Nuclear Engineering 150: Midterm 2 Study Guide

Disclaimer: This is not an official study guide. Stuff ~~might~~ **is** wrong. Use the lecture notes and book!

Note: Everything in this guide is from the text () or lecture, or office hours and should be cited as completely as possible.

Contents

1 Neutron Slowing Down

1.1 Lethargy

- Lethargy (u) is a measure of the amount that a neutron has slowed down *relative* to an energy E_0. It is important to note that this is a relative measure, it only tells us about the neutron's energy when compared to our reference value.[1, Lec. 9]

$$u = \ln\left(\frac{E_0}{E}\right) = \ln(E_0) - \ln(E)$$

- As the neutron slows down, relative to E_0, the value of lethargy goes *up*. This is why we call it a measure of the slowing down. This is why we call it lethargy because it's like a measure of the neutron's sleepiness or something; some nuclear engineer obviously thought he was being super cute.[1, Lec. 9]

- Every collision causes a decrease in neutron energy (and increase in neutron lethargy). If the neutron goes from E_i to E_f after a scattering event, we can solve for the difference in the initial and final energies.

$$\Delta u = \ln\left(\frac{E_0}{E_f}\right) - \ln\left(\frac{E_0}{E_i}\right) = \ln\left(\frac{E_i}{E_f}\right)$$

  This is also called the logarithmic energy loss.[1, Lec.9]

1.1.1 Average Logarithmic Energy Loss per collision

- If we want to know the average logarithmic energy loss per collision, we can take the average. We call that squiggle (written ξ). We will use the energy loss per collision from the last midterm, and denote the original energy E and the final energy E':

$$\xi = \overline{\ln\left(\frac{E}{E'}\right)} = \int_{\alpha E}^{E} dE' \ln\left(\frac{E}{E'}\right) p(E \to E') = \int_{\alpha E}^{E} dE' \ln\left(\frac{E}{E'}\right) \frac{1}{(1-\alpha)E}$$

  Finally, you get:

$$\xi = 1 + \frac{\alpha}{1-\alpha}\ln(\alpha)$$

  Note that this doesn't depend on the original energy E. This makes sense, because it's just an average value, we integrated over all the possible energies. In the end, it's just a function of what it's colliding with, because:

$$\alpha = \left(\frac{A-1}{A+1}\right)^2$$

  [1, Lec. 9]

- Note that for hydrogen ($A = 1$), the value of α is 0. This makes ξ undefined ($\ln(0)$ is undefined), so we just set ξ for hydrogen at 1 (there might be a mathematical way to justify this but who cares). This is the highest ξ can be:

$$\xi = 1 = \ln\left(\frac{E}{E'}\right) \to E' = \frac{E}{e}$$

- You might be saying "But study guide, can't the neutron lose *all* of its energy in a collision with a proton? Like billiard balls?" and I'd say "who says billiard balls, they're pool balls, get it together." Remember this is the *average* energy lost per collision so its between the maximum lost (all of it) and the minimum lost (none of it).

- The value of ξ gets smaller and smaller with increasing A; bouncing a ping-pong ball off a basketball won't do much to slow the ping-pong ball down.

$$\xi \approx \frac{2}{A + 2/3}, \text{for } A > 10$$

  Smaller $A \rightarrow$ larger ξ (more effective at slowing). [1, Lec. 9]

- We can use this to figure out the average number of elastic collisions required to go from an energy E_1 to E_2 (Note that we use higher numbers for lower energies, this is pretty standard notation in neutronics; this is why E_0 was our *highest* relative energy when defining lethargy).

$$n = \frac{\Delta u}{\xi} = \frac{\ln\left(\frac{E_1}{E_2}\right)}{\xi}$$

  This is just the total logarithmic difference of our two energies, divided by the logarithmic amount of energy lost in each collision (distance divided by rate).[1, Lec. 9]

1.1.2 ξ for Molecules

- If you have a molecule, you sum over the individual values of ξ, weighted by their cross-sections. This is to take into account that there's a different probability that the neutron will scatter off of the different elements in the molecule.

$$\bar{\xi} = \frac{1}{\Sigma_s} \sum_i \xi_i \Sigma_{si}$$

  Where Σ is the total scattering cross-section for the molecule and Σ_{Si} is the scattering cross-section for each element i.[1, Lec. 9]

- Remember to get the total scattering cross-section for the molecule, you just sum based on how many of each you have. For example, for water:

$$\Sigma_s^{H_2O} = N_{H_2O}(2\sigma^H + \sigma^O)$$

  The number densities should all cancel out, letting you just use microscopic cross-sections.

2 Reactor Criticality

2.1 Neutron Population

- We generally view the neutron population in our reactor in "generations." This is an artificial construct, but is a good way to understand the inner mechanics of a reactor. We model our fission chain reaction as generating a whole bunch of neutrons at once (a generation) that then go on and live their lives. Ultimately, the neutrons will either leak, be absorbed, go on to cause more fission, etc. The fissions will then create the next generation of neutrons.

- Leakage and absorption are examples of loss mechanisms, and fission is a production mechanism. Overall, if we could take the whole loss rate $L(t)$ and the whole production rate $P(t)$, we could calculate the change in neutrons between generations:

$$\frac{dn(t)}{dt} = P(t) - L(t)$$

2.1.1 Multiplication Factor

- The multiplication factor is the ratio of neutrons in the current generation, to those in the last generation.[1, Lec. 10]

$$k \equiv \frac{\text{Number of neutrons in this generation}}{\text{Number of neutrons in the last generation}}$$

- There are three possible situations[1, Lec. 10]:

  $k < 1$: Subcritical; there are less neutrons in this generation than last, population is decreasing

  $k = 1$: Critical: there are the same number of neutrons in this generation as the last, the neutron population is steady.

