculation requires knowledge of the wavefunctions of the nuclear levels involved in the decay process. The value of the matrix element will depend on overlap of the wavefunctions of the two states involved in decay process. If the orbitals occupied by the protons and the neutrons in parent and daughter nuclei are same, the overlap of the wavefunction is expected to be maximum and for such nuclei $|M_{fi}|$ will be the largest. These decay transitions are called *super-allowed* transitions for which $|M_{fi}|^2 = 2$ and $ft \simeq 3000$. These transitions are mostly available in lighter nuclei ($0^+ \rightarrow 0^+$) where neutrons and protons occupy same set of orbitals. For heavier nuclei, number of neutrons is large compared to protons and the valence neutrons

occupy different set of orbitals and overlap of the wavefunctions will be less. For *allowed* transitions $|M_{fi}|^2 = 1$.

Estimation of strength of weak interaction g

For super allowed transition $|M_{fi}|^2 = 2$ and $ft \simeq 3000$. We can then calculate strength g of weak interaction process using the expression of ft.

$$g^2 = \frac{0.693 \times 2\pi^3\hbar^7}{ft|M_{fi}|^2 m_e^5 c^4}$$

This gives a value of $g = 0.88 \times 10^{-4}$ MeV fm^3. In order to compare the parameter with those of other interactions, it is expressed in dimensionless form

$$G = g\frac{M^2 c}{\hbar^3}$$

In this expression M is the nucleon mass. Considering $M = 939$ MeV in the above expression, the dimensionless strength parameter is $G = 1 \times 10^{-5}$. This can be compared with strength parameter of strong interaction which is of the order unity and that of electromagnetic interaction is 10^{-2}.

Lecture 4

Decay selection rules

In the allowed approximation, the wave functions of the electron and the neutrino are replaced by its value at $r = 0$, i.e. they do not carry orbital angular momentum ($l = 0$ as $r = 0$). Therefore, change in the angular momentum of the parent and daughter nuclei must result from spins of electron and neutrino, each of which is $\frac{1}{2}$.

With $s_e = \frac{1}{2}$ and $s_\nu = \frac{1}{2}$, total spin will be either $S = 0$ (anti-parallel combination of electron and neutrino spins) or $S = 1$ (parallel combination of electron and neutrino spins).

Note that total orbital angular momentum $L = 0$ for *allowed* transitions and, if $S = 0$ then the total angular momentum ($J = L + S$) carried by the electron and the neutrino is zero. Therefore, $|J_f - J_i| = 0$, where J_f and J_i are total angular momentum of the final and the initial states. This transition is called as *Fermi* decay.

If $S = 1$, then the electron and neutrino together will carry one unit of angular momentum ($J = L + S$). Therefore, $\Delta J = |J_f - J_i| = 0$ or 1. [with vector addition of $J_i = J_f + (J = 1)$, J_f can take values $J_i + 1$, J_i and $J_i - 1$]. This transition is called as *Gamow-Teller* decay

As parity is decided by the orbital angular momentum, which is zero in this case, the parties of initial and final states of the nuclei must be identical, i.e $\Delta\pi =$ no change.

There is a special group of transitions among the *allowed* transitions, called as super-allowed transitions, which are mostly from $0^+ \rightarrow 0^+$ decay and are pure Fermi transitions.

Therefore, selection rules for allowed transitions are
$\Delta\pi =$ no change
Fermi transition : $\Delta J = 0$
Gamow-Teller (GT) transition : $\Delta J = 0$ or 1 (excluding $0^+ \rightarrow 0^+$)

Examples :

^{14}O $\rightarrow$ ^{14}N: the decay is between $0^+ \rightarrow 0^+$ and it is a pure Fermi transition ($\Delta J = 0$).

^{6}He $\rightarrow$ ^{6}Li: the decay is between $0^+ \rightarrow 1^+$ and it will be a pure GT transition ($\Delta J = 1$).

Decay of free neutron to proton : $\frac{1}{2}^+ \rightarrow \frac{1}{2}^+$ (spin =1/2), i.e. $\Delta J = 0$ decay. This is a mixed transition as $\Delta J = 0$ rule is for both *Fermi* and GT transitions. The exact amount of mixing can be determined by using their wavefunction.

In order to estimate the mixing in this transition, let us define a ratio Y of Fermi and GT amplitudes (nuclear matrix element)

$$Y = \frac{g_F M_F}{g_{GT} M_{GT}}$$

Here, g is the strength of the interaction and M is the nuclear matrix element. Note that they appear in squared form in the expression of transition rate. As the neutron decay is a mixed transition, the factor $g^2|M_{fi}|^2$ in the expression of transition rate must be replaced by $g_F^2|M_F|^2 + g_{GT}^2|M_{GT}|^2$, where the subscripts F and GT represent Fermi and GT transitions, respectively. The strength g_F can be estimated from super-allowed decay transitions (Fermi decay).

For $n \rightarrow p + \beta^- + \bar{\nu}$, the Q value is

$$Q = (m_n - m_p - m_e -)c^2 \;=\; (939.56 - 938.28 - 0.511) \;=\; 0.779 \text{ MeV}.$$

Also, by using

$$Q = \sqrt{p_{max}^2 c^2 + m_e^2 c^4} - m_e c^2$$

we get $p_{max}c = 1.185$ and $\epsilon_0 = \dfrac{p_{max}}{m_e c} = 2.317$ and then $f(\epsilon_0) = 1.605$

The ft value can be calculated using

$$ft = \frac{constant}{g_F^2|M_F|^2 + g_{GT}^2|M_{GT}|^2}$$

$$= \frac{constant}{g_F^2 |M_F|^2 \left(1 + Y^{-2}\right)}$$

The constant factor in the above expression can be obtained from ft values of super-allowed transition with $ft = 3000$. Note that they are pure Fermi transition.

$$3000 = \frac{constant}{g_F^2 |M_F|^2}$$

For super-allowed transition, $|M_F|^2 = 2$ and, hence,

$$constant = 6000 \times g_F^2$$

For neutron decay, $t_{1/2} = 630$ sec, therefore, we can write ft of neutron decay as

$$1.605 \times 630 = \frac{6000 \times g_F^2}{g_F^2 \left(1 + Y^{-2}\right)}$$

here, $|M_F|^2 = 1$ has been used for neutron decay (allowed transition) transition. The above equation gives $Y = 0.45$. Therefore, the fraction

$$\frac{g_F^2 |M_F|^2}{g_F^2 |M_F|^2 + g_{GT}^2 |M_{GT}|^2} = \frac{1}{1 + Y^{-2}} \simeq 0.17$$

Similarly,

$$\frac{g_{GT}^2 |M_{GT}|^2}{g_F^2 |M_F|^2 + g_{GT}^2 |M_{GT}|^2} = \frac{1}{1 + Y^2} \simeq 0.83$$

Therefore, the neutron beta decay is 83% GT and 0.17% Fermi transitions.

First forbidden transitions

The selection rules for allowed transitions are $\Delta J = 0$ or 1 and no change in parity. If the difference in angular momenta of the initial and final states is larger than one or there is a change in parity, allowed transitions are not

possible. The beta decay transitions within these states are called as *Forbidden* transitions. Note that *Forbidden* transitions are not strictly forbidden, the transition rate for them is hindered or reduced. Furthermore, the electron and neutrino must carry orbital angular momentum as the difference in angular momenta is more than one and that can not be generated through spins only.

In the calculation of transition rate for allowed transitions, the wave functions of electron and neutrino at $r = 0$ were considered. In case *allowed* transition are forbidden due to angular momentum or parity selection rules, we need to consider the second term of the wavefunction

$$e^{i\mathbf{k}.\mathbf{r}} = 1 + i\mathbf{k}.\mathbf{r} + \frac{1}{2}(i\mathbf{k}.\mathbf{r})^2.....$$

The second term $i\mathbf{k}.\mathbf{r} = i\frac{\mathbf{p}}{\hbar}.\mathbf{r}$. If we take momentum $\mathbf{p}$ along the z direction,

$$i\frac{\mathbf{p}}{\hbar}.\mathbf{r} = i\frac{p}{\hbar}rcos\theta$$

The $cos\theta$ can be replaced by Y_l^0, where

$$Y_l^0 = \sqrt{\frac{3}{4\pi}}cos\theta.$$

Thus, the second term represents a decay with $l = 1$. These transitions are called as *first forbidden* transition. We have seen that the parameter $kr = 0.1 << 1$ for 1.5 keV β particle. In the calculation of nuclear matrix element, the kr factor will come from both the wavefunctions of the electron and the neutrino. Since square of the matrix element appears in the calculation of transition rate, a factor of 10^{-4} will be multiplied in the transition rate of first forbidden transition compared to that of allowed transition.

Similarly, if the first and second terms does not contribute to transition rate (not allowed by selection rules), the third term has to be considered. As shown above, one can show that it represents decays with orbital angular momentum $l = 2$ and are called *second forbidden* transitions. Successive higher terms are representing decays with higher angular momentum, like *third forbidden* and so on. Note that contribution from successive terms are smaller by a factor of 10^{-4} due to higher powers of kr.

Selection rules

For the first forbidden transition, orbital angular momentum carried by the electron and the neutrino is $L = 1$. As parity is decided by $(-1)^L$, there will be a change of parity of the nuclear states, i.e. $\Delta\pi$=Yes .

If spin S=0, it is called Fermi transition, total angular momentum carried by electron and neutrino is $J = L + S = 1$. Therefore, ΔJ should be 0 or 1 [with vector addition of $J_i = J_f + (J = 1)$, J_f can take values $J_i + 1$, J_i and $J_i - 1$].

If spin S=1, it is called as GT transition. The selection rule is then ΔJ should be 0, 1 or 2

Therefore, the selection rules for *First forbidden* transition are

ΔJ should be 0, 1 or 2 and $\Delta\pi$ is Yes

Examples :

^{17}N $\rightarrow$ ^{17}O, from $\frac{1}{2}^- \rightarrow \frac{3}{2}^+$

^{76}Br $\rightarrow$ ^{76}Se, from $1^- \rightarrow 0^+$

Second Forbidden transitions

In case the first two terms of the wavefunctions of electron and neutrino do not contribute to the transition rate, we need to consider the third term. This can happen if the difference of spins of parent and daughter nuclei is more than 2. Note that a maximum of $\Delta J = 1$ is for the allowed and $\Delta J = 2$ is for the first forbidden transitions.

Selection rules

In this case, orbital angular momentum carried by the electron and the neutrino is $L = 2$. As parity is decided by $(-1)^L$, there will be NO change of parity, i.e. $\Delta\pi$=No. This is same as the allowed transitions.

If spin S=0, it is called Fermi transition. Therefore, ΔJ should be 0, 1 or 2 for the decay.

If spin S=1, it is called as GT transition. The selection rule is then ΔJ should be 0, 1, 2 or 3.

