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M 365C - Real Analysis

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Contents

The Real and Complex Number Systems

Theorem 1.1. Some baloney proofs for $0 = 1$.

Proof. Let $a = b$. Then

$$\begin{aligned} a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= (a-b)b \\ a+b &= b \\ b+b &= b \\ 2b &= b \\ 2 &= 1 \\ 1 &= 0. \end{aligned}$$

□

Proof. Consider the series $S = 1 - 1 + 1 - 1 + \dots$. So

$$\begin{aligned} S &= (1-1) + (1-1) + \dots \\ S &= 0 + 0 + \dots \\ S &= 0. \end{aligned}$$

But also,

$$\begin{aligned} S &= 1 + (-1+1) + (-1+1) + \dots \\ S &= 1 + 0 + 0 + \dots \\ S &= 1. \end{aligned}$$

Therefore, $S = 0 = 1$. □

1.1 Ordered Sets

$\mathbb{Q}$ is an ordered field, or a field that is also an ordered set. It also has "holes".

Theorem 1.2. There is no rational number x such that $x^2 = 2$.

Proof. By contradiction, suppose $x \in \mathbb{Q}$ and $x^2 = 2$. Then $x^2 = \frac{m}{n}$ with $m, n \in \mathbb{Z}$ and $n \neq 0$. We may assume that it is not the case that both m and N are even (prove this as an exercise). So,

$$\begin{aligned} (\frac{m}{n})^2 &= 2 \\ \frac{m^2}{n^2} &= 2 \\ m^2 &= 2n^2, \end{aligned}$$

and m^2 is even.

Lemma 1.3. If m^2 is even, then m is even

Proof. By contrapositive, we show that if m is odd, then m^2 is odd. Assuming m is odd, that is $m = 2n+1$ for an integer n. So, $m^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$, and m^2 is odd. Therefore, if m^2 is even, then m is even. □

By our lemma, m is also even. So, $m = 2k$ for $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}(2k)^2 &= 2n^2 \\ 4k^2 &= 2n^2 \\ 2k^2 &= n^2,\end{aligned}$$

and by the same logic, n^2 and n is even. But this is a contradiction, as it was said that at least one of m and n are not even. Therefore, there is no rational number x such that $x^2 = 2$. □

So what? Consider $A = \{x \in \mathbb{Q}^+ \mid x^2 < 2\}$ and $B = \{x \in \mathbb{Q}^+ \mid x^2 > 2\}$. Then $\mathbb{Q}^+ = A \cup B$. Also, for all $a \in A$ and $b \in B$, $a < b$. Observe that if $x \in \mathbb{Q}^+$, then $y = x - \frac{x^2-2}{x+2} = \frac{2x+2}{x+2} \in \mathbb{Q}^+$. Is y in A or B?

$$\begin{aligned}y^2 &= \frac{4x^2 + 8x + 4}{x^2 + 4x + 4} \\ y^2 - 2 &= \frac{2x^2 - 4}{(x+2)^2} = \frac{2(x^2-2)}{(x+2)^2}\end{aligned}$$

Therefore, $x \in A \implies y \in A$ and $x \in B \implies y \in B$. Also, $x \in A \implies y > x$, and $x \in B \implies y < x$. That is, A is bounded above, but it has no *least upper bound*, and that is a problem in analysis. Real numbers will "fill" the holes in $\mathbb{Q}$ to fix this problem.

Ordered Sets. An order on a set S is a relation, denoted $<$, such that

(1) If $x \in S, y \in S$, then exactly one of the following is true: $x < y$, $x = y$, and $y < x$.

(2) If $x, y, z \in S$ with $x < y$ and $y < z$, then $x < z$.

Note, $y > x$ means $x < y$, and $x \leq y$ means $x < y$ or $x = y$. An *ordered set* is an ordered pair $(S, <)$.

(1) $\mathbb{Z}$ with the usual order: $m < n$ if $n - m \in \mathbb{Z}^+$.

(2) $\mathbb{Q}$ with the usual order: $\frac{p}{q} < \frac{p'}{q'}$ if $\frac{p'}{q'} - \frac{p}{q} \in \mathbb{Q}^+$.

Bounded above. Suppose S is an ordered set and $E \subseteq S$ nonempty. If $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, then β is an *upper bound* for E, and E is called *bounded above.*

Similarly, we can define *lower bound* and *bounded below.*

Least Upper Bound. With S an ordered set and $E \subseteq S$, assume E is bounded above. Suppose there exists some $\alpha \in S$ such that α is an upper bound and if $\gamma < \alpha$ then γ is not an upper bound. Then α is called the *least upper bound* of E or the *supremum* of E, denoted $\alpha = \sup E$.

Similarly, if E is bounded below, then a lower bound $\beta \in S$ such that for all $\gamma > \beta$, γ is not a lower bound, then β is called the *greatest lower bound* of E, denoted $\beta = \inf E$.

From out example, $A \subseteq \mathbb{Q}^+$ has no least upper bound in $\mathbb{Q}^+$. However, some subsets of $\mathbb{Q}$ do have least upper bounds, for example $\sup\{x \in \mathbb{Q} \mid x \leq 2\} = 2$.

In summary, if $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$, C has no least upper bound in $\mathbb{Q}$.

Least Upper Bound Property. Let S be an ordered set. S has the *least upper bound property* if every non-empty subset $E \subseteq S$ which is bounded above has a least upper bound. So $\sup E$ exists in S.

A non-example is $\mathbb{Q}$, in light of our set C.

Theorem 1.4. Suppose S has the least upper bound property. Then S also has the *greatest lower bound property*: for any $B \subseteq S$ non-empty and bounded below, B has a greatest lower bound in S.

Proof. Let L be the set of all lower bounds of B in S. We show that $\alpha = \sup L$ exists and $\alpha = \inf B$. Since B is bounded below, L is non-empty. Also L is bounded above: let x be an element of B, then for $y \in L$, y is a lower bound for B, and so $y \leq x$. By the LUB property, L has a least upper bound, call it $\alpha = \sup L$.

