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4-Manifolds via Surfaces

UGA REU Notes

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Contents

1.1 Week of 6/20 and 6/27: Mainly Lots of Examples

Goals and Overview

Two of our main goals were to

(1) Put together the Heegaard diagrams for integer homology spheres we had found to form new multisection diagrams for which we can compute invariants (e.g. intersection form).

(2) Find methods for generating many such examples quickly, either in the form of Heegaard diagrams or multisections.

These goals led to fruitful examples and inquiries into the structures we're dealing with; in particular, we now better understand the construction of intersection forms for multisections and the linear symplectic geometry underlying our ideas.

Investigating a Nontrivial 4-section

Until now, we have found a number of families of Heegaard diagrams representing integer homology spheres. We managed to string them together in such a way that they formed an interesting 4-section diagram.

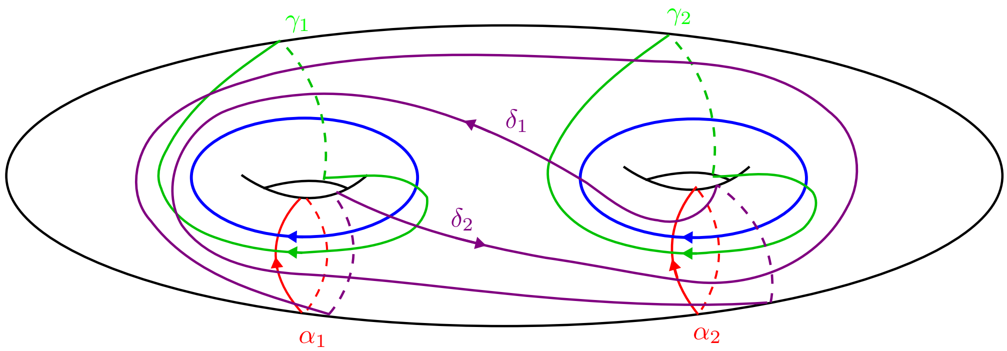

Figure 1.1: A nontrivial 4-section

If we denote $M_{\alpha,\beta} = (\langle \alpha_i, \beta_i \rangle)_{i,j}$ (and similarly for the other cut systems), we have the following intersection data:

$$M_{\alpha,\beta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M_{\alpha,\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M_{\alpha,\delta} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$M_{\beta,\gamma} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad M_{\beta,\delta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$M_{\gamma,\delta} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

One can verify that these matrices are unimodular, hence each "adjacent" pair of cut systems is a Heegaard diagram for an integer homology sphere. In this case, each of these diagrams represent S^3. Using the algorithm described before, we compute the intersection form to be [1]

$$Q = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix}$$

This form is indefinite odd with signature -2.

[1]todo: Something is wrong here, this matrix is not symmetric; check intersection numbers and program for bugs

Brieskorn Spheres and The Poincaré Homology Sphere

Having explored a few nontrivial multisections, we were retrospectively curious about nontrivial Heegaard diagrams. Multisections are defined so as to allow us to "fill in" the holes left after gluing with fake 4-balls, namely those whose boundary is an integer homology sphere. All of our examples so far used only copies of S^3 as these boundaries, that is, our adjacent cut systems were Heegaard diagrams for S^3. What examples are there where this is not the case?

This question led us to investigate the homology spheres that are already well known, as well as their Heegaard splittings. In particular, we focused on a family of homology spheres known as *Brieskorn spheres*.

Brieskorn Manifold. Let $\Sigma(p, q, r)$ be the smooth, compact 3-manifold obtained by intersecting the complex hypersurface

$$x_1^p + x_2^1 + x_3^r = 0$$

with the unit sphere $|x_1|^2 + |x_2|^2 + |x_3|^2 = 1$ with p, q, r integers ≥ 2. For some fixed p, q, and r, we call $\Sigma(p, q, r)$ a Brieskorn manifold.[2]

Theorem 1.1. If $\Sigma = \Sigma(p, q, r)$ where p, q, r are pairwise coprime, then $H_1(\Sigma)$ is trivial. That is, Σ is an $\mathbb{Z}HS^3$. We call these *Brieskorn spheres*. If one of p, q, r is 1, then Σ is homeomorphic to S^3.

