Griffith School of Engineering and Built Environment
Griffith University

6321ENG Discrete Time Signal Processing
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Lab Report: Classical Spectral Estimation Techniques

Jessy Barber, s5138877

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TABLE OF CONTENTS

1 Periodogram Method and Effect of Data-Record Size on Spectral Estimation

Experiment one involves the use of the following signal:

$$x(n) = \cos[2\pi(0.135)n - \frac{\pi}{2}] + \cos[2\pi(0.145)n], \qquad 0 \leq n \leq N-1$$

It is expected that the signals power spectral density (PSD) function should have two peaks at $f_1 = 0.135$ and $f_2 = 0.145$.
The periodogram method estimates the PSD function as follows:

$$\hat{P}(k) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{N}}\right|^2, \qquad 0 \leq k \leq N-1$$

Here, $\hat{P}(k)$ denotes the value of the estimated power spectrum at normalized frequency $\frac{k}{N}$.

The periodogram estimate can be obtained by computing the discrete Fourier transform (DFT) of the signal $x(n), n = 0, 1, ..., N-1$, through the fast Fourier transform (FFT) algorithm.

Part 1 of this experiment is to use the periodogram method to estimate its PSD function for data-record length N = 128, 256, 512 and 1024, and to plot them.

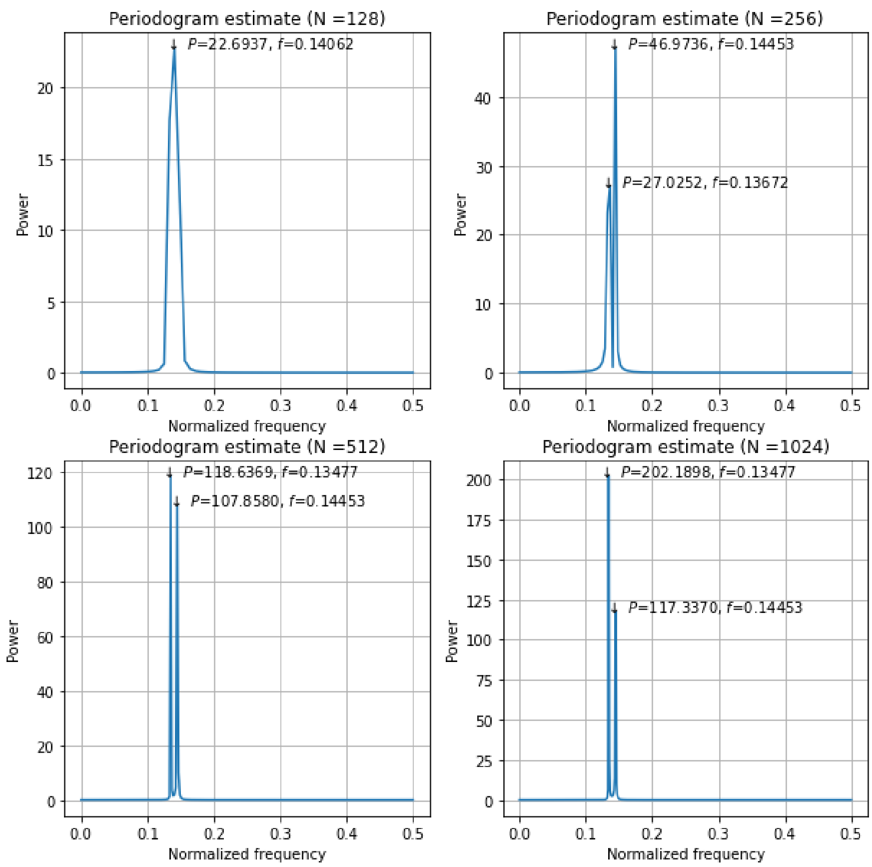

Figure 1: Periodogram Estimates for N = 128, 256, 512, 1024

For the periodogram estimates N = 256, 512 and 1024, the f_1 and f_2 values approximate the expected values of 0.135 and 0.145 respectively. However, only one peak is present in the N = 1 periodogram estimate. This is because the resolution is too small to distinguish the second peak. In other words, there is not enough data at N = 128 to detect the separation between the peaks of $\frac{1}{N}$.

After implementing the peak detection algorithm, it is clear that the periodogram method is useful for detecting the frequency location of the periodogram estimate peaks. However, in the x(n) signal, both cosine components have an amplitude of one and should therefore have an equal contribution in power. In the periodogram method, the peaks are not the same height and vary with N. This means that the power value at these peak locations is unreliable.

2 Effect of Zero-Padding on Spectral Estimation

Experiment two involves the use of the following signal:

$$x(n) = \cos[2\pi(0.135)n - \frac{\pi}{2}] + \cos[2\pi(0.135 + \Delta f)n], \qquad 0 \leq n \leq N - 1$$

This signal is to be padded with (L-N) zeros producing a sequence of length L. The FFT algorithm is then used to periodogram estimate as follows:

$$\hat{P}(k) = \frac{1}{N} \left| \sum_{n=0}^{L-1} x(n) e^{\frac{-j2\pi kn}{L}} \right|^2, \qquad 0 \leq k \leq L - 1$$

Here, $\hat{P}(k)$ denotes the value of the estimated power spectrum at normalized frequency $\frac{k}{L}$.

