Classical Dynamics and Special Relativity (PH21201) - Mini Project - Chaos: An Introduction

Chua's circuit with a Quintic Nonlinearity

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Abstract

The chua's circuit is one of the simplest electronic circuits which exhibits chaotic behavior. The simplest form of the circuit consists of several linear components including resistors, capacitors, and inductors and one nonlinear component called the Chua's diode which is essentially a nonlinear resistor with negative resistance near the origin. The basic circuit, which has been widely studied, consists of a nonlinear resistor with a piecewise linear voltage-current characteristic. In this paper, we explore the chaotic phenomenon in a Chua's circuit with a nonlinear resistor with a smooth nonlinearity and compare it with the piecewise linear case. Since the cubic nonlinearity has been studied before, here as an example, we study the Quintic nonlinearity, derive the differential equations, calculate the Jacobian matrix and study the phase plots and attractors obtained with different parameter values through MATLAB simulations and pinpoint the points of Bifurcation.

Introduction - Basic Circuit and Derivation of Equations

The basic Chua's circuit is given in Figure 1.

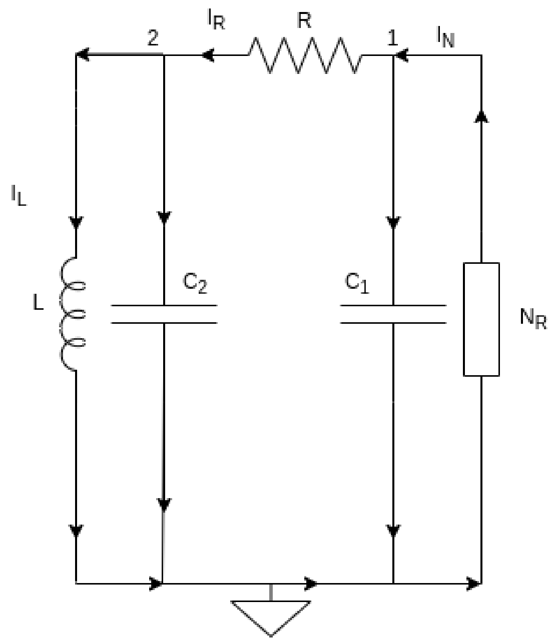


Figure 1 - Chua's circuit. The arrows indicate the direction of flow of current.

The nonlinear resistor must have a negative resistance, in order to function as a power source in the circuit. Using Kirchoff's voltage and current laws, we derive the following equations. Let the voltage across the capacitor C_1 be V_{C_1}, voltage across capacitor C_2 be V_{C_2} and the current flowing through the inductor L be I_L. Let the voltage-current characteristic of the nonlinear resistor N_R be represented by function $g(V_{C_1})$ since V_{C_2} is also the voltage across the nonlinear resistor.

Using KVL at the node 1,

$$I_{C_1} = I_R - I_N$$
$$\therefore C_1 \frac{dV_1}{dt} = \frac{1}{R}(V_{C2} - V_{C1}) - g(V_{C_1})$$

Similarly, using KVL at node 2,

$$I_{C_2} = I_R - I_L$$
$$\therefore C_2 \frac{dV_2}{dt} = \frac{1}{R}(V_{C_2} - V_{C_1}) + I_L$$

At the inductor we can write,

$$L \frac{dI_L}{dt} = -V_{C_1}$$

Let $V_{C_1} = x,\ V_{C_2} = y,\ I_L = z$. Then we get the following system of nonlinear differential equations,

$$\begin{aligned}\dot{x} &= \frac{1}{RC_1}(y - x) - \frac{1}{C_1} g(x)\\ \dot{y} &= \frac{1}{RC_2}(y - x) + \frac{1}{C_2} z\\ \dot{z} &= -\frac{1}{L} y\end{aligned}$$

Let as take,

$$\alpha = \frac{C_2}{C_1},\ \beta = \frac{R^2 C_2}{L}, \tau = \frac{t}{RC_2}, z' = Rz$$

Then we can write the above differential equations as,

$$\begin{aligned}\frac{dx}{d\tau} &= \alpha(y - x - Rg(x))\\ \frac{dy}{d\tau} &= y - x + z\\ \frac{dz'}{d\tau} &= -\beta y\end{aligned}$$

Rewriting the above equations for ease of understanding, we get the below final equations in dimensionless form,

$$\begin{aligned}\dot{x} &= \alpha(y - h(x))\\ \dot{y} &= y - x + z\\ \dot{z} &= -\beta y\end{aligned}$$

We can further study these equations to observe the characteristics of chaos in the chua's circuit.

Calculation of Jacobian Matrices

We have the following set of equations-

$$\dot{x} = \alpha(y - x - g(x))$$
$$\dot{y} = x - y + z$$
$$\dot{z} = -\beta y$$

For fixed points, $(x_\star, y_\star, z_\star)$-

$$\dot{x} = 0, \dot{y} = 0, \dot{z} = 0$$
$$\dot{z} = 0 \Rightarrow y_\star = 0$$
$$\dot{y} = 0 \Rightarrow x_\star = -z_\star$$
$$\dot{z} = 0 \Rightarrow x_\star + g(x) = 0$$

We are studying a **'Quintic'** nonlinearity of the form $g(x) = x^5 + (c - 1)x,\ c \in R$

$$\dot{x} = \alpha[y - x - (x^5 + (c - 1)x)]$$
$$\dot{x} = \alpha(y - x - x^5 - cx + x)$$
$$\dot{x} = \alpha(y - x^5 - cx)$$
$$\dot{x} = \alpha(y - f(x))$$

$$f(x) = x^5 + cx$$
$$f(x_\star) = x_\star + g(x_\star) = 0$$

$$\therefore,\ x_\star^5 + cx_\star = 0 \Rightarrow x_\star = 0\ or\ x_\star = (-c)^{\frac{1}{4}}$$

$\therefore$, for $c \geq 0$, one fixed point exists at (0, 0, 0).
for $c < 0$, three fixed points exist at -

$$\begin{cases} \rightarrow (0, 0, 0) & O \\ \rightarrow \left(|c|^{\frac{1}{4}}, 0, -|c|^{\frac{1}{4}}\right) & +P \\ \rightarrow \left(-|c|^{\frac{1}{4}}, 0, |c|^{\frac{1}{4}}\right) & -P \end{cases}$$

For small perturbations,

$x = x_\star + \Psi$

$y = y_\star + \eta$

$z = z_\star + \epsilon$

$\dot{\Psi} = \alpha[y_\star + \eta - (x_\star + \Psi)^5 - c(x_\star + \Psi)]$

$\dot{\Psi} = \alpha[y_\star + \eta - x_\star^5 - 5x_\star^4\Psi - cx_\star - c\Psi)]$

$\dot{\Psi} = \alpha[\eta - (5x_\star^4 + c)\Psi]$

$\dot{\eta} = x_\star + \Psi - y_\star - \eta + z_\star + \epsilon$

$\dot{\eta} = \Psi - \eta + \epsilon$

$\dot{\epsilon} = -\beta(y_\star + \eta) = -\beta\eta$

$\dot{\epsilon} = -\beta\eta$

$$\begin{bmatrix} \dot{\Psi} \\ \dot{\eta} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} -\alpha(5x_\star^4 + c) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \eta \\ \epsilon \end{bmatrix}$$

$$J = \begin{bmatrix} -\alpha(5x_\star^4 + c) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$$

→ For fixed point O, J=M_0 -

$$M_o = \begin{bmatrix} -\alpha c & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$$

→ For fixed points, ±P, J=$M_{\pm P}$ -

$$M_{\pm P} = \begin{bmatrix} 4\alpha c & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$$

The above matrices can be used to determine the eigenvalues. By the roots of the characteristic equation Det[λI - M] = 0. Depending on the eigenvalues, we can determine the type of fixed point. This will depend on the parameter values α, c, β.

