Basic Concepts on Probability : Review

Probability means whether a certain event has a good chance of occurring or not.
Its value lies between 0 and 1.

Rules for combining probabilities

1) ***Independent Events:***
Two events are said to be independent if the occurrence of one event does not affect the probability of occurrence of the other event.
Example: Throwing a dice and tossing coin are independent events.

2) ***Mutually exclusive events:***
Two events are said to be mutually exclusive or disjoint if they cannot happen at the same time.
Example: (i) When throwing a single die, the events 1, 2, 3, 4, 5 and 6 spots are all mutually exclusive because two or more cannot occur simultaneously
(ii) Similarly success and failure of a device are mutually exclusive events since they cannot occur simultaneously.

3) ***Complimentary Events:***
Two outcomes of an event are said to be complementary, if when one outcome does not occur, the other must occur.
If the outcomes A & B have probabilities P(A) and P(B), then
$P(A) + P(B) = 1$ $\qquad$ $P(B) = P(\bar{A})$

Example: When tossing a coin, the outcomes head and tail are complementary since
$P(head) + P(tail) = 1$ or
$P(head) = P(\overline{tail})$
$P(tail) = P(\overline{head})$
Therefore we can say that two events that are complementary events are mutually exclusive also. But the converse is not necessarily true i.e, two mutually exclusive events are not necessarily complementary.

4) ***Conditional Events;***
Conditional events are events which occur conditionally on the occurrence of another event or events.
Consider two events A & B and also consider the probability of event A occurring under the condition that event B has occurred.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

5) ***Simultaneous occurrence of events:***
Occurrence of both A & B - Mathematically it is represented as $A \cap B$, A AND B, AB.

Case (i) Independent, then the probability of occurrence of each event is not influenced by the probability of occurrence of the other.

$P(A/B) = P(A)$
$P(A \cap B) = P(A) - P(B)$
And $P(B/A)- = P(B)$
Case (ii) Events are dependent
If two events are not independent, then the probability of occurrence of one event is influenced by the probability of occurrence of the other

Therefore, $P(A \cap B) = P(B/A).P(A)$
$= (PA/B).P(B)$

Numerical Problem - 1
An engineer selects two components A & B. The probability that component A is good is 0.9 & the probability that component B is good is 0.95.What is the probability of both components being good.

$P(A \text{ good} \cap B \text{ good}) = P(A \text{ good}) (B \text{ good})$
$= 0.9 \times 0.95 = 0.80$

Numerical Problem - 2
One card is drawn from a standard pack of 52 playing cards. Let A be the event that it is a red card and B be the event that it is a face card. What is the probability that both A & B occur.

$P(A) = 26/52$
Given that 'A' has occurred
Then the sample space for B is 26 states, out of which 6 are those of a face card.
Therefore, $P(B/A = 6/26$
$P(A \cap B) = 6/26 \times 26/52 = 6/52$
$P(A \cap B) = P(B/A)\ P(A)$

6) *Occurrence of atleast one of two events*:
The occurrence of atleast one of two events A and B is the occurrence of A or B or BOTH. Mathematically it is the union of the two events and is expressed as (AUB), (A or B) or (A x B)

Case (i) – Events are independent but not mutually exclusive.
$P(AUB) = P(A \text{ OR } B \text{ OR BOTH } A \text{ AND } B)$
$= 1 - P(\text{NOT } A \text{ AND NOT } B)$
$= 1 - P(\overline{A} \cap \overline{B})$
$= 1 - P(\overline{A}) . P(\overline{B})$
$= 1 - (1-P(A)) (1-P(B))$
$= P(A) + P(B) - P(A).P(B)$

Using Venn diagram
$P(AUB) = P(A) + P(B) - P(A \cap B)$
If $P(A) = 0.9$ and $P(B) = 0.95$
$P(AUB) = P(A) + P(B) - P(A).P(B)$
$= 0.9 + 0.95 - 0.9 \times 0.95 = 0.995$

Case (ii) – Events are independent and mutually exclusive In the case of events A & B being mutually exclusive, then the probability of their simultaneous occurrence P(A).P(B) must be zero by definition.

$P(A \cup B) = P(A) + P(B)$

Case (iii) – Events are not independent
If two events are not independent then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= P(A) + P(B) - P(B/A).P(A)$
$= P(A) + P(B) - P(A/B).P(B)$

Numerical Problem - 3

A cinema hall gets electric power from a generator run by a diesel engine. On any ~~give~~ given day, the probability that the generator is down (event A) is 0.025 and the probability that the diesel engine is ~~done~~ down (event B) is 0.04. What is the probability that the cinema house will have power on any given day.

Assume that the occurrence of events A & B are independent of each other.
Probability that the
Cinema hall does not have power given by the probability of the event that either the diesel engine or generator is down.

$Q = P_r(A \cup B) = P(A) + P(B) - P(A)\,P(B)$
$= 0.025 + 0.04 - 0.025 \times 0.04 = 0.064$

Therefore, the probability that the cinema house have power
$= R = 1 - 0.064 = 0.936$

Numerical Problem - 4

In a sample of 60 mails, 10 of them contains only defective heads, five contain only defective tails and and five contain both the defects. What is the probability that a mail that is selected randomly contains either defective head or a defective tail?

Let X denote the event that a mail contains a defective head and Y denote the event containing a defective tail.
Than $P(X) = \frac{10+5}{60} = 0.25$
$P(Y) = \frac{5+5}{60} = 0.1667$
$P(X \cap Y) = 5/60 = 0.0833$

The probability that a mail contains either of the two defects is $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = 0.334$

Therefore, probability that a mail contains no defect is $40/60 = 0.6667 = 1 - P(X \cup Y)$

Random Variables

To study about a system's behavior for the application of probability theory to reliability evaluation, a series of experiments must be performed or a data collection scheme should be deduced.
To apply the probability theory to occurrence of these values or events which are random in nature, we need to study these variables called as Random Variables.

∴ Random variable is a variable quantity which denotes the result or outcome of a given random experiment.
A random variable is one that can have only a discrete number of states or countable values.

