1. Let X denote the number on a chip drawn at random from an urn containing three chips, numbered 1, 2, and 3. Let Y be the number of heads that occur when a fair coin is tossed X times. Find $p_{X,Y}(x,y)$.

$P_X(x) = \frac{1}{3}$ $\quad X = \{1, 2, 3\}$

$Y|X \sim Bin(x, \frac{1}{2})$

$P_{Y|X}(y|x) = \binom{x}{y} (\frac{1}{2})^y (\frac{1}{2})^{x-y}$ $\qquad P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$

$P_{XY}(x,y) = P_{Y|X}(y|x) \cdot P_X(x)$

$= \frac{1}{3} \cdot \binom{x}{y} (\frac{1}{2})^y \cdot (\frac{1}{2})^{x-y}$

$= \frac{1}{3} \binom{x}{y} (\frac{1}{2})^x$ $\quad X = \{1, 2, 3\}$

$Y \leq X$

4. Let Z be a standard normal random variable, $Z \sim N(0,1)$, and $X = \mu + \sigma Z$. Then $X \sim N(\mu, \sigma^2)$. The pdf of Z is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

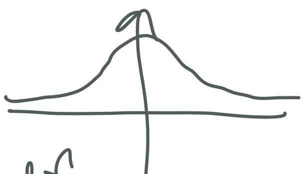
and the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(a) Find the mgf of Z.

(b) Noting that X is a linear transformation of Z, find the mgf of X.

(c) Using (b), show that a random variable with distribution $N(\mu, \sigma^2)$ has mean μ and variance σ^2.



$Z \sim N(0,1)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, p.d.f.

MGF $\quad$ r.v. X

$M_X(t) = E_X(e^{t \cdot X})$

conti. $E_X(e^{tX}) = \int e^{tx} \cdot f_X(x) dx$

discr. $E_X(e^{tX}) = \sum e^{tx} \cdot P_X(x)$

$M_Z(t) = E_Z(e^{tz})$

$= \int_{-\infty}^{\infty} \boxed{e^{t \cdot z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}} dz$

r.v. X

$\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2 \cdot t \cdot z}{2}} dz$

$z^2 - 2 \cdot t \cdot z + t^2 - t^2$.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2} - t^2} \, dx.$$

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.} \quad N(\mu, \sigma^2)$$

$$\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2}}$$

$$= e^{\frac{t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}} \, dz.}_{=1}$$

p.d.f of $N(t, 1)$

$$= e^{\frac{t^2}{2}}$$

$Z \sim N(t, 1)$ $\sigma^2 = 1$

$\mu = t$

$$f_Z(z) = \frac{1}{\sqrt{2\pi} \cdot 1} \cdot e^{-\frac{(z-1)^2}{2}}$$

(b)

$$\boxed{X = \mu + \sigma Z.} \quad \mu, \sigma \text{ fixed.}$$

$\tau = t \cdot \sigma.$

$$Z \quad M_Z(t) = e^{\frac{t^2}{2}} = E_Z(e^{\tau \cdot z}) = M_Z(\tau)$$

$$M_Z(t \cdot \sigma) = E_Z(e^{t \cdot \sigma z}) = e^{\frac{(t \cdot \sigma)^2}{2}}$$

$$M_X(t) = E_X(e^{t \cdot X}) = E_Z(e^{t \cdot (\mu + \sigma \cdot z)})$$

$$= E_Z(\boxed{e^{t \cdot \mu}} \, e^{t \cdot \sigma \cdot z})$$

$$= e^{t \cdot \mu} \cdot E_Z(e^{t \cdot \sigma \cdot z}) = M_Z(t \cdot \sigma)$$

$$= e^{t \cdot \mu} \cdot e^{\frac{(t \cdot \sigma)^2}{2}}$$

(c) $M'_X(0) = E(X)$

$M''_X(0) = E(X^2)$

$$M_X(t) = e^{t \cdot \mu + \frac{t^2 \cdot \sigma^2}{2}}$$

$$M'_X(t) = e^{t \cdot \mu + \frac{t^2 \cdot \sigma^2}{2}} \cdot \left(\mu + \frac{\cancel{2} \cdot t \cdot \sigma^2}{\cancel{2}}\right)$$

$$M'_X(0) = \mu = E(X)$$

$$M''_X(t) = e^{t \cdot \mu + \frac{t^2 \sigma^2}{2}} (\mu + t \cdot \sigma^2)^2 + e^{t \cdot \mu + \frac{t^2 \cdot \sigma^2}{2}} \cdot \sigma^2$$

$$M''_X(0) = \mu^2 + \sigma^2 = E(X^2)$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$= M_X''(0) - (M_X'(0))^2$$

$$= \mu^2 + \sigma^2 - \mu^2 \quad = \sigma^2$$