Chapter 7: Quantum Algorithms

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1 Introduction

Circuit complexity is defined as the least number of quantum gates that are needed for implementing a quantum algorithm w.r.t universal quantum gates. Query complexity is the number of times a function is being called to solve a particular problem, and we can define bounds for the same. Quantum oracle is similar to a black box which takes in the inputs and gives the corresponding outputs. Upon measurement of the output, we will get a linear combination of states corresponding to the function *f(x)*.

In a **Phase oracle**, the inputs acquire a phase change of $(-1)^{f(x)}$ while the ancilla qubits remain the same. This is called *phase kickback*. For example, after applying a phase oracle (U_f) on the input qubit - $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, we will get $(-1)^{f(0)}\frac{\sqrt{3}}{2}|0\rangle + (-1)^{f(1)}\frac{1}{2}|1\rangle$. For design of quantum algorithms, we need a quantum register, oracle, measurement.

2 Simple Aglorithms

In **Deutsch's Algorithm**, using quantum algorithm, only one query is required to determine if a univariate function *f(x)* is constant or balanced unlike its classical counterpart where two queries are needed. The function $f : \{0, 1\} \to \{0, 1\}$ can be one of the following possibilities:

1. $f(0) = f(1) = 1$ which is a constant function.
2. $f(0) = f(1) = 0$ which is a constant function.
3. $f(0) = 0, f(1) = 1$ which is a balanced function.
4. $f(0) = 1, f(1) = 0$ which is a balanced function.

Using classical algorithm, it can be done in the following way using *if-else* statements.

```
if f(0) = 0:
   if f(1) = 0:
      print("Constant")
   else:
      print("Balanced")
else:
   if f(1) = 0:
      print("Balanced")
   else:
      print("Constant")
```

Using quantum algorithm, we can tackle this issue for a univariate function in one query although we cannot determine the value of the given function. Let the input register be initialised to $|0\rangle$ while ancilla register be initialised to $|1\rangle$ This can be done in the following steps:

1. Apply Hadamard gate to both input and output registers each of a single qubit. This transforms to -
   $|x\rangle\,|y\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$

2. Upon applying U_f on both the registers, we get $|y\rangle$ as $|y \oplus f(x)\rangle$ and the input register qubits remain the same. If y takes the value of 0, $|y \oplus f(x)\rangle$ transforms to $|f(x)\rangle$, else we get complement of $f(x)$.

   Therefore, $|y\rangle$ transforms to
   $\frac{1}{\sqrt{2}}(|0, f(0)\rangle + |0, \neg f(0)\rangle - |1, f(1)\rangle - |1, \neg f(1)\rangle)$

3. If $f(0) = f(1)$, then we will get $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|f(0) - \neg f(0)\rangle)$, else we will get $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(|f(0) - \neg f(0)\rangle)$. Then apply the hadamard gate only to the input register.

4. Upon measurement of the first register, if we get $|0\rangle$, then f is a constant function, else $|1\rangle$, then f is a balanced function. Measuring of the ancilla register will not give us the value of f.

Deutsch's Algorithm is useful for univariate functions, let us tackle the same issue for a mutlivariate function $f : \{0,1\}^n \rightarrow \{0,1\}$. An algorithm for this is known as **Deutsch-Jozsa Algorithm**.

In **Deutsch-Jozsa Algorithm**, the input register consist of n-qubits initialised to $|0\rangle^{\otimes n}$ and ancilla register as $|1\rangle$. We follow the same procedure as that of the Deutsch's Algorithm. On doing so, we write a general form for the input register n-qubits after applying Hadamard gate on them, as -

$$\frac{1}{2^{\frac{n}{2}}} \sum_{j=0}^{2^n-1} |j\rangle$$

where $|j\rangle = |j_{n-1}.....j_0\rangle$.

