Presentation-4

Grover's Algorithm and Shor's Code

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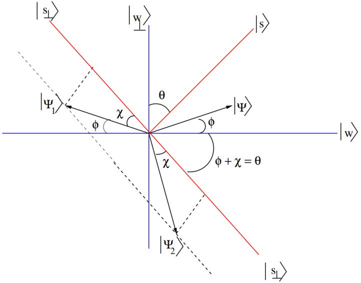
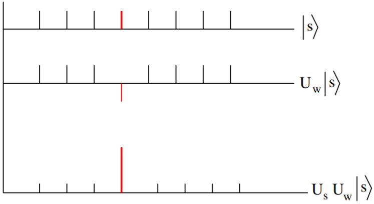
Grover's Algorithm

Procedure

- We apply H-gate on the input register of n-qubits and HX on oracle qubit, and then we apply for Grover's operator(R_g) for $m \approx \frac{\pi}{4}\sqrt{N}$
- $R_g = U_s U_w$ - a product of sign-flip and reflection operators, where $U_s = 2\left|s\right\rangle\left\langle s\right| - I$, where s is the superposition of all basis vectors with equal probabilities.
- U_w is used for changing the sign of the marked state using $U_w = I - 2\left|w\right\rangle\left\langle w\right|$, where w is the marked state.
- After iterating m times, we will measure the input register to get the marked state whose probability is higher than the remaining states. In m times, we have $\left|s\right\rangle$ come closer to $\left|w\right\rangle$ and the angle between them is $\frac{\pi}{2} - \theta$.

Grover's Algorithm
Procedure



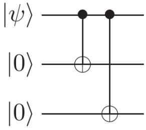


U_s leaves the component of $|\psi\rangle$ along $|s\rangle$ undisturbed but flips the sign of the component perpendicular to $|s\rangle$. The new amplitude becomes $2\bar{a} - a_x$. U_s is defined by the diffusion operator $D = -I + 2\frac{J}{N}$ where J is a N X N matrix where each element is 1.

Quantum Error Correction

Quantum errors are a combination of Z, X gates.
We are unable to directly creat QEC codes same as that of the classical due to -

- No-cloning principle forbids the copying of quantum states.
- Measurement destroys quantum information.
- Quantum states are continuous.



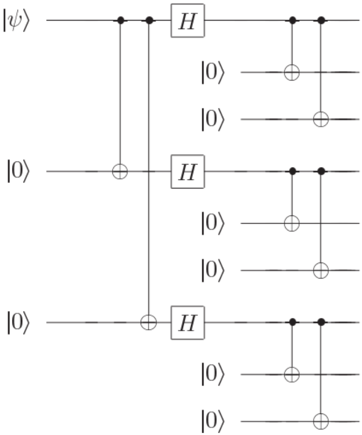
However, we can encode $|0\rangle \rightarrow |000\rangle$ and $|1\rangle \rightarrow |111\rangle$ using the circuit above.

Quantum Error Correction

We can detect error on qubits by comparing the measurement after applying this $Z_j Z_k = (-1)^{b_j+b_k}$ on two consecutive qubits.
For example, $Z_1 Z_2 = 1$ and $Z_2 Z_3 = 1$, then no error has occurred, else if $Z_1 Z_2 = -1$ and $Z_2 Z_3 = -1$, then on qubit-2, error has occurred, else if $Z_1 Z_2 = -1$ and $Z_2 Z_3 = 1$, then on qubit-1, error has occurred or else if $Z_1 Z_2 = 1$ and $Z_2 Z_3 = -1$, then on qubit-3, error has occurred. This related to classical bit parity.
For phase flip, we use the operator - $X_j X_k = (-1)^{b_j+b_k}$.
We can recover the states by applying appropriate operations on the state. For example, for a bit-flip error, we apply X gate and for a phas-flip error, we apply Z gate.

Quantum Error Correction

Shor's code



$$|0\rangle \to |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \to |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Shor Code Circuit Diagram Concatenation Method - Obtained by encoding the single-qubit quantum state using the phase flip code and then encoding each of these qubits using the bit flip code.

Quantum Error Correction

Shor's code

Name	Stabilizer
g_1	$Z_1Z_2I_3I_4I_5I_6I_7I_8I_9$
g_2	$I_1Z_2Z_3I_4I_5I_6I_7I_8I_9$
g_3	$I_1I_2I_3Z_4Z_5I_6I_7I_8I_9$
g_4	$I_1I_2I_3I_4Z_5Z_6I_7I_8I_9$
g_5	$I_1I_2I_3I_4I_5I_6Z_7Z_8I_9$
g_6	$I_1I_2I_3I_4I_5I_6I_7Z_8Z_9$
g_7	$X_1X_2X_3X_4X_5X_6I_7I_8I_9$
g_8	$I_1I_2I_3X_4X_5X_6X_7X_8X_9$

For example, the fourth qubit is flipped and a phase in the second block of the encoded state of $\chi = \alpha|0\rangle + \beta|1\rangle$. Then we apply the bit-flip stabilizers to each of the three blocks (or to all nine qubits, in general). On doing so, we obtain -1 for g_3 and 1 for the remaining first six stabilizers. For this bit-flip, we apply X_4 to the fourth qubit.

Quantum Error Correction

Shor's code

Name	Stabilizer
g_1	$Z_1Z_2I_3I_4I_5I_6I_7I_8I_9$
g_2	$I_1Z_2Z_3I_4I_5I_6I_7I_8I_9$
g_3	$I_1I_2I_3Z_4Z_5I_6I_7I_8I_9$
g_4	$I_1I_2I_3I_4Z_5Z_6I_7I_8I_9$
g_5	$I_1I_2I_3I_4I_5I_6Z_7Z_8I_9$
g_6	$I_1I_2I_3I_4I_5I_6I_7Z_8Z_9$
g_7	$X_1X_2X_3X_4X_5X_6I_7I_8I_9$
g_8	$I_1I_2I_3X_4X_5X_6X_7X_8X_9$

For the phase-flip, we apply the last two stablilizers and we obtain -1 in both the cases, indicating that the phase has been changed in the second block. Therefore, for this, we apply Z gate to any one of the qubits in the second block or to all of them. Below is the table consisting of the eight stabilizers needed to detect and recovering the encoded state.

Using shor code, we can correct any arbitrary errors and this is done by expanding a trace-preserving quantum operation ϵ in an operator-sum representation with operation elements as E_i.
To understand how this works, we can consider only a single term in this sum - $\epsilon(|\chi\rangle\langle\chi|) = \sum_i E_i |\chi\rangle\langle\chi| E_i^\dagger$.
As an operator on a single qubit, E_i can be written as a linear combination I, X, Z and XZ, i.e,
$E_i = e_{i0}I + e_{i1}X_1 + e_{i2}Z_1 + e_{i3}X_1Z_1$. $E_i|\chi\rangle$ is an un-normalized quantum state.
Measuring the error syndrome, the state will collapse into any one of the four states. Depending on the result of the measurement, we can apply relevant operation on the result we obtained to get the original state χ.