# Nuclear Engineering 150: Midterm 2 Study Guide

**Disclaimer:** This is not an official study guide. Stuff ~~might~~ **is** wrong. Use the lecture notes and book!

**Note:** Everything in this guide is from the text () or lecture, or office hours and should be cited as completely as possible.

# Contents



# 1 Neutron Slowing Down

## 1.1 Lethargy

- Lethargy ($u$) is a measure of the amount that a neutron has slowed down *relative* to an energy $E_0$. It is important to note that this is a relative measure, it only tells us about the neutron's energy when compared to our reference value.[1, Lec. 9]

$$u = \ln\left(\frac{E_0}{E}\right) = \ln(E_0) - \ln(E)$$

- As the neutron slows down, relative to $E_0$, the value of lethargy goes *up*. This is why we call it a measure of the slowing down. This is why we call it lethargy because it's like a measure of the neutron's sleepiness or something; some nuclear engineer obviously thought he was being super cute.[1, Lec. 9]
- Every collision causes a decrease in neutron energy (and increase in neutron lethargy). If the neutron goes from $E_i$ to $E_f$ after a scattering event, we can solve for the difference in the initial and final energies.

  $$\Delta u = \ln\left(\frac{E_0}{E_f}\right) - \ln\left(\frac{E_0}{E_i}\right) = \ln\left(\frac{E_i}{E_f}\right)$$

  This is also called the logarithmic energy loss.[1, Lec.9]

#### 1.1.1 Average Logarithmic Energy Loss per collision

- If we want to know the average logarithmic energy loss per collision, we can take the average. We call that squiggle (written $\xi$). We will use the energy loss per collision from the last midterm, and denote the original energy $E$ and the final energy $E'$:

  $$\xi = \overline{\ln\left(\frac{E}{E'}\right)} = \int_{\alpha E}^{E} dE' \ln\left(\frac{E}{E'}\right) p(E \to E') = \int_{\alpha E}^{E} dE' \ln\left(\frac{E}{E'}\right) \frac{1}{(1-\alpha)E}$$

  Finally, you get:

  $$\xi = 1 + \frac{\alpha}{1-\alpha}\ln(\alpha)$$

  Note that this doesn't depend on the original energy $E$. This makes sense, because it's just an average value, we integrated over all the possible energies. In the end, it's just a function of what it's colliding with, because:

  $$\alpha = \left(\frac{A-1}{A+1}\right)^2$$

  [1, Lec. 9]
- Note that for hydrogen ($A = 1$), the value of $\alpha$ is 0. This makes $\xi$ undefined (ln(0) is undefined), so we just set $\xi$ for hydrogen at 1 (there might be a mathematical way to justify this but who cares). This is the highest $\xi$ can be:

  $$\xi = 1 = \ln\left(\frac{E}{E'}\right) \to E' = \frac{E}{e}$$

- You might be saying "But study guide, can't the neutron lose *all* of its energy in a collision with a proton? Like billiard balls?" and I'd say "who says billiard balls, they're pool balls, get it together." Remember this is the *average* energy lost per collision so its between the maximum lost (all of it) and the minimum lost (none of it).
- The value of $\xi$ gets smaller and smaller with increasing $A$; bouncing a ping-pong ball off a basketball won't do much to slow the ping-pong ball down.

  $$\xi \approx \frac{2}{A + 2/3}, \text{for } A > 10$$

  Smaller $A \to$ larger $\xi$ (more effective at slowing). [1, Lec. 9]

- We can use this to figure out the average number of elastic collisions required to go from an energy $E_1$ to $E_2$ (Note that we use higher numbers for lower energies, this is pretty standard notation in neutronics; this is why $E_0$ was our *highest* relative energy when defining lethargy).

$$n = \frac{\Delta u}{\xi} = \frac{\ln\left(\frac{E_1}{E_2}\right)}{\xi}$$

This is just the total logarithmic difference of our two energies, divided by the logarithmic amount of energy lost in each collision (distance divided by rate).[1, Lec. 9]