Since ΔJ =0 and 1 with no change in parity is already for allowed transition, the selection rules for *second forbidden* transitions are $\Delta J = 2$ and 3 with no change in parity.

Therefore, the selection rules for *Second forbidden* transitions are

ΔJ should be 2 or 3 and $\Delta\pi$ is No

Example :

$^{22}Na \rightarrow ^{22}\ Ne$, from $3^+ \rightarrow 0^+$

Calculation of logft values

In this section we will calculate logft values for some of the beta decay transitions.

^{60}Co $\rightarrow$ ^{60}Ni decay

The decay scheme of radioactive ^{60}Co source is shown in Fig. 10. The ground state spin of ^{60}Co is 5^+. It decays to 4^+ excited state of ^{60}Ni through β^-. The half-life of the decay is 5.27 years. The $Q(gs - gs)$ is the difference between measured atomic masses of parent and daughter nuclei. Here, $gs-gs$

represents ground state of parent to ground state of daughter nuclei. The excited daughter nucleus decays to its ground state by emission of two gamma ray photons of energy 1173 and 1332 keV.

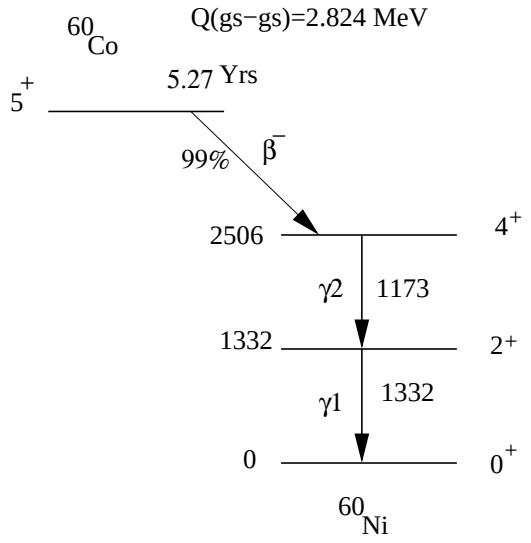

Figure 10: Decay scheme of ^{60}Co source. The Q value given in the diagram is the difference of atomic masses of parent and daughter nuclei.

Using the selection rules of beta decay you should be able to classify the decay. As the spin difference is one and there is no change in parity of the states involved ($5^+ \rightarrow 4^+$), it is an allowed transition of GT type.

The end-point energy of β^- is then equal to 2.824-2.506=0.318 MeV

The $t_{1/2} = 5.27$ years needs to converted in seconds and then log$t_{1/2}$ comes out be 8.2. Using the end-point energy of beta, one can calculate ϵ_0 and then $f(\epsilon_0)$ using Fermi function. The value of log$f$ is calculated as -0.8. You should verify all the data by yourself.

Therefore, logft= log$f$+log$t$=8.2-0.8=7.4

^{22}Na $\rightarrow$ ^{22}Ne decay

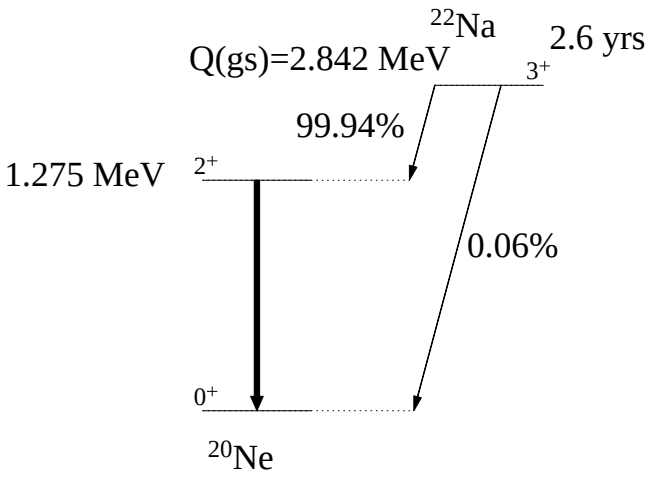

Figure 11: Decay scheme of ^{22}Na source. The Q value given in the diagram is the difference of atomic masses of parent and daughter nuclei.

The decay scheme of radioactive ^{22}Na source is shown in Fig. 11. The ground state spin of ^{22}Na is 3^+. It has two decay branches. It decays to 2^+ excited state of ^{22}Ne through β^+ with 99.94% branching. The second decay is directly to the ground state of ^{22}Ne with 0.06% branching. The the half-life of the decay is 2.6 years. The $Q(gs-gs)$ value is 2.842 MeV and is the difference between measured atomic masses of parent and daughter nuclei. The excited daughter nucleus decays to its ground state by emission of a gamma ray photon of energy 1.275 MeV.

Question : Using selection rules classify the above two decay transitions .

We would like to calculate $\log ft$ values for both the β^+ transitions.

We know that total

$$\lambda = \lambda_1 + \lambda_2$$

Here, λ_1 and λ_2 are transition rates of decay (1) $3^+ \rightarrow 2^+$ and (2) $3^+ \rightarrow 0^+$ decays, respectively

Therefore, we can write

$$1 = \frac{\lambda_1}{\lambda} + \frac{\lambda_2}{\lambda}$$

Here, $\frac{\lambda_1}{\lambda} = 0.9994$ and $\frac{\lambda_2}{\lambda} = 0.0006$ are the branching ratios. If t_1 and t_2 are the partial half-life of the state for (1) and (2) branches, respectively

Therefore, using branching ratios, we can write $\frac{t}{t_1} = 0.9994$ and $\frac{t}{t_2} = 0.0006$. Here, t is the total half-life of the decaying state.

We have then $t_1 = \frac{t}{0.9994} = 8.2 \times 10^7$s and $t_2 = \frac{t}{0.0006} = 1.37 \times 10^{11}$s

The available decay energy for β^+ decay is equal to (2.842-1.022) = 1.82 MeV.

Note that in the calculation of Q-value for β^+ decay, we need to subtract twice the mass of electron from the atomic mass difference of parent and daughter nuclei.

Therefore, for branch (1), the end-point energy of beta is 1.82-1.274 = 0.546 MeV. Using this, the log$f$ is calculated 1.6

Using partial half-life t_1 of branch (1), logft= log$f$+log$t$= 1.6+7.914 = 9.5

For branch (2), logft= log$f$+log$t$= 1.6+11.14 = 12.74

Lecture 5

Neutrino

The neutrino was introduced in beta decay to conserve energy. It was clear from the conservation of charges that it is a neutral particle and in order to satisfy statistical requirements, the neutrino must be a fermion. Conservation of angular momentum in neutron decay limits spin of anti-neutrino to $\frac{1}{2}$ or $\frac{3}{2}$. However, observation of beta decays from $0^+ \rightarrow 0^+$ with nearly equal strength as that of neutron decay suggest that they are allowed decay and restricts its spin to $\frac{1}{2}$. The balance of energy in the case of the ${}^3H \rightarrow {}^3He$ transformation shows that the neutrino rest mass is less than 200 eV. In recent experiments limits has been put on its mass to be less than an eV. In the standard model, it was assumed to be a massless particle and, therefore, its velocity is equal to the velocity of light.

The antiparticle of neutrino is antineutrino with same mass, spin, and charge. It has been confirmed in experiment that neutrino and anti-neutrino are different particle. The spin of neutrino is always antiparallel to its momentum and the spin of antineutrino is parallel to its momentum. This property gives to the neutrino a "handedness", left-handed to neutrino and right-handed to antineutrino. This is understood in the way that if you look in the direction of the momentum of neutrino, its spin will be directed towards you and has the sense similar to that of a left-handed screw. Similarly, the spin of antineutrino is along the direction of momentum and can be described by a right-handed screw. The handedness of neutrino can be described using the concept of helicity. Helicity is the scalar product of the spin and the momentum divided by the product of the modulus of these quantities.

$$\mathcal{H} = \frac{\mathbf{p} \cdot \sigma}{|\mathbf{p}||\sigma|}$$

For the neutrino the helicity has the value -1, which means that its spin vector is antiparallel to its momentum direction. The antineutrino has helicity +1 and the spin vector in along the direction of its momentum. Helicity is fixed for a particle of mass zero as the particle moves with velocity c. For massive particle we can always define a reference frame moving faster than

the particle and in that frame direction of its momentum will be opposite. Therefore, in this case direction of momentum will depend upon the choice of reference frame and that means it is not well defined.

Non-conservation of parity

When the theory of beta decay was formulated by Fermi, non-conservation of parity in beta decay was not known. It was assumed that parity is conserved in the weak interaction like in the electromagnetic interaction. Therefore, the interaction Hamiltonian chosen by him did not have parity non-conserving term. In the year 1956, Lee and Yang proposed that parity is not conserved in weak interaction and it was verified experimentally by C.S. Wu. In this section we will discuss non-conservation of parity in beta decay process.

We know that conservation of a quantity follows from invariance principle. A typical example is that invariance of the Hamiltonian under translation leads to the conservation of the corresponding momentum. Similarly, if a system remains invariant under rotation, angular momentum is a conserved quantity. The parity conservation follows when the Hamiltonian of the system remains invariant with respect to the operation $\mathbf{r} \rightarrow -\mathbf{r}$. This is achieved by reflecting coordinate axes at the origin.

