Title of Qualification Exam Talk

J. S. Rehak

Qualification Exam
September 4$^{\text{th}}$, 2019

2019-08-20

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Outline

1. Motivation
2. Background
3. Acceleration Methods

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Steady-state Boltzman Transport Equation

Our problem of interest is the time-independent transport equation for a critical system on a domain of interest $\mathbf{r} \in \mathbb{D}, E \in \mathbb{E}$ [1]

$$\left[\hat{\Omega} \cdot \nabla + \Sigma_t(\mathbf{r}, E)\right] \psi(\mathbf{r}, E, \hat{\Omega})$$
$$= \int_0^\infty dE' \int_{4\pi} d\hat{\Omega}' \Sigma_s(\mathbf{r}, E' \to E, \hat{\Omega}' \to \hat{\Omega}) \psi(\mathbf{r}, E', \hat{\Omega}')$$
$$+ Q(\mathbf{r}, E, \hat{\Omega})$$

with a given boundary condition,

$$c_0 \psi(\mathbf{r}, \hat{\Omega}, E) + c_1 \frac{\partial \psi(\mathbf{r}, \hat{\Omega}, E)}{\partial \mathbf{r}} = f(\mathbf{r}, \hat{\Omega}, E), \quad \hat{n} \cdot \hat{\Omega} < 0, \mathbf{r} \in \partial \mathbb{D}$$

Energy discretization

Introduce a discretization of the energy domain $\mathbb{E}$ into G non-overlapping elements, such that

$$E_h = \{E_1, E_2, \ldots, E_G\}, \quad \mathbb{E} = \bigcup_{g=1}^{G} E_g$$

Assume that the energy-dependent angular flux can be separated into a group angular flux and a energy function within each of these groups

$$\psi(\mathbf{r}, E, \hat{\Omega}) \approx \psi_g(\mathbf{r}, \hat{\Omega}) f_g(E), \quad E \in E_g$$

Finally, assume that

$$\int_{E_{g'} \in E_h} f_g(E) dE = \delta_{g,g'}$$

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Self-adjoint angular flux equation (SAAF)

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The Self-adjoint angular flux equation (SAAF) is a second-order from of the transport equation introduced by Morel and McGhee in 1999. To derive, consider scattering term part of tte source. Properties of SAAF

- +Can solve using standard CFEM methods, which give SPD matrices (can use CG instead of GMRES)
- +Full angular flux is obtained by solve (unlike Even/Odd parity)
- +BCs only coupled in one direction when reflective
- -General sparse matrix, not block lower-triangular (no sweeping)
- -Pure scattering causes issues like odd-parity

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Start with the single-group first-order transport equation [3]:

$$\hat{\Omega} \cdot \nabla\psi + \Sigma_t\psi = S\psi + q \ . \quad (1)$$

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$$-\hat{\Omega} \cdot \nabla \frac{1}{\Sigma_t} \hat{\Omega} \cdot \nabla \psi + \Sigma_t \psi = S\psi + q - \hat{\Omega} \cdot \nabla \frac{S\psi + q}{4\pi}$$

With boundary conditions, for all $\mathbf{r} \in \partial D$:

$$\psi = f, \quad \hat{\Omega} \cdot \hat{n} < 0$$

$$\hat{\Omega} \cdot \nabla \psi + \Sigma_t \psi = S\psi + q, \quad \hat{\Omega} \cdot \hat{n} > 0$$

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Nonlinear Diffusion Acceleration (NDA)

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- Uses a lower order diffusion solve to accelerate a higher order solve.
- Start with the same single-group first-order transport equation, multiply by and integrate over angle, giving the "neutron continuity equation."
- We need closure for this problem, so often we use Fick's law, we will introduce a correction onto Fick's Law based on a higher order solve.
- We will introduce an additive correction based on our two definitions of the current.

Nonlinear Diffusion Acceleration (NDA)

Start, again, with the single-group first-order transport equation [2], integrated over angle:

$$\nabla \cdot J_g + \left(\Sigma_{t,g} - \Sigma_s^{g\to g}\right)\phi_g = \sum_{g'\neq g} \Sigma_s^{g'\to g}\phi_{g'} + q_g , \quad J_g \equiv \int d\hat{\Omega}\hat{\Omega}\psi_g \ .$$

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$$J_g = -D\nabla\phi_g \;.$$

Construct an additive correction to the current using information from an angular solve:

$$\begin{aligned} J_g &= -D\nabla\phi_g + J_g^{\mathsf{ang}} - J_g^{\mathsf{ang}} \\ &= -D\nabla\phi_g + \int_{4\pi} d\hat{\Omega}\hat{\Omega}\psi_g + D\nabla\phi_g \end{aligned}$$

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Nonlinear Diffusion Acceleration (NDA)