Now we show that α is also the greatest lower bound for B. So,

(1) We first show that α is a lower bound. So, if $x \in B$, then x is an upper bound for L, and $\alpha \leq x$. Hence, α is a lower bound for B.

(2) Let $\gamma > \alpha$. Since α is an upper bound for L, $\gamma \notin L$, and γ is not a lower bound for B.

Hence, $\alpha = \inf B$. □

1.2 Fields

Field. A *field* is a set F with two binary operations, called addition and multiplication, which satisfy the following axioms:

(1) $(F, +)$ is an abelian group with identity 0 and $x^{-1} = -x$ for $x \in (F, +)$.

(2) $(F\backslash\{0\}, \cdot)$ is an abelian group with identity 1 and $x^{-1} = \frac{1}{x}$ for $x \in (F\backslash\{0\}, \cdot)$.

(3) *Distributivity:* For all $x, y, z \in F$, $x(y + z) = xy + xz$.

Some examples of fields include:
(1) $\mathbb{Q}$.
(2) $\mathbb{F}_2 = \{0, 1\}$.

Proposition 1.5. The addition axioms imply:

(1) If $x + y = x + z$, then $y = z$.

(2) If $x + y = x$, then $y = 0$.

(3) If $x + y = 0$, then $y = -x$.

(4) $-(-x) = x$.

Proof.

(1)

$$\begin{aligned} x+y &= x+z \\ -x+(x+y) &= -x+(x+z) \\ (-x+x)+y &= (-x+x)+z \\ 0+y &= 0+z \\ y &= z. \end{aligned}$$

(2)

$$\begin{aligned} x+y &= x \\ x+y &= x+0 \\ y &= 0. \end{aligned}$$

(3)

$$\begin{aligned} x+y &= 0 \\ x+y &= x+(-x) \\ y &= -x. \end{aligned}$$

(4) $(-x)+x=0$, so $x=-(-x)$.

□

Proposition 1.6.

(1) $0 \cdot x = 0$.
(2) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
(3) $(-x)y = -(xy) = x(-y)$.
(4) $(-x)(-y) = xy$.

Ordered Field. An *ordered field* is a field F with an order relation $<$ such that

(1) $x+y < x+z$ if $y < z$.
(2) $xy > 0$ if $x > 0$ and $y > 0$.

Proposition 1.7. In an ordered field F,

(1) $x > 0$ then $-x < 0$, $x < 0$ then $-x > 0$.
(2) $x > 0$ and $y < z$ then $xy < xz$.
(3) $x < 0$ and $y < z$ then $xy > xz$.
(4) If $x \neq 0$ then $x^2 > 0$. In particular, $1 = 1^1 > 0$.
(5) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

Theorem 1.8. There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as an ordered subfield. $\mathbb{R}$ is the real numbers, and is unique up to field automorphisms.

Theorem 1.9. $\mathbb{R}$ has:

(1) The *Archimedean property:* if $x, y \in \mathbb{R}$ with $x > 0$, there exists some $N \in \mathbb{Z}^+$ such that $nx > y$.
(2) $\mathbb{Q}$ is *dense* in $\mathbb{R}$: if $x, y \in R$ and $x < y$, then there exists $z \in \mathbb{Q}$ such that $x < z < y$.

Proof.

(1) Let $x, y \in \mathbb{R}$ with $x > 0$. Suppose for contradiction that for all $n \in \mathbb{Z}^+$ such that $nx \leq y$. Let $E = \{nx \in \mathbb{R} \mid n \in \mathbb{Z}^+\}$. E is then non-empty and bounded above by y. Since $\mathbb{R}$ has the LUB property, E has a least upper bound $\alpha = \sup E$. Consider $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound. Thus for some $mx \in E$ with $m \in \mathbb{Z}^+$, $mx > \alpha - x$, and therefore

$$\alpha < mx + x = (m+1)x \in E.$$

But this is a contradiction, because α was said to be an upper bound for E.

□

Theorem 1.10. There exists a unique positive $x \in \mathbb{R}$ such that $x^2 = 2$.

Proof.

Lemma. If $x, y \in \mathbb{R}^+$ and $x^2 < y^2$, then $x < y$.

Proof. Suppose $x, y > 0$ and $x^2 < y^2$. Then $(x-y)(x+y) < 0$, so either $x - y < 0$ or $x + y < 0$. $x + y > 0$ because $x, y > 0$, so $x - y < 0$, and $x < y$. □

To show existence, let $E = \{y \in \mathbb{R} \mid y^2 \leq 2\}$. E is non-empty (e.g $1^2 < 2$), and E is bounded above by our lemma, so let $\alpha = \sup A$. We show that $\alpha^2 = 2$. Suppose $\alpha^2 \neq 2$. Then (i.) $\alpha^2 < 2$ or (ii.) $\alpha^2 > 2$.

(i.) $\alpha^2 < 2$. Let $0 < h < 1$. Consider $\alpha + h$. Then

$$\begin{aligned}(\alpha+h)^2 - \alpha^2 &= \alpha^2 + 2\alpha h + h^2 - \alpha^2 \\ &= 2\alpha h = h^2 > 2\alpha h.\end{aligned}$$

Note, $h^2 < h$, and

$$(\alpha+h)^2 - \alpha^2 = 2\alpha h + h^2 < 2\alpha h + h = (2\alpha+1)h.$$

In particular, $(\alpha+h)^2 < \alpha^2 + (2\alpha+1)h$. Arrange for $\alpha^2 + (2\alpha+1)h = 2$, and solve for $h = \frac{2-\alpha^2}{2\alpha+1}$. This will be true for $\alpha > 1$, so we need to show this. For this h, we have that $(\alpha+h)^2 < \alpha^2 + \frac{(2\alpha+1)(2-\alpha^2)}{2\alpha+1} = 2$. So, $\alpha + h \in E$. But $\alpha + h > \alpha$, so α is not an upper bound. This is a contradiction, and $\alpha^2 \not< 2$.