Bifurcation with respect to Alpha

Alpha	Beta	c	Figure No.	Type of phase plot
6.0	16.0	-0.142	1(a)	2 stable fixed points placed symmetrically about the origin. 1 unstable fixed point at origin.
7.2	16.0	-0.142	1(b)	2 asymmetric orbits with period (n), 1 unstable fixed point at origin.
8.9	16.0	-0.142	1(c)(i),1(c)(ii)	2 asymmetric orbits across origin with period (n), 1 unstable fixed point at origin, 1 Large stable orbit
9.8	16.0	-0.142	1(d)(i),1(d)(ii)	2 Chua's spiral attractors across origin, 1 unstable fixed point at origin, 1 Large stable orbit
11.0	16.0	-0.142	1(e)(i),1(e)(ii)	Chua's double scroll attractor, 1 unstable fixed point at the origin, 1 stable

				large orbit
17.0	16.0	-0.142	1(f)	1 stable attractor, 1 unstable fixed point at origin

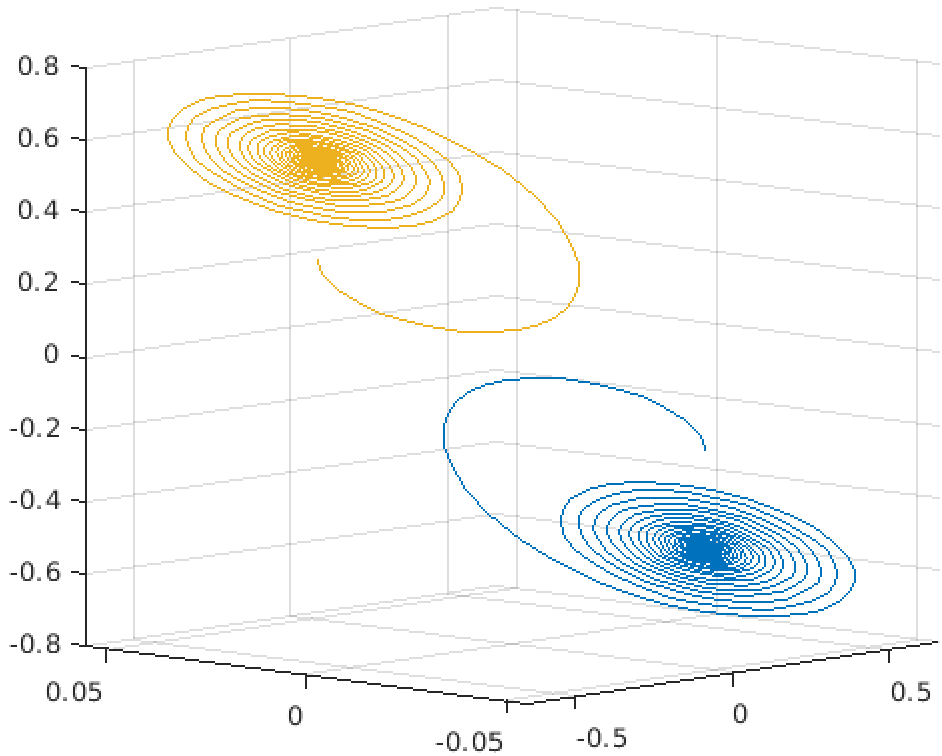


Figure 1(a)

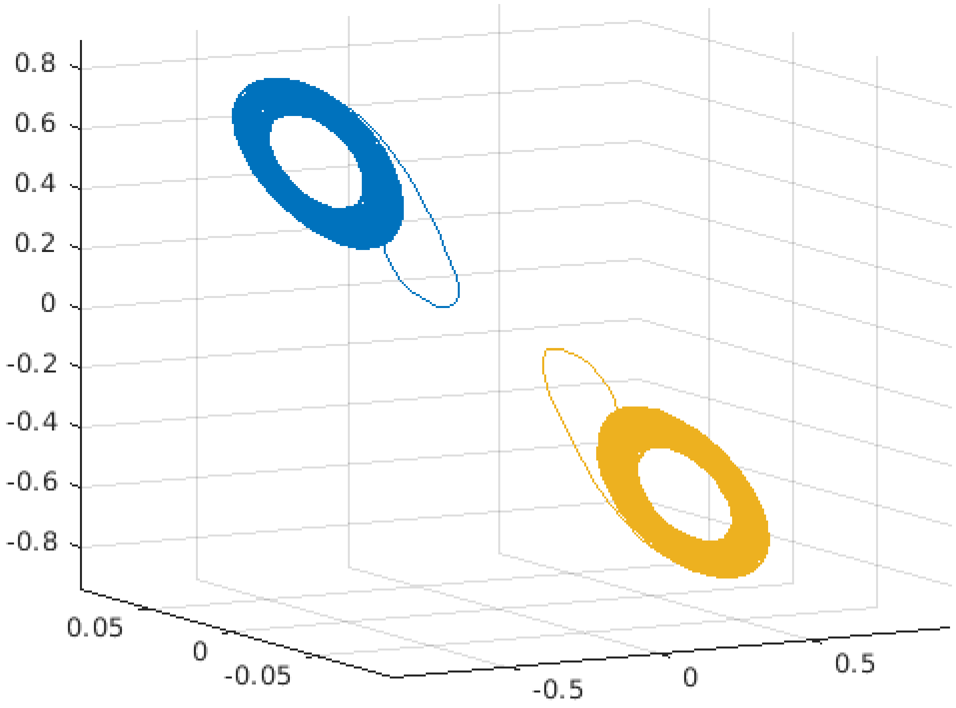

0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0.05
0
-0.05
-0.5
0
0.5

Figure 1(b)

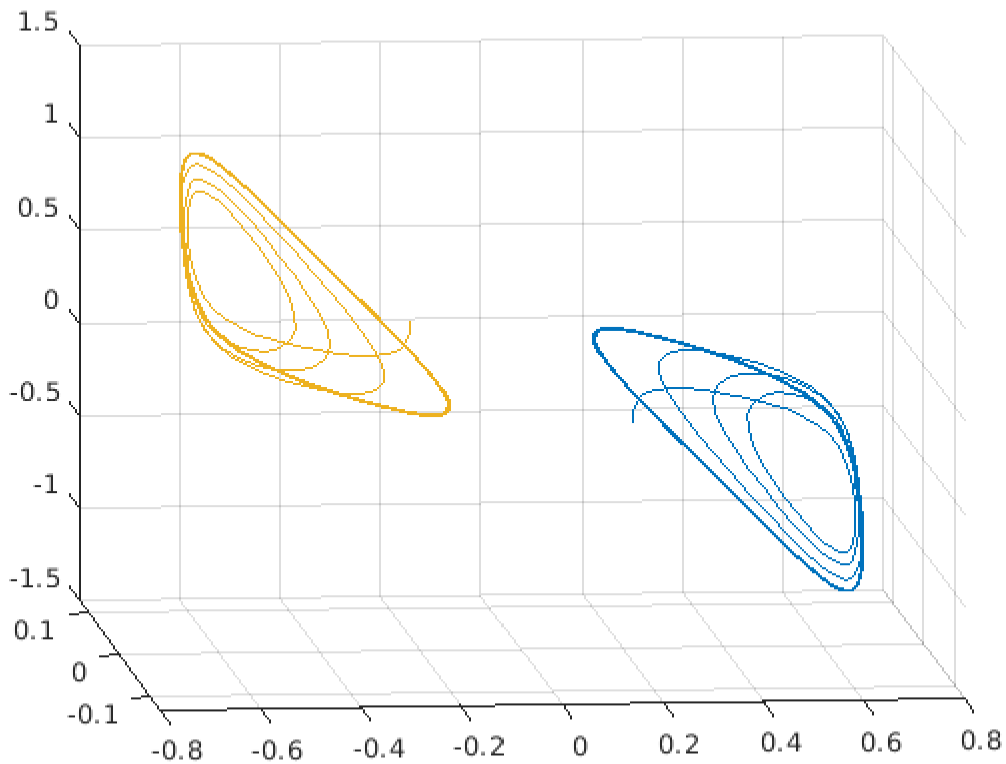

1.5
1
0.5
0
-0.5
-1
-1.5
0.1
0
-0.1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Figure 1(c)(i) - zoomed in picture of orbits

