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1. Suppose U is a uniform random variable over $[0,1]$. Show that $Y = (b-a)U + a$ is uniform over $[a,b]$ and determine its variance.

$U \sim \mathrm{Unif}(0,1)$

$f_U(u) = 1 \quad \boxed{F_U(u) = u}$

$Y = (b-a)U + a$

$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}((b-a)\cdot U + a \leq y)$

$= \mathbb{P}((b-a)\cdot U \leq y-a)$

$= \mathbb{P}\left(U \leq \frac{y-a}{b-a}\right)$

$= F_U\left(\frac{y-a}{b-a}\right)$

$= \frac{y-a}{b-a}$

$f_Y(y) = F_Y'(y) = \frac{1}{b-a} \Rightarrow Y \sim \mathrm{Unif}(a,b) \qquad \mathrm{Var}(Y) = \frac{(b-a)^2}{12}$

$\mathrm{Var}(Y) = \mathrm{Var}((b-a)U + a)$

$= \mathrm{Var}((b-a)\cdot U)$

$= (b-a)^2 \cdot \mathrm{Var}(U) = \frac{(b-a)^2}{12}$

r.v. $X \qquad Y = a\cdot X + b \qquad P_Y(y) = P_X\left(\frac{y-b}{a}\right)\cdot \frac{1}{|a|}$

$= P_U\left(\frac{y-a}{b-a}\right)\cdot \frac{1}{|b-a|}$

$= \frac{1}{b-a}\cdot 1$

3. Suppose that the random variable Y is described by the pdf $f_Y(y) = cy^{-4}$, $y > 1$. Find c and determine the highest moment of Y that exists.

$$\int_1^\infty c \cdot y^{-4}\, dy = 1$$

$$\Rightarrow c = 3$$

$\mathbb{E}(Y^r) \leftarrow r^{th}$ moment of Y

$$= \int_1^\infty y^r \cdot 3 \cdot y^{-4}\, dy$$

$$= \int_1^\infty 3 \cdot y^{r-4}\, dy$$

$r = 3$ $\int_1^\infty 3 \cdot y^{-1}\, dy = 3 \cdot \log(y)\Big|_1^\infty$ $\to \infty$

$r \neq 3$ $\int_1^\infty 3 \cdot y^{r-4}\, dy$

$$= 3 \cdot \frac{1}{r-3} \cdot y^{r-3}\Big|_1^\infty$$

$r < 3$ moment exists

5. Let $f_X(x) = xe^{-x}$, $x \geq 0$, and $f_Y(y) = e^{-y}$, $y \geq 0$, where X and Y are independent. Find the pdf of $X + Y$.

$W = X + Y$ $\quad f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$

$W = X + Y$

$X = W - Y$

$X \leq W$

$$f_W(w) = \int_{-\infty}^{\infty} x \cdot e^{-x} \cdot e^{-(w-x)}\, dx$$

$$= \int_0^w x \cdot e^{-x} \cdot e^{-(w-x)}\, dx$$

$$= \int_0^w x \cdot \cancel{e^{-x}} \cdot e^{-w} \cdot \cancel{e^{x}}\, dx$$

$$= e^{-w} \cdot \frac{1}{2} x^2 \Big|_0^w$$

$$= \frac{1}{2} \cdot w^2 \cdot e^{-w}$$

6. Let Y be a continuous non-negative random variable. Show that $W = Y^2$ has pdf $f_W(w) = \frac{1}{2\sqrt{w}} f_Y(\sqrt{w})$. (*Hint:* First find $F_W(w)$.)

$$F_W(w) = \mathbb{P}(W \le w) = \mathbb{P}(Y^2 \le w)$$

$$= \mathbb{P}(Y \le \sqrt{w})$$

$$= F_Y(\sqrt{w})$$

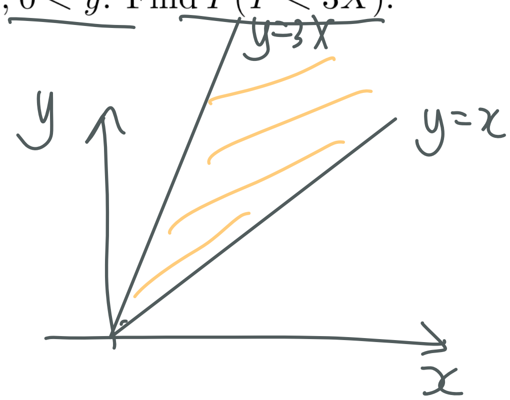
$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{dF_Y(\sqrt{w})}{dw}$$

$$= \frac{dF_Y(\sqrt{w})}{d\sqrt{w}} \cdot \frac{d\sqrt{w}}{dw}$$

$$= f_Y(\sqrt{w}) \cdot \frac{1}{2} \cdot w^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{w}} \cdot f_Y(\sqrt{w})$$

9. Let X and Y have the joint pdf $f_{X,Y}(x,y) = 2e^{-(x+y)}$, $0 < x < y$, $0 < y$. Find $P(Y < 3X)$.



$$\mathbb{P}(Y < 3X) = \int\int f_{XY}(x,y)\, dx\, dy$$

$$= \int_0^\infty \int_x^{3x} 2 \cdot e^{-(x+y)}\, dy\, dx$$

$$= \int_0^\infty 2 \cdot e^{-x} \int_x^{3x} e^{-y}\, dy\, dx$$

$$= \int_0^\infty 2 \cdot e^{-x} \left(-e^{-y} \Big|_x^{3x} \right) dx$$

$$= \int_0^\infty 2 \cdot e^{-x} \left(-e^{-3x} + e^{-x} \right) dx$$

$$= \int_0^\infty -2 \cdot e^{-4x} + 2 \cdot e^{-2x}\, dx$$

$$= \frac{2}{4} e^{-4x} - e^{-2x} \Big|_0^\infty$$

$$= 1/2$$

10. A man and a woman decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 12 noon and 1pm, find the probability that the first to arrive has to wait longer than 10 minutes.

$X \sim \text{Unif}(0, 60)$ $\quad f_X(x) = 1/60$

$Y \sim \text{Unif}(0, 60)$ $\quad f_Y(y) = 1/60$

$X \perp\!\!\!\perp Y$

$f_{XY}(x, y) = f_X(x) \cdot f_Y(y) = \left(\frac{1}{60}\right)^2$

$\mathbb{P}(Y + 10 \le X) + \mathbb{P}(X + 10 \le Y)$

$= 2 \cdot \mathbb{P}(X + 10 \le Y)$

$= 2 \int_{10}^{60} \int_{0}^{y-10} \left(\frac{1}{60}\right)^2 dx\, dy$

$= 2 \cdot \int_{10}^{60} \cdot \frac{1}{3600} \cdot (y - 10)\, dy$

$= \frac{2}{3600} \cdot \left(\frac{1}{2} \cdot y^2 - 10y \right) \Big|_{10}^{60}$

$= 0.694$