(ii.) $\alpha^2 > 2$. Again, let $h > 0$ and consider $\alpha - h$. We want to find a small enough h so that $(\alpha - h)^2 > 2$. This will contradict that α is the least upper bound because by the lemma, $(\alpha - h)$ is an upper bound for E. So,

$$\begin{aligned}\alpha^2 - (\alpha-h)^2 &= \alpha^2 - (\alpha^2 - 2\alpha h + h^2) \\ &= 2\alpha h - h^2 < 2\alpha h,\end{aligned}$$

amd $(\alpha-h)^2 > \alpha^2 - 2\alpha h$. Choose h to make $\alpha^2 - 2\alpha h = 2$: so $h = \frac{2-\alpha^2}{-2\alpha} = \frac{\alpha^2-2}{2\alpha}$. Note, $h > 0$. Then $(\alpha-h)^2 > \alpha^2 - 2\alpha h = \alpha^2 - 2\alpha(\frac{\alpha^2-2}{2\alpha})$. By the lemma, $\alpha - h$ is then an upper bound for E, but $\alpha - h < \alpha$, contradicting that $\alpha = \sup E$, and $\alpha^2 \not> 2$.

Therefore, $\alpha^2 \not< 2$ and $\alpha^2 \not> 2$, and $\alpha^2 = 2$.

For uniqueness, suppose $\alpha^2 = \beta^2 = 2$ with $\alpha, \beta < 0$ and $\alpha \neq \beta$. WLOG, $\alpha < \beta$. Then $\alpha^2 < \beta^2$, but this is a contradiction. Therefore, $\alpha = \beta$. □

Theorem 1.11. In general, for $x \in \mathbb{R}^+, n \in \mathbb{Z}^+$, there exists a unique $\alpha \in \mathbb{R}^+$ such that $\alpha^n = x$, denoted $\alpha = x^{\frac{1}{n}}$.

Proof. Find in Rudin. □

1.3 Euclidean Geometry

I literally do not care enough to write this down again.

Basic Topology

Metric Space. A *metric space* is a set X together with a function $d : X \times X \to \mathbb{R}$ called the *metric* with

(1) $d(x, y) > 0$ if $x \neq y$, and $d(x, x) = 0$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, x)$.

Neighborhood and Limit Point. Let (X, d) be a metric space. For $p \in X$ and $E \subset X$, an *r-neighborhood* of p is $N_r(p) = \{x \in X \mid d(x, p) < r\}$. p is a *limit point* of E if for all $r > 0$, $N_r(p) \cap E$ contains a point other than p.

For $X = \mathbb{R}$, the ordinary metric is given by $d(x, y) = |x - y|$. Additionally, if $E = [0, 1)$, the limit points for E are $[0, 1]$. If $E = [0, 1) \cup \{3\}$, the limit points are once again $[0, 1]$. Here, $3 \in E$ is called an *isolated point*, that is, it is not a limit point.

Closed and Perfect. E is *closed* if every limit point of E is in E. E is *perfect* if it is closed and every point of E is a limit point of E.

Clearly, $E = [0, 1)$ is not closed, as $1 \notin E$. Also, $E = [0, 1]$ is definitely closed. The set $\{1\}$ is closed, but 1 is not a limit point, and so it is not perfect. Also, $[0, 1] \cap 3$ is closed but not perfect, and in general, perfect sets are closed sets with no isolated points. $[0, 1]$ by itself is perfect.

Interior and Open. p is in the *interior* of E if for some $r > 0$, $N_r(p) \subset E$. E is *open* if every point of E is an interior point of E.

For $E = [0, 1]$, $p = 1/2$ is an interior point for $r = 1/2$, that is $N_{1/2}(1/2) = (0, 1) \subset [0, 1]$. Additionally, $\mathbb{R}$ is open, $(0, 1)$ is open, $[0, 1]$ is not open, etc.

Bounded. E is *bounded* if there is some $M \in \mathbb{R}^+$ and $q \in X$ such that $d(x, q) < M$ for all $x \in E$.

Dense. E is *dense* in X if every point of X is either a limit point of E or a point of E (or both).

Naturally, $E = \mathbb{R} - \{0\}$ is dense in $\mathbb{R}$. Also, more interestingly, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 2.1. For any metric space (X, d), $N_r(p)$ is open.

Proof. Let $q \in N_r(p)$. We want to show that q is an interior point. That is, there exists some $h > 0$ such that $N_h(q) \subset N_r(p)$. Note, $d(p, q) < r$. Choose $0 < h \leq r - d$. So, we check if this holds. Let $x \in N_h(q)$.

Then

$$\begin{aligned} d(p,x) &\leq d(p,q) + d(q,x) \\ &< d + h \leq d + (r-d) \\ &= d + r - d \\ &= r. \end{aligned}$$

by the triangle inequality. This shows that $x \in N_r(p)$, and $N_h(q) \subset N_r(p)$. □

Theorem 2.2. If p is a limit point of E, then any neighborhood of p contains infinitely many points of E.

Closure. Two operations we can define on a metric space (X,d) is the *closure*, $\overline{E} \coloneqq E \cap E'$ where E' is the set of limit points of E, and the *interior*, E^0 is the set of all interior points of E. Note, $\overline{E} \supset E$ and $E^0 \subset E$.

Theorem 2.3.

(1) $\overline{E}$ is closed.

(2) $E = \overline{E}$ if and only if E is closed.

(3) $\overline{E} \subset F$ for every closed set F that contains E.

(4) E^0 is open.

(5) $E = E^0$ if and only if E is open.

(6) $G \subset E^0$ for every open set G such that $G \subset E$.

Proof.

