1. (Proof of Theorem 3.6.2,, continuous case) Let X and Y be any two random variables with joint pdf $f_{X,Y}(x,y)$ and let a and b be any two constants. Show that $E(aX+bY) = aE(X)+bE(Y)$.

$$\mathbb{E}(g(X,Y)) = \iint g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

$$\mathbb{E}(aX+bY)$$
$$= \iint (aX+bY)\cdot f_{XY}(x,y)\, dx\, dy$$
$$= a\cdot \iint x\cdot f_{XY}(x,y)\, dx\, dy + b\cdot \iint y\cdot f_{XY}(x,y)\, dx\, dy$$
$$= a\int x \int f_{XY}(x,y)\, dy\, dx + b\cdot \int y \int f_{XY}(x,y)\, dx\, dy$$
$$= a\int x\cdot f_X(x)\, dx + b\cdot \int y\cdot f_Y(y)\, dy$$
$$= a\cdot \mathbb{E}(X) + b\cdot \mathbb{E}(Y) \quad \#$$

4. Show that

$$\operatorname{Cov}(aX+b, cY+d) = ac\operatorname{Cov}(X,Y)$$

for any constants a, b, c, d.

X, Y

$\operatorname{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y)$

$\operatorname{cov}(aX+b, cY+d)$ $\quad a\cdot\mathbb{E}(X)+b$

$$= \mathbb{E}\big((aX+b)\cdot(cY+d)\big) - \mathbb{E}(aX+b)\cdot\mathbb{E}(cY+d)$$
$$= \mathbb{E}\big(a\cdot c\cdot \boxed{X\cdot Y} + a\cdot d\cdot X + b\cdot c\cdot Y + b\cdot d\big) - (a\cdot\mathbb{E}(X)+b)(c\cdot\mathbb{E}(Y)+d)$$
$$= a\cdot c\cdot\mathbb{E}(XY) + \cancel{a\cdot d\cdot\mathbb{E}(X)} + \cancel{b\cdot c\cdot\mathbb{E}(Y)} + \cancel{b\cdot d} - a\cdot c\cdot\mathbb{E}(X)\cdot\mathbb{E}(Y) - \cancel{a\cdot d\cdot\mathbb{E}(X)} - \cancel{b\cdot c\cdot\mathbb{E}(Y)} - \cancel{b\cdot d}$$
$$= a\cdot c\cdot\mathbb{E}(XY) - a\cdot c\cdot\mathbb{E}(X)\cdot\mathbb{E}(Y)$$
$$= a\cdot c\big(\mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y)\big)$$
$$= a\cdot c\cdot\operatorname{cov}(X,Y)$$

5. Suppose the length of time, in minutes, that you have to wait at a bank teller's window is uniformly distributed over the interval $(0, 10)$. If you go to the bank four times during the next month, what is the probability that your second longest wait will be less than five minutes?

X

$X_1, \dots X_4 \overset{i.i.d}{\sim} U(0,10)$

$X_{(1)}, \dots X_{(4)}$ $\qquad P(X_{(3)} \le 5)$

$f_X(x) = 1/10 \quad F_X(x) = x/10$

$Y_1 \dots Y_n$

i^{th} order Y_i

$f_{Y_i}(y) = \frac{n!}{(i-1)!(n-i)!} \cdot [F_Y(y)]^{i-1} \cdot [1-F_Y(y)]^{n-i} \cdot f_Y(y).$

$$f_{X_{(3)}}(x) = \frac{4!}{2!\,1!} \cdot \left(\frac{x}{10}\right)^2 \cdot \left(1-\frac{x}{10}\right)^1 \cdot 1/10$$

$$\int_0^5 \frac{4!}{2!\,1!} \cdot \left(\frac{x}{10}\right)^2 \cdot \left(1-\frac{x}{10}\right)^1 \cdot 1/10 \, dx$$

$$= \int_0^5 \frac{24}{20} \frac{x^2}{100} \left(1-\frac{x}{10}\right) dx$$

$$= \int_0^5 \frac{6}{5} \cdot \left(\frac{x^2}{100} - \frac{x^3}{1000}\right) dx$$

$$= 5/16$$

9. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2, and let

$$S^2 = \sum_{i=1}^n \frac{(X_i - \overline{X})^2}{n-1}$$

be the sample variance, where $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Find the $E(S^2)$.

$$S^2 = \sum_{i=1}^n \frac{(X_i-\overline{X})^2}{n-1} \Rightarrow (n-1)S^2 = \sum_{i=1}^n (X_i - \overline{X})^2$$

$$\sum_{i=1}^n (X_i - \overline{X})^2 = \sum_{i=1}^n (X_i - \mu + \mu - \overline{X})^2$$

$$= \sum_{i=1}^n \left((X_i-\mu)^2 + (\mu-\overline{X})^2 + 2\cdot(X_i-\mu)\cdot(\mu-\overline{X})\right)$$

$$= \sum_{i=1}^n (X_i-\mu)^2 + n\cdot(\mu-\overline{X})^2 + 2\cdot(\mu-\overline{X})\left(\sum_{i=1}^n (X_i-\mu)\right)$$

$$\sum_{i=1}^n (X_i-\mu) = \sum_{i=1}^n X_i - n\mu$$

$$= n\cdot\frac{1}{n}\sum_{i=1}^n X_i - n\mu$$

$$= n\cdot\bar{X} - n\cdot\mu$$

$$= n(\bar{X}-\mu)$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 + n\cdot(\mu-\bar{X})^2 + 2\cdot(\mu-\bar{X})\cdot n\cdot(\bar{X}-\mu)$$

$$- 2\cdot n\cdot(\mu-\bar{X})^2$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 - n(\mu-\bar{X})^2$$

$$E(S^2) = \frac{1}{n-1}\cdot E\left(\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$$

$$= \frac{1}{n-1}\cdot E\left(\sum_{i=1}^{n}(X_i-\mu)^2 - n\cdot(\mu-\bar{X})^2\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}\underbrace{E((X_i-\mu)^2)}_{Var(X_i)} - n\cdot\underbrace{E((\mu-\bar{X})^2)}_{Var(\bar{X})}\right)$$

$$= \frac{1}{n-1}\left(n\cdot\sigma^2 - n\cdot\frac{\sigma^2}{n}\right)$$

$$= \cancel{\frac{1}{n-1}}(\cancel{(n-1)}\cdot\sigma^2)$$

$$= \sigma^2$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\cdot\sum X_i\right)$$

$$= \frac{1}{n^2}\cdot\sum Var(X_i)$$

$$= \frac{1}{n^2}\cdot n\cdot\sigma^2$$

$$= \sigma^2/n$$