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4-Manifolds via Surfaces

UGA REU Notes

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Contents

Invariants

Intersection Form. For some closed oriented topological 4-manifold M, the intersection form $Q_M : H^2(M) \times H^2(M) \to \mathbb{Z}$ is a symmetric unimodular bilinear form given by the cup product:

$$Q_M(a,b) = \langle a \smile b, [M] \rangle .$$

If M is smooth, then $Q_M(a,b) = \mathrm{int}_M\, \alpha \wedge \beta$ where a and b are represented by 2-forms α and β in $H^2_{dR}(M;\mathbb{R})$.

Dually, this is equivalent to taking the sums of oriented intersection numbers of 2-cycles, thus we also have such a bilinear pairing on $H_2(M)$. Once we understand how to compute the homology of M given a (g,k)-multisection, it is possible to give Q_M in terms of the intersection pairing $\langle \cdot, \cdot \rangle_\Sigma$ on $H_1(\Sigma)$.

Let $(\Sigma, X_1, \ldots, X_n)$ denote a (g,k)-multisection of a closed oriented 4-manifold X, and consider the handlebodies $H_{i,j} = X_i \cap X_j$. Let $\iota_i : \Sigma \to H_{i,i+1 \pmod n}$ be the inclusion map, so that we have the induced inclusions $\iota_{i*} : H_1(\Sigma) \to H_1(H_{i,i+1 \pmod n})$ for all $1 \le i \le n$. Then suppose $L_i = \ker(\iota_{i*})$, so that L_i is a Lagrangian subspace of $H_1(\Sigma)$ generated by any choice of oriented defining curves for $H_{i,i+1 \pmod n}$.

Theorem 1.1. *The homology of X with coefficients in $\mathbb{Z}$ canonically identifies with the homology of the following complex [1]:*

$$0 \to \bigoplus_{i=1}^{n} (L_{i-1} \cap L_i) \xrightarrow{\partial_2} \bigoplus_{i=1}^{n} L_i \xrightarrow{\partial_1} H_1(\Sigma) \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

where

$$\partial_2\left((x_i)_{1\le i\le n}\right) = \left((x_i - x_{i+1})_{1\le i\le n}\right) \quad \text{and} \quad \partial_1\left((x_i)_{1\le i\le n}\right) = \left(\bigoplus_{i=1}^{n} \iota_{i*}\right)\left((x_i)_{1\le i\le n}\right) = \sum_{i=1}^{n} x_i.$$

Denoting ∂_1 by ι, we have that $H_2(X) \cong \ker(\iota)$[1]*.*

Theorem 1.2. *Suppose $c_1 = ((x_i)_{1\le i\le n})$ and $c_2 = ((y_i)_{1\le i\le n})$ are elements of $H_2(X)$ with $(x_i)_{1\le i\le n}, (y_i)_{1\le i\le n} \in \bigoplus_i L_i$. Then the intersection form*

$$Q_X(c_1,c_2) = \sum_{1\le i<j\le n} \langle x_i, y_j \rangle_\Sigma$$

It follows that Q_X is represented as a bilinear form by the symmetric unimodular matrix $Q = (Q_X(e_i,e_j))_{ij}$ where $(e_i)_{1\le i\le g(n-2)}$ generate $H_2(X)$. That is, $Q_X(c_1,c_2) = c_1^T Q c_2$ for all $c_1, c_2 \in H_2(X)$.[2]

Motivation and Background

In this paper, we are concerned with classifying the intersection forms arising from $(2,0)$-multisections. [3]

Intersection Forms Arising from $(2,0)$-Multisections

Standard Examples

Computing Intersection Forms

Theorem 1.3. *For cut systems $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ defining a $(g,0)$-multisection of a manifold X, the intersection form Q_X of X can be computed by the following algorithm:*

[1] todo: Is this only true in the $(g,0)$ case?
[2] todo: Prove that Q is symmetric?
[3] todo: State big theorems and relevance

(1) We can write

$$\gamma_{i,j} = \left(\sum_{k=1}^{g} a_k^{i,j}\alpha_k\right) + \left(\sum_{k=1}^{g} b_k^{i,j}\beta_k\right)$$

for $\gamma_{i,j} \in \overline{\alpha_i}$ and for $a_k^{i,j}, b_k^{i,j} \in \mathbb{Z}$. Then set

Generating examples using symplectic structure and generators of symplectic group geometric realization

Requisite Results

We first begin with some basic results from linear symplectic geometry over (free) $\mathbb{Z}$-modules. Consider the symplectic module $(V = \mathbb{Z}^{2g}, \omega)$[4] .