(1) We must show that $\overline{E}$ contains all limit point of $\overline{E}$. Note, $\overline{E}$ contains all limit points of E by definition. That is, can $\overline{E}$ have a limit point which is not a limit point of E. No, but let's prove this. Let p be a limit point of $\overline{E}$. If we show p is also a limit point of E, then we're done. Let $r > 0$, then $N_r(p)$ contains a point $q \in \overline{E}$ with $q \neq p$. Either $q \in E$ or q is a limit point of E. Assume the latter. Then $d(p,q) < r$, so choose h so that $h < r - d(p,q)$. Additionally, choose $h < d(p,q)$. Since q is a limit point of E, there is a point $x \in N_h(q) \cap E$. Check that $x \neq p$. Since $d(x,q) < h$, $d(p,q) \leq d(p,x) + d(x,q)$, and so

$$\begin{aligned} d(p,x) &\geq d(p,q) - d(x,q) \\ &\geq d(p,q) - h \\ &> 0. \end{aligned}$$

So, $x \neq p$. This shows that p is a limit point of E, and that limit point of $\overline{E}$ are contained in the limit points of E. Hence, $\overline{E}$ is closed.

□

As an exercise, prove (4), (5), (6) in this order: (6), (4), (5).

Theorem 2.4. A set $E \subset X$ is open if and only if its complement E^c is closed.

Proof. Observe, $p \in E^0$ if and only if $p \notin \overline{E^c}$. In other words, $E^0 = \left(\overline{E^c}\right)^c$. So,

$$E \text{ is open} \iff E = E^0 \iff E = \left(\overline{E^c}\right)^c \iff E^c = \left(\left(\overline{E^c}\right)^c\right)^c = \overline{E^c} \iff E^c \text{ closed.}$$

□

Theorem 2.5.

(1) The union of any family of open sets is open.

(2) The intersection of any family of closed sets is closed.

(3) The intersection of finitely many open sets is open.

(4) The union of finitely many closed sets is closed.

Proof.

(1) For $\{G_\alpha\}_{\alpha \in A}$ open sets, let $p \in \bigcup_{\alpha \in A} G_\alpha$ be an interior point. Since p is in the union, $p \in G_\beta$ for some $\beta \in A$. By Homework 4 2.9c, $G_\beta = (\bigcup G_\alpha)^o$. So, p is an interior point, and $\bigcup_{\alpha \in A}$ is open.

(2) For $\{F_\alpha\}_{\alpha \in A}$. This follows from (1) and the lemma: $\left(\bigcap_{\alpha \in A} F_\alpha\right)^c = \bigcup_{\alpha \in A} F_\alpha^c$. Since each F_α is closed, F_α^c is open. So, $\bigcup_{\alpha \in A} F_\alpha^c$ is open, and its complement is closed.

(3) Let $p \in \bigcap_{i=1}^n G_i$. Thus, for each $i \in \{1, \ldots, n\}$, $p \in G_i$ which is open. So, there exists some $r_i > 0$ with $N_{r_i}(p) \subset G_i$. Thus, $\bigcap_{i=1}^n N_{r_i}(p) = N_r(p) \subset \bigcap_{i=1}^n G_i$ where $r = \min\{r_1, \ldots, r_n\} > 0$.

(4) Follows from (3) and the set theory lemma using the same proof as (2).

□

Metric Subspace. Let (X, d) be a metric space. Let $Y \subset X$. Recall that Y becomes a metric space by restricting d to $Y \times Y$. Let $E \subset Y$. Then

(1) E is *open relative to* X: for each $p \in E$, there is an $r > 0$ so that all $q \in X$ with $d(p, q) < r$, denoted $N_r^X(p)$, are in E.

(2) E is *open relative to* Y: for each $p \in E$, there is another $r > 0$ so that all $q \in Y$ with $d(p, q) < r$, denoted $N_r^Y(p)$, are in E.

Note, $N_r^Y(p) = N_r^X(p) \cap Y$.

Theorem 2.6. For $E \subset Y$, E is open relative to Y if and only if $E = G \cap Y$ for some open set $G \subset X$.

2.1 Compactness

Let (X, d) be a metric space with $E \subset X$.

Open Cover. An *open cover* of E is a collection $\{G_\alpha\}_{\alpha \in A}$ of open subsets of X such that $E \subset \bigcup \alpha \in AG_\alpha$.

Compact Subsets. A subset $K \subset X$ is compact if any open cover $\{G_\alpha\}_{\alpha \in A}$ of K has a finite subcover $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ where $\alpha_i \in A$. That is, $K \subset \bigcup_{i=1}^n G_i$.

For $X = \mathbb{R}$ and $E = (0, 1]$. Is E compact? Note, $(0, 1] \subset (-1, \frac{1}{2}) \cup (\frac{1}{3}, 2)$ is a finite open cover. However, this does not prove E is compact; we need to consider all covers to show compactness. In fact, in this case, E is not compact. Consider the collection of open subsets $\{(\frac{1}{n}, 2)\}_{n \in \mathbb{Z}^+}$. Then $E \subset \bigcup_{n \in \mathbb{Z}^+} (\frac{1}{n}, 2) = (0, 2)$, and so this is an open cover, but it has no finite subcover.

Any singleton set $\{x\}$ is a compact subset of $\mathbb{R}$. To prove this, suppose $\{G_\alpha\}_{\alpha \in A}$ is an open cover of $\{x\}$. Since $x \in \bigcup_{\alpha \in A} G_\alpha$, it must be that $x \in G_\beta$ for some $\beta \in A$. So, the set $\{G_\beta\}$ is a finite subcover.

Theorem 2.7. Compact subsets of a metric space are closed.

Proof. Let K be a compact subset of the metric space (X, d). Lets show the complement of K is open. Let $p \in K^c$, that is some $p \in X$ such that $p \notin K$. Observe, $\{N_{r_q}(q)\}_{q \in K}$ is an open cover of K for $q \in K$. Because K is compact, there is a finite subcover $\{N_{r_{q_1}}(q_1), \ldots, N_{r_{q_m}}(q_m)\}$. Then $K \subset \bigcup_{i=1}^m N_{r_{q_i}}(q_i)$. Therefore, $N_{r_{q_i}}(p) \cap N_{r_{q_i}}(q) = \emptyset$. So, $\bigcap_{i=1}^n N_{r_{q_i}}(p)$ is disjoint from $K \subset \bigcup_{i=1}^m N_{r_{q_i}}(q_i)$ for $r = \min\{r_{q_1}, \ldots, r_{q_m}\}$. This shows that p is an interior point of K^c, hence K^c is open and K is closed. □

***Proposition* 2.8.**

(1) Show that if E is compact, then E is bounded.