  $k > 1$: Supercritical: there are more neutrons in this generation than the last, population is increasing.

- Alternatively, we can also define k using the production rate of neutrons and the loss rate:

$$k \equiv \frac{P(t)}{L(t)}$$

  You can see how this has the same three situations discussed above.[1, Lec. 10]

2.1.2 Neutron Population Lifetime

- Using this loss rate $L(t)$, we can figure out how long our generation would survive (with no production). At a given time t, we have $n(t)$ neutrons, so using number over loss rate:

$$\ell \equiv \frac{n(t)}{L(t)}$$

  This is our neutron generation lifetime.[1, Lec 10.]

- We can combine this with our value of k above by examining the change in our neutron population over time using the equation from above:

$$\frac{dn(t)}{dt} = P(t) - L(t) = \left[\frac{P(t)}{L(t)} - 1\right] L(t) = [k-1]\frac{L(t)}{n(t)}n(t) = \frac{k-1}{\ell}n(t)$$

  We can solve this using an initial condition $n(0) = n_0$:

$$n(t) = n_0 \text{Exp}\left(\frac{k-1}{\ell}t\right)$$

2.2 Four and Six Factor Formulas

The four and six factors are values that multiply the number of neutrons in the last generation (n) to get the number of neutrons in the next generation. For this, we only assume that there are two populations of neutrons, fast and thermal.

2.2.1 Those Six Factors

Start with n fast neutrons created from fission.

Fast non-leakage probability There is a chance these fast neutrons will leak out of the reactor. The chance that this *doesn't* happen is P_{FNL}, or the **fast non-leakage probability**. This means that nP_{FNL} fast neutrons are left, and $n(1 - P_{\text{FNL}})$ leak out of the reactor.

Resonance escape probability The fast neutrons that don't leak will collide with the moderator and slow down. While slowing down, they must pass through the resonance region where there is an enhanced chance of capture in the fuel. The probably of *not* being absorbed while slowing down to thermal energies is p, the **resonance escape probability**. Therefore, npP_{FNL} fast neutrons reach thermal energies.

> Resonance escape probability is strongly affected by moderator/fuel ratio and temperature. As temperature rises and the moderator gets less dense (or we have less moderator somehow), neutrons moderate slower, spending more time in the resonance region. Therefore, **p will decrease** because there is a higher probability that they'll be absorbed. Also, as fuel temperature increases, **p will decrease** because Doppler broadening increases the probability of capture.[1, Lec. 10]

Thermal non-leakage probability These thermal neutrons can also escape, just like fast neutrons. Again, the chance this *doesn't* happen is the **thermal non-leakage probability**, P_{TNL}. This gives us $npP_{\text{FNL}}P_{\text{TNL}}$ thermal neutrons that don't leak out.

Thermal utilization factor If a thermal neutron doesn't leak out, it has to go somewhere: it gets absorbed. But, just because the thermal neutron is absorbed, doesn't mean that it's absorbed in the fuel, or *utilized*. It could be absorbed anywhere else in the reactor. This gives rise to the **thermal utilization factor**:

$$f = \frac{\Sigma_a^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{non-fuel}}} = \frac{\propto \text{Probability of absorption in fuel}}{\propto \text{Probability of absorption in anything}}$$

Remember that cross-sections are characteristic of (proportional to) probabilities, so dividing cross sections gives us an actual probability. All f is is the probability that given a neutron is absorbed, it is absorbed in fuel. We knew that our thermal neutrons were absorbed, so we can just tack this onto the end of our growing expression to get the number of thermal neutrons absorbed in the fuel material: $fnpP_{\text{FNL}}P_{\text{TNL}}$.

> **Important Note:** The above formulation assumes you have a homogeneous reactor. That is, the flux in the fuel and non-fuel materials are the same, and the volume of the fuel is the same as the volume of the non-fuel materials. If this isn't the case (heterogeneous), you have to do more. For example, if we have only fuel and moderator of different volumes and with different flux:
>
> $$f = \frac{\Sigma_a^{\text{fuel}} V^{\text{fuel}} \phi^{\text{fuel}}}{\Sigma_a^{\text{fuel}} V^{\text{fuel}} \phi^{\text{fuel}} + \Sigma_a^{\text{mod}} V^{\text{mod}} \phi^{\text{mod}}}$$
>
> Or, we can divide through by $V^{\text{fuel}}\phi^{\text{fuel}}$ to get:
>
> $$f = \frac{\Sigma^{\text{fuel}}}{\Sigma^{\text{fuel}} + \Sigma_a^{\text{mod}} \frac{V^{\text{mod}}}{V^{\text{fuel}}} \frac{\phi^{\text{mod}}}{\phi^{\text{fuel}}}}$$
>
> The important thing to note here is that if $V^{\text{fuel}} > V^{\text{mod}}$ then the fraction is > 1 and f can be higher than for the homogeneous case.

