STAT 2011
Tutorial 10

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Conditional Distribution

- Definition:

Definition 3.8.1 (Conditional pmf for discrete RVs). *Let X and Y be discrete random variables. The **conditional probability mass function of** Y **given** x - that is the probability that Y takes on the value y given that X is equal to x - is denoted by $p_{Y|x}(y)$ and given by*

$$p_{Y|x}(y) = P(Y = y|X = x) = \frac{p_{X,Y}(x, y)}{p_X(x)}$$

for $p_X(x) \neq 0$.

- Continuous:

$$P(Y \leq y|X = x) = \int_{-\infty}^{y} \frac{f_{X,Y}(x, u)}{f_X(x)} du$$

Moment-Generating Functions

- Definition:

Definition 3.9.1 (Moment-generating function). *Let W be a random variable. The* ***moment-generating function*** *for W is denoted $M_W(t)$ and given by*

$$M_W(t) = E(e^{tW}) = \begin{cases} \sum_{all\ k} e^{tk} p_W(k) & \text{if } W \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tw} f_W(w) dw & \text{if } W \text{ is continuous.} \end{cases} \quad (8)$$

at all values of t for which the expected value exists.

Theorem 3.9.1. *Let W be a random variable with probability mass function $p_W(w)$ or probability density function $f_W(w)$. [If W is continuous, $f_W(w)$ must be sufficiently smooth to allow the order of differentiation and integration to be interchanged]. Let $M_W(t)$ be the moment-generating function for W. Then, provided the r-th moment exists,*

$$M_W^{(r)}(0) = {}^{`}E(W^r).$$

where $M_W^{(r)}$ is the r-th derivative of the moment generating function M_W.

Central Limit Theorem

- Definition:

Theorem 3.10.1 (Central Limit Theorem). *Let $W_1, W_2, \ldots$ be an infinite sequence of independent random variables, each with identical distribution. Suppose that the mean μ and the variance σ^2 of $f_{W_i}(w)$ are both finite. For any number $a, b \in \mathbb{R}$*

$$\lim_{n\to\infty} P\left[a \leq \frac{W_1 + W_2 + \ldots + W_n - n\mu}{\sqrt{n\sigma^2}} \leq b\right] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-z^2/2} dz.$$

Normal Distribution

- Properties:

Theorem 3.10.2. *Let $Y_1 \sim N(\mu_1, \sigma_1^2)$ and $Y_2 \sim N(\mu_2, \sigma_2^2)$. Define $Y = Y_1 + Y_2$. If Y_1 and Y_2 are independent, Y is normally distributed with $Y \sim N(\mu_1 + \mu_2, \sigma^2 + \sigma^2)$.*

Theorem 3.10.3. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2, i.e. $Y_i \sim N(\mu, \sigma^2)$ for $i = 1, \ldots, n$. Then the sample mean*

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Theorem 3.10.4. *Let $Y_1, Y_2, \ldots, Y_n$ be a set of independent normal random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.*
Let $a_1, a_2, \ldots, a_n$ be any set of constants.
Then $Y = a_1Y_1 + a_2Y_2 + \ldots a_nY_n$ is normally distributed with mean

$$\mu = \sum_{i=1}^{n} a_i\mu_i$$

and variance

$$\sigma^2 = \sum_{i=1}^{n} a_i^2\sigma_i^2.$$