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4-Manifolds via Surfaces

UGA REU Notes

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Contents

6/6: Surfaces and Homology

1.1 Surfaces

***n*-Manifold.** A topological space that locally looks like an open n-ball.

(1) $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

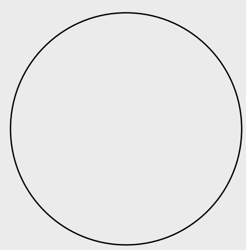


Figure 1.1: S^1

(2) $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

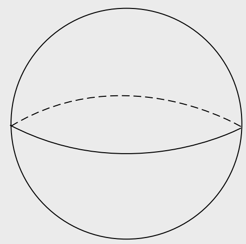


Figure 1.2: S^2

(3) $S^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 = 1\}$

(4) $T^2 = S^1 \times S^1$

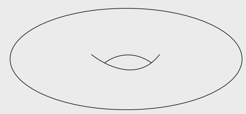


Figure 1.3: T^2

(5) Möbius Band (Mb).

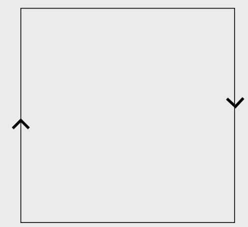


Figure 1.4: Möbius Band

The Möbius band is an example of a *non-orientable* manifold. Also note, $\partial(\mathrm{Mb}) = S^1$.

Connected Sum. To form the *connected sum* of two surfaces, remove a disk from both surfaces and glue along the boundary of these disks:

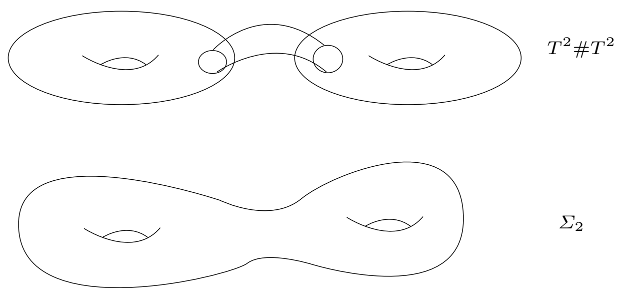


Figure 1.5: Connected Sum

Orientation. A surface is *non-orientable* if there exists a circle on the surface such that a neighborhood of that circle on the surface is a Möbius band.

Theorem 1.1 (Classification of Surfaces)**.** Every orientable closed surface is homeomorphic to a genus g surface for some $g \geq 0$. Every non-orientable closed surface is homeomorphic to

$$(\#^m \, \mathbb{R}P^2) \, \# \, (\#^n \, T^2),$$

for $m > 0$ and $n \geq 0$.

The Klein bottle is given by $\mathbb{R}P^2 \# \mathbb{R}P^2$.

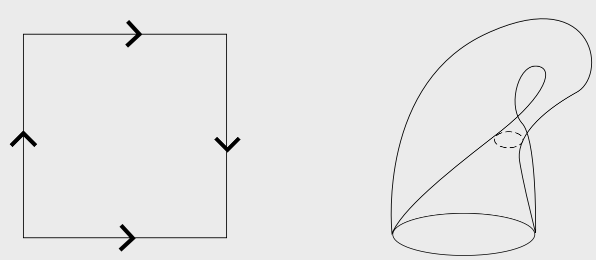


Figure 1.6: The Klein Bottle

1.2 Homology

An n-manifold M gives rise to n abelian groups denoted $H_i(M)$ for $0 \leq i \leq n$. However, only those index by $0 < i < n$ are actually "interesting"; as special cases, if M has m connected components,

$$H_0(M) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m \text{ times}},$$

and if M is orientable, $H_n(M) = \mathbb{Z}$. In the case of surfaces, $H_i(M)$ are going to be finitely generated abelian groups, hence for some s and t,

$$H_i(M) = \mathbb{Z}^s \oplus \mathbb{Z}/p_1^{e_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_t^{e_t}$$

where each p_j is prime. Note also, $H_i(M)$ is invariant under homotopy equivalence.

(1) For $n = 1$, we have for S^1 that
$$H_0(S^1) = \mathbb{Z} = H_1(S^1).$$

(2) For $n = 2$, we have for Σ_g that
$$H_0(\Sigma_g) = \mathbb{Z} = H_2(\Sigma_g),$$
but what about $H_1(\Sigma_g)$.

Homology on Surfaces.

(1) Start by drawing a graph G on a surface Σ such that the edges are oriented in some way and $\Sigma \setminus G$ looks like the disjoint union of polygons (i.e. balls).

(2) Denote by C_i the i-dimensional "cells" induced by this graph, so C_0 is the $\mathbb{Z}$-module $\mathbb{Z}\langle v_1, \ldots, v_V\rangle$ generated by the vertices of G, C_1 is $\mathbb{Z}\langle e_1, \ldots, e_E\rangle$ where the e_i are edges of G, and C_2 is $\mathbb{Z}\langle r_1, \ldots, r_R\rangle$, where the r_i are the 2-dimensional regions bounded by the edges, i.e. the components of $\Sigma \setminus G$.

