Final Report

Dirichlet Domains and Deformations of Convex Projective Structures

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1 Background

This project arose out of a research problem I am thinking about under the guidance of a professor in the math department, Dr. Jeffrey Danciger. It falls loosely into the subject of *geometric topology*.

There is a deep idea in mathematics that the geometry of a space can be encoded by the symmetries of that space that preserve the geometry. This goes back to Klein, with his Erlangen program. The idea is to define geometry by an abstract characterization of the symmetries of that geometry, called the *isometry group*. For example,

- Euclidean geometry is given by the data of the space $\mathbb{R}^n$ and the Euclidean isometry group $\mathrm{Euc}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$, consisting of translations, rotations, and reflections.
  - Think of a triangle in high school geometry. Two triangles are the same if they can be "transformed" into each other using the above operations.
- Spherical geometry in 2-dimensions is given on the sphere S^2 with isometry group $O(3)$.
- Hyperbolic geometry is given by hyperbolic space $\mathbb{H}^n$ with the isometry group $\mathrm{PSL}_2\,\mathbb{R}$.

Additionally, it's possible to "equip" other spaces called *manifolds* with these geometries. If you glue up a square in $\mathbb{R}^2$ by Euclidean isometries, e.g. translation by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, you get a Euclidean or flat torus [1].

Mathematicians, in many areas of math, study *classification problems*. They seek to classify families of mathematical objects. In this case, we might want to classify all of the possible geometries we can put on manifolds or all of the unique ways to put a given geometry on a manifold. This project focuses on the latter, looking at the specific case of putting *convex projective structures* on surfaces, or 2-dimensional manifolds. These structures come from a canonical metric on properly convex domains Ω living in $\mathbb{RP}^d$, called the Hilbert metric d_Ω.

One way to get a handle on this is via *deformation theory*. The Ehressman-Thurston principle says that geometric structures are stable under small deformation. This project seeks to visualize a number of these deformations by drawing fundamental domains (like the square gluing to the torus) for actions of discrete subgroups $\Gamma \subset \mathrm{PSL}_3\,\mathbb{R}$ on properly convex domains $\Omega \subset \mathbb{RP}^2$. One way to do that is by drawing Voronoi diagrams with respect to the Hilbert metric. In the case of the unit disk, the Hilbert metric corresponds to the ordinary hyperbolic metric, so it is also of interest to draw hyperbolic Voronoi diagrams. Also, it would potentially be novel to draw bisectors in the Hilbert metric — another goal of this project.

The research project is specifically interested in answering the question, "when are these fundamental domains finitely sided?"[2]

2 Accomplishments

So far, the following are working:

1. Drawing Hyperbolic geodesics.
   (a) This is just getting our feet wet, so to speak. It would be nice to have a fully modeled hyperbolic space, including the ability to compute distances and geodesics.
   (b) This is implemented both for the Poincaré and Klein disk models.
2. Drawing Hilbert bisectors.
   (a) Start with some convex domain Ω and compute the Hilbert metric d_Ω using the cross-ratio. Then given two points $x, y \in \Omega$, draw the bisector $\{z \in \Omega : d_\Omega(x, z) = d_\Omega(z, y)\}$. There were a number of design decisions to be made:
      i. There are multiple ways to commute d_Ω. One idea is to keep track of the attracting hyperplanes when drawing the domain and just find which two best approximate where the endpoints a, b for the chord through x, y would land. Ultimately, I ended up just keeping track of the boundary edges while generating Ω. This is not as mathematically accurate, since the points we solve for aren't actually on the boundary, but it is a good approximation.
      ii. At first, I tried the naïve algorithm: sample random points in the domain and compute distances, and if $d_\Omega(x, z) - d_\Omega(z, y) < \epsilon$ for some small ϵ, draw them. Since it is relatively slow to compute d_Ω, it turned out to be beneficial to be a bit more clever. The final project uses a box walking technique to sample points along the bisector and drawing points if the above condition is met. I also added an option to draw the boxes as gradients instead of a simple curve.
      iii. It is non-trivial to define these convex domains, and the best way to do so may be to deform a discrete representation of a group G into $\mathrm{PSL}_2 \mathbb{R}$ into $\mathrm{SL}_3 \mathbb{R}$. For example, via a bulging or earthquake deformation. This is what I ended up doing for this project. I chose to draw the bisectors on domains gotten by bulging deformations of triangle reflection groups.
      Here is how this works technically [3]: a subgroup

$$\Gamma_1 *_\Lambda \Gamma_2 \subset \mathrm{PSL}_{d+1} \mathbb{R}$$

      can be deformed as follows: let $(B_t)_t \subset \mathrm{PSL}_{d+1} \mathbb{R}$ be a path of elements starting at the identity and that commute with Λ. Then we can deform $i : \Gamma \to \mathrm{PSL}_{d+1} \mathbb{R}$ to representations ρ_t which are the identity on Γ_1 and send $\gamma_2 \in \Gamma_2$ to $B_t \gamma_2 B_t^{-1}$. If Γ divides some Ω, then representations ρ_t are injective, and $\rho_t(\Gamma)$ divides a properly convex domain Ω_t. By the Ehresmann-Thurston principle, we have that

$$\Omega_t/\rho_t(\Gamma) \approx_{\text{Diffeo}} \Omega/\Gamma$$

for all t.
Observe, ρ_t is well-defined since B_t is in the identity component of the centralizer $\mathcal{Z}(\mathrm{PSL}_3 \mathbb{R})$, hence both components of the representation agree on Λ. We can construct B_t as follows: $B_t = e^{tB}$ where

$$B = \begin{pmatrix} -1 & & & & \\ & d & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}.$$