Reproduction Factor Finally, we the number of neutrons that are actually absorbed in fuel and we've almost come full circle. Now we just need to figure out how many neutrons we get out of those absorptions to start the next generation. For this, we use the **reproduction factor**:

$$\eta = \frac{\nu\Sigma_f}{\Sigma_f + \Sigma_\gamma} = \frac{\nu\Sigma_f}{\Sigma_a}$$

These are all cross-sections for the fuel. This isn't exactly equal to the average number of neutrons from fission (ν) because sometimes an absorption event doesn't result in fission. η depends on the fuel, and ranges from 1.34 for natural uranium to 2.08 for pure uranium 235. Now, we have the number of fission neutrons due to thermal fission: $\eta fnpP_{\text{FNL}}P_{\text{TNL}}$.

The difference between η and ν can be confusing:

- ν: The number of fast neutrons per *fission*.
- η: The number of fast neutrons per *thermal neutron absorbed* by fuel. This will always be lower than ν because some amount of neutrons will be absorbed and not cause fission (Σ_a above).

Fast Fission Factor Some of the fast neutrons from the beginning might have caused fission before thermalizing. To account for this, we use a **fast fission factor**. This is essentially a correction to what we ended up with after the last step (the number of fission neutrons due to thermal fission):

$$\epsilon = \frac{\text{Number of fission neutrons due to fast and thermal fission}}{\text{Number of fission neutrons due to thermal fission}}$$

So in the end, we have the total number of neutrons due to both fast and thermal fission from our original population of neutrons n: $\epsilon\eta f n p P_{\text{FNL}} P_{\text{TNL}}$. The value of ϵ ranges from 1.00 to 1.10.

The fast fission factor is affected by the moderator temperature and moderator to fuel ratio. As the moderator temperature rises and the moderator gets less dense, neutrons moderate slower, spending more time fast. This enhances the probability that they'll interact with fuel and create fast fission, so ϵ **will increase.**[1, Lec. 13]

2.2.2 The Six-Factor Formula

Now, we can use this with our definition of the multiplication factor:

$$k \equiv \frac{\text{Number of neutrons in this generation}}{\text{Number of neutrons in the last generation}}$$
$$k = \frac{\epsilon\eta f n p P_{\text{FNL}} P_{\text{TNL}}}{n}$$
$$k = \epsilon\eta f p P_{\text{FNL}} P_{\text{TNL}}$$

This is our six-factor formula. This allows us to multiply these factors together and determine if the reactor is critical ($k = 1$), subcritical ($k < 0$) or supercritical ($k > 0$).

2.2.3 Infinite Reactor (k_∞)

If we have an infinite reactor, there is no leakage: $P_{\text{FNL}} = P_{\text{TNL}} = 1$. Therefore, our equation reduces down to four factors. We call this k_∞ because it's for an infinite reactor.[1, Lec. 10]

$$k_\infty = \epsilon p f \eta$$

Leakage is impossible to avoid unless your reactor is infinite, so k_∞ represents the upper bound of k.[1, Lec. 10]

We can also represent k_∞ as a straight-up ratio of the production rate of neutrons to the absorption rate (our only loss rate)[1, Lec. 13]

$$k_\infty = \frac{\text{Production rate of neutrons}}{\text{Absorption rate of neutrons}} = \frac{\nu R_f}{R_a} = \frac{\nu \Sigma_f \Phi}{\Sigma_a \Phi}$$

I assume that the numerator includes both fast and thermal fission (as k_∞ has ϵ in it) so this formula is somewhat confusing. Then I guess the flux in the denominator has to be broken up as a sum of thermal and fast times their own cross-sections? In reality, I think this equation just plays loosey-goosey with fast fission and kind of ignores it (as it's a small factor). Just make sure you understand what it's saying and why it's weird.

2.2.4 Infinite Homogenous Reactor

An infinite *homogenous* reactor we still have no leakage because the reactor is infinite. The homogenous part means that our reactor is a completely uniform mixture of fuel and moderator.[1, Lec. 10] For whatever reason, we assume this means that all neutrons are thermal neutrons. I like to think of it as the *perfect* mixture for thermalizing. If a neutron isn't thermal, it will always then scatter off a moderator and slow down more. If a neutron *is* thermal, it will always find a fuel nuclide to interact with. What this means is that we don't have any fast fission (all fast neutrons always scatter) and there is no resonance absorption (for the same reason). This means our fast fission factor (ϵ) and resonance escape probability (p) are both 1.[1, Lec. 10, 13]

$$k_\infty = \epsilon p f \eta = f\eta = \frac{\Sigma_a^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{non-fuel}}} \frac{\nu\Sigma_f^{\text{fuel}}}{\Sigma_a^{\text{fuel}}} = \frac{\nu\Sigma_f^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{non-fuel}}}$$

2.3 Energy Dependence

Everything up until this point has assumed that we have only two neutron energies (fast and thermal). If we want to treat these with actual energy, we need to reintroduce the idea of flux weighted cross sections. This is just averaging over the neutron energy spectrum.[1, Lec. 13]

$$\overline{\Sigma}_x = \frac{\int_0^\infty \Sigma_x(E)\Phi(E)dE}{\int_0^\infty \Phi(E)dE}$$

We can use these in that equation for k_∞ that I don't really like.[1, Lec. 13]

$$k_\infty = \frac{\nu R_f}{R_a} = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a}$$

2.3.1 Thermal Disadvantage Factor

Armed with average cross sections, we can go back to the thermal utilization factor (f) for heterogeneous reactors without energy dependence (from the framed section):

$$f = \frac{\Sigma_a^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{mod}} \frac{V^{\text{mod}}}{V^{\text{fuel}}} \frac{\phi^{\text{mod}}}{\phi^{\text{fuel}}}}$$