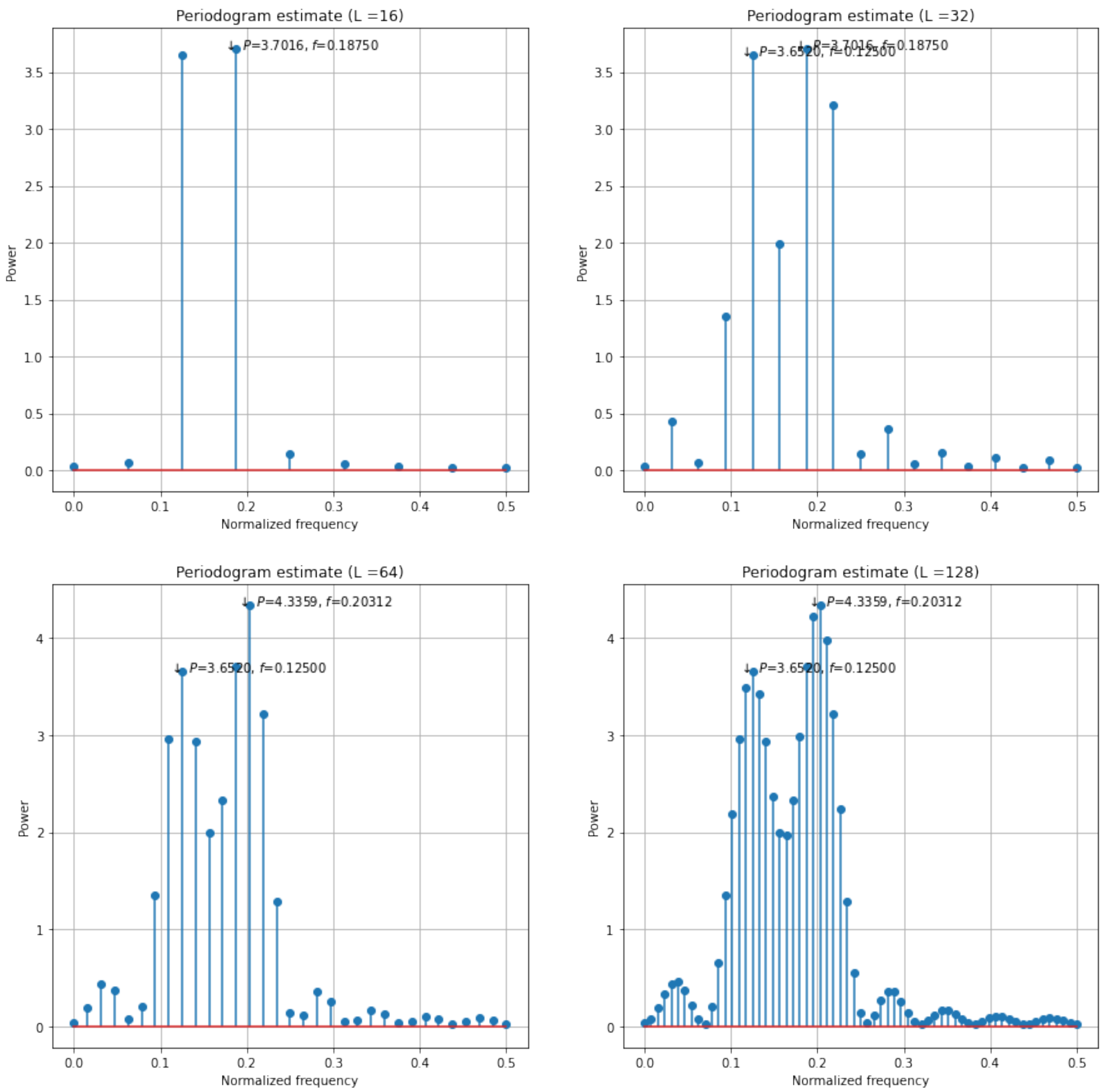

Figure 2: Periodogram Estimates for N = 16, L = 16, 32, 64, 128 and $\Delta f = 0.06$

Padding the N-point sequence with L-N zeros results in the L-point FFT, and also more closely spaced frequency points. The separation between two consecutive points becomes $\frac{1}{L}$.

The periodogram estimate can be rewritten as:

$$\hat{P}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)w(n)e^{\frac{-j2\pi fn}{n}}\right|^2, \qquad -0.5 \leq f \leq 0.5$$

Where w(n) is a rectangular window function given by:

$$w(n) = \begin{cases} 1, & if\ 0 \leq n \leq N-1, \\ 0, & otherwise \end{cases}$$

Thus, a rectangular window function is implicit in the periodogram method. Due to the rectangular windowing function, peak merging can occur when there is a low frequency resolution. This occurs when the main lobe width of the windowing function is too wide to distinguish the two peaks. Therefore, it is ideal to reduce the width of the main lobe to increase the frequency resolution. Zero-padding the signal does nothing to increase this resolution as it does not add any extra information to help the periodogram resolve the two peaks. The following is the repeated experiment for $\Delta f = 0.01$.

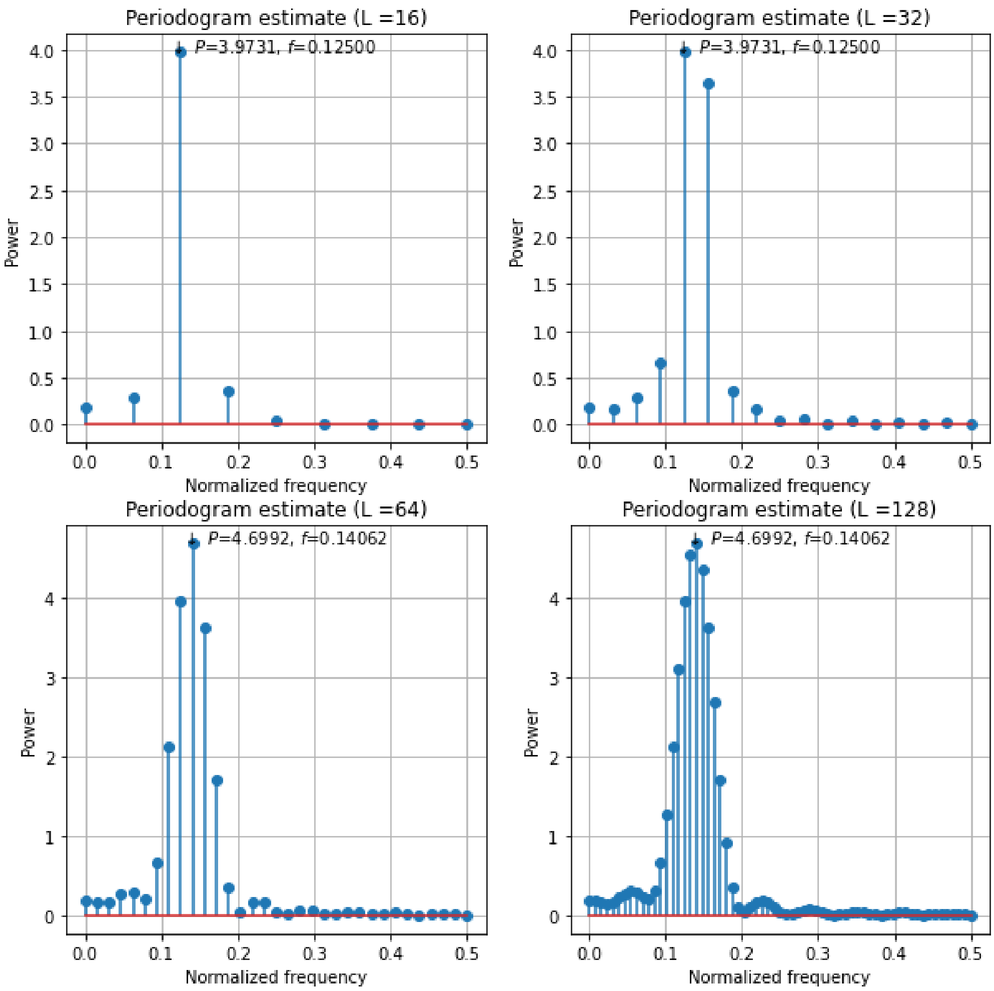

Figure 3: Periodogram Estimates for N = 16, L = 16, 32, 64, 128 and $\Delta f = 0.01$

Since Δf has been reduced to 0.01, the distance between signal peaks has been reduced and the periodogram method is no longer able to resolve both peaks. Smoothing has occurred in this instance and the two peaks are blurred together. It can be seen that the resolution does not increase as L increases, but the signal only becomes more smoothed and contains more data points. The only way to improve the frequency resolution is to decrease the main lobe width by increasing the number of data points N.