A random variable can be either "discrete" or "continuous".

A discrete random variable is one that can have only a discrete number of states or countable values.
Ex: 1.Tossing a coin - Outcomes are heads or tails.
2. Rolling a dice - Outcomes are 1,2,3,4,5 or 6.

A continuous random variable is one which takes an infinite number of values or if its range forms a continuous set of real numbers. This does not mean that the range extends from $-\infty$ to $+\infty$. It only means that there are infinite number of possibilities of the value.

Ex: 1. The life time of a light bulb.
2. If electric current have values between 5A and 10 A, then it indicates a continuous random variable.
~~variable.~~

Probability Density Function

The probabilities associated with the random variables can be described by a formula called Probability density function or Probability mass function.

We use the notation f(x) for the probability density function.

Ex : 1. Consider the throw of a dice
Let the random variable associated with the outcome be 'X'.
The value of X are 1, 2, 3, 4, 5 and 6.
$f(1) = P(x=1) = 1/6$
$f(2) = P(x=2) = 1/6$
$\therefore f(1) = f(2) = \ldots\ldots\ldots\ldots = f(6) = 1/6$
$f(x) = 1/6$ = Constant density function.

2. Consider the rolling of two dice. What is f(x) for the sum of dots facing up?
X= Total Sum of dots
$P(x) = f(x) = ?$

$P(x=2) = f(2)$

$\therefore f(x) = \frac{x-1}{36}$ for x= 2,3,4,5,6,7

$\therefore f(x) = \frac{13-x}{36}$ for x= 8,9,10,11,12

Probability Distribution Function

If 'x' is a random variable, then for any real number x, the probability that 'x' will assume a value less than or equal to x is called Probability distribution functions.
It is indicated as F(x)

$f(x) = P(x)$

$F(x) = P(X \leq x)$

Ex: Consider the rolling of a single dice.

$f(x) = 1/6$

$f(1) = f(2) = \ldots\ldots\ldots\ldots f(6) = 1/6$

$F(1) = P(X \leq 1) = f(1) = 1/6$

$F(2) = P(X \leq 2) = f(1) + f(2)$
$= 1/6 + 1/6 = 2/6$

$F(3) = P(X \leq 3) = f(1) + f(2) + f(3)$
$= 1/6 + 1/6 + 1/6 = 3/6$

$F(4) = 4/6$

$F(5) = 5/6$

$F(6) = 6/6 = 1$

Suppose a random variable X has the following density function.

X	0	1	2	3	4	5
f(x)	1/32	5/32	10/32	10/32	5/32	1/32

Then, the Probability distribution function is given by

X	0	1	2	3	4	5
F(x)	1/32	1/32 + 5/32 =6/32	6/32 + 10/32 =16/32	16/32 + 10/32 =26/32	26/32 + 5/32 =31/32	31/32 + 1/32 =32/32

Relation between Probability density function and distribution function:

$F(x) = \sum f(x)$ (Discrete Random variable)

$F(x) = \int f(x)dx$ (Continuous Random variable)

A random variable 'x' and the corresponding distribution function F(x) are said to be continuous if the following condition is satisfied for all 'x'.

$$f(x) = \frac{d}{dx} f(x)$$

Mathematical Expectation

It is useful to describe the random behavior of a system by one or more parameters rather than as a distribution. This is particularly useful in the case of system reliability evaluation.

This parametric description can be achieved using numbers known mathematically as moments of distribution.
The most important of these moments is the expected value, which is also referred to as average mean value.

Mathematically it is the first moment of the distribution.

Consider a Probability model with outcome $x_1, x_2, x_3 \ldots\ldots x_r$ and the probability of each is $P_1, P_2, P_3 \ldots\ldots\ldots\ldots P_n$, then the expected value of the variable is $E(x) = P_1x_1 + P_2X_2 + P_3X_3 + \ldots\ldots\ldots P_nX_n = \sum_{i=1}^{n} x_i P_i$

Expected value E(x) of a discrete random variable x having 'n' outcomes x_i each with a probability of occurrence P_i is $E(x) = \sum_{i=1}^{n} x_i P_i$ where $\sum_i^n P_i = 1$

In case of continuous random variable, the equation can be modified from the summation to integration.

$$E(x) = \int x f(x)dx$$

Expected value is the weighted mean of the possible value using their Probability of occurrence as the weighing factor.

Variance and Standard Deviation

The expected value is the most important distribution parameters in reliability evaluation. But to know the amount of 'spread' or 'dispersion' of a distribution, the second moment of distribution. i.e., ~~variane~~ Variance V(x) should be deduced.

The ~~variane~~ Variance of a random variable 'x' is defined as the expectation of the square of deviation of 'x' from E(x).

$m = E(x) = \int x f(x)dx$
Variance $= \sigma^2 = \int (x-m)^2 f(x)dx$
The quantity 'σ' is called standard Deviation.
The K^{th} moment of a random variable 'x' about its expectation is defined as $M_k = E[x - E(x)]^K$.
The second moment of distribution is known as variance V(x) (K=2)

$$\begin{aligned} V(x) &= E[x - E(x)]^2 \\ &= E[x^2 - 2x\,E(x) + E^2(x)] \\ &= E(x^2) - E(2x\,E(x)) + E[\,E^2(x)] \\ &= E(x^2) - 2\,E(x)\,E(x) + E^2(x) \end{aligned}$$

$= E(x^2) - 2\,E^2(x) + E^2(x)$
$= E(x^2) - E^2(x)$
$= \sum_{i=1}^{n} x_i^2 P_i - E^2(x)$

Properties of the binomial distribution

The binomial distribution can be represented by the general expression:

$$(p+q)^n$$

For the expression to be applicable, four specific conditions are required. These are:

(a) There must be a fixed number of trials, i.e. n is known
(b) Each trial must result in either a success or a failure, i.e., only two outcomes are possible and p + q = 1.
(c) All trials must have identical probabilities of success and therefore of failure, i.e., the values of p and q remain constant, and
(d) All trials must be independent (this property follows from (c) since the probabilities of success in trial i must be constant and not affected by the outcome of trials 1, 2, ..., (i-1)).