We can now express these all as energy averaged cross-sections and fluxes by very carefully drawing lines over the sigmas and replacing ϕ with Φ (thank god for find and replace):

$$f = \frac{\overline{\Sigma}_a^{\text{fuel}}}{\overline{\Sigma}_a^{\text{fuel}} + \overline{\Sigma}_a^{\text{mod}} \frac{V^{\text{mod}}}{V^{\text{fuel}}} \frac{\overline{\Phi}^{\text{mod}}}{\overline{\Phi}^{\text{fuel}}}}$$

Remember that this is the *thermal* utilization factor. All the neutrons in these absorptions have been thermalized and didn't leak out, so the fluxes shown are averaged over the thermal energies. We define the **thermal disadvantage factor** ζ as that flux term in the denominator:

$$\zeta \equiv \frac{\overline{\Phi}_{\text{th}}^{\text{mod}}}{\overline{\Phi}_{\text{th}}^{\text{fuel}}}$$

Where "th" has been added to remind us that these are average *thermal* flux.[1, Lec. 13]

2.4 Conversion and Breeding

Fissile isotopes are created in a reactor during the fission process, from the (n,γ) reaction on fertile isotopes. For example, the (n,γ) on uranium 238 (fertile, non-fissile) leads to the creation of plutonium 239 (fissile). The rate at which uranium 238 is converted to plutonium 239 is given by:

$$C = \frac{\Sigma_a^{238}}{\Sigma_a^{235}} + \eta^{235}\epsilon(1-p)P_{\text{FNL}}P_{\text{TNL}}$$

If $C < 1$ then this is still called the conversion factor (C). If $C > 1$, this is the breeding factor (B):

$$B \equiv \frac{\text{Number of new fissile nuclides produced}}{\text{Number of fissile nuclides consumed}}$$

3 Kinetics

Some key assumptions for reactor kinetics are:

- Quantities of interest are averaged over neutron energy.
- Infinite medium (no leakage) without spatial dependence (homogeneous)

[1, Lec. 14]
Using the average neutron speed (in cm/s), $\overline{v}$ and the energy-averaged cross section for a reaction x (discussed in the last section), we can find the reaction rate for that reaction:

$$R_x = \Sigma_x \overline{v} n(t)$$

Where $n(t)$ is the neutron population at time t, and Σ_x is the energy averaged cross section (note we dropped the line, but for all this kinetics stuff it's all energy averaged) [1, Lec. 14].

3.1 Infinite non-multiplying medium

This section is from Lecture 14 [1].

We defined the neutron population as a function of time in the last section:

$$\frac{dn(t)}{dt} = P(t) - L(t)$$

For an infinite, non-multiplying (no fission) medium, the production is only due to sources, and loss is only due to absorption because we are in an finite medium (no leakage, see assumptions above). So replacing the appropriate terms:

$$\begin{aligned}\frac{dn(t)}{dt} &= S(t) - R_a \\ &= S(t) - \Sigma_a \overline{v} n(t)\end{aligned}$$

3.1.1 No Sources

With no sources ($S(t) = 0$), we can solve this using an initial neutron population $n(0) = n_0$:

$$\frac{dn(t)}{dt} = -\Sigma_a \overline{v} n(t)$$

With solution:

$$n(t) = n_0 e^{-\Sigma_a \overline{v} t}$$

This makes sense, because we have no sources, and only absorption. We expect that the neutron population will decay away at a time proportional to the chance of absorption ($\propto \Sigma_a$). The probability that a neutron survives from time 0 to time t without absorption is:

$$\frac{n(t)}{n_0} = e^{-\Sigma_a \overline{v} t}$$

Much like a decay constant, the factor $-\Sigma_a \overline{v}$ gives the probability per unit time that an absorption event will occur. We therefore define something like the mean life time from decay theory but here we call it the **average neutron lifetime**:

$$\ell_\infty = \frac{1}{\Sigma_a \overline{v}}$$

3.1.2 Alternate Derivation of ℓ_∞

We can also find the average neutron lifetime using the probability that a neutron will be absorbed in a small time step dt. From above, we know that the probability that a neutron survives from time 0 to time t without absorption is:

$$\frac{n(t)}{n_0} = e^{-\Sigma_a \overline{v} t}$$

The probability that the neutron will then be absorbed in dt (after living so long from 0 to t) is:

$$p(t)dt = (\Sigma_a \overline{v}) e^{-\Sigma_a \overline{v} t} dt$$

The factor in front is a normalization so if you integrate from 0 to ∞ you get 1, which we need for a probability density function. We can use this to find the expectation value of t:

$$\overline{t} = \int_0^\infty t p(t) dt = \int_0^\infty t (\Sigma_a \overline{v}) e^{-\Sigma_a \overline{v} t} = \frac{1}{\Sigma_a \overline{v}} = \ell_\infty$$

Yay we made it, now we know this is really the average time a neutron will bounce around before being absorbed.

3.1.3 Sources

If we have a source of constant strength: S_0, we get (using our new-fangled average neutron lifetime):

$$\begin{aligned} \frac{dn(t)}{dt} &= S_0 - \Sigma_a \overline{v} n(t) \\ &= S_0 - \frac{1}{\ell_\infty} n(t) \end{aligned}$$

With solution:

$$n(t) = \ell_\infty S_0 (1 - e^{-t/\ell_\infty})$$

This looks a lot like an exponential build-up that at $t = \infty$ goes to a value of $\ell_\infty S_0$.

