Universidad Autonoma de Nuevo León

Facultad de Ciencias Físico Matemáticas

Proyecto 2

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Resumen

En este documento nuestro equipo presenta el Proyecto 2 del curso de mecánica teórica, donde encontramos la ecuación de Euler-Lagrange para una variable independiente y dos dependientes con una restricción de ligadura.

1. Euler-Lagrange

Solución.

Consideremos la integral:

$$J = \int_{x_1}^{x_2} f\left[y, z, y_x, z_x; x\right] dx \tag{1}$$

Donde tenemos:

$$y_x = \frac{\partial y}{\partial x} \tag{2}$$

$$z_x = \frac{\partial z}{\partial x} \tag{3}$$

Y también una restricción de la forma:

$$g\left[y, z; x\right] = 0 \tag{4}$$

Consideremos la condición para que J sea un valor extremo:

$$\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0 \tag{5}$$

Primeramente definamos funciones axiliares de y,z, en términos del parámetro de variación α y la variable independiente x:

$$y(\alpha, x) = y(0, x) + \alpha\eta_1(x) \tag{6}$$

$$z(\alpha, x) = z(0, x) + \alpha\eta_2(x) \tag{7}$$

Con las siguientes condiciones:

1. En los extremos $\eta_1(x)$ y $\eta_2(x)$ son iguales a cero.
2. $\eta_1(x)$ y $\eta_2(x)$ son diferenciables en (x_1, x_2)

Aplicamos la condición (5) a (1):

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\int_{x_1}^{x_2} f\left[y, z, y_x, z_x; x\right] dx\right]$$

Como en el lado derecho de la ecuación los límites de la integral están fijos, es posible aplicar la derivada parcial con respecto a alfa al integrando mediante la regla de la cadena:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y_x}\frac{\partial y_x}{\partial \alpha} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \alpha} + \frac{\partial f}{\partial z_x}\frac{\partial z_x}{\partial \alpha}\right] dx \tag{8}$$

Derivando parcialmente respecto a α las ecuaciones de (6) y (7)

$$\begin{aligned}
\frac{\partial}{\partial \alpha} y(\alpha, x) &= \frac{\partial}{\partial \alpha}\left[y(0, x) + \alpha\eta_1(x)\right] \\
&= \frac{\partial}{\partial \alpha} y(0, x) + \frac{\partial}{\partial \alpha}\alpha\eta_1(x) \\
&= \eta_1(x)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \alpha} z(\alpha, x) &= \frac{\partial}{\partial \alpha}\left[z(0, x) + \alpha\eta_2(x)\right] \\
&= \frac{\partial}{\partial \alpha} z(0, x) + \frac{\partial}{\partial \alpha}\alpha\eta_2(x) \\
&= \eta_2(x)
\end{aligned}$$

Por tanto, tenemos las siguientes ecuaciones:

$$\frac{\partial y}{\partial \alpha} = \eta_1(x) \tag{9}$$

$$\frac{\partial z}{\partial \alpha} = \eta_2(x) \tag{10}$$

Ahora hallamos $\frac{\partial y_x}{\partial \alpha}$ y $\frac{\partial z_x}{\partial \alpha}$:

$$\begin{aligned}
\frac{\partial y_x}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{dy}{dx} \\
&= \frac{\partial}{\partial \alpha}\frac{d}{dx}\left[y(0,x) + \alpha\eta_1(x)\right] \\
&= \frac{\partial}{\partial \alpha}\left[y^{'}(0,x) + \alpha\eta_1^{'}(x)\right] \\
&= \eta_1^{'}(x) \\
&= \frac{d\eta_1}{dx}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z_x}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{dz}{dx} \\
&= \frac{\partial}{\partial \alpha}\frac{d}{dx}\left[z(0,x) + \alpha\eta_2(x)\right] \\
&= \frac{\partial}{\partial \alpha}\left[z^{'}(0,x) + \alpha\eta_2^{'}(x)\right] \\
&= \eta_2^{'}(x) \\
&= \frac{d\eta_2}{dx}
\end{aligned}$$

Por tanto tenemos las siguientes ecuaciones:

$$\frac{\partial y_x}{\partial \alpha} = \frac{d\eta_1}{dx} \tag{11}$$

$$\frac{\partial z_x}{\partial \alpha} = \frac{d\eta_2}{dx} \tag{12}$$

Sustituyendo (9), (10), (11) y (12) en (8):

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta_1 + \frac{\partial f}{\partial y_x}\frac{d\eta_1}{dx} + \frac{\partial f}{\partial z}\eta_2 + \frac{\partial f}{\partial z_x}\frac{d\eta_2}{dx} \right] dx$$

Separamos la integral en sumas de integrales:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta_1 dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y_x}\frac{d\eta_1}{dx}dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial z}\eta_2 dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial z_x}\frac{d\eta_2}{dx}dx \tag{13}$$

Prestemos atención en los siguientes términos

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_x}\frac{d\eta_1}{dx}dx$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial z_x}\frac{d\eta_2}{dx}dx$$

de la ecuación (13).