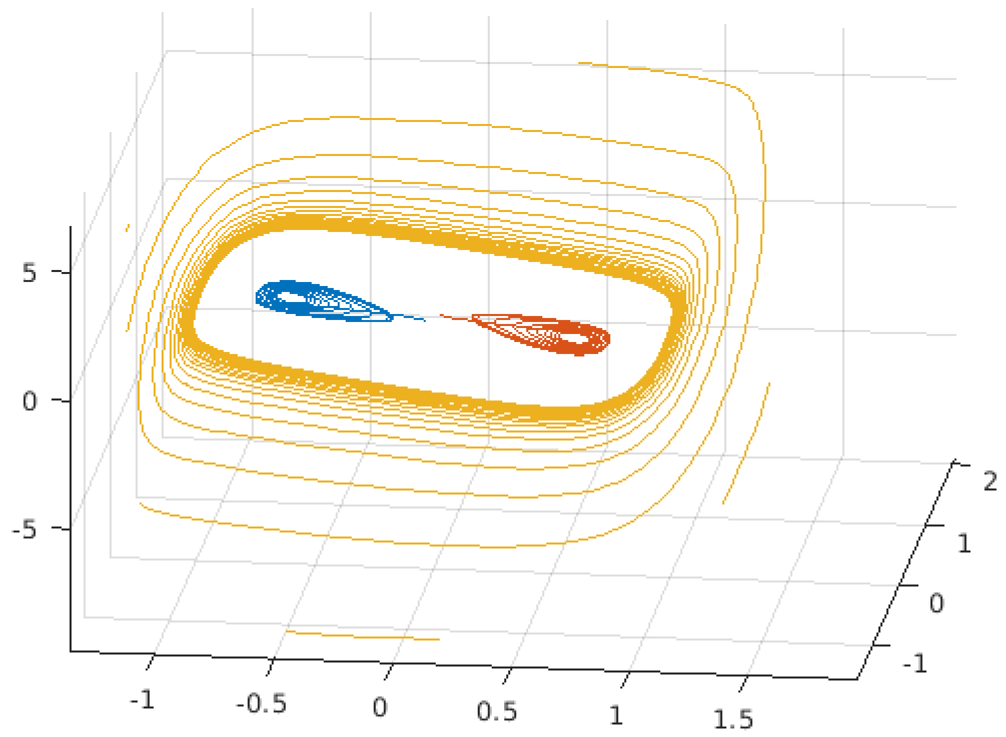


Figure 1(c)(ii)

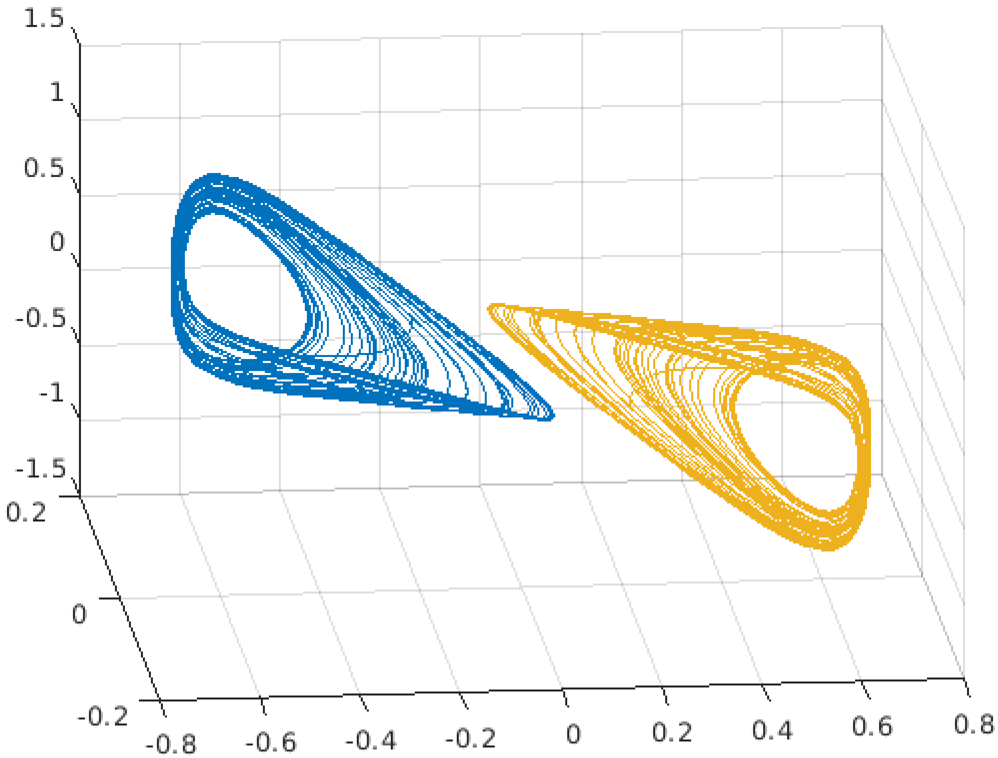


Figure 1(d)(i)

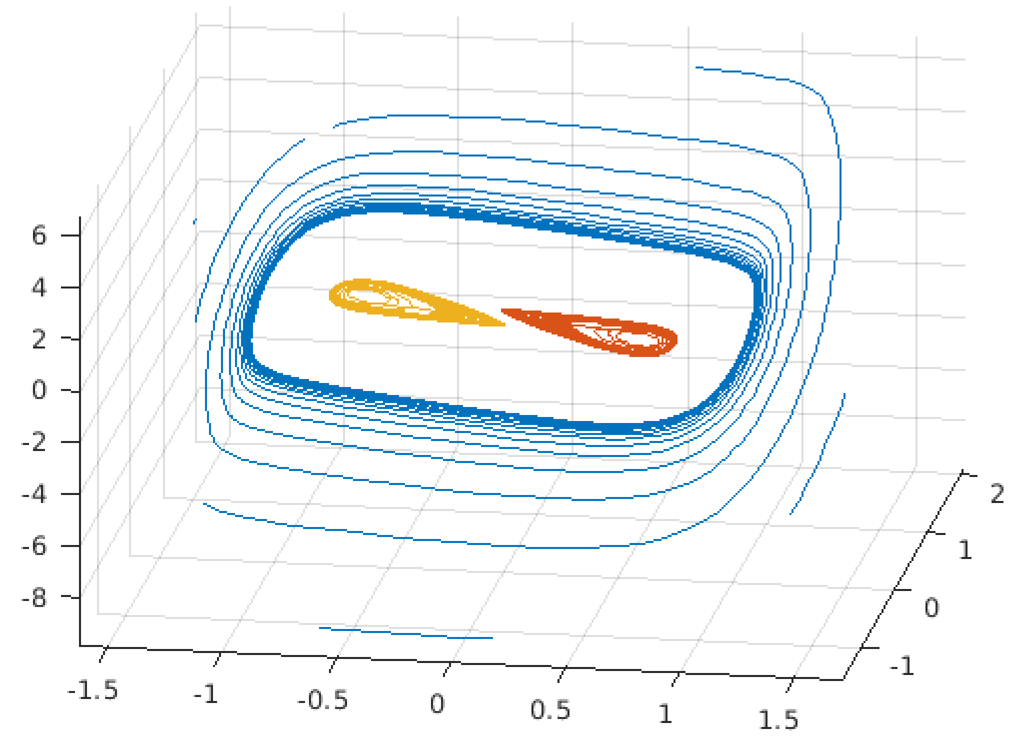


Figure 1(d)(ii)