3.2 Infinite Multiplying Medium

This section is from Lecture 14 [1].

If we have an infinite multiplying medium, our equation becomes a little more complicated.

$$\begin{aligned} \frac{dn(t)}{dt} &= S(t) - L(t) \\ &= S(t) + \nu \Sigma_f \overline{v} n(t) - \Sigma_a \overline{v} n(t) \end{aligned}$$

Note that we still only lose due to absorption (infinite means no leakage) and now we have an extra source term from fission. We can clean this up a bit using our definitions of k_∞ and ℓ_∞:

$$k_\infty = \frac{\nu \Sigma_f}{\Sigma_a}$$

$$\begin{aligned} \frac{dn(t)}{dt} &= S(t) + \Sigma_a \overline{v} n(t) \left(\frac{\nu \Sigma_f}{\Sigma_a} - 1 \right) \\ &= S(t) + \frac{(k_\infty - 1)}{\ell_\infty} n(t) \end{aligned}$$

We saw almost this exact equation earlier, but just with k and ℓ (the neutron generation time), in section 2.1.2. There, we had leakage (see next section), here we don't so we have ∞ on the terms.

3.2.1 No Source

As above, you don't have a source, you end up with:

$$n(t) = n_0 \text{Exp}\left(\frac{k_\infty - 1}{l_\infty} t\right)$$

You can see how k_∞ controls criticality here.

3.2.2 Sources

This isn't in the lecture, but if you do have a time independent source, the solution is found the same was as in the previous section:

$$n(t) = \frac{\ell_\infty S_0}{(k_\infty - 1)} \left[1 - \text{Exp}\left(-\frac{(1 - k_\infty)}{\ell_\infty} t\right)\right]$$

3.3 Finite Multiplying Medium

This section is from Lecture 14 [1].
As previously described, if we have a finite medium, then we have to re-introduce leakage:

$$k = P_{\text{NL}} k_\infty$$
$$\ell = P_{\text{NL}} \ell_\infty$$

Where I assume P_{NL} is just $P_{\text{FNL}} P_{\text{PNL}}$, but it isn't specified. You'd probably pick the right ones for the energy groups that you assume your problem has. This gives us (same as above):

$$\frac{dn(t)}{dt} = S(t) + \frac{(k-1)}{\ell} n(t)$$

3.3.1 No source

No source is exactly the same as above, just get rid of the ∞s.

3.3.2 Source

Solving this equation with $n(0) = 0$ gives us:

$$n(t) = \frac{\ell S_0}{(k-1)} \left[1 - \text{Exp}\left(-\frac{(1-k)}{\ell} t\right)\right]$$

For a subcritical system, $k < 1$, as $t \to \infty$ the exponential is negative and goes to 0:

$$n(\infty) = \frac{\ell S_0}{(k-1)}$$

As k approaches 1, the value $(1 - k)$ gets very small, and you can expand the exponential:

$$n(t) = \frac{\ell S_0}{(k-1)} \left[1 + \frac{(k-1)}{\ell} t + \frac{1}{2} \frac{(k-1)^2}{\ell^2} t^2 + \ldots - 1\right]$$

Which gives us, via some black magic:

$$n(t) = S_o t$$

I think what we do here is cancel the first 1 and the final -1, then if you distribute in the $\frac{\ell}{(k-1)}$, you end up with t plus stuff on the order of $(k-1)$. You can make the argument that because that is a very small quantity, we can just ignore all of them, so you're just left with t. This is speculation.

3.4 Multiplying Medium Takeaways

This section is taken from Lecture 14[1].

You'll notice that the results for the finite case and infinite case are exactly the same, with the addition of some ∞ symbols. The take-aways from both are also the same. There are two conditions in which a time-independent neutron population can be established:

- If you **do not have a source**, you get the exponential decays from above. The only way to make this not decay is if $k = 1$ (or the ∞ variation if you're getting paid by the ∞, which I am not).
- If you **have a source**, you get the buildup equation. If $k < 1$, then you approach a constant value at time ∞ (swear they're not paying me for these). As k approaches ∞, the neutron population blows up. Go reread that section and it'll make sense.

To reiterate, the two cases of a stable neutron population in a multiplying medium are:

1. $k = 1$ and no source.
2. $k < 1$ and a source.

4 Delayed Neutrons

This section is from Lecture 14 [1].

In addition to the prompt neutrons created immediately with fission, there are delayed neutrons that are the result of the decay of fission products. The specifics ones that make these neutrons are called **Delayed Neutron Precursors (DNPs)**. Following fission, these fission products hang around for a bit waiting to β-decay. Once they do, they almost immediately release a neutron. We divide these up into 6 groups (based on their half-lives), not to mention that each group produces a different number of neutrons. The neutrons that come from these Delayed Neutron Precursors are called **delayed neutrons**. If we look at all the neutrons in a particular generation, some of them were born in fission (prompt) and some of them were born from this decay (delayed). We express the fraction of all neutrons born delayed as β:

$$\beta = \frac{\text{Neutrons born delayed}}{\text{Neutrons born delayed and prompt}} = \sum_{i=1}^{6} \beta_i$$

