$Var(U) = 1/12$

1. Suppose U is a uniform random variable over $[0,1]$. Show that $Y = (b-a)U + a$ is uniform over $[a,b]$ and determine its variance.

$U \sim Unif(0,1)$

$f_U(u) = 1$

$Y = (b-a)\cdot U + a$

$\mathbb{P}(Y \le y) = \mathbb{P}((b-a)\cdot U + a \le y)$

$= \mathbb{P}((b-a)\cdot U \le y - a)$

$= \mathbb{P}\left(U \le \frac{y-a}{b-a}\right)$

$= \frac{y-a}{b-a}$

$f_Y(y) = \frac{1}{b-a}$ $\quad Y \sim Unif(a,b)$

$Var(Y) = \frac{(b-a)^2}{12}$

$Var(Y) = Var((b-a)\cdot U + a)$

$= (b-a)^2 \cdot Var(U)$

$= (b-a)^2 \cdot \frac{1}{12}$

$X \quad Y = aX + b \Rightarrow \boxed{P_Y(y) = P_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}}$

$= \frac{1}{b-a} \cdot P_X\left(\frac{y-a}{b-a}\right)$

$= \frac{1}{b-a} \cdot 1$

3. Suppose that the random variable Y is described by the pdf $f_Y(y) = cy^{-4}$, $y > 1$. Find c and determine the highest moment of Y that exists.

$$\int_1^\infty c \cdot y^{-4}\, dy = 1$$

$$c \cdot \frac{1}{-3} \cdot y^{-3} \Big|_1^\infty = 1$$

$$c = 3$$

$$f_Y(y) = 3 \cdot y^{-4}$$

$E(Y^r) \leftarrow r^{th}$ moment of Y

$$= \int_1^\infty y^r \cdot f_Y(y)\, dy$$

$$= \int_1^\infty y^r \cdot 3 \cdot y^{-4}\, dy = \int_1^\infty 3 \cdot y^{r-4}\, dy$$

$r = 3$ $\quad \int_1^\infty 3 \cdot y^{-1}\, dy = 3 \cdot \log(y) \Big|_1^\infty = \infty$

$r \neq 3$ $\quad \int_1^\infty 3 \cdot y^{r-4}\, dy = \frac{3}{r-3} \cdot y^{r-3} \Big|_1^\infty$

if $r > 3$ $\quad E(Y^r) = \infty$

if $\boxed{r < 3}$ $\qquad$ have defi.

5. Let $f_X(x) = xe^{-x}$, $x \geq 0$, and $f_Y(y) = e^{-y}$, $y \geq 0$, where X and Y are independent. Find the pdf of $X + Y$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w - x)\, dx \qquad W = X + Y$$

$f_X(x) = x \cdot e^{-x}$ $\quad x \geq 0$

$f_Y(y) = e^{-y}$ $\quad y \geq 0$

$\boxed{W = X + Y}$

$X \leq W - Y$

$X \leq W$

$$f_W(w) = \int_{-\infty}^{\infty} x \cdot e^{-x} \cdot e^{-(w-x)}\, dx$$

$$= \int_0^w x \cdot e^{-x} \cdot e^{-(w-x)}\, dx$$

$$= \int_0^w x \cdot \cancel{e^{-x}} \cdot e^{-w} \cdot \cancel{e^{x}}\, dx = e^{-w} \cdot \frac{1}{2} \cdot x^2 \Big|_0^w$$

$= \frac{1}{2} w^2 \cdot e^{-w} \qquad w \geq 0$

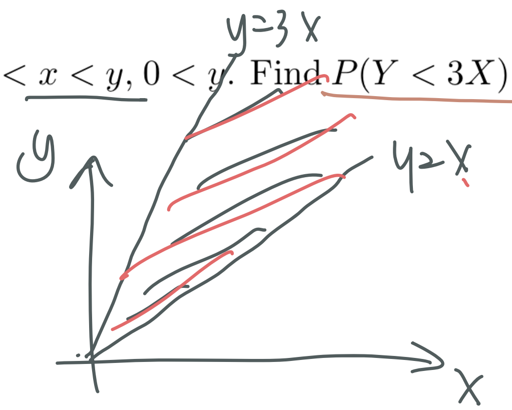
$\sim Ga(3.1)$

6. Let Y be a continuous non-negative random variable. Show that $W = Y^2$ has pdf $f_W(w) = \frac{1}{2\sqrt{w}} f_Y(\sqrt{w})$. (*Hint:* First find $F_W(w)$.)

$$F_W(w) = \mathbb{P}(W \leq w) = \mathbb{P}(Y^2 \leq w)$$
$$= \mathbb{P}(Y \leq \sqrt{w})$$
$$= F_Y(\sqrt{w})$$

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{dF_Y(\sqrt{w})}{dw} = f_Y(\sqrt{w}) \cdot \frac{d\sqrt{w}}{dw}$$
$$= f_Y(\sqrt{w}) \cdot \frac{1}{2} \cdot w^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{w}} \cdot f_Y(\sqrt{w})$$

9. Let X and Y have the joint pdf $f_{X,Y}(x, y) = 2e^{-(x+y)}$, $0 < x < y$, $0 < y$. Find $P(Y < 3X)$.



$$\mathbb{P}(Y < 3X) = \boxed{\int\int} f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^\infty \int_x^{3x} 2e^{-(x+y)}\, dy\, dx$$

$$= \int_0^\infty 2 \cdot e^{-x} \cdot \left(-e^{-y} \Big|_x^{3x}\right) dx$$

$$= \int_0^\infty -2 \cdot e^{-4x} + 2 \cdot e^{-2x}\, dx$$

$$= \frac{2}{4} \cdot e^{-4x} - e^{-2x} \Big|_0^\infty$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$