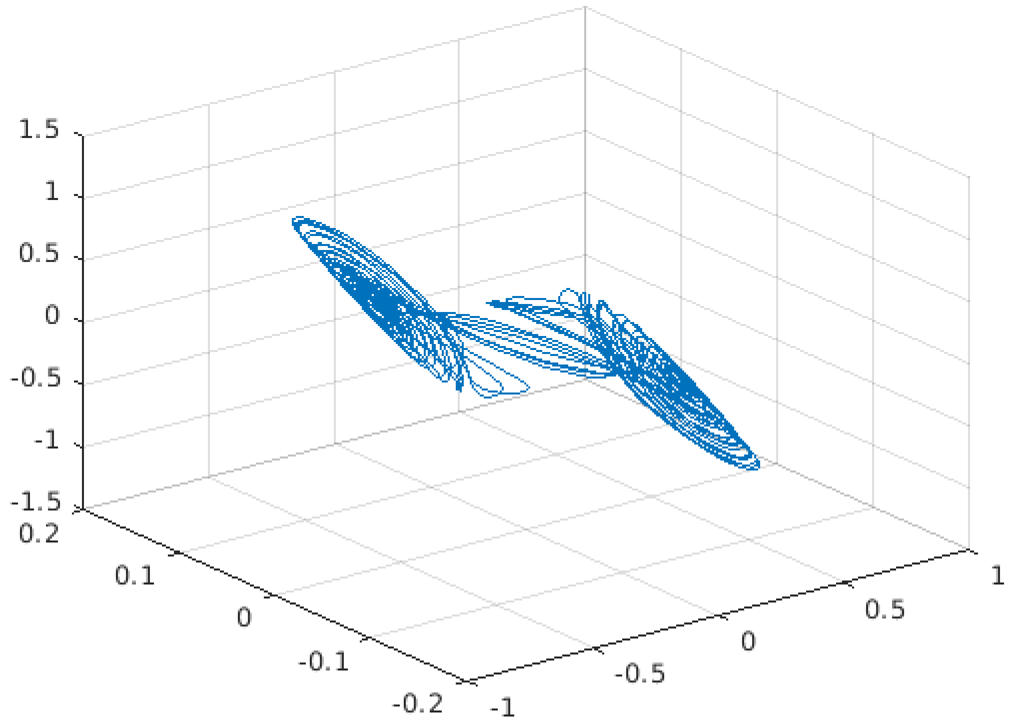


Figure 1(e)(i)

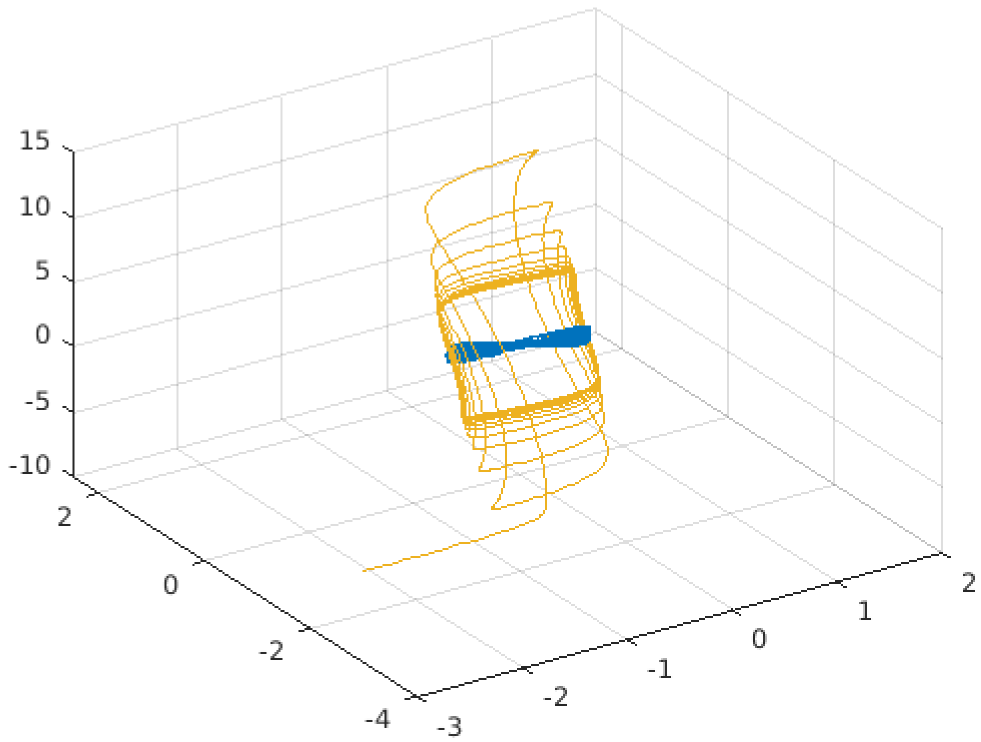


Figure 1(e)(ii)

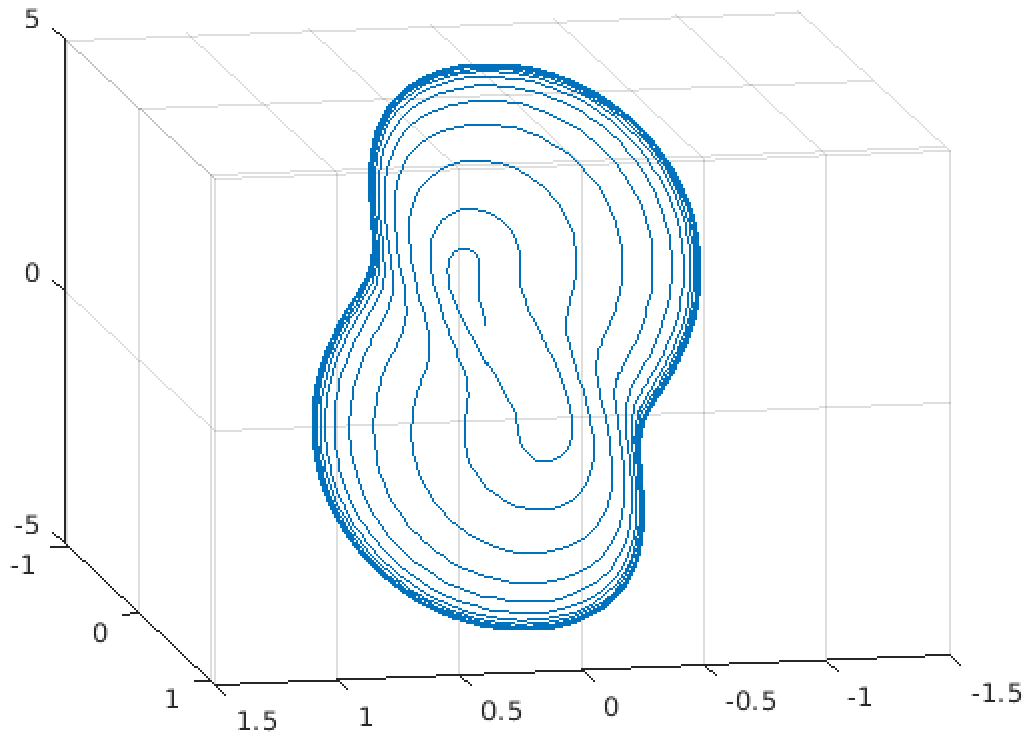


Figure 1(f)

In the given case, we observe a ‘period doubling’ route to chaos. We start with 2 stable fixed points, and through a ‘Hopf Bifurcation’. A Hopf Bifurcation is a critical point where a system's stability switches and a periodic solution arises. We then observe the addition of a larger obit. Slowly, the smaller orbits double in period, and a period-doubling cascade is observed until the 2 orbits turn into Chua’s Spiral attractor and then converge into a Chua’s Double scroll attractor. On increasing alpha further, the double scroll vanishes and only the larger periodic orbit remains.

Below, we will do a similar analysis for Beta. We observe an opposite flow of the phase plot, starting from a single large orbit, to Chua’s double scroll, and a reverse period doubling, followed by a reverse Hopf Bifurcation. Finally 2 stable fixed points are obtained.