Where β_i is the contribution from the i-th group. Using this, we can weight the half-lives of each of these groups using their values of β_i over the total β. This weighs the DNP groups that make more neutrons more importantly, giving us an average half-life:

$$t_{1/2} = \frac{1}{\beta} \sum_{i=1}^{6} \beta_i t_{i,1/2}$$

We can also define the average decay constant and plug it into the same formula:

$$t_{i,1/2} = \frac{\ln 2}{\lambda_{d,i}}$$

$$\frac{1}{\lambda_d} = \frac{1}{\beta} \sum_{i=1}^{6} \beta_i \frac{1}{\lambda_{d,i}}$$

We can add these into our kinetics equations before, they will provide another source of neutrons by decaying.

$$\frac{dn(t)}{dt} = P(t) - L(t)$$

$$\begin{aligned}\frac{dn(t)}{dt} &= \text{Source} + \text{Prompt} + \text{Delayed} - \text{Absorption} \\ &= S(t) + \frac{(1-\beta)k-1}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t)\end{aligned}$$

Important things to note are that the delayed source term is just from the decay of the i-th group, with concentration C_i. There is an extra $(1-\beta)$ in front of the fission term because we only want prompt neutrons, and β is the fraction born delayed, meaning $(1-\beta)$ is the fraction born prompt. The loses are still just absorption. The values of $C_i(t)$ are given by:

$$\frac{dC_i(t)}{dt} = \beta_i \frac{k}{\ell} n(t) - \lambda_i C_i(t)$$

5 Point Reactor Kinetics Equations

This section is from Lecture 14 [1].

First, we have to define the **mean neutron generation time**:

$$\Lambda \equiv \frac{\ell}{k}$$

This is the mean time between the birth of a neutron and subsequent absorption inducing fission. You can see that when the reactor is critical ($k = 1$), this is equal to the average neutron lifetime (ℓ). As $k > 1$, the value of Λ goes *down*, so neutrons are being absorbed faster. This is a way to adjust the neutron lifetime for different criticality conditions in the core. So we plug that into the equation we found for our neutron population with delayed neutrons in Section 4:

$$\begin{aligned}\frac{dn(t)}{dt} &= S(t) + \frac{(1-\beta)k-1}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\ &= S(t) + \frac{k(1-\beta-1/k)}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\ &= S(t) + \frac{\frac{k-1}{k}-\beta}{\Lambda} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t)\end{aligned}$$

That mess in the numerator is defined as **reactivity:**

$$\rho(t) \equiv \frac{k(t)-1}{k(t)}$$

Which is a measure of the criticality of the reactor:

$\rho > 0$: supercritical

$\rho = 0$: critical

$\rho < 0$: subcritical

Finally, plugging that in and bringing in the equations for C_i give us the **point reactor kinetics equations** (PKEs):

$$\begin{aligned}\frac{dn(t)}{dt} &= S(t) + \frac{\rho(t)-\beta}{\Lambda} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\ \frac{dC_i(t)}{dt} &= \frac{\beta_i}{\Lambda} n(t) - \lambda_i C_i(t) \quad i = 1, 2, \ldots, 6\end{aligned}$$

To get these in terms of power, multiply by $\overline{v}\Sigma_f$ to get the reaction rate for fission, then multiply by w_f, the energy per fission:

$$\frac{dP(t)}{dt} = S(t) + \frac{\rho(t) - \beta}{\Lambda} P(t) + \sum_{i=1}^{6} \lambda_i C_i(t)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda} P(t) - \lambda_i C_i(t) \quad i = 1, 2, \ldots, 6$$

The values of C_i must be different here, because we multiplied the whole thing through by a constant, but no attempt is made to explain this. It's even easier if you replace $\rho(t)$ by just ρ_0 if reactivity isn't a function of time, and drop the independent source $S(t)$. I'm not writing out the equations *again*.

5.1 One Effective Delayed Group

Motivation We'd like to understand how a reactor's power changes over time. To do this, we want to solve the point reactor kinetics equations (PRKEs) from the previous section. We'll be expecting some kind of exponential behavior (because that's what decays give us). In particular, we want to know how our reactor changes in *time*, because these are time-dependent equations and we don't want our reactor to blow up.

This section is taken from Lecture 14 [1].

Remember we had six groups of DNPs? Let's assume instead that we can lump them all together into a single group. In this case, we can get rid of the summations and indexes in the previous equations (also dropping the $S(t)$ from the first equation):

$$\frac{dP(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda} P(t) + \lambda C(t)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} P(t) - \lambda C(t)$$

More assumptions:

- Prior to $t = 0$, the reactor was operating at some steady state power level P_0 with steady state DNP concentration C_0.
- At $t = 0$ we inserted an instantaneous amount of reactivity ρ_0. Or mathematically, replace $\rho(t)$ with:

$$\rho(t) = \begin{cases} 0 & t < 0 \\ \rho_0 & t \geq 0 \end{cases}$$

First step: $t < 0$ First, we solve the above equations for $t < 0$. We know that we were in a steady-state situation; this means that power wasn't changing over time, nor were the amount of DNPs:

$$\frac{dP}{dt} = \frac{dC}{dt} = 0$$

We also know that we assumed we were operating at some steady state power level P_0 and DNP concentration C_0.

$$P(t) = P_0$$
$$C(t) = C_0$$

If you set the equations equal to each other and plug 0 in for $\rho(t)$:

$$-\frac{\beta}{\Lambda} P_0 + \lambda C_0 = \frac{\beta}{\Lambda} P_0 - \lambda C_0$$

$$C_0 = \frac{\beta}{\Lambda \lambda} P_0$$

This gives us the initial condition that we need to solve for $t \geq 0$.