#### 1.1.2 $\xi$ for Molecules

- If you have a molecule, you sum over the individual values of $\xi$, weighted by their cross-sections. This is to take into account that there's a different probability that the neutron will scatter off of the different elements in the molecule.

$$\bar{\xi} = \frac{1}{\Sigma_s}\sum_i \xi_i \Sigma_{si}$$

Where $\Sigma$ is the total scattering cross-section for the molecule and $\Sigma_{Si}$ is the scattering cross-section for each element $i$.[1, Lec. 9]

- Remember to get the total scattering cross-section for the molecule, you just sum based on how many of each you have. For example, for water:

$$\Sigma_s^{H_2O} = N_{H_2O}(2\sigma^H + \sigma^O)$$

The number densities should all cancel out, letting you just use microscopic cross-sections.

# 2 Reactor Criticality

## 2.1 Neutron Population

- We generally view the neutron population in our reactor in "generations." This is an artificial construct, but is a good way to understand the inner mechanics of a reactor. We model our fission chain reaction as generating a whole bunch of neutrons at once (a generation) that then go on and live their lives. Ultimately, the neutrons will either leak, be absorbed, go on to cause more fission, etc. The fissions will then create the next generation of neutrons.

- Leakage and absorption are examples of loss mechanisms, and fission is a production mechanism. Overall, if we could take the whole loss rate $L(t)$ and the whole production rate $P(t)$, we could calculate the change in neutrons between generations:

$$\frac{dn(t)}{dt} = P(t) - L(t)$$

#### 2.1.1 Multiplication Factor

- The multiplication factor is the ratio of neutrons in the current generation, to those in the last generation.[1, Lec. 10]

$$k \equiv \frac{\text{Number of neutrons in this generation}}{\text{Number of neutrons in the last generation}}$$

- There are three possible situations[1, Lec. 10]:

  $k < 1$: Subcritical; there are less neutrons in this generation than last, population is decreasing

  $k = 1$: Critical: there are the same number of neutrons in this generation as the last, the neutron population is steady.

$k > 1$: Supercritical: there are more neutrons in this generation than the last, population is increasing.

- Alternatively, we can also define $k$ using the production rate of neutrons and the loss rate:

$$k \equiv \frac{P(t)}{L(t)}$$

You can see how this has the same three situations discussed above.[1, Lec. 10]

### 2.1.2 Neutron Population Lifetime

- Using this loss rate $L(t)$, we can figure out how long our generation would survive (with no production). At a given time $t$, we have $n(t)$ neutrons, so using number over loss rate:

$$\ell \equiv \frac{n(t)}{L(t)}$$

This is our neutron generation lifetime.[1, Lec 10.]

- We can combine this with our value of $k$ above by examining the change in our neutron population over time using the equation from above:

$$\frac{dn(t)}{dt} = P(t) - L(t) = \left[\frac{P(t)}{L(t)} - 1\right] L(t) = [k-1]\frac{L(t)}{n(t)}n(t) = \frac{k-1}{\ell}n(t)$$

We can solve this using an initial condition $n(0) = n_0$:

$$n(t) = n_0 \text{Exp}\left(\frac{k-1}{\ell}t\right)$$

## 2.2 Four and Six Factor Formulas

The four and six factors are values that multiply the number of neutrons in the last generation ($n$) to get the number of neutrons in the next generation. For this, we only assume that there are two populations of neutrons, fast and thermal.

### 2.2.1 Those Six Factors

Start with $n$ fast neutrons created from fission.

**Fast non-leakage probability** There is a chance these fast neutrons will leak out of the reactor. The chance that this *doesn't* happen is $P_{\text{FNL}}$, or the **fast non-leakage probability**. This means that $nP_{\text{FNL}}$ fast neutrons are left, and $n(1 - P_{\text{FNL}})$ leak out of the reactor.

