④ Sea $\{f_n\}$ la sucesión de funciones de $\mathbb{R}_0^+$ en $\mathbb{R}$ definida por:

$$f_n(x) = \frac{2nx^2}{1+n^2x^2} \quad \forall x \in \mathbb{R}_0^+, \ \forall n \in \mathbb{N}$$

a) Estudiar la convergencia puntual de $\{f_n\}$

Para x fijo

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2nx^2}{1+n^2x^4} = 0 \qquad \left| \quad \frac{2\cancel{n}\cancel{x^2}}{n^{\cancel{2}}x^{\cancel{4}}} = \frac{2}{nx^2} = \frac{2}{x^2}\cdot\frac{1}{n} = c\cdot 0 = 0 \right.$$

$\{f_n\}$ converge puntualmente a 0 en $\mathbb{R}_0^+$

b) Dado $\delta \in \mathbb{R}^+$, probar que $\{f_n\}$ converge uniformemente en $[\delta, +\infty[$, pero no en el intervalo $[0, \delta]$

$f_n(x)$ es derivable en $\mathbb{R}^+$

$$f_n'(x) = \frac{4nx\cdot(1+n^2x^4) - (2nx^2\cdot 4n^2x^3)}{(1+n^2x)^2} =$$

$$= \frac{4nx\,(1+\cancel{n^2x^4} - 2n^2x^4)}{(1+n^2x^4)^2} = \frac{4nx\,(1-n^2x^4)}{(1+n^2x^4)^2}$$

$$4nx\,(1-n^2x^4) = 0 \quad < \begin{array}{l} x=0 \\ x = \sqrt[4]{\frac{1}{n^2}} \end{array}$$

$$f\left(\sqrt[4]{\tfrac{1}{n^2}}\right) = \frac{2n\cdot\sqrt[4]{\frac{1}{n^4}}}{1+n^2\cdot\frac{1}{n^2}} = \frac{2\cdot n\cdot\frac{1}{n}}{1+1} = \frac{2}{2} = 1$$

$0 < x < \sqrt[4]{\frac{1}{n^2}}$ $f'_n(x) > 0 \rightarrow f_n(x)$ es creciente

$\sqrt[4]{\frac{1}{n^2}} < x$ $f'_n(x) < 0 \rightarrow f_n(x)$ es decreciente

Sea $\delta \in \mathbb{R}^+$, $\exists m \in \mathbb{N}: n \geq m$ $\sqrt[4]{\frac{1}{n^2}} < \delta$ $x_n = \sqrt[4]{\frac{1}{n^2}}$ $n \geq m$

$x_n = 0$ si $n < m$

$x_n \in [0, \delta]$ $\forall n \in \mathbb{N}$ $\{f_n(x_n)\} \not\rightarrow 0$

$\{f_n\}$ no converge uniformemente en el intervalo $[0, \delta]$

$m \geq n$ $\sqrt[4]{\frac{1}{n^2}} < \delta$ f_n decreciente en $[\delta, +\infty[$

$0 < f_n(x) \leq f_n(\delta)$ $\{f_n(\delta)\} \rightarrow 0 \Longrightarrow$

$\exists m \in \mathbb{N}: n \geq m$ $|f_n(x)| \leq \delta_n$ $\forall x \in [\delta, +\infty[$

$\{f_n\}$ converge uniformemente en $[\delta, +\infty[$

(5) Para cada $n \in \mathbb{N}$ sea $g_n : [0, \pi/2] \to \mathbb{R}$ la función definida por

$$g_n(x) = n(\cos x)^n \operatorname{sen}(x) \quad \forall x \in [0, \pi/2]$$

Fijado un $\rho \in \mathbb{R}$ con $0 < \rho < \pi/2$, probar que la sucesión $\{g_n\}$ converge uniformemente en el intervalo $[\rho, \pi/2]$, pero no en el intervalo $[0, \rho]$