Bifurcation with respect to Beta

Alpha	Beta	c	Figure No.	Type of phase plot
11.0	0.1	-0.142	2(a)	1 large stable orbit, and unstable fixed point at origin
11.0	12.0	-0.142	2(b)(i),2(b)(ii)	Chua’s double scroll attractor, unstable fixed point at origin, 1 large stable orbit
11.0	16.0	-0.142	2(c)(i),2(c)(ii)	2 Chua’s spiral attractors, 1 unstable fixed point at origin, 1 large stable orbit
11.0	21.0	-0.142	2(d)(i),2(d)(ii)	2 Chua’s spiral

				attractors, 1 unstable fixed point at origin, 1 large stable orbit
11.0	25.0	-0.142	2(e)	2 asymmetric orbits with period (n), 1 unstable fixed point at origin
11.0	40.0	-0.142	2(f)	2 stable fixed points, symmetric about the origin. 1 unstable fixed point at origin.

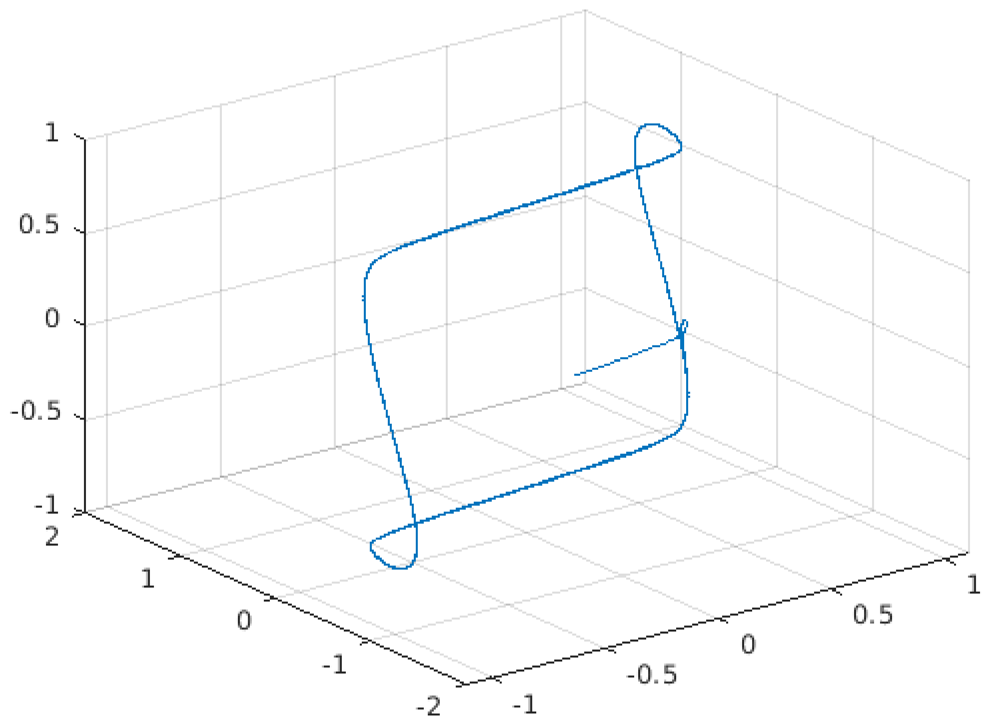


Figure 2(a)

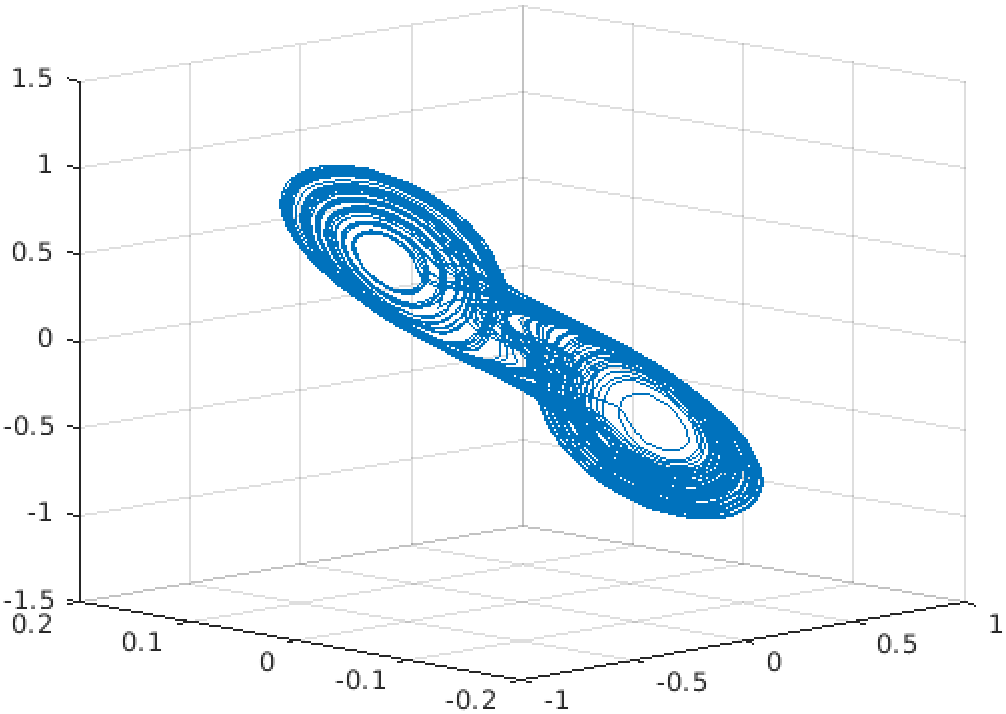


Figure 2(b)(i)

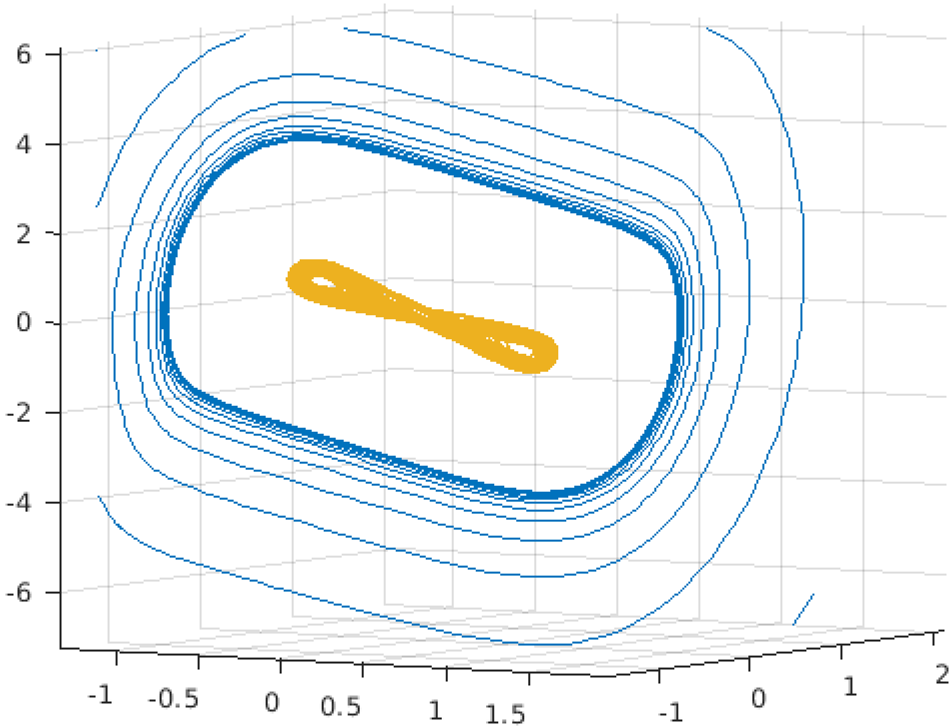


Figure 2(b)(ii)