**Resonance escape probability** The fast neutrons that don't leak will collide with the moderator and slow down. While slowing down, they must pass through the resonance region where there is an enhanced chance of capture in the fuel. The probably of *not* being absorbed while slowing down to thermal energies is $p$, the **resonance escape probability**. Therefore, $npP_{\text{FNL}}$ fast neutrons reach thermal energies.

> Resonance escape probability is strongly affected by moderator/fuel ratio and temperature. As temperature rises and the moderator gets less dense (or we have less moderator somehow), neutrons moderate slower, spending more time in the resonance region. Therefore, **p will decrease** because there is a higher probability that they'll be absorbed. Also, as fuel temperature increases, **p will decrease** because Doppler broadening increases the probability of capture.[1, Lec. 10]

**Thermal non-leakage probability** These thermal neutrons can also escape, just like fast neutrons. Again, the chance this *doesn't* happen is the **thermal non-leakage probability**, $P_{\text{TNL}}$. This gives us $npP_{\text{FNL}}P_{\text{TNL}}$ thermal neutrons that don't leak out.

**Thermal utilization factor** If a thermal neutron doesn't leak out, it has to go somewhere: it gets absorbed. But, just because the thermal neutron is absorbed, doesn't mean that it's absorbed in the fuel, or *utilized*. It could be absorbed anywhere else in the reactor. This gives rise to the **thermal utilization factor**:

$$f = \frac{\Sigma_a^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{non-fuel}}} = \frac{\propto \text{Probability of absorption in fuel}}{\propto \text{Probability of absorption in anything}}$$

Remember that cross-sections are characteristic of (proportional to) probabilities, so dividing cross sections gives us an actual probability. All $f$ is is the probability that given a neutron is absorbed, it is absorbed in fuel. We knew that our thermal neutrons were absorbed, so we can just tack this onto the end of our growing expression to get the number of thermal neutrons absorbed in the fuel material: $fnpP_{\text{FNL}}P_{\text{TNL}}$.

**Important Note:** The above formulation assumes you have a homogeneous reactor. That is, the flux in the fuel and non-fuel materials are the same, and the volume of the fuel is the same as the volume of the non-fuel materials. If this isn't the case (heterogeneous), you have to do more. For example, if we have only fuel and moderator of different volumes and with different flux:

$$f = \frac{\Sigma_a^{\text{fuel}} V^{\text{fuel}} \phi^{\text{fuel}}}{\Sigma_a^{\text{fuel}} V^{\text{fuel}} \phi^{\text{fuel}} + \Sigma_a^{\text{mod}} V^{\text{mod}} \phi^{\text{mod}}}$$

Or, we can divide through by $V^{\text{fuel}}\phi^{\text{fuel}}$ to get:

$$f = \frac{\Sigma^{\text{fuel}}}{\Sigma^{\text{fuel}} + \Sigma_a^{\text{mod}} \frac{V^{\text{mod}}}{V^{\text{fuel}}} \frac{\phi^{\text{mod}}}{\phi^{\text{fuel}}}}$$

The important thing to note here is that if $V^{\text{fuel}} > V^{\text{mod}}$ then the fraction is $> 1$ and $f$ can be higher than for the homogeneous case.

**Reproduction Factor** Finally, we the number of neutrons that are actually absorbed in fuel and we've almost come full circle. Now we just need to figure out how many neutrons we get out of those absorptions to start the next generation. For this, we use the **reproduction factor**:

$$\eta = \frac{\nu \Sigma_f}{\Sigma_f + \Sigma_\gamma} = \frac{\nu \Sigma_f}{\Sigma_a}$$

These are all cross-sections for the fuel. This isn't exactly equal to the average number of neutrons from fission ($\nu$) because sometimes an absorption event doesn't result in fission. $\eta$ depends on the fuel, and ranges from 1.34 for natural uranium to 2.08 for pure uranium 235. Now, we have the number of fission neutrons due to thermal fission: $\eta fnpP_{\text{FNL}}P_{\text{TNL}}$.