(3) Form the complex

$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

by defining the maps ∂_1, ∂_2 in the following way: send the generators of the codomain to linear combinations of the generators of the domain with coefficients determined by the orientations of the latter. For example, if we choose the standard counter-clockwise orientation on a region r_i, the coefficient of some e_j is 1 if the orientation "agrees" with that on r_i and is -1 otherwise.

(4) Define $H_n(\Sigma) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$.

Here we compute the homology of S^1.

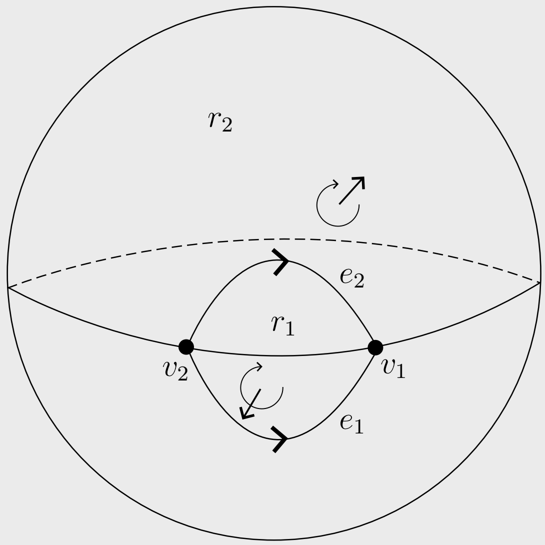


Figure 1.7: Cell structure on S^1

So, we have that $C_0 = \mathbb{Z}\langle v_1, v_2\rangle$, $C_1 = \mathbb{Z}\langle e_1, e_2\rangle$, and $C_2 = \mathbb{Z}\langle r_1, r_2\rangle$ (in other words, three copies of $\mathbb{Z}\oplus\mathbb{Z}$). Looking at orientations, we find that

$$0 \to \mathbb{Z}\langle r_1, r_2\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1, e_2\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v_1, v_2\rangle \to 0$$

$$\partial_2 : \begin{cases} r_1 \mapsto e_2 - e_1 \\ r_2 \mapsto e_1 - e_2 \end{cases} \qquad \partial_1 : \begin{cases} e_1 \mapsto v_1 - v_2 \\ e_2 \mapsto v_1 - v_2 \end{cases}$$

So, $H_1(S^1) = 0$.

Problem 1.2.1.

(1) Compute $H_1(\Sigma_g)$ for all $g \geq 2$.

(2) Compute $H_1(\mathbb{R}P^2)$.

(3) Draw a closed loop γ on the torus such that its homology class $[\gamma] = 4e_1 + 3e_2$.

Proof.

(1) Consider Σ_g for $g \geq 2$. One way we can view Σ_g is as the identification of edges of a $4g$-gon:

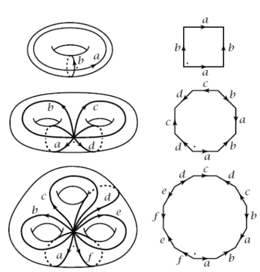


Figure 1.8: Gluing edges of a $4g$-gon to construct a genus g surface (Hatcher)

This is the cell structure we impose on Σ_g; namely, $C_0 = \mathbb{Z}\langle v\rangle$ where v is the single vertex under this identification, $C_1 = \mathbb{Z}\langle e_1, \ldots, e_{2g}\rangle$ where $e_1, \ldots, e_{2g}$ are the edges connected to v with their inherited orientation, and $C_2 = \mathbb{Z}\langle r\rangle$ where $r \cong D^2$ is the interior of the polygon we started with. Then

$$\partial_2(r) = e_1 - e_1 + e_2 - e_2 + \cdots + e_{2g} - e_{2g} = 0$$

and $\partial_1 = 0$, hence

$$H_1(\Sigma_g) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \mathbb{Z}^{2g}/0 = \mathbb{Z}^{2g}.$$

(2) We can compute $H_1(\mathbb{R}P^2)$ similarly. We can see $\mathbb{R}P^2$ as the identification of a square as follows:

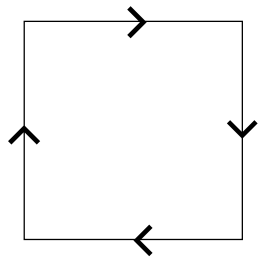


Figure 1.9: $\mathbb{R}P^2$ Identification

But of course, in this context, opposite edges on the square represent the same edge under the identification. Therefore,

$$C_0 = \mathbb{Z}\left\langle v_1, v_2\right\rangle, \quad C_1 = \mathbb{Z}\left\langle e_1, e_2\right\rangle, \quad \text{and} \quad C_2 = \mathbb{Z}\left\langle r\right\rangle$$

and

$$\partial_2 : \begin{cases} r \mapsto 2e_1 + 2e_2 \end{cases} \quad \partial_1 : \begin{cases} e_1 \mapsto v_1 - v_2 \\ e_2 \mapsto v_2 - v_1 \end{cases}$$