As a result of the rectangular window, there are two types problems in the spectral estimation. These are the spectral resolution, and the spectral leakage.

3 Finding Resolution of a PSD Estimate

Experiment three involves the use of the following signal:

$$x(n) = \cos[2\pi(0.135)n - \frac{\pi}{2}] + \cos[2\pi(0.135 + \Delta f)n], \qquad 0 \leq n \leq N - 1$$

In this experiment L = 8N, and the signal is padded with (L-N) zeros. After the periodogram is computed, the task is to find Δf such that two peaks in the PSD estimate are resolved for N = 16. This minimum value is the resolution. This is then repeated for N = 32, 64, 128 and 256.

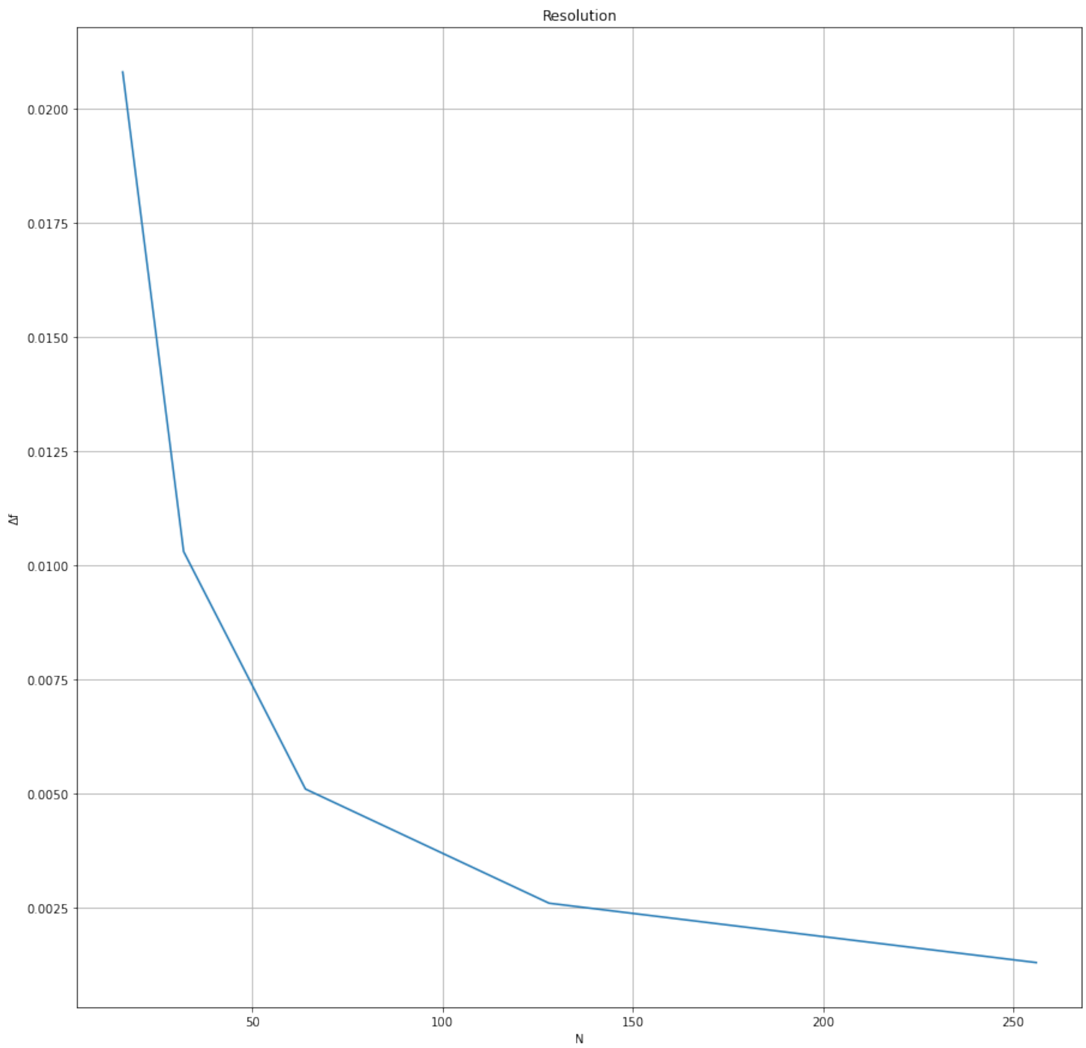

Figure 4: Resolution as a Function of N

As seen in the figure above, as N increases the frequency resolution also increases since Δf become much smaller at N = 256. Since Δf is a much smaller value, the periodogram method will be able to detect peaks that are much closer together. Increasing the value of N adds more data points to the FFT which decreases the windowing function's main lobe width and increases the resolution of the peak detection method.

4 Observation of Spectral Leakage

Experiment four involves the use of the following signal:

$$x(n) = \cos[2\pi(0.135)n - \frac{\pi}{2}] + 0.01\cos[2\pi(0.195)n], \qquad 0 \le n \le N-1$$

In this experiment N = 256, and L = 4N. The signal is padded with (L-N) zeros and the periodogram estimate is computed. The PSD estimate is then plotted on a logarithmic scale.