The above statement is best given by calculating the time dependence of the expectation value of a given observable $\hat{O}$.

$$\frac{d}{dt}\langle\hat{O}\rangle = \frac{i}{\hbar}\langle\Big[H, \hat{O}\Big]\rangle.$$

if $\hat{O}$ does not depend on time, The operator $\hat{O}$ commute with Hamiltonian H. Therefore, invariance of a Hamiltonian under parity operation is equivalent to the commutator $[P, H] = 0$. Since $H = T + V$, where T is the kinetic energy and V is the potential energy operator then $[P, V] = 0$

This means that in a transition from an initial state ψ_i of parity π_i to a final state ψ_f of parity π_f due to an interaction described by the potential V, then $\pi_i = \pi_f$ as can be seen below. The transition matrix element is given by

$$M_{fi} = \int \psi_f^* V \psi_i d\tau$$

$$= \int \psi_f^* V P^2 \psi_i d\tau \qquad \text{As } \mathrm{P}^2 = 1$$

$$= \int \psi_f^* P V P \psi_i d\tau \qquad \text{As } [\mathrm{P}, \mathrm{V}] = 0$$

$$= \int (P\psi_f)^* P V P \psi_i d\tau \qquad \text{As P is Hermitian}$$

$$\text{or } M_{fi} = \pi_f \pi_i M_{fi} \qquad \pi_\mathrm{f} \text{ and } \pi_\mathrm{i} \text{ are eigenvalues of P}$$

So either $M_{fi} = 0$ or $\pi_f = \pi_i$

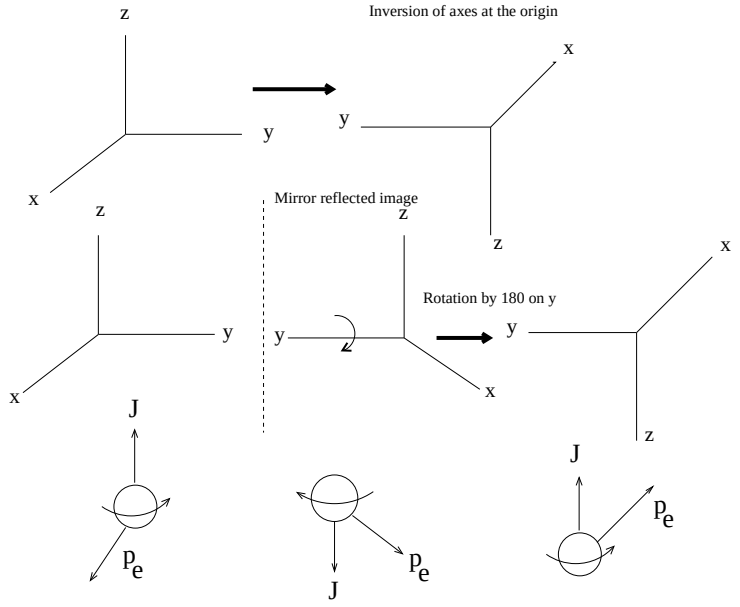

Figure 12: Coordinate axes under parity operation. The parity operation is equivalent to 180° rotation about the y axis on the mirror reflected image of the axes. In the lower panel, orientation of the angular momentum J of a nucleus and momentum p_e of an electron are shown under different operation. Note that, if the momentum of beta particle is in opposite direction to the nuclear spin J, it will be in the direction of J in the mirror reflected image. This picture will be required to understand the results of experiment performed by Wu et al.

Reflection of coordinates at the origin is equivalent to its mirror image and then rotation by 180° about the Y-axis. This is shown in Fig. 12. As angular

momentum is conserved here, the system will be invariant under rotation. Therefore, if the mirror reflection of the system can not be distinguished from the real system, we can say that parity is conserved.

Let us see how physical observables behave under parity operation. We know that vectors used in physics are of two kinds. A vector which under parity transformation changes sign, i.e. $\mathbf{r} \rightarrow -\mathbf{r}$ or $\mathbf{p} \rightarrow -\mathbf{p}$ are called polar vector.

Now what about angular momentum? It is a vector quantity $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ but this does not change sign under parity operation as both position and momentum vectors change sign. The same is true for spin. They are called axial vector or pseudo vector.

What about Magnetic Field B? Well, the equation of Lorentz force $F = eE + e(v \times B)$ should be same in both the system. Under parity operation, electric field E, force F and velocity v change sign as they are polar vector. Therefore, in order that the force relation to be the same, B should not change sign. That is B is an axial vector.

The scalar product of two polar vectors or axial vectors is a number, invariant upon reflection of coordinate system, and is called a true scalar or simply scalar. The scalar product of a polar vector and an axial vector is a number that changes sign upon inversion of the coordinate system. They are called pseudo scalar.

The result of a physical measurement is always a number, which, however, may behave as a scalar or as a pseudoscalar. The mirror reflection mentioned above leaves true scalars unchanged, but it changes the sign of pseudoscalars. Hence if the image of the system must be indistinguishable, all pseudoscalars must vanish. The observation of pseudoscalars different from zero thus implies the breakdown of parity conservation.

To show why the expectation value E of a pseudoscalar operator $\hat{O}$ is zero in a state ψ_f whose parity eigenvalue is π_f consider :

$$E = \int \psi_f^* \hat{O} \psi_f d\tau \text{ or } \langle f|\hat{O}|f\rangle$$

$$
\begin{aligned}
&= \langle f|\hat{O}P^2|f\rangle && \text{as } \mathrm{P}^2 = 1 \\
&= -\langle f|P\hat{O}P|f\rangle && \text{as OP} = -\text{PO} \\
&= -\pi_f^2\langle f|\hat{O}|f\rangle && \text{as P is Hermitian} \\
&= -E
\end{aligned}
$$

So E is zero. This expectation value would not necessarily be zero if the state ψ_f did not have good parity. Therefore, if in a transition from a state ψ_i of good parity π_i to a state ψ_f the expectation value of a pseudoscalar quantity in the final state is not zero then the parity has been violated in the transition. Note that the parity of the final state is the product of the intrinsic parities of the particles and parities of the wave functions describing the relative motions of the particles. Therefore, until a pseudoscalar is observed in experiment, no information on the conservation of parity is available.

In 1956, an experiment was designed by We et al. to verify non-conservation of parity in beta decay transition. A ^{60}Co source was used as β^- source. The decay scheme of the source is shown in Fig. 13. The beta transition takes between $5^+ \rightarrow 4^+$ state. Thus the transition is a GT decay and the electron and the neutrino together carries the difference of angular momentum, i.e. one unit. Therefore, the spins of the emitted electron and neutrino are in the same direction. Note that they do not carry orbital angular momentum as it is an allowed transition.

In this experiment the number of beta particles is measured relative to nuclear spin $(J \cdot p)$ and for that we need to fix the direction of nuclear spin J of the ^{60}Co. The source was transplanted into a host material to use transient magnetic filed of the atoms. The sample was then placed in an external magnetic field. The alignment of nuclear spin is best obtained by cooling the sample to a sub-Kelvin temperature 0.01K. This was achieved using adiabatic demagnetization process. It is a discrete process of cooling where temperature of the sample is brought to very low temperature in a short time by demagnetization under adiabatic condition. The temperature of the sample gradually increases with time. The cycle works for a few minutes, as shown in Fig 13 (c) and (d). The alignment of nuclear spin can be monitored through anisotropy of the gamma rays following the beta emission (see Fig. 13 (c). As you can see that the anisotropy disappears after 6 minutes due to disorientation of nuclear spin due to rise in temperature of the sample.

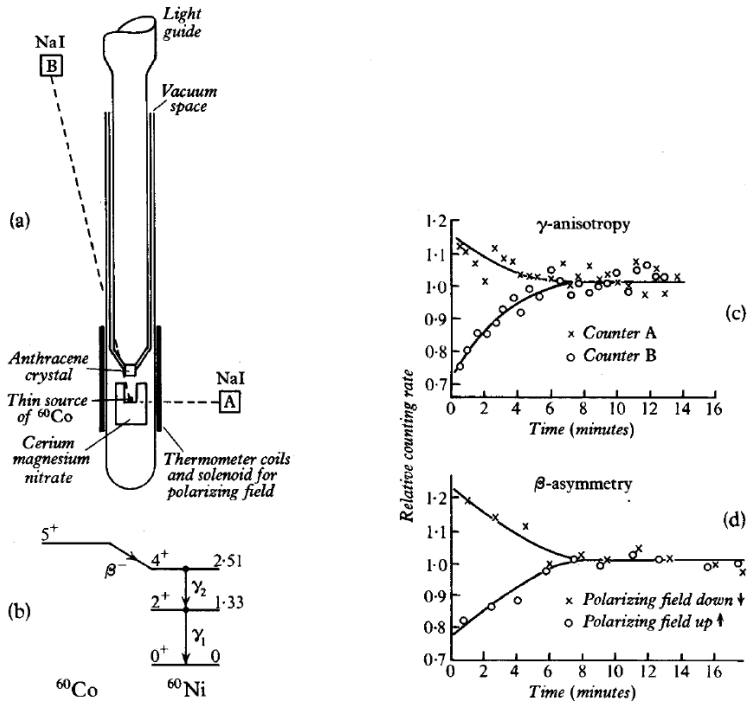

Figure 13: Experimental test of parity conservation in β decay. (a) Apparatus (b) Decay scheme of ^{60}Co (c) Gamma-ray anisotropy obtained from counters A and B at different times as the crystal warms up. The difference between the curves measures the net polarisation of the nuclei. (d) Beta asymmetry shown by counting rate in the anthracene crystal for two direction of polarizing field.

The number of beta particles was measured as a function of time for both the direction of polarizing magnetic field. If parity would have been conserved, the intensity of the beta particles is expected to be same for both the direction of the field. The asymmetry observed for β particles (see Fig. 13 (d)) for up and down direction of the magnetic field clearly suggests violation of parity in the beta decay process. The same was confirmed later for another such process, the $\pi \to \mu \to e$ decay.

The asymmetric angular distribution always involves interference between amplitudes of opposite symmetry. The β-asymmetry therefore means that the transition between the ground state of ^{60}Co and the second state of ^{60}Ni,

both of definite parity, can take place by emission of electron-antineutrino pair in both odd and even parity states and the two corresponding amplitudes interfere leading to an angular distribution of electron emission of the form

$$a + bcos\theta$$

with respect to the nuclear spin axis.

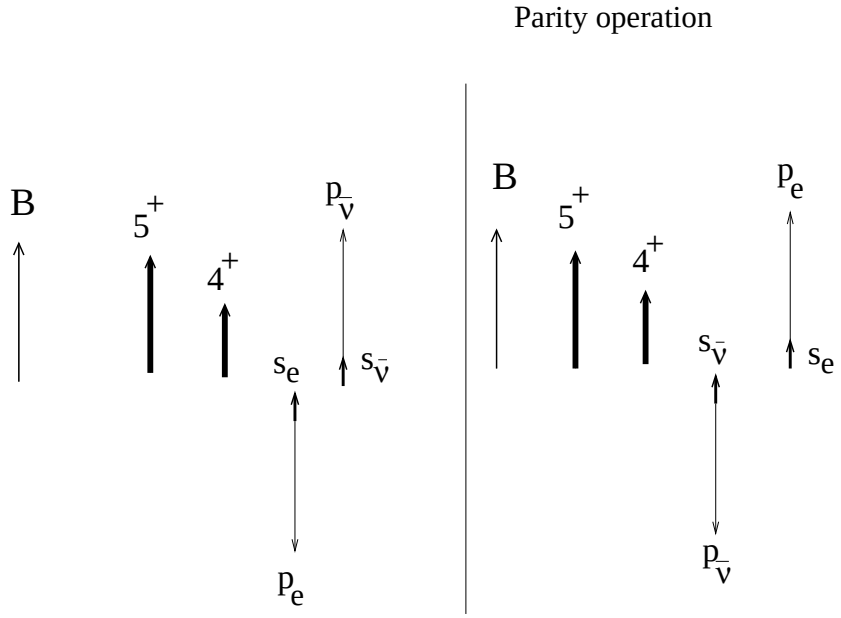

Figure 14: Angular momenta of the nucleus and the emitted electron and neutrino in beta decay. Mirror image of the same is also shown in the right.