This gives us that

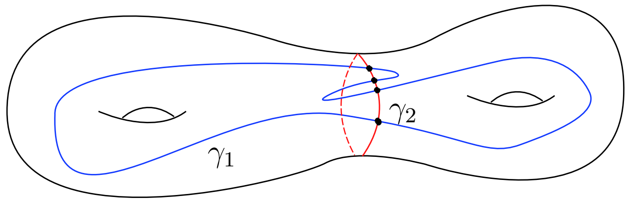
$$H_1(\mathbb{R}P^2) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \mathbb{Z}/2\mathbb{Z}.$$

Figure 1.10: closed-loop-homology

(3)

□

1.3 Intersection Number



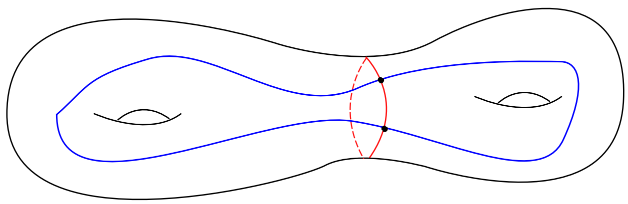


Figure 1.11: Isotopy of a curve on a surface

Geometric Intersection Number. The *geometric intersection number* between two curves γ_1, γ_2 on a surface Σ is

$$\iota(\gamma_1, \gamma_2) = \min\{\#\text{ intersections of }\gamma_1, \gamma_2\}.$$

Algebraic Intersection Number. The *algebraic intersection number* is the pairing

$$\gamma_1 \cdot \gamma_2 = \sum_{P \in \gamma_1 \cap \gamma_2} \mathcal{O}(P)$$

where $\mathcal{O}(P)$ is the sign of the orientation on P induced by the orientation of γ_1, γ_2.

Problem 1.3.1. *How does $\gamma_1 \cdot \gamma_2$ change if*

(1) we reverse the orientation of γ_1, γ_2, or both?

(2) we reverse the orientation of Σ?

Additionally, show that $\gamma_1 \cdot \gamma_2 = -\gamma_2 \cdot \gamma_1$.

(1) If we reverse the orientation of either γ_1 or γ_2, then $\gamma_1 \cdot \gamma_2$ changes by a sign. Thus, if we change both orientations, $\gamma_1 \cdot \gamma_2$ remains the same.

(2) Reversing the orientation of Σ again changes $\gamma_1 \cdot \gamma_2$ by a sign.

Proposition 1.2. The algebraic intersection number gives a bilinear pairing $H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ defined by $\langle [\gamma_1], [\gamma_2] \rangle \mapsto \gamma_1 \cdot \gamma_2$.

Problem 1.3.2. *Pick a basis for $H_1(\Sigma_2)$ and compute the intersection matrix.*

Suppose that $H_1(\Sigma_2)$ has basis given by $\{[e_i]\}_{i=1}^4$, where the e_i are the same curves identified in the computation of $H_1(\Sigma_g)$.

Symplectic Forms and 4-Manifolds

1.4 Bilinear Forms

Recall, as computed in a previous exercise, $H_1(\Sigma_g) = \mathbb{Z}^{2g}$. Additionally, we have defined the *intersection pairing* $\langle , \rangle : H_1(\Sigma_g) \times H_1(\Sigma_g) \to \mathbb{Z}$ given by the algebraic intersection number on homology class representatives. For a general $(2n)$-manifold X, there exists an *intersection product* $H_n(X) \times H_n(X) \to \mathbb{Z}$ such that

$$\langle a, b \rangle = (-1)^n \langle b, a \rangle .$$

Symmetric Bilinear Forms

Theorem 1.3. If B is a symmetric bilinear form on $\mathbb{R}^n$, then up to base change, its matrix representation is of the form

$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && \ddots &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& \ddots && \\ &&&&&& -1 & \\ &&&&&&& -1 \end{pmatrix}$$

containing n_+ ones, n_0 zeros, and n_- negative ones, so that the matrix is $n \times n$ for $n = n_+ + n_0 + n_-$.

Non-Degenerate. We say that $B(n)$ is *non-degenerate* if $n_0 = 0$.

Index/Signature. The *index/signature* is defined as either the difference $\sigma = n_+ - n_-$ or the ordered pair (n_+, n_-). If B is non-degenerate, then it is determined by σ.

However, this story is not as clear-cut over $\mathbb{Z}$.