The difference between $\eta$ and $\nu$ can be confusing:

- $\nu$: The number of fast neutrons per *fission*.
- $\eta$: The number of fast neutrons per *thermal neutron absorbed* by fuel. This will always be lower than $\nu$ because some amount of neutrons will be absorbed and not cause fission ($\Sigma_a$ above).

**Fast Fission Factor** Some of the fast neutrons from the beginning might have caused fission before thermalizing. To account for this, we use a **fast fission factor**. This is essentially a correction to what we ended up with after the last step (the number of fission neutrons due to thermal fission):

$$\epsilon = \frac{\text{Number of fission neutrons due to fast and thermal fission}}{\text{Number of fission neutrons due to thermal fission}}$$

So in the end, we have the total number of neutrons due to both fast and thermal fission from our original population of neutrons $n$: $\epsilon\eta f n p P_{\text{FNL}} P_{\text{TNL}}$. The value of $\epsilon$ ranges from 1.00 to 1.10.

The fast fission factor is affected by the moderator temperature and moderator to fuel ratio. As the moderator temperature rises and the moderator gets less dense, neutrons moderate slower, spending more time fast. This enhances the probability that they'll interact with fuel and create fast fission, so $\epsilon$ **will increase.**[1, Lec. 13]

### 2.2.2 The Six-Factor Formula

Now, we can use this with our definition of the multiplication factor:

$$k \equiv \frac{\text{Number of neutrons in this generation}}{\text{Number of neutrons in the last generation}}$$
$$k = \frac{\epsilon\eta f n p P_{\text{FNL}} P_{\text{TNL}}}{n}$$
$$k = \epsilon\eta f p P_{\text{FNL}} P_{\text{TNL}}$$

This is our six-factor formula. This allows us to multiply these factors together and determine if the reactor is critical ($k = 1$), subcritical ($k < 0$) or supercritical ($k > 0$).

### 2.2.3 Infinite Reactor ($k_\infty$)

If we have an infinite reactor, there is no leakage: $P_{\text{FNL}} = P_{\text{TNL}} = 1$. Therefore, our equation reduces down to four factors. We call this $k_\infty$ because it's for an infinite reactor.[1, Lec. 10]

$$k_\infty = \epsilon p f \eta$$

Leakage is impossible to avoid unless your reactor is infinite, so $k_\infty$ represents the upper bound of $k$.[1, Lec. 10]

We can also represent $k_\infty$ as a straight-up ratio of the production rate of neutrons to the absorption rate (our only loss rate)[1, Lec. 13]

$$k_\infty = \frac{\text{Production rate of neutrons}}{\text{Absorption rate of neutrons}} = \frac{\nu R_f}{R_a} = \frac{\nu \Sigma_f \Phi}{\Sigma_a \Phi}$$

I assume that the numerator includes both fast and thermal fission (as $k_\infty$ has $\epsilon$ in it) so this formula is somewhat confusing. Then I guess the flux in the denominator has to be broken up as a sum of thermal and fast times their own cross-sections? In reality, I think this equation just plays loosey-goosey with fast fission and kind of ignores it (as it's a small factor). Just make sure you understand what it's saying and why it's weird.

### 2.2.4 Infinite Homogenous Reactor

An infinite *homogenous* reactor we still have no leakage because the reactor is infinite. The homogenous part means that our reactor is a completely uniform mixture of fuel and moderator.[1, Lec. 10] For whatever reason, we assume this means that all neutrons are thermal neutrons. I like to think of it as the *perfect* mixture for thermalizing. If a neutron isn't thermal, it will always then scatter off a moderator and slow down more. If a neutron *is* thermal, it will always find a fuel nuclide to interact with. What this means is that we don't have any fast fission (all fast neutrons always scatter) and there is no resonance absorption (for the same reason). This means our fast fission factor ($\epsilon$) and resonance escape probability ($p$) are both 1.[1, Lec. 10, 13]

$$k_\infty = \epsilon p f \eta = f\eta = \frac{\Sigma_a^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{non-fuel}}} \frac{\nu \Sigma_f^{\text{fuel}}}{\Sigma_a^{\text{fuel}}} = \frac{\nu \Sigma_f^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{non-fuel}}}$$