In order to apply the binomial distribution and to evaluate the outcomes and their probability of occurrence of a given experiment or set of trials, the expression $(p+q)^n$ must be expanded into the form of equations and

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2!}p^{n-2}q^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}p^{n-r}q^r + \dots + q^n$$

If equation is compared with, it is seen that the coefficient of the (r+1)th term in the binomial expansion represents the number of ways, i.e., combinations, in which exactly r failures and therefore (n-r) successes can occur in n trials and is equal to ${}_nC_r$. Therefore each coefficient in equation can be directly evaluated from the definition of ${}_nC_r$ as discussed and the probability of exactly r successes or (n-r) failures in n ~~trails~~ trials can be evaluated from

$$p_r = \frac{n!}{r!\,(n-r)!}p^r q^{n-r}$$

$$= {}_nC_r p^r q^{n-r}$$

$$= {}_nC_r p^r (1-p)^{n-r}$$

Substituting of equations gives

$$(p+q)^n = \sum_{r=0}^{n} c_r p^r q^{n-r} = 1$$

Numerical example – I

A coin is tossed 5 times. Evaluate the probability of each possible outcome and draw the probability mass (density) function and the probability distribution function.

Solution

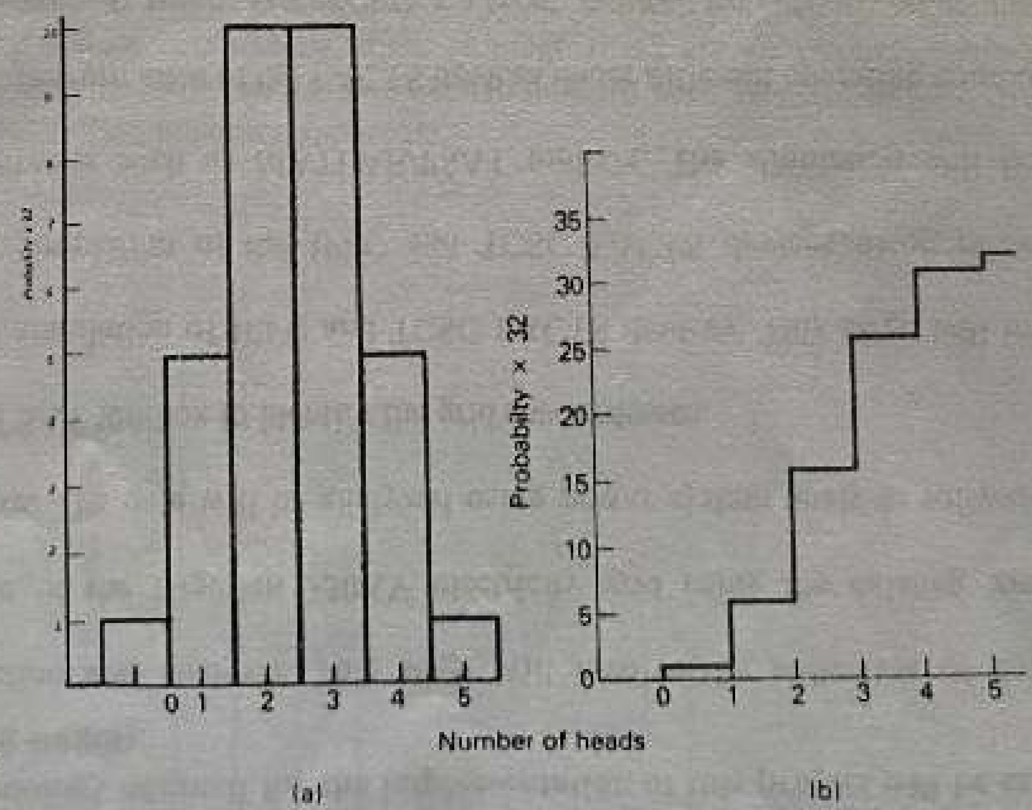
In this example n=5, p=q=1/2. Using the binomial expansion the outcomes, the probability of exactly r heads or (n-r) tails and the cumulative probability are determined as shown in below table.

Number of heads	Number of Tails	Individual probability Expression	Individual probability Value	Cumulative probability
0	5	${}_5C_0(1/2)^0(1/2)^5$	1/32	1/32
1	4	${}_5C_1(1/2)^1(1/2)^4$	5/32	6/32
2	3	${}_5C_2(1/2)^2(1/2)^3$	10/32	16/32
3	2	${}_5C_3(1/2)^3(1/2)^2$	10/32	26/32
4	1	${}_5C_4(1/2)^4(1/2)^1$	5/32	31/32
5	0	${}_5C_5(1/2)^5(1/2)^0$	1/32	32/32
			$\sum = 1$	

In the above table the values of individual probability have been summated and a value of unity obtained.

The results are plotted as a probability mass (density) function and probability distribution functions in figures.

The probability density function is symmetrical. This only occurs when p=q=1/2 since in this case the success and failure events can be interchanged without any alteration in the numerical value of any of the individual outcomes. This will not be the case when p and q are unequal.



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Results for Example (a) Probability density (mass) function. (b) Probability distribution function

Numerical example – II

Consider the case in which the probability of success in a single trial is ¼ and four trials are to be made. Evaluate the individual and cumulative probabilities of success in this case and draw the two respective probability functions.

