Planning

March 26, 2023

Goals:

1. Draw Hilbert bisectors.
   (a) Start with some convex domain Ω and compute the Hilbert metric d_H using the cross-ratio. Then given two points $x, y \in \Omega$, draw the bisector $\{z \in \Omega : d_H(x, z) = d_H(z, y)\}$.
      i. There are multiple ways to commute d_H. One idea is to keep track of the attracting hyperplanes when drawing the domain and just find which two best approximate where the endpoints a, b for the chord through x, y would land.
      ii. Since this isn't a huge priority and is kind of just for fun, the naïve algorithm should work: sample random points in the domain and compute distances, and if $d_H(x, z) - d_H(z, y) < \epsilon$ for some small ϵ, draw them.
      iii. It is non-trivial to define these convex domains, and the best way to do so may be to deform a discrete representation of a group G into $\mathrm{PSL}_2 \mathbb{R}$ into $\mathrm{SL}_3 \mathbb{R}$. For example, via a bulging or earthquake deformation.
2. Draw different convex domains.
   (a) As in 1(a)ii., it would be nice to visualize different types of deformations.
   (b) For example, bending deformations is a good place to start. A subgroup

$$\Gamma_1 *_\Lambda \Gamma_2 \subset \mathrm{PSL}_{d+1} \mathbb{R}$$

can be deformed as follows: let $(B_t)_t \subset \mathrm{PSL}_{d+1} \mathbb{R}$ be a path of elements starting at the identity and that commute with Λ. Then we can deform $i : \Gamma \to \mathrm{PSL}_{d+1} \mathbb{R}$ to representations ρ_t which are the identity on Γ_1 and send $\gamma_2 \in \Gamma_2$ to $B_t \gamma_2 B_t^{-1}$. If Γ divides some Ω, then representations ρ_t are injective, and $\rho_t(\Gamma)$ divides a properly convex domain Ω_t. By the Ehresmann-Thurston principle, we have that

$$\Omega_t / \rho_t(\Gamma) \approx_{\mathrm{Diffeo}} \Omega / \Gamma$$

for all t.

Observe, ρ_t is well-defined since B_t is in the identity component of the centralizer $\mathbb{Z}(\mathrm{PSL}_3 \mathbb{R})$, hence both components of the representation agree on Λ. We can construct B_t as follows: $B_t = e^{tB}$ where

$$B = \begin{pmatrix} -1 & & & & \\ & d & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}.$$

3. Draw the Dirichlet domain for these deformed representations on Ω. This includes when Ω is the Klein model for $\mathbb{H}^n$.
   (a) This will involve drawing a Voronoi tessellation of Ω according to the orbit $\mathcal{O} = \Gamma \cdot x_0$ of some point $x_0 \in \Omega$ with respect to d_H.
      i. There are a number of algorithms to do this:
         A. The naïve one. Loop through all of the points in Ω and find the closest point. Then color accordingly.
         B. Something else.
4. Draw the intersection of the Dirichlet domain of the induced action of Γ on the symmetric space $\mathcal{P}_n = \mathrm{SL}_n \mathbb{R} / \mathrm{SO}_n \mathbb{R}$ and a quadratically embedded copy of $\mathbb{RP}^2 \subset \partial\mathcal{P}_3$ (via Sym^2). This will give a "fundamental domain" of the original action of Γ on Ω but with much nicer bisectors (quadratic curves).
   (a) In particular, we want to know when these Dirichlet domains are finitely sided. Use https://arxiv.org/pdf/2302.00643.pdf for a reference on known cases, and understand the cyclic ones.

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Make progress on 1., 4(a), and 2. The order might be wacky, since we need a domain to draw Hilbert bisectors on in the first place. Basically, in order, we want to be able to deform $\mathbb{H}^2$ into some convex domain Ω on which we can draw Hilbert bisectors and eventually Voronoi tessellations. Probably best to start with bending deformations.