Documentation

The general idea is as follows. Let the method we use be represented as python List: $[m_0, m_1, ..., m_{n-1}]$. In the example of 4-3-4, the list is [4, 3, 4]. The trajectory is required to go through $\theta_0, \theta_1, ..., \theta_n$ at times $t_0, t_1, ..., t_n$, respectively. The initial and final velocities are $\dot{\theta_0}$ and $\dot{\theta_n}$, and accelerations are $\ddot{\theta_0}$ and $\ddot{\theta_n}$.

The trajectory consists of n polynomials:

$$y_0 = a_{00} + a_{01}t + a_{02}t^2 + ... + a_{0m_0}t^{m_0}$$
$$y_1 = a_{10} + a_{11}t + a_{12}t^2 + ... + a_{1m_1}t^{m_1}$$
$$...$$
$$y_{n-1} = a_{(n-1)0} + a_{(n-1)1}t + a_{(n-1)2}t^2 + ... + a_{(n-1)m_{n-1}}t^{m_{n-1}}$$

The unknown coefficients, X, is a vector that we need to find:

$$X = [a_{00} \quad a_{01} \quad ... \quad a_{0m_0} \quad a_{10} \quad ... \quad a_{(n-1)0} \quad ... \quad a_{(n-1)m_{n-1}}]$$

There are 3 equations for the initial and final points since the position, velocity and acceleration have to match. For other points, there are 4 equations: 2 for matching the position for the polynomial on left and right, 1 for continuous velocity and 1 for continuous acceleration. The number of equations we have is $4(n-1) + 3 \cdot 2 = 4n + 2$.

The number of equations calculated above can, in some cases, exceeds the number of unknowns. One example is 3-3-3-3-3, where number of unknowns is only 20 whereas number of equations is $4n + 2 = 22$. Therefore, the python script allows omitting certain velocities and/or accelerations by entering colon(:) instead of a value. This reduces the number of equations and ensure the linear system has unique solution.

Now we list all the equations. For $t = t_0$, the initial position must match:

$$a_{00} + a_{01}t_0 + a_{02}t_0^2 + ... + a_{0m_0}t_0^{m_0} = \theta_0$$

This is equivalent to

$$(1 \quad t_0 \quad t_0^2 \quad ... \quad t_0^{m_0} \quad 0 \quad ... \quad 0)X = \theta_0 \tag{1}$$

Equation for initial velocity and accelerations are similar:

$$0 + a_{01} + 2a_{02}t_0 + ... + m_0 a_{0m_0}t_0^{m_0-1} = \dot{\theta_0}$$
$$0 + 0 + 2a_{02} + ... + m_0(m_0-1)a_{0m_0}t_0^{m_0-2} = \ddot{\theta_0}$$

Or,

$$(0 \quad 1 \quad 2t_0 \quad ... \quad m_0 t_0^{m_0-1} \quad 0 \quad ... \quad 0)X = \dot{\theta_0} \tag{2}$$

$$(0 \quad 0 \quad 2 \quad ... \quad m_0(m_0-1)t_0^{m_0-2} \quad 0 \quad ... \quad 0)X = \ddot{\theta_0} \tag{3}$$

For $t = t_1$, we have equations for matching position for polynomial y_0

$$(1 \quad t_1 \quad t_1^2 \quad ... \quad t_1^{m_0} \quad 0 \quad ... \quad 0)X = \theta_1 \tag{4}$$

Also for y_1,

$$a_{10} + a_{11}t_1 + a_{12}t_1^2 + ... + a_{1m_1}t_1^{m_1} = \theta_1$$

The above is equivalent to

$$(\underbrace{0 \quad ... \quad 0}_{m_0+1 \text{ zeros}} \quad 1 \quad t_1 \quad t_1^2 \quad ... \quad t_1^{m_1} \quad 0 \quad ... \quad 0)X = \theta_1 \tag{5}$$

Equation for matching velocity is

$$0 + a_{01} + 2a_{02}t_1 + ... + m_0 a_{0m_0} t_1^{m_0-1} = 0 + a_{11} + 2a_{12}t_1 + ... + m_1 a_{1m_1} t_1^{m_1-1}$$

or,

$$(0 \quad 1 \quad 2t_1 \quad ... \quad m_0 t_1^{m_0-1} \quad 0 \quad -1 \quad -2t_1 \quad ... \quad -m_1 t_1^{m_1-1} \quad 0 \quad ... \quad 0)X = 0 \tag{6}$$

Equation for matching acceleration is similar

$$(0 \quad 0 \quad 2 \quad ... \quad m_0(m_0-1)t_1^{m_0-2} \quad 0 \quad 0 \quad -2 \quad ... \quad -m_1(m_1-1)t_1^{m_1-2} \quad 0 \quad ... \quad 0)X = 0 \tag{7}$$

With equations like (1) - (7), put the vectors on the LHS as rows of a matrix, call A, and put the RHS in a vector Y. We have a linear system $AX = Y$. The polynomial coefficients X is easily solved.