

## QCD and Monte Carlo event generators

Monte Carlo course seminar - Milan, February 2021



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## ① Hadron collisions and strong interactions

- Hadron collisions and strong interactions
- Renormalization group
- QCD factorization

## ② MC and Parton Showers

- Collinear factorization
- Final state radiation
- Initial state radiation

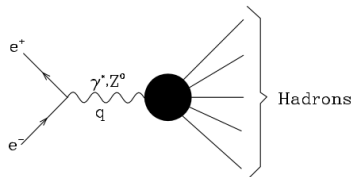
## ③ Hadronization: some basics

- Large number of colors approximation
- Hadronization models

# Strong interactions

## QCD from $e^+e^-$ annihilation

Quantum Chromodynamics (QCD)  $\rightarrow$  theory describing the interaction between quarks and gluons (strong interactions)



QCD arises already from  $e^+e^-$  annihilation  $\rightarrow R_0$  ratio

$$R_0 = \frac{\sigma(\gamma^* \rightarrow \text{hadrons})}{\sigma(\gamma^* \rightarrow \mu^+ \mu^-)} = 3 \sum_f c_f^2$$

- ❶ Color factor (3 color for each quark)
- ❷ Sum over charges of different flavors
- ❸ Threshold and higher order corrections

# Strong interactions

## Renormalization group

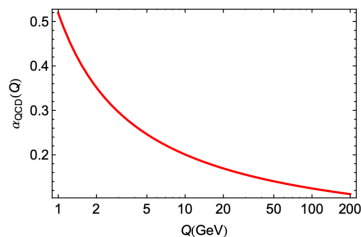
- Running coupling given by Renormalization Group Equation (RGE)

$$\mu \frac{d\alpha_s(\mu)}{d\mu} = \beta(\alpha_s(\mu)) = - \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{\pi} \right)^{n+1}$$

- Coupling  $\alpha_s$  evolves with scale  $\mu$  as given by RGE  $\rightarrow$  LO behavior driven by  $\beta_0$
- $\beta_0^{\text{QED}} < 0 \implies$  strongly coupled at large energies, UV divergent
- $\beta_0^{\text{QCD}} > 0 \implies$  weakly coupled at large energies, IR divergent

# Strong interactions

## Renormalization group



- Running coupling given by Renormalization Group Equation (RGE)

$$\alpha_s(\mu) = \frac{1}{\beta_0 \log\left(\frac{\mu^2}{\Lambda_{\text{QCD}}^2}\right)}$$

- $\beta_0$  LO of the  $\beta$  function, is  $> 0$
- $\Lambda_{\text{QCD}}$ , parameter that defines value of the coupling at large scales

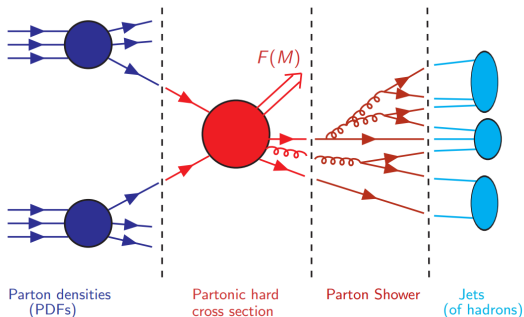
QCD is weakly coupled for  $\mu \gg \Lambda_{\text{QCD}} \rightarrow$  asymptotically free

Perturbative Quantum Chromodynamics (pQCD)

# Factorization theorem

## QCD factorization

LHC processes  $H_1 + H_2 \rightarrow F$



Separate process **PDFs** and **partonic (hard) interaction**

$$\sigma^F(p_1, p_2) = \sum_{\alpha, \beta} \int_0^1 dx_1 dx_2 f_{\alpha}(x_1, \mu_F^2) * f_{\beta}(x_2, \mu_F^2) * \hat{\sigma}_{\alpha\beta}^F(x_1 p_1, x_2 p_2, \alpha_s(\mu_R^2), \mu_F^2)$$

# Parton showers

## MC Parton showers

Partons in the initial and final state emit radiation. Initial state Radiation (ISR) and Final State Radiation (FSR) model by Monte Carlo (MC) shower algorithms

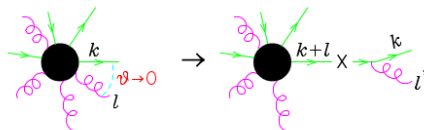
### Shower Monte Carlo programs (HERWIG, PYTHIA)

- Libraries for computing SM and BSM cross sections
- Shower algorithms produce a number of enhanced coloured parton emissions to be added to the hard process
- Hadronization models, underlying event, decays of unstable hadrons, etc