One of the most important examples of a Brieskorn sphere is $\Sigma(2, 3, 5)$, called the *Poincaré homology sphere*.

A Brief Tangent on E_8

In this project, we are interested in the properties of the intersection forms of different genus g multisections, especially the minimal g required to get certain forms. It would be fascinating if we could find a genus 2 multisection whose intersection form is E_8, as according to the classification of symmetric bilinear forms, this would tell us a lot about the previous question.

Freedman tells us some essential facts about intersection forms and which manifolds might have such forms. In particular,

Theorem 1.2 (Freedman's Theorem)**.** For each symmetric bilinear unimodular form Q over $\mathbb{Z}$, there exists a closed oriented simply-connected topological 4-manifold with Q as its intersection form. If Q is even, there is exactly one, and if Q is odd, there are exactly two, at least one of which is nonsmoothable[3].

Furthermore, there is a well-known topological manifold whose intersection form is (the Cartan matrix for) E_8, called the E_8 *manifold*. Given the above theorem and the fact that the E_8 manifold has intersection form E_8 by construction, this is the unique such manifold. The E_8 manifold is constructed as follows: take Freedman's plumbing construction to obtain M_{E_8} such that ∂M_{E_8} is a homology sphere, and glue a fake 4-ball along this boundary. It turns out that this homology sphere is the Poincaré homology sphere[4]; this is one of the reasons it is such an important example in this project.

Although we have not yet found Heegaard splittings for general Brieskorn spheres, there is a Heegaard diagram for Poincaré that we hope to make use of in the future[5]:

Figure 1.2: Two-bridge Heegaard diagram for the Poincaré homology sphere

[2]todo: Cite Milnor
[3]todo: Cite Morgan Weiler's notes and/or Freedman's paper
[4]todo: Cite Morgan Weiler's notes
[5]todo: Finish diagram and cite Ozsváth and Szabó's PCMI notes

We are still hoping to pursue these ideas further to create more examples, especially those relating to E_8.

Generating Examples with Linear Symplectic Geometry

Motivation

Given that we want to find many examples of multisection diagrams, it would be convenient if there were a way for us to generate compatible cut systems that yield multisections. The ideas for this section come from the observation that, given some Lagrangian subspace L of a symplectic vector space V and some symplectic transformation $T : V \to V$, the image of L under T is again Lagrangian.

It is a natural question then, does the converse hold? That is, between any two Lagrangian subspaces $L, L' \subseteq V$, does there exists a symplectic transformation T such that $T(L) = L'$? If this is the case, then we can consider the seemingly stronger condition that there exists a symplectic map bringing complementary pairs of Lagrangian subspaces to different pairs.

Requisite Results

We first begin with some basic results from linear symplectic geometry over (free) $\mathbb{Z}$-modules. Consider the symplectic module $(V = \mathbb{Z}^{2g}, \omega)$[6] .

***Theorem* 1.3.** For any Lagrangian subspace L_1 of V, there exists a *complementary* or *dual* Lagrangian subspace L_2 such that $L_1 \oplus L_2 = V$. Moreover, a choice of basis $x_1, \ldots, x_g$ of L_1 determines a dual basis $y_1, \ldots, y_g$ for a complement L_2 by $\omega(x_i, y_j) = \delta_{ij}$. Taken together, these bases form a *symplectic basis* for V.

***Proposition* 1.4.** Observe, any Lagrangian subspace L of V spans a sublattice of a unimodular g-dimensional lattice Λ (since the dimension of any Lagrangian subspace is $g/2 = g$). If L is Lagrangian, then $L = \Lambda$; that is, L is full[7].

Proof. Suppose for contradiction that $L \subsetneq \Lambda$ and that L is Lagrangian. Since Λ is unimodular, there exists a basis $a_1, \ldots, a_g$ of Λ such that the Gram matrix of Λ in terms of this basis has determinant ± 1. Of course, there must exist some $a_i \notin L$, as otherwise $L = \Lambda$. Moreover, there exists some $\lambda \in \mathbb{Z}$ such that $\lambda a_i \in L$. Thus, since $\omega(x, \lambda a_i) = 0$ for all $x \in L$, we can conclude that $\omega(x, a_i) = 0$, and thus $a_i \in L^\omega$. But this is a contradiction, as since $a_i \notin L$ but $a_i \in L^\omega$, we have that $L \neq L^\omega$, and L is not Lagrangian. □

***Corollary* 1.5.** For any two Lagrangian subspaces L, L' of V, there exists some $T \in \mathrm{GL}_{2g}(\mathbb{Z})$ such that $T(L) = L'$.