The project allows you to tune this t, and we draw Ω_t.

iv. It is also currently possible to view the fundamental domain coming from the Voronoi tessellation of the triangle reflection group I chose, but not yet under the deformation. This should be an easy thing to add [4].

3. Hyperbolic Voronoi diagrams.

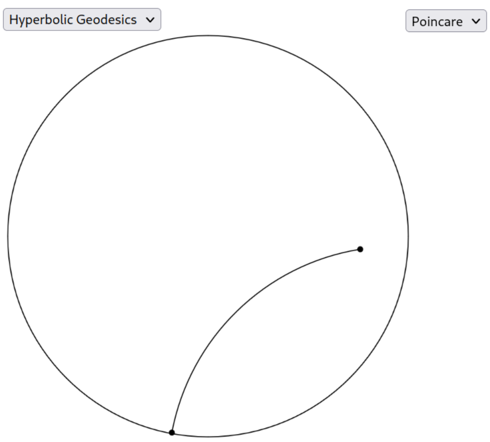
   (a) Again, there were a number of options for the implementation here. I implemented Bowyer-Watson to compute the hyperbolic Delaunay triangulation (to then compute the dual Voronoi tessellation), but ultimately wasn't able to finish debugging this for the final project. The algorithm in use currently is the naive one, but it works pretty fast up to 100 points.

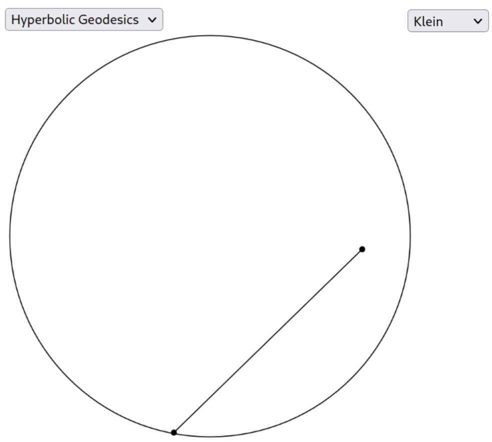
   (b) This works for both the Poincaré and Klein models, and thanks to caching, it is very fast to switch between them.

   (c) For the Voronoi diagrams with respect to the Hilbert metric, I was planning on working through [4].