Theorem 1.4. *For any Lagrangian subspace L_1 of V, there exists a complementary or dual Lagrangian subspace L_2 such that $L_1 \oplus L_2 = V$. Moreover, a choice of basis $x_1, \ldots, x_g$ of L_1 determines a (non-canonical) dual basis $y_1, \ldots, y_g$ for a complement L_2 by $\omega(x_i, y_j) = \delta_{ij}$. Taken together, these bases form a symplectic basis for V.*

Proposition 1.5. *Observe, any Lagrangian subspace L of V spans a sublattice of a unimodular g-dimensional lattice Λ (since the dimension of any Lagrangian subspace is $g/2 = g$). If L is Lagrangian, then $L = \Lambda$; that is, L is full*[5]*.*

Proof. Suppose for contradiction that $L \subsetneq \Lambda$ and that L is Lagrangian. Since Λ is unimodular, there exists a basis $a_1, \ldots, a_g$ of Λ such that the Gram matrix of Λ in terms of this basis has determinant ± 1. Of course, there must exist some $a_i \notin L$, as otherwise $L = \Lambda$. Moreover, there exists some $\lambda \in \mathbb{Z}$ such that $\lambda a_i \in L$. Thus, since $\omega(x, \lambda a_i) = 0$ for all $x \in L$, we can conclude that $\omega(x, a_i) = 0$, and thus $a_i \in L^\omega$. But this is a contradiction, as since $a_i \notin L$ but $a_i \in L^\omega$, we have that $L \neq L^\omega$, and L is not Lagrangian. □

Corollary 1.6. *For any two Lagrangian subspaces L, L' of V, there exists some $T \in \mathrm{GL}_{2g}(\mathbb{Z})$ such that $T(L) = L'$.*

Proposition 1.7. *If L_1, L_2 and L_1', L_2' are pairs of complementary Lagrangian subspaces, then there exists some $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $T(L_1) = L_1'$ and $T(L_2) = L_2'$.*

Proof. Fix a symplectic basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ and $x_1', \ldots, x_g', y_1', \ldots, y_g'$ for $L_1 \oplus L_2$ and $L_1' \oplus L_2'$, respectively. Then since each of L_1, L_2, L_1', and L_2' are unimodular, the map T with $x_i \mapsto x_i'$ and $y_i \mapsto y_i'$ is clearly symplectic, as T is a linear isomorphism and $T^*\omega = \omega$. □

Corollary 1.8. *Between any two Lagrangian subspaces L, L' of V, there exists a symplectic map $T : L \to L'$. Conversely, for any symplectic transformation $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$, the image of some Lagrangian subspace L under T is again Lagrangian.*

Geometric Realization. A *geometric realization* of a symplectic basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ is an ordered sequence $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of oriented simple closed curves such that

(1) $[\alpha_i] = x_1$ and $[\beta_i] = y_i$ for all $1 \leq i \leq g$.
(2) $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{i,j}$ for all $1 \leq i, j \leq g$.

Informally, this means that the curves on Σ_g and their algebraic intersection numbers agrees with their homology classes and intersection pairing.

Primitive Element. An element $x \in H_1(\Sigma_g)$ is *primitive* if it cannot be written as $x = \lambda x'$ for some $x' \in H_1(\Sigma)$ and $n \geq 2$.

[4]todo: Background? How much of this has been covered already?
[5]Right word?

Lemma 1.9. *A nonzero $x \in H_1(\Sigma_g)$ can be written as $x = [\gamma]$ for some simple closed curve γ if and only if x is primitive.*

Proof. [6] □

Theorem 1.10. *Every symplectic basis for $H_1(\Sigma_g)$ has a geometric realization.*

Proof. This is a consequence of Theorem 1.9 and Theorem 1.5 — A basis of a unimodular lattice necessarily consists of primitive elements, hence these basis elements have a geometric realization. An alternative, full proof can be seen in [7]. □

[6] Cite Putman
[7] Cite Putman

Bibliography

[1] Peter Feller, Michael Klug, Trenton Schirmer, and Drew Zemke. Calculating the homology and intersection form of a 4-manifold from a trisection diagram. *Proceedings of the National Academy of Sciences*, 115(43):10869–10874, oct 2018. 1