The above description can also be understood through the angular momenta plot of different particles shown in Fig. 14. As you can see that the difference of one unit of angular momentum of parent and daughter nuclei is equal to sum of the spins of electron and neutrino. Antineutrino has helicity +1 and, therefore, its momentum will be in the direction of nuclear spin. In order to conserve the linear momentum, the electron momentum will be opposite to the nuclear spin. Under parity operation magnetic field, angular momenta, and spin do not change direction as they are axial vector. The momentum of the electron will now be in the direction of nuclear spin. Therefore, if the number of beta particles along the direction of nuclear spin and opposite to nuclear spin is equal then the parity is conserved. In this experiment it was found different and hence parity is not conserved in beta decay.

Electron capture

In electron capture an atomic electron interacts weakly with a proton in the nucleus and initiates the reaction

$$e + p \rightarrow n + \nu. \tag{23}$$

The most probable capture is from the K shell because a K electron has the greatest probability of being inside the nucleus. The expression for the Q value of the decay has been introduced in the first lecture. In this process, the whole energy is carried away by the neutrino, i.e. $Q_{EC} = T_\nu$. The signature of this process is the emission of x-rays due to transition of electrons from the higher orbits to the vacancy.

The transition rate λ_{EC} for electron capture can be calculated by the same theory as presented above. Since there is one particle in the final state, the density of states needs to be calculated only for the neutrino. Following in the same way of phase space calculation of beta particle we have

$$\frac{dn}{dE_f} = \frac{Vp_\nu^2}{2\pi^2\hbar^3}\frac{dp_\nu}{dE_f}$$

Assuming $m_\nu = 0$ and $E_\nu = p_\nu c$,

$$\frac{dp_\nu}{dE_f} = \frac{1}{c}$$

Therefore,

$$\frac{dn}{dE_f} = \frac{VE_\nu^2}{2\pi^2\hbar^3c^3}$$

The matrix element in this case for allowed transition will be

$$M_{fi} = g\int u_n(r)^*\psi_\nu^*(0)u_p(r)\psi_e(0)d\tau$$

Capture is most likely from a $1s$-state electron and the wave function of electron is

$$\psi_e(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Zm_ec^2}{4\pi\epsilon_0\hbar^2}\right)^{3/2} e^{-\frac{Zr}{a_0}}$$

where a_0 is the Bohr radius. The wave function at the origin $r = 0$ is then

$$\psi_e = \frac{1}{\sqrt{\pi}} \left(\frac{Z m_e c^2}{4\pi\epsilon_0 \hbar^2} \right)^{3/2}$$

The transition rate is then give by

$$\lambda_{EC} = \frac{E_\nu^2}{\pi^2 \hbar^4 c^3} g^2 |M_{fi}|^2 \left(\frac{Z m_e c^2}{4\pi\epsilon_0 \hbar^2} \right)^3$$

As you can see that $\lambda_{EC} \propto E_\nu^2$. Note that the ratio $\dfrac{\lambda_{EC}}{\lambda_{\beta^+}}$ is practically independent of any nuclear effects, as the M_{fi} canceled out, and so forms a good check of the theory of beta decay. This process competes with β^+ decay when both decay modes are possible. Note that the energy released in EC is greater than it is in β^+ emission by $2m_e c^2$. This means that, for decay energies near the threshold for β^+, kinetic energy of beta particle $T_e \simeq 0$, the relative importance of EC will become very large, because the phase-space factor for it will remain finite whereas that of β^+ emission will be close to zero (as it is $\propto (Q - T_e)^2$).

Gamma Decay

Lecture-I

Introduction

A nucleus in an excited state (bound state) comes down to the lower energy state usually by emission of gamma rays if the excitation energy is below the particle emission threshold. The spontaneous decay takes place due to interaction of the nucleus with an external electromagnetic field. Even in free space, the interaction can take place due to non-zero value of electromagnetic field intensity in the vacuum state (ground state of the field). In this interaction, the nucleus transfer its energy to the field and this excitation of the field appears as a gamma ray.

The nucleus can also interact with the field of atomic electrons and energy can be exchanged. In this process the energy is transferred to an atomic electron without emission of radiation. This is called as *internal conversion* process and it competes with gamma emission. The above process is favorable in high-Z nuclei and the probability of conversion is more for low energy transitions. The electron which comes out has unique energy, equal to the excitation energy minus binding energy of the electron. This is in contrast to the electron coming out of the nucleus due to weak interaction with a continuous energy distribution. For unbound states well above the particle threshold particle emissions predominates. Near threshold, charged particle emission is hindered due to Coulomb barrier, neutron emission can take place and can compete with gamma decay. The energies of the γ rays are in the range of to say 100 keV to 10 MeV.

The gamma ray spectroscopy is an important tool to study nuclear structure. The advantage with the gamma rays is that they can be easily detected (without much attenuation, contrary to α and β particles) and their energies can be measured precisely leading to accurate energy measurement of the excited levels. The gamma transition probability is related to the width of energy level. Higher the width higher is the transition probability. Using uncertainty principle we can estimate order of the lifetime of a nuclear state.

$$\Delta E \Delta t \geq \hbar$$

Here, ΔE is the width Γ of the energy level and is of the order 0.1 eV and $\Delta t \sim \tau$, lifetime of the state. Therefore,

$$\tau = \frac{\hbar}{\Gamma} \;=\; \frac{1 \times 10^{-34}}{0.1 \times 1.6 \times 10^{-19}} \;=\; 6.6 \times 10^{-15} s$$

which shows that the nuclear excited states have half-lives of the order 10^{-15}s for gamma decay. This can be compared with 10^{-8} s for atomic transition.

The mean life of the nuclear states varies between nanosecond to femtosecond. Some excited states have unusually higher lifetime values, i.e around microsecond, millisecond or higher. These are called isomeric states and results from unfavored transition of higher multipole order due to selection rules. This will be discussed later in a separate section.

Energetics of γ decay

In the γ decay process there is no change in the number of protons or the neutrons, only the excitation energy of the same nucleus is changed. If a nucleus of mass M decays from initial energy E_i to a final energy state E_f by the emission of gamma ray of energy E_γ then we can write

$$E_i = E_f + E_\gamma + E_R$$

Here, E_R is the recoil energy received by the nucleus due to emission of gamma ray. Therefore the energy difference between the two levels is

$$\Delta E = E_i - E_f \;=\; E_\gamma + E_R$$

We can write the recoil energy E_R as

$$E_R = \frac{P_R^2}{2M}$$

where P_R is the momentum of the recoiling nucleus. Since gamma decay is a two-body decay process, conservation of momentum gives

$$|P_R| = |P_\gamma|.$$

Therefore,

$$\Delta E = E_\gamma + \frac{P_R^2}{2M} \;=\; E_\gamma + \frac{P_R^2 c^2}{2Mc^2} \;=\; E_\gamma + \frac{E_\gamma^2}{2Mc^2}$$

or we can write

$$E_\gamma^2 + 2Mc^2 E_\gamma - 2Mc^2 \Delta E = 0$$

$$\text{or } E_\gamma = -Mc^2 \pm Mc^2 \sqrt{1 + \frac{2\Delta E}{Mc^2}}$$

As $\Delta E \sim 1$ MeV $<< Mc^2$, we can expand the square root. Considering the positive sign we have

$$E_\gamma = \Delta E - \frac{1}{2}\frac{(\Delta E)^2}{Mc^2}$$

The factor $\frac{(\Delta E)^2}{Mc^2} \sim 10^{-5}$ which is much smaller than the resolution of a detector. Therefore, for most of the transitions we take $E_\gamma = \Delta E$. For a high energy 5-10 MeV radiation gives recoil in 100 eV range. This is called as Mössbaur effect, i.e. whole energy is not available in the form of γ radiation.

Classification of gamma decays

The γ radiation can be generated either by the oscillating charge, which causes an oscillation in the external electric field, or by the varying current or magnetic moment, which sets up a varying magnetic field. Radiation emitted by the former mechanism is called electric (E) radiation and the latter is

said to give rise to magnetic (M) radiation. The classification of the different processes leading to the emission of photons is based on conservation of angular momentum and parity between the radiating system and the radiation field.

We know from classical electromagnetism that the potential at a large distance from a charge distribution can be expressed in the form of potentials due to different multipoles of order L. Here, L can take integer values starting from zero. The periodic variation of the charge distribution can then be expressed as oscillation of each term in the expansion and resulting radiation can be thought of as being due to the various terms in the expansion, e.g. dipole, quadrupole etc. In quantum mechanical calculations each multipole moment of order $L > 0$ is found to produce radiation which carries off an angular momentum $L\hbar$. For radiation, the multipole order $L = 0$ is excluded as photons with zero angular momentum do not exist. Note that the photon is described by the vector electromagnetic field and has intrinsic spin $s = 1$ and parity negative.

The gamma radiations can be then categorized into two categories

(1) Electric type of multipole order L (EL)

For examples :
E1 : Electric dipole transition
E2 : Electric quadrupole transition
E3 : Electric octupole transition
and so on

(2) Magnetic type of multipole order L (ML)

For examples :
M1 : Magnetic dipole transition
M2 : Magnetic quadrupole transition
M3 : Magnetic octupole transition
and so on

If a transition occurs between the nuclear states with spins J_i (initial) and

J_f (final), then using the conservation of angular momentum we can write

$$J_i = J_f + L$$

which leads to a selection rule that L can have any integer value between

$$|J_i - J_f| \leq L \leq |J_i + J_f|$$

since gamma emission with $L = 0$ is not allowed, single-photon emission is forbidden by angular momentum conservation between two spin-zero states. In principle, two-photons emission are allowed but its transition probability is very small. In this case, the nucleus decays through internal conversion process.

Parity is conserved in electromagnetic transitions an parity selection rule must be obeyed. The radiation field can have even or odd parity for a given L, depending on whether the transition is of electric or magnetic type. The electric multipole radiation has parity $(-1)^L$ and magnetic multipole radiation has parity $(-1)^{L+1}$.

For Example, the electric dipole moment is $D = e\mathbf{r}$, where e is the charge separated by a distance r. Under parity it is $-e\mathbf{r}$ and, hence, has odd parity. A magnetic dipole is like a circulating charge which forms a current loop. The magnetic moment of a charged particle of charge e moving in a circular orbit of radius r with velocity v is $\mu = \frac{e}{2}\mathbf{r} \times \mathbf{v}$. Under parity operation both $\mathbf{r}$ and $\mathbf{v}$ changes sign and therefore the magnetic moment has even parity.