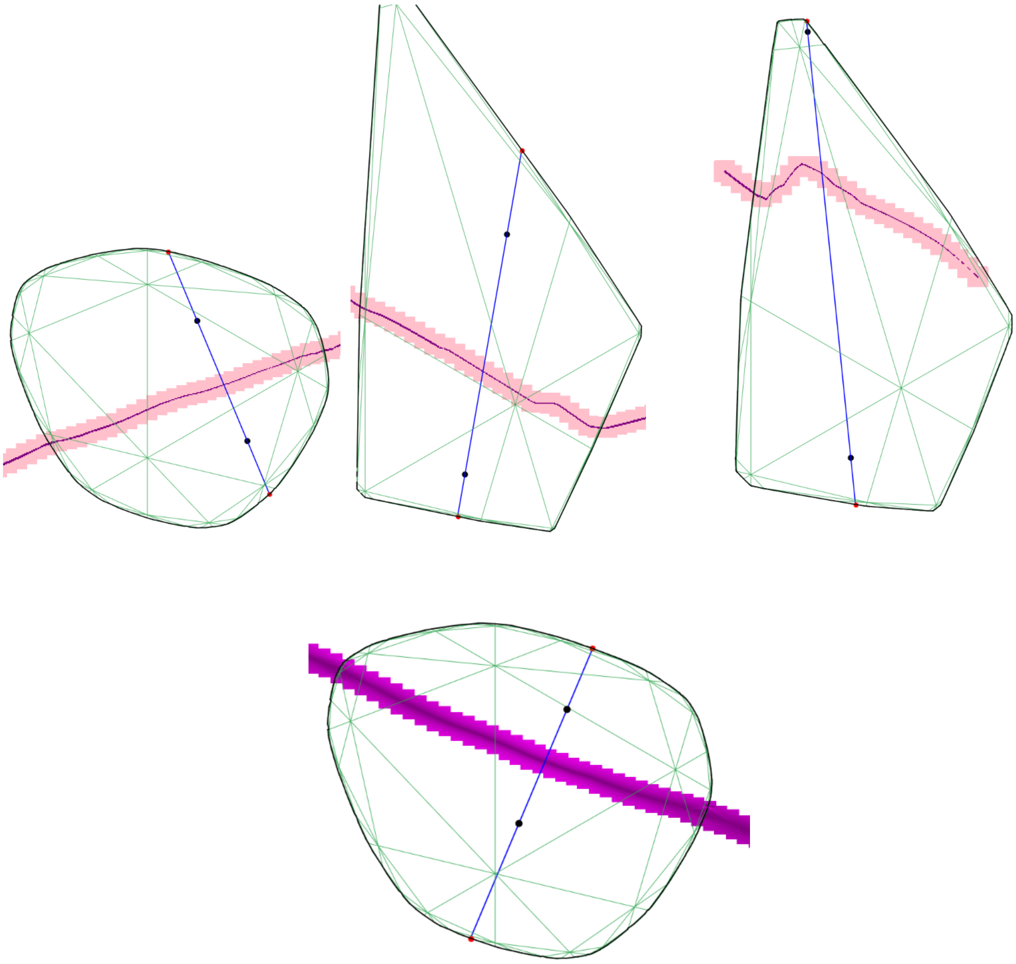
3 Artifacts

1. Hyperbolic Geodesics

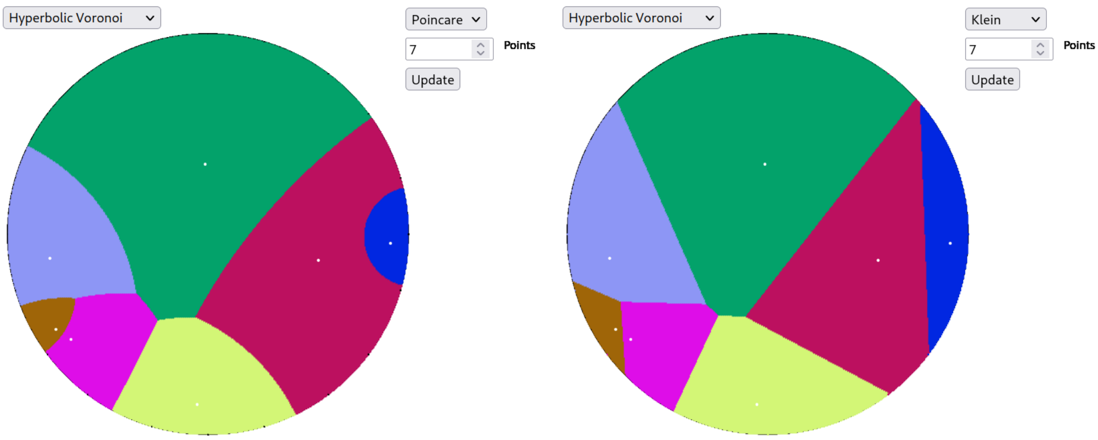


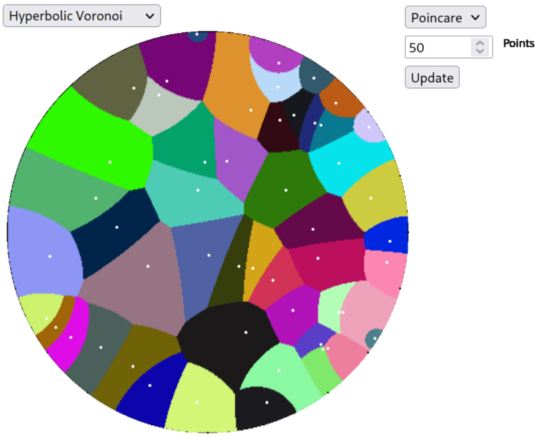


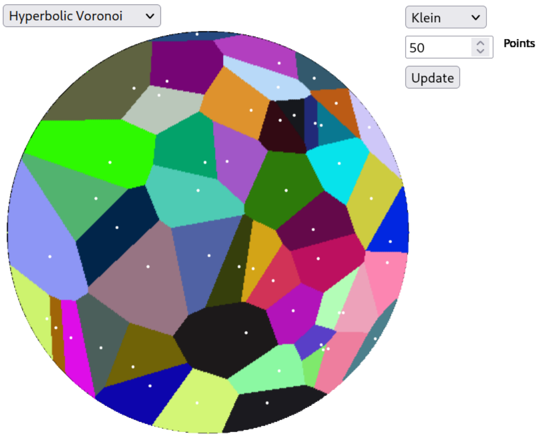
2. Hilbert Bisectors



3. Hyperbolic Voronoi diagrams







References

[1] Sam Ballas and Daryl Cooper. 209p: Projective structures on manifolds, 2016.

[2] Yukun Du. Geometry of selberg's bisectors in the symmetric space $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R})$, 2023.

[3] William Goldman. Bulging deformations of convex $\mathbb{R}P^2$-manifolds. 02 2013.

[4] Auguste H. Gezalyan and David M. Mount. Voronoi diagrams in the hilbert metric, 2021.