***Proposition* 1.6.** If L_1, L_2 and L'_1, L'_2 are pairs of complementary Lagrangian subspaces, then there exists some $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $T(L_1) = L'_1$ and $T(L_2) = L'_2$.

Proof. Fix a symplectic basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ and $x'_1, \ldots, x'_g, y'_1, \ldots, y'_g$ for $L_1 \oplus L_2$ and $L'_1 \oplus L'_2$, respectively. Then since each of L_1, L_2, L'_1, and L'_2 are unimodular, the map T with $x_i \mapsto x'_i$ and $y_i \mapsto y'_i$ is clearly symplectic, as T is a linear isomorphism and $T^*\omega = \omega$. □

***Corollary* 1.7.** Between any two Lagrangian subspaces L, L' of V, there exists a symplectic map $T : L \to L'$. Conversely, for any symplectic transformation $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$, the image of some Lagrangian subspace L under T is again Lagrangian.

A *geometric realization* of a symplectic basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ is an ordered sequence $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of oriented simple closed curves such that

(1) $[\alpha_i] = x_1$ and $[\beta_i] = y_i$ for all $1 \leq i \leq g$.

(2) $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{i,j}$ for all $1 \leq i, j \leq g$.

[6] todo: Background? How much of this has been covered already?

[7] Right word?

Informally, this means that the curves on Σ_g and their algebraic intersection numbers agrees with their homology classes and intersection pairing.

An element $x \in H_1(\Sigma_g)$ is *primitive* if it cannot be written as $x = \lambda x'$ for some $x' \in H_1(\Sigma)$ and $n \geq 2$.

Lemma 1.8. A nonzero $x \in H_1(\Sigma_g)$ can be written as $x = [\gamma]$ for some simple closed curve γ if and only if x is primitive.

Proof. [8] □

Theorem 1.9. Every symplectic basis for $H_1(\Sigma_g)$ has a geometric realization.

Proof. This is a consequence of Theorem 1.8 and Theorem 1.4 — A basis of a unimodular lattice necessarily consists of primitive elements, hence these basis elements have a geometric realization. An alternative, full proof can be seen in [9]. □

Knowing now that we can go from one Lagrangian subspace to any other via a symplectic transformation (and that any symplectic transformation sends Lagrangians to Lagrangians) by Theorem 1.7, the structure of $\mathrm{Sp}_{2g}(\mathbb{Z})$ becomes very important to us. These observations yield a promising avenue for generating diagrams algorithmically; certain symplectic transformations may now give us geometric realizations of complementary Lagrangian subspaces, and these may correspond to valid multisection diagrams. First though, we must understand which symplectic transformations give us these diagrams, and how we might go about choosing them.

Fix the standard L_1, L_2. Observe, by the reasoning above, these are geometrically realized by the standard α and β cut systems[10]. For any $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$, we know now that $T(L_1)$ and $T(L_2)$ are necessarily complementary Lagrangian subspaces. This is one direction we might pursue: the only condition placed on T is that it is not the identity map, and so a computer could generate these examples fairly quickly. However, in order to form a valid diagram, $L_2, T(L_1)$ and $T(L_2), L_1$ must also be complementary. This is much harder to enforce, especially if we were to choose more than one symplectic transformation.

As such, it may be better to generate Lagrangian subspaces in an "ascending" order rather than randomly, first generating a complement to L_2 via some map T_1, then a complement to $T_1(L_2)$ via some T_2, and so on. This way, we need only generate Lagrangian subspaces whose union is unimodular, until the last space which we must guarantee is complementary to L_1. This places significantly less restriction on the symplectic transformations we can use, and turns out to be computationally feasible.

