4. Let Z be a standard normal random variable, $Z \sim N(0,1)$, and $X = \mu + \sigma Z$. Then $X \sim N(\mu, \sigma^2)$. The pdf of Z is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

and the pdf of X is given by

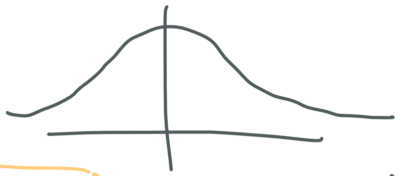
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

$N(\mu.\sigma^2)$

(a) Find the mgf of Z.

(b) Noting that X is a linear transformation of Z, find the mgf of X.

(c) Using (b), show that a random variable with distribution $N(\mu, \sigma^2)$ has mean μ and variance σ^2.



r.v. X

(a) $M_Z(t) = E(e^{Z\cdot t})$

$$= \int_{-\infty}^{\infty} \boxed{e^{Z\cdot t}\cdot\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2-2zt+t^2-t^2}{2}}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}+\frac{t^2}{2}}\, dz$$

$$e^{-\frac{(z-t)^2}{2}}\ \boxed{e^{\frac{t^2}{2}}}$$

$$= e^{t^2/2}\cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}}\, dz}_{=1}$$

$$= e^{t^2/2}$$

$M_X(t) = E(e^{X\cdot t})$

$f_X(x)$ p.d.f.

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

p.d.f. $N(t,1)$

$\int_{-\infty}^{\infty} f_X(x)\,dx$

(b) $X = \mu + \sigma Z.$

$Z \sim N(0,1)$

$X \sim N(\mu, \sigma^2)$

$M_X(t) = E_X(e^{t\cdot X}) = E_Z(e^{t\cdot(\mu+\sigma\cdot Z)})$

$= E_Z(e^{t\cdot\mu}\cdot e^{t\cdot\sigma\cdot Z})$

$M_Z(t) = E(e^{t\cdot Z})$
$= e^{t^2/2}$

$$= e^{t\cdot\mu} \cdot E_Z\left(e^{t\cdot\sigma\cdot z}\right) = M_Z(t\cdot\sigma)$$

$$\boxed{= e^{t\cdot\mu} \cdot e^{(t\cdot\sigma)^2/2}} \qquad = e^{(t\cdot\sigma)^2/2}$$

. Prove that the mgf of a $N(\mu, \sigma^2)$ random variable is $e^{\mu t+\frac{\sigma^2 t^2}{2}}$.

r.v. $X \sim N(\mu, \sigma^2)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

$$M_X(t) = E(e^{t\cdot X}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx.$$

Q6. $X \sim N(\mu, \sigma)$.

$$M_X(t) = E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2-2\mu x+\mu^2-2\sigma^2\cdot x\cdot t}{2\sigma^2}}\, dx.$$

$x^2 - 2\cdot(\mu+\sigma^2 t)x$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2-2(\mu+\sigma^2 t)x+\mu^2}{2\sigma^2}}\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2-2(\mu+\sigma^2 t)\cdot x+(\mu+\sigma^2 t)^2}{2\sigma^2}}\, e^{\frac{(\sigma^2 t)^2+2\mu\sigma^2 t}{2\sigma^2}}\, dx$$

$$= e^{\frac{\sigma^2 t^2}{2}+\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\cdot\sigma^2}}\, e^{-\frac{(x-\mu+\sigma^2 t)^2}{2\sigma^2}}\, dx$$

$N(\mu+\sigma^2\cdot t,\ \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(c)

$$M_X(t) = e^{t\cdot\mu} \cdot e^{(t\cdot\sigma)^2/2}, \qquad X \sim N(\mu, \sigma^2)$$

$$M_X^{(r)}(0) = E(X^r) \qquad E(X) = M'_X(t)\big|_{t=0}$$

$$M'_X(t) = e^{t\cdot\mu + (t\cdot\sigma)^2/2} \cdot (\mu + \sigma^2 t) = \mu = E(X).$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\mathbb{E}(X^2) = M_X''(0)$$

$$M_X''(t) = e^{t\cdot\mu + t^2\sigma^2/2} \cdot (\mu + \sigma^2 t)^2 + e^{t\cdot\mu + (t\cdot\sigma)^2/2} \cdot \sigma^2$$

$$= \mu^2 + \sigma^2$$

$$\mathrm{Var}(x) = \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

1. Let X denote the number on a chip drawn at random from an urn containing three chips, numbered 1, 2, and 3. Let Y be the number of heads that occur when a fair coin is tossed X times. Find $p_{X,Y}(x, y)$.

$P_X(x) = \frac{1}{3}$ $\quad X = \{1, 2, 3\}$

$Y | X \sim \mathrm{Bino}(X, \frac{1}{2})$ $\quad P_{Y|X}(y|x) = \binom{x}{y} \cdot (\frac{1}{2})^y \cdot (\frac{1}{2})^{x-y}$

$$= \frac{P_{XY}(x,y)}{P_X(x)}$$

$P_{XY}(x,y)$

$= P_X(x) \cdot P_{Y|X}(y|x)$

$= \frac{1}{3} \cdot \binom{x}{y} \cdot (\frac{1}{2})^y \cdot (\frac{1}{2})^{x-y}$

$= \frac{1}{3} \binom{x}{y} \cdot (\frac{1}{2})^x$ $\quad x = \{1, 2, 3\}$

$y \leq x \quad y \in \mathbb{Z}$

2. Given the joint pdf

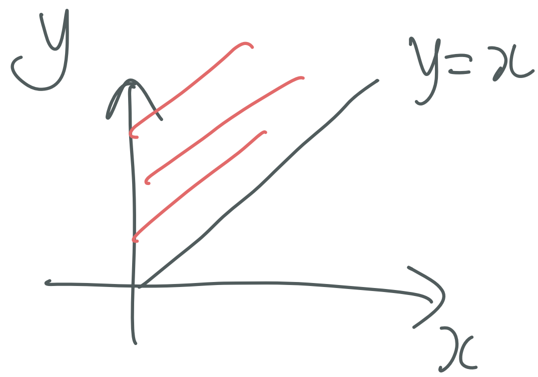
$$f_{X,Y}(x,y) = 2e^{-(x+y)}, \qquad 0 \le x \le y, \qquad y \ge 0,$$

find

(a) $\boxed{P(Y < 1 | X < 1)} = \dfrac{\mathbb{P}(Y<1, X<1)}{\mathbb{P}(X<1)}$

(b) $P(Y < 1 | X = 1)$

(c) $f_{Y|x}(y)$



$$f_X(x) = \int f_{XY}(x,y)\,dy$$

$$= \int_x^\infty 2\cdot e^{-(x+y)}\,dy = 2\cdot e^{-2x}$$

$$\mathbb{P}(X<1) = \int_0^1 f_X(x)\,dx = \int_0^1 2\cdot e^{-2x}\,dx = 1-e^{-2}$$

$$\mathbb{P}(Y<1, X<1) = \int_0^1 \int_x^1 2\cdot e^{-(x+y)}\,dy\,dx.$$