## 2.3 Energy Dependence

Everything up until this point has assumed that we have only two neutron energies (fast and thermal). If we want to treat these with actual energy, we need to reintroduce the idea of flux weighted cross sections. This is just averaging over the neutron energy spectrum.[1, Lec. 13]

$$\overline{\Sigma}_x = \frac{\int_0^\infty \Sigma_x(E)\Phi(E)dE}{\int_0^\infty \Phi(E)dE}$$

We can use these in that equation for $k_\infty$ that I don't really like.[1, Lec. 13]

$$k_\infty = \frac{\nu R_f}{R_a} = \frac{\nu \overline{\Sigma}_f}{\overline{\Sigma}_a}$$

### 2.3.1 Thermal Disadvantage Factor

Armed with average cross sections, we can go back to the thermal utilization factor ($f$) for heterogeneous reactors without energy dependence (from the framed section):

$$f = \frac{\Sigma_a^{\text{fuel}}}{\Sigma_a^{\text{fuel}} + \Sigma_a^{\text{mod}} \frac{V^{\text{mod}}}{V^{\text{fuel}}} \frac{\phi^{\text{mod}}}{\phi^{\text{fuel}}}}$$

We can now express these all as energy averaged cross-sections and fluxes by very carefully drawing lines over the sigmas and replacing $\phi$ with $\Phi$ (thank god for find and replace):

$$f = \frac{\overline{\Sigma}_a^{\text{fuel}}}{\overline{\Sigma}_a^{\text{fuel}} + \overline{\Sigma}_a^{\text{mod}} \frac{V^{\text{mod}}}{V^{\text{fuel}}} \frac{\overline{\Phi}^{\text{mod}}}{\overline{\Phi}^{\text{fuel}}}}$$

Remember that this is the *thermal* utilization factor. All the neutrons in these absorptions have been thermalized and didn't leak out, so the fluxes shown are averaged over the thermal energies. We define the **thermal disadvantage factor** $\zeta$ as that flux term in the denominator:

$$\zeta \equiv \frac{\overline{\Phi}_{\text{th}}^{\text{mod}}}{\overline{\Phi}_{\text{th}}^{\text{fuel}}}$$

Where "th" has been added to remind us that these are average *thermal* flux.[1, Lec. 13]

## 2.4 Conversion and Breeding

Fissile isotopes are created in a reactor during the fission process, from the (n,$\gamma$) reaction on fertile isotopes. For example, the (n,$\gamma$) on uranium 238 (fertile, non-fissile) leads to the creation of plutonium 239 (fissile). The rate at which uranium 238 is converted to plutonium 239 is given by:

$$C = \frac{\Sigma_a^{238}}{\Sigma_a^{235}} + \eta^{235}\epsilon(1-p)P_{\text{FNL}}P_{\text{TNL}}$$

If $C < 1$ then this is still called the conversion factor ($C$). If $C > 1$, this is the breeding factor ($B$):

$$B \equiv \frac{\text{Number of new fissile nuclides produced}}{\text{Number of fissile nuclides consumed}}$$

# 3 Kinetics

Some key assumptions for reactor kinetics are:

- Quantities of interest are averaged over neutron energy.
- Infinite medium (no leakage) without spatial dependence (homogeneous)

[1, Lec. 14]
Using the average neutron speed (in cm/s), $\overline{v}$ and the energy-averaged cross section for a reaction $x$ (discussed in the last section), we can find the reaction rate for that reaction:

$$R_x = \Sigma_x \overline{v} n(t)$$

Where $n(t)$ is the neutron population at time $t$, and $\Sigma_x$ is the energy averaged cross section (note we dropped the line, but for all this kinetics stuff it's all energy averaged) [1, Lec. 14].