Next Step: $t \geq 0$ For $t \geq 0$, we can use the following initial conditions:

$$P(0) = P_0$$
$$C(0) = \frac{\beta}{\Lambda\lambda} P_0$$

To solve the following one-group PRKE equations:

$$\frac{dP(t)}{dt} = \frac{\rho_0 - \beta}{\Lambda} P(t) + \lambda C(t)$$
$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} P(t) - \lambda C(t)$$

We will use the time-tested differential equation solving technique of knowing the right answer when we start. It's not too ridiculous though. We know that equations that look like $N' \pm cN = 0$ can usually be solved with exponentials of some kind, so we guess exponentials with an arbitrary constant in the exponent that we call s and some arbitrary constants out front P and C.

$$P(t) = Pe^{st} \text{ and } C(t) = Ce^{st}$$

If we substitute these into the one-group PRKE equations above, we get:

$$sP = \left(\frac{\rho_0 - \beta}{\Lambda}\right) P + \lambda C$$
$$sC = \frac{\beta}{\Lambda} P - \lambda C$$

So we have two equations, and two unknowns. Lets bring everything to one side:

$$\left(\frac{\rho_0 - \beta}{\Lambda} - s\right) P - \lambda C = 0$$
$$-\frac{\beta}{\Lambda} P + (\lambda + s) C = 0$$

Now, we have to equations, with two unknowns, and we can put it into a matrix:

$$\begin{bmatrix} \frac{\rho_0 - \beta}{\Lambda} - s & -\lambda \\ -\frac{\beta}{\Lambda} & \lambda + s \end{bmatrix} \begin{bmatrix} P \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only way that this has a solution is if the matrix is singular, i.e. its determinant is 0. This gives us:

$$\left(\frac{\rho_0 - \beta}{\Lambda} - s\right)(\lambda + s) - (-\lambda)\left(-\frac{\beta}{\Lambda}\right) = 0$$
$$\frac{\lambda\rho_0}{\Lambda} - \frac{\beta\lambda}{\Lambda} - s\lambda + \left(\frac{\rho_0 - \beta}{\Lambda}\right) s - s^2 + \frac{\beta\lambda}{\Lambda} = 0$$
$$\Lambda s^2 + (\lambda\Lambda + \rho_0 - \beta) s - \lambda\rho_0 = 0$$

This is a quadratic equation that we can use the quadratic formula to solve, giving us this messy equation:

$$s = \frac{1}{2\Lambda}\left[-(\Lambda\lambda + \beta - \rho_0) \pm \sqrt{(\Lambda\lambda + \beta - \rho_0)^2 + 4\rho_0\Lambda\lambda}\right]$$

The $\pm$ gives us two values of s, which makes sense because it's a quadratic formula. We, very originally, call these s_1 and s_2. Both of these solutions will contribute to our equations for P and C that we guessed earlier:

$$P(t) = P_1 e^{s_1 t} + P_2 e^{s_2 t}$$
$$C(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

Note that we still have no idea what the constants in front of the exponentials are.

5.1.1 Simplified Case

One simplifying assumption is that the value of reactivity is much less than the value of β, $|\rho_0| << \beta$. This means that of the two terms under the square root:

$$(\Lambda\lambda + \beta - \rho_0)^2 >> 4\rho_0\Lambda\lambda$$

because the value of the first term is going to give us a β^2, which – because of our assumption above – is going to be *much* larger than the second term, whose value is controlled by ρ_0. This gives us the following two values of s:

$$s_1 \approx \frac{\lambda\rho_0}{\beta - \rho_0} \qquad s_2 \approx \frac{\beta - \rho_0}{\Lambda}$$

We still don't know anything about the constants in front of our exponentials, but we can use these values of s to write:

$$P(t) = P_1 e^{\frac{\lambda\rho_0}{\beta-\rho_0}t} + P_2 e^{\frac{\beta-\rho_0}{\Lambda}}$$

$$C(t) = C_1 e^{\frac{\lambda\rho_0}{\beta-\rho_0}t} + C_2 e^{\frac{\beta-\rho_0}{\Lambda}}$$

5.2 In-Hour Equation

This section is taken from Lecture 15. [1]

5.2.1 One DNP Group

The values of s that we found are particularly interesting, and not just because we could actually solve for them. The values s in the exponential determines if P and C blow up (if positive) or decay away (if negative). That's the difference between a bomb and a shutdown reactor. Let's go back to that quadratic equation we had:

$$\Lambda s^2 + (\lambda\Lambda + \rho_0 - \beta)s - \lambda\rho_0 = 0$$

As the reactor conditions change (supercritical, subcritical), the values of s will change too: our reactivity ρ_0 and mean generation time Λ both change when k changes. It would be interesting to see what values of s we get for different reactor conditions. So, we can solve for s in terms of reactivity ρ_0:

$$\rho_0 = \frac{s\ell}{s\ell + 1} + \frac{1}{s\ell + 1}\frac{s\beta}{s + \lambda}$$

Now, if we know our reactors reactivity $\rho_0 = \frac{k-1}{k}$, we can solve for the values of s and figure out how our reactor power P (and C for that matter) changes over time. It's easiest to solve this equation graphically, but first we can examine some of the properties of the equation itself:

- This equation just came from the manipulation of a quadratic equation. So we know that there will be **two** values of s for every ρ_0 that we select.
- The denominator of the first fraction goes to 0 as $s \to -\frac{1}{\ell}$, sending it to ∞, so there must be a vertical asymptote there.
- The denominator of the second term blows up also as $s \to -\lambda$. So there's also a vertical asymptote there.
- If you set $\rho_0 = 0$, then $s = 0$ is a solution. This is actually easier to see from the quadratic form above. Therefore, no matter what the values of λ or β, the graph will always cross at the origin (when plotting ρ_0 vs. s).