Let us consider a transition between states with spin and parity $J_i^{\pi_i}$ (initial state) and $J_f^{\pi_f}$ (final state). Since parity is conserved for electromagnetic transitions, we have

$$\pi_i \pi_f = (-1)^L \quad \text{for electric transitions}$$

and

$$\pi_i \pi_f = (-1)^{L+1} \quad \text{for magnetic transitions}$$

This gives

$$\pi_i \pi_f = -1 \quad for\ E1,\ M2,\ E3,\ M4....$$

and

$$\pi_i \pi_f = +1 \quad for\ M1,\ E2,\ M3,\ E4....$$

Transition rate

According to classical theory of electromagnetic radiation, the power emitted by an antenna is

$$P(\sigma L) = \frac{2(L+1)c}{\epsilon_0 L[(2L+1)!!]^2} \left(\frac{\omega}{c}\right)^{2L+2} \left[\mathcal{M}(\sigma L)\right]^2 \tag{24}$$

where σ denotes type of radiation, electric E or magnetic M, ω is the angular frequency of radiation and the double factorial $(2L+1)!! = (2L+1) \times (2L-1) \times ... \times 3 \times 1$ and $\mathcal{M}(\sigma L)$ is the amplitude of the oscillating electric or magnetic multipole moment.

In quantum mechanics, the radiation is considered to be emitted as discrete photons, and the multipole moment amplitude is replaced by a matrix element of the form

$$\mathcal{M}_{fi}(\sigma L) = \int \psi_f^*\ m(\sigma L)\ \psi_i d\tau,$$

where the integral is carried out over the nuclear volume. The wave functions ψ_i and ψ_f are for initial and final nuclear states. The multipole operator $m(\sigma L)$ is related to the multipole moment.

Since emitted photons have energy $\hbar\omega$, the transition rate can be obtained by dividing equation 24 by $\hbar\omega$.

$$\lambda_\gamma(\sigma L) = \frac{P(\sigma L)}{\hbar\omega} \ = \ \frac{2(L+1)}{\epsilon_0 L[(2L+1)!!]^2} \hbar \left(\frac{\omega}{c}\right)^{2L+1} \mathcal{B}(\sigma L)$$

where $\mathcal{B}(\sigma L)$ is called the reduced transition probability and is the square of the modulus of the matrix element $\mathcal{M}_{fi}(\sigma L)$, i.e.

$$\mathcal{B}(\sigma L) = |\mathcal{M}_{fi}(\sigma L)|^2$$

Calculation of reduced transition probability requires knowledge of the nuclear wave functions involved in the transitions and can be very complicated.

A single-particle estimate, can be obtained by assuming that the transition is due to a single proton making a transition between two shell model states. The calculated transition rates are called as *Weisskopf estimates* and will be discussed in the next lecture.

Comparison of transition rates

The detail calculation of transition rate based on quantum mechanical approach is discussed in the next lecture. It follows from the calculation that for electric transition (EL), transition rate

$$\lambda_\gamma(EL) \sim \left(\frac{1}{\lambda\!\!\!^{-}}\right)\left(\frac{R}{\lambda\!\!\!^{-}}\right)^{2L}.$$

For magnetic transition (ML), transition rate

$$\lambda_\gamma(ML) \sim \left(\frac{1}{\lambda\!\!\!^{-}}\right)\left(\frac{R}{\lambda\!\!\!^{-}}\right)^{2L+2}$$

Here $\lambda\!\!\!^{-} = \dfrac{\lambda}{2\pi} = \dfrac{c}{2\pi\nu} = \dfrac{\hbar c}{E_\gamma}$. For a photon of energy 1 MeV $\lambda\!\!\!^{-} = 197fm \simeq 10^{-13}$m. Considering nuclear radius $R = 10$ fm, $\dfrac{R}{\lambda\!\!\!^{-}} = 0.1 << 1$. Therefore, as multipolarity L increases, the transition rate of a particular type of radiation decreases. For example :

$$\frac{\lambda_\gamma(E(L+1))}{\lambda_\gamma(EL)} = \left(\frac{R}{\lambda\!\!\!^{-}}\right)^2 \simeq 10^{-2},$$

i.e. the transition rate for the next oder transition of same type will be less by a factor of 100.

For transitions of same L

$$\frac{\lambda_\gamma(ML)}{\lambda_\gamma(EL)} = \left(\frac{R}{\lambda\!\!\!^{-}}\right)^2 \simeq 10^{-2}.$$

i.e. the transition rate of magnetic transition is hindered by a factor of 10^{-2} compared to electric transition of same multipolarity.

Now if you compare transition rate of ML and next multipole order of electric type then

$$\frac{\lambda_\gamma(ML)}{\lambda_\gamma(E(L+1))} = 1.$$

i.e. they are equally probable. Therefore, magnetic transition of multipolarity L competes with electric transition of $L+1$ multipolarity. For example, $M1$ competes with $E2$ and so on.

Examples :

(1) A gamma ray transition between the states with $J_i^{\pi_i} = \frac{3}{2}^-$ and $J_f^{\pi_f} = \frac{1}{2}^-$.
(i) What are their multipole orders?
(ii) what type of transitions are allowed between these two states?
(iii) Which one is most probable ?

From the conservation of angular momentum

$$|J_i - J_f| \leq L \leq |J_i + J_f|,$$

possible values of L are 1 and 2. The parity conservation gives

$$\pi_i \pi_f = +$$

Therefore, the type of transitions of the above multipoles satisfying party rule are $M1$ and $E2$. Since both the transitions compete with each other, they are equally probable (see above for transition rate).

(2) A gamma ray transition between the states with $J_i^{\pi_i} = \frac{3}{2}^-$ and $J_f^{\pi_f} = \frac{1}{2}^+$.
(i) What are their multipole orders?
(ii) what type of transitions are allowed between these two states?
(iii) Which one is most probable ?

From the conservation of angular momentum

$$|J_i - J_f| \leq L \leq |J_i + J_f|,$$

possible values of L are 1 and 2. The parity conservation gives

$$\pi_i \pi_f = -$$

Therefore, the type of transitions of the above multipoles satisfying party rule are $E1$ and $M2$. The ratio of the transition rates

$$\frac{\lambda_\gamma(M2)}{\lambda_\gamma(E1)} = \left(\frac{R}{\lambda\!\!\!^{-}}\right)^4 \simeq 10^{-4}$$

i.e. $M2$ is hindered by a factor of 10^{-4} compared to $E1$. Therefore, most probable transition is of $E1$ type.

Lecture-II

Interaction of a nucleus with electromagnetic field

In this lecture, a detail calculation of transition rates based upon quantum mechanical formalism has been presented. This formalism you may have come across in time-dependent perturbation theory of Quantum Mechanics and also in Atomic Physics lectures.

Let us consider a nucleus placed in an electromagnetic field. We assume that only one proton of mass m and charge e is involved in the interaction with the field. The electromagnetic field is described by a scalar potential ϕ and a vector potential $\mathbf{A}$ such that electric and magnetic fields satisfy the relation

$$\mathbf{B} = \nabla \times \mathbf{A} \text{ and}$$
$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

In the absence of the perturbating electromagnetic field, the Hamiltonian of the proton is

$$H_0 = \frac{p^2}{2m} + V(r)$$

where $V(r)$ is the static potential representing interaction of the proton with other nucleons.

The interaction of the proton with the electromagnetic field is obtained by replacing the momentum

$$\mathbf{p} \rightarrow (\mathbf{p} - e\mathbf{A})$$

Therefore, the perturbed Hamiltonian becomes

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}\right)^2 + e\phi + V(r)$$

$$H = \frac{1}{2m}\left(p^2 - e\mathbf{A}\cdot\mathbf{p} - e\mathbf{p}\cdot\mathbf{A} + e^2\mathbf{A}.\mathbf{A}\right) + e\phi + V(r) \tag{25}$$

Let us take the term

$$\mathbf{p}\cdot\mathbf{A}\psi = -i\hbar\nabla\cdot\mathbf{A}\psi \;=\; -i\hbar\left[(\nabla\cdot\mathbf{A})\psi + \mathbf{A}\cdot\nabla\psi\right]$$

As the vector potential $\mathbf{A}$ satisfies Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ we can write

$$\mathbf{p} \cdot \mathbf{A}\psi = -i\hbar(\mathbf{A} \cdot \nabla\psi) \;=\; \mathbf{A} \cdot \mathbf{p}\psi$$

Therefore, the term $\mathbf{p} \cdot \mathbf{A}$ in equation (25) is equivalent to $\mathbf{A} \cdot \mathbf{p}$. Also, $\mathbf{A} \cdot \mathbf{A}$ term is neglected as it represents two photon emission and its contribution is very small. Since the potential $\phi = 0$ for the radiation with no electrostatic source, we can write the Hamiltonian

$$H = \frac{p^2}{2m} + V(r) - \frac{e}{m}\mathbf{A} \cdot \mathbf{p} \;= H_0 + H_{\text{int}}$$

where

$$H_{\text{int}} = -\frac{e}{m}\mathbf{A} \cdot \mathbf{p}$$

represents interaction of the particle with electromagnetic field. The matrix element that describes the transition from initial state i to final state f under the influence of electromagnetic field is

$$M_{fi} = \int \psi_f^* H_{\text{int}} \psi_i d\tau$$

where ψ_i and ψ_f are the wave functions of the stationary states representing initial and final states, respectively. The matrix element of the interaction H_{int} can have two possible forms

$$M_{fi} = -\int \psi_f^* \frac{e}{m}\mathbf{p} \cdot \mathbf{A}\psi_i d\tau \tag{26}$$

$$M_{fi} = -\int \psi_f^* \frac{e}{m}\mathbf{A} \cdot \mathbf{p}\psi_i d\tau \tag{27}$$

These two integrals are representing absorption and emission of a photon. We can regard $\mathbf{A}$ as the wave function of the photon and $\mathbf{A}\psi_i$ as the wave function of a state containing the photon and the nucleus in initial state ψ_i. Then, in equation (26), $\mathbf{p}$ is the operator that causes the transition to the final state of the nucleus ψ_f and no photon. This represents an absorption process. Similarly, in equation (27), the momentum operator $\mathbf{p}$ operates on the wave function ψ_i of the nucleus in initial state and produces a photon and the nucleus in final state representing emission process of a photon. Both the matrix elements representing emission and absorption processes are equal.

A photon state is represented by its frequency ω and direction of travel in the plane wave form

$$\mathbf{A}(\mathbf{r},t) = a_0\hat{\varepsilon}e^{(i\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{28}$$

Here, $\hat{\varepsilon}$ is the polarization vector, a_0 is the amplitude normalized to a volume V and $\mathbf{k}$ is the propagation vector. It is clear from the relation $\nabla\cdot\mathbf{A}=0$ that $\hat{\varepsilon}\cdot\mathbf{k}=0$, i.e. polarization vector is perpendicular to propagation direction.

Since $\mathbf{A}$ satisfies the wave equation

$$\nabla^{\mathbf{2}}\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0$$

we have $k = \dfrac{\omega}{c}$.

