4. The exponential pdf is a special case of the Weibull distribution, which measures time to failure of devices where the probability of failure increases as time does. A Weibull random variable Y has pdf

$$f_Y(y; \alpha, \beta) = \alpha\beta y^{\beta-1} e^{-\alpha y^\beta}, \quad 0 < y, 0 < \alpha, 0 < \beta.$$

(a) Given a random sample $y_1, \ldots, y_n$, find the maximum likelihood estimator for α assuming that β is known.

(b) Suppose α and β are both unknown. Write down the equations that would be solved simultaneously to find the maximum likelihood estimators of α and β.

$y_1, \ldots, y_n \overset{i.i.d}{\sim} f_Y(y; \alpha.\beta)$

MLE: $\mathcal{L}(y_1, \ldots, y_n; \alpha, \beta) = f_Y(y_1; \alpha, \beta) \times \cdots \times f_Y(y_n; \alpha, \beta)$

$$= \prod_{i=1}^{n} f_Y(y_i; \alpha.\beta)$$

$$= \prod_{i=1}^{n} \alpha \cdot \beta \cdot y_i^{\beta-1} e^{-\alpha \cdot y_i^\beta}$$

$e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

$$\log\left(\alpha^n \cdot \beta^n \prod_{i=1}^{n} y_i^{\beta-1} \cdot e^{\sum -\alpha \cdot y_i^\beta}\right)$$

$$\ell(y_1, \ldots, y_n; \alpha.\beta) = \log(\mathcal{L})$$

$$= n \cdot \log\alpha + n \cdot \log\beta + (\beta-1) \cdot \sum_{i=1}^{n} \log(y_i) - \alpha \cdot \sum y_i^\beta$$

$$\log\left(\prod_{i=1}^{n} y_i^{\beta-1}\right) = \sum_{i=1}^{n} \log(y_i^{\beta-1}) = \sum (\beta-1) \cdot \log(y_i) = (\beta-1) \cdot \sum \log(y_i)$$

MLE for α; given β is known.

$$\frac{d\ell}{d\alpha} = \frac{n}{\alpha} - \sum y_i^\beta = 0$$

$$\hat{\alpha}_{MLE} = \frac{n}{\sum y_i^\beta}$$

$$\frac{d^2\ell}{d\alpha^2} = -\frac{n}{\alpha^2} \leq 0$$

α, β both unknown

$$\begin{cases} \dfrac{\partial \ell}{\partial \alpha} = 0 \\ \dfrac{\partial \ell}{\partial \beta} = 0 \end{cases}$$

5. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the pdf

$$f_Y(y,\theta) = \frac{2y}{\theta^2}, \quad \boxed{0 \le y \le \theta.}$$

Find the maximum likelihood estimate and the method of moments estimate for θ. Compare the values of the method of moments estimate and the maximum likelihood estimate if a random sample of size 5 consists of the numbers 17, 92, 46, 39, and 56.

$$\mathcal{L}(y_1, \ldots, y_n) = \prod_{i=1}^{n} \frac{2y_i}{\theta^2}$$

$$= \theta^{-2n} \cdot 2^n \cdot \prod_{i=1}^{n} y_i$$

$$\ell(y_1, \ldots, y_n) = -2n \cdot \log(\theta) + n \cdot \log(2) + \sum \log(y_i)$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{2n}{\theta} = 0$$

$\theta \in [y, \infty)$

$\theta \uparrow \quad \ell \downarrow \quad \mathcal{L} \downarrow$

$\theta \downarrow \quad \mathcal{L} \uparrow$

$$\hat{\theta}_{MLE} = \max_i y_i$$
$$= 92$$

MoM: $\mathbb{E}(Y) = \frac{1}{n} \sum y_i$

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$$
$$= \int_0^\theta y \cdot \frac{2y}{\theta^2}\, dy$$
$$= \int_0^\theta \frac{1}{\theta^2} \cdot 2y^2\, dy$$
$$= \frac{2}{3}\theta$$

$$\frac{2}{3}\theta = \frac{1}{n}\sum y_i$$

$$\hat{\theta}_{MoM} = \frac{3}{2} \cdot \frac{1}{n}\sum y_i$$
$$= \frac{3}{2} \cdot \bar{y}$$
$$= 75$$

17, 92, 46, 39, and 56.

In this case, MLE is better.

$$P(x) = e^{-\lambda} \cdot \lambda^x$$

$f_X(x) = \frac{\quad}{x!}$

6. (a) Let $X \sim \text{Pois}(\lambda)$. Find the moment generating function of this distribution and use it to find its first moment.

   (b) A criminologist is searching through FBI files to document the prevalence of a rare double-whorl fingerprint. Among six consecutive sets of 100,000 prints scanned by a computer, the numbers of persons having the abnormality are 3, 0, 3, 4, 2, and 1, respectively. Assume that double whorls are Poisson events. Use the method of moments to estimate their occurrence rate, λ.

   (c) Find the MLE of λ. How would your answer change from (b) if λ were estimated using the method of maximum likelihood?

$$M_X(t) = \mathbb{E}(e^{t \cdot X})$$

$$= \sum_{k=0}^{\infty} e^{t \cdot k} \cdot \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

$$= e^{-\lambda} \underline{\sum_{k=0}^{\infty} \frac{e^{tk} \lambda^k}{k!}} \qquad \underline{\sum_{k=0}^{\infty} \frac{(\boxed{e^t \lambda})^k}{k!} = e^{e^t \lambda}}$$

$$= e^{-\lambda} \cdot e^{e^t \lambda}$$

$$\mathbb{E}(X) = \left. \frac{d M_X(t)}{dt} \right|_{t=0} = e^{-\lambda} \cdot e^{e^t \lambda} \cdot \lambda \cdot e^t \Big|_{t=0} = \lambda$$

(b) $\mathbb{E}(X) = \frac{1}{n} \sum X_i$

$$\underline{\hat{\lambda}_{mom} = \frac{1}{n} \sum X_i = \bar{X}}$$

(c) $X_1, \ldots X_n \overset{i.i.d}{\sim} \text{Pois}(\lambda)$

$$\mathcal{L} = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x!} = e^{-n\lambda} \cdot \lambda^{\sum X_i} \cdot \prod (x_i!)^{-1}$$

$$\ell = -n\lambda + \sum X_i \cdot \log(\lambda) - \underline{\log\left(\prod_{i=1}^{n} (x_i!)\right)}$$

$$\frac{\partial \ell}{\partial \lambda} = -n + \frac{\sum X_i}{\lambda} = 0$$

$$\ell'' = \frac{-\sum X_i}{\lambda^2} \leq 0$$

$$\hat{\lambda}_{MLE} = \frac{\sum X_i}{n} = \bar{X}$$