1. $m(x) = \frac{1}{N}\sum_{i=1}^{N} x_i$

Transform X using the function $a+bx$: $y_i = a + bx_i$

$$m(a+bX) = \frac{1}{N}\sum_{i=1}^{N}(a+bx_i)$$

$$= \frac{1}{N}\left(\sum_{i=1}^{N} a + \sum_{i=1}^{N} bx_i\right)$$

$$= \frac{1}{N}\left(Na + b \cdot \sum_{i=1}^{N} bx_i\right)$$

$$= a + b \times m(X)$$

2. $cov(X, Y) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$

$z = a + bY$, so for each observation $z_i = a + by_i$

$m(z) = m(a+bY) = a + bm(Y)$

$$cov(X, z) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(z_i - m(Z))$$

$$= \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))((a+by_i) - (a+bm(Y)))$$

$$= \frac{1}{N}\sum_{i=1}^{N}(x_i - m(x))(by_i - bm(Y))$$

$$= \frac{1}{N}\sum_{i=1}^{N} b(x_i - m(X))(y_i - m(Y))$$

$$= b \times \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$$

$$cov(x, z) = b \times cov(X, Y)$$

3. $\operatorname{cov}(U, V) = \frac{1}{N}\sum_{i=1}^{N}(u_i - m(U))(v_i - m(V))$

Set $U = a+bX$, $V = a+bX$

from part 1 , we showed that $m(a+bX) = a+bm(X)$

the deviation for each observation is :

$(a+bx_i) - (a+bm(X)) = b(x_i - m(X))$

$\operatorname{cov}(a+bX, a+bX) = \frac{1}{N}\sum_{i=1}^{N}((a+bx_i)-(a+bm(X)))\cdot((a+bx_i)-(a+bm(X)))$

$$= \frac{1}{N}\sum_{i=1}^{N}(b(x_i - m(X)))(b(x_i - m(X)))$$

$$= \frac{1}{N}\sum_{i=1}^{N} b^2 (x_i - m(X))^2$$

$$= b^2 \left(\frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2\right)$$

Since $\frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2 = \operatorname{cov}(X, X) = s^2$

so $\operatorname{cov}(a+bX, a+bX) = b^2 s^2$

4. The non-decreasing function preserves the ordering of values in a dataset. Applying $g(X)$ to all values keeps the ranking intact. The median index remains the same.

Since the middle value remains at the same position after applying $g(X)$, the median of the transformed variable is just the transformation of the median of X:

$$\text{Median}(g(X)) = g(\text{Median}(X))$$

Since non-decreasing transformations do not change rankings, the result extends to all quantiles.

$$Q_p(g(X)) = g(Q_p(X))$$

$$IQR(g(X)) = |b| \times IQR(X)$$

$$\text{Range}(g(X)) = |b| \times \text{Range}(X)$$

IQR and Range are affected by the transformation, but in the case of linear transformations, they scale by $|b|$.

5.

$$g(m(X)) \Rightarrow g\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

$$m(g(X)) \Rightarrow \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$

For these to be equal, we would need:

$$\frac{1}{N}\sum_{i=1}^{N} g(x_i) = g\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

This only holds in special cases

The equality hold if $g(X)$ is a linear function

then $g(m(X)) = a + b\,m(X)$

and $m(g(X)) = \frac{1}{N}\sum_{i=1}^{N}(a + b x_i) = a + b\frac{1}{N}\sum_{i=1}^{N} x_i$

$$= a + b\,m(X)$$

These are equal, so the property holds for linear functions.

For convex functions, $m(g(X)) \geq g(m(X))$

For concave functions, $m(g(X)) \leq g(m(X))$

$\therefore$ $m(g(X)) = g(m(X))$ only if $g(X)$ is linear.