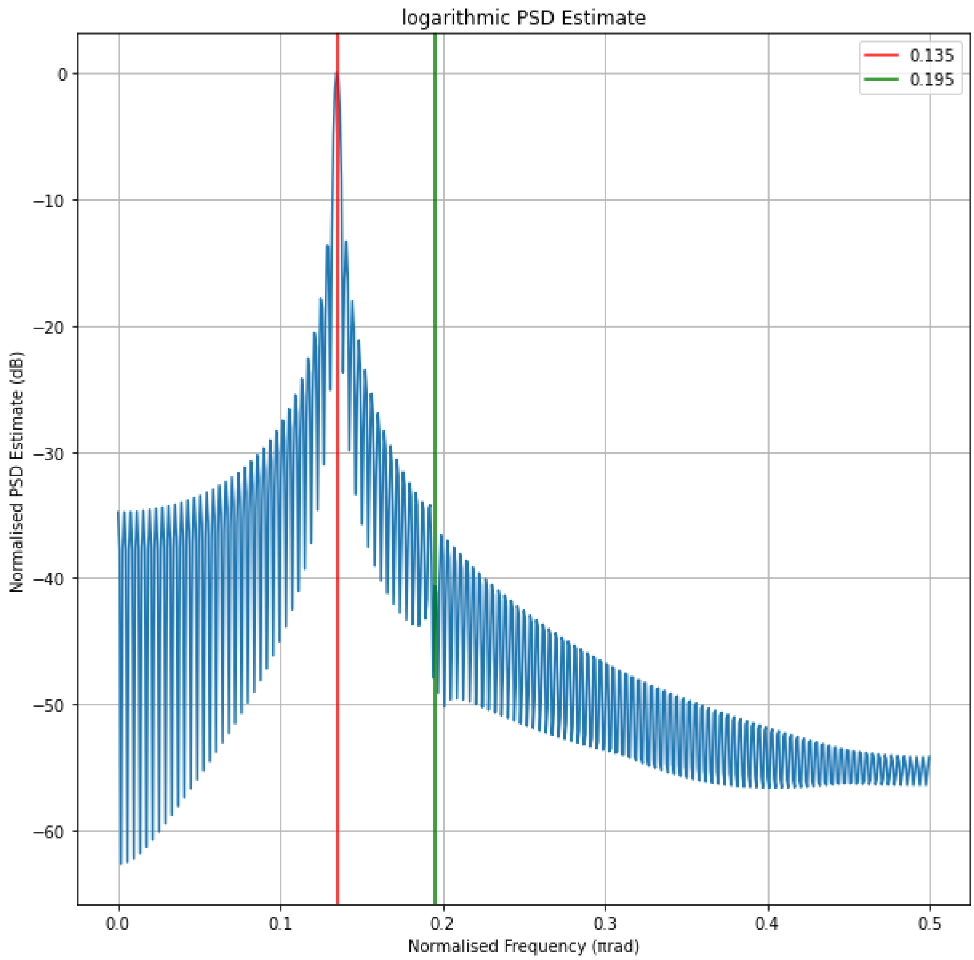

Figure 5: Logarithmic PSD Estimate

As seen in the figure above, the peak at 0.135 is resolved but the peak at 0.195 is barely. This is because the implicit rectangular window function has high side lobe peaks which cause spectral leakage. The peak at 0.195 is therefore masked by the spectral leakage of the strong side lobe peaks of the window function. To fix this, a tapered window function can be used to give lower side lobes.

5 Reducing the Spectral Leakage by Data Windowing

Experiment five involves the use of the following signal:

$$x(n) = \cos[2\pi(0.135)n - \frac{\pi}{2}] + 0.01\cos[2\pi(0.195)n], \qquad 0 \leq n \leq N-1$$

In this experiment N = 256, L = 4N and the signal is padded with (L-N) zeros. This time, the following signal is formed:

$$y(n) = x(n)w(n), \qquad 0 \leq n \leq N-1$$

Where w(n) is a window function defined as:

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \qquad 0 \leq n \leq N-1$$

The PSD estimate is then plotted on a logarithmic scale.

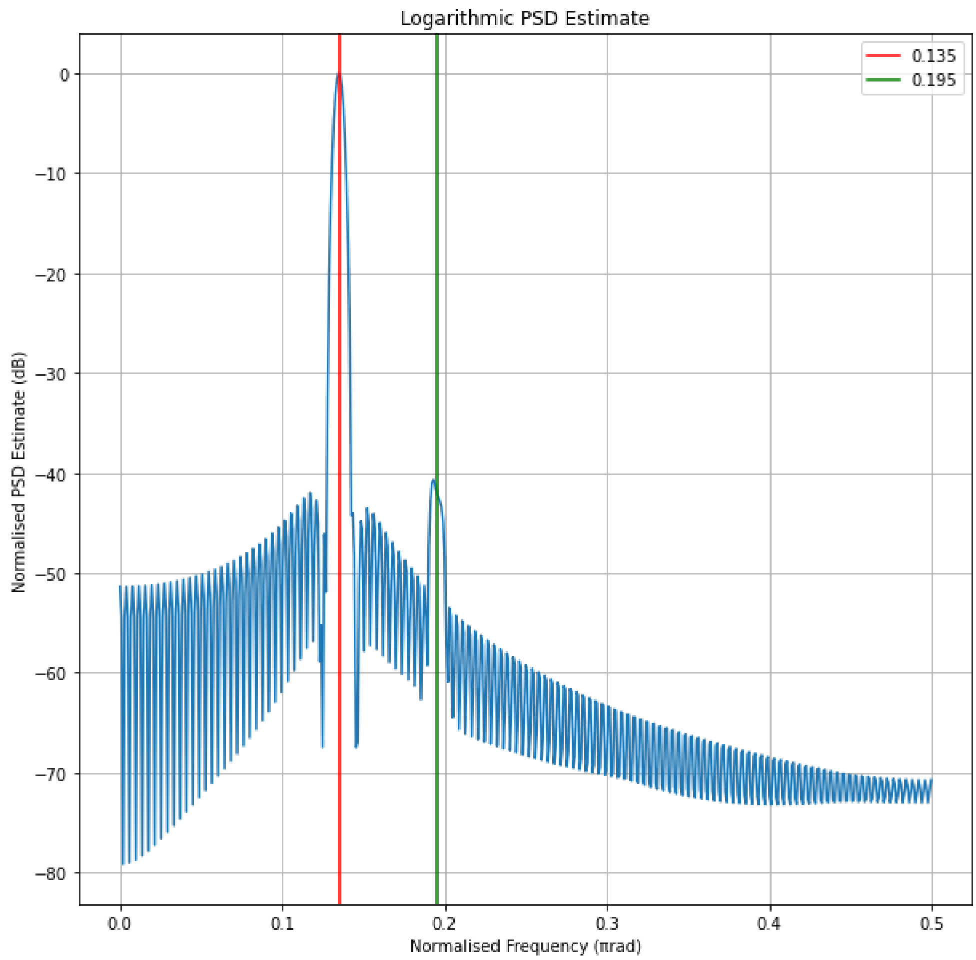

Figure 6: Logarithmic PSD Estimate

As seen in the figure above, the second peak at 0.195 is clearly resolved after implementing the tapered window function. This window function has a much lower initial side lobe than the rectangular window function, meaning that the spectral leakage is no longer masking the peak at 0.195.

6 Effect of Data Windowing on Resolution

Experiment six involves repeating experiment three using the rectangular and Hamming window functions.