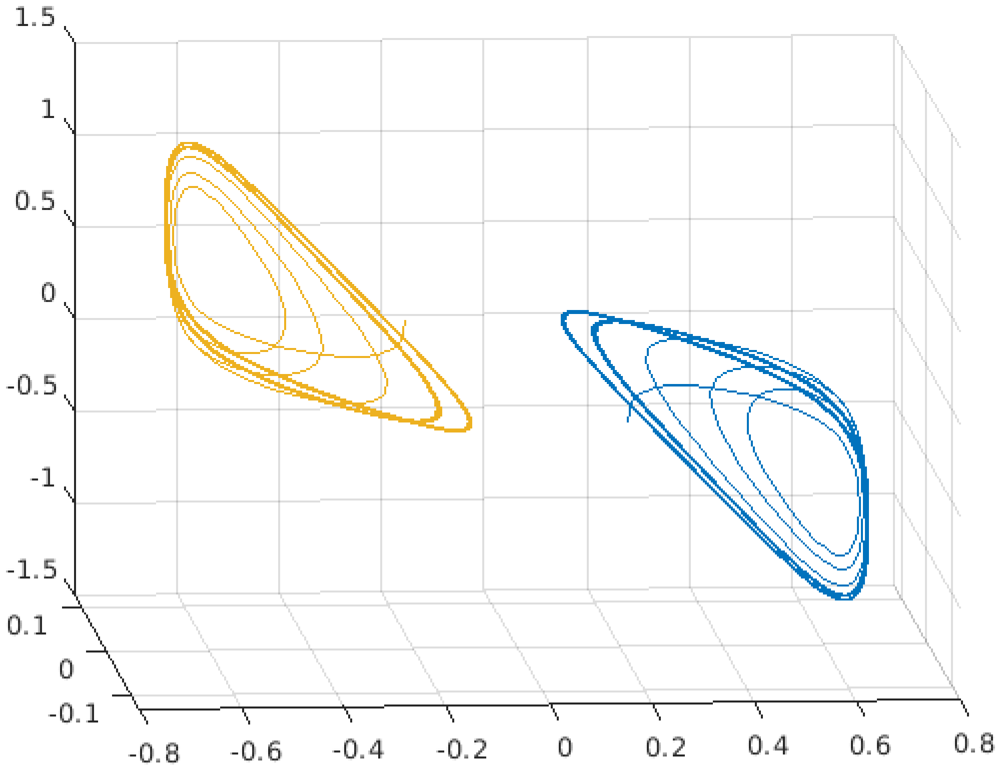


Figure 2(c)(i)

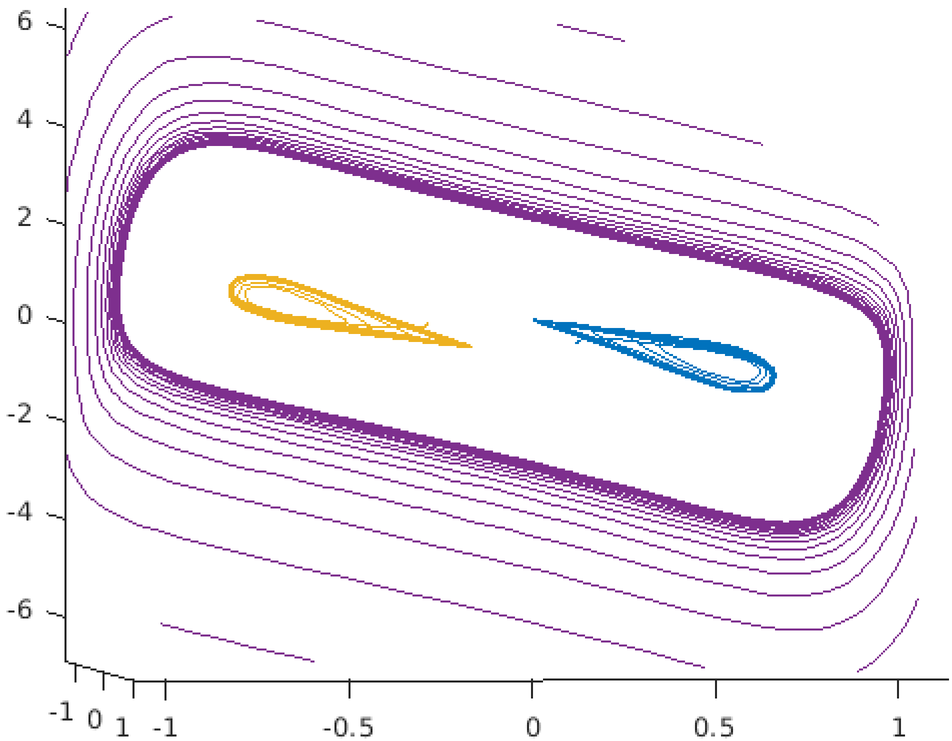


Figure 2(c)(ii)

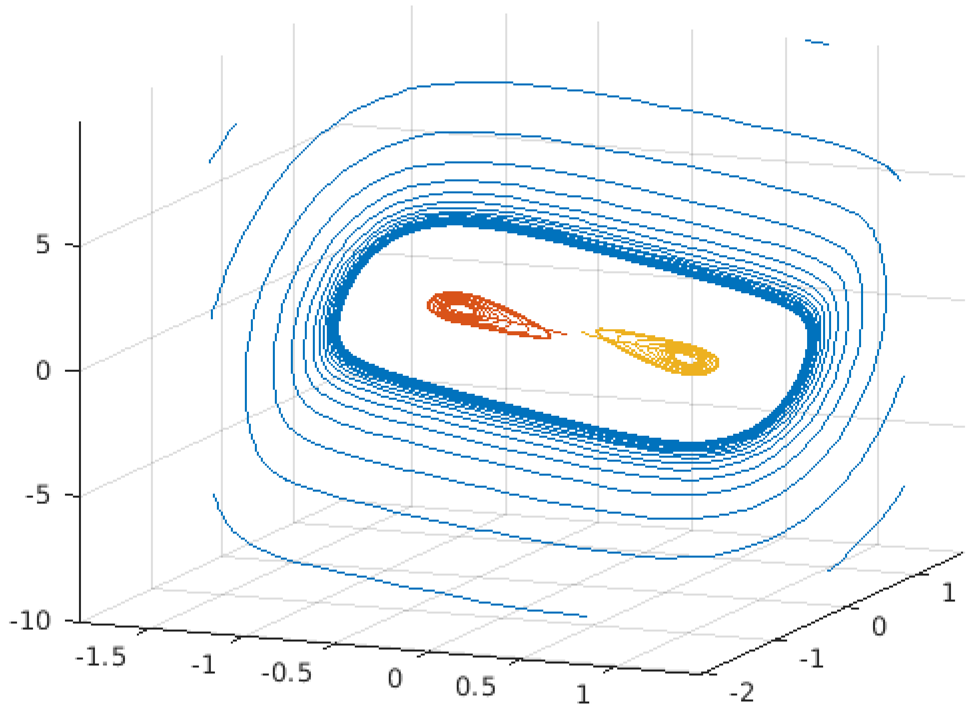


Figure 2(d)(i)