# Parton showers

## Collinear limit

- QCD emission processes are enhanced in the collinear limit ( $\theta$  small)
- $\sigma$  dominated by collinear splittings  $q \rightarrow qg, g \rightarrow gg, g \rightarrow q\bar{q}$



Collinear factorization  $\longrightarrow$  The cross section factorizes into the product of a tree-level cross section and a splitting probability

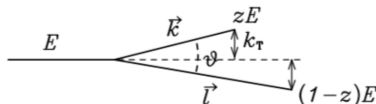
$$|M_{n+1}|^2 d\Phi_{n+1} \Rightarrow |M_n|^2 d\Phi_n \frac{\alpha_S}{2\pi} \frac{dt}{t} P_{q,qg}(z) dz \frac{d\phi}{2\pi}.$$

$$d\Phi_n = (2\pi)^4 \delta^4 \left( \sum_i^n k_i - q \right) \prod_i^n \frac{d^3 k_i}{(2\pi)^3 2k_i^0}$$



# Parton showers

## Kinematics of splitting



Kinematics of splitting given by  $(t, z, \phi)$

- $t$ : parameter with dimensions of energy that vanish in the collinear limit
  - Virtuality  $t = (k + l)^2 \approx z(1 - z)E^2\theta^2$
  - Transverse momentum  $t = k_{\perp}^2 = l_{\perp}^2 \approx z^2(1 - z)^2E^2\theta^2$
  - Hardness  $t = E^2\theta^2$
- $z$ : fraction of energy of radiated parton  $z = \frac{k^0}{k^0 + l^0}$
- $\phi$  represents azimuth of the  $k, l$  plane

# Parton showers

## AP splitting functions

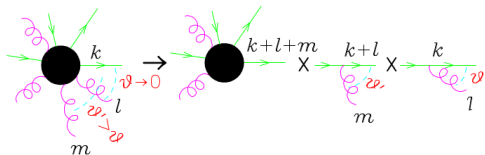
Factorization holds for small angles  $\rightarrow$  small  $t$  variable

Difference in the splitting  $\rightarrow$  Altarelli-Parisi splitting functions (singular in  $z \rightarrow 0, 1$ )

$$P_{q,qg}(z) = C_F \frac{1+z^2}{1-z}$$

$$P_{g,gg}(z) = C_A \left( \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right)$$

$$P_{g,q\bar{q}}(z) = T_F(z^2 + (1-z)^2)$$



We can proceed in an iterative way

$$|M_{n+2}|^2 d\Phi_{n+2} = |M_n|^2 d\Phi_n \frac{\alpha_s(t')}{2\pi} P_{q,qg}(z') \frac{dt'}{t'} dz' \frac{d\phi'}{2\pi} \frac{\alpha_s(t)}{2\pi} P_{q,qg}(z) \frac{dt}{t} dz \frac{d\phi}{2\pi}$$

Angles become small, maintaining a strong ordering relation  $\theta' \gg \theta \rightarrow 0$

# Parton showers

## Exclusive final state

To describe exclusive final state  $\rightarrow$  sum perturbative expansion to all orders in  $\alpha_s$

$$\sigma_0 \alpha_s^n \int \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n} \theta(Q^2 > t_1^2 > \dots > t_n^2 > \Lambda_S^2) = \sigma_0 \frac{\alpha_s^n}{n!} \log^n \left( \frac{Q^2}{\Lambda_S^2} \right)$$

Limit to the most singular terms, in ordered sequence of angles

Collinear approximation  $\rightarrow$  Leading log approximation

# Final state radiation MC

Formal representation of a shower

$$S_i(t, E) = \text{---} \overset{t, E}{\underset{i}{\bullet}} \text{---}$$

- Ensemble of all possible branchings from parton  $i$  at scale  $t$
- Shower Unitarity  $\sum_{\mathcal{F}} S_i(t, E) = 1$

- Splitting weighted by AP

$$\frac{\alpha_S(t)}{2\pi} P_{i,jl}(z) \frac{dt}{t} dz \frac{d\phi}{2\pi}$$

- Each line weighted by Sudakov factor

$$\Delta_i(t', t'') = \exp \left[ - \sum_{(jl)} \int_{t''}^{t'} \frac{dt}{t} \int_0^1 dz \frac{\alpha_S(t)}{2\pi} P_{i,jl}(z) \right]$$