Fold the additive correction into a *drift-diffusion vector*:

$$\begin{aligned}
J_g &= -D\nabla\phi_g + \int_{4\pi} d\hat{\Omega}\hat{\Omega}\psi_g + D\nabla\phi_g \\
&= -D\nabla\phi_g + \left[\frac{\int_{4\pi} d\hat{\Omega}\hat{\Omega}\psi_g + D\nabla\phi_g}{\phi_g}\right]\phi_g \\
&= -D\nabla\phi_g + \hat{D}_g\phi_g\ .
\end{aligned}$$

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- We combine these corrections into a drift diffusion vector.
- This gives us the LONDA equation, which is just the same integrated transport equation with a corrected current term.
- Presumably, the "higher order" angular solve will have better current information, so we can use it to calculate the drift diffusion vector.

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\end{aligned}
$$

Plugging this into our integrated transport equation gives the low-order non-linear diffusion acceleration equation (LONDA),

$$
\nabla\cdot\left[-D\nabla + \hat{D}_g\right]\phi_g + \left(\Sigma_{t,g} - \Sigma_s^{g\to g}\right)\phi_g = \sum_{g'\neq g}\Sigma_s^{g'\to g}\phi_{g'} + q_g
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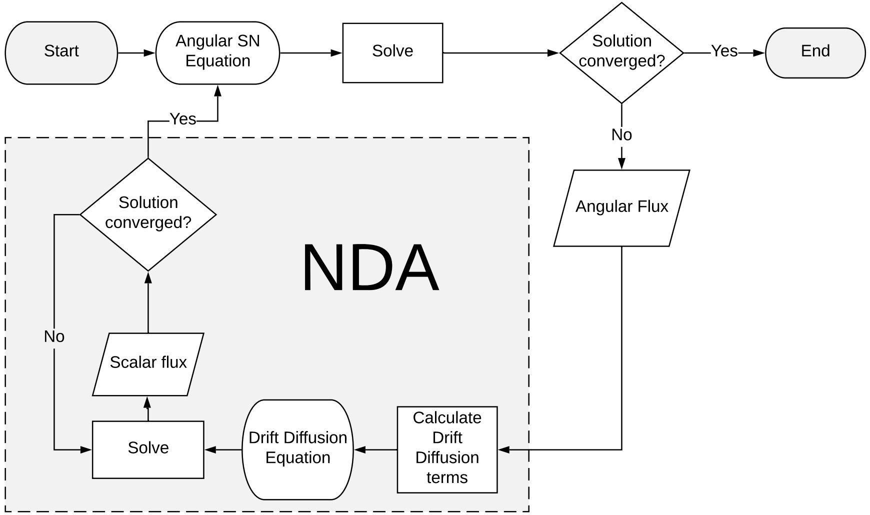
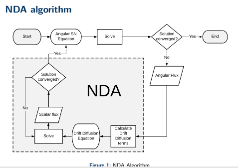
NDA algorithm





- NDA algorithm showing inner low order loop, and outer high order loop.
- In general, outer loop updates both scattering and fission source, checking for k convergence. Inner loop updates fission source, also checking k convergence.

Single-group steady-state transport equation

$$\left[\hat{\Omega} \cdot \nabla + \Sigma_t(\mathbf{r}, E)\right] \psi_g(\mathbf{r}, \hat{\Omega}) \\ = \sum_{g'=1}^{G} \int_{4\pi} d\hat{\Omega}' \Sigma_s(\mathbf{r}, E_{g'} \to E_g, \hat{\Omega}' \to \hat{\Omega}) \psi_{g'}(\mathbf{r}, \hat{\Omega}') + Q_g(\mathbf{r}, \hat{\Omega})$$

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Transport equation second-order forms

Consider the mono-energetic form of the transport equation, using the scattering operator $S\psi(\mathbf{r},\hat{\Omega}) = \int_{4\pi} d\hat{\Omega}'\Sigma_s(\mathbf{r},\hat{\Omega}' \rightarrow \hat{\Omega})\psi(\mathbf{r},\hat{\Omega}')$:

$$\left[\hat{\Omega}\cdot\nabla + \Sigma_t(\mathbf{r})\right]\psi(\mathbf{r},\hat{\Omega}) = S\psi(\mathbf{r},\hat{\Omega}) + Q \tag{2}$$

Substitute $-\hat{\Omega}$ for $\hat{\Omega}$, add to Eq. (2), and divide by two to get a function of even- and odd-parity angular fluxes.

$$\hat{\Omega}\cdot\nabla\psi^- + \Sigma_t\psi^+ = S^+\psi^+ + Q^+$$

where,

$$\psi^+ = \frac{1}{2}\left(\psi(\hat{\Omega}) + \psi(-\hat{\Omega})\right)$$
$$\psi^- = \frac{1}{2}\left(\psi(\hat{\Omega}) - \psi(-\hat{\Omega})\right)$$

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Acknowledgments

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