Quantum entanglement and its applications

Introduction:

Correlation is a statistical term used to predict the extent to which two bodies under observation can move in coordination with one another. Quantum correlation is the expected change in physical characteristics as one quantum system passes through an interaction site and is one of the most known quantum communication schemes. Quantum correlation is considered to be a significant area within the quantum information theory and entanglement is one of the important correlations among several other relevant ones. The quantum information theory can be divided into three parts roughly - the first part is the qualitative part which addresses the question “Is this state entangled or not?”, the second part is the comparative part which asks, “Is this state more entangled than that state?” and the third part is the quantitative part which asks, “How entangled state is this state?” and a value is given to every in the form of entanglement measures. The quantative aspect is needed whenever entanglement is used as a resource for quantum information processing tasks. For instance, entangled states act as a major source when it comes to teleportation and dense coding processes.

Using quantum information theory, classical information can be transferred over quantum states with and without security - quantum key distribution and quantum dense-coding protocols, quantum state transfer by using finite amount of classical communication as it is possible in quantum teleportation and factorization of large integers into their corresponding prime numbers using Shor’s Algorithm. These inventions are beneficial for the society such as in case of having a secure cryptography using quantum states. Besides, these can be attributed to the success of quantum information science due to successful experiments being conducted at the laboratories by using photons, ions, etc. Bipartite entangled states shared between two distinctly located parties can be achieved in the laboratories unlike in multiple particle system which is considered to be a challenge. For better performance of quantum computer, quantum error correcting codes than their classical counterparts, entanglement of multiple parties becomes essential.

Entangled States:

- <u>Entanglement definition-</u>

  Let us consider a system of two parties, Alice and Bob be scientists located in distant locations, and be denoted by A and B respectively and quantum states made by them are respectively, $|\psi_A>$ and $|\psi_B>$ belonging to the complex Hilbert Space C_A and C_B . Then the pure state prepared by them using local operations is -

$$|\phi_{AB}> \; = \; |\psi_A> \otimes \; |\psi_B>$$

And, such a pure state is called the product state in $C_A \otimes C_B$ since this joint state shared between Alice and Bob is separable. And, a bipartite pure state is said to be an entangled state provided that it is not possible to be prepared by local operations, i.e -

$$|\phi_{AB}> \; \neq \; |\psi_A> \otimes \; |\psi_B> \text{ in } C_A^2 \otimes C_B^2$$

An example of an entangled state is the singlet state which is given by -

$$|\psi^-> \; = \; \frac{1}{\sqrt{2}}(|01> \; - \; |10>)$$

Where $|0>$ and $|1>$ represent eigenvectors of σ_Z, with σ_α, $\alpha = x, y, z$ being the Pauli Spin matrices.

A bipartite state prepared using quantum mechanically allowed local operations and classical communication (LOCC) is considered to be a separable state and is given by -

$$\rho_{AB} \; = \; \sum_{i=1}^{d} p_i \rho_A^i \; \otimes \; \rho_B^i$$

Where

$$p_i \geq 0 \; for\; all\; i\; and \sum_i p_i \; = \; 1, \; \rho_B^i = |\psi_B^i><\psi_B^i| \; and \; \; \rho_A^i = |\psi_A^i><\psi_A^i| \; for\; i \; = \; 1, .. , d$$

Separable states are quantum states belonging to a composite space that can be factored into individual states belonging to separate subspaces. The problem of deciding whether a state is separable in general is sometimes called the separability problem in quantum information theory. It is considered to be a difficult problem. It has been shown to be NP-hard in many cases.

A state is considered to be entangled if it cannot be written as convex combination of the product of local projectors. A werner state is considered to be an example of an entangled mixed state. As mentioned earlier, to achieve an entanglement for a N-parties system located at distant laboratories becomes a challenging task. To understand this, we can consider a tripartite state shared between Alice (A), Bob (B) and Claire (C) and then their joint shared state is given by -

$$|\phi_{ABC}> \; = \; |\psi_A> \otimes \; |\psi_B> \otimes \; |\psi_C>$$

Which is a fully separable state. Now, if A and B are in an entangled state and they are not entangled with C, then this is considered to be biseparable states and this is given by -

$$|\Psi_{A:BC}\rangle = |\phi_{AB}\rangle|\psi_C\rangle$$

We can also have a pure state which is multipartite entangled if its not a product across any bipartitions and two such states are the Greenberger-Horne-Zeilinger (GHZ) and the W and these are given by -

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

$$|\psi_{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

Also, in case of pure two-qubit entangled states, it is always possible to transform an entangled state to another by LOCC with some non-zero probability. But such an equivalence does not exist in non-bipartite state.

- <u>Entanglement measures-</u>

Entanglement measure quantifies how much entanglement is contained in a quantum state. Formally it is any non-negative real function of a state which can not increase under local operations and classical communication (LOCC), and is zero for separable states. Measurement of entanglement content in case of a bipartite pure state is done by using the vonNeumann entropy of local density matrices of a given state. Therefore, entanglement can be quantified as the following for any bipartite pure state -

$$\mathcal{E}(|\psi_{AB}\rangle) = S(\rho_A),$$

Where S(.) denotes the von-Neumann entropy.
In case of pure states, other known quantum correlations measures (other than entanglement) also reduce to entropy of local density matrices as given in the above equation.

According to the entanglement theory, the basic properties that an entanglement measure should satisfy are the following -

1. For a given state, ρ_{AB} , $E(\rho_{AB}) \geq 0$ and vanishes if ρ_{AB} is separable.
2. If ρ_{AB} transforms to an ensemble $\{p_i , \sigma_A^i\}$ by LOCC, then -

   $$\mathcal{E}(\rho_{AB}) \geq \sum_i p_i \mathcal{E}(\sigma^i_{AB}),$$

   ensuring non-increasing nature of entanglement using LOCC, known as monotonicity property of entanglement.

Based on the geometry of quantum states, a generalized geometric measure can be used. For an N-party pure state, it is defined as -

$$\mathcal{G}(|\psi_{A_1 \ldots A_N}\rangle) = \max(1 - |\langle\phi_{A_1 \ldots A_N}|\psi_{A_1 \ldots A_N}\rangle|^2),$$

where maximization is taken over the set of non-genuinely multiparty entangled states, for instance, in case of three-party, maximization must be done over the set of inseparable states.

References:

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