1. (Proof of Theorem 3.6.2,, continuous case) Let X and Y be any two random variables with joint pdf $f_{X,Y}(x, y)$ and let a and b be any two constants. Show that $E(aX + bY) = aE(X) + bE(Y)$.

$$E(aX+bY) = \iint (ax+by)\cdot f_{XY}(x,y)\,dx\,dy$$

$$= a\iint x\cdot f_{XY}(x,y)\,dx\,dy + b\cdot\iint y\cdot f_{XY}(x,y)\,dx\,dy$$

$$= a\int x \underline{\int f_{XY}(x,y)\,dy}\,dx + b\cdot\int y \underline{\int f_{XY}(x,y)\,dx}\,dy$$

$$= a\cdot\int x\, f_X(x)\,dx + b\cdot\int y\, f_Y(y)\,dy$$

$$= a\cdot E(X) + b\cdot E(Y)$$

4. Show that

$$\mathrm{Cov}(aX + b, cY + d) = ac\,\mathrm{Cov}(X, Y)$$

for any constants a, b, c, d.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

$$E(aX+b) = a\cdot E(X)+b$$

$$\mathrm{Cov}(aX+b,\ cY+d) = E\big((aX+b)(cY+d)\big) - E(aX+b)\cdot E(cY+d)$$

$$= E(a\cdot c\cdot X\cdot Y + a\cdot d\cdot X + b\cdot c\cdot Y + b\cdot d) - (a\cdot E(X)+b)(c\cdot E(Y)+d)$$

$$= a\cdot c\cdot E(XY) + \cancel{a\cdot d\cdot E(X)} + \cancel{b\cdot c\cdot E(Y)} + \cancel{b\cdot d} - (a\cdot c\cdot E(X)\cdot E(Y) + \cancel{a\cdot d\cdot E(X)} + \cancel{b\cdot c\cdot E(Y)} + \cancel{b\cdot d})$$

$$= a\cdot c\cdot E(XY) - a\cdot c\cdot E(X)\cdot E(Y)$$

$$= a\cdot c\,\big(E(XY) - E(X)\cdot E(Y)\big)$$

$$= a\cdot c\cdot \mathrm{Cov}(X,Y)$$

5. Suppose the length of time, in minutes, that you have to wait at a bank teller's window is uniformly distributed over the interval $(0, 10)$. If you go to the bank four times during the next month, what is the probability that your second longest wait will be less than five minutes?

X

$X_1, \ldots, X_4 \overset{i.i.d}{\sim} U(0,10)$

$Y_1, \ldots, Y_n$. ith order

$X_{(1)} \ldots X_{(4)}$ $\quad$ $X_{(3)}$ $\quad$ $\mathbb{P}(X_{(3)} \leq 5)$

$f_{Y_{(i)}}(y) = \frac{n!}{(i-1)!(n-i)!} \cdot [F_Y(y)]^{i-1} [1-F_Y(y)]^{n-i} f_Y(y)$

$n=4 \quad i=3 \quad f_Y(y) = 1/10 \quad F_Y(y) = x/10$

$f_{X_{(3)}}(x) = \frac{4!}{2!\,1!} \cdot \left(\frac{x}{10}\right)^2 \cdot \left(1-\frac{x}{10}\right)^1 \cdot 1/10$

$\mathbb{P}(X_{(3)} \leq 5) = \int_0^5 \frac{4!}{2!\,1!} \cdot \left(\frac{x}{10}\right)^2 \cdot \left(1-\frac{x}{10}\right)^1 \cdot 1/10 \, dx$

$= \int_0^5 \frac{24}{20} \cdot \frac{x^2}{100} \left(1-\frac{x}{10}\right) dx$

$= 5/16$

9. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having expected value μ and variance σ^2, and let

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

be the sample variance, where $\boxed{\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i}$. Find the $\mathrm{E}(S^2)$.

$n \cdot \overline{X} = \sum X_i$

$\mathbb{E}(S^2) = \mathbb{E}\left(\sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}\right)$

$(n-1) \cdot \mathbb{E}(S^2) = \mathbb{E}\left(\boxed{\sum_{i=1}^{n} (X_i - \overline{X})^2}\right)$

$\sum_{i=1}^{n} (X_i - \overline{X})^2 = \sum_{i=1}^{n} (X_i - \mu + \mu - \overline{X})^2$

$= \sum_{i=1}^{n} \left((X_i - \mu)^2 + (\mu - \overline{X})^2 + 2 \cdot (X_i - \mu)(\mu - \overline{X}) \right)$

$= \sum_{i=1}^{n} (X_i - \mu)^2 + n \cdot (\mu - \overline{X})^2 + 2 \cdot (\mu - \overline{X}) \boxed{\sum_{i=1}^{n} (X_i - \mu)}$

$$= \sum_{i=1}^{n}(x_i-\mu)^2 + n\cdot(\mu-\bar{x})^2 \; -2\cdot(\mu-\bar{x})\cdot n\cdot(\bar{x}-\mu)$$

$$(\sum x_i - n\cdot\mu) \quad \frac{}{n\cdot\bar{x}} \quad (n\cdot\bar{x}-n\cdot\mu) \quad n\cdot(\bar{x}-\mu)$$

$$(\cdot(\mu-\bar{x}))$$

$$-2\cdot n\cdot(\mu-\bar{x})^2$$

$$= \sum_{i=1}^{n}(x_i-\mu)^2 - n\cdot(\mu-\bar{x})^2$$

$$E\left(\sum_{i=1}^{n}(x_i-\bar{X})^2\right) = E\left(\sum_{i=1}^{n}(x_i-\mu)^2 - n\cdot(\mu-\bar{x})^2\right)$$

$$= \sum_{i=1}^{n}\underbrace{E\left((x_i-\mu)^2\right)}_{Var(x_i)} - n\cdot\underbrace{E\left((\bar{X}-\mu)^2\right)}_{Var(\bar{X})}$$

$$= n\cdot\sigma^2 - n\cdot\frac{\sigma^2}{n}$$

$$= (n-1)\cdot\sigma^2$$