Solution

n=4, p=1/4, q=3/4

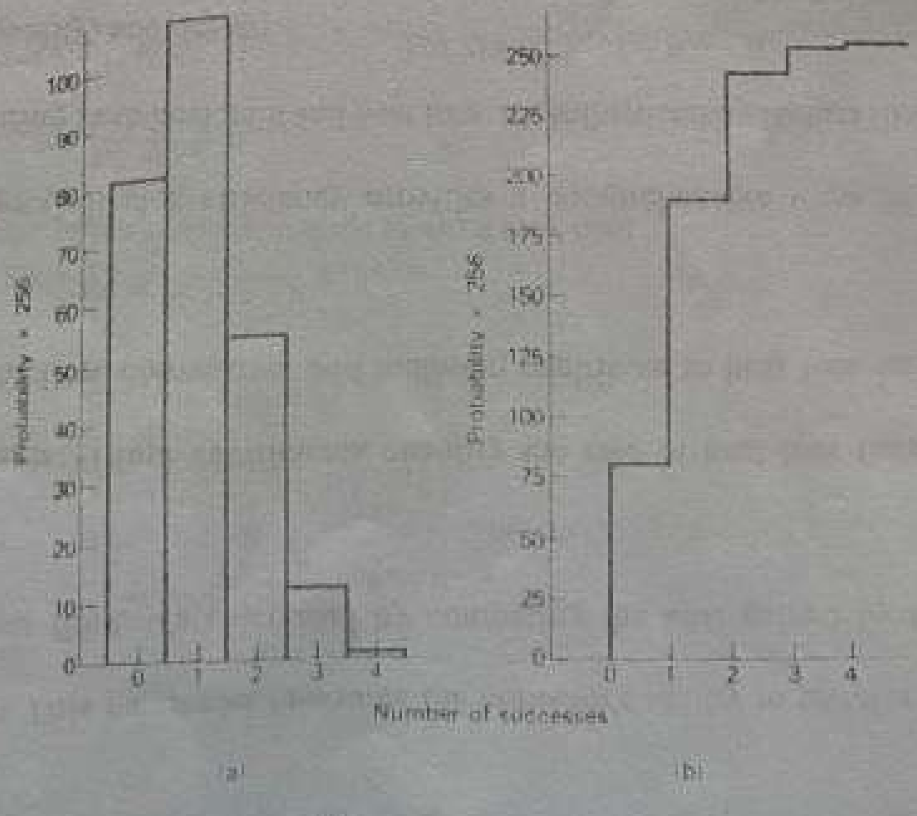
Number of successes	Number of Failures	Individual probability	Cumulative probability
0	4	$(3/4)^4 = 81/256$	81/256
1	3	$4(1/4)(3/4)^3 = 108/256$	189/256
2	2	$6(1/4)^2(3/4)^2 = 54/256$	243/256
3	1	$4(1/4)^3(3/4) = 12/256$	255/256
4	0	$(1/4)^4 = 1/256$	256/256
		$\sum = 1$	

Numerical example – III

A die is thrown in 6 times. Evaluate the probability of getting 2 spots on the upper face 0, 1, 2, . . . , 6 times and draw the probability mass (density) function and the probability distribution function.

Solution:

On each throw, the probability of getting 2 spots on the upper face is 1/6 and the probability of not getting 2 spots is 5/6. If these two events are defined as success and failure respectively, then, although there are six possible outcomes on each throw, the problem has been constrained to have two outcomes and the binomial distribution becomes applicable.



Consequently n=6, p=1/6 and q=5/6. The probability results are shown in below table.

Number of successes	Individual probability	Cumulative probability
0	$(5/6)^6 = 15625/46656$	15625/46656
1	$6(1/6)\ (5/6)^5 = 18750/46656$	34375/46656
2	$15(1/6)^2(5/6)^4 = 9375/46656$	43750/46656
3	$20(1/6)^3(5/6)^3 = 2500/46656$	46250/46656
4	$15(1/6)^4(5/6)^2 = 375/46656$	46625/46656
5	$6(1/6)^5(5/6) = 30/46656$	46655/46656
6	$(1/6)^6 = 1/46656$	46656/46656
	$\sum = 1$	

Expected value and Standard Deviation for Binomial Distribution:

The two most important parameters of a distribution are the expected or mean value and the standard deviations.
The binomial distribution is a discrete random variable and therefore the expected value and standard deviation can be evaluated using equation.

$$E(x) = \sum_{i=1}^{n} x_i P_i \quad \text{where } \sum_{i}^{n} P_i = 1$$

$$= \sum_{x=0}^{n} x\,(n_{c_x}\, p^x q^{n-x}) \qquad [P(r,n) = n_{c_r}\, p^r (1-p)^{n-r})]$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

As the contribution to this summation made by x=0 is zero, then

$$E(x) = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}\ p^x q^{n-x}$$

$$E(x) = \sum_{x=1}^{n} \frac{xn(n-1)!}{x(x-1)!(n-x)!}\ p\, p^{x-1} q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-x)!}\ p\, p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!}\ p^{x-1} q^{n-x}$$

Let n-1 = m and x-1 = y

$E(x) = np \sum_{y=0}^{m} \frac{(n-1)!}{y!(n-1-y)!} p^y q^{n-1-y}$

$E(x) = np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y q^{m-y}$

$Since,\ \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y q^{m-y} = 1$

E(x) = np

Expected value or Mean value and Standard Deviation for Exponential Distribution:

The expected value of a continuous random variable having a range (0 , ∝) is given by

$E(x) = \int_0^{\infty} t.f(t)\,dt$

$E(x) = \int_0^{\infty} t.\lambda\, e^{-\lambda t}\,dt$

This can be integrate by parts

Let $u = t$ and $v = -e^{-\lambda t}$

$du = dt$ $dv = \lambda\, e^{-\lambda t}\,dt$

$\therefore E(t) = \int u dv = [uv]_0^{\infty} - \int_0^{\infty} v du$

$= [-t\, e^{-\lambda t}]_0^{\infty} - \int_0^{\infty} e^{-\lambda t}\,dt$

$= \left[-t\, e^{-\lambda t}\right]_0^{\infty} - \left[\frac{1}{\lambda} e^{-\lambda t}\right]_0^{\infty}$

$= 0 + \frac{1}{\lambda}$

$\therefore E(t) = \frac{1}{\lambda}$

$$\sigma^2 = \int_0^{\infty} t^2.\lambda\, e^{-\lambda t}\,dt - E^2(t)$$

$u = t^2$ $v = -e^{-\lambda t}$

$du = 2tdt$ $dv = \lambda\, e^{-\lambda t}\,dt$

$$\sigma^2 = \int u dv - E^2(t)$$

Integrating by parts

$= [uv]_0^{\infty} - \int_0^{\infty} v du - E^2(t)$

$= [-t^2 e^{-\lambda t}]_0^{\infty} - \int_0^{\infty} -2t\, e^{-\lambda t}\,dt - E^2(t)$

$\sigma^2 = 0 + \frac{2}{\lambda}.\frac{1}{\lambda} - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$\sigma^2 = \frac{1}{\lambda^2} \Rightarrow \sigma = 1/\lambda$

∴ Expected value and standard deviation of an Exponential distribution are equal.

