Revised 12/24/2015

LAB 5: Eigenvalues and Eigenvectors

In this lab you will use MATLAB to study these topics:

- The geometric meaning of eigenvalues and eigenvectors of a matrix
- Determination of eigenvalues and eigenvectors using the characteristic polynomial of a matrix
- Use of eigenvectors to transform a matrix to diagonal form.
- Steady-state eigenvector for a transition matrix
- Applications of eigenvalues and eigenvectors to study Markov chains.

Preliminaries

Reading from Textbook: In connection with this Lab, read through Sections 5.1, 5.2, 5.3, and pp. 334-336 of Section 5.5 of the text and work the suggested problems for each section.

Script Files and T-codes: For this lab you will need the the m-file `rmat.m` from Lab 2 and the Teaching Code `nulbasis.m`.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Be sure to answer all the questions in the lab assignment. Be sure your write-up begins with the following comment lines, filling in your information where appropriate.

```
% [Name]
% [Last 4 digits of RUID]
% Section [Section number]
% Math 250 MATLAB Lab Assignment #5
```

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

5 Question 1. Graphic Demonstration of Eigenvectors and Eigenvalues

(a) Type `eigshow` at the MATLAB prompt. A graphics window should open. Underneath the graph the statement

```
Make A*x parallel to x
```

should appear (if it does not, then click on the `eig` button to get this statement).
Click on the pull-down bar above the graph and select the matrix `[1 3; 4 2]/4`. Move the cursor onto the vector $\mathbf{x}$, and make $\mathbf{x}$ go around a full circle. The transformed vector $A\mathbf{x}$ then moves around an ellipse. Search for the *special lines* through zero that contain both $A\mathbf{x}$ and $\mathbf{x}$. When $\mathbf{x}$ lies on such a line, it is an *eigenvector* of the matrix A (the word *eigen* means *special* in German). For any $\mathbf{x}$ lying on these *special lines*, $A\mathbf{x} = \lambda\mathbf{x}$, where λ is an *eigenvalue* of A. Since $\mathbf{x}$ is a unit vector, the length of $A\mathbf{x}$ is $|\lambda|$. If $A\mathbf{x}$ points in the *same* direction as $\mathbf{x}$, then $\lambda > 0$. If $A\mathbf{x}$ points in the *opposite* direction to $\mathbf{x}$, then $\lambda < 0$.
From your graphical experimentation answer the following questions (no algebraic calculations needed):

(1) *(i)* How many different positive eigenvalues does A have? (This occurs when the $A\mathbf{x}$ arrow points in the *same* direction as the $\mathbf{x}$ arrow.)

(1) *(ii)* How many different negative eigenvalues does A have? (This occurs when the $A\mathbf{x}$ arrow points in the *opposite* direction to the $\mathbf{x}$ arrow.)

(1) *(iii)* What are the (approximate) numerical values of the eigenvalues? (Estimate these values using the relative lengths of the $\mathbf{x}$ arrow and the $A\mathbf{x}$ arrow.)

Caution: Be careful in counting eigenvalues; if $\mathbf{x}$ is an eigenvector with eigenvalue λ, then $-\mathbf{x}$ is also an eigenvector with the *same* eigenvalue λ. (Don't try to print the eigshow window.)

(b) Click on pull-down matrix selection bar again and select `[3 1; -2 4]/4`. Move $\mathbf{x}$ around the circle with the cursor and observe what happens, as in part **(a)**. Use your graphical experimentation to answer the following questions (no algebraic calculations needed):

(1) *(i)* Are there any lines through zero that contain both $A\mathbf{x}$ and $\mathbf{x}$?

(1) *(ii)* Does A have any *real* eigenvectors or eigenvalues? Explain.

5 Question 2. Characteristic Polynomial

At the MATLAB prompt type `A = [1 3; 4 2]/4` (this is the matrix in part **(a)** of Question #1). The eigenvalues of A are the roots of the *characteristic polynomial* of A.

(a) Use MATLAB to calculate its characteristic polynomial $p(t)$ by

```
syms t; I = eye(2); p = det(A - t*I)
```

(1) Verify by hand calculation that the constant term in the polynomial $p(t)$ is $\det(A)$.

(1) **(b)** Use the MATLAB command `solve(p)` to get the roots of $p(t)$ (the eigenvalues of A).

(1) Compare these values with your graphical estimates for the eigenvalues from Question #1(a).

(c) Now at the MATLAB prompt type `A = [3 1 ; -2 4 ]/4` (the matrix in part **(b)** of Question #1).

(1) Calculate the characteristic polynomial $p(t)$ of A as in part **(a)** and find its roots as in part **(b)**.

(1) Are the eigenvalues of A real? How does this explain what you observed Question #1(b).

5 Question 3. Eigenvectors and Diagonalization

Random Seed: Initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your student ID number. This will ensure that you generate your own particular random vectors and matrices in the following parts of the lab.
BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

(a) Generate a random 3×3 integer matrix and test whether its eigenvalues are all real by the commands

```
A = rmat(3,3), z = eig(A) - real(eig(A))
```

If any entry in the vector `z` is not zero, then the eigenvalues of A are not all real. In this case repeat these commands until you get an A for which `z` has all zeros. If you generated any matrices with complex eigenvalues delete these from the diary file when you create your lab report.
Now calculate the characteristic polynomial $p(t)$ of your matrix A by

```
syms t; I = eye(3); p = det(A - t*I)
```

(1) Plot the characteristic polynomial of A in a graphics window by

```
figure; ezplot(p, [-10, 10]), grid
```

Adjust the horizontal range of the plot (change $[-10, 10]$ as needed) to show that $p(t)$ has three real roots (zoom in as necessary, using the magnifying glass button on the top of the graph) .