If you look at the graph, you can see each of those properties. For any chosen value of ρ_0, there are two possible values of s, which is what we expected from our quadratic equation. One of the values will always be somewhere between $-\frac{1}{\ell}$ and $-\lambda$, and therefore *always* negative. The second value will always be greater than $-\lambda$ but can be negative *or* positive. As a reminder, these two values of s are what will dictate the evolution of power and DNP concentration with time:

$$P(t) = P_1 e^{s_1 t} + P_2 e^{s_2 t}$$
$$C(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

Interpreting this graph is aided by looking at three cases for ρ_0:

Case 1: $\rho_0 = 0$ When $\rho_0 = 0$, we know that one of the values of s is 0 (because it crosses the origin. The second is still a negative number between $-\frac{1}{\ell}$ and $-\lambda$. This means we get a power equation that looks like:

$$\begin{aligned} P(t) &= P_1 e^{s_1 t} + P_2 e^{s_2 t} \\ &= P_1 e^{0} + P_2 e^{-|s_2| t} \\ &= P_1 + P_2 e^{-|s_2| t} \end{aligned}$$

As you can see, our power equation has become a constant plus a term that will decay away over time. This is what we expect, because we chose $\rho_0 = 0$ which we know is a **critical** system.

Case 2: $\rho_0 < 0$ When ρ_0 is less than zero, we are in a **subcritical** system and therefore expect all of our power to go away over time. Plotting a line where $\rho_0 < 0$, we can see that our value of s_1 will be some negative number between $-\lambda$ and 0. We know s_2 is always negative, giving us a power equation:

$$\begin{aligned} P(t) &= P_1 e^{s_1 t} + P_2 e^{s_2 t} \\ &= P_1 e^{-|s_1| t} + P_2 e^{-|s_2| t} \end{aligned}$$

As we expected, both terms decay away, and power goes to 0 at $t = \infty$.

Case 3: $\rho_0 > 0$ When ρ_0 is greater than zero, we have a **supercritical** system and expected our power to go to infinity over time. This time when we plot a line for $\rho_0 > 0$, we see that our value of s_1 is positive. This gives us:

$$\begin{aligned} P(t) &= P_1 e^{s_1 t} + P_2 e^{s_2 t} \\ &= P_1 e^{|s_1| t} + P_2 e^{-|s_2| t} \end{aligned}$$

Now we have a term that goes to infinity as $t \to \infty$, which is what we expected.

5.2.2 Reactor Period

The value of s is in units of per time, so we can invert it to get a time period that we can the **reactor period T**:

$$T = \frac{1}{|s|}$$

This period defines how quickly the reactor power will change over time. As ρ_0 gets large (negative or positive) the value of this period gets smaller, and power changes more rapidly.

- **Critical:** When critical $s_1 = 0$ and $T = \infty$ because we have a stable power, so it will never change.

- **Supercritical:** As $\rho_0 \to 1$ (it's maximum value), there is a horizontal asymptote, and therefore $s_1 \to \infty$. This gives us a reactor period of 0 seconds, or a bomb.
- **Subcritical:** As ρ_0 gets very largely negative, we approach the value of the first vertical asymptote: as $\rho_0 \to -\infty$, $s_1 \to -\lambda$. This means that no matter how much negative reactivity we put into our reactor, we can never shutdown faster than on the period $T = \frac{1}{\lambda}$; that group of DNPs is spitting out neutrons, and won't decay away any faster than their decay constant.

5.3 Six DNP Groups

If we go back to our formulation in which we included all 6 DNP groups:

$$\frac{dP(t)}{dt} = S(t) + \frac{\rho(t) - \beta}{\Lambda} P(t) + \sum_{i=1}^{6} \lambda_i C_i(t)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda} P(t) - \lambda_i C_i(t) \quad i = 1, 2, \ldots, 6$$

We can again solve by using our exponential form:

$$P(t) = \sum_{j=1}^{7} P_j e^{s_j t}$$

Where we have $j = 1, 2, 3 \ldots, 7$ instead of just 1 and 2, now that we have all six DNP groups. When we solve this, we get:

$$\rho_0 = \frac{s\ell}{s\ell + 1} + \frac{1}{s\ell + 1} \sum_{i=1}^{6} \frac{s\beta_i}{s + \lambda_i}$$

Instead of a quadratic equation, this one came from a 7th degree polynomial, so there will be 7 values of s. We expected that from the form our power equation took. The behavior is very similar to the one-group version. We still have vertical asymptotes, one of which is at $s = -\frac{1}{\ell}$. Due to the summation, we now also have asymptotes at each λ_i of each DNP group.

The overall behavior is very similar to the behavior of the one group. Now we have seven values of s, six of which are always negative. The largest value acts identically as in the one-group version — positive for supercritical, negative for subcritical, 0 for critical.

6 Diffusion Approximation

References

[1] Jasmine Vujic. Nuclear engineering class lectures. Spring 2015.