$\sigma = \frac{1}{\lambda}$

∴ Expected value and Standard deviation of an Exponential Distribution are equal.

Mean Time To Failure (MTTF)

The expected value of a failure density function is often designated as the mean time to failure MTTF.

In case of exponential distribution this is equal to the reciprocal of the failure rate λ.

E(t) $= \int_o^p t\, f(t)dt$

$f(t) = -\frac{d}{dt}R(t)$

MTTF $= -\int_o^p t\, dR(t)$

$= \int_o^p R(t)dt$

$= [-t\, R(t)]_o^p + \int_o^p R(t)dt$

MTTF = $\int_o^p R(t)dt$

Reliability Analysis of series networks using exponential distribution

Let $R_1(t)$, $R_2(t)$ . . . . $R_n(t)$ be the reliabilities of 'n' components connected in series.

$R_s(t) = R_1(t)\, R_2(t) \,..\, R_n(t)$

$= \prod_{i=1}^{n} R_i t$

Let $R_1(t) = e^{-\lambda_1 t}$ & $R_2(t) = e^{-\lambda_2 t}$ ...

$R_n(t) = e^{-\lambda_n t}$

∴ $R_s(t) = e^{-(\lambda_1 \lambda_2 \lambda_3 \ldots + \lambda_n)t}$

$= e^{-\sum_{i=1}^{n} \lambda_i t}$

Total failure rate

$$\lambda_s(t) = \sum_{i=1}^{n} \lambda_i t$$

Therefore, Hazard rate function for the system is determined by summing the hazard rate function of the 'n' independent components.

MTTF = $\int_o^p R_s(t)dt = \int_o^p e^{-\sum_{i=1}^{n} \lambda_i t}\, dt$

MTTF = $\frac{1}{\lambda} = \frac{1}{\sum_{i=1}^{n} \lambda_i} = \frac{1}{\sum_{i=1}^{n} \frac{1}{MTTF_i}}$

If all the components connected in series have the same failure rates

$\lambda_1 = \lambda_2 = \lambda_3 \ldots\ldots \lambda$ then $\lambda = n\lambda_1$

MTTF = $\frac{1}{n\lambda 1}$

Parallel configuration

Let $R_1(t)$ $R_2(t)$ . . . $R_n(t)$ be the reliabilities of the components connected in parallel.

$R_s(t) = 1 - [(1-R_1(t))\,(1-R_2(t))\,.\,.\,.\,.\,.\,(1-R_n(t))]$

$= 1 - \prod_{i=1}^{n}[1 - R_i(t)]$

$Rs(t) = 1 - \prod_{i=1}^{n}[1 - e^{-\lambda_i t}]$

Where λ_i = failure rate of its component

Two components in parallel

$$R_s(t) = 1 - \left[\left(1 - e^{-\lambda_1(t)}\right)\left(1 - e^{-\lambda_2(t)}\right)\right]$$
$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$\text{MTTF} = \int_o^P R_s(t) = \int_o^P e^{-\lambda_1 t}dt + \int_o^P e^{-\lambda_2 t}dt - \int_o^P e^{-(\lambda_1+\lambda_2)t}$$
$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$$

If $\lambda_1 = \lambda_2 = \lambda$

Then $R_s(t) = 2e^{-\lambda t} - e^{-2\lambda t}$

$\text{MTTF} = \frac{2}{\lambda} - \frac{1}{2\lambda}$

Numerical example 1

An aircraft engine consists of three modules having constant failure rates of $\lambda_1 = 0.002$ $\lambda_2 = 0.015$ and $\lambda_3 = 0.0025$ failures per operating hour. What is the reliability function for the engine and what is the MTTF?

Solution:

$R(t) = e^{-(0.002+0.015+0.0025)t}$
$= e^{-0.0195t}$

$\text{MTTF} = \frac{1}{0.0195} = 51.28\ \textit{operating hours}$

Numerical example 2

Consider a four component system of which the components are independent and identically distributed with Constant Failure Rate (CFR). If $R_s(100) = 0.95$, find the individual component MTTF?

Solution:

$R_s(100) = e^{-100\lambda_s} = e^{-100(4x\lambda)} = 0.95$

$\lambda = \frac{-\ln(0.95)}{400} = 0.000128$

$\text{MTTF} = \frac{1}{0.000128} = 7812.5$

Numerical example 3

A Simple electronic circuit consists of 6 transistors each having a failure rate of 10^{-6} f/hr, 4 diodes each having a failure rate of 0.5×10^{-6}f/hr, 3 capacitor each having a failure rate of 0.2×10^{-6}f/hr, 10 resistors each having a failure rate of 5×10^{-6}f/hr. Assuming connectors and wiring are 100% reliable (these can be included if considered significant), evaluate the equivalent failure rate of the system and the probability of the system surviving 1000hr if all components must operate for system success.

Solution

λ_s = Equivalent failure rate of the system

$= 6 \times (1 \times 10^{-6}) + 4 \times (0.5 \times 10^{6}) + 3 \times (0.2 \times 10^{-6}) + 10 \times (5 \times 10^{-6}) + 2 \times (2 \times 10^{-6})$

$= 6.26 \times 10^{-5}$ f/hr

$R_s(t) \quad = e^{\lambda_s t}$

$R_s(1000) \quad = \exp(-6.26 \times 10^{-5} \times 1000)$

$= 0.9393$

Since $Q_s(t) \quad = 1-R_s(t)$

$Q_s(1000) \quad =1- 0.9393$

$= 0.0707$

Poisson distribution

Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time.

