1. Show that the function $f_Y(y) = 6y(1-y)$, $0 \le y \le 1$ is a pdf.

Q1: $\int_y f_Y(y)\,dy = 1$?

$$\int_0^1 6y(1-y)\,dy$$
$$= \int_0^1 6y - 6y^2\,dy$$
$$= 3y^2 - 2y^3 \Big|_0^1$$
$$= 3 - 2 - 0 = 1$$

3. If the pdf for W is

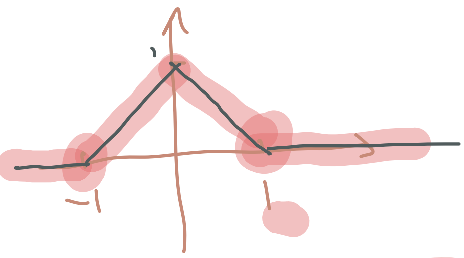
$$f_W(w) = \begin{cases} 0, & |w| > 1 \\ 1 - |w|, & |w| \le 1 \end{cases}$$

$1+w \quad -1 \le w \le 0$

$1-w \quad 0 \le w \le 1$

find and graph $F_W(w)$.

if we know p.d.f. $f_X(x)$, for c.d.f. $F(x) = \int_{-\infty}^{x} f_X(x)\,dx$.



$-1 \le w \le 0$

$$F_W(w) = \int_{-1}^{w} 1 + w\,dw$$
$$= w + \tfrac{1}{2}w^2 \Big|_{-1}^{w}$$
$$= w + \tfrac{1}{2}\cdot w^2 + \tfrac{1}{2}$$

$$\boxed{\begin{cases} 0 & \text{if } w \le -1 \\ w + \frac{1}{2}\cdot w^2 + \frac{1}{2} & -1 \le w \le 0 \\ w - \frac{1}{2}w^2 + \frac{1}{2} & 0 \le w \le 1 \\ 1 & \text{if } w \ge 1 \end{cases}}$$

$0 \le w \le 1$

$$F_W(w) = \int_{-\infty}^{w} f_W(w)\,dw$$
$$= \int_{-\infty}^{-1} 0\,dw + \int_{-1}^{0} 1 + w\,dw + \int_{0}^{w} 1 - w\,dw$$
$$= \int_{-1}^{0} 1 + w\,dw + \int_{0}^{w} 1 - w\,dw$$
$$= w + \tfrac{1}{2}w^2 \Big|_{-1}^{0} + w - \tfrac{1}{2}w^2 \Big|_{0}^{w}$$
$$= w - \tfrac{1}{2}\cdot w^2 + \tfrac{1}{2}$$

6. Consider a geometric random variable.

(a) Show that the cdf for a geometric random variable is given by $F_X(t) = P(X \le t) = 1 - (1-p)^{\lfloor t \rfloor}$, where $\lfloor t \rfloor$ denotes the greatest integer in t, $t \ge 0$.

(b) Suppose three fair dice are tossed repeatedly. Let the random variable X denote the roll on which a sum of 4 appears for the first time. Use the expression for $F_X(t)$ given in (a) to evaluate $P(65 \le X \le 75)$.

(a) $X \sim \text{Geometric}(p)$

$p_X(k) = \mathbb{P}(X=k) = (1-p)^{k-1} \cdot p$

$F_X(t) = \mathbb{P}(X \le t) = \mathbb{P}(X \le \lfloor t \rfloor)$

$= \sum_{k=1}^{\lfloor t \rfloor} (1-p)^{k-1} \cdot p$

$= \cancel{p} \dfrac{1-(1-p)^{\lfloor t \rfloor}}{\cancel{1-(1-p)}}$

$= 1-(1-p)^{\lfloor t \rfloor}$

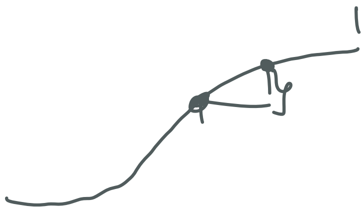
(b) $X \sim \text{Geometric}(p)$ $\quad \{(1,1,2)\ (1,2,1)\ (2,1,1)\}$

$p = \dfrac{3}{6^3} = \dfrac{1}{72}$

$X \sim \text{Geometric}(\frac{1}{72})$

$\mathbb{P}(65 \le X \le 75) = \sum_{x=65}^{75} (1-p)^{x-1} \cdot p$

$= F_X(75) - F_X(64)$



$= 1-(1-p)^{75} - (1-(1-p)^{64})$

$= 1-(1-\frac{1}{72})^{75} - (1-(1-\frac{1}{72})^{64})$

$= 0.058$

8. When a machine is improperly adjusted, it has probability 0.15 of producing a defective item. Each day, the machine is run until three defective items are produced. When this occurs, it is stopped and checked for adjustment. What is the probability that an improperly adjusted machine will produce five or more items before being stopped? What is the average number of items an improperly adjusted machine will produce?

X: # of item the machine produced until three defective items are produced

$X \sim \text{NegBino}(3, 0.15)$ $\qquad \mathbb{P}(X=k) = P_X(k) = \binom{k-1}{r-1} \cdot p^r \cdot (1-p)^{k-r}$

$r=3 \quad p=0.15$

$\mathbb{P}(X \ge 5) = 1 - \mathbb{P}(X<5)$

$= 1 - \mathbb{P}(X=3) - \mathbb{P}(X=4)$

$$= 1 - \sum_{k=3}^{4} \binom{k-1}{3-1} \cdot 0.15^{3} \cdot (1-0.15)^{k-3}$$

$$= 0.988$$

$$E(X) = \frac{r}{p} = \frac{3}{0.15} = 20$$

9. Suppose that X is a continuous random variable whose probability density function is given by

$$f_X(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

- What is the value of C?
- Find $P(X > 1)$.

(a) $\int_x c\,(4x - 2x^2)\,dx = 1$

$$\int_0^2 c \cdot (4x - 2x^2)\,dx = 1$$

$$c \cdot \int_0^1 (4x - 2x^2)\,dx = 1$$

$$c \cdot \left(2x^2 - \frac{2}{3} \cdot x^3 \Big|_0^2\right) = 1$$

$$c \cdot \left(2 \cdot 4 - \frac{2}{3} \cdot 8\right) = 1$$

$$c = 3/8$$

(b) $P(X > 1) = \int_1^\infty f_X(x)\,dx$

$$= \int_1^2 \frac{3}{8} \cdot (4x - 2x^2)\,dx$$

$$= \int_1^2 \frac{3}{2}x - \frac{3}{4}x^2\,dx$$

$$= \frac{3}{4}x^2 - \frac{3}{12}x^3 \Big|_1^2$$

$$= \frac{3}{4} \cdot 4 - \frac{3}{12} \cdot 8 - \left(\frac{3}{4} - \frac{3}{12}\right)$$

$$= \frac{1}{2}$$