(1) Print the graph with a range that shows all three real roots, and include the graph in your lab report.

(1) Use the graph to obtain approximate values for the three real roots of $p(t)$.

(b) Use the MATLAB command

```
[P  D] = eig(A)
```

to generate a matrix P and a diagonal matrix D.

(1) Compare the diagonal entries of D with your graphical estimates for the eigenvalues of A in part **(a)**.

Use MATLAB to define

```
p1 = P(:,1),   p2 = P(:,2),   p3 = P(:,3)
```

(the columns of P). Calculate

```
A*p1 - D(1,1)*p1,  A*p2 - D(2,2)*p2,  A*p3 - D(3,3)*p3
```

(1) What does this calculation tell you about the eigenvalues and eigenvectors of A? (See Theorem 5.2, page 315.)

(c) Let A, P, D be as in part **(b)**. Verify by MATLAB that `A = P*D*inv(P)`.

(1) Use this formula for A to express A^5 and A^{10} symbolically in terms of P, P^{-1}, D^5 and D^{10}.

Verify your answer to this question numerically using MATLAB.

5 Question 4. Steady-State Eigenvector for a Transition Matrix

(a) The square matrix A is called a *transition matrix* if its entries are nonnegative and the sum of the entries in each column is one. Generate a random 2×2 transition matrix A by

```
A = eye(2); B = rand(2);
A(:,1) = B(:,1)/sum(B(:,1)); A(:,2) = B(:,2)/sum(B(:,2))
```

(1) Calculate `[1  1]*A`.

(1) Explain (by a hand calculation) why the answer to this calculation shows that A is a transition matrix.

(b) Transition matrices such as A with all entries positive are called *regular*. For a regular transition matrix, 1 is the largest eigenvalue, and the corresponding eigenspace is one-dimensional. Use the T-code `nulbasis` to calculate a normalized eigenvector for the matrix A you generated in part **(a)**.

```
u = nulbasis(A - eye(2)), v = u/sum(u)
```

(1) The vector $\mathbf{v}$ should have components that are positive and sum to 1. Verify by MATLAB that $A\mathbf{v} = \mathbf{v}$.

Thus $\mathbf{v}$ is an eigenvector for A with eigenvalue 1, called the *steady-state vector* for A. Plot this vector (as a solid line) by

```
figure; plot([0,v(1)], [0, v(2)]), hold on
```

Leave the graphic window open for the next part.

(c) A general result about regular transition matrices (Theorem 5.4 on page 335) asserts that if $\mathbf{p}$ is any initial choice of a probability vector in $\mathbf{R}^2$, then the sequence of vectors $A^k\mathbf{p}$ converges to the steady-state vector $\mathbf{v}$ as $k \to \infty$. To demonstrate this graphically for your matrix A, generate a random initial probability vector

```
w = rand(2,1), p = w/sum(w)
```

(1) Graph the vector $A\mathbf{p}$ (as a dotted line) in the same window from part **(b)**:

```
p = A*p, plot([0,p(1)], [0, p(2)], ':'), hold on
```

To plot the sequence of vectors $A^2\mathbf{p}$, $A^3\mathbf{p}, A^4\mathbf{p}, \ldots$ in the graphics window, just use the up-arrow key ↑ to repeat this last command. Do this as many times as needed until the vector $\mathbf{p}$ has converged numerically (to three decimal places in each component) to the steady-state vector $\mathbf{v}$ that you plotted in part **(b)**.

(1) Print the graphics window and include it in your lab report. Be sure that you used the `hold on` command so that all the different vectors that you generated appear in your printed graph.

5 Question 5. Markov Chains

Read the section *Markov Chains* in Section 5.5 and look at Practice Problem 1 (page 336) and its solution (page 353). The following questions refer to Exercise #32 on page 349 of the text. Enumerate the states as 1 = city, 2 = suburbs, 3 = country.

(a) Determine the transition matrix A for Exercise #32 (page 349) and enter it into your MATLAB workspace.

(1) Let $\mathbf{u}$ be the row vector $[1,\ 1,\ 1]$. Verify that each column of A sums to 1 by calculating that $\mathbf{u} * A = \mathbf{u}$.

(b) Determine the initial probability vector $\mathbf{p}$ from the description given in Exercise #32. Verify that the entries of $\mathbf{p}$ sum to 1 by calculating that $\mathbf{u} * \mathbf{p} = 1$.

(2) Now use powers of the matrix A and the vector $\mathbf{p}$ to find the percentage of people living in the city, suburbs, and country after 1, 2, 3, 5, and 8 years. Label the numbers in your answers (city, suburbs, country).

(c) The steady-state probability vector $\mathbf{v}$ is an eigenvector for A with eigenvalue 1.

(1) Use MATLAB to find $\mathbf{v}$ by the same method as Question #4(b).

(1) What is the relation between the vector $\mathbf{v}$ and the population distribution vector in part **(b)** after 8 years?

Final Editing of Lab Write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. *Remove all typing errors. Your write-up should only contain the input commands that you typed and the output results generated by* MATLAB, *together with your answers to the questions.* In particular, remove from your diary file any of the results generated by the `load, save, clear, format, help` commands that you used. Preview the document before printing and remove unnecessary page breaks and blank space.

Lab write-up submission guidelines: You will submit your diary file `lab5.txt` online via Sakai. In your web browser, go to Sakai and select your tab for this course. You will find the application `Assignments 2`. Open this application, and submit the file `lab1.txt` there. *Lab write-ups will only be accepted if they are in plain text format or PDF format.* **A lab writeup in any other format will not be graded!** In future assignments, you will submit images along with your diary file. Image files will only be graded if they are submitted in `.jpg` or `.png` format!