Forward evolution equation

$$S_i(t, E) = \Delta_i(t, t_0) S_i(t_0, E) + \sum_{jl} \int_{t_0}^t \frac{dt'}{t'} \int_0^1 dz \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\alpha_S(t')}{2\pi} \Delta_i(t', t_0) S_j(t', zE) S_l(t', (1-z)E)$$

$$\text{---} \overset{t, E}{\underset{i}{\bullet}} \text{---} = \text{---} \overset{t}{\underset{i}{\bullet}} \text{---} \overset{t_0}{\underset{i}{\bullet}} \text{---} + \text{---} \overset{t}{\underset{i}{\bullet}} \text{---} \overset{t'}{\underset{i}{\bullet}} \text{---} \overset{t', zE}{\underset{j}{\bullet}} \text{---} \overset{t', (1-z)E}{\underset{l}{\bullet}} \text{---}$$

# Final state radiation MC

## Probabilistic interpretation

$$\frac{\alpha_s(t')}{2\pi} P_{i,jl}(z') \frac{dt'}{t'} dz' \frac{d\phi'}{2\pi}$$

Probability of branching in the infinitesimal volume  $dt' dz' d\phi'$

$$dP_{br} = \frac{\alpha_s(t')}{2\pi} \frac{dt'}{t'} \int_0^1 dz' P_{i,jl}(z') \int_0^{2\pi} \frac{d\phi'}{2\pi}$$

Probability of branching in the interval  $dt'$

$$dP_{nobr} = 1 - dP_{br} = 1 - \frac{\alpha_s(t')}{2\pi} \frac{dt'}{t'} \int_0^1 dz' P_{i,jl}(z') \int_0^{2\pi} \frac{d\phi'}{2\pi}$$

Probability of no branching in the interval  $dt'$

$$\Delta_i(t, t') = 1 - dP_{br} = \prod_i \left( 1 - \frac{\alpha_s(t_i)}{2\pi} \frac{\delta t}{t_i} \int_0^1 dz' \int_0^{2\pi} P_{i,jl}(z') \frac{d\phi'}{2\pi} \right)$$

Sudakov form factor in  $[t, t']$

$$dP_{fbr} = \Delta_i(t, t') \frac{\alpha_s(t')}{2\pi} P_{i,jl}(z') \frac{dt'}{t'} dz' \frac{d\phi'}{2\pi}$$

Probability of, starting at  $t$ , first branching in the phase space element  $dt' dz' d\phi'$

# Final state radiation MC

## Shower algorithm

Generate hard process with probability proportional to its parton level cross section  
For each final state colored parton:

- ❶ Set scale  $t = Q$ , hard scale of the process
- ❷ Generate random number  $0 < r < 1$
- ❸ Solve  $r = \Delta_i(t, t')$  for  $t'$
- ❹ i) if  $t' < t_0$ , no further branching and stop shower
- ❺ ii) if  $t' \geq t_0$ , generate  $j, l$  with energies

$$E_j = zE_i \quad \text{and} \quad E_l = (1 - z)E_i,$$

with  $z$  following a distribution given by  $P_{i,jl}(z)$  and with azimuth  $\phi$  uniformly distributed in the interval  $[0, 2\pi]$ .

- ❻ For each branched partons set  $t = t'$  and start from (2)

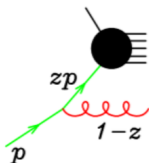
# Initial state radiation MC

## General structure

- QCD coupling much larger than QED
- Coupling grows for small momentum transfer

We can never neglect QCD ISR

Radiation from some initial state leading to some hard collision



- Incoming parton is on shell  $p^2 = 0$
- ISR showers are spacelike  $q^2 = (k - k')^2 < 0$
- Parton with scaled momentum  $zp$  acquires negative virtuality

Factorization formula

$$d\sigma_j^{\text{ISR}}(p, \dots) = \frac{\alpha_S}{2\pi} \frac{dt}{t} dz P_{ij}(z) d\sigma_i(zp, \dots)$$

# Initial state radiation MC

## Formal representation

$$S_i(m, x, t, E) = \frac{t_0 E}{i} \text{S} \begin{array}{c} m, t, xE \end{array}$$

$\delta x S_i(m, x, t, E)$  represents ensemble of all possible states containing a spacelike parton  $m$  with energy between  $x E$  and  $(x + \delta x) E$  and scale  $t$