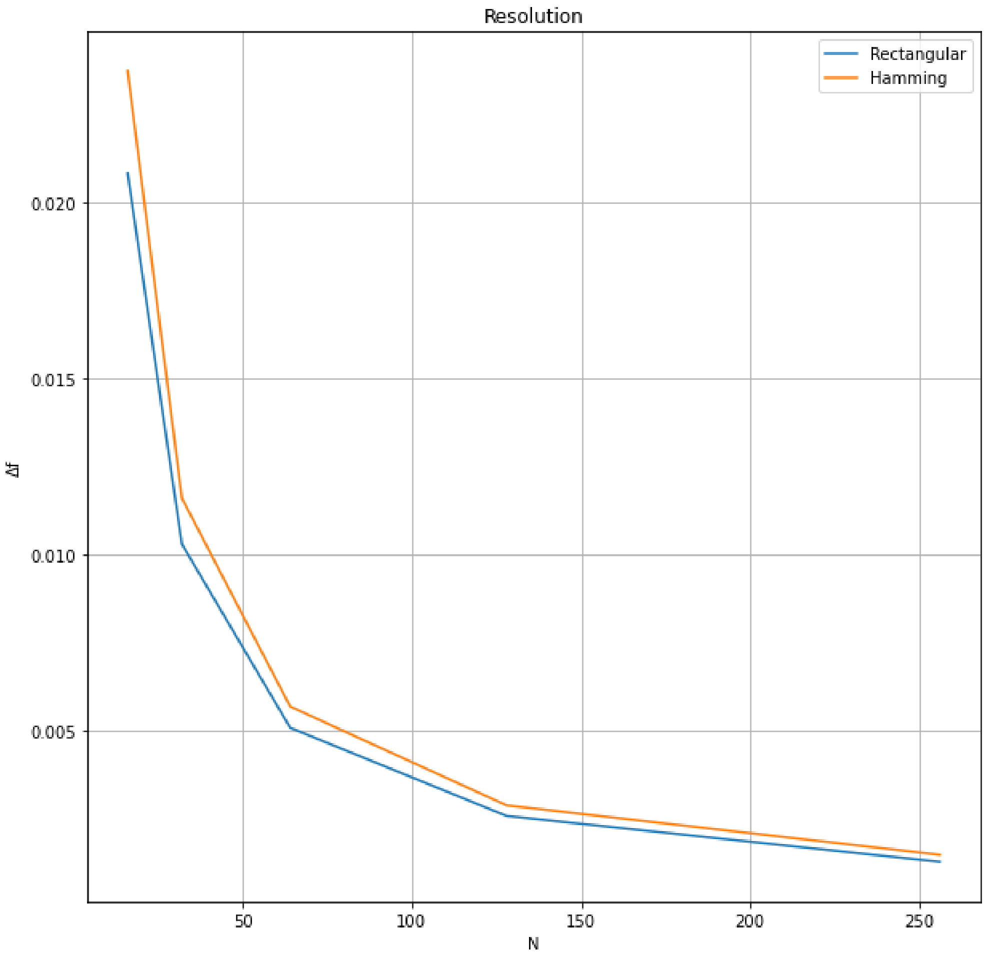

Figure 7: Resolution as a Function of N for Rectangular and Hamming Window Functions

The trade-off with using window functions is that by decreasing the size of the initial side lobes, as present in the Hamming window, to fix masking from the spectral leakage, the width of the main lobe is increased. Inversely, a smaller main lobe width is achieved with higher initial side lobes as seen in the rectangular window function. This means although using the Hamming window allows for the second peak to be resolved, the PSD estimate will have an overall lower frequency resolution compared to the rectangular window function. The Hamming window can be improved by increasing the number of data points N.

7 Bias and Variance of the PSD Estimate for the Periodogram Method

Experiment seven involves the use of the following signal:

$$x(n) = \cos[2\pi(0.135)n - \frac{\pi}{2}] + \cos[2\pi(0.195)n] + u(n), \qquad 0 \leq n \leq N-1$$

Where u(n) represents a white Gaussian noise sequence with zero-mean and variance equal to 16. In this experiment N = 128, L = 2N and R = 40 realisations of this signal are taken. The signal is to be padded with (L-N) zeros and the PSD estimate using the periodogram method is to be computed. The experiment then involves finding the expected value of the PSD estimate and its variance.

The expected value of the PSD estimate is given as:

$$E[\hat{P}(k)] = \frac{1}{R}\sum_{i=1}^{R}\hat{P}_i(k)$$

The variance of the PSD estimate is given as:

$$Var[\hat{P}(k)] = \frac{1}{R}\sum_{i=1}^{R}\left(\hat{P}_i(k) - E[\hat{P}(k)]\right)^2$$

The normalized variance is given as:

$$NormalizedVariance = \frac{Var[\hat{P}(k)]}{(E[\hat{P}(k)])^2}$$

The average normalized variance is given as:

$$ANV = \frac{1}{L}\sum_{k=0}^{L-1}\frac{Var[\hat{P}(k)]}{(E[\hat{P}(k))^2}$$

These values are to be calculated for N = 256, 512 and 1024. The ANV is then to be plotted as a function of N.

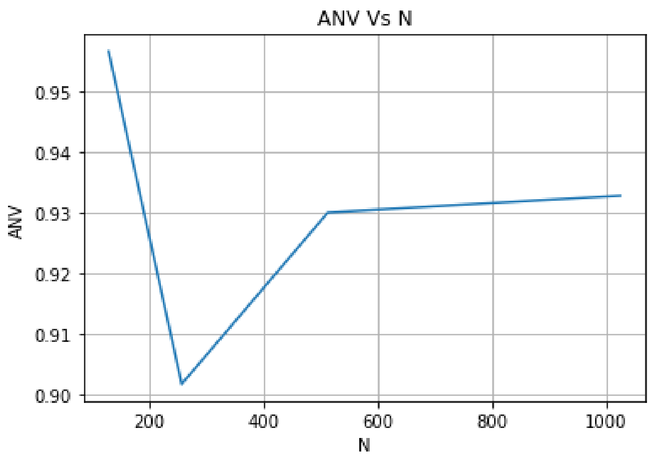

Figure 8: ANV as a Function of N for N = 128, 256, 512, 1024

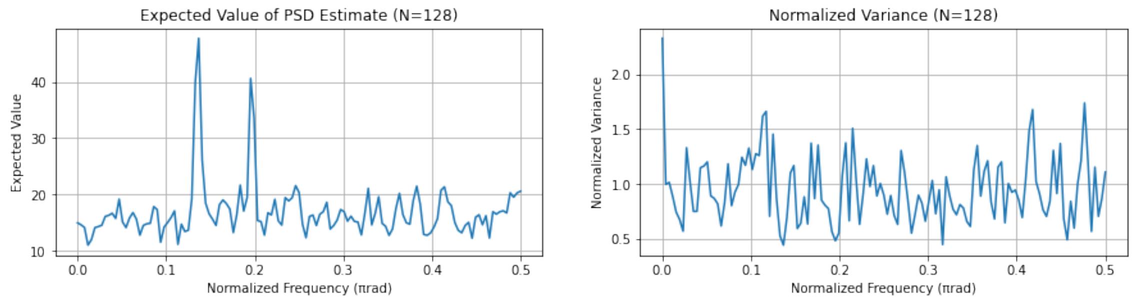

Figure 9: PSD Estimate and Normalized Variance as a Function of Frequency for N = 128

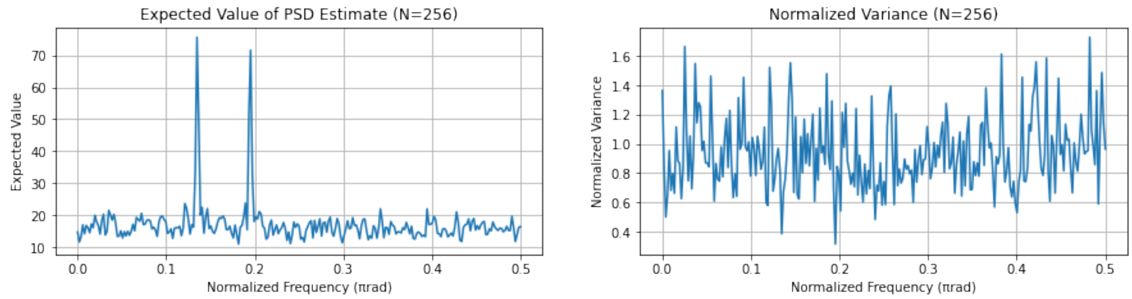

Figure 10: PSD Estimate and Normalized Variance as a Function of Frequency for N = 256

