STAT 2011

Q1

$$\mathbb{P}(k_1 \text{ outcome } 1,\ k_2 \text{ outcome } 2,\ n-k_1-k_2 \text{ outcome } 3)$$

$\uparrow p_1$ $\uparrow p_2$ $1-p_1-p_2$

$$= p_1^{k_1} \cdot p_2^{k_2} \cdot (1-p_1-p_2)^{n-k_1-k_2} \cdot \frac{n!}{k_1!\, k_2!\, (n-k_1-k_2)!}$$

Multinomial Distribution.

Q2

2. Repair calls for central air conditioners fall into three general categories: coolant leakage, compressor failure, and electrical malfunction. Experience has shown that the probabilities associated with the three are 0.5, 0.3, and 0.2, respectively. Suppose that a dispatcher has logged in ten service requests for tomorrow morning. Use the answer to Question 3.2.18 to calculate the probability that three of those ten will involve coolant leakage and five will be compressor failures.

$n=10$ $k_1=3$ $k_2=5$ $p_1=0.5$ $p_2=0.3$ $n-p_1-p_2=0.2$

$$\mathbb{P} = 0.5^3 \times 0.3^5 \cdot 0.2^2 \cdot \frac{10!}{3!\cdot 5! \cdot 2!}$$

$$= 0.031$$

3. Find the probability mass function (pmf) for the discrete random variable X whose cumulative distribution function (cdf) at the points $x = 0, 1, \ldots, 6$ is given by $F_X(x) = x^2/36$.

Q3:

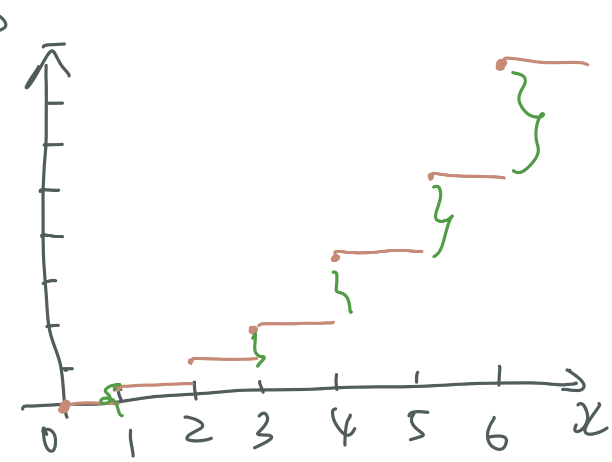
$F_X(x) = \dfrac{x^2}{36}$ $\mathbb{P}(X \le x)$

$x = 0, 1, 2, \ldots, 6$

$F_X(0) = 0$ $F_X(1) = 1/36$ $F_X(2) = 4/36$

$F_X(3) = 9/36$ $F_X(4) = 16/36$ $F_X(6) = 36/36$

$F_X(5) = 25/36$



$P_X(x) = \mathbb{P}(X=x) = \mathbb{P}(X \le x) - \mathbb{P}(X \le x^-)$

$P_X(0) = \mathbb{P}(X=0) = \mathbb{P}(X \le 0) = 0$

$P_X(1) = \mathbb{P}(X=1) = \mathbb{P}(X \le 1) - \mathbb{P}(X \le 0) = 1/36 - 0$

$P_X(2) = \mathbb{P}(X \le 2) - \mathbb{P}(X \le 1) = 4/36 - 1/36 = \cancel{1/36} = 3/36$

$P_X(3) = 5/36 \quad P_X(4) = 7/36 \quad P_X(5) = 9/36 \quad P_X(6) = 11/36$

7. Independent trials consisting of the flipping of a coin having probability p of coming up heads are continually performed until either a head occur or a total of n flips is made. Let X denote the number of times the coin is flipped, find $p_X(k)$, and check that it is a pmf.

Q7 X: # of times the coin flipped.

$P_X(k) = \mathbb{P}(X=k) = (1-p)^{k-1} \cdot p \qquad k<n$

$n \qquad \{T \ldots\ldots T, H\} \quad \le n.$

$\mathbb{P}_X(k) = (1-p)^{n-1} \cdot p + (1-p)^n \qquad k=n$

$\{\underbrace{T \ldots\ldots T}_{n-1} H\}$ n

$\{\underbrace{T \ldots\ldots}_{n-1} T\}$

Check : $\sum_k P_X(k) = 1$?

$\sum_k P_X(k)$

$= \sum_{k=1}^{n-1} (1-p)^{k-1} \cdot p + (1-p)^{n-1} \cdot p + (1-p)^n$

$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p + (1-p)^n$

$p + p(1-p) + p(1-p)^2 + \cdots + p \cdot (1-p)^n$

$= p \frac{1 \cdot (1-(1-p)^n)}{1-(1-p)} + (1-p)^n$

$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{p}} + (1-p)^n \quad = 1 \cancel{- (1-p)^n} + \cancel{(1-p)^n}$

$= 1$

8. Find E(X) and Var(X), where X is the outcome when we roll a fair die.

X can be $\{1, 2, 3, 4, 5, 6\}$

X^2: $\{1, 4, 9, 16, 25, 36\}$

$P_X(k) = \mathbb{P}(X=k) = 1/6$

$$\mathbb{E}(X) = \sum_x x \cdot P_X(x)$$

$$= \frac{1}{6} \times (1 + \cdots + 6)$$

$$= 3.5$$

$$\mathrm{Var}(X) = \mathbb{E}\left((X-\mu)^2\right)$$

$$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 91/6 - \left(\frac{7}{2}\right)^2 = 35/12$$

$$\mathbb{E}(X^2) = \frac{1}{6} \times (1+4+9+16+25+36) = 91/6$$

10. Suppose there are m days in a year, and that each person is independently born on day r with probability p_r, with $r = 1, \ldots, m$ and $\sum_{k=1}^{m} p_k = 1$. Let $A_{i,j}$ be the event that person i and person j are born on the same day.

(a) Find $P(A_{1,3})$.

(b) Find $P(A_{1,3}|A_{1,2})$.

(c) Show that $P(A_{1,3}|A_{1,2}) \geq P(A_{1,3})$.

(a) $P(A_{1,3}) = P_1 \cdot P_1 + P_2 \cdot P_2 + \cdots + P_m P_m$

$$= \sum P_r^2$$

(b) $$\mathbb{P}(A_{13} | A_{12}) = \frac{\mathbb{P}(A_{13}, A_{12})}{\mathbb{P}(A_{12})} = \frac{\mathbb{P}(A_{1,2,3})}{\mathbb{P}(A_{12})}$$

$$= \frac{\sum P_r^3}{\sum P_r^2}$$