STAT 2011

Q2

2. The great English diarist Samuel Pepys asked his friend Sir Isaac Newton the following question: Is it more likely to get A at least one 6 when six dice are rolled, at least two 6's when twelve dice are rolled, or at least three 6's when eighteen dice are rolled? After considerable correspondence Newton convinced the skeptical Pepys that the first event is the most likely. Compute the three probabilities.

A: X_1: # of 6 in trails

$X_1 \sim \text{bino}(6, 1/6)$

$$\mathbb{P}(X_1 \geq 1) = 1 - \mathbb{P}(X_1 \leq 0)$$
$$= 1 - \mathbb{P}(X_1 = 0)$$
$$= 1 - \binom{6}{0} \cdot \left(\tfrac{1}{6}\right)^0 \cdot \left(1 - \tfrac{1}{6}\right)^6$$
$$= 0.6651$$

1 - pbinom(0, 6, 1/6)

B $X_2 \sim \text{bino}(12, 1/6)$

$$\mathbb{P}(X_2 \geq 2) = 1 - \mathbb{P}(X_2 = 0) - \mathbb{P}(X_2 = 1)$$
$$= 0.6187$$

C $X_3 \sim \text{bino}(18, 1/6)$

$$\mathbb{P}(X_3 \geq 3) = 0.5973$$

8. A fair coin is tossed four times. Let X be the number of heads in the tosses minus the number of tails. Find $p_X(k)$.

Y: # of heads

X: # of heads - # of tails

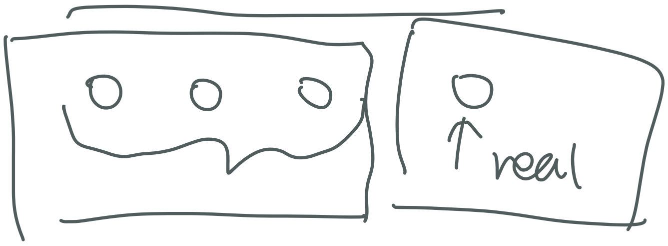
$Y \sim \text{binom}(4, 1/2)$

$Y =$	0	1	2	3	4
	4	3	2	1	0
$X:$	-4	-2	0	2	4

$$P_Y(y) = \binom{n}{y} \cdot \left(\tfrac{1}{2}\right)^y \cdot \left(\tfrac{1}{2}\right)^{n-y}$$

Y	$P_Y(y)$	X	$P_X(x)$
0	0.0625	-4	0.0625
1	0.25	-2	0.25
2	0.375	0	0.375
3	0.25	2	0.25
4	0.0625	4	0.0625

5. A display case contains thirty-five gems, of which ten are real diamonds and twenty-five are fake diamonds. A burglar removes four gems at random, one at a time and without replacement. What is the probability that the last gem she steals is the second real diamond in the set of four?



A: get one real in first three

$$\mathbb{P}(A) = \frac{\binom{10}{1}\binom{25}{2}}{\binom{35}{3}}$$

$= 0.458$

32 → 9 real, 23 fake.

$\mathbb{P}(B) = 9/32$ B: get real one in the last time

$$\mathbb{P}(A)\cdot\mathbb{P}(B) = \frac{9}{32} \times 0.458$$
$$= 0.129$$

10. Determine the expected value and the variance of a hypergeometric random variable.

$X \sim \text{Hypergeometric}(N, n, r)$

$$\mathbb{P}_X(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$\mathbb{E}(X) = \sum_x x \cdot P_X(x)$$

$$\mathbb{E}(X^2) = \sum_{x=0}^{n} x^2 \cdot \mathbb{P}(X = x)$$

$$= \sum_{x=0}^{n} x^2 \cdot \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$\binom{r}{x} = \frac{r!}{x!(r-x)!} \cdot \cancel{x}$$

$$\frac{r!}{(x-1)!\,(r-x)!}$$

$$\frac{r \cdot (r-1)!}{(x-1)!\,(r-x)!} \quad \binom{r-1}{x-1}$$

$$= r \sum_{x=0}^{n} x \frac{\binom{r-1}{x-1} \cdot \binom{N-r}{n-x}}{\binom{N}{n}}$$

$$= r \cdot \sum_{x=1}^{n} x \frac{\binom{r-1}{x-1}\binom{N-r}{n-x}}{\binom{N-1}{n-1} \cdot \frac{N}{n}}$$

$$\frac{N!}{(N-n)! \cdot n!} \qquad \frac{N\,(N-1)!}{n\,(N-n)!\,(n-1)!}$$

$$= \frac{r \cdot n}{N} \cdot \sum_{x=1}^{n} x \frac{\binom{r-1}{x-1}\binom{N-r}{n-x}}{\binom{N-1}{n-1}}$$

$$= \frac{r \cdot n}{N} \cdot 1$$

$$\boxed{= \frac{r \cdot n}{N}}$$

$X - 1 \sim$ Geometric $(N-1, n-1, r-1)$

$$= \frac{r \cdot n}{N} \cdot \mathbb{E}(\boxed{X-1} + 1)$$

$$= \frac{r \cdot n}{N} \left(\mathbb{E}(X-1) + 1 \right)$$

$$= \frac{r \cdot n}{N} \left(\frac{(n-1) \cdot (r-1)}{N-1} + 1 \right)$$

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$= \frac{r \cdot n}{N} \left(\frac{(n-1) \cdot (r-1)}{N-1} + 1 \right) - \left(\frac{r \cdot n}{N} \right)^2$$

13. If a typist averages one misspelling in every 3250 words, what are the chances that a 6000-word report is free of all such errors?

(a) Find this probability using exact binomial distribution.

(b) Find this probability using the Poisson approximation. Compare the answer with (a) and comment on the similarity (or dissimilarity).

(a) X : # of errors in 6K-word report.

$X \sim \text{binom}\left(6000, \frac{1}{3250}\right)$

$\mathbb{P}(X=0) = \binom{6000}{0} \cdot \left(\frac{1}{3250}\right)^0 \times \left(\frac{3249}{3250}\right)^{6000} = \underline{0.1578}$

(b) $\lambda = n \cdot p = 6000 \times \frac{1}{3250}$

$\mathbb{P}(X=0) = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-\lambda} = e^{-\frac{6000}{3250}}$

$= 0.1578$