(2) Show that $E = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$ is a compact subset of $\mathbb{R}$.

Proof.

(1) Let $p \in X$. Then $\bigcup_{n=1}^{\infty} N_n(p) = X$. So in particular, $\{N_n(p)\}_{n\in\mathbb{Z}^+}$ is an open cover of E. Therefore, since E is compact, there is a finite subcover of this cover $\{N_{n_1}(p), \dots, N_{n_m}(p)\}$. So for $M = \max\{n_1, \dots, n_m\}$, we have $E \subset N_M(p)$, so E is bounded.

(2) Let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of E. Let G_{α_0} be an open set from the cover that contains 0. Let $r > 0$ be such that $N_r(0) \subset G_{\alpha_0}$. Then $\frac{1}{n} \in N_r(0)$, and so $\frac{1}{a} < r$. This is true for all but finitely many $n \in \mathbb{Z}^+$. So, G_α contains 0 and all but finitely many points from E. For some $M \in \mathbb{Z}^+$ and $n > M$, then $\frac{1}{n} < \frac{1}{m} < r$. Pick a G_{α_i} containing $\frac{1}{i}$ for each $i = 1, \dots, M$. Therefore, $\{G_{\alpha_0}, \dots, G_{\alpha_m}\}$ is a finite subcover.

□

Heine-Borel

***k*-Cell.** A k-Cell is given by

$$[a_1, b_1] \times \cdots \times [a_k, b_k] = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_i \in [a_i, b_i] \text{ for all } i\}.$$

***Theorem* 2.9.** Every k-cell in $\mathbb{R}^k$ is compact.

Proof. We proceed inductively. Let $k = 1$. Then we show that $[a, b]$ is a compact subset of $\mathbb{R}$. Let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of $[a, b]$. Assume for contradiction that there are no finitely many of G_α that cover $[a, b]$. Let $c = \frac{a+b}{2}$. Then $[a, b] = [a, c] \cup [c, b]$. It must be the case that no finitely many of the G_α cover $[a, c]$ or no finitely many cover $[c, b]$; call this I_1. Note, the length of I_1 is given by $\frac{b-a}{2}$. We continue subdividing inductively. This gives us a sequence of nested closed intervals $I_1 \supset I_2 \supset I_3 \supset \dots$ such that I_n has length $\frac{b-a}{2^{n+1}}$, and no finitely many of the G_α cover I_n.

As a brief lemma, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Denote $I_n = [a_n, b_n]$. Let $x = \sup\{a_n\}_{n\in\mathbb{Z}^+}$. Then $a \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$, as $a_n \le x$ for all n and $x \le b_n$ for all n since for a fixed n, $b_n \ge a_k$ for all k. This proves the lemma.

Hence, $x \in G_\beta$ for some $\beta \in A$. Since G_β is open, $N_r(x) \subset G_\beta$. Choose M such that $\frac{b-a}{2^{M+1}} < r$. Then since $x \in I_m$ and the length of $I_m = \frac{b-a}{2^{M+1}}$, and $I_m \subset N_r(x) \subset G_\beta$. This contradicts that I_m can not be covered by finitely many G_α. Therefore, $[a, b]$ is compact. □

***Theorem* 2.10.** In any metric space (X, d): if $K \subset X$ is compact, $F \subset X$ is closed, and $K \subset K$, then F is compact.

Proof. Let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of F. To cover K, take $\{G_\alpha\}_{\alpha\in A} \cup \{F^c\}$. This is open since F^c is open. Since K is compact, there is some finitely many $\{G_{\alpha_1}, \dots, G_{\alpha_m}\} \cup \{F^c\}$. So, $\{G_{\alpha_1}, \dots, G_{\alpha_m}\}$ has to cover F. This shows that F is compact. □

***Theorem* 2.11 (Heine-Borel).** In the metric space R^k, a subset E is compact if and only if E is closed and bounded (this is *not* true in a general metric space).

Proof. If E is compact, then it is closed and bounded. Otherwise, suppose E is closed and bounded. If E is bounded, then E is contained in a k-cell. By Theorem 2.40, that k-cell is compact, so E is a closed subset of a compact set, hence is compact by Theorem 2.35. □

Other Notions of Compactness

***Proposition* 2.12.** Let E be a subset of a metric space (X, d). Then E is compact if and only if every infinite subset $S \subset E$ has a limit point in E (this is called *limit point compact*).

Proof. See Rudin for the case $X = \mathbb{R}^k$. $\square$

2.2 Perfect Sets

Perfect. A subset P in a metric space (X, d), P is *perfect* if P is closed and every point in P is a limit point of P. For example, $[a, b]$ is perfect.

***Theorem* 2.13.** Let $K_1 \supset K_1 \supset \dots$ be a "decreasing" sequence of non-empty compact subsets of a metric space. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Proof. Suppose for contradiction that $\bigcap K_n = \emptyset$. Then $\bigcup K_n^c = X$. So, $\{K_n\}$ are an open cover of X, thus this covers every subset of X, including K_1. Since K_1 is compact, it has a finite subcover $\{K_{n_1}^c, \dots, K_{n_m}^c\}$. Take $N = \max\{n_1, \dots, n_m\}$. Then $\bigcap_{i=1}^m K_{n_i}^c = K_N^c$, so K_N^c covers K_1. So, $K_1^C \supset K_N$, and $K_N \subset K_1 \cap K_1^c = \emptyset$, a contradiction that K_N is nonempty. Therefore, $\bigcap_{n \in \mathbb{Z}^+} K_n \neq \emptyset$. $\square$