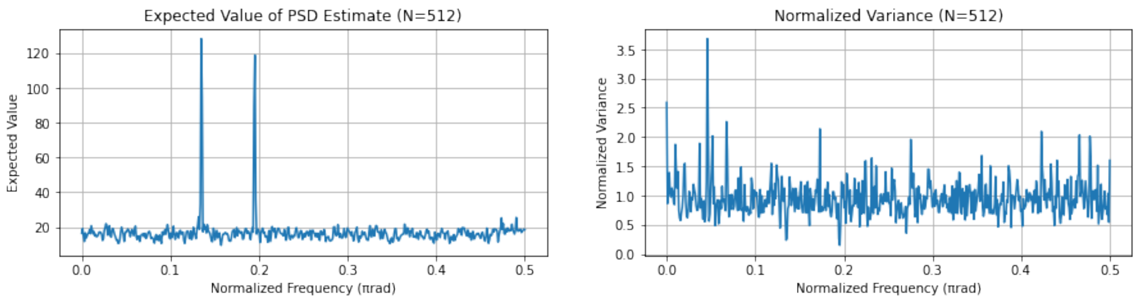

Figure 11: PSD Estimate and Normalized Variance as a Function of Frequency for N = 512

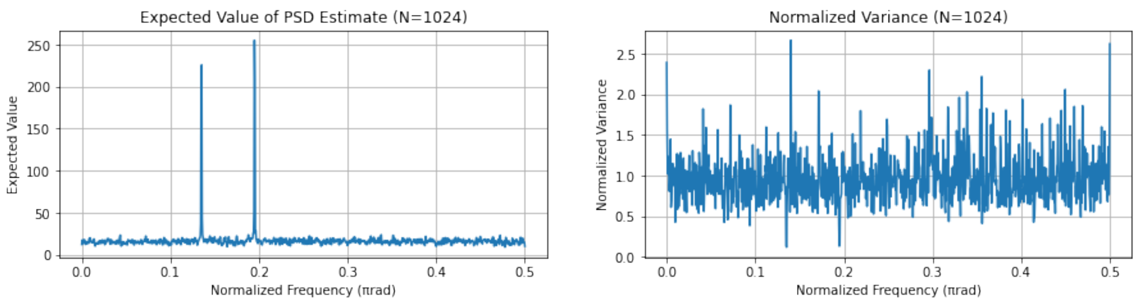

Figure 12: PSD Estimate and Normalized Variance as a Function of Frequency for N = 1024

As seen in the figures above, it is clear that the periodogram estimate is not a consistent estimator since the variance does not decrease as N increases. This is seen in the graphs as the normalized variance hovers around 1 for all values of N. This is a bad property of the periodogram method since the variance is not dependent on the data length N, hence the periodogram method is biased and its reliability is reduced.

8 Bias and Variance of the PSD Estimate for the Bartlett Method

Experiment 8 involves repeating experiment 7 but instead using the Bartlett method in place of the periodogram method. M = 64 is used for this experiment.

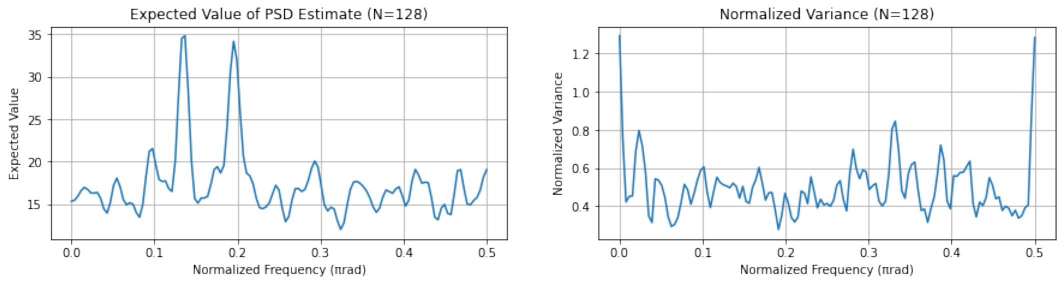

Figure 13: PSD Estimate and Normalized Variance as a Function of Frequency for N = 128

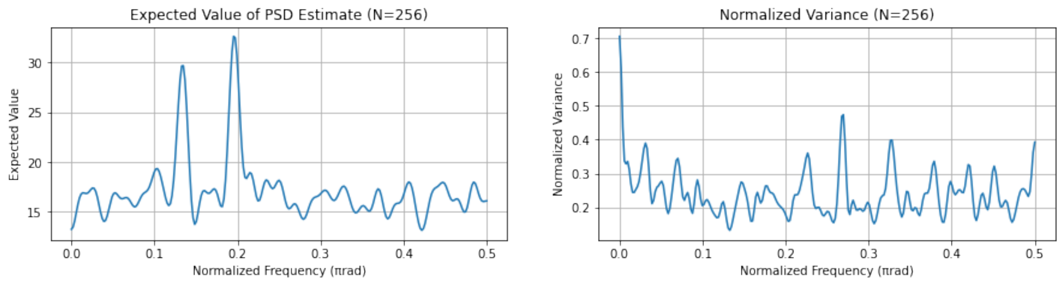

Figure 14: PSD Estimate and Normalized Variance as a Function of Frequency for N = 256

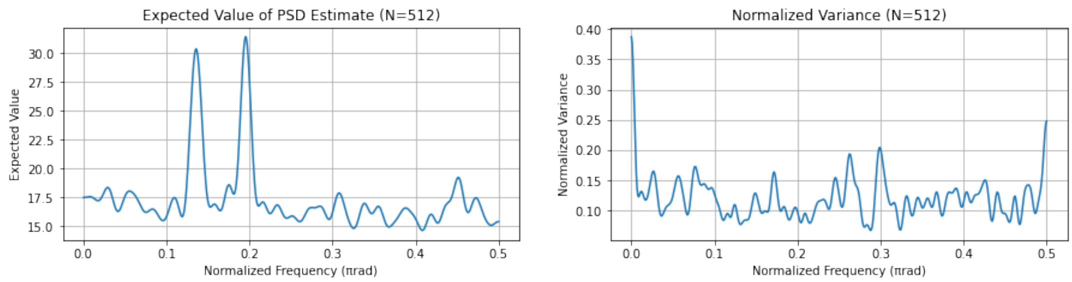

Figure 15: PSD Estimate and Normalized Variance as a Function of Frequency for N = 512