## 3.1 Infinite non-multiplying medium

This section is from Lecture 14 [1].

We defined the neutron population as a function of time in the last section:

$$\frac{dn(t)}{dt} = P(t) - L(t)$$

For an infinite, non-multiplying (no fission) medium, the production is only due to sources, and loss is only due to absorption because we are in an finite medium (no leakage, see assumptions above). So replacing the appropriate terms:

$$\begin{aligned}\frac{dn(t)}{dt} &= S(t) - R_a \\ &= S(t) - \Sigma_a \overline{v} n(t)\end{aligned}$$

### 3.1.1 No Sources

With no sources ($S(t) = 0$), we can solve this using an initial neutron population $n(0) = n_0$:

$$\frac{dn(t)}{dt} = -\Sigma_a \overline{v} n(t)$$

With solution:

$$n(t) = n_0 e^{-\Sigma_a \overline{v} t}$$

This makes sense, because we have no sources, and only absorption. We expect that the neutron population will decay away at a time proportional to the chance of absorption ($\propto \Sigma_a$). The probability that a neutron survives from time 0 to time $t$ without absorption is:

$$\frac{n(t)}{n_0} = e^{-\Sigma_a \overline{v} t}$$

Much like a decay constant, the factor $-\Sigma_a \overline{v}$ gives the probability per unit time that an absorption event will occur. We therefore define something like the mean life time from decay theory but here we call it the **average neutron lifetime**:

$$\ell_\infty = \frac{1}{\Sigma_a \overline{v}}$$

### 3.1.2 Alternate Derivation of $\ell_\infty$

We can also find the average neutron lifetime using the probability that a neutron will be absorbed in a small time step $dt$. From above, we know that the probability that a neutron survives from time 0 to time $t$ without absorption is:

$$\frac{n(t)}{n_0} = e^{-\Sigma_a \overline{v} t}$$

The probability that the neutron will then be absorbed in $dt$ (after living so long from 0 to $t$) is:

$$p(t)dt = (\Sigma_a \overline{v}) e^{-\Sigma_a \overline{v} t} dt$$

The factor in front is a normalization so if you integrate from 0 to $\infty$ you get 1, which we need for a probability density function. We can use this to find the expectation value of $t$:

$$\overline{t} = \int_0^\infty t p(t) dt = \int_0^\infty t(\Sigma_a \overline{v}) e^{-\Sigma_a \overline{v} t} = \frac{1}{\Sigma_a \overline{v}} = \ell_\infty$$

Yay we made it, now we know this is really the average time a neutron will bounce around before being absorbed.

### 3.1.3 Sources

If we have a source of constant strength: $S_0$, we get (using our new-fangled average neutron lifetime):

$$\begin{aligned} \frac{dn(t)}{dt} &= S_0 - \Sigma_a \overline{v} n(t) \\ &= S_0 - \frac{1}{\ell_\infty} n(t) \end{aligned}$$

With solution:

$$n(t) = \ell_\infty S_0 (1 - e^{-t/\ell_\infty})$$

This looks a lot like an exponential build-up that at $t = \infty$ goes to a value of $\ell_\infty S_0$.

## 3.2 Infinite Multiplying Medium

This section is from Lecture 14 [1].

If we have an infinite multiplying medium, our equation becomes a little more complicated.

$$\begin{aligned} \frac{dn(t)}{dt} &= S(t) - L(t) \\ &= S(t) + \nu \Sigma_f \overline{v} n(t) - \Sigma_a \overline{v} n(t) \end{aligned}$$

Note that we still only lose due to absorption (infinite means no leakage) and now we have an extra source term from fission. We can clean this up a bit using our definitions of $k_\infty$ and $\ell_\infty$:

$$k_\infty = \frac{\nu \Sigma_f}{\Sigma_a}$$

$$\begin{aligned} \frac{dn(t)}{dt} &= S(t) + \Sigma_a \overline{v} n(t) \left( \frac{\nu \Sigma_f}{\Sigma_a} - 1 \right) \\ &= S(t) + \frac{(k_\infty - 1)}{\ell_\infty} n(t) \end{aligned}$$

We saw almost this exact equation earlier, but just with $k$ and $\ell$ (the neutron generation time), in section 2.1.2. There, we had leakage (see next section), here we don't so we have $\infty$ on the terms.