- Procedure for evolution equation as in FSR (with spacelike showers)
- Forward evolution equation

$$\text{S} \begin{array}{c} t_0, E \\ i \end{array} \begin{array}{c} m, t, xE \end{array} = \frac{t_0}{i} \begin{array}{c} t \\ i \end{array} \begin{array}{c} t' \\ i \end{array} \begin{array}{c} m \\ m \end{array} \delta_{im} \delta(1-x) + \frac{t_0}{i} \begin{array}{c} t' \\ i \end{array} \begin{array}{c} j \\ i \end{array} \begin{array}{c} l \\ i \end{array} \begin{array}{c} t', zE \\ m, t, xE \end{array} \begin{array}{c} t', (1-z)E \end{array} + \frac{t_0}{i} \begin{array}{c} t' \\ i \end{array} \begin{array}{c} j \\ i \end{array} \begin{array}{c} l \\ i \end{array} \begin{array}{c} t', zE \end{array} \begin{array}{c} t', (1-z)E \\ m, t, xE \end{array}$$

Difference from FSR in Sudakov factors and Splitting functions arise at NLO

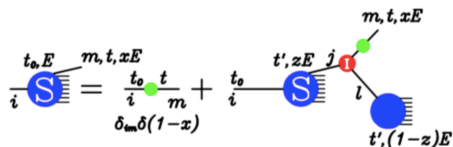


# Initial state radiation MC

## Backwards evolution equation

Great amount of computation time to hard scattering that we want

Moderns MC programs → recursive procedure starting at the large scale



$$\sum_{\mathcal{F}} S_i(m, t, x, E) = f_m^{(i)}(x, t)$$

- The **blob l** at the splitting vertex is given by the inclusive splitting kernel  $P_{jm}$
- Backwards evolution equation (scale dependent parton density)  
Sum over all final states (yields 1 for timelike blobs,  $f_m^{(i)}(x, t)$  for spacelike)

$$f_m^{(i)}(x, t) = \delta_{im} \delta(1-x) \Delta_m(t, t_0) + \int_{t_0}^t \frac{dt'}{t'} \int_x^1 \frac{dz}{z} \sum_j f_j^{(i)}(z, t') \frac{\alpha_S(t')}{2\pi} \hat{P}_{jm} \left( \frac{x}{z} \right) \Delta_m(t, t')$$

# Initial state radiation MC

## Shower algorithm

Generate hard process with probability proportional to its parton level cross section  
For each initial state colored parton:

- 1 Set scale  $t = Q$ , hard scale of the process
- 2 Generate random number  $0 < r < 1$
- 3 Solve

$$r = \frac{f_i^{(h)}(t', x) \Delta_i(t, t')}{f_i^{(h)}(t, x)} \quad \text{for } t'$$

- 4 i) if  $t' < t_0$ , no further branching and stop shower
- 5 ii) if  $t' \geq t_0$ ,  $j, l$  generate  $j$  and  $z$  following a distribution given by  $P_{ij}(z)$ , and azimuth  $\phi$  uniformly distributed in the interval  $[0, 2\pi]$   
Call  $l$  radiated parton, assign energies

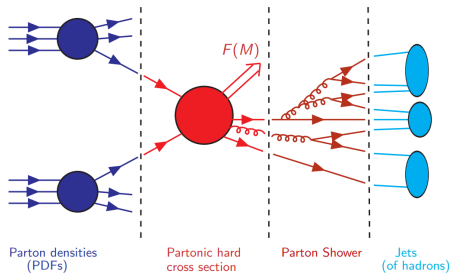
$$E_j = zE_i \quad \text{and} \quad E_l = (1 - z)E_i,$$

- 6 For parton  $j$ , set  $t = t'$  and start from (2)
- 7 For parton  $l$ , set  $t = t'$  and proceed with timelike shower as described in FSR

# Hadronization

## Some basics

Emit QCD radiation until becoming a measurable hadron  $\rightarrow$  non-perturbative physics



# Hadronization

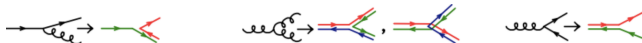
## Some basics

### Hadronization models

- Independent fragmentation model
  - Each final state particle treated independently from all the others
  - Gluon forced to split in  $q\bar{q}$  pair
  - Introduce diquarks for formation of baryons
- Cluster models
- Lund string model

### Large number of color approximation (planar limit)

- Color indices ranging from 1 to  $N_c$ , replace  $(N_c^2 - 1) \rightarrow N_c^2$
- Represent color flow  $\rightarrow$  t'Hooft double line notation



Hadronization models have unavoidably a large number of parameters  
They are one of the most complex aspects of Shower Monte Carlo algorithms

- LHC processes factorization in perturbative and non-perturbative components
- Perturbative QCD applied only at high energies
- Monte Carlo shower algorithms describe the enhanced emissions produced by initial and final state partons
- Hadronization models describing the non-perturbative physics

Thank you!