Let us find the amplitude a_0 in equation (28). From classical electrodynamics, the energy density of a plane wave averaged over a large region is given by

$$\bar{\mathcal{E}} = \frac{ka_0^2}{2\mu_0}.$$

Therefore, the amplitude a_0 can be determined by equating the energy of the plane wave inside a volume V to $\hbar\omega$, i.e.

$$\frac{ka_0^2}{2\mu_0}V = \hbar\omega$$

$$\text{or, } a_0^2 = \frac{2\hbar\omega\mu_0}{k^2V}.$$

Using the relation $\omega^2 = c^2k^2$, where c is the velocity of the photon, and $\mu_0\epsilon_0 = \dfrac{1}{c^2}$ we have

$$a_0^2 = \frac{2\hbar^2}{\epsilon_0\hbar\omega V} = \frac{2\hbar^2}{\epsilon_0 E_\gamma V}$$

$$\text{or,} a_0 = \sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}$$

We will now drop the time-dependent term from the expression of $\mathbf{A}$ given in equation (28). Note that, in the evaluation of the matrix element M_{fi} similar time-dependent terms in exponential form will appear from the wave functions of initial and final states, ψ_i and ψ_f, respectively and from the conservation of energy, $E_f - E_i = \hbar\omega$, we can write

$$e^{i\omega_f t}e^{-i\omega t}e^{-i\omega_i t} = e^{i(\omega_f-\omega_i)t}\ e^{-i\omega t} = 1$$

Here, ω_i and ω_f are the frequencies related to the initial and the final energy states, respectively

Using the expression of the amplitude a_0, we can write the vector potential as

$$\mathbf{A}(\mathbf{r}) = \sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}\hat{\varepsilon}e^{i\mathbf{k}\cdot\mathbf{r}}$$

Here, we have dropped the time-dependent term in the expression of the vector potential $\mathbf{A}$.

Therefore, the matrix element of the interaction Hamiltonian

$$\begin{aligned} M_{fi} &= -\langle\psi_f|\frac{e}{m}\mathbf{A}\cdot\mathbf{p}|\psi_i\rangle \\ &= -\langle\psi_f|\frac{e}{m}\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\varepsilon}\cdot\mathbf{p}|\psi_i\rangle \end{aligned} \tag{29}$$

Now, let us estimate the magnitude of kr for a typical gamma ray energy $E_\gamma = 1MeV$. We know that

$$p = \hbar k \text{ or } k = \frac{p}{\hbar}$$

Taking $r = R$ as radius of a nucleus

$$\text{or } kr = \frac{pR}{\hbar}\text{or } = \frac{E_\gamma R}{\hbar c}$$

For $E_\gamma = 1MeV$ and radius $R = 10$ fm, $kr \simeq 0.1 << 1$

Therefore, we can expand the exponential $e^{i\mathbf{k}\cdot\mathbf{r}}$ in equation (29) as

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 1 + i\mathbf{k}\cdot\mathbf{r} + \frac{1}{2}(i\mathbf{k}\cdot\mathbf{r})^2.... \tag{30}$$

First term of the expansion

Using the first term of the expansion given in equation (30), the matrix element of the interaction can be written as

$$M_{fi} = -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}\langle\psi_f|\frac{e}{m}(\hat{\varepsilon}\cdot\mathbf{p})|\psi_i\rangle$$

let us take $\hat{\varepsilon}$ along the x direction and propagation direction $\mathbf{k}$ along the z direction. Note that $\hat{\varepsilon}\cdot\mathbf{k} = \mathbf{0}$. We can then write the matrix element as

$$M_{fi} = -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}\langle\psi_f|\frac{e}{m}p_x|\psi_i\rangle \tag{31}$$

Using the Heisenberg equation of motion for an operator $\hat{O}$

$$\frac{d}{dt}\langle\hat{O}\rangle = \frac{i}{\hbar}\langle\Big[H_0, \hat{O}\Big]\rangle.$$

we can replace $\frac{d}{dt}x$ of p_x in equation (31) by the commutator of x with H_0

$$M_{fi} = -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}\langle\psi_f|\frac{ie}{\hbar}\left[H_0, x\right]|\psi_i\rangle.$$

$$= -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}\langle\psi_f|\frac{ie}{\hbar}\left(H_0 x - x H_0\right)|\psi_i\rangle$$

As the ψ_i and ψ_f are the eigen states of H_0 with eigen values E_i and E_f, respectively, we can write

$$= -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}\frac{i}{\hbar}(E_f - E_i)\langle\psi_f|ex|\psi_i\rangle$$

since $E_f - E_i = \hbar\omega$, we have

$$M_{fi} = -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}}i\omega\langle\psi_f|ex|\psi_i\rangle$$

$$= -\sqrt{\frac{2\hbar^2}{\epsilon_0 E_\gamma V}} i\omega D_{fi}$$

where, D_{fi} is the matrix element of the x-component of dipole moment operator

The transition rate for the gamma decay can be obtained by using the Fermi Golden rule

$$\lambda_\gamma = \frac{2\pi}{\hbar}|M_{fi}|^2 \frac{dn}{dE}$$

We already have calculated density of states in beta decay chapter. Following the same we can write

$$\frac{dn}{dE} = \frac{V}{2\pi^2\hbar^3} p^2 \frac{dp}{dE} \tag{32}$$

For a photon $E = pc$ or $\frac{dp}{dE} = \frac{1}{c}$ and also $p = \frac{\hbar\omega}{c}$ Therefore,

$$\lambda_\gamma = \frac{2\pi}{\hbar}|M_{fi}|^2 \frac{dn}{dE}$$

$$= \frac{2\pi}{\hbar} \frac{V}{2\pi^2\hbar^3 c} p^2 |M_{fi}|^2$$

Using the value of matrix element M_{fi} we have

$$\lambda_\gamma = \frac{2\omega^3}{\pi c^3 \epsilon_0 \hbar}|D_{fi}|^2$$

Let us approximate the matrix element of the dipole moment operator $D_{fi} = eR$, where maximum value of the x has been taken as nuclear radius R.

$$\lambda_\gamma = \frac{2\omega^3}{\pi c^3 \epsilon_0 \hbar} e^2 R^2$$

Using the relation $R = R_0 A^{1/3}$ and fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$, and $E_\gamma = \hbar\omega$

$$\lambda_\gamma = \frac{8\alpha R_0^2}{c^2\hbar^3} A^{2/3} E_\gamma^3$$

Note that the transition rate for electric dipole transition is proportional to E_γ^3. Using the relation $\omega = ck$ and $k = \dfrac{2\pi}{\lambda} = \dfrac{1}{\lambda\!\!\!^{-}}$ we can write the equation

$$\lambda_\gamma = \frac{2\omega^3}{\pi c^3 \epsilon_0 \hbar} e^2 R^2 = \frac{2e^2}{\pi \epsilon_0 \hbar} \frac{1}{\lambda\!\!\!^{-}} \left(\frac{R}{\lambda\!\!\!^{-}}\right)^2$$

Second term of the expansion

Considering the second term of the expansion of

$$e^{(i\mathbf{k}\cdot\mathbf{r})} = 1 + i\mathbf{k}\cdot\mathbf{r} + ...$$

the matrix element of the interaction Hamiltonian H_{int}

$$M_{fi} = \langle\psi_f|\frac{e}{m}(\mathbf{A}\cdot\mathbf{p})|\psi_i\rangle$$

can be written as

$$M_{fi} = a_0\langle\psi_f|\frac{e}{m}(\hat{\varepsilon}\cdot\mathbf{p})((i\mathbf{k}\cdot\mathbf{r})|\psi_i\rangle \tag{33}$$

let us again fix the direction of $\hat{\varepsilon}$ along the x direction and propagation direction $\mathbf{k}$ along the z direction. The matrix element is then

$$\begin{aligned}
M_{fi} &= a_0\langle\psi_f|\frac{e}{m}p_x ikz|\psi_i\rangle \\
&= ika_0\langle\psi_f|\frac{e}{m}p_x z|\psi_i\rangle \\
&= ika_0\langle\psi_f|\frac{e}{2m}(zp_x + p_x z)|\psi_i\rangle, \text{ As } [p_x, z] = 0, p_x z = zp_x \\
&\text{In the above we have added } zp_x \text{ and divided by a factor 2} \\
&= ika_0\left[\langle\psi_f|\frac{e}{2m}\left(zp_x - xp_z\right)|\psi_i\rangle + \langle\psi_f|\frac{e}{2m}\left(p_x z + xp_z\right)|\psi_i\rangle\right] \\
&\text{Here, } xp_z \text{ has been added and subtracted.} \\
&= ika_0\left[\langle\psi_f|\frac{e}{2m}L_y|\psi_i\rangle + \langle\psi_f|\frac{e}{2m}\left(p_x z + xp_z\right)|\psi_i\rangle\right], \text{ As } zp_x - xp_z = L_y \\
&= ika_0\left[\langle\psi_f|\mu_y|\psi_i\rangle + \langle\psi_f|\frac{e}{2m}\left(p_x z + xp_z\right)|\psi_i\rangle\right] \qquad (34)
\end{aligned}$$

The first term in the above equation produces a radiation of magnetic dipole transition. Using the Heisenberg equation of motion, the second term in the equation (34) can also be written as

$$
\begin{aligned}
&ika_0\langle\psi_f|\frac{e}{2m}\left(p_x z + x p_z\right)|\psi_i\rangle \\
&= ika_0\langle\psi_f|\frac{ie}{\hbar}[H_0, x]z + \frac{ie}{\hbar}x[H_0, z]|\psi_i\rangle \\
&= -\frac{k}{2\hbar}\langle\psi_f|e(H_0 xz - xH_0 z) + e(xH_0 z - xzH_0)|\psi_i\rangle
\end{aligned}
$$

In the above expression the second and the third terms cancel out. Using the eigenvalues of H_0 and $E_f - E_i = \hbar\omega$, the second term in equation (34)

$$
\begin{aligned}
&= -\frac{k}{2\hbar}\hbar\omega\langle\psi_f|exz|\psi_i\rangle \\
&= -\frac{k\omega}{2}\langle\psi_f|Q_{zx}|\psi_i\rangle
\end{aligned}
$$

here, $Q_{zx} = exz$ represent the zx-component of electric quadrupole moment tensor. Therefore the the matrix element M_{fi}, as given in equation (34), can be written as

$$M_{fi} = ika_0\langle\mu_y\rangle_{fi} - \frac{1}{2}k\omega\langle Q_{zx}\rangle_{fi} \tag{35}$$

The transition rate for *magnetic dipole transition* can be calculated using the first term of the M_{fi}, see equation (35), and the density of states given in equation (32). Note that $E = pc$ or $\dfrac{dp}{dE} = \dfrac{1}{c}$ and also $p = \dfrac{\hbar\omega}{c}$.

$$\lambda_\gamma(M1) = \frac{2\pi}{\hbar}\frac{V}{2\pi^2\hbar^3}\left(\frac{\hbar\omega}{c}\right)^2\frac{1}{c}\frac{2\hbar^2}{\epsilon_0\hbar\omega V}\frac{\omega^2}{c^2}|\mu_{fi}|^2$$

Here, different terms in the expression of transition rate have been put explicitely. After simplification we have

$$\lambda_\gamma(M1) = \frac{2}{\pi\hbar}\frac{\omega^3}{\epsilon_0 c^5}|\mu_{fi}|^2 \;=\; \frac{2\mu_0}{\pi\hbar}\frac{\omega^3}{c^3}|\mu_{fi}|^2$$

here, we have used $\mu_0\epsilon_0 = \dfrac{1}{c^2}$ and $\omega = ck$.