#### 3.2.1 No Source

As above, you don't have a source, you end up with:

$$n(t) = n_0 \text{Exp}\left(\frac{k_\infty - 1}{l_\infty} t\right)$$

You can see how $k_\infty$ controls criticality here.

#### 3.2.2 Sources

This isn't in the lecture, but if you do have a time independent source, the solution is found the same was as in the previous section:

$$n(t) = \frac{\ell_\infty S_0}{(k_\infty - 1)}\left[1 - \text{Exp}\left(-\frac{(1-k_\infty)}{\ell_\infty} t\right)\right]$$

### 3.3 Finite Multiplying Medium

This section is from Lecture 14 [1].
As previously described, if we have a finite medium, then we have to re-introduce leakage:

$$k = P_{\text{NL}} k_\infty$$
$$\ell = P_{\text{NL}} \ell_\infty$$

Where I assume $P_{\text{NL}}$ is just $P_{\text{FNL}} P_{\text{PNL}}$, but it isn't specified. You'd probably pick the right ones for the energy groups that you assume your problem has. This gives us (same as above):

$$\frac{dn(t)}{dt} = S(t) + \frac{(k-1)}{\ell} n(t)$$

#### 3.3.1 No source

No source is exactly the same as above, just get rid of the $\infty$s.

#### 3.3.2 Source

Solving this equation with $n(0) = 0$ gives us:

$$n(t) = \frac{\ell S_0}{(k-1)}\left[1 - \text{Exp}\left(-\frac{(1-k)}{\ell} t\right)\right]$$

For a subcritical system, $k < 1$, as $t \to \infty$ the exponential is negative and goes to 0:

$$n(\infty) = \frac{\ell S_0}{(k-1)}$$

As $k$ approaches 1, the value $(1-k)$ gets very small, and you can expand the exponential:

$$n(t) = \frac{\ell S_0}{(k-1)}\left[1 + \frac{(k-1)}{\ell} t + \frac{1}{2}\frac{(k-1)^2}{\ell^2} t^2 + \ldots - 1\right]$$

Which gives us, via some black magic:

$$n(\infty) = S_o t$$

I think what we do here is cancel the first 1 and the final -1, then if you distribute in the $\frac{\ell}{(k-1)}$, you end up with $t$ plus stuff on the order of $(k-1)$. You can make the argument that because that is a very small quantity, we can just ignore all of them, so you're just left with $t$. This is speculation.

### 3.4 Multiplying Medium Takeaways

This section is taken from Lecture 14[1].

You'll notice that the results for the finite case and infinite case are exactly the same, with the addition of some $\infty$ symbols. The take-aways from both are also the same. There are two conditions in which a time-independent neutron population can be established:

- If you **do not have a source**, you get the exponential decays from above. The only way to make this not decay is if $k = 1$ (or the $\infty$ variation if you're getting paid by the $\infty$, which I am not).
- If you **have a source**, you get the buildup equation. If $k < 1$, then you approach a constant value at time $\infty$ (swear they're not paying me for these). As $k$ approaches $\infty$, the neutron population blows up. Go reread that section and it'll make sense.

To reiterate, the two cases of a stable neutron population in a multiplying medium are:

1. $k = 1$ and no source.
2. $k < 1$ and a source.

# 4 Delayed Neutrons

This section is from Lecture 14 [1].