The Poisson distribution can also be used for the number of events in other specified intervals such as distance, area or volume.

Poisson distribution is an approximation to binomial distribution. It is used for large values of n and small p

$$P(x, \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

λ = shape parameter which indicates the **average** number of events in the given time interval.

= Mean value

Numerical example - 1

A rare disease has an incidence of 1 in 1000 person-years. Assuming that members of the population are affected independently, find the probability of 'k' cases in a population of 10,000 for k=0, 1, 2

Solution:

Expected mean $\lambda = 0.001 \times 10{,}000$

$= 10$

$$P(x = 0) = \frac{e^{-10}10^0}{0!} = 0.0000454$$

$P(x = 1) = \frac{e^{-10}10^1}{1!} = 0.000454$

$P(x = 2) = \frac{e^{-10}10^2}{2!} = 0.00227$

Numerical example - 2

In a large system the average number of cable faults per year per 100 km of cable is 0.5. Consider a specified piece of cable 10km long and evaluate the probabilities of 0, 1, 2 etc, faults occurring in (a) a 20 year period, and (b) a 40 year period

Solution:

Assuming the average failure rate data to be valid for the 10km cable and for the two periods being considered, the expected failure rate λ is,

$$\lambda = \frac{0.5 \, x \, 10}{100} = 0.05 \, f/yr$$

(a) For a 20 year period,
E(x) = 0.05 X 20 = 1.0

And

$P_x = \frac{1.0^x e^{-1.0}}{x!}$ for x = 0, 1, 2,...

(b) For a 40 year period,
E(x) = 0.05 X 40 = 2.0

And

$P_x = \frac{2.0^x e^{-2.0}}{x!}$ for x = 0, 1, 2,...

Reliability [redacted]

Reliability : The reliability of a device is considered high if it had repeatedly performed its function with success and low if it had tended to fail in repeated trials.

The reliability of a system is defined as the probability of performing the intended function over a given period of time under specified operating conditions.

The above definition can be broken do on into four parts:

i) Probability
ii) Intended function

iii) Given period of time
iv) Specified operating conditions

Probability: Because, the reliability is a Probability, the reliability of system R_i is governed by the equation $0 \leq R_s \leq 1$. The equality sign hold good in case of equipment called one shot equipment.

Intended function: It is also defined to as the successful operation.

Example-1: As an example, let us consider the building up of voltage by a dc shunt generator. For some reasons, let us assume that the voltage is not build up.
We say that the dc shunt generator has failed to do its job. The failure in this contest doesn't imply any physical failure, but only the operational failure.

Example-2: Lightning arrester : The lightning arrestor should burst in the event of occurrence of a lightning stroke. On the occurrence of a lightning stroke, if the lightning arrestor bursts, there is the physical failure or damage but operationally it is successful.
On the other hand, if it doesn't burst there is no physical failure, but yet there is an operational failure and we say that the lightning arrestor has failed (operationally).

Given period of time: Any component has some useful life period, within which time the component should operate successfully. For example, a power transformer has a useful life of at least 20 to 25 years.
If, it fails within this time period, then the instrument is said to be unreliable and if it fails after its useful life period, then we say it is reliable.

Specified operating or environmental conditions: Any equipment is supposed to perform its duty satisfactorily under contain specified operating condition such as temperature, humidity, pressure and altitude.
Though an equipment is able to perform its duty satisfactorily in a cold country yet it may fail when used under hot climatic conditions.

Component Reliability

It is usual for a large system to be divided into components for the purpose of reliability evaluation. A component is that part of a system which is treated as a single entity for the purpose of reliability evaluation. There is no clear distinction between component and system. The same unit can be considered as component (or) system depending on circumstances. For instance, a generating unit is considered as a component while dealing with the reliability of entire power system. Same can be treated as a complex system consisting of several components like the boiler, Turbine and Generator, etc.

Components can be classified into two groups – Non-repairable components and Repairable components.
Non-repairable components are components that cannot be repaired or the repair is uneconomical.
Repairable components are components which can be repaired upon failure and thus their life histories consist of alternating operating and repair periods.
In the reliability evaluation of power systems, it is the repairable type that is of greater interest.

For several reasons, a component put into service fails after sometime, called the TIME TO FAILURE (T), this can be recognized as a random variable and the reliability of a component at any time can be expressed as

$$\begin{aligned} R(t) &= P(T>t) \\ &= 1 - F(t) \\ &= 1 - P(T \leq t) \end{aligned}$$

Reliability Function:

f(x) = Probability density function

F)x) = Probability distribution

All components have a different failure rate, hence these time-to-failure obey a probability distribution, thus probability value is a function of time that is specified or considered.

f(t)= density function which indicate the rate of failures per hour.

R(t)= Reliability function

An additional function which is one of the most extensively used function in reliability evaluation is the hazard rate h(t).

In terms of failure the hazard rate is a measure of the rate at which failures occur or the instantaneous failures/hour.

$$f(t) = \frac{\text{no of failures}}{\text{no of component x operating hours}}$$

$$h(t) = \frac{\text{no of failures}}{\text{no of component at the begining of internal x operating hours}}$$

Thus the hazard rate is dependent on the number of failures in a given time period and the number of components exposed to failures.

PROBLEMS:

1. The field test data in respect of 172 components is as given below. Calculate failure density rate and hazard rate

Time internal hrs	0-1000	1000-2000	2000-3000	3000-4000	4000-5000	5000-6000
Failure in internal	59	24	29	30	17	13

Solution :

f(t)	h(t)
59/172 X 1000	59/172X1000
24/172X100	24/113x1000
29/172X100	29/89x1000
30/172X100	30/60X100
17/172X100	17/30X100
13/172X100	13/13X100

2. The component failure data for ten components subjected to a life test are given below. Find the failure density rate and hazard rate.