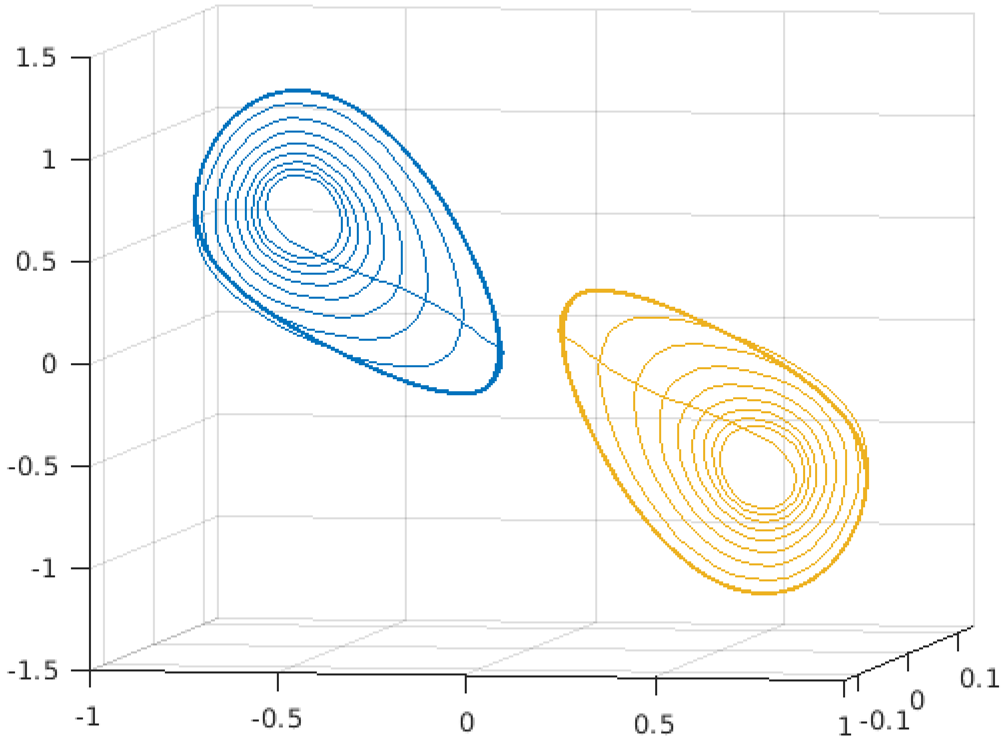


Figure 2(d)(ii)

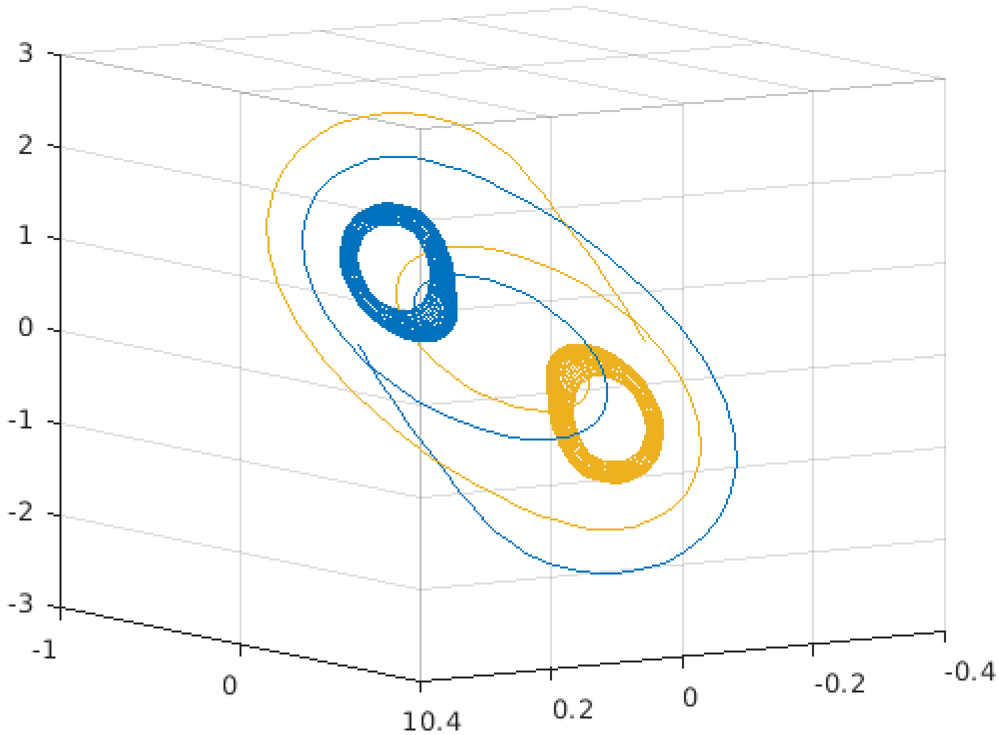


Figure 2(e)

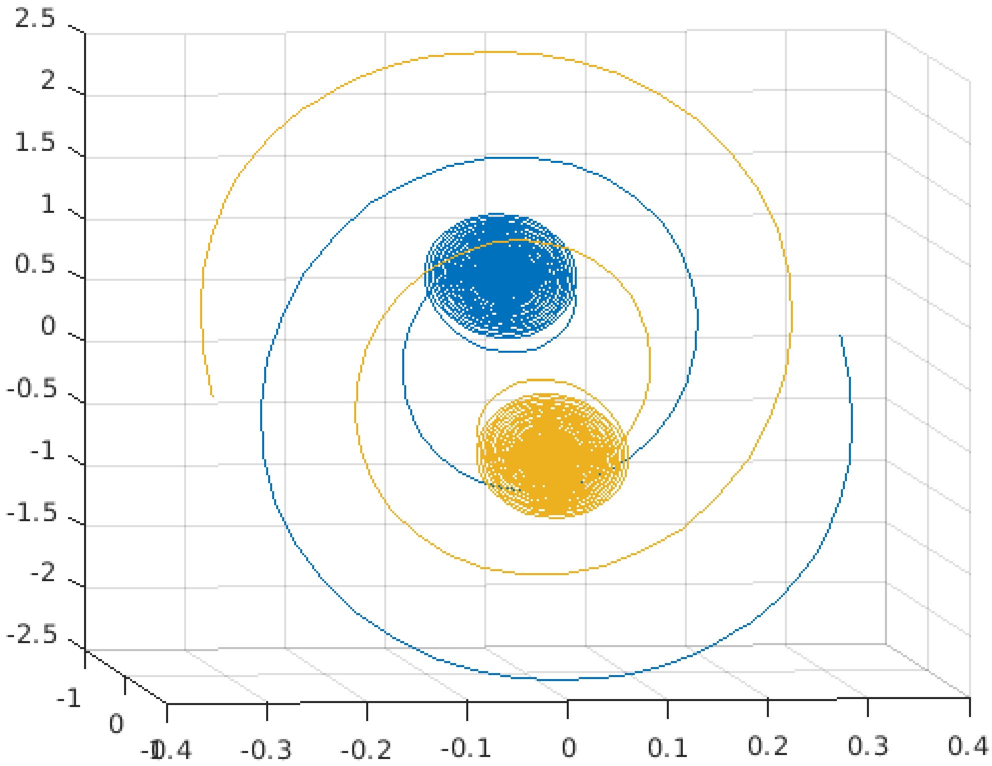
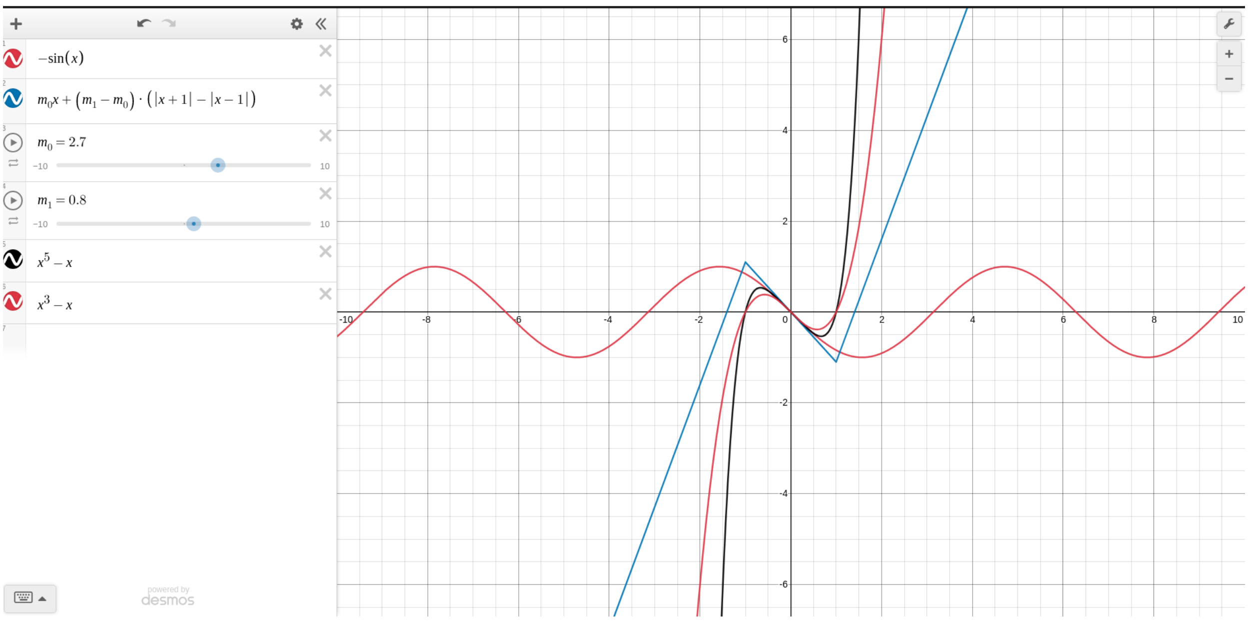


