1. (Proof of Theorem 3.6.2,, continuous case) Let X and Y be any two random variables with joint pdf $f_{X,Y}(x,y)$ and let a and b be any two constants. Show that $E(aX+bY) = aE(X)+bE(Y)$.

$$\underline{E(g(x)) = \iint g(x)\cdot f_{XY}(x,y)\,dx\,dy}$$

$$E(aX+bY) = \iint (aX+bY)\cdot f_{XY}(x,y)\,dx\,dy$$

$$= a\iint x f_{XY}(x,y)\,dx\,dy + b\cdot\iint y\cdot f_{XY}(x,y)\,dx\,dy$$

$$= a\int x \boxed{\int f_{XY}(x,y)\,dy}\,dx + b\cdot\int y \underline{\int f_{XY}(x,y)\,dx}\,dy$$

$$= a\underline{\int x f_X(x)\,dx} + b\underline{\int y f_Y(y)\,dy}$$

$$= a\cdot E(X) + b\cdot E(Y)$$

4. Show that

$$\mathrm{Cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y)$$

for any constants a, b, c, d.

Any r.v. $X\,Y$

$\mathrm{Cov}(XY) = E(XY) - E(X)\cdot E(Y)$

$$\mathrm{Cov}(aX+b, cY+d)$$

$$= E\big((aX+b)\cdot(cY+d)\big) - \underbrace{E(aX+b)}_{a\cdot E(X)+b}\cdot E(cY+d)$$

$$= E\big(a\cdot c\cdot X\cdot Y + a\cdot d\cdot X + b\cdot c\cdot Y + b\cdot d\big) - \big(aE(X)+b\big)\big(c\cdot E(Y)+d\big)$$

$$= a\cdot c\cdot E(XY) + \cancel{a\cdot d\cdot E(X)} + \cancel{b\cdot c\cdot E(Y)} + \cancel{b\cdot d} - a\cdot c\cdot E(X)\cdot E(Y) - \cancel{a\cdot d\cdot E(X)} - \cancel{b\cdot c\cdot E(Y)} - \cancel{b\cdot d}$$

$$= a\cdot c\big(E(XY) - E(X)\cdot E(Y)\big)$$

$$= a\cdot c\cdot \mathrm{Cov}(X, Y)$$

5. Suppose the length of time, in minutes, that you have to wait at a bank teller's window is uniformly distributed over the interval $(0, 10)$. If you go to the bank four times during the next month, what is the probability that your second longest wait will be less than five minutes?

X

$X_1, \ldots, X_4 \overset{i.i.d.}{\sim} U(0,10)$

$X_{(1)}, \ldots, X_{(4)}$ $\quad \mathbb{P}(X_{(3)} \le 5)$

$Y_1, \ldots, Y_n$

i^{th} order

$f_{Y_{i}}(y) = \frac{n!}{(i-1)!(n-i)!} [F_Y(y)]^{i-1} [1-F_Y(y)]^{n-i} f_Y(y)$

$n=4 \quad i=3 \quad f_X(x) = 1/10$

$F_X(x) = x/10$

$f_{X_{(3)}}(x) = \frac{4!}{2! \cdot 1!} \cdot \left(\frac{x}{10}\right)^2 \cdot \left(1 - \frac{x}{10}\right)^1 \cdot \frac{1}{10}$

$\mathbb{P}(X_{(3)} \le 5) = \int_0^5 \frac{4!}{2! \cdot 1!} \cdot \left(\frac{x}{10}\right)^2 \cdot \left(1 - \frac{x}{10}\right)^1 \cdot \frac{1}{10} \, dx$

$= \int_0^5 \frac{24}{20} \cdot \frac{x^2}{100} \cdot \left(1 - \frac{x}{10}\right) dx$

$= 5/16$

9. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2, and let

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

be the sample variance, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the $E(S^2)$.

$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1} \qquad (n-1) \cdot S^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2$

$\sum_{i=1}^{n} (X_i - \overline{X})^2 = \sum_{i=1}^{n} (X_i - \mu + \mu - \overline{X})^2$

$= \sum_{i=1}^{n} \left((X_i - \mu)^2 + (\mu - \overline{X})^2 + 2 \cdot (X_i - \mu) \cdot (\mu - \overline{X}) \right)$

$= \sum_{i=1}^{n} (X_i - \mu)^2 + n \cdot (\mu - \overline{X})^2 + 2 \cdot (\mu - \overline{X}) \boxed{\sum_{i=1}^{n} (X_i - \mu)}$

$n \cdot (\overline{X} - \mu)$

$$\sum_{i=1}^{n}(x_i-\mu)=\sum_{i=1}^{n}x_i-n\cdot\mu$$

$$=n\cdot\frac{1}{n}\cdot\sum_{i=1}^{n}x_i-n\mu.$$

$$=n\cdot\bar{X}-n\cdot\mu$$

$$=\underline{n(\bar{X}-\mu)}$$

$$-2n(\mu-\bar{X})^2$$

$$=\sum_{i=1}^{n}(x_i-\mu)^2-n\cdot(\mu-\bar{X})^2$$

$$E\left(\sum_{i=1}^{n}(x_i-\bar{X})^2\right)=\underline{E\left((n-1)\cdot S^2\right)}$$

$$=E\left(\sum_{i=1}^{n}(x_i-\mu)^2-n\cdot(\mu-\bar{X})^2\right)$$

$$=\underline{\sum_{i=1}^{n}E\left((x_i-\mu)^2\right)}-n\cdot E\left((\boxed{\bar{X}}-\mu)^2\right)$$

$$=\sum_{i=1}^{n}Var(x_i)-n\,Var(\bar{X})$$

$$=\sum_{i=1}^{n}\sigma^2-n\cdot\frac{\sigma^2}{n}$$

$$=n\cdot\sigma^2-\sigma^2=\underline{(n-1)\cdot\sigma^2}$$

$$E(S^2)=\sigma^2$$

$$\bar{X}=\frac{1}{n}\cdot\sum x_i$$

$$E(\bar{X})=\frac{1}{n}\cdot\sum_i E(x_i)$$

$$=\frac{1}{n}\cdot\sum\mu.$$

$$=\frac{1}{n}\cdot n\cdot\mu$$

$$=\mu.$$

$$Var(\bar{X})=Var\left(\frac{1}{n}\sum x_i\right)$$

$$=\left(\frac{1}{n}\right)^2\cdot Var\left(\sum x_i\right)$$

$$=\left(\frac{1}{n}\right)^2\cdot\sum Var(x_i)$$

$$=\frac{1}{n^2}\cdot n\cdot\sigma^2$$

$$=\underline{\sigma^2/n}$$