$Var(U) = 1/12$

1. Suppose U is a uniform random variable over $[0, 1]$. Show that $Y = (b-a)U + a$ is uniform over $[a, b]$ and determine its variance.

$U \sim Unif(0,1)$

$f_U(u) = 1 \quad F_U(u) = u$

$Y = (b-a)\cdot U + a$

$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}((b-a)\cdot U + a \le y)$

$= \mathbb{P}((b-a)\cdot U \le y-a)$

$= \mathbb{P}(U \le \frac{y-a}{b-a})$

$= F_U(\frac{y-a}{b-a})$

$= \frac{y-a}{b-a}$

$f_Y(y) = F_Y'(y) = \frac{1}{b-a}$

$Y \sim Unif(a,b) \qquad Var(Y) = \frac{(b-a)^2}{12}$

$Var(Y) = Var((b-a)\cdot U + a)$

$= (b-a)^2 \cdot Var(U)$

$= (b-a)^2/12$

r.v. X $\qquad Y = aX + b \qquad f_Y(y) = \frac{1}{|a|} \cdot f_X(\frac{y-b}{a})$

$= \frac{1}{|b-a|} \cdot f_X(\frac{y-a}{b-a})$,

$= \frac{1}{b-a}$

3. Suppose that the random variable Y is described by the pdf $f_Y(y) = cy^{-4}$, $y > 1$. Find c and determine the highest moment of Y that exists.

$$\int_1^\infty c \cdot y^{-4}\, dy = 1$$

$$c \cdot \frac{1}{-3} \cdot y^{-3} \Big|_1^\infty = 1$$

$$c = 3$$

$E(Y^r) \leftarrow r^{th}$ moment

$$= \int_1^\infty y^r \cdot 3 \cdot y^{-4}\, dy$$

$$= \int_1^\infty 3 \cdot y^{r-4}\, dy$$

$r=3$ $\quad \int_1^\infty 3 \cdot y^{-1}\, dy = 3 \cdot \log(y) \Big|_1^\infty$

$r \neq 3$ $\quad \int_1^\infty 3 \cdot y^{r-4}\, dy$

$$= 3 \cdot \frac{1}{r-3} \cdot y^{r-3} \Big|_1^\infty$$

$r < 3$ $\quad E(Y^r) < \infty$

5. Let $f_X(x) = xe^{-x}$, $x \geq 0$, and $f_Y(y) = e^{-y}$, $y \geq 0$, where X and Y are independent. Find the pdf of $X + Y$.

$$f_W(w) = \int_{-\infty}^\infty f_X(x) \cdot f_Y(w-x)\, dx$$

$W = X + Y$ $\qquad X = W - Y$ $\qquad Y \geq 0$ $\qquad X \leq W$ $\qquad 0 \leq X \leq W$

$$f_W(w) = \int_{-\infty}^\infty x \cdot e^{-x}\, e^{-(w-x)}\, dx$$

$$= \int_0^w x \cdot e^{-x} \cdot e^{-(w-x)}\, dx$$

$$= \int_0^w x \cdot \cancel{e^{-x}} \cdot e^{-w} \cdot \cancel{e^{x}}\, dx$$

$$= e^{-w} \cdot \frac{1}{2} \cdot x^2 \Big|_0^w$$

$$= e^{-w} \cdot \frac{1}{2} \cdot w^2 \sim Ga(3, 1)$$

6. Let Y be a continuous non-negative random variable. Show that $W = Y^2$ has pdf $f_W(w) = \frac{1}{2\sqrt{w}} f_Y(\sqrt{w})$. (*Hint:* First find $F_W(w)$.)

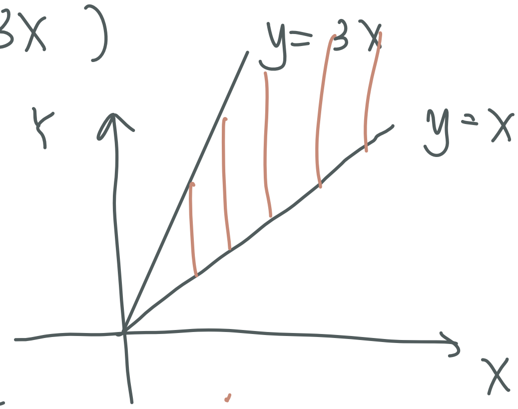
$$F_W(w) = \mathbb{P}(W \le w)$$
$$= \mathbb{P}(Y^2 \le w)$$
$$= \mathbb{P}(Y \le \sqrt{w})$$
$$= F_Y(\sqrt{w})$$

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{dF_Y(\sqrt{w})}{dw} = f_Y(\sqrt{w}) \cdot \frac{d\sqrt{w}}{dw}$$
$$= f_Y(\sqrt{w}) \cdot \frac{1}{2} \cdot w^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{w}} \cdot f_Y(\sqrt{w})$$

9. Let X and Y have the joint pdf $f_{X,Y}(x,y) = 2e^{-(x+y)}$, $0 < x < y$, $0 < y$. Find $\boxed{P(Y < 3X)}$.

$$Y - 3X < 0$$

$$\iint f_{XY}(x,y)\,dx\,dy = \mathbb{P}(Y < 3X)$$



$$\int_0^\infty \int_x^{3x} 2 \cdot e^{-(x+y)}\,dy\,dx$$
$$= \int_0^\infty 2 \cdot e^{-x} \int_x^{3x} e^{-y}\,dy\,dx$$
$$= \int_0^\infty 2 \cdot e^{-x} \left(-e^{-y} \Big|_x^{3x} \right) dx$$
$$= \int_0^\infty -2 \cdot e^{-4x} + 2 \cdot e^{-2x}\,dx$$
$$= \frac{2}{4} \cdot e^{-4x} - e^{-2x} \Big|_0^\infty$$
$$= 1 - \frac{1}{2} = 1/2$$