- $$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_x}\frac{d\eta_1}{dx}dx$$

Hacemos $u = \frac{\partial f}{\partial y_x}$ y $dv = \frac{d\eta_1}{dx}dx$

Entonces $du = \frac{d}{dx}\frac{\partial f}{\partial y_x}dx$ y $v = \eta_1(x)$

Por lo tanto

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_x}\frac{d\eta_1}{dx}dx = \left[\frac{\partial f}{\partial y_x}\eta_1(x)\right]\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta_1(x)\frac{d}{dx}\frac{\partial f}{\partial y_x}dx$$

Como $\eta_1(x_1) = \eta_1(x_2) = 0$ implica que:

$$\left[\frac{\partial f}{\partial y_x}\eta_1(x)\right]\Bigg|_{x_1}^{x_2} = 0$$

Sustituyendo esta expresión en la integral anterior:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_x}\frac{d\eta_1}{dx}dx = -\int_{x_1}^{x_2} \eta_1(x)\frac{d}{dx}\frac{\partial f}{\partial y_x}dx \tag{14}$$

- $$\int_{x_1}^{x_2} \frac{\partial f}{\partial z_x} \frac{d\eta_2}{dx} dx$$

De manera análoga, aplicando el mismo procedimiento de cambio de variable con $u = \frac{\partial f}{\partial z_x}$ y $dv = \frac{d\eta_2}{dx} dx$

Se obtiene que:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial z_x} \frac{d\eta_2}{dx} dx = -\int_{x_1}^{x_2} \eta_2(x) \frac{d}{dx} \frac{\partial f}{\partial z_x} dx \tag{15}$$

Sustituyendo las ecuaciones (14), (15) en (13):

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta_1 dx - \int_{x_1}^{x_2} \eta_1 \frac{d}{dx} \frac{\partial f}{\partial y_x} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial z} \eta_2 dx - \int_{x_1}^{x_2} \eta_2 \frac{d}{dx} \frac{\partial f}{\partial z_x} dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta_1 - \eta_1 \frac{d}{dx} \frac{\partial f}{\partial y_x} \right) dx + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial z} \eta_2 - \eta_2 \frac{d}{dx} \frac{\partial f}{\partial z_x} \right) dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_x} \right) \eta_1(x) dx + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z_x} \right) \eta_2(x) dx \tag{16}$$

Obs. En (16) $\frac{\partial y}{\partial \alpha}$ y $\frac{\partial z}{\partial \alpha}$ no son independientes, ya que y,z están relacionadas por la restricción (4) $g\left[y, z; x\right] = 0$ y por tanto no podemos separar la integral y hacer los términos en paréntesis iguales a cero.

Ahora derivamos parcialmente a g respecto a α:

$$\frac{\partial}{\partial \alpha} g\left[y, z; x\right] = \frac{\partial}{\partial \alpha} 0$$

Por regla de la cadena:

$$\frac{\partial g}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial \alpha} = 0 \tag{17}$$

Sustituyendo (9), (10) en (17):

$$\frac{\partial g}{\partial y} \eta_1(x) + \frac{\partial g}{\partial z} \eta_2(x) = 0$$

$$\frac{\partial g}{\partial y} \eta_1(x) = -\frac{\partial g}{\partial z} \eta_2(x) \tag{18}$$

Retomando la ecuación (16) y uniendo la integral:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_x} \right) \eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z_x} \right) \eta_2(x) \right] dx$$

Ahora factorizamos un $\eta_1(x)$:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \frac{\eta_2(x)}{\eta_1(x)} \right] \eta_1(x) dx \tag{19}$$

Despejando $\frac{\eta_2(x)}{\eta_1(x)}$ de (18):

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial z}\eta_2(x)$$

$$\frac{\eta_2(x)}{\eta_1(x)} = -\frac{\partial g/\partial y}{\partial g/\partial z} \tag{20}$$

Sustituyendo (20) en (19):

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \left(-\frac{\partial g/\partial y}{\partial g/\partial z} \right) \right] \eta_1(x) dx \tag{21}$$

Obs. En la ecuación (21) vemos que $\eta_1(x) \neq 0$ para toda x en el intervalo (x_1, x_2) y no depende de α.

Aplicamos la condición (5) a (21):

$$\int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \left(-\frac{\partial g/\partial y}{\partial g/\partial z} \right) \right] \eta_1(x) dx = 0 \tag{22}$$

La observación anterior implica lo siguiente:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \left(-\frac{\partial g/\partial y}{\partial g/\partial z} \right) = 0$$

Haciendo algo de álgebra:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \left(\frac{\partial g/\partial y}{\partial g/\partial z} \right)$$

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \left(\frac{\partial g}{\partial z} \right)^{-1} \tag{23}$$

Obs. Ambos lados de la ecuación (23) son funciones de la variable x.

Hacemos (23) igual a una función $\lambda'(x)$.

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} \right) \left(\frac{\partial g}{\partial y} \right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} \right) \left(\frac{\partial g}{\partial z} \right)^{-1} = \lambda'(x) \tag{24}$$

De (24) se desprenden las siguientes ecuaciones:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \lambda'(x)$$

$$\left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} = \lambda'(x)$$

Haciendo algo de álgebra:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = \lambda'(x)\frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} = \lambda'(x)\frac{\partial g}{\partial z}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} - \lambda'(x)\frac{\partial g}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} - \lambda'(x)\frac{\partial g}{\partial z} = 0$$

Haciendo el cambio de variable $\lambda(x) = -\lambda'(x)$:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} + \lambda(x)\frac{\partial g}{\partial y} = 0 \tag{25}$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} + \lambda(x)\frac{\partial g}{\partial z} = 0 \tag{26}$$

Es así finalmente como las ecuaciones (25) y (26) representan las ecuaciones de Euler-Lagrange con una variable independiente, dos variables dependientes y una restricción de ligadura.