Failure	1	2	3	4	5	6	7	8	9	10
Operating time hrs	8	20	34	46	63	86	111	141	186	206

Solution :

Time internal	0-8	8-20	20-34	34-46	46-63	63-86	86-111	111-141	141-186	186-206
f(t)	$\frac{1}{10x8}$ = 0.0125	$\frac{1}{10x12}$	$\frac{1}{10x14}$	$\frac{1}{10x12}$	$\frac{1}{10x17}$	$\frac{1}{10x23}$	$\frac{1}{10x25}$	$\frac{1}{10x20}$	$\frac{1}{10x45}$	$\frac{1}{10x20}$

h(t)	$\frac{1}{10\times8}$ =0.0125	$\frac{1}{9\times12}$	$\frac{1}{8\times14}$	$\frac{1}{7\times12}$	$\frac{1}{6\times17}$	$\frac{1}{5\times23}$	$\frac{1}{4\times25}$	$\frac{1}{3\times20}$	$\frac{1}{2\times45}$	$\frac{1}{1\times20}$

HAZARD FUNCTION : h(t)

The probability function of the random variable 'T' can be determined by

$$f(t) = \lim_{\Delta t\to 0} \frac{P(t<T \leq t+\Delta t)}{\Delta t} \quad \Delta t = \text{increment time}$$

Suppose at t=0, N(0) components are put to work. At any time 't' suppose the number of components is N(t),

then $f(t) = \frac{N(t)-N(t+\Delta t)}{N(0).\Delta t}$

If the probability in the above case is calculated as conditional probability, condition being that the component should be working at t, then the function is Hazard function h(t).

$$h(t) = \lim_{\Delta t\to 0} \frac{P(t<T \leq t+\Delta t)}{\Delta t} \Big/_{\text{at } (T > t)}$$

$$h(t) = \frac{N(t)-N(t+\Delta t)}{N(t).\Delta t}$$

General Reliability functions

All components have a different failure rate, hence these times to failure obey a probability distribution. This Probability value is a function of time that is specified or considered.

Let f(t) = failure density function which indicates the rate of failures per hour

R(t) = Reliability function

An additional function which is one of the most extensively used in reliability evaluation is the hazard rate h(t).
In terms of failure, the hazard rate is a measure of the rate at which failures occur. It indicates the instantaneous failures / hour.

$$f(t) = \frac{\text{Number of failures}}{\text{Number of components X operating hours}}$$

$$h(t) = \frac{\text{Number of failures}}{\text{Number of components at the begining of interval X operating hours}}$$

The hazard rate is dependent on the number of failures in a given time period and the number of components exposed to failure.

Failure density function f(t) is the rate of failures per hour.
Hazard rate h(t) is the instantaneous failures per hour.

Derivation of Reliability function R(t) in terms of hazard rate h(t)

A non - repairable component is of use only till the failure occurs and if the component fails, we have to replace it with a new component

Such a component is described by its life time T, a random variable.
Since 'R' is a function of 't' (operating time), the reliability can be defined as

$$R(t) = P(T>t) \quad \text{---------} (1)$$

$$= 1 - P(T \leq t)$$

But $P(T \leq t) = F(t) =$ failure distribution function

$$\therefore R(t) = 1 - F(t) \text{ -------}(2)$$

$$\text{Failure density function } f(t) = \frac{d\,F(t)}{dt} = -\frac{d\,R(t)}{dt} \text{ --------}(3)$$

Consider a case in which fixed number N_0 of identical component are tested.

Let $N_s(t)$ = Number of components surviving at time 't'
$N_f(t)$ = Number of components failed at time 't'

$$N_s(t) + N_f(t) = N_0$$

At any time 't' the reliability function

$$R(t) = \frac{N_s(t)}{N_o} \text{ -------}(4)$$

$$= \frac{N_o - N_f(t)}{N_o} = 1 - \frac{N_f(t)}{N_o} \text{ ------ } (5)$$

Similarly the probability of failure or cumulative failure distribution

$$F(t) = \frac{N_f(t)}{N_o} \text{ -------}(6)$$

From equations 5 and 6 we get equation 2

$$R(t) = 1 - F(t)$$

$$\frac{d\,R(t)}{dt} = -\frac{d\,F(t)}{dt} = -\frac{1}{N_0}\frac{d\,N_f(t)}{dt} \text{ -------- } (7)$$

$$f(t) = -\frac{d\,R(t)}{dt}$$

$$= \frac{1}{N_o} \frac{d\,N_f(t)}{dt} \text{ -------- (8)}$$

The failure density function and hazard rate are identical only at t= 0.

∴ The general expression for hazard rate at time 't' is

$$h(t) = \frac{1}{N_s(t)} \frac{d\,N_f(t)}{dt} \text{ --------- (9)}$$

$$= \frac{N_o}{N_o} \cdot \frac{1}{N_s(t)} \frac{d\,N_f(t)}{dt}$$

$$= \frac{N_o}{N_s(t)} \frac{1}{N_o} \frac{d\,N_f(t)}{dt}$$

$$= \frac{1}{R(t)} f(t) = \frac{f(t)}{R(t)}$$

$$\therefore\ h(t) = \frac{f(t)}{R(t)} \text{ -------- (10)}$$

From equation 8 $\quad f(t) = \frac{-d}{dt} R(t)$

$$h(t) = \frac{-1}{R(t)} \frac{d\,R(t)}{dt} \text{ -------- (11)}$$

Let us consider

$$\frac{d}{dt}[\ln R(t)] = \frac{1}{R(t)} \frac{d\,R(t)}{dt}$$

$$= \frac{1}{R(t)} \frac{d}{dt}[1 - F(t)]$$

$$= \frac{1}{R(t)} - \frac{d\,F(t)}{dt} = -\frac{f(t)}{R(t)} = -h(t)$$

$$\therefore \frac{d}{dt} \ln R(t) = -h(t)$$

$$\ln R(t) = -\int_0^t h(t)\,dt$$

$$R(t) = e^{-\int_0^t h(t)\,dt} \text{ -------(12)}$$

For a constant hazard rate, h(t) = λ= number of areas

$$\therefore R(t) = e^{-\int_0^t \lambda\,dt} = e^{-\lambda t}$$

$$R(t) = e^{-\lambda t} \text{ --------- (13)}$$

Relation between R(t), Q(t), F(t) , f(t) and h(t)

