4. The exponential pdf is a special case of the Weibull distribution, which measures time to failure of devices where the probability of failure increases as time does. A Weibull random variable Y has pdf

$$f_Y(y;\alpha,\beta) = \alpha\beta y^{\beta-1}e^{-\alpha y^\beta}, \quad 0 < y, 0 < \alpha, 0 < \beta.$$

(a) Given a random sample $y_1, \ldots, y_n$, find the maximum likelihood estimator for α assuming that β is known.

(b) Suppose α and β are both unknown. Write down the equations that would be solved simultaneously to find the maximum likelihood estimators of α and β.

$y_1, \ldots, y_n \overset{i.i.d}{\sim} f_Y(y_i \alpha, \beta)$

$\mathcal{L}(y_1, \ldots, y_n; \alpha, \beta) = \prod_{i=1}^{n} f_Y(y_i; \alpha, \beta)$

$= \prod_{i=1}^{n} \alpha \cdot \beta \cdot y_i^{\beta-1} e^{-\alpha \cdot y_i^\beta}$

$e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

$\log(a \cdot b) = \log(a) + \log(b)$

$\log\left(= \alpha^n \cdot \beta^n \prod_{i=1}^{n} y_i^{\beta-1} \cdot e^{\sum -\alpha \cdot y_i^\beta}\right)$

$\ell(y_1, \ldots, y_n; \alpha, \beta) = n \cdot \log(\alpha) + n \cdot \log(\beta) + (\beta-1) \cdot \sum_{i=1}^{n} \log y_i + \sum -\alpha \cdot y_i^\beta$

$\log \prod_{i=1}^{n} y_i^{\beta-1}$

$\log y_1^{\beta-1} + \log y_2^{\beta-1} \ldots$

$(\beta-1) \cdot \log y_i$

$(\beta-1) \cdot \sum \log y_i$

$\frac{d\ell}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} y_i^\beta = 0$

$\frac{n}{\alpha} = \sum_{i=1}^{n} y_i^\beta$

$\boxed{\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n} y_i^\beta}}$

$\frac{d^2\ell}{d\alpha^2} = -\frac{n}{\alpha^2} \le 0$

$\begin{cases} \frac{\partial \ell}{\partial \alpha} = 0 \\ \frac{\partial \ell}{\partial \beta} = 0 \end{cases} \Rightarrow \begin{cases} \frac{n}{\alpha} - \sum_{i=1}^{n} y_i^\beta = 0 \\ \frac{n}{\beta} + \sum \log y_i + \alpha \cdot \sum y_i^\beta \cdot \log y_i = 0 \end{cases}$

3. If the random variable Y denotes an individual's income, Pareto law claims that $P(Y \geq y) = (\frac{k}{y})^\theta$, where k is the entire population's minimum income. It follows that $F_Y(y) = 1 - (\frac{k}{y})^\theta$, and, by differentiation

$$f_Y(y;\theta) = \theta k^\theta \left(\frac{1}{y}\right)^{\theta+1}, \quad y \geq k; \theta \geq 1.$$

Assume k is known. Find the maximum likelihood estimate of θ if income information has been collected on a random sample of 25 individuals.

$y_1, \dots, y_n \overset{i.i.d}{\sim} f_Y(y;\theta)$ $\qquad n = 25$

$\mathcal{L}(y_1, \dots, y_n; \theta) = f_Y(y_1;\theta) \cdots f_Y(y_n;\theta)$

$= \prod_{i=1}^{n} f_Y(y_i;\theta)$

$\prod_{i=1}^{n} y_i = y_1 \times y_2 \times \cdots \times y_n$

$= \prod_{i=1}^{n} \theta \cdot k^\theta \left(\frac{1}{y_i}\right)^{\theta+1}$ $\quad y_i^{-(\theta+1)}$

$\underbrace{k^\theta \, k^\theta \cdots k^\theta}_{n}$

$\log = \left(\theta^n \cdot k^{n \cdot \theta} \prod_{i=1}^{n} y_i^{-(\theta+1)} \right)$

θ^n

$\max_\theta \mathcal{L}(y_1, \dots, y_n; \theta) = \max_\theta \ell(y_1, \dots, y_n; \theta)$

$(k^\theta)^n = k^{n \cdot \theta}$

$\ell(y_1, \dots, y_n; \theta) = n \cdot \log(\theta) + n \cdot \theta \cdot \log(k) + (-(\theta+1)) \cdot \cancel{\log(\prod_{i=1}^{n} y_i)}$ $\quad \sum \log(y_i)$

find Maximum

$\frac{d\ell}{d\theta} = 0$ $\qquad \frac{n}{\theta} + n \cdot \log(k) - \sum \log y_i = 0$

$\hat{\theta}_{MLE} = \frac{n}{\sum \log y_i - n \cdot \log(k)}$

$\frac{d^2\ell}{d\theta^2} \leq 0$

5. Let $Y_1, Y_2, \dots, Y_n$ be a random sample of size n from the pdf

$$f_Y(y, \theta) = \frac{2y}{\theta^2}, \qquad 0 \le y \le \theta.$$

Find the maximum likelihood estimate and the method of moments estimate for θ. Compare the values of the method of moments estimate and the maximum likelihood estimate if a random sample of size 5 consists of the numbers 17, 92, 46, 39, and 56.

MoM

$$E(Y) = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$E(Y^2) = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

$$\vdots$$

$$E(Y^k) = \frac{1}{n}\sum_{i=1}^{n} y_i^k$$

$y_1, \dots, y_n$

$$E(Y) = \boxed{\frac{1}{n}\sum y_i}$$

k.

$$E(Y) = \int_0^\theta y \cdot f_Y(y;\theta)\, dy$$

$$= \int_0^\theta y \cdot \frac{2y}{\theta^2}\, dy$$

$$= \frac{2}{3}\theta$$

$$\frac{2}{3}\theta = \frac{1}{n}\sum y_i$$

$$\hat{\theta}_{MoM} = \frac{3}{2} \cdot \frac{1}{n}\sum y_i$$

MLE

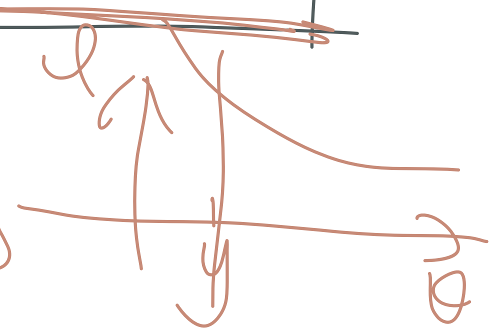
$$L = \prod_{i=1}^{n} \frac{2y_i}{\theta^2} = \theta^{-2n} \cdot 2^n \cdot \prod_{i=1}^{n} y_i$$

$$\boxed{l = -2n \cdot \log(\theta) + n \cdot \log(2) + \sum \log(y_i)}$$

$$l' = \frac{-2n}{\theta} = 0$$

$y < \theta$

$\theta \in (92, \infty)$



Max l = Min θ

$= 92$

$$\hat{\theta}_{MLE} = 92$$