Si $x = 0$ $\quad g_n(0) = n(\cos(0))^n \operatorname{sen}(0) = n \cdot 1^n \cdot 0 = 0$

$g_n(x) = 0 \quad \forall n$

Si $x \in]0, \pi/2[$ $\quad \left.\begin{array}{l} 0 < \cos(x) < 1 \\ 0 < \operatorname{sen}(x) < 1 \end{array}\right\} \Rightarrow n \cdot (\cos(x)^n) \cdot \operatorname{sen} x \leq$

$\leq n \cdot \cos(x)^n \qquad 0 \leq \lim\limits_{n\to\infty} g_n(x) \leq \lim\limits_{n\to\infty} n \cos(x)^n = 0$

Si $x = \frac{\pi}{2}$ $\quad g_n\left(\frac{\pi}{2}\right) = n \cdot \cos\left(\frac{\pi}{2}\right)^n \operatorname{sen}\left(\frac{\pi}{2}\right) = n \cdot 0 = 0 \quad \forall n \in \mathbb{N}$

$g_n(x)$ converge puntualmente a 0 $\forall x \in [0, \pi/2]$

$g_n(x)$ es derivable

$$g'_n(x) = n \cdot \left(-n \operatorname{sen}^2(x) \cdot \cos^{n-1}(x) + \cos^{n+1}(x)\right)$$

$$g'_n(x) = 0 \iff \cos^{n-1}(x)\left(-n \operatorname{sen}^2(x) + \cos^2(x)\right) = 0$$

$x \in [0, \pi/2] \to \cos(x)$ es inyectiva $\quad \cos^{n-1}(x) = 0 \iff x = \frac{\pi}{2}$

No nos aporta nada útil este valor

$-n\sin^2(x)+\cos^2(x)=0 \qquad \sin^2(x)=\frac{1}{n+1} \qquad \sin(x)=\sqrt{\frac{1}{n+1}} \quad x=\arcsin\sqrt{\frac{1}{n+1}}$

$-(n+1) \quad -\operatorname{sen}^2(x)+\operatorname{sen}^2(x) \quad =1$

Si $0<x<\arcsin\sqrt{\frac{1}{n+1}}$ $\quad g_n'(x)>0 \longrightarrow g_n(x)$ es creciente

Si $\arcsin\sqrt{\frac{1}{n+1}}<x<\frac{\pi}{2}$ $\quad g_n'(x)<0 \longrightarrow g_n(x)$ es decreciente

$g_n\left(\arcsin\sqrt{\frac{1}{n+1}}\right)\neq 0$

Sea $\rho\in\left]0,\frac{\pi}{2}\right[$ $\quad \exists\, m\in\mathbb{N}: n\geq m \quad \arcsin\sqrt{\frac{1}{n+1}}<\rho$

$x_n=\arcsin\sqrt{\frac{1}{n+1}} \quad \forall\, n\geq m$

$x_n=0$ si $n<m$

$\Big\}\ x_n\in[0,\rho] \quad \{g_n(x_n)\}\not\to 0$

g_n no converge uniformemente a 0 en $[0,\rho]$

$n\geq m \quad \arcsin\sqrt{\frac{1}{n+1}}<\rho \quad g_n$ es decreciente en $\left[\rho,\frac{\pi}{2}\right]$

$0<g_n(x)\leq g_n(\rho) \quad \{g_n(\rho)\}\longrightarrow 0 \implies$

$\exists\, m\in\mathbb{N}: n\geq m \quad \{g_n(x)\}\leq \rho_n \quad \forall x\in\left[\rho,\frac{\pi}{2}\right]$

$g_n(x)$ converge uniformemente en $\left[\rho,\frac{\pi}{2}\right]$

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$$-n\,\mathrm{sen}^2(x)+\cos^2(x)=-(n+1)\,\mathrm{sen}^2(x)+\mathrm{sen}^2(x)+\cos^2(x)=0$$