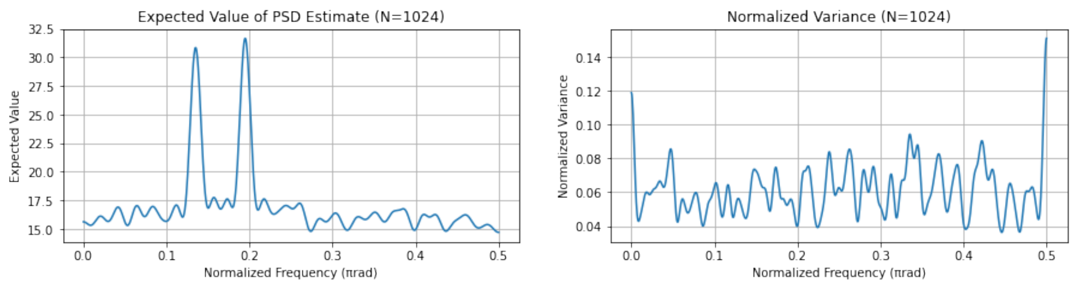

Figure 16: PSD Estimate and Normalized Variance as a Function of Frequency for N = 1024

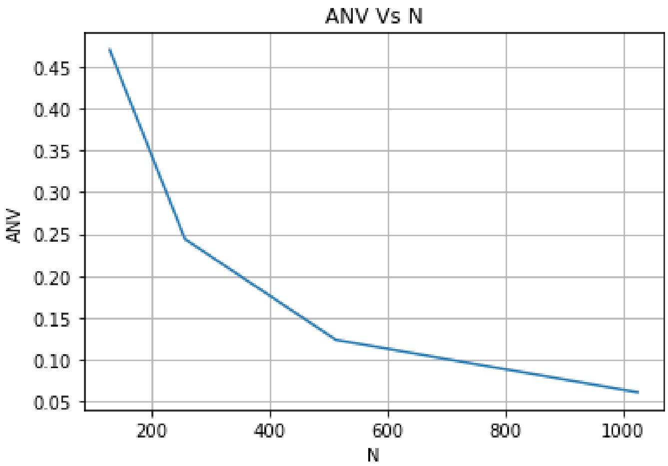

Figure 17: ANV as a Function of N for N = 128, 256, 512, 1024

The Bartlett method overcomes the inconsistent estimator of the periodogram method by dividing the data record into a number of non overlapping segments of equal length, where segment length is M and number of segments is $S = \frac{N}{M}$, and averaging the periodograms of the individual segments. The power spectral estimate of the data record $[x(n) = 0, 1, 2, ..., N-1]$ is obtained by averaging the S periodograms assuming the periodogram estimate for the i^{th} segment is $\hat{P}_i(f)$.

$$\hat{P}(f) = \frac{1}{S}\sum_{i=1}^{S} P_i(f), \qquad -0.5 \leq f \leq 0.5$$

This averaging operation reduces the variance as seen in the normalized variance figures above. As N increases, the normalized variance decreases which means that it is a consistent estimator. The problem is that with the reduction in variance, there is also a decrease in spectral resolution due to the shorter data records used in each periodogram calculation of length M.

9 Bias and Variance of the PSD Estimate for the Welch Method

Experiment 9 involves repeating experiment 7 but instead using the Welch method in place of the periodogram method. M = 64 and O = 32 are used for this experiment.

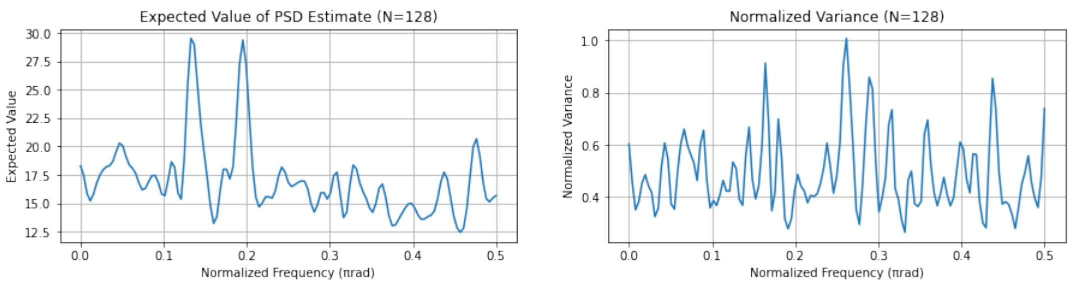

Figure 18: PSD Estimate and Normalized Variance as a Function of Frequency for N = 128

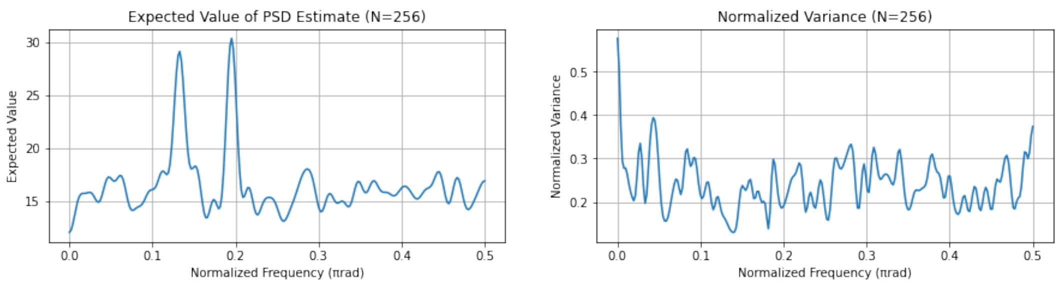

Figure 19: PSD Estimate and Normalized Variance as a Function of Frequency for N = 256

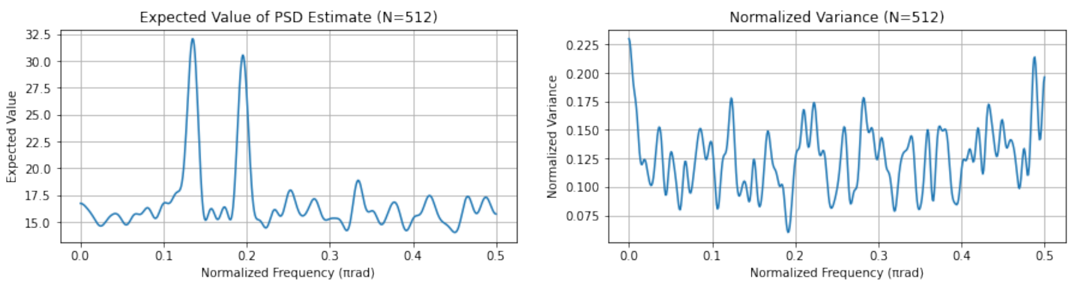

Figure 20: PSD Estimate and Normalized Variance as a Function of Frequency for N = 512

