1. Let X denote the number on a chip drawn at random from an urn containing three chips, numbered 1, 2, and 3. Let Y be the number of heads that occur when a fair coin is tossed X times. Find $p_{X,Y}(x,y)$.

$P_X(x) = \frac{1}{3} \quad x = 1.2.3$

$Y | X \sim Bin(x. \frac{1}{2})$

$P_{Y|X}(y|x) = \binom{x}{y} \cdot (\frac{1}{2})^y (\frac{1}{2})^{x-y}$

$P_{X,Y}(x.y) = P_{Y|X}(y|x) \cdot P_X(x)$

$= \binom{x}{y} (\frac{1}{2})^y (\frac{1}{2})^{x-y} \cdot \frac{1}{3}$

$= \frac{1}{3} \binom{x}{y} (\frac{1}{2})^x$ $\quad x = \{1.2.3\}$

$y \leq x$

$P_{Y|X}(y|x) = \frac{P_{XY}(x.y)}{P_X(x)}$

4. Let Z be a standard normal random variable, $Z \sim N(0,1)$, and $X = \mu + \sigma Z$. Then $X \sim N(\mu, \sigma^2)$. The pdf of Z is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

$\int f_X(x)\, dx = 1$

and the pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(a) Find the mgf of Z. $\mathbb{E}(e^{t \cdot Z}) = M_Z(t)$

(b) Noting that X is a linear transformation of Z, find the mgf of X.

(c) Using (b), show that a random variable with distribution $N(\mu, \sigma^2)$ has mean μ and variance σ^2.

(a). $M_Z(t) = \mathbb{E}(e^{t \cdot Z}) = \int_{-\infty}^{\infty} e^{t \cdot z} f_Z(z)\, dz.$

$= \int_{-\infty}^{\infty} e^{t \cdot z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2 - 2t \cdot z + t^2 - t^2}{2}}\, dz.$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2} + \frac{t^2}{2}}\, dz.$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}} dz$$

$T \sim N(t, 1)$

$$= e^{t^2/2}$$

$X = \mu + \sigma Z.$

(b). $M_X(t) = E(e^{t\cdot X}) = E(e^{t\cdot(\mu + \sigma\cdot Z)})$

$$= E(e^{t\cdot\mu + t\cdot\sigma\cdot Z})$$

$$= e^{t\cdot\mu}\, E(e^{t\cdot\sigma\cdot Z})$$

$$= e^{t\cdot\mu}\cdot e^{(t\cdot\sigma)^2/2}$$

$X \sim N(\mu, \sigma^2)$

$E(e^{a\cdot Z}) = e^{a^2/2}$

$E(e^{t\cdot\sigma\cdot Z}) = e^{(t\cdot\sigma)^2/2}$

6. Prove that the mgf of a $N(\mu, \sigma^2)$ random variable is $e^{\mu t + \frac{\sigma^2 t^2}{2}}$. Suppose $X_i \sim N(\mu_i, \sigma_i^2)$, for $i = 1, 2$, where X_1, X_2 are independent. Identify the distribution of $Y = X_1 + X_2$ by deriving its mgf first.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

$X \sim N(\mu, \sigma^2)$

$$M_X(t) = E(e^{X\cdot t}) = \int_{-\infty}^{\infty} e^{x\cdot t} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Q6. $X \sim N(\mu, \sigma)$.

$$M_X(t) = E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2 - 2\mu x + \mu^2 - 2\sigma^2 \cdot x \cdot t}{2\sigma^2}} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2 - (2\mu + 2\sigma^2 t)x + \mu^2}{2\sigma^2}} dx.$$

$(x - (\mu + \sigma^2 t))^2$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2 - 2(\mu + \sigma^2 t)x + (\mu + \sigma^2 t)^2}{2\sigma^2}} \cdot e^{\frac{(\sigma^2 t)^2 + 2\mu\sigma^2 t}{2\sigma^2}} dx$$

$-(\mu + \sigma^2 t)^2$

$$= e^{\frac{\sigma^2 t^2}{2} + \mu t}$$

(c) Using (b), show that a random variable with distribution $N(\mu, \sigma^2)$ has mean μ and variance σ^2.

$$M_X(t) = \quad = e^{t \cdot \mu} \cdot e^{(t \cdot \sigma)^2 / 2} \qquad X \sim N(\mu, \sigma^2)$$

$$M_X'(0) = \mathbb{E}(X) \qquad M_X''(0) = \mathbb{E}(X^2)$$

$$M_X'(t) = e^{\frac{t^2 \cdot \sigma^2}{2} + \mu \cdot t} (\mu + \sigma^2 \cdot t)$$

$$\mathbb{E}(X) = M_X'(0) = \mu$$

$$M_X''(t) = e^{\frac{t^2 \cdot \sigma^2}{2} + \mu t} (\mu + \sigma^2 t)^2 + e^{\frac{t^2 \sigma^2}{2} + \mu t} \cdot (\sigma^2)$$

$$\mathbb{E}(X^2) = M_X''(0) = \mu^2 + \sigma^2$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2.$$