***Theorem* 2.14.** Let $P \subset \mathbb{R}^k$ be a non-empty perfect set. Then P is uncountable.

Proof. P has limit points, so P is infinite. Assume for contradiction that P is countable. Then $x_1, x_2, \dots$ are all points in P. Note, for closed neighborhoods, $\overline{N_r(p)} = \{x \in \mathbb{R}^k \mid d(p, x) \leq r\}$. Clearly, these are also bounded, hence compact. Choose $y_1 \neq x_1$ with $y_1 \in P$ and $r_1 > 0$ such that $\overline{y_1} \not\ni x_1$, denote $V = N_{r_1}(y_1)$. Since y_1 is a limit point of P, there is a $y_2 \in E$ inside a small neighborhood of y_1. Choose $r_2 > 0$ such that $\overline{N_r(y_2)} \not\ni y_2$. Let $V_2 = N_{r_2}(y_2)$. Then $y \in P$ such that $\overline{V_n} \subset V_{n-1}$ and $x_n \notin \overline{V_m}$. So, we have a decreasing sequence of compact subsets of $\mathbb{R}^k$ with $x_n, x_{n-1}, \dots \notin \overline{V_n}$. Take $K_n = \overline{V_n} \cap P$. Then $K_n \ni yN$, hence is nonempty. K_n is also compact because it has a closed subset of a compact set. Hence $K_1 \supset K_2 \supset \dots$. Applying our lemma, $\cap K_n \neq \emptyset$, and $\bigcap K_n \ni p$. Also, $p \in P$, but $p \neq x_n$ for all $x \in \mathbb{Z}^+$ because $p \in \overline{V_n}$. So, p is not in the list, and the list does not contain all of P. Hence, P is uncountable. $\square$

***Corollary* 2.15.** $[a, b]$ is uncountable and $\mathbb{R}$ is uncountable. The cantor set is uncountable.

Connected Sets

Separated. For (X, d) a metric space, $A, B \subset X$ are *separated* if $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$.

$A = [0, 1]$ and $B = [2, 3]$ are separated. $A = (0, 1)$ and $B = (1, 2)$ are *also* separated because $[0, 1] \cap (1, 2) = \emptyset$ and $(0, 1) \cap [1, 2] = \emptyset$. If $A = (0, 1)$ and $B = [1, 2]$, A and B are *not* separated, as $\overline{a} \cap B \ni 1$.

Disconnected and Connceted. A subset $E \subset X$ is *disconnected* if $E = A \cap B$ for A, B nonempty and separated. E is *connected* if it is not disconnected.

$(0, 1) \cup (1, 2)$ is disconnected.

***Theorem* 2.16.** A subset $E \subset \mathbb{R}$ is connected if and only if it satisfies the following property: if $x, y \in E$ with $x < z < y$, then $z \in E$. That is, E is *convex*.

Series and Sequences

3.1 Sequences

Converging Sequence. A sequence (p_n) in a metric space (X,d) *converges* if there exists a $p \in X$ such that for every $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $d(p_n, p) < \epsilon$. This is commonly denoted $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

Bounded. (p_n) is *bounded* if $\{p_n \mid n \in \mathbb{Z}^+\}$ is bounded.

(1) $p_n = \frac{1}{n}$. Then $\lim_{n\to\infty} p_n = 0$. To verify, let $\epsilon > 0$. Then choose $N > \frac{1}{\epsilon}$. So, $\frac{1}{N} < \epsilon$. Then if $n \geq N$, we have that $p - n = \frac{1}{n} \leq \frac{1}{N} < \epsilon$, and so $d(p_n, p) = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$. This shows that $p - n$ converges to 0.

(2) $p_n = n$ does not converge. Let $p \in \mathbb{R}$. Choose $\epsilon = \frac{1}{2}$ and let $N \in \mathbb{Z}^+$. Choose $n \geq N$ and $n \geq \frac{1}{2}$. Then $d(p_n, p) = |n - p| = n - p \geq (p + \frac{1}{2}) - p = \frac{1}{2} = \epsilon$. Thus, p_n does not converge to p, and since p was arbitrary, p_n does not converge.

Theorem 3.1. If (p_n) converges, then (p_n) is bounded. Also, if $E \subseteq X$ and $p \in X$ is a limit point of E, then there is a sequence (p_n) of points in E that converge to p.

Proof. Suppose $p_n \to p$. Choose $\epsilon = 1$. Since $p_n \to p$, there is some $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $d(p_n, p) < 1$. Take $x = p$ and $M = \max\{1, d(p, p_1), \ldots, d(p, p_{N-1})\} + 1$. Then $d(p, p_n) < m$ for all n.

For the second statement, recall that since p is a limit point of E, for all $\epsilon > 0$, $|N_\epsilon(p) \cap E|$ is infinite. Choose $\epsilon = 1$. Let p_1 be any point in $N_1(p)$. Next, choose $\epsilon = \frac{1}{2}$ and let $p - 2 \in N_{1/2}(p)$, and so on. In general, let p_n be any of the infinitely many points in $N_{1/2}(p) \cap E$. So, we have a sequence p_n, and we show that $p_n \to p$. So, let $\epsilon > 0$. Then choose $N > \frac{1}{\epsilon}$. So, if $n \geq N$, we have that $d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$, and we're done. $\square$

Theorem 3.2. Let $(s_n), (t_n)$ be sequences with $s_n \to s$ and $t_n \to s$. Then for some scalar c,

(1) $s_n + t_n \to s + t$.

(2) $cs_n \to cs$.

(3) $c + s_n \to c + s$.

(4) $s_n t_n \to sn$.

(5) $\frac{1}{s_n} \to \frac{1}{s}$ if s_n and s are not 0.

Proof. Let $\epsilon > 0$. Since $s_n \to s$, there is $N_1 \in \mathbb{Z}^+$ such that for all $n \geq N_1$, $|s_n - s| < \frac{\epsilon}{2}$. SImilarly, since $t_n \to t$, there is $N_2 \in \mathbb{Z}^+$ such that for all $n \geq N_2$, $|t_n - t| < \frac{\epsilon}{2}$. Choose $N = \max\{N_1, N_2\}$. Then for $n \geq N$,

$$\begin{aligned} d(s_n + t_n) &= |s_n + t_n - (s+t)| = |(s_n - s) + (t_n - t)| \\ &\leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The rest of these proofs are in Rudin. □

Theorem 3.3. Let (x_n) be a sequence in $\mathbb{R}^k$. Write $x_n = (x_{n,1}, \ldots, x_{n,k})$ Then $x_n \to x = (x_1, \ldots, x_k)$ if and only if $\lim_{n\to\infty} x_{n,i} = x_i$ for all $i = 1, \ldots, k$.