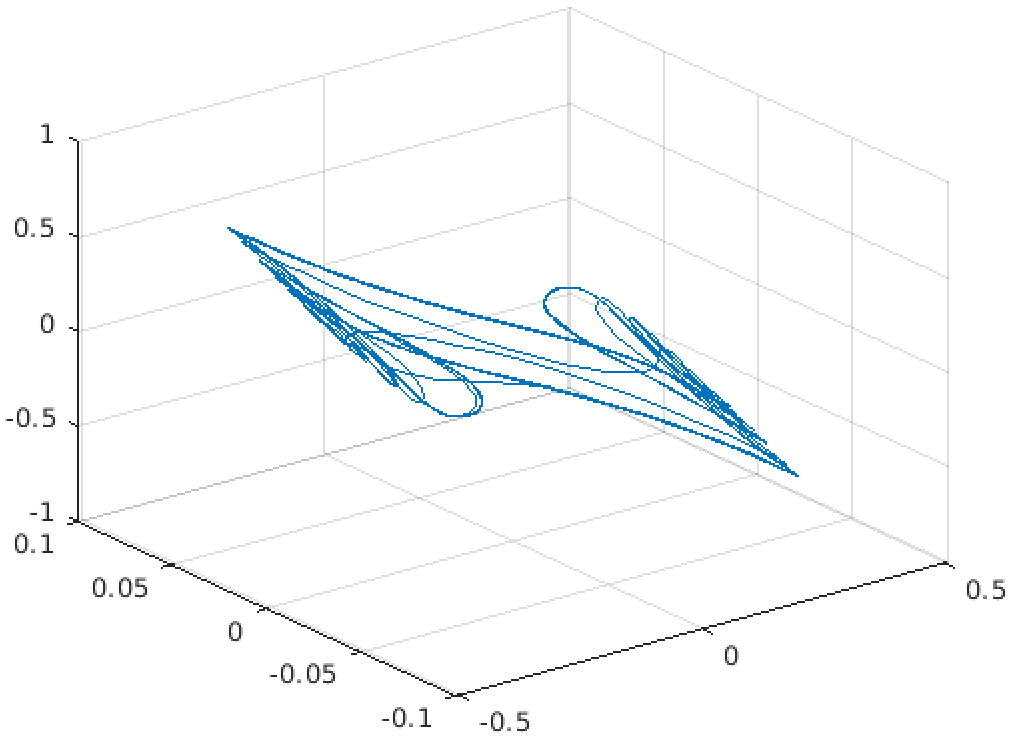
Figure 2(f)

Other nonlinearities which exhibit chaos and parallels with normal chua's circuit

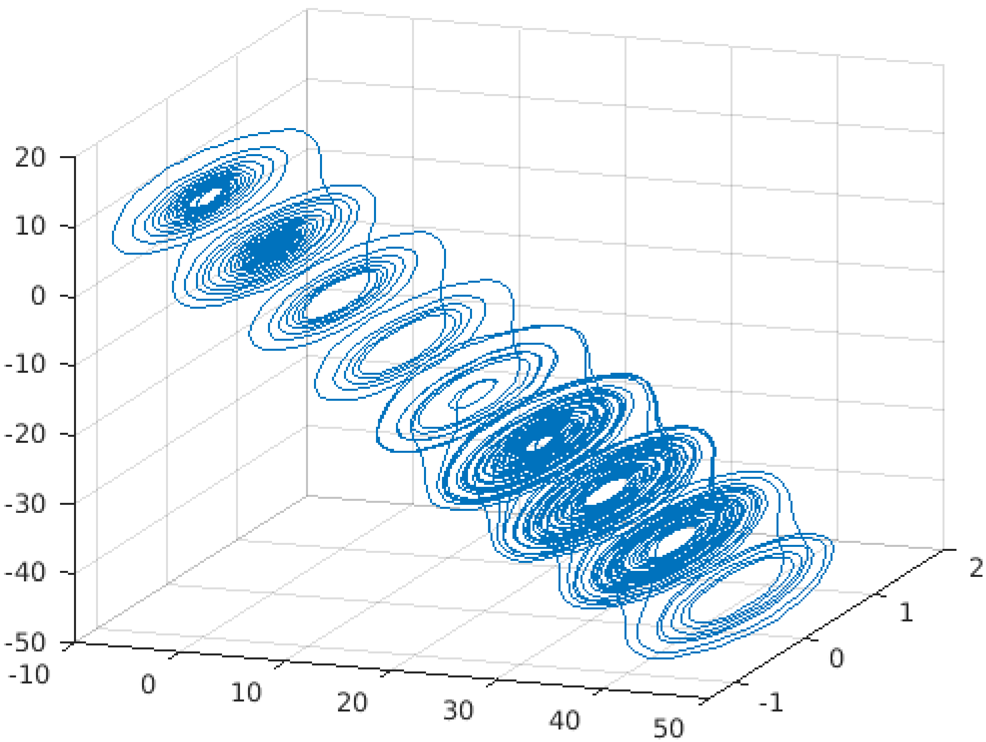


Functions matching the general shape of the piecewise function in the original circuit are shown to exhibit chaos. These include odd-powered polynomials, sin(x), piecewise combinations of other functions, etc. We can observe the chaotic phenomenon as shown below,

(i) x^3-x



(ii) piecewise sin(x)



MATLAB Code

Chua.m

```
function out = chua(t,in)
x = in(1);
y = in(2);
z = in(3);
alpha  = 11;
beta   = 16;
c = -0.142;
h= x^5+c*x
xdot = alpha*(y-h);
ydot = x - y+ z;
zdot  = -beta*y;
out = [xdot ydot zdot]';
```

Chua_sim.m

```
[t,y] = ode45(@chua,[0 300],[-0.03 -0.03 -0.03]);
plot3(y(:,1),y(:,2),y(:,3))
Grid

(Adapted from [6])
```

Conclusion

The Chua's circuit has a wide range of applications, involving secure communication, and in biomedical application (for providing small amplitude random currents). It is therefore beneficial to study the effects of smooth nonlinear resistors on the Chua's circuit since many real world applications have smooth V-I Characteristics. The smooth nonlinearity is observed to showcase all the chaotic phenomena which are shown by the piecewise nonlinearity. In this project we studied the period doubling route to chaos of a Chua's circuit with a Quintic nonlinearity. Similar results as the basic Chua's circuit were obtained.

References

[1] Implementation of Chua's circuit with a Cubic Nonlinearity - Guo Qun Zhong
Implementation of Chua's circuit with a cubic nonlinearity - Circuits and Systems I: Fundamental Theory and Applications, IE
[2] Chua's Equation with Cubic nonlinearity - Anshan Huang, Ladislav Pivka, Chai Wah Wu, Martin Franz
CHUA'S EQUATION WITH CUBIC NONLINEARITY
[3] NPTEL - Chaos, Fractals and Dynamical Systems Lecture 7 - Prof.S.Banerjee IIT Kharagpur
https://www.youtube.com/watch?v=I7e238zjXt0
[4] https://adipandas.github.io/posts/2021/03/fixed-point-high-dim/
[5] Python Applications - Dynamical Systems
https://github.com/springer-math/dynamical-systems-with-applications-using-python
[6] https://www.chuacircuits.com/matlabsim.php