STAT 2011
Tutorial 9

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Mean and variance

- Further Properties:

Theorem 3.6.1. *For RVs X and Y and some function $g(X,Y) = W$*

$$\mathrm{E}[g(X,Y)] = \begin{cases} \sum_{\text{all } x} \sum_{\text{all } y} g(x,y) p_{X,Y}(x,y) & X,Y \text{ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy & X,Y \text{ continuous} \end{cases} \qquad \square$$

Alternatively to Theorem 3.6.1, we could also work more directly with the pdf of W to calculate $\mathrm{E}[g(X,Y)] = \mathrm{E}(W)$.

Theorem 3.6.2. *Let $W_1, W_2, \ldots, W_n$ be any RVs with $\mathrm{E}(|W_i|) < \infty$ and let $a_1, a_2, \ldots, a_n$ be any set of constants. Then*

$$\mathrm{E}(a_1 W_1 + a_2 W_2 + \ldots + a_n W_n) = a_1 \mathrm{E}(W_1) + a_2 \mathrm{E}(W_2) + \ldots + a_n \mathrm{E}(W_n) \qquad \square$$

Theorem 3.6.3. *If X and Y are independent RVs then*

$$\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y) \qquad \square$$

Covariance

- Definition:

Definition 3.6.1. *The covariance of X and Y (assuming they have finite variance) is*

$$\operatorname{Cov}(X, Y) = \operatorname{E}(XY) - \operatorname{E}(X)\operatorname{E}(Y) \qquad \square$$

Theorem 3.6.4. *If X and Y are independent then*

$$\operatorname{Cov}(X, Y) = 0 \qquad \square$$

Theorem 3.6.5.

a. $\operatorname{Var}(aX + bY) = a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y)$

b. $\operatorname{Var}(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i^2 \operatorname{Var}(X_i) + 2 \sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j)$ $\square$

Order Statistics

- Definition:

Definition 3.7.1 (Order statistics). *Let Y be a continuous random variable for which $y_1, \ldots, y_n$ are the values of a random sample of size n. Reorder the y_i's from smallest to largest:*

$$y'_1 < \ldots < y'_n$$

- PDF:

Theorem 3.7.1 (Pdfs of Y'_{min} and Y'_{max}). *Suppose that $Y_1, Y_2, \ldots, Y_n$ is a i.i.d. sample of continuous random variables each having pdf $f_Y(y)$ and cdf $F_Y(y)$. Then*

(i) The pdf of the largest order statistic is

$$f_{Y_{max}}(y) = f_{Y'_n}(y) = n[F_Y(y)]^{n-1} f_Y(y)$$

(ii) The pdf of the smallest order statistic is

$$f_{Y_{min}}(y) = f_{Y'_1}(y) = n[1 - F_Y(y)]^{n-1} f_Y(y)$$

Order Statistics

Theorem 3.7.2 (pdfs of Y_i'). *Let* $Y_1, Y_2, \ldots, Y_n$ *be an i.i.d. sample of continuous random variables drawn from a distribution having* $f_Y(y)$ *and cdf* $F_Y(y)$. *The pdf of the i-th order statistic, for* $1 \leq i \leq n$, *is given by*

$$f_{Y_i'}(y) = \frac{n!}{(i-1)!(n-i)!}[F_Y(y)]^{i-1}[1-F_Y(y)]^{n-i} f_Y(y).$$

- Joint Distribution:

$$f_{Y_i',Y_j'} = \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!} \cdot [F_Y(u)]^{i-1} \cdot$$
$$\cdot [F_Y(v) - F_Y(u)]^{j-i-1} \cdot [1-F_Y(v)]^{n-j} \cdot f_Y(u) \cdot f_Y(v)$$

for $i < j$ and $u < v$.