The matrix element $|\mu_{fi}|^2$ can be approximated for a transition due to single proton as

$$\mu_{fi} = \frac{e\hbar}{2m_p} = \text{one nuclear magneton}$$

The m_p represents mass of proton. Therefore, the transition rate is

$$\lambda_\gamma(M1) = \frac{2}{\pi\hbar}\frac{\omega^3}{\epsilon_0 c^5}\frac{e^2\hbar^2}{4m_p^2} \;=\; \frac{2\alpha}{c^4\hbar m_p^2}E_\gamma^3$$

where α is fine structure constant.

We can calculate the constant in front of E_γ^3 as

$$\lambda_\gamma(M1) = 2.52 \times 10^{13} E_\gamma^3$$

Weisskopf estimate using better estimate of the matrix element is given by

$$\lambda_\gamma(M1) = 5.6 \times 10^{13} E_\gamma^3$$

Similarly, transition rate for electric quadrupole transition can be obtained from the second term of the matrix element given in equation 35

$$\lambda_\gamma(E2) = \frac{2\pi}{\hbar}\frac{V}{2\pi^2\hbar^3}\left(\frac{\hbar\omega}{c}\right)^2\frac{1}{c}\frac{2\hbar^2}{\epsilon_0\hbar\omega V}\frac{\omega^4}{4c^2}|(Q_{zx})_{fi}|^2$$

$$= \frac{1}{2\epsilon_0\pi\hbar}\left(\frac{\omega}{c}\right)^5|(Q_{zx})_{fi}|^2 \;=\; \frac{1}{2\epsilon_0\pi\hbar^6 c^5}E_\gamma^5|(Q_{zx})_{fi}|^2$$

we can make a rough estimate of the matrix element by considering maximum value of z and x each as R, radius of the nucleus

$$\langle\psi_f|ezx|\psi_i\rangle = eR^2$$

Therefore,

$$\lambda_\gamma(E2) = \frac{1}{2\epsilon_0\pi\hbar}\left(\frac{\omega}{c}\right)^5 e^2R^4 \;=\; \frac{e^2}{2\epsilon_0\pi\hbar}\left(\frac{1}{\lambda\!\!\!^{-}}\right)^5 R^4 = \frac{e^2}{2\epsilon_0\pi\hbar}\left(\frac{1}{\lambda\!\!\!^{-}}\right)\left(\frac{R}{\lambda\!\!\!^{-}}\right)^4$$

We can also write

$$\lambda_\gamma(E2) = \frac{e^2 R_0^4}{2\epsilon_0 \pi \hbar^6 c^5} A^{4/3} E_\gamma^5$$

Taking the ratio of the transition rates for electric quadrupole and for electric dipole transitions, we have

$$\frac{\lambda_\gamma(E2)}{\lambda_\gamma(E1)} \sim \left(\frac{R}{\lambda\!\!\!^{-}}\right)^2$$

For a nucleus of radius $R = 10$ fm and gamma energy $\hbar\omega = 1$ MeV

$$\frac{\lambda_\gamma(E2)}{\lambda_\gamma(E1)} \sim 10^{-4}$$

That means $\lambda_\gamma(E2) << \lambda_\gamma(E1)$

The Weisskopf estimates for some of the multipoles are given in the following. Here, $E_\gamma$is in MeV.

$$\begin{aligned}
\lambda_\gamma(E1) &= 1.0 \times 10^{14} A^{2/3} E_\gamma^3 \\
\lambda_\gamma(E2) &= 7.3 \times 10^{7} A^{4/3} E_\gamma^5 \\
\lambda_\gamma(E3) &= 34 A^2 E_\gamma^7 \\
\lambda_\gamma(E4) &= 1.1 \times 10^{-5} A^{8/3} E_\gamma^9
\end{aligned}$$

$$\begin{aligned}
\lambda_\gamma(M1) &= 5.6 \times 10^{13} E_\gamma^3 \\
\lambda_\gamma(M2) &= 3.5 \times 10^{7} A^{2/3} E_\gamma^5 \\
\lambda_\gamma(M3) &= 16 A^{4/3} E_\gamma^7 \\
\lambda_\gamma(M4) &= 4.5 \times 10^{-6} A^2 E_\gamma^9
\end{aligned}$$

Lecture-III

Angular distribution of gamma rays

If the spins and parities of the initial ($J_i^{\pi_i}$) and the final ($J_f^{\pi_f}$) states of a nucleus is known then using conservation of angular momentum one can predict possible multipole orders L of the gamma transition between these two levels. In experiment, for most of the states, we just do the opposite. We measure multipole order L of the transition and try to fix J_i if J_f is known. The multipole order of a gamma transition can be measured through angular distribution of the γ ray, i.e. measuring intensities of the gamma ray in detectors placed at different angle relative to a fixed axis.

Now, consider a γ ray of dipole character from $J_i = 1$ to $J_f = 0$ states. The initial states includes $m_i = \pm 1$ and 0 and the final state has $m_f = 0$, as shown in Fig. 15. The angular distribution of the gamma ray will depend upon the value of m_i and m_f. For $m_i = 0$ to $m_f = 0$, the emission probability is proportional to $sin^2\theta$, where θ is measured from the z-axis through which m_i are defined. For the transitions between $m_i = \pm 1$ and $m_f = 0$, the emission probability is proportional to $\frac{1}{2}(1 + cos^2\theta)$. In the experiment, if we can fix the m values then we should see the characteristic distribution.

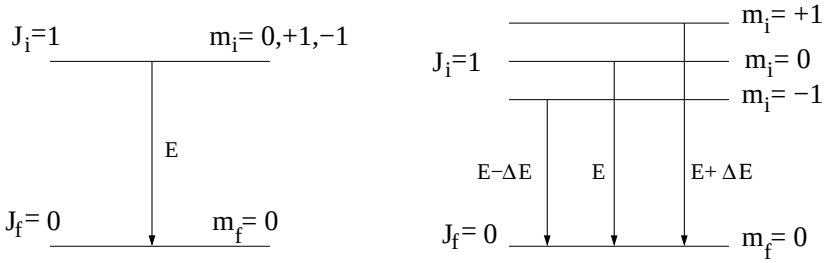

Figure 15: Gamma decay between $J_i = 1$ and $J_f = 0$ states. The splitting of the levels under the magnetic field is also shown on the right hand side.

The simplest way to achieve this is to place the nucleus in a magnetic field such that magnetic moment μ of the nucleus interacts with the magnetic field B producing splitting in energy levels according to their m values (similar to Zeeman splitting). This is shown in Fig. 15 where transitions from the

levels of different m-values are shown with energies E and $E \pm \Delta E$. The ΔE is the magnitude of splitting due to interaction and equal to $\mu \cdot B$. Thus, if we can pick one of the transition, say $E + \Delta E$, then the angular distribution is $\frac{1}{2}(1 + cos^2\theta)$ relative to the direction of the field.

For a typical magnetic moment of $1\mu_N$ of a nucleus placed in a magnetic field of 10 Tesla, one can calculate

$$\Delta E = 3.15 \times 10^{-8} eV/T \times 10T \;=\; 3.15 \times 10^{-7} eV$$

$$\text{where, } 1\mu_N = 5.05 \times 10^{-27} J/T \;=\; 3.15 \times 10^{-8} eV/T$$

has been used.

As one can see that the splitting of energy levels is of the order 10^{-6} eV which is too small to be measured by a detector. The resolution of the best detector is of the order of keV. Therefore, we will observe a mixture of all the three transitions in the angular distribution measurement, i.e from $m_i = +1 \rightarrow m_f = 0$, $m_i = 0 \rightarrow m_f = 0$, and $m_i = -1 \rightarrow m_f = 0$. If we let $W(\theta)$ to represent angular distribution then

$$W(\theta) = \sum_{m_i} P(m_i) w_{m_i \rightarrow m_f}(\theta)$$

where $P(m_i)$ is the population of initial state, i.e. the fraction of the nuclei that occupies each sub level m_i. Under normal circumstances

$$P(+1) \;=\; P(0) \;=\; P(-1) \;=\; \frac{1}{3}$$

, i.e. all the m states are equally populated. Therefore,

$$W(\theta) \propto \frac{1}{3}\left[\frac{1}{2}(1 + cos^2\theta) + sin^2\theta + \frac{1}{2}(1 + cos^2\theta)\right]$$

This gives a constant value of $W(\theta)$, i.e the angular distribution of the gamma ray will be isotropic, i.e independent of θ. It is clear that if we can make population $P(m_i)$ of the sub levels unequal then the $W(\theta)$ will not show a isotropic distribution.

The unequal population of the levels can be achieved in two ways. In the first case, one can place the nucleus in a strong magnetic field and reduce

the temperature to very low (sub-Kelvin). The population of the levels is described by the Boltzmann distribution

$$P(m_i) = e^{-m_i\left(\frac{\Delta E}{KT}\right)}.$$

In order to have different $P(m_i)$, the fraction $\frac{\Delta E}{KT}$ should be approximately one. At room temperature $KT = \frac{1}{40}$ eV and, hence $\Delta E << KT$. To make $\Delta E \simeq KT$, the temperature has to be brought down to $T \simeq 0.01K$. The angular distribution plots under different conditions of temperature are shown in Fig. 16.

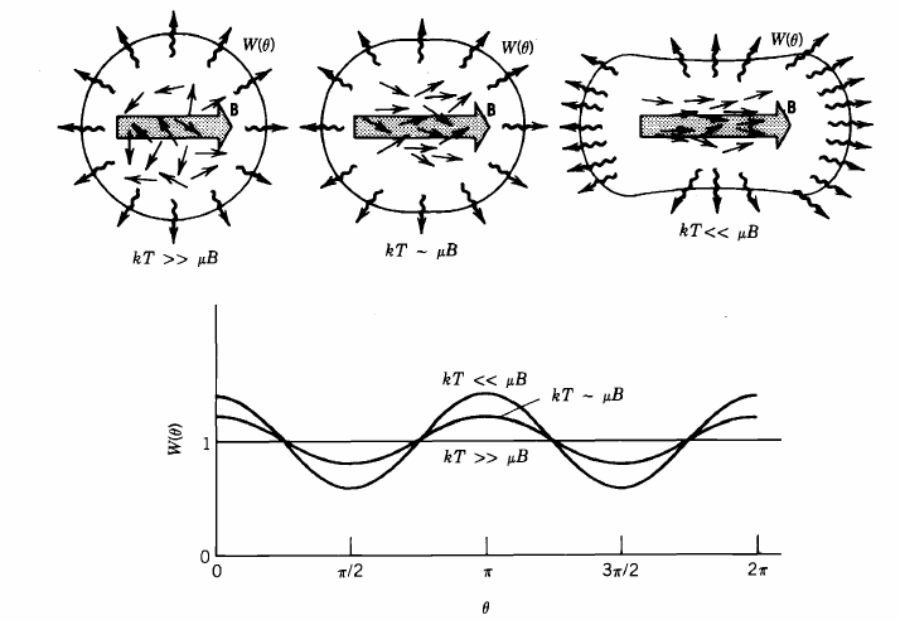

Figure 16: Angular distribution of gamma rays from nuclei with spin oriented at low, intermediate and high temperatures.