Now, applying the oracle, we get the state to be transformed as

$$\frac{1}{2^{\frac{n}{2}}} \sum_{j=0}^{2^n-1} |j\rangle\,|y \oplus f(j)\rangle$$

Upon simplifying this further, we get it as

$$\frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j, f(j) - \neg f(j)\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} (-1)^{f(j)} |j\rangle \frac{1}{\sqrt{2}} |0\rangle - |1\rangle$$

On applying the Hadamard gate on the input n-qubit registers, we will get -

$$H^{\otimes n} |j\rangle = \frac{1}{2^{\frac{n}{2}}} \sum_{k=\{0,1\}^n} (-1)^{k.j} |k\rangle$$

where $k \cdot j$ is $k_0 j_0 + ...k_{n-1} j_{n-1}$ $(mod\ 2)$

The state becomes

$$\frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=\{0,1\}^n}^{2^n-1} (-1)^{k \cdot j + f(j)} |k\rangle |-\rangle$$

In case, the function is balanced, then the amplitude for the case $|k\rangle = |0\rangle^{\otimes n}$ will be zero, then it means that when we measure the input registers we will get any state other than $|00...0\rangle$. If the function is constant, then the amplitude for the state $|00..0\rangle$ will be one.

In case of **Bernstein-Vazirani Problem**, we need to determine the value of a for a function $f = a \cdot x$. The algorithm is same as that of Deutsch-Jozsa Algorithm. The final state before measurement is the following -

$$\frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=\{0,1\}^n}^{2^n-1} (-1)^{(k+a) \cdot j} |k\rangle |-\rangle$$

Only for the case when $a = k$, the probability amplitude of the state will be one, which means for all other states, the probability amplitude will be zero. Hence, we can find the value of the unknown string a. To solve this problem using classical algorithm, we will need to call the function n times where n is the length of the unknown string a using a given input x such that $f(100..00) = a_{n-1}$ and so on.

3 Simon's Algorithm

In **Simon's problem**, we need to find the value of a non-zero unknown string s. Let a function f be $f : \{0,1\}^n \to \{0,1\}^n$ such that $f(x) = f(y)$ if $x = y \oplus s$. If the function is one-to-one, we will get s as $|00..0\rangle$, and if the function is two-to-one, then we will get a non-zero value for s. Let there be two registers consist of n-qubits each. We apply Hadamard gate on the first register and then apply the query function, then we will get the state as

$$\frac{1}{2^{\frac{n}{2}}} \sum_{x=\{0,1\}^n} |x\rangle\,|f(x)\rangle$$

. On measuring the second register, we will get a certain value of $f(x)$, then the first register is reduced to the linear combination of $|x\rangle$ and $|x \oplus s\rangle$, i.e, $\frac{1}{\sqrt{2}}(\ |x\rangle + |x \oplus s\rangle)$ and this happens if f is two-to-one. The measurement of the second register causes the two registers to be entangled with each other. We then pass the first register qubits to the hadamard gate. Then we obtain the state as -

$$\frac{1}{2^{\frac{n+1}{2}}} \sum_{y=\{0,1\}^n} [(-1)^{x\cdot y} + (-1)^{(x\oplus s)\cdot y}]\,|y\rangle$$

If $(-1)^{x\cdot y} = (-1)^{(x\oplus s)\cdot y}$, we get $s \cdot y = 0$, i.e the inner product of s and y must be zero, they must be orthogonal to each other, and on measurement of the first register, we will get a value of y which satisfies this condition. In this case, the state will become -

$$\frac{1}{2^{\frac{n-1}{2}}} \sum_{y=\{0,1\}^n} [(-1)^{x\cdot y}]\,|y\rangle$$

and its probability amplitude is $\frac{1}{2^{n-1}}$. If $(-1)^{x\cdot y}$ is not equal to $(-1)^{(x\oplus s)\cdot y}$, then the coefficient of $|y\rangle$ will be zero and hence its probability amplitude. To find the unknown non-zero string s, we have to run this algorithm n times to obtain linearly independent vectors such that they are orthonormal to s and also we may consider that y is not equal to zero. Upon solving each of those linearly independent vectors by taking inner product with s will give us its value.

4 References

1. Chapter 7 from "Introduction to Classical and Quantum Computing"

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3. Lecture Videos by Professor D K Ghosh
4. Simon's Algorithm from QuIC Seminar
5. Simon's Algorithm - from Stackexchange Discussion
6. Simon's Algorithm and Simple Algorithms
7. Deutsch-Jozsa Algorithm - Nielson and Chuang
8. Simon's Algorithm - Lecture Videos by Advanced Maths