Let $(x_n), (y_n)$ be sequences in $\mathbb{R}^k$ and (β_n) a sequence in $\mathbb{R}$. Suppose $x_n \to x$, $y_n \to y$, and $\beta_n \to \beta$. Then $\lim_{n\to\infty}(x_n + y_n) = x + y$, $\lim_{n\to\infty}(x_n y_n) = xy$, and $\lim_{n\to\infty} \beta_n x_n = \beta x$.

Subsequences

Subsequence. Let (p_n) be a sequence in a metric space XS. let $n_1, n_2, \dots \in \mathbb{Z}^+$ be a sequence of positive integers that is increasing, that is $n_1 < n_2 < \ldots$, then the sequence $p_{n_1}, p_{n_2}, \ldots$ is called a *subsequence*, denoted by $(p_{n_k})_{k=1}^{\infty}$ or (p_{n_k}). Formally, thinking of a sequence as a function $f : \mathbb{Z}^+ \to X$, then a subsequence is some $f \circ g : \mathbb{Z}^+ \to X$ given by right composition with $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $k \mapsto n_k$ such that $g(k+1) > g(k)$ with $k \mapsto n_k \mapsto p_{n_k}$ for all k.

Subsequential Limit. *Subsequential limits* are the limits of convergent subsequences.

Proposition 3.4. $p_n \to p$ if and only if every subsequence (p_{n_k}) of (p_n) converges to p.

Theorem 3.5.

(1) Let (p_n) be a sequence in a compact metric space X. Then there is a subsequence (p_{n_k}) that converges to a point of X.

(2) If (p_n) is a sequence in $\mathbb{R}^k$ which is bounded, then there is a sequence (p_{n_k}) that converges to a point of $\mathbb{R}^k$.

Proof. Clearly, $(a) \implies (b)$ by Heine-Borel.

(1) To show (a), let $E = \{p_n \mid n \in \mathbb{Z}^+\}$ (the "range" of p_n). We consider the following cases:
   1. E is finite. Then some $p \in E$ shows up infinitely many times for $n_1 < n_2 < \ldots$, and so $(p_{n_k})_{k=1}^{\infty} = \{p\}$. So, this converges to p.
   2. E is infinite. By a theorem from chapter 2, E has a limit point $p \in X$. Then for all $\epsilon > 0$, $N_\epsilon(p) \cap E$ is infinite. Thus, there exists an n_1 such that $d(p, n_1) < 1$, an n_2 such that $d(p, n_2) < \frac{1}{2}$ with $n_2 > n_1$, and so on. Inductively, we find $n_1 < n_2 < \ldots$ with $d(p_{n_k}, p) < \frac{1}{k}$. This sequence $(p_{n_k})_{k=1}^{\infty}$ converges to p. □

Upper and Lower Limits

We're gonna work in $\mathbb{R}$ now.

Diverging Sequence. If (s_n) is a sequence in $\mathbb{R}$ with $s_n \to +\infty$, what does this mean? For every $M \in \mathbb{R}$, there is some $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $s_n \geq M$. Similarly, if $s_n \to -\infty$, for all $M \in \mathbb{R}$, there exists $N \in \mathbb{Z}$ such that for all $n \geq N$, $s_n \leq M$.

Extended Reals. To make the notion of $+\infty, -\infty$ more formal, we denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ as an ordered set. If $E \subset \mathbb{R}$ is not bounded, then $\sup E = +\infty$. Similarly for $E \subset \mathbb{R}$ not being bounded below.

Upper and Lower Limits. Let (s_n) be a sequence in $\mathbb{R}$. Let

$$E = \{x \in \overline{\mathbb{R}} \mid x \text{ is a subsequential limit of } (s_n)\}$$

and E be nonempty. Then $\limsup_{n\to\infty} s_n = \sup E \in \overline{\mathbb{R}} = S^*$ and $\liminf_{n\to\infty} = \inf E = S_*$. In fact, $S^k = S_k \in E$.

Recall, (p_n) in X a metric space is *Cauchy* if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$. Alternatively, the same holds if and only if $\operatorname{diam} E_n \xrightarrow{N\to\infty} 0$ with $E_n = \{p_n \mid n \geq N\}$.

Theorem 3.6. (1) In X, any convergent sequence is Cauchy.

(2) If X is compact, every Cauchy sequence converges.

(3) In $\mathbb{R}^k$, every Cauchy sequence converges (hence $(b) \implies (c)$).

Proof. For (b): let X be compact and (p_n) be Cauchy. So, $\operatorname{diam} E_n \to 0$. Thus, $E_1 \supset E_2 \supset \dots$. For each N_1, $\overline{E_N}$ is closed, hence is compact. Also, $\operatorname{diam} \overline{E} = \operatorname{diam} E$ by 3.10(a). Thus, $\overline{E_1} \supset \overline{E_2} \supset \dots$ is a sequence of compact nested sets with $\operatorname{diam} E_n \to 0$, thus by 3.10(b), $\bigcap_{N=1}^{\infty} \overline{E_n} = \{p\}$. We show (p_n) converges to p. let $\epsilon > 0$, then there exists $N_0 \in \mathbb{Z}^+$ such that for all $N \geq N_0$, $\operatorname{diam} \overline{E_n} < \epsilon$. Then for all $n \geq N_0$, $p_n \in E_{N_0}$, $p \in \overline{E_{N_0}}$. Therefore, $d(p_n, p) < \epsilon$. □

Complete. A metric space X is called *complete* if every Cauchy sequence converges. In particular, compact spaces are complete, $\mathbb{R}^k$ is complete, and any space which is proper is complete.