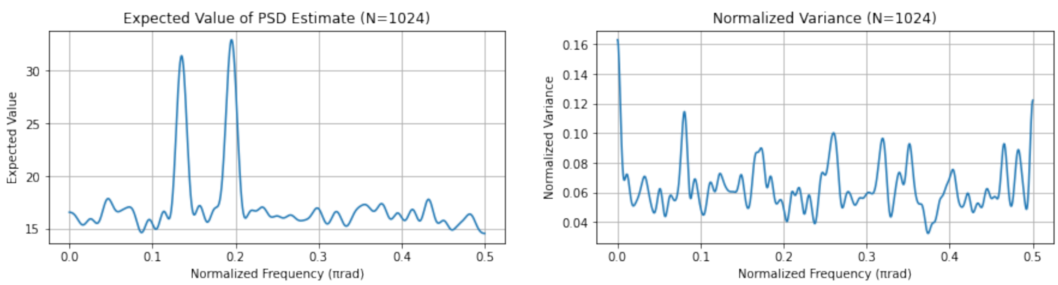

Figure 21: PSD Estimate and Normalized Variance as a Function of Frequency for N = 1024

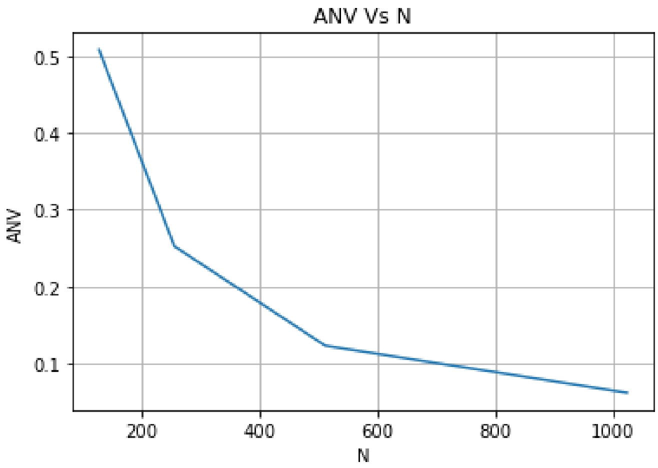

Figure 22: ANV as a Function of N for N = 128, 256, 512, 1024

The Welch method is similar to the Bartlett method except for the use of overlapping segments instead of non-overlapping segments and the application of a tapered window to each segment. For the sake of this experiment and comparing the Welch and Bartlett methods, the tapered window was not applied to each segment. The data record $[x(n) = 0, 1, 2, ..., N-1]$ is divided into S number of segments where $S = \frac{N}{M-O}$, the length of each segment is M and the overlap of each given segment with the preceding segment is O. By overlapping the segments, the compromise between the variance reduction and frequency resolution can be controlled.

Similar to the Bartlett method, the variance of the Welch method decreases as N increases which means that it is a consistent estimator. This gives the method an advantage over the inconsistent periodogram method. In comparison with the Bartlett method, the Welch method offers an advantage in terms of a reduced variance if M is the same value in both methods. Likewise, the Welch method offers a better spectral resolution than the Bartlett method if S is the same in both methods. The disadvantage of the Welch method is that by using too much overlap between segments, the independence of each realisation will be violated and the segments will become highly correlated.

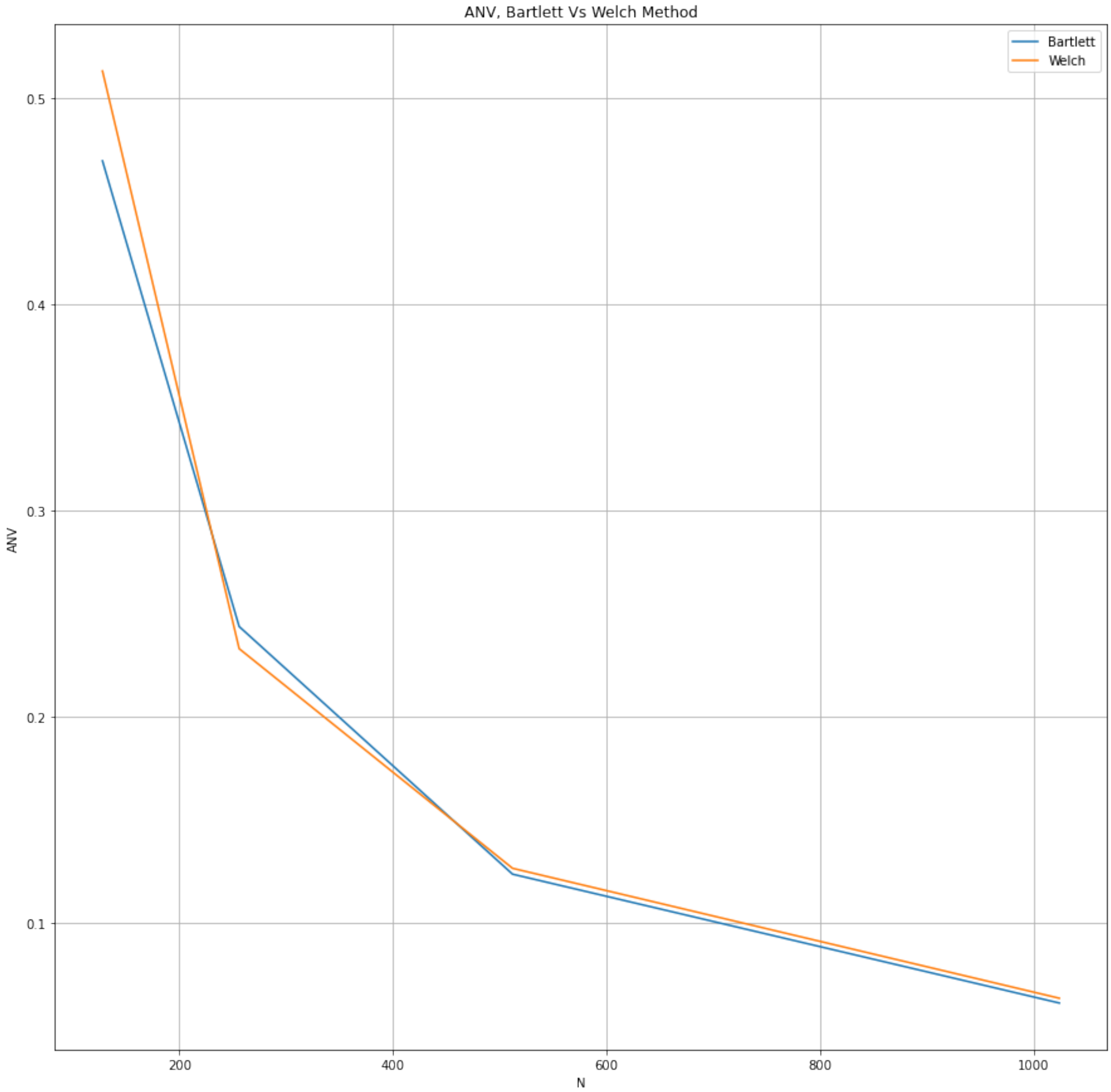

Figure 23: ANV as a Function of N for N = 128, 256, 512, 1024, Bartlett Vs Welch