Structure of $\mathrm{Sp}_{2g}(\mathbb{Z})$

With this plan in mind, we need to be able to choose random elements of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for any $\mathbb{Z}$. Luckily, in 1890, Burkhardt gave the following generators for $\mathrm{Sp}_4(\mathbb{Z})$:

Swap $(x_1, x_2, y_1, y_2) \mapsto (x_1 + y_1, x_2, y_1, y_2)$

Rotation $(x_1, x_2, y_1, y_2) \mapsto (y_1, x_2, -x_1, y_2)$

Mix $(x_1, x_2, y_1, y_2) \mapsto (x_1 - y_2, x_2 - y_1, y_1, y_2)$

Transvection/Shear $(x_1, x_2, y_1, y_2) \mapsto (x_1 + y_1, x_2, y_1, y_2)$.

Additionally, it is not difficult to generalize this finite set of generators to $\mathrm{Sp}_{2g}(\mathbb{Z})$ for arbitrary g.

[8] Cite Putman
[9] Cite Putman
[10] Insert diagram

Similar to how $\mathrm{SL}_n(\mathbb{Z})$ is generated by elementary matrices, it turns out that $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by *elementary symplectic matrices.* Let $\sigma \in S_{2g}$ be given by $2i \leftrightarrow 2i-1$ for each $1 \leq i \leq g$. Then if $E_{i,j}$ is the matrix with a 1 in the (i,j)-th entry and 0s elsewhere, the symplectic elementary matrices is the set of matrices

$$SE_{i,j} = \begin{cases} I_{2g} + e_{i,j} & \text{If } i = \sigma(j) \\ I_{2g} + e_{i,j} - (-1)^{i+j} e_{\sigma(j),\sigma(i)} & \text{Otherwise} \end{cases}$$

for $1 \leq i, j \leq 2g$ and $i \neq j$.

***Theorem* 1.10.** $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by elementary symplectic matrices[11].

We are now ready to describe the algorithm for generating random genus g n-sections.

(1) Fix the standard L_1, L_2 corresponding to the standard cut systems.

(2) Choose some symplectic transformation $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $L_2 \oplus T(L_2) = H_1(\Sigma_g)$. Computationally, this is not terribly difficult, since the majority of Lagrangian subspaces will be complementary to L_2.

(3) Repeat this process with the image of L_2 under T, until we have the algebraic multisection $L_1, \ldots, L_n$.

(4) If L_n and L_1 are not complementary, choose different symplectic transformations to apply to L_{n-1} until this condition is met.

(5) Take the geometric realization of each adjacent pair of Lagrangian subspaces to arrive at our genus g n-section.

We can now take the algebraic data we have and compute the intersection form.

Results

So far, we have found many examples of genus g multisections. We are particularly interested in what non-isomorphic intersection forms are achievable in the genus 2 case. The following table are some of the generated examples:

[11] Cite Farb's Mapping Class Group book

#	n	n-section	Q	Signature	Parity
I	3	$\gamma_{3,1} = -\alpha_1 + 2\beta_1 + 3\beta_2$ $\gamma_{3,2} = -2\alpha_1 + \alpha_2 + \beta_1 + \beta_2$	$\begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$	$(0,2)$	Odd
II	3	$\gamma_{3,1} = \alpha_2 + \beta_1$ $\gamma_{3,2} = \alpha_1 + \alpha_2 + \beta_2$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$(1,1)$	Odd
III	3	$\gamma_{3,1} = \alpha_1 + 2\beta_1 + \beta_2$ $\gamma_{3,2} = \alpha_2 + \beta_1 + \beta_2$	$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$	$(1,1)$	Odd
IV	3	$\gamma_{3,1} = \alpha_2 + \beta_1$ $\gamma_{3,2} = \alpha_1 + \beta_2$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$(1,1)$	Even
V	4	$\gamma_{3,1} = -\alpha_1 - 2\alpha_2 + 3\beta_2$ $\gamma_{3,2} = -\alpha_1 - \alpha_2 + \beta_1 + \beta_2$ $\gamma_{4,1} = 2\alpha_1 - 3\alpha_2 - \beta_1 + 3\beta_2$ $\gamma_{4,2} = -\alpha_2 + \beta_2$	$\begin{bmatrix} -6 & -3 & -5 & -2 \\ -3 & -2 & -2 & -1 \\ -5 & -2 & -11 & -3 \\ -2 & -1 & -3 & -1 \end{bmatrix}$	$(0,4)$	Odd