Another way to produce unequal mixture to populations $P(m_i)$ is to observe a previous radiation in coincidence with the radiation of interest. This is shown in the Fig. 17 where a cascade of two γ rays is drawn from the levels with spin sequence $0 \rightarrow 1 \rightarrow 0$. With this assignment of spins both the radiations are dipole transitions. Let us observe the γ_1 in a certain direction called z-direction and the γ_2 is observed at an angle θ_2 with respect to the z direction. For γ_1, the angular distribution will be according to the the decay

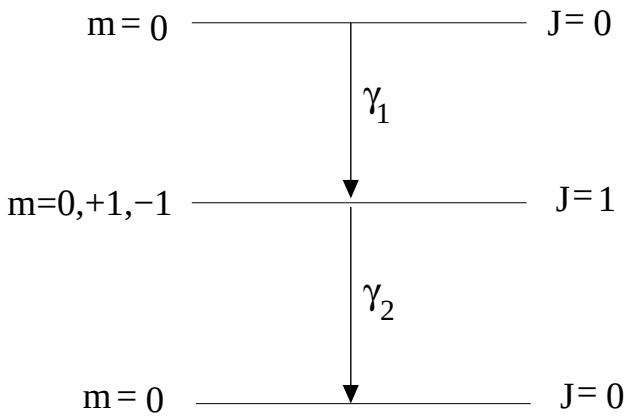

Figure 17: A cascade of gamma rays, γ_1 and γ_2, between $0 \to 1 \to 0$ states.

from $m_i = 0 \to m_f = 0$ and $m_i = 0 \to m_f = \pm 1$. Since we have freedom to choose the z axis and has been defined as the direction of γ_1, the angle θ_1 in the angular distribution function will be zero. Therefore, $0 \to 0$ transition can not be emitted in that direction as $sin\theta_1 = 0$. That means the nuclei for which γ_2 is observed following γ_1 must have a population of $P(m_i) = 0$ for $m_i = 0$. Therefore, angular distribution of γ_2 relative to γ_1 is only due to $m_i = 0 \to m_f = \pm 1$ decays and is given by

$$W(\theta) \propto \frac{1}{2}\left[\frac{1}{2}(1 + cos^2\theta) + \frac{1}{2}(1 + cos^2\theta)\right]$$

or

$$W(\theta) \propto \left(1 + cos^2\theta\right)$$

This type of experiment is called angular correlation. In general, angular distribution or angular correlation of multipole radiation is of the form of a polynomial in even powers of $cos\theta$

$$W(\theta) = 1 + \sum_{k=1}^{L} a_{2k} cos^{2k}\theta$$

Note that only even powers of $cos\theta$ is included in the above expression because of symmetry of $W(\theta)$ and $W(\pi - \theta)$ must be equal if parity is conserved. The values of coefficients a depend on multipolarity of the transition. For a cascade of transitions, $4^+ \to 2^+ \to 0^+$, both γ_1 and γ_2 are $E2$ transitions

and in this case $a_2 = \frac{1}{8}$ and $a_4 = \frac{1}{24}$. For mixed transition like $2^+ \rightarrow 2^+ \rightarrow 0^+$ where γ_1 is a mixture of $M1$ and $E2$, the value of the coefficients depend on mixing ratio δ which is defined as ratio of transition matrix element $\frac{M_{fi}(E2)}{M_{fi}(M1)}$. In this case fraction of $E2$ radiation is given by $\frac{\delta^2}{(1+\delta^2)}$ and the fraction of $M1$ is $\frac{1}{(1+\delta^2)}$. The measured values of the coefficients are $a_2 = -0.06 \pm 0.22$ and $a_4 = 0.89 \pm 0.24$ and the mixing ratio is $\delta = -1.2 \pm 0.2$

Internal conversion

In this process an excited nucleus transfers its excitation energy to an atomic electron. The electron comes out of the atom with kinetic energy equal to the excitation energy minus its binding energy. Characteristic x-ray is emitted after ejection of electron. The conversion probability is more for the electrons moving in an orbit closest to the nucleus, i.e. K shell. The internal conversion process is an electromagnetic interaction process where electromagnetic field of the nucleus interacts with the atomic electrons and causes electrons to be ejected from the atom. Thus, internal conversion process competes with γ emission.

The total transition rate λ from an excited state of a nucleus is then equal to sum of the transition rates due to gamma emission and internal conversion process, i.e.

$$\lambda = \lambda_\gamma + \lambda_e$$

Let us define conversion coefficient

$$\alpha = \frac{\lambda_e}{\lambda_\gamma}.$$

Therefore,

$$\lambda = \lambda_\gamma (1 + \alpha)$$

Here, α is the total internal conversion coefficient and is equal to

$$\alpha = \alpha_K + \alpha_L + \alpha_M + ...$$

For the calculation of transition rate due to internal conversion process, we have to proceed in the same way as for the gamma decay process. The matrix element of the interaction Hamiltonian is similar to that of the γ decay process with an exception that the initial state wavefunction should also include wavefunction of bound electron apart from that of the nucleus, i.e.

$$\psi_i = \psi_{iN}\psi_{ie}.$$

Similarly

$$\psi_f = \psi_{fN}\psi_{fe}$$

where ψ_{fe} is the wave function of free electron

$$\psi_{fe} \sim e^{ik\cdot r_e}.$$

With an assumption that ψ_{fe} varies very little over the nuclear volume, we can approximate

$$\psi_{fe} \to \psi_{fe}(r_e = 0).$$

The nuclear information is contained in ψ_{iN} and ψ_{fN}. In the calculation of internal conversion coefficient $\alpha = \dfrac{\lambda_e}{\lambda_\gamma}$, the nuclear matrix element cancels out and, thus, it is independent of details of nuclear structure. Non-relativistic calculation of the conversion coefficient gives an expression

$$\alpha(EL) = \frac{Z^3}{n^3}\left(\frac{L}{L+1}\right)\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^4\left(\frac{2m_ec^2}{E}\right)^{L+5/2}$$

$$\alpha(ML) = \frac{Z^3}{n^3}\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^4\left(\frac{2m_ec^2}{E}\right)^{L+3/2}$$

Here, Z is the atomic number of the atom in which conversion takes place, L is multipolarity of the transition, E is excitation energy of the nucleus and n is principal quantum number. The factor $\dfrac{Z^3}{n^3}$ is coming from $|\psi_{fe}(r_e = 0)|^2$. Following observations can be made

(i) α is proportional to Z^3
(ii) α is decreasing with transition energy
(iii) α is increasing with multipole order L
(iv) α decreases for higher values of n

Lifetimes for γ emission and partial decay rate

Consider the energy level diagram of ^{72}Se shown in Fig. 18. The excitation energy, spin, parity and lifetimes of each level are shown in the figure. We would like to calculate transition rate of the gamma transitions. let us consider the level at 1317 keV with spin and parity 2^+ and half-life 8.7 ps.

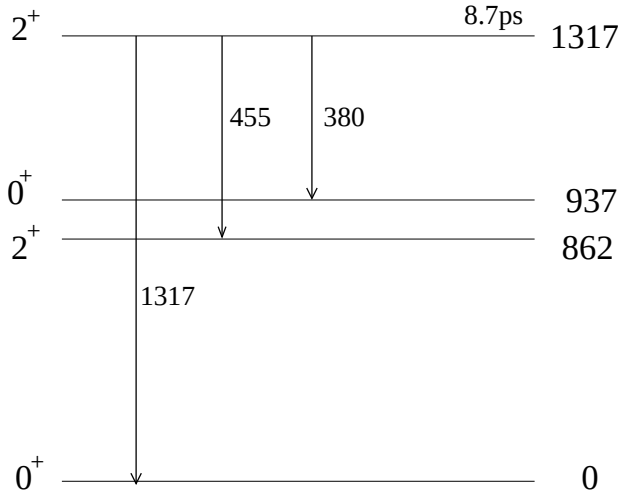

Figure 18: Energy level diagram of ^{72}Se.

Therefore, total transition rate of the level is

$$\lambda_t = \frac{0.693}{t_{1/2}} \;=\; 8.0 \times 10^{10} s^{-1}$$

The above level decay through three transitions of energies, 1317, 455 and 380 keV. Therefore, total transition rate is sum of the transitions rates of these transitions.

$$\lambda_t = \lambda_{t,1317} + \lambda_{t,455} + \lambda_{t,380}$$

Considering conversion coefficient of the transitions we can write

$$\lambda_t = \lambda_{\gamma,1317}(1 + \alpha_{1317}) + \lambda_{\gamma,455}(1 + \alpha_{455}) + \lambda_{\gamma,380}(1 + \alpha_{380})$$

Usually, the conversion coefficients are small and for the above transitions we will neglect here in comparison to the factor 1. The relative intensities of the three γ rays have been measured as

$$\lambda_{\gamma,1317} : \lambda_{\gamma,455} : \lambda_{\gamma,380} = 51 : 39 : 10$$

Therefore, partial decay rates of the gamma rays are

$$\lambda_{\gamma,1317} = 0.51 \times 8.0 \times 10^{10} s^{-1} \;=\; 4.1 \times 10^{10} s^{-1}$$

$$\lambda_{\gamma,455} = 0.39 \times 8.0 \times 10^{10} s^{-1} \;=\; 3.1 \times 10^{10} s^{-1}$$

$$\lambda_{\gamma,380} = 0.10 \times 8.0 \times 10^{10} s^{-1} \;=\; 0.8 \times 10^{10} s^{-1}$$

These can be compared with those calculated using Weisskopf estimates

$$\lambda_{E2,1317} = 8.7 \times 10^{10} s^{-1}$$

$$\lambda_{E2,455} = 4.3 \times 10^{8} s^{-1}$$

$$\lambda_{E2,380} = 1.7 \times 10^{8} s^{-1}$$

It has been observed that the measured transition rates are at least one order higher than the Weisskopf estimates. One should note that Weisskopf estimates are based on motion of a single nucleon. The disagreement suggests collective aspects of nuclear structure in which several nucleon participates.