3.2 Series

Series. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. For completeness of notation, $\sum_{k=p}^{q} a_k = a_p + \cdots + a_q$. For $n \in \mathbb{Z}^+$, let $s_n = \sum_{i=1}^{n} a_i$ be called the *n-th partial sum*. Then $(s_n)_{n=1}^{\infty}$ is a new sequence, so $\sum_{n=1}^{\infty} a_n = (s_n)$. A *series* is just this sequence of partial sums, or $\sum a_n$. So, $\sum a_n = L$ converges if and only if (s_n) converges to L, and $\sum a_n$ diverges if and only if (s_n) diverges.

Theorem 3.7. $\sum a_n$ converges if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $|\sum_{k=n}^{m} a_k| < \epsilon$ for all $m \geq n \geq N$.

Corollary 3.8. If $\sum a_n$ converges, then $a_n \to 0$, but the converse is not true.

Theorem 3.9 (Monotone Convergence Theorem)***.*** If $a_n > 0$ for all n, then $\sum a_n$ converges if and only if (s_n) is bounded.

Theorem 3.10 (Direct Comparison Test)***.*** (1) If $|a_n| \leq c_n$ for all $n \geq N_0$ for some constant N_0, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(2) As the contrapositive, if $a_n \geq d_n \geq 0$ for all $n \geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof. Assume $\sum a_n$ converges. let $\epsilon > 0$. Since $\sum a_n$ converges, there is some $N \in \mathbb{Z}^+$ such that for all $m \geq n \geq N$, $|\sum_{k=n}^{m} c_k| < \epsilon$. WLOG, take $N \geq N_0$. So,

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \epsilon.$$

Thus, $\sum a_n$ converges by Theorem 3.22. Thus, (b) holds as the contrapositive. □

Theorem 3.11. If $|x| \leq 1$, then $\sum_{n=p}^{\infty} x^n = \frac{1}{1-x}$ diverges if $x \geq 1$.

Proof. $s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n-p}}{1-x}$, then use the fact that (x^n) converges (to 0) if $|x| < 1$, (to 1) if $x = 1$, and diverges otherwise. □

Theorem 3.12. Suppose $a_1 \geq a_2 \geq \dots$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof. Since all terms ar positive, it suffices to consider boundedness of the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ and $t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$. If $n \leq 2^k$, then

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + \cdots + a_{s^{n+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_1 + \cdots + 2^k a_{2^k} = t_k. \end{aligned}$$

Therefore, $\sum s^k a_{2^k}$ converges implies $\sum a_n$ converges, and the converse holds as well. ☐

Theorem 3.13. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. If $p < 0$, then $\frac{1}{n^p} \not\to 0$, hence it diverges (duh). Assume $p \geq 0$. Then $\frac{1}{1^p} \geq \frac{1}{2^p} \geq \frac{1}{3^p} \geq \cdots \geq 0$. Now, apply 3.27: $\sum \frac{1}{n^p}$ converges if and only if $\sum 2^k \frac{1}{2^{kp}} = \sum \left(\frac{1}{2^{p-1}}\right)^k$, a geometric series with $x = \frac{1}{2^{p-1}}$. This converges if and only if $\frac{1}{2^{p-1}} < 1$, which is true if and only if $p > 1$. ☐

Similarly, $\sum \frac{1}{n(\log n)^2}, \sum \frac{1}{n \log n \log \log n}$, and so on converges.

3.3 The Number e

The Number e***.*** $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$. If $n \geq 2$, then $\frac{1}{n!} = \frac{1}{n(n-1)(n-2)\ldots 1} \leq \frac{1}{(n-1)^2}$, so this series converges by the comparison test.

Theorem 3.14. e is irrational.

Proof. For $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, the error is given by

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots \\ &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \ldots\right) \\ &\leq \frac{1}{(n+1)!}\underbrace{\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \ldots\right)}_{\text{Geometric series with } x = \frac{1}{n+2}} \\ &= \frac{1}{(n+1)!}\left(\frac{n+2}{n+1}\right) = \frac{1}{n!n}. \end{aligned}$$

Suppose e is rational, that is $e = \frac{p}{q}$ for $p, q \in \mathbb{Z}^+$. Take $0 < e - s_q < \frac{1}{q!q}$. Then $0 < \underbrace{p(q-1)!}_{\in \mathbb{Z}} + \underbrace{s_q q!}_{\in \mathbb{Z}} < \frac{1}{q}$, but this is a contradiction (this method of arguing irrationality is very common, and called *Diophantine approximation*). ☐

Theorem 3.15 (Root Test)***.*** Given $\sum a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt{|a_n|}$. Then

(1) If $\alpha > 1$, a_n diverges.

(2) If $\alpha < 1$, a_n converges.

(3) If $\alpha = 1$, we have no information.

Theorem 3.16 (Ratio Test)***.*** $\sum a_n$ converges if $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ and diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$ for some n_0 a fixed index.

Absolute Convergence

Absolute Convergence. $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges.

As a non-example, $\sum \frac{(-1)^n}{n}$ does not converge absolutely.

Theorem 3.17. If $\sum |a_n|$ converges, $\sum a_n$ converges.

Proof. We use the Cauchy criterion. Let $\epsilon > 0$. We want N such that for all $m \geq n \geq N$, $|\sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} |a_k| < \epsilon$. So, $\sum a_n$ converges. $\square$

Addition and Multiplication of Series

Theorem 3.18. If $\sum a_n = \alpha$ and $\sum b_n = \beta$, then $\sum (a_n + b_n) = \alpha + \beta$.

Proof. $\sum a_n = \alpha$ means that $A_n \to \alpha$ where $A_n = \sum_{k=1}^{n} a_k$, and $\sum b_n = \beta$ means that $B - n \to \beta$ where $B_n = \sum_{k=1}^{n} a_k$. Observe that the n-th partial sum for $\sum (a_n + b_n)$ is

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} + \sum_{k=1}^{n} b_k = \alpha + \beta = A_n + B_n.$$

$\square$

3.4 Multiplication of Series