In addition to the prompt neutrons created immediately with fission, there are delayed neutrons that are the result of the decay of fission products. The specifics ones that make these neutrons are called **Delayed Neutron Precursors (DNPs)**. We divide these up into 6 groups (based on half-life), and each group produces a different number of neutrons. We express the fraction of all neutrons born delayed as $\beta$:

$$\beta = \frac{\text{Neutrons born delayed}}{\text{Neutrons born delayed and prompt}} = \sum_{i=1}^{6} \beta_i$$

Where $\beta_i$ is the contribution from the $i$-th group. Using this, we can weight the half-lives of each of these groups using their values of $\beta_i$ over the total $\beta$. This weighs the DNP groups that make more neutrons more importantly, giving us an average half-life:

$$t_{1/2} = \frac{1}{\beta} \sum_{i=1}^{6} \beta_i t_{i,1/2}$$

We can also define the average decay constant and plug it into the same formula:

$$t_{i,1/2} = \frac{\ln 2}{\lambda_{d,i}}$$

$$\frac{1}{\lambda_d} = \frac{1}{\beta} \sum_{i=1}^{6} \beta_i \frac{1}{\lambda_{d,i}}$$

We can add these into our kinetics equations before, they will provide another source of neutrons by decaying.

$$\frac{dn(t)}{dt} = P(t) - L(t)$$

$$\frac{dn(t)}{dt} = \text{Source} + \text{Prompt} + \text{Delayed} - \text{Absorption}$$

$$= S(t) + \frac{(1-\beta)k - 1}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t)$$

Important things to note are that the delayed source term is just from the decay of the $i$-th group, with concentration $C_i$. There is an extra $(1-\beta)$ in front of the fission term because we only want prompt neutrons, and $\beta$ is the fraction born delayed, meaning $(1-\beta)$ is the fraction born prompt. The loses are still just absorption. The values of $C_i(t)$ are given by:

$$\frac{dC_i(t)}{dt} = \beta_i \frac{k}{\ell} n(t) - \lambda_i C_i(t)$$

# 5 Point Reactor Kinetics Equations

This section is from Lecture 14 [1].

First, we have to define the **mean neutron generation time**:

$$\Lambda \equiv \frac{\ell}{k}$$

And we plug that into the equation we found for our neutron population with delayed neutrons in Section 4:

$$\begin{aligned}
\frac{dn(t)}{dt} &= S(t) + \frac{(1-\beta)k - 1}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\
&= S(t) + \frac{k(1-\beta-1/k)}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\
&= S(t) + \frac{\frac{k-1}{k} - \beta}{\Lambda} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t)
\end{aligned}$$

That mess in the numerator is defined as **reactivity:**

$$\rho(t) \equiv \frac{k(t) - 1}{k(t)}$$

Which is a measure of the criticality of the reactor:

$\rho > 0$: supercritical

$\rho = 0$: critical

$\rho < 0$: subcritical

Finally, plugging that in and bringing in the equations for $C_i$ give us the **point reactor kinetics equations** (PKEs):

$$\begin{aligned}
\frac{dn(t)}{dt} &= S(t) + \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\
\frac{dC_i(t)}{dt} &= \frac{\beta_i}{\Lambda} n(t) - \lambda_i C_i(t) \quad i = 1, 2, \ldots, 6
\end{aligned}$$

To get these in terms of power, multiply by $\overline{v}\Sigma_f$ to get the reaction rate for fission, then multiply by $w_f$, the energy per fission:

$$\begin{aligned}
\frac{dP(t)}{dt} &= S(t) + \frac{\rho(t) - \beta}{\Lambda} P(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \\
\frac{dC_i(t)}{dt} &= \frac{\beta_i}{\Lambda} P(t) - \lambda_i C_i(t) \quad i = 1, 2, \ldots, 6
\end{aligned}$$

The values of $C_i$ must be different here, because we multiplied the whole thing through by a constant, but no attempt is made to explain this. It's even easier if you replace $\rho(t)$ by just $\rho_0$ if reactivity isn't a function of time, and drop the independent source $S(t)$. I'm not writing out the equations *again*.

### 5.1 One Effective Delayed Group

This section is taken from Lecture 14 [1].

## References

[1] Jasmine Vujic. Nuclear engineering class lectures. Spring 2015.