R(t) = Reliability function Q(t) = Unreliability function h(t) = hazard rate function

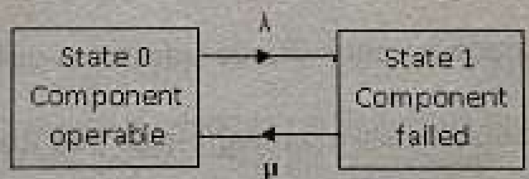
F(t) = failure Distribution function

f(t) = failure density function which indicates the rate of failures per hour

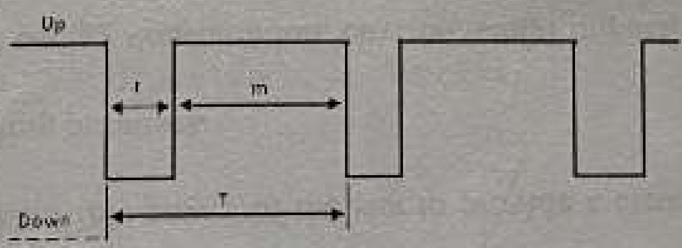
	R(t)	Q(t) = F(t)	f(t)	h(t)
R(t)	–	1- Q(t)	$-\int f(t)\,dt$	$e^{-\int_0^t h(t)\,dt}$
Q(t)	1- R(t)	–	$\int f(t)\,dt$	$1-e^{-\int_0^t h(t)\,dt}$
f(t)	$-\frac{d\,R(t)}{dt}$	$\frac{d\,Q(t)}{dt}$	–	$h(t)e^{-\int_0^t h(t)\,dt}$
h(t)	$-\frac{d}{dt}\ln R(t)$	$\frac{\frac{d\,Q(t)}{dt}}{1-Q(t)}$	$\frac{f(t)}{-\int f(t)dt}$	–

Measures of Reliability:

Consider a single repairable component for which the failure rate and repair rate are constant. The state transition diagram for this component is shown below.



(a)



(b)

Single component system (a) State space diagram (b) Mean time/ state diagram

Let, λ = failure rate of the component

μ = repair rate of the component

m = mean operation time of the component

r = mean repair time of the component

The two system states and their associated transitions can be shown chronologically on a time graph. The mean values of up and down times can be used to give the average performance of this two state system. This is shown in figure b.

In figure b, the period T is the system cycle time and is equal to the sum of the mean time to failure (MTTF) and mean time to repair (MTTR). This cycle time is defined as the mean time between failures (MTBF). Some times, MTBF is used in place of MTTF. It is evident however that there is a significant conceptual difference between MTTF and MTBF. The numerical difference between them will depend on the value of MTTR. In practice the repair time is usually very small compared with the operating time and therefore the numerical values of MTTF and MTBF are usually very similar.

The following relationships can therefore be defined

$$m = MTTF = 1/\lambda \qquad r = MTTR = 1/\mu$$
$$T = MTBF = m + r = 1/f$$

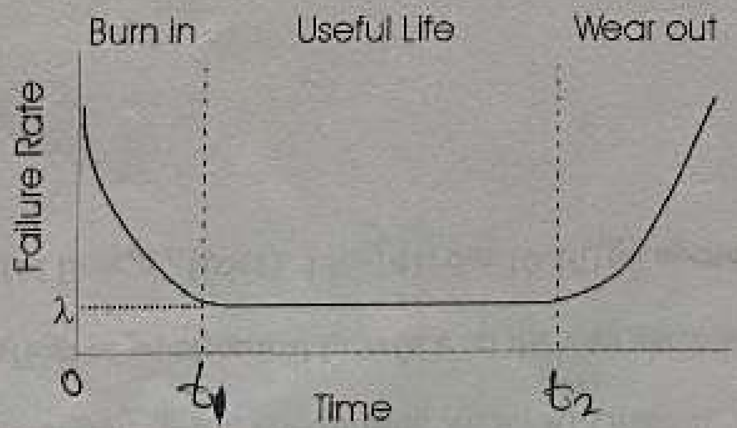
Where f = cycle frequency, i.e., the frequency of encountering a system state.
The failure rate λ is the reciprocal of the mean time to failure, MTTF, with the times to failure counted from the moment the component begins to operate to the moment it fails. Similarly, the repair rate μ is the reciprocal of the mean time to repair, MTTR, with these times counted from the moment the component fails to the moment it is returned to an operable condition.

$$\lambda = \frac{\text{number of failures of a component in the given period of time}}{\text{total period of time the component was operating}}$$

$$\mu = \frac{\text{number of repairs of a component in the given period of time}}{\text{total period of time the component was repaired}}$$

Bath – Tub Curve

The plot of hazard rate versus time is referred to as Bath – tub curve. Most of the components have a high failure density rate at the beginning and the failure rate decreases with time. The failures in the beginning are mainly due to defects in design and due to the improper manufacturing techniques, etc.



- These will be detected and corrected so that failure rate or hazard rate decreases with time. This is indicated by the portion $0 - t_1$ of the curve.
- Between t_1 and t_2 the hazard rate is more or less constant and beyond t_2 the hazard rate increases with time due to normal wear and tear.

- The period 0- t_1 is referred to as **debugging period** or **burn – in** period and the failures are referred ~~defined~~ to as infant mortality.
- The period $t_1 - t_2$ is called as the **useful life** period. In this period the failures are chance failures or random failures.
- The period beyond t_2 is called the **wear-out period** and the failures are mainly due to aging effect. These failures are called wear-out failures.
- Because of the shape of the curve, it is called as Bath- Tub- Curve.
- The Bath – Tub – Curve can be divided into three regions namely

  i) Decreasing hazard rate region
  ii